Canonical Dimension of a Quadric

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Abstract

A quadric is the zero-set of a quadratic form in a suitable projective space. Starting with a brief introduction to the algebraic theory of quadratic forms, we shall introduce the *canonical dimension* of a scheme, which behaves better than the classical dimension with respect to splitting properties. We shall then discuss an algebraic description of the canonical dimension of a quadric.

These are the notes based on a talk given in the Mathematics Students' Seminar at TIFR on 29th October, 2010.

1 Quadratic Forms

In this section, we give an outline of results in quadratic forms refer the reader to the books of Lam [7] and Elman-Karpenko-Merkurjev [2] for a detailed exposition.

Definition 1.1. Let V be a vector space over F. A *quadratic form* on V is a map $\varphi : V \to F$ satisfying
(1) $\varphi(av) = a^2\varphi(v)$, for all $v \in V$ and for all $a \in F$; and
(2) (Polar identity) $B_\varphi : V \times V \to F$ defined by

$$B_\varphi(v, w) = \varphi(v + w) - \varphi(v) - \varphi(w)$$

is a bilinear form.

The bilinear form B_φ is called the *polar form* of φ. We call $\dim_F(V)$ the *dimension* of the quadratic form φ and also write it as $\dim(\varphi)$. A quadratic form of dimension n is said to be *n-ary*.

Let φ be a quadratic form on a vector space V of dimension n. Fix a basis $\{v_1, \dots, v_n\}$ of V. If we put $a_{ii} := \varphi(v_i)$ and

$$a_{ij} := \begin{cases} B_\varphi(v_i, v_j), \text{ if } i < j; \\ 0, \text{ if } i > j, \end{cases}$$

we see that

$$\varphi\left(\sum_{i=1}^{n} x_i v_i\right) = \sum_{i,j} a_{ij} x_i x_j,$$

for all $x_i \in F$. Note that the scalars a_{ij} completely determine φ. Thus, an n-ary quadratic form over a field F can be regarded as a homogeneous polynomial of degree 2 in n variables over F. If M_φ denotes the matrix (a_{ij}), then we have

$$\varphi(x_1, \dots, x_n) = x^t M_\varphi x,$$

where x stands for $(x_1, \dots, x_n)$, viewed as a column vector. In case M_φ is a diagonal matrix with diagonal $(a_1, \dots, a_n)$, we abbreviate the form φ by $\langle a_1, \dots, a_n \rangle$.

Definition 1.2. Two n-ary quadratic forms φ and ψ are said to be equivalent if there exists a nonsingular linear transformation $A \in GL_n(F)$ such that $\varphi(x) = \psi(Ax)$ for all $x \in F^n$; in such a case, we write $\varphi \cong \psi$.

Equivalence of two quadratic forms is clearly an equivalence relation. Note that two quadratic forms φ and ψ are equivalent if and only if $M_\psi = A^t M_\varphi A$, for some $A \in GL_n(F)$.

If φ is a quadratic form over F, then we can naturally regard it as quadratic form over any field extension K of F, and denote it by φ_K.

Definition 1.3. Let φ be a quadratic form on an F-vector space V. Then the *radical* of φ is defined to be

$$\operatorname{rad}\varphi := \{v \in \operatorname{rad} B_\varphi \mid \varphi(v) = 0\},$$

where $\operatorname{rad} B_\varphi$, the radical of B_φ, is the subspace $\{v \in V \mid B_\varphi(v, w) = 0, \text{ for all } w \in V\}$. The quadratic form φ is said to be *regular* if $\operatorname{rad}\varphi = 0$.

Definition 1.4. A quadratic form φ on V is said to be *isotropic* if there exists a nonzero vector $v \in V$ such that $\varphi(v) = 0$. φ is called *anisotropic* if it is not isotropic.

Note that every anisotropic quadratic form is regular.

Definition 1.5. A quadratic form φ over F is said to be *nondegenerate* if $\varphi_{\bar{F}}$ is regular, where $\bar{F}$ is an algebraic closure of F.

The definition the notion of nondegeneracy for a quadratic form is motivated by the following lemma.

Lemma 1.6. *Let φ be a quadratic form over F. Then the following are equivalent:*
(1) *φ_K is regular for every field extesion K/F.*
(2) *$\varphi_{\bar{F}}$ is regular, where $\bar{F}$ denotes an algebraic closure of F.*
(3) *φ is regular and* $\dim_F \operatorname{rad} B_{\operatorname{rad} B_\varphi} = 1$.

Proof. See [2, Lemma 7.16]. □

There are two basic operations on quadratic forms. If φ is an n-ary quadratic form and ψ an m-ary quadratic form, then we define their *orthogonal direct sum* $\varphi \oplus \psi$ to be the $(n+m)$-ary quadratic form associated with the matrix

$$M_{\varphi\oplus\psi} := \begin{pmatrix} M_\varphi & 0 \\ 0 & M_\psi \end{pmatrix}.$$

The *tensor product* of φ and ψ is defined to be the the (nm)-ary quadratic form $\varphi \otimes \psi$ associated with the matrix

$$M_{\varphi\otimes\psi} := \begin{pmatrix} a_{11}M_\psi & a_{12}M_\psi & \cdots & a_{1n}M_\psi \\ a_{21}M_\psi & a_{22}M_\psi & \cdots & a_{2n}M_\psi \\ \vdots & \vdots & \ddots & M_{q'} \\ a_{n1}M_\psi & a_{n2}M_\psi & \cdots & a_{nn}M_\psi \end{pmatrix}, \text{ where } M_\varphi = (a_{ij}).$$

Thus, $M_{\varphi\otimes\psi}$ is just the *Kronecker product* of matrices M_φ and M_ψ.

