

Hartshorne's Connectedness Theorem

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Abstract

The Zariski topology on a locally noetherian scheme has special features near a point whose local ring is Cohen-Macaulay; for instance, such a point cannot lie on two irreducible components of different dimensions. Hartshorne's Connectedness Theorem, which says that a variety must be locally connected in codimension 1 at a Cohen-Macaulay point (that is, removing a subvariety of codimension 2 or more cannot disconnect it), puts a further restriction on the Zariski topology near a Cohen-Macaulay point. We shall give a proof of Hartshorne's Connectedness Theorem in this article, which is based on the talk given in the Mathematics Students' Semianr at TIFR on 18th February, 2011.

1 Introduction

The most fundamental notion in geometry and topology is that of dimension. The first precise definitions of dimension were given by Brouwer in 1913 and by Urysohn and Menger independently in 1922. It is a local property, inductively defined to be the smallest n such that arbitrarily small neighbourhoods of a point have boundary of dimension less than n .

Algebraic geometers at the end of 19th century intuitively knew that dimension of an algebraic variety (which for them was a curve or a surface defined over \mathbb{C}) should be defined as the smallest number of independent parameters needed to define it. The formal definition was given by Noether in 1925. She defined the dimension of irreducible variety in an affine n -space over a field k (which for Noether was \mathbb{C}) to be the transcendence degree of the field of rational functions on the variety over k , which is just the quotient field of $k[x_1, \dots, x_n]/I$, where I is the ideal of functions vanishing on the variety. This definition agrees with the modern definition given by Krull for affine domains, but not in general because it is not local in nature. In particular, it does not work in the case of the power series ring $k[[x]]$, which represents the affine line in the local analytic case and hence, should have dimension 1, but has infinite transcendence degree. Krull came up with the following definition of dimension in 1937.

Definition 1.1. The *Krull dimension* of a ring A is defined to be the supremum of the lengths of chains of distinct prime ideals in A .

$$\dim A := \sup \{ n \in \mathbb{Z} \mid \exists \text{ a chain } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \dots \subsetneq \mathfrak{p}_n \text{ of prime ideals of } A \}.$$

This definition is *correct* because it satisfies the natural axioms a notion of dimension of a ring should satisfy. For instance, this definition is local in nature and does not change under taking completions. It is not affected by presence of nilpotents, it is preserved by a map with finite fibres and agrees with the local analytic case: $\dim(k[[x_1, \dots, x_n]]) = n$. For a nice and detailed explanation of this, we refer the reader to [1, Chapter 8].

We now state the original version of Krull's principal ideal theorem in modern language, which serves as a starting point for a way towards an algebraic approach to the notion of dimension.

Theorem 1.2 (Principal ideal theorem - General case). *Let (A, \mathfrak{m}) be a noetherian local ring. Then the Krull dimension of A is the smallest n such that there exist $f_1, \dots, f_n \in \mathfrak{m}$ such that not all of f_1, \dots, f_n are contained in any prime other than \mathfrak{m} .* \square

The elements f_1, \dots, f_n , for $n = \dim A$ as in Theorem 1.2 are said to form a *system of parameters* for A , because the closed point \mathfrak{m} in $\text{Spec } A$ is determined by vanishing of f_1, \dots, f_n , thought of as functions near the point \mathfrak{m} . Theorem 1.2 gives us a characterization of the dimension of a local ring. If one wishes to determine the dimension of a local ring, that is, the height of its maximal ideal, then finding such f_i 's is a reasonable way to going about it, as compared to the geometric way of finding chains. How does one find these f_i 's. This leads us to the concepts of regular sequences and depth, an algebraic notion parallel to the geometric notion of height. We shall introduce these in the next section and discuss *Cohen-Macaulay rings*, for which the notions of depth and height coincide.

2 Cohen-Macaulay Rings

We shall require some concepts from basic commutative algebra. We only give an outline of basic concepts and results in this section and refer the reader to Eisenbud [1] and Matsumura [4] for a detailed exposition.

