We give a survey of results on restriction of cohomology classes on locally symmetric spaces to smaller locally symmetric spaces; these results are closely connected with cohomological representations of semi-simple Lie groups associated with the locally symmetric spaces and we describe the connection.

If $S(\Gamma) = \Gamma \backslash X$ is an arithmetic quotient of a Hermitian symmetric domain $X$ (a connected component of a "Shimura Variety") then a natural class of subvarieties that one can construct explicitly are quotients of Hermitian subdomains by smaller arithmetic subgroups ("Shimura Subvarieties"). It is easy to see from the "homotopy version" of the Lefschetz hyperplane section theorem that these subvarieties are not intersections of hyperplane sections.

However, one may consider all the translates of these Shimura Subvarieties under Hecke operators and ask for (a cohomological version of) a weaker Lefschetz property for the collection of these Hecke translates.

In [Oda], it is shown that Hecke translates of the Jacobian of a fixed Shimura curve span the Albanese of a quotient of the unit ball in $\mathbb{C}^n$ by an arithmetic group of the group $SU(n,1)$ of automorphisms of the unit ball in $\mathbb{C}^n$. This proves a version of the Lefschetz Theorem on the injection of the cohomology to Shimura curves.

There are a number of criteria developed in recent years to determine if Hecke translates a given cohomology class on a Shimura Variety, restricts non-trivially to a given Shimura subvariety. We give a survey of these results. These results are formulated in terms of the "representation type ("$A_q"\) to which the cohomology class belongs. The criteria can be extended even to non-hermitian cases, and are expressed in terms of the compact dual of the symmetric space under consideration.

1. Notation and Statements

Fix two semi-simple algebraic groups $H$ and $G$ defined over $\mathbb{Q}$ and a morphism $j : H \to G$ of algebraic groups defined over $\mathbb{Q}$, with finite kernel. Fix a maximal compact subgroup $K_H$ of $H(\mathbb{R})$ and extend $j(K_H)$ to a maximal compact subgroup $K_\infty$ of $G(\mathbb{R})$. We have then an embedding $j : X_H \to X_G$ of the symmetric spaces.
$X_H = H(\mathbb{R})/K_H$ and $X_G = G(\mathbb{R})/K_\infty$.

If $\Gamma \subset G(\mathbb{Q})$ is a torsion-free congruence arithmetic group, then the quotient $S(\Gamma) = \Gamma \backslash X_G$ is a manifold covered by $X_G$. Denote by $\mathbb{A}_f$ the ring of finite adeles over $\mathbb{Q}$ and by $G(\mathbb{A}_f)$ the group of $\mathbb{A}_f$ rational points. The group $G(\mathbb{R})$ acts on $X_G$ and $G(\mathbb{A}_f)$ acts on the left on $G(\mathbb{A}_f)$; hence $G(\mathbb{Q}) \subset G(\mathbb{R}) \times G(\mathbb{A}_f)$ acts diagonally on $X_G \times G(\mathbb{A}_f)$. Also, $G(\mathbb{A}_f)$ acts (by right multiplication on the second factor) on $X \times G(\mathbb{A}_f)$. Hence $G(\mathbb{A}_f)$ acts on the quotient $S_G = G(\mathbb{Q}) \backslash X_G \times G(\mathbb{A}_f)$. Moreover, $S_G$ is the inverse limit $S_G(K) = S_G/K$ where $K \subset G(\mathbb{A}_f)$ is a compact open subgroup. The space $S_G(K)$ is a finite union of locally symmetric manifolds $S(\Gamma)$ for a finite set of $\Gamma$.

Denote by $H^*(S_G)$ the cohomology of $S_G$ with complex coefficients. Then (by [Rohlf]), the cohomology ring $H^*(S_G)$ is the direct limit over $K \subset G(\mathbb{A}_f)$ of the cohomology groups $H^*(S_G(K), \mathbb{C})$ on which $G(\mathbb{A}_f)$ acts via its right action on $S_G$. If $g \in G(\mathbb{A}_f)$ and $\omega \in H^*(S_G)$, then we denote by $g^* (\omega)$ the action of $g$ on $\omega$.