It is not difficult to verify that the operations $\oplus$ and $\otimes$ are preserved under equivalence of quadratic forms. This makes the set of equivalence classes of quadratic forms into a commutative semiring. The *Grothendieck completion* of this semiring, obtained by adding the additive inverses of elements to the semiring *formally* is a ring, called the *Grothendieck-Witt ring* of F and denoted by $\widehat{W}(F)$. The quotient ring of $\widehat{W}(F)$ by the hyperbolic forms is a ring with the form $\langle 1, -1\rangle$ serving as the zero and the form $\langle 1\rangle$ as the unity, called the *Witt ring* of F.

Definition 1.7. A quadratic form of the type $\langle 1, -1\rangle \oplus \cdots \oplus \langle 1, -1\rangle$ is said to be *hyperbolic* and is denoted by $\mathbb{H}$. Up to isometry, $\mathbb{H}$ is the unique nondegenerate isotropic quadratic form on a two-dimensional F-vector space.

We end this section by stating the Witt decomposition theorem, which gives a decomposition of a quadratic form into quadratic forms of special types defined above, making it easier to study.

Theorem 1.8 (Witt decomposition theorem)**.** *Let φ be a quadratic form on a vector space V over F. Then there exist subspaces V_1 and V_2 of V such that there is an orthogonal deomposition*

$$\varphi \simeq \varphi|_{\operatorname{rad}\varphi} \oplus \varphi|_{V_1} \oplus \varphi|_{V_2}$$

where $\varphi|_{V_1}$ is anosotropic and $\varphi|_{V_2}$ is hyperbolic, that is, $\varphi|_{V_2} \simeq n\mathbb{H}$, for some n. Moreover, $\varphi|_{V_1}$ and $\varphi|_{V_2}$ are unique up to isometry.

Proof. See [2, Theorem 8.5]. □

2 The Projective Quadric of a Quadratic Form

Let φ be a quadratic form on a vector space V over F. Viewing φ as an element of $\operatorname{Sym}^2(V^*)$, we define the *projective quadric associated to φ* to be the closed subscheme

$$X_\varphi := \operatorname{Proj}(\operatorname{Sym}(V^*)/(\varphi))$$

of the projective space $\mathbb{P}(V) = \operatorname{Proj}(\operatorname{Sym}(V^*)$. If $\varphi \neq 0$ and $\dim V \geq 2$, then the scheme X_φ is equidimensional of dimension $\dim V - 2$. By construction, for any field extension L/F, the set of L-valued points of X corresponds to the set of isotropic lines in $V_L := V \otimes_F L$.

Proposition 2.1. *Let φ be a nonzero quadratic form of dimension at least 2. Then the quadric X_φ is smooth if and only if φ is nondegenerate.*

Proof. (Taken from [2, Proposition 22.1].) Since a variety over F is smooth if and only if it is smooth over an algebraic closure of F, we may assume that F is algebraically closed. Let p be an F-rational point of X_φ. Then p corresponds to an isotropic line $L = Fu \subseteq V$. Let $F[\varepsilon]/(\varepsilon^2)$ be the ring of dual numbers over F. Recall that (see Appendix B) a tangent vector to X at p is represented by a vector of the form $u + \varepsilon v$, where $v \in V$ (modulo the equivalence relation that identifies all the scalar multiples of a vector), satisfying $\tilde{\varphi}(u + \varepsilon v) = 0$, where $\tilde{\varphi}$ is the extension of φ to $V[\varepsilon] := V \otimes_F F[\varepsilon]/(\varepsilon^2)$. Now, since $\varepsilon^2 = 0$, $\tilde{\varphi}(u+\varepsilon v) = B_{\tilde{\varphi}}(u, \varepsilon v) + \tilde{\varphi}(v) + \tilde{\varphi}(\varepsilon v) = 0$ if and only if $B_{\tilde{\varphi}}(u, \varepsilon v) = 0$, that is, $B_{\tilde{\varphi}}(u, v) = 0$. Thus, $\tilde{\varphi}(u + \varepsilon v) = 0$ if and only if $v \in L^\perp$. Therefore, the tangent space $T_p(X)$ to X at p is the subspace $\operatorname{Hom}_F(L, L^\perp/L)$ of the tangent space $T_p(\mathbb{P}(V)) = \operatorname{Hom}(L, V/L)$. Now, the point p is nonsingular if and only if $\dim T_p(X) = \dim X = \dim V - 2$, which holds if and only if $L^\perp \neq V$, that is, if and only if $L \not\subseteq \operatorname{rad}\varphi$. Consequently, X_φ is smooth if and only if $\operatorname{rad}\varphi = 0$, which is the case here since φ is nondegenerate. □

Remark 2.2. Note that the proof of Proposition 2.1 also implies that the singular locus of the quadric X_φ is $\mathbb{P}(\operatorname{rad}\varphi)$.

We say that the quadratic form φ on V is irreducible if φ is irreducible as an element of $\mathrm{Sym}(V^*)$. If φ is irreducible, X_φ is an integral scheme. The function field $F(X_\varphi)$ of X_φ is called the *function field of* φ and is denoted by $F(\varphi)$. By definition, $F(\varphi)$ is the subfield of degree zero elements in the quotient field of the integral domain $\mathrm{Sym}(V^*)/(\varphi)$.