Given any ideal in a noetherian ring A , a prime ideal \mathfrak{p} of A is said to be an *associated prime* of I if \mathfrak{p} is the annihilator of some element of A/I . There are only finitely many such primes associated to a given ideal I . The prime ideals of A minimal over I occur among the primes associated to I . An ideal $Q \subseteq \mathfrak{p}$ is said to be \mathfrak{p} -*primary* if $\sqrt{Q} = \mathfrak{p}$ and if for any $a, b \in A$ with $ab \in Q$ and $a \notin Q$, we have $b^n \in Q$ for some n . Any ideal I of A can be expressed as the intersection of primary ideals. Since the intersection of ideals primary to a given prime \mathfrak{p} is again \mathfrak{p} -primary, I can be expressed as an intersection of ideals that are primary to distinct primes. If this is done so that none of the associated primes of I are left out, the expression is called a *primary decomposition* of I and the primary ideals involved are called *primary components* of I . The associated primes of I are thus the radicals of the primary components. The primary component of I corresponding to a given prime \mathfrak{p} is uniquely determined by I if \mathfrak{p} is minimal over I . Such primary components are called *isolated components* and the rest are called *embedded components*. If A is a commutative noetherian ring, then primary decomposition of any ideal I exists.

Let A be a commutative local ring with maximal ideal \mathfrak{m} and M be an A -module. Let a_1, \dots, a_n be a sequence of elements in \mathfrak{m} . We say that a_1, \dots, a_n is an *M -regular sequence* in \mathfrak{m} if for each $1 \leq i \leq n$, a_i is not a zerodivisor in $M/(a_1, \dots, a_{i-1})M$, and $M \neq \sum a_i M$. If, in addition, A is noetherian, and M is a finitely generated A -module, every M -regular sequence in \mathfrak{m} can be extended to a *maximal M -regular sequence* in \mathfrak{m} . In this setting, any two maximal M -regular sequence in \mathfrak{m} have the same length. The length of the maximal M -regular sequence in \mathfrak{m} under these assumptions is called the *depth* of M and is denoted by $\text{depth}_A(M)$. If $M = A$, we just write $\text{depth } A$ for the depth of A . If $M \neq 0$, it can be proved that $\text{depth } A \leq \dim A$.

Definition 2.1. Let A be a noetherian local ring with maximal ideal \mathfrak{m} . A is said to be a *Cohen-Macaulay local ring* if $\text{depth } A = \dim A$, or if $A = 0$. A noetherian ring A is said to be a *Cohen-Macaulay ring* if $A_{\mathfrak{p}}$ is a Cohen-Macaulay local ring, for every prime ideal \mathfrak{p} of A . This is equivalent to saying that $A_{\mathfrak{m}}$ is a Cohen-Macaulay local ring, for every maximal ideal \mathfrak{m} of A .

Examples 2.2. We now state some of the examples of Cohen-Macaulay rings. A nice collection of examples with some insight into the geometric meaning of Cohen-Macaulay property can be found in Hochster [3].

1. All 0-dimensional noetherian local rings are Cohen-Macaulay.
2. All 1-dimensional noetherian local rings without nilpotents are Cohen-Macaulay.

3. All 2-dimensional normal noetherian domains are Cohen-Macaulay.
4. If A is a Cohen-Macaulay local ring and $I \subseteq A$ is a *set-theoretic complete intersection*, that is, an ideal of A of height n generated by n elements, then A/I is Cohen-Macaulay. In general, if x_1, \dots, x_n is an A -regular sequence, then $A/(x_1, \dots, x_n)$ is Cohen-Macaulay.
5. (Hochster-Roberts) If k is a field and G is a linear reductive group acting k -rationally by k -algebra automorphisms on $k[x_1, \dots, x_n]$, then the ring of invariants $k[x_1, \dots, x_n]^G$ is Cohen-Macaulay.

Let A be a noetherian ring and I an ideal of A ; let $\text{Ass}_A(A/I) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_l\}$. We say that I is *unmixed* if $\text{ht}(\mathfrak{p}_i) = \text{ht}(I)$, for all i . Note that I is unmixed if and only if A/I has no embedded primes. We say that the *unmixedness theorem holds in A* if the following is true: every ideal of height r , which is generated for r elements, is unmixed. It is not hard to see that the unmixedness theorem holds in A if it holds in $A_{\mathfrak{m}}$ for every maximal ideal \mathfrak{m} of A . The most important characterization of Cohen-Macaulay rings is the following:

Theorem 2.3. *Let A be a noetherian ring. Then A is Cohen-Macaulay if and only if the unmixedness theorem holds in A .*

Proof. See Matsumura [4, Theorem 32]. □

Theorem 2.3 was proved by Macaulay for polynomial rings and by Cohen for regular local rings; hence, the name *Cohen-Macaulay* was given to rings satisfying this property.