We have similarly the space $S_H = H(\mathbb{Q}) \backslash X_H \times H(\mathbb{A}_f)$ and a map $j : S_H \to S_G$.

We can now define the “Oda restriction map” (see [Oda])

$$Res : H^*(S_G) \to \prod_{g \in G(\mathbb{A}_f)} H^*(S_H),$$

defined by $Res(\omega) = (j^* g^* (\omega))_{g \in G(\mathbb{A}_f)}$.

In this survey we are concerned with describing the kernel of $Res$ in terms of representation theory.

If $G$ is anisotropic over $\mathbb{Q}$, then $S_G$ is compact and by the Matsushima formula we have the decomposition

$$H^*(S_G) = \oplus m(\pi) H^*(\mathfrak{g}, K_\infty, \pi_\infty) \otimes \pi_f.$$  

In this formula, $\pi = \pi_\infty \otimes \pi_f$ is a representation of the group $G(\mathbb{A}) = G(\mathbb{R}) \times G(\mathbb{A}_f)$ which occurs in $L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))$ and $\pi_\infty$ is a cohomological representation, i.e. the relative Lie algebra cohomology space $H^*(\mathfrak{g}, K_\infty, \pi_\infty) \neq 0$, where $\mathfrak{g}$ is the complexification of the Lie algebra of $G(\mathbb{R})$, and $m(\pi)$ is the multiplicity of the representation $\pi = \pi_\infty \otimes \pi_f$ of $G(\mathbb{R}) \times G(\mathbb{A}_f) \cong G(\mathbb{A})$ in $L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))$.

The representations with cohomology, of $G(\mathbb{R})$ are classified (by the work of Parthasarathy, Kumareshan, Vogan and Zuckerman) in terms of the $\theta$-stable parabolic subalgebras $\mathfrak{q}$ of the complex semi-simple Lie algebra $\mathfrak{g}$, with $\theta$ being the Cartan involution on $G(\mathbb{R})$ with respect to the maximal compact subgroup $K_\infty$. If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the associated Cartan decomposition, we have the $\theta$-stable Levi decomposition $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ of the parabolic subalgebra $\mathfrak{q}$ and the decomposition $\mathfrak{u} = \mathfrak{u} \cap \mathfrak{k} \oplus \mathfrak{u} \cap \mathfrak{p}$. Put $R = dim(\mathfrak{u} \cap \mathfrak{p})$. 


The Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a decomposition of $K_\infty$ modules. The line $\wedge^R(u \cap \mathfrak{p})$ generates an irreducible representation $V(\mathfrak{q})$ of $K_\infty$ in $\wedge^R \mathfrak{p}$.

The classification of unitary irreducible cohomological representations $\pi_\infty$ of $(\mathfrak{g}, K_\infty)$ now says that to each $\theta$-stable parabolic subalgebra $\mathfrak{q}$ as above, there exists a cohomological representation $A_\mathfrak{q}$ characterised by the property that the only irreducible $K_\infty$ representation common to $\wedge^\mathfrak{q}$ and $A_\mathfrak{q}$ is the representation $V(\mathfrak{q})$. Moreover, every cohomological representation $\pi_\infty$ is an $A_\mathfrak{q}$.

If $\omega \in H^R(S_G)$, and under the Matsushima decomposition, $\omega$ lies in the component $H^R(\mathfrak{g}, K_\infty, \pi_\infty) \otimes \pi_f m(\pi)$, where $\pi_\infty = A_\mathfrak{q}$ and $R = \dim(u \cap \mathfrak{p})$, we will then refer to $\omega$ as a strongly primitive class of type $A_\mathfrak{q}$.

Denote by $\hat{X}_G$ and $\hat{X}_H$ the compact dual symmetric spaces of $X_G$ and $X_H$. The Matsushima component corresponding to the trivial representation of $G(\mathbb{A})$ is isomorphic to $H^\ast(\hat{X}_G)$. The submanifold $\hat{X}_H$ yields a cohomology class (its fundamental class) in $H^\ast(\hat{X}_G) \subset H^\ast(S_G)$, denoted $[\hat{X}_H]$.