The quotient field of $\mathrm{Sym}(V^*)/(\varphi)$ is a purely transcendental extension of $F(\varphi)$ of transcendence degree 1. Since φ is isotropic over the quotient field of $\mathrm{Sym}(V^*)/(\varphi)$, it is isotropic over $F(\varphi)$.

We now state two of the important properties of the projective quadric associated to a quadratic form φ.

Proposition 2.3. *Let φ be an irreducible regular quadratic form over F. Then $F(\varphi)$ is a purely transcendental extension of F if and only if φ is isotropic.*

Proof. See [2, Propostion 22.9] □

Proposition 2.4. *Let φ be an anisotropic quadratic form over F. Let K/F be a finite field extension of odd degree. Then φ_K is anisotropic. In other words, every closed point of X_φ is of even degree.*

Proof. See [2, Corollary 18.5] □

Witt indices

Let φ be a quadratic form over F. Then by Theorem 1.8, there exists a unique positive integer n_φ such that

$$\varphi \simeq \varphi|_{\mathrm{rad}\,\varphi} \oplus \varphi_1 \oplus n_\varphi \mathbb{H}. \qquad (*)$$

where φ_1 is anisotropic.

Definition 2.5. Given a quadratic form φ, the unique positive integer n_φ described above in $(*)$ is called the 0^{th} *Witt index* of φ and is denoted by $i_0(\varphi)$. The anisotropic quadratic form φ_1 in $(*)$ is isotropic over the function field $F(\varphi)$ of φ. The *first Witt index* of φ is the defined to be

$$i_1(\varphi) := i_0({\varphi_1}_{F(\varphi)}).$$

Note that the first Witt index is well defined since φ_1 is unique up to isometry. If X_φ is the projective quadric associated to φ, then we define the 0^{th} and the *first Witt indices* of X_φ to be $i_0(\varphi)$ and $i_1(\varphi)$ respectively, and denote them by $i_0(X_\varphi)$ and $i_1(X_\varphi)$ respectively.

The following lemma gives an alternate description of the first Witt index. It also gives us a relation between the 0^{th} and first witt indices of a quadratic form.

Lemma 2.6. *Let φ be an anisotropic nondegenerate quadratic form over F of dimension ≥ 2. Then*
(1) $i_1(\varphi) = \min\ \{\ i_0(\varphi_E) \mid E$is a field extension of Fsuch that $\varphi_E$is isotropic$\}$.
(2) *Let ψ be a nondegenerate subform of φ of codimension r and let E/F be a field extension. Then $i_0(\psi_E) \geq i_0(\varphi_E) - r$. In particular, we have $i_1(\psi) \geq i_1(\varphi) - r$.*

Proof. See [2, Lemma 74.1]. □

Definition 2.7. The *Izhboldin dimension* of a quadratic form φ over F, denoted by $\dim_{\mathrm{Izh}}(\varphi)$, is defined to be $\dim(\varphi) - i_1(\varphi) + 1$.

3 Places

Let K be a field. A *valuation ring* R of K is a subring $R \subseteq K$ such that for every $x \in K \backslash R$, we have $x^{-1} \in R$. Thus, a valuation ring is a local domain.

Definition 3.1. Given two fields K and L, a *place*

$$\pi : K \rightharpoonup L$$

is a local homomorphism $f : R \to L$ of a valuation ring R of K. In this case, we say that the place π is defined over R. The place $\pi : K \rightharpoonup L$ is said to be *surjective* if f is surjective.

Examples 3.2. (1) An inclusion of fields is clearly an example of a place.
(2) If R is a valuation ring of a field K with maximal ideal $\mathfrak{m}$ and residue field L, then the quotient map $R \to R/\mathfrak{m} = L$ determines a place $K \rightharpoonup L$.

If K and L are extensions of a field F, then we say that a place $\pi : K \rightharpoonup L$ is an *F-place* if π is defined on F and is the identity on F.

Let $K \rightharpoonup L$ and $L \rightharpoonup E$ be two places, given by local homomorphisms $f : R \to L$ and $g : S \to E$ respectively, where $R \subseteq K$ and $S \subseteq L$ are valuation rings. Then $T = f^{-1}(S)$ is a valuation ring of K and the composition $f|_T \circ g : T \to S \to E$ defines the *composition place* $K \rightharpoonup E$. In particular, any place $L \rightharpoonup E$ can be restricted to a subfield K of L. It can be similarly verified that composition of F-places is an F-place.

Definition 3.3. A place $K \rightharpoonup L$ is said to be *geometric* if it is a composition of finitely many places, each of which is either defined on a discrete valuation ring or given by a field embedding.

We now obtain a way to relate the concepts of the L-valued points of a smooth projective variety over F and the F-places of the function field of the variety in L.

Let Y be a complete variety over F and let $\pi : F(Y) \rightharpoonup L$ be an F-place defined by a local homomorphism $f : R \to L$. By definition, the structure morphism $Y \to \operatorname{Spec} F$ is proper. Let η be the generic point of Y, which is an $F(Y)$-valued point of Y. This gives a canonical morphism $\operatorname{Spec} F(Y) \to Y$. By the valuative criterion of properness ([4, Theorem 4.7]), there exists a unique map $h : \operatorname{Spec} R \to Y$ making the following diagram commute:

$$\begin{array}{ccc} \operatorname{Spec} F(Y) & \longrightarrow & Y \\ \downarrow & \overset{\exists\, !\, h}{\nearrow} & \downarrow \\ \operatorname{Spec} R & \longrightarrow & \operatorname{Spec} F \end{array}$$

Thus, R dominates a unique point $y \in Y$ (which is a specialization of η, the image of $\operatorname{Spec} F(Y)$), that is, the local ring $\mathcal{O}_{Y,y}$ is contained in R with its maximal ideal $\mathfrak{m}_{Y,y}$ contained in the unique maximal ideal $\mathfrak{m}$ of R. The induced homomorphism of fields

$$F(y) = \mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y} \to R/\mathfrak{m} \xrightarrow{\bar{f}} L$$

gives an L-valued point of Y. The uniquely determined point y is called the *centre* of the place π.