We now state some of the important properties of Cohen-Macaulay rings without proof. For the proofs, we refer the reader to Eisenbud [1, Chapter 18] and Matsumura [4, Chapter 6].

Theorem 2.4. *Let A be a commutative ring. For (2) – (4), assume that A is Cohen-Macaulay.*

- (1) *A is Cohen-Macaulay if and only if $A[x]$ is Cohen-Macaulay, where x is an indeterminate.*
- (2) *Cohen-Macaulay rings are universally catenary, that is, for any A -algebra B , any two maximal chains of primes between given two prime ideals of B have the same length.*
- (3) *If A is local, any two maximal chains of primes have the same length and $\text{Ass}(A)$ consists of all the minimal primes of A .*
- (4) *Every local Cohen-Macaulay ring is equidimensional (that is, all its maximal ideals have the same height and all its minimal primes have the same dimension).* □

3 Hartshorne's Connectedness Theorem

The following proposition is the key to proving the algebraic version of Hartshorne's Connectedness Theorem.

Proposition 3.1. *Let (A, \mathfrak{m}) be a noetherian local ring and let $X = \text{Spec } A$. If $X \setminus \{\mathfrak{m}\}$ is disconnected in the Zariski topology, then $\text{depth}(A) \leq 1$.*

Proof. Since $X \setminus \{\mathfrak{m}\}$ is disconnected, and since $\{\mathfrak{m}\}$ is a closed set, we can find two closed subsets $Y, Z \subseteq X$ such that $X = Y \cup Z$ and $Y \cap Z = \{\mathfrak{m}\}$. Now, write the irredundant primary decomposition of the zero ideal in A :

$$0 = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_n.$$

Let $\mathfrak{p}_i := \sqrt{\mathfrak{q}_i}$ be the associated primes of A . If \mathfrak{m} is an associated prime of A , then $\mathfrak{m} = \text{Ann}_A(x)$, for some nonzero $x \in A$; whence, it consists of zerodivisors. Therefore, $\text{depth}(A) = 0$, and we are done.

We thus assume henceforth that $\mathfrak{m} \notin \text{Ass}(A)$. Now, the associated primes \mathfrak{p}_i of A fall into two disjoint subsets, formed by those lying in Y and those lying in Z . Since all the minimal primes of A ,

which are the generic points of the irreducible components, are all in $\text{Ass}(A)$, none of these subsets is empty. Say, $\mathfrak{p}_1, \dots, \mathfrak{p}_t \in Y$ and $\mathfrak{p}_{t+1}, \dots, \mathfrak{p}_n \in Z$. Put

$$I = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_t$$

and

$$J = \mathfrak{q}_{t+1} \cap \dots \cap \mathfrak{q}_n.$$

We then have $IJ = 0$.

We claim that $\sqrt{I+J} = \mathfrak{m}$. Recall that $\sqrt{I+J}$ is the intersection of all the prime ideals containing $I+J$. If \mathfrak{p} is any prime containing $I+J$, then it follows that $\mathfrak{p} \in Y \cap Z = \{\mathfrak{m}\}$. Hence, \mathfrak{m} the only prime containing $I+J$, proving the claim.

Now, observe that $I \not\subseteq \mathfrak{p}_j$, for $t+1 \leq j \leq n$, since any prime ideal containing I must be in Y . Hence, by the prime avoidance lemma, there exists an element $x \in I$ such that $x \notin \mathfrak{p}_{t+1} \cup \dots \cup \mathfrak{p}_n$. Similarly, we can find an element $y \in J$ such that $y \notin \mathfrak{p}_1 \cup \dots \cup \mathfrak{p}_t$. Hence, $x+y \notin \mathfrak{p}_1 \cup \dots \cup \mathfrak{p}_n$, and consequently, is a nonzerodivisor in A . We next prove that $x+y$ constitutes a maximal A -sequence in A , which amounts to proving that the maximal ideal $\bar{\mathfrak{m}}$ of $\bar{A} := A/(x+y)$ consists of zerodivisors in \bar{A} .