The Levi subgroup $L(\mathbb{C}) \subset Q(\mathbb{C}) \subset G(\mathbb{C})$ is defined over $\mathbb{R}$ and is $\theta$-stable. We have an associated map of compact symmetric spaces $\hat{X}_L \subset \hat{X}_G$, and the restriction map $\hat{\text{Res}} : H^\ast(\hat{X}_G) \rightarrow H^\ast(\hat{X}_L)$. We have then the following criterion for the non-vanishing of the Oda-restriction purely in terms of the compact dual of $X_G$ ([V1]):

**Theorem 1.** If $\omega$ is a strongly primitive cohomology class of type $A_\mathfrak{q}$ in $H^R(S_G)$, and if $\hat{\text{Res}}([\hat{X}_H]) \neq 0$ in $H^\ast(\hat{X}_L)$, then the Oda restriction of $\omega$ is non-zero.

As a corollary, we get the following result ([V1]) (conjectured by M.Harris and J-S.Li ([H-L]), and proved by them in degrees $i \leq 2$).

**Theorem 2.** If $G(\mathbb{R}) = SU(n, 1)$ and $H(\mathbb{R}) = SU(m, 1)$ up to compact factors and $j : H \rightarrow G$ induces the standard embedding of $SU(m, 1)$ in $SU(n, 1)$, then the restriction map

$$\text{Res} : H^\ast(S_G) \rightarrow \prod_{g \in G(\mathbb{A}_f)} H^\ast(S_H)$$

is injective for $i \leq m$.

The criterion of Theorem 1 is especially useful in the case when both $X_G$ and $X_H$ are Hermitian symmetric domains and the embedding $j : X_H \rightarrow X_G$ is holomorphic. Then we have the $K_\infty$-equivariant decomposition of the complexified tangent space $\mathfrak{p}$ into holomorphic and anti-holomorphic tangent spaces $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$. Similarly, for the subalgebra $\mathfrak{h}$ we have $\mathfrak{h} = \mathfrak{h} \cap \mathfrak{t} \oplus \mathfrak{h} \cap \mathfrak{p}$, and $\mathfrak{p}_H = \mathfrak{h} \cap \mathfrak{p}$ decomposes into a direct sum of $\mathfrak{p}^+_H$ and $\mathfrak{p}^-_H$.

The embedding $j$ induces a map $\mathfrak{p}^+_H \rightarrow \mathfrak{p}^+$. 
Moreover, \( u \cap p = u \cap p^+ \oplus u \cap p^- \). We have \( g = u \oplus l \oplus u^- \) where \( u^- \) is the nilradical opposite to \( u \) and stable under \( l \).

Write \( R^\pm = \dim u \cap p^\pm \). Then \( R = R^+ + R^- \). Set \( V^+(q) = \text{span of } K_\infty \text{ translates of the line } \wedge R^+(u \cap p^+) \wedge \wedge R^-(u^- \cap p^+) \) in the \( K_\infty \)-representation \( \wedge^R p^+ \).

Denote by \( E(G,H,R) \) the \( K_\infty \)-span of the subspace \( \wedge^R p^+ \). When \( X_G \) is Hermi-tian, note that \( S_G \) is a projective limit of algebraic varieties.

We have then the necessary condition ([V1]) for the non-vanishing of \( \text{Res} \):

**Theorem 3.** If \( G \) is anisotropic over \( \mathbb{Q} \) and is of Hermitian type, and if \( \omega \) is a strongly primitive class on \( S_G \) of Hodge type \((R^+,R^-)\) and of type \( A_4 \), then \( \text{Res}(\omega) \neq 0 \) provided \( V^+(q) \cap E(G,H,R) \neq 0 \).