Let X be a nonsingular (smooth) variety over F and let x be an L-valued point of X. That is, there is a morphism $f : \operatorname{Spec} L \to X$ of schemes over F with image $\{x\}$. Since X is nonsingular, $\mathcal{O}_{X,x}$

is a regular local ring.Choose a regular system of parameters $a_1, \ldots, a_n$ in the local ring $R = \mathscr{O}_{X,x}$. $\{a_1, \ldots, a_n\}$ is a minimal set of generators for the maximal ideal $\mathfrak{m}_{X,x}$. Put $\mathfrak{m}_i = (a_1, \ldots, a_n)$, $R_i = R/\mathfrak{m}_i$ and $\mathfrak{p}_i = \mathfrak{m}_{i+1}/\mathfrak{m}_i$, for each i. Since R is regular local, so is R_i. Consequently, each R_i is a UFD. Let F_i denote the quotient field of R_i. Note, in particular, that $F_0 = F(X)$ and $F_n = F(x)$. Now, the localization $(R_i)_{\mathfrak{p}_i}$ is a one-dimensional regular local ring, and hence, is a discrete bvaluation ring. Moreover, $(R_i)_{\mathfrak{p}_i}$ has quotient field F_i and residue field F_{i+1}. This determines a place $F_i \rightharpoonup F_{i+1}$. Therefore, the composition

$$F(X) = F_0 \rightharpoonup F_1 \rightharpoonup \cdots \rightharpoonup F_n = F(x) \hookrightarrow L$$

gives a geometric place constructed out of the L-valued point x of X. Note that this construction is noncanonical as it involves a choice of a regular system of parameters.

Since projective varieties are complete, we can summerize the discussion in the preceeding two paragraphs as follows:

Lemma 3.4. *If X is a smooth projective variety over F, then an L-valued point of X gives an F-place $F(X) \rightharpoonup L$, and vice versa.* □

4 Canonical Dimension

Let F be a field and $\mathscr{C}$ be a class of field extensions of F. $E \in \mathscr{C}$ is said to be *generic* for the class $\mathscr{C}$ if for any $L \in \mathscr{C}$, there exists an F-place $E \rightharpoonup L$.

Let X be a scheme over F. A field extension L of F is called an *isotropy field* of X if X has an L-valued point, that is, $X(L) = \mathrm{Hom}_{\mathrm{Sch}/F}(\mathrm{Spec}\, L, X) \neq \emptyset$. With this terminology, Lemma 3.4 can be interpreted as follows: *if X is a smooth projective variety, then its function field $F(X)$ is generic in the class of all isotropy fields of X.*

Definition 4.1. Let F be a field, X be a scheme over F and $\mathscr{C}_X$ be the class of isotropy fields of X. The *canonical dimension* of X is defined to be the minimum of the transcendence degrees over the generic isotropy fields of X and is denoted by cdim(X).

$$\mathrm{cdim}(X) := \min\ \{\mathrm{tr.deg}_F(E) \mid E \text{ is generic in the class } \mathscr{C}_X\}.$$

If X is a smooth projective variety over F, then by Lemma 3.4, we have

$$\mathrm{cdim}(X) \leq \mathrm{tr.deg}_F(F(X)) = \dim(X).$$

Let p be a prime and let $\mathscr{C}$ be a class of field extensions of F. $E \in \mathscr{C}$ is said to be *p-generic* for the class $\mathscr{C}$ if for any $L \in \mathscr{C}$, there exists an F-place $E \rightharpoonup L'$ for some finite extension L'/L of degree coprime to p. The *canonical p-dimension*, $\mathrm{cdim}_p(X)$, of a scheme X over F is then defined similarly;

$$\mathrm{cdim}_p(X) := \min\ \{\mathrm{tr.deg}_F(E) \mid E \text{ is } p\text{-generic in the class } \mathscr{C}_X\}.$$

We are now ready to state the main theorem.

Theorem 4.2. *For an arbitrary anisotropic smooth projective quadric $X = X_\varphi$, we have*

$$\mathrm{cdim}_2(X) = \mathrm{cdim}(X) = \dim_{\mathrm{Izh}}(X).$$

Proof. We shall divide the proof into several steps, each describing a key idea. We shall use several results, which are not developed above.

Step 1. There exists a smooth subquadric Y of X of dimension $\dim_{\mathrm{Izh}}(X)$.

Let V' be a linear subspace of V of codimension $i_1(X) - 1$. Let $\psi := \varphi|_{V'}$. Then ψ is a subform of φ of codimension $i_1(X) - 1$, which is anisotropic since φ is anisotropic. Therefore, the quadric $Y := X_\psi$ is smooth by Proposition 2.1 and satisfies $\dim(Y) = \dim_{\mathrm{Izh}}(X)$.

Step 2. For the smooth subquadric Y as above, $i_1(Y) = 1$.

This follows immediately from [2, Corollary 74.3].

Step 3. $F(Y)$ is a generic isotropy field of X.