We first note that $\bar{x} \neq 0$ in \bar{A} ; for, otherwise, $x = a(x+y)$ for some $a \in A$. Since $x \in \mathfrak{q}_i$ and $x+y \notin \mathfrak{p}_i$, for $1 \leq i \leq t$, we must have $a \in \mathfrak{q}_i$, \mathfrak{q}_i being \mathfrak{p}_i -primary. Thus, $a \in \bigcap_{i=1}^t \mathfrak{q}_i = I$. Since $IJ = 0$, we then have $(1-a)x = ay = 0$. But this is impossible, since $x \neq 0$ and $1-a$ is a unit in A , it being a local ring.

Now, let $z \in \mathfrak{m}$ be arbitrary. Since $\mathfrak{m} = \sqrt{I+J}$, there exist $a \in I$, $b \in J$ and $n \in \mathbb{N}$ (which can be chosen to be the least) such that $z^n = a + b$. Since $IJ = 0$, we obtain

$$xz^n = xa + xb = xa = xa + ya = a(x+y),$$

so $\bar{x}\bar{z}^n = 0$ in \bar{A} . Since $\bar{x}\bar{z}^{n-1} \neq 0$, we conclude that \bar{z} is a zerodivisor in \bar{A} . Therefore, it follows that $\text{depth}(A) = 1$, completing the proof. \square

Lemma 3.2. *Let (A, \mathfrak{m}) be a Cohen-Macaulay local ring. If I is a set-theoretic complete intersection of height r , then A/I is Cohen-Macaulay of dimension $\dim(A) - r$.*

Proof. Let $I = (x_1, \dots, x_r)$. We can choose a maximal regular sequence in \mathfrak{m} that starts with x_1, \dots, x_r . Hence, it follows that $\text{depth}(A/I) = \text{depth}(A) - r = \dim(A) - r$. Note that for any $1 \leq i \leq n$, x_i does not lie in any of the primes minimal over (x_1, \dots, x_{i-1}) . Hence, we have $\dim(A/I) \leq \dim(A) - n$, which in turn, implies that $\text{depth}(A/I) = \dim(A/I)$, completing the proof. \square

We are now ready to state the algebraic version of Hartshorne's Connectedness Theorem.

Theorem 3.3 (Hartshorne's Connectedness Theorem - Algebraic Version). *Let (A, \mathfrak{m}) be a Cohen-Macaulay local ring. Let $V = V(I)$ be a set-theoretic complete intersection of dimension greater than 1 in $\text{Spec } A$. Then $V \setminus \{\mathfrak{m}\}$ is connected in the Zariski topology.*

Proof. Since I is a set-theoretic complete intersection of dimension > 1 , A/I is Cohen-Macaulay of dimension > 1 , by Lemma 3.2. Now, it follows immediately from Proposition 3.1 that $V \setminus \{\mathfrak{m}\}$ is connected. \square

There is also a geometric version of Hartshorne's connectedness theorem, which is intuitively more enlightening. Before we discuss the geometric version, we state a lemma from point-set topology, which gives us a way to determine connectedness of the space after removing a closed subspace from the connectedness of certain local spaces.

Lemma 3.4. *Let X be a connected topological space and Y be a closed subspace of X . For every $y \in Y$, let*

$$X^y := \{x \in X \mid y \in \{x\}^-\}.$$

If $X^y \setminus \{y\}$ is nonempty and connected for all y , then $X - Y$ is connected.

Proof. See [2, Lemma 1.2]. □

We now prove the geometric version of Hartshorne's connectedness theorem.

Theorem 3.5 (Hartshorne's Connectedness Theorem - Geometric Version). *Let X be a connected locally noetherian scheme. Let Y be a closed subset of X such that for every $y \in Y$, the local ring $\mathcal{O}_{X,y}$ has depth at least 2. Then $X \setminus Y$ is connected.*

Proof. We first observe that $X^y = \{x \in X \mid y \in \{x\}^- \}$ is the same as $\text{Spec } \mathcal{O}_{y,X}$. In view of Lemma 3.4, it suffices to prove that for every $y \in Y$, $\text{Spec } \mathcal{O}_{y,X} \setminus \{\mathfrak{p}_y\}$ is nonempty and connected, where \mathfrak{p}_y is the maximal ideal of $\text{Spec } \mathcal{O}_{y,X}$. But this is immediate from Theorem 3.3. Clearly, $\text{Spec } \mathcal{O}_{y,X} \setminus \{\mathfrak{p}_y\}$ is nonempty, since $\dim \mathcal{O}_{X,y} \geq \text{depth}(\mathcal{O}_{X,y}) \geq 2$. This completes the proof. □

References

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