Since the subquadric Y of X has an $F(Y)$-rational point, so does X. Thus, $F(Y)$ is an isotropy field of X. If L is any isotropy field of X, then we have $i_0(\varphi_L) > 0$. By Lemma 2.6, we see that

$$\begin{aligned} i_0(\psi_L) &\geq i_0(\varphi_L) - \mathrm{codim}(\psi) \\ &\geq i_1(\varphi) - \mathrm{codim}(\psi) \\ &= 1. \end{aligned}$$

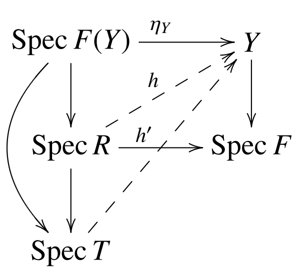
Therefore, Y has an L-valued point. By the discussion in Section 3, this gives rise to an F-place $F(Y) \rightharpoonup L$ and proves that $F(Y)$ is a generic isotropy field of X.

Step 4. We conclude from Step 3 that

$$\mathrm{cdim}_2(X) \leq \mathrm{cdim}(X) \leq \mathrm{tr.deg}_F F(Y) = \dim(Y) = \dim_{\mathrm{Izh}}(X).$$

Hence, to prove the theorem, it suffices to prove that for any 2-generic isotropy field E of X, we have $\mathrm{tr.deg}_F E \geq \dim(Y)$.

Step 5. Since $F(Y)$ is a generic isotropy field of X and E is a 2-generic isotropy field for X, there exist F-places $\pi : F(Y) \rightharpoonup E$ and $\varepsilon : E \rightharpoonup E'$, for some finite extension $E'/F(Y)$ of odd degree. Let y and y' be the centres of the F-places π and $\varepsilon \circ \pi$ respectively. Suppose that π and $\varepsilon \circ \pi$ are given by local homomorphisms $f : R \to E$ and $g : S \to E'$ respectively, where $R \subseteq F(Y)$ and $S \subseteq E$ are valuation rings. Then the place $\varepsilon \circ \pi$ is defined on the valuation ring $T = f^{-1}(S)$ of $F(Y)$, contained in R. We thus have the following commutative diagram:



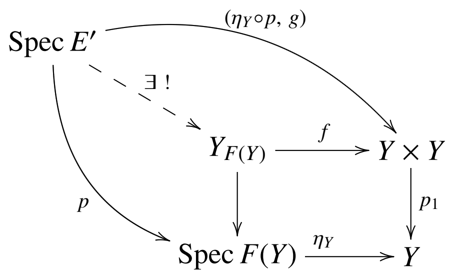
Here the three morphisms on the left are all induced by inclusions, the arrows $\mathrm{Spec}\, R \to \mathrm{Spec}\, F$ and $Y \to \mathrm{Spec}\, F$ are the structure morphisms, $\eta_Y : \mathrm{Spec}\, F(Y) \to Y$ is given by the generic point η of Y, and the dotted arrows are the unique morphisms obtained by the valuative criterion for properness as in Section 3. Note that y is the image of the maximal ideal $\mathfrak{m}_R$ of R under h, whereas y' is the image of the maximal ideal $\mathfrak{m}_T$ of T under h'. Since $\mathfrak{m}_R \cap T \subseteq \mathfrak{m}_T$, it follows that y' is a specialization of y, that is, $y' \in \overline{\{y\}}$. Moreover, since $F(y) \hookrightarrow E$, we have

$$\dim y' \leq \dim y \leq \mathrm{tr.deg}_F E.$$

We shall prove that y' is the generic point of Y, which will complete the proof, in view of Step 4.

Step 6. There exists a prime correspondence $\delta : Y \rightsquigarrow Y$ with odd multiplicity such that $(p_2)_*(\delta) = [y']$, where $p_2 : Y \times Y \to Y$ is the second projection.

Observe that y' is an E'-valued point of Y, so it induces a morphism $g : \operatorname{Spec} E' \to Y$. The odd degree field extension $F(Y) \hookrightarrow E'$ induces a morphism $p : \operatorname{Spec} E' \to \operatorname{Spec} F(Y)$. The morphisms $\eta_Y \circ p : \operatorname{Spec} F(Y) \to Y$ and $g : \operatorname{Spec} Y \to Y$, making the digram below commute, give rise to morphism $(\eta_Y \circ p,\ g) : \operatorname{Spec} E' \to Y \times Y$. Therefore, by the universal property of the fibred product, there exists a uque morphism $\operatorname{Spec} E' \to Y_{F(Y)}$ making the diagram below commute.



This gives an E'-valued point of $Y_{F(Y)}$, which is a 0-cycle on $Y_{F(Y)}$ of odd degree, since $[E' : F(Y)]$ is odd. By Corollay A.4, this gives a correspondence $\delta : X \rightsquigarrow X$ of odd multiplicity. Moreover, it follows from the proof of Proposition A.3 that the pushforward of δ under the second projection $p_2 : Y \times Y \to Y$ is $(p_2)_*(\delta) = [y']$.

Step 7. Since $(p_2)_*(\delta) = [y']$, we have $(p_1)_*(\delta^t) = \operatorname{mult}(\delta^t)[y']$. Since $\operatorname{mult}(\delta)$ is odd, by Theorem A.5, $\operatorname{mult}(\delta^t)$ is odd; in particular, it is nonzero. Therefore, we conclude that y' is the generic point of Y. This completes the proof of the theorem, in view of Step 5. □

We end this section with a short discussion of a geometric way of looking at the canonical dimension of a quadric, following [6]. Recall Springer's theorem (Theorem 2.4), which says that all the closed points on an anisotropic quadric have even degree. Also recall from elementary algebraic geometry that if X and Y are varieties and if Y has an $F(X)$-rational point, then there exists a rational map $X \dashrightarrow Y$. One can then consider the following question: What is the minimal dimension of a complete variety Y, whose all closed points are of even degree, to which X can be compressed rationally, that is, there is a rational map $X \dashrightarrow Y$?

Let $X = X_\varphi$, where φ is a nondegenerate, anisotropic quadratic form on V. We know that $\varphi_{F(X)} = \varphi' \oplus i_1(X) \cdot \mathbb{H}_{F(X)}$, where φ' is an anisotropic quadratic form over $F(X)$. Let V' be a linear subspace of V of codimension $i_1(X) - 1$. Now, $\psi := \varphi|_{V'}$ is an anisotropic quadratic form over F, which becomes isotropic over $F(X)$. This is because $V' \otimes_F F(X)$ has to intersect a maximal isotropic subspace of $V \otimes_F F(X)$ (which has dimension $i_1(X)$). Hence, there exists a rational map $X \dashrightarrow X_\psi$. Note that all the closed points of X_ψ have even degree and that $\dim X_\psi = \dim X - i_1(X) + 1 = \dim_{\mathrm{Izh}}(X) = \operatorname{cdim}(X)$. Karpenko and Merkurjev ([6]) prove that $\operatorname{cdim}(X)$ is the minimal dimension of a complete variety Y, whose all closed points are of even degree, to which X can be compressed rationally.

Theorem 4.3. *Let X be an anisotropic projective quadric over F and let Y be a complete variety over F with all closed points of even degree. If Y has a closed point of odd degree over $F(X)$, then*

$$\dim_{\mathrm{Izh}}(X) \leq \dim(Y).$$

Moreover, if $\dim_{\mathrm{Izh}}(X) = \dim(Y)$*, then X is isotropic over $F(Y)$.*

Proof. See [6, Theorem 3.1]. □

Corollary 4.4. [6, Corollary 3.4] *Let X be an anisotropic projective quadric over F and let Y be a complete variety over F with all closed points of even degree. If* $\dim_{\mathrm{Izh}}(X) > \dim(Y)$*, then there are no rational morphisms $X \dashrightarrow Y$.* □

Application of Theorem 4.3 in the case Y is also a projective quadric yields the following.

Theorem 4.5. *Let X and Y be anisotropic projective quadrics over F and suppose that Y has a closed point of over $F(X)$, then*

$$\dim_{\mathrm{Izh}}(X) \leq \dim_{\mathrm{Izh}}(Y).$$

Moreover, $\dim_{\mathrm{Izh}}(X) = \dim_{\mathrm{Izh}}(Y)$ *if and only if X is isotropic over $F(Y)$.* □

Appendices

A Correspondences

Definition A.1. Let X and Y be schemes over a field F. Let $d = \dim X$. A *correspondence* $\alpha : X \rightsquigarrow Y$ *of degree* 0 is an element of the Chow group $\mathrm{CH}_d(X \times Y)$ of cycles of dimension d modulo rational equivalence. α is said to be *prime* if it is represented by a prime cycle, that is, a d-dimensional subvariety of $X \times Y$.

Thus, every correspondence $X \rightsquigarrow Y$ is a linear combination of prime correspondences $X \rightsquigarrow Y$ with integer coefficients.

Definition A.2. Let X and Y be varieties over F and let Y be complete. Then the first projection $p_X : X \times Y \to X$ is proper, so the pushforward homomorphism $(p_X)_* : \mathrm{CH}_d(X \times Y) \to \mathrm{CH}_d(X) \simeq \mathbb{Z} \cdot [X]$ is defined. For a correespondence $\alpha : X \rightsquigarrow Y$, the integer $\mathrm{mult}(\alpha)$ satisfying

$$(p_X)_*(\alpha) = \mathrm{mult}(\alpha) \cdot [X]$$

is called the *multiplicity* of α.

The fibre of the first projection $p_X : X \times Y \to X$ over the generic point η of X is called the *generic fibre* of p_X, and is denoted by $Y_{F(X)}$. Thus, $Y_{F(X)}$ is the fibre product $\mathrm{Spec}\, F(X) \times_X (X \times Y)$:

$$\begin{array}{ccc} Y_{F(X)} & \xrightarrow{\ i\ } & X \times Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_X} \\ \mathrm{Spec}\, F(X) & \xrightarrow{\ \eta_X\ } & X \end{array}$$

Observe that the morphism $\eta_X : \mathrm{Spec}\, F(X) \to X$ giving the generic point is flat. This immedietly follows follows from the fact that the inclusion of any integral domain in its quotient field is a flat map. Hence, its base change $i : Y_{F(X)} \to X \times Y$ is also flat. Also, since p_X is proper, its base change p is proper.

Proposition A.3. *Let X be a variety and Y be a complete scheme over F. Let* $\dim X = d$*. Let $Y_{F(X)}$ denote the generic fibre of the first projection $p_X : X \times Y \to X$.*

$$\begin{array}{ccc} Y_{F(X)} & \xrightarrow{\ i\ } & X \times Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_X} \\ \mathrm{Spec}\, F(X) & \xrightarrow{\ \eta_X\ } & X \end{array}$$

Then the composition $\mathrm{CH}_d(X \times Y) \overset{i^*}{\to} \mathrm{CH}_0(Y_{F(X)}) \overset{\deg}{\to} \mathbb{Z}$ *is given by* $\alpha \mapsto \mathrm{mult}(\alpha)$.

Proof. Since both the vertical arrows in the above diagram are proper and both the horizontal arrows are flat, we get a commutative square (by [3, Proposition 1.7]):

$$\begin{array}{ccc} \mathrm{CH}_0(Y_{F(X)}) & \xleftarrow{\ i^*\ } & \mathrm{CH}_d(X \times Y) \\ \Big\downarrow{\scriptstyle \deg} & & \Big\downarrow{\scriptstyle (p_X)_*} \\ \mathbb{Z} \simeq \mathrm{CH}_0(\mathrm{Spec}\, F(X)) & \xleftarrow{\ (\eta_X)^*\ } & \mathrm{CH}_d(X) \end{array}$$

Therefore, for any correspondence $\alpha : X \rightsquigarrow Y$, we have

$$\deg \circ i^*(\alpha) = (\eta_X)^* \circ (p_X)_*(\alpha) = (\eta_X)^*(\mathrm{mult}(\alpha) \cdot [X]) = \mathrm{mult}(\alpha),$$

proving the proposition. □

We now use this proposition to obtain a bijection between correspondences $X \rightsquigarrow Y$ and 0-cycles on $Y_{F(X)}$, which are easier to work with.

Corollary A.4. *Let* X *be a variety and* Y *be a complete scheme over* F. *Let* $\dim X = d$. *Let* $Y_{F(X)}$ *denote the generic fibre of the first projection* $p_X : X \times Y \to X$. *Then there is a bijection between*
(1) 0*-cycles of degree* r *on* $Y_{F(X)}$*; and*
(2) *Correspondences* $X \rightsquigarrow Y$ *of multiplicity* r.

Proof. This follows from Proposition A.3, if we prove that f^* is surjective. Recall that the pullback $i* : \mathrm{CH}_d(X \times Y) \to \mathrm{CH}_0(Y_{F(X)})$ is given by the following rule:

$$i^*([y]) = \begin{cases} [y]_{Y_{F(X)}}, & \text{if } y \in Y_{F(X)}; \\ 0, & \text{if } y \notin Y_{F(X)}, \end{cases}$$

for any point $y \in X \times Y$. Now, given any of the generators $\alpha = [y] \in \mathrm{CH}_0(Y_{F(X)})$, set $\beta := [i(y)]$, so we have $i^*(\beta) = \alpha$, proving the surjectivity of i^*. □

Now, assume that X and Y are varities over F with $\dim X = \dim Y = d$. Then the natural *switch* isomorphism $X \times Y \simeq Y \times X$ induces isomorphisms on the Chow groups. The image of a correspondence $\alpha : X \rightsquigarrow Y$ under this isomorphism gives another correspondence $Y \rightsquigarrow X$, which we call the *transpose* of α and denote by α^t. In the special case where $X = Y$ is a quadric, we have the following relation between the multiplicities of α and α^t.

Theorem A.5. *Let* X *be the projective quadric associated to an anisotropic quadratic form* φ *of dimension* ≥ 3 *with* $i_1(\varphi) = 1$. *Then for any correspondence* $\alpha : X \rightsquigarrow X$, *we have*

$$\mathrm{mult}(\alpha) \equiv \mathrm{mult}(\alpha^t) \pmod 2.$$

Proof. See [5, Theorem 6.4]. □

B Tangent Space to $\mathbb{P}(V)$

Let V be a finite dimensional vector space over a field F. In this section, we give a nice description of the tangent space to the projective space $\mathbb{P}(V)$ at an F-rational point of it. A reference for the material in this appendix is [1].

Recall that for a scheme X and an F-rational point p of X, the Zariski tanget space to X at p is defined to be $T_p(X) := \mathrm{Hom}_F(\mathfrak{m}/\mathfrak{m}^2, F)$, where $\mathfrak{m} = \mathfrak{m}_{X,p}$ is the maximal ideal of the local ring $\mathscr{O}_{X,p}$ at p.

Lemma B.1. *An $F[\varepsilon]/(\varepsilon^2)$-valued point of a scheme X is equivalent to an F-rational point of X together with an element of the Zariski tangent space to X at that point.*

Proof. An $F[\varepsilon]/(\varepsilon^2)$-valued point of X is a morphism $\mathrm{Spec}\, F[\varepsilon]/(\varepsilon^2) \to X$. The F-algebra homomorphism $F[\varepsilon]/(\varepsilon^2) \to F$ given by $\varepsilon \mapsto 0$ determines a morphism $\mathrm{Spec}\, F \to \mathrm{Spec}\, F[\varepsilon]/(\varepsilon^2) \to X$, giving an F-rational point of X, say p. An extension of such a morphism to $\mathrm{Spec}\, F[\varepsilon]/(\varepsilon^2)$ involves lifting of the the F-algebra homomorphism $\mathscr{O}_{X,p} \to F = \mathscr{O}_{X,p}/\mathfrak{m}_{X,p}$ to a local homomorphism $\mathscr{O}_{X,p} \to F[\varepsilon]/(\varepsilon^2)$; thus, the image of $\mathfrak{m}_{X,p}$ is caontained on (ε). Since $\varepsilon^2 = 0$, this induces a map of F-vector spaces $\mathfrak{m}_{X,p}/\mathfrak{m}^2_{X,p} \to (\varepsilon) \cong F$, giving an element of the Zariski tangent space to X at p.

Conversely, given an F-rational point $p \in X$ and $t \in \mathrm{Hom}_F(\mathfrak{m}_{X,p}/\mathfrak{m}^2_{X,p}, F)$, consider the quotient map $\pi : \mathscr{O}_{X,p} \to \mathfrak{m}_{X,p}/\mathfrak{m}^2_{X,p} = F$. This composed with the structure map $F \to \mathscr{O}_{X,p}$ gives a splitting

$$\mathscr{O}_{X,p}/\mathfrak{m}^2_{X,p} \cong F \oplus \mathfrak{m}_{X,p}/\mathfrak{m}^2_{X,p}.$$

Define $\mathscr{O}_{X,p}/\mathfrak{m}^2_{X,p} \to F[\varepsilon]/(\varepsilon^2)$ by $\mathrm{id}_F \oplus t$ with respect to the above splitting and get an $F[\varepsilon]/(\varepsilon^2)$-valued point of X by composing it with the natural projection:

$$\mathscr{O}_{X,p} \to \mathscr{O}_{X,p}/\mathfrak{m}^2_{X,p} \to F[\varepsilon]/(\varepsilon^2).$$

□

Lemma B.2. *Let V be an $n+1$-dimensional vector space over F. Then the morphisms*

$$\mathrm{Spec}\, F[\varepsilon]/(\varepsilon^2) \to \mathbb{P}(V) \cong \mathbb{P}^{n+1}_F$$

are in one to one correspondence with the set of $(n+1)$-tuples $[\alpha_0, \ldots, \alpha_n] \in (F[\varepsilon]/(\varepsilon^2))^{n+1}$ such that at least one α_i is a unit, modulo the equivalence relation $[\alpha_0, \ldots, \alpha_n] \sim [\alpha\alpha_0, \ldots, \alpha\alpha_n]$, for all units α.

Proof. Write $\mathbb{P}(V) = \mathrm{Proj}\, F[x_0, \ldots, x_n]$ and let U_i denote the affine open set given by the complement of the hyperplane $x_i = 0$. Given an n-tuple $[\alpha_0, \ldots, \alpha_n] \in (F[\varepsilon]/(\varepsilon^2))^{n+1}$ such that at least one α_i is a unit, define a morphism $\mathrm{Spec}\, F[\varepsilon]/(\varepsilon^2) \to \mathbb{P}(V)$ by mapping $\mathrm{Spec}\, F[\varepsilon]/(\varepsilon^2)$ to U_i via the morphism corresponding to the F-algebra homomorphism

$$F[x_0/x_i, \ldots, x_n/x_i] \to F[\varepsilon]/(\varepsilon^2)$$

given by $x_j/x_i \mapsto \alpha_j/\alpha_i$. Conversely, given a morphism $\varphi : \mathrm{Spec}\, F[\varepsilon]/(\varepsilon^2) \to \mathbb{P}(V)$ of schemes over F, let p be the closed point of $\mathrm{Spec}\, F[\varepsilon]/(\varepsilon^2)$ and assume that $\varphi(p) \in U_i$. Then $\varphi^{-1}(U_i)$ is open in $\mathrm{Spec}\, F[\varepsilon]/(\varepsilon^2)$ and hence, is the whole of $\mathrm{Spec}\, F[\varepsilon]/(\varepsilon^2)$. Therefore, φ maps $\mathrm{Spec}\, F[\varepsilon]/(\varepsilon^2)$ to U_i, and consequently, is given by a map of F-algebras

$$\varphi^\sharp : F[x_0/x_i, \ldots, x_0/x_n] \to F[\varepsilon]/(\varepsilon^2).$$

Associate the $(n+1)$-tuple $[\alpha_0, \ldots, \alpha_n]$ to φ, where $\alpha_j = \varphi^\sharp(x_j/x_i)$. Note that if $varphi(p) \in U_j$, then we get an $(n+1)$-tuple $[\varphi^\sharp(x_0/x_j), \ldots, \varphi^\sharp(x_n/x_j)]$, which is equivalent to $[\alpha_0, \ldots, \alpha_n]$ as the former tuple is obtained by mutiplying the latter one by the unit $\varphi^\sharp(x_i/x_j)$. This completes the proof of the lemma. □

We now prove the main result of this appendix.

Proposition B.3. *Let V be a finite dimensional vector space over a field F and let L be a line in V. Then L corresponds to an F-valued point of the projective space $\mathbb{P}(V)$. Then the tangent space $T_{\mathbb{P}(V),L}$ to $\mathbb{P}(V)$ at L is given by* $\mathrm{Hom}_F(L, V/L)$.

Proof. We shall give a surjective F-linear map $\psi : \mathrm{Hom}_F(L, V) \to T_{\mathbb{P}(V),L}$ whose kernel is the subspace $\mathrm{Hom}_F(L, L)$ of $\mathrm{Hom}_F(L, V)$. Let $L = Fu$. Extend u to a basis $\{u, v_1, \ldots, v_n\}$ of V. Given a linear map $h \in \mathrm{Hom}_F(L, V)$, define $\psi(h)$ to be the tuple $[\alpha_0, \ldots, \alpha_n]$ in $(F[\varepsilon]/(\varepsilon^2))^{n+1}$, where $u + h(u)\varepsilon = \alpha_0 u + \sum_{i=1}^{n} \alpha_i v_i$. This corresponds to tangent vector to $\mathbb{P}(V)$ at L in view of Lemma B.1 and Lemma B.2, since α_0 is a unit. ψ is clearly surjective. Since the zero element of the tangent space $T_{\mathbb{P}(V),L}$ corresponds to $L \oplus L\varepsilon$, it is easy to see that $\ker\psi = \mathrm{Hom}_F(L, L)$. Therefore, $T_{\mathbb{P}(V),L} \simeq \mathrm{Hom}_F(L, V)/\mathrm{Hom}_F(L, L) \simeq \mathrm{Hom}_F(L, V/L)$, as desired. □

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