When the class \( \omega \) is of Hodge type \( (m,0) \) (i.e. \( R = R^+ \)), this criterion is necessary and sufficient ([Cl-V]):

**Theorem 4.** If \( G \) is anisotropic over \( \mathbb{Q} \) and is of Hermitian type, and if \( \omega \) is a holomorphic form of degree \( R \) on \( S_G \) of type \( A_q \), then \( \text{Res}(\omega) \neq 0 \) if and only if \( E(G,H,R) \supset V(q) \).

The case when the cohomology classes are not of holomorphic type is more involved and this is the result of Theorem 3; however, in this case, the criterion of Theorem 3 is only proved to be sufficient.

1.1. Applications to cup-products. We take \( G/\mathbb{Q} \) as before, of Hermitian type. Replace the pair \((H,G)\) by the diagonal embedding \((G,G \times G)\). The restriction to the diagonal \( G \) of a tensor product class \( \omega_1 \otimes \omega_2 \in H^*(S_G) \otimes H^*(S_G) = H^*(S_G \times G) \) is simply the cup product, and from Theorem 3 we get

**Theorem 5.** If \( \omega_1 \) and \( \omega_2 \) are two strongly primitive classes on \( S_G \) of type \( A_{q_1} \) and \( A_{q_2} \), then for some \( g \in G(\mathbb{A}_f) \) the cup product \( g^*(\omega_1) \wedge \omega_2 \neq 0 \) if \( V^+(\mathbb{q}_1) \wedge V^+(\mathbb{q}_2) \neq 0 \subset \wedge^R p^+ \).

In case the classes are holomorphic, this is actually necessary and sufficient thanks to a result of Clozel ([Clo 2]) in Parthasarathy ([Par]) a sufficient condition for the vanishing of cup products is given.

As an application of Theorem 5, we have([V1]): if \( G/\mathbb{Q} \) is such that \( G(\mathbb{R}) = SU(n,1) \) up to compact factors, then given \( \omega_1 \in H^1(S_G) \) and \( \omega_2 \in H^2(S_G) \) (not necessarily primitive), the cup product \( g^*(\omega_1) \wedge \omega_2 \neq 0 \) for some \( g \in G(\mathbb{A}_f) \). Analogous results were proved earlier by Kudla ([Ku]).

1.2. Cycles on Shimura Varieties. The results (Theorem 3 and Theorem 1) may be used to prove some results on cycles on compact Shimura varieties ([V2] and [V3]).
Let $G/\mathbb{Q}$ be an anisotropic semi-simple group such that $G(\mathbb{R})$ is, up to compact factors, isomorphic to $SU(n, 2)$. It can be shown that $H^4(S_G)$ is a direct sum of $H^{4,0} \oplus H^{0,4}$ and $H^{2,2}$ as $\mathbb{Q}$-Hodge structures. Moreover, we may write $H^2(S_G) = H^4(\tilde{X}_G) \oplus W$ where $W$ consists of non-$G(\mathbb{A}_f)$ invariant classes. Using the foregoing criteria, one may prove that $W$ restricts injectively into $H^2(SU(2, 1)) \otimes H^2(SU(1, 2))$; one may even prove that $W$ restricts into a product of Hodge classes: $W \subset H^1(SU(1, 2)) \otimes H^1(SU(1, 2))$. Using the Lefschetz $(1,1)$ Theorem, we now get ([V2])

**Corollary 1.** All the Hodge classes in $H^{2n-2, 2n-2}(S_G)$ are generated by $G(\mathbb{A}_f)$-translates of fundamental classes of products of curves i.e. classes of the form $[C_1 \times C_2]$ where $C_1$ and $C_2$ are curves and $C_1 \times C_2$ embeds in $S_G/K$ for some compact open subgroup $K \subset G(\mathbb{A}_f)$.

The criteria of Theorem 3 and Theorem 1 can be applied to prove non-triviality of certain cycle classes as well as the occurrence of certain cohomological representations in the automorphic spectrum. The following can be shown.

**Corollary 2.** If $X_H \subset X_G$ is an embedding of Hermitian domains, there exists a holomorphic cohomology class on $S_G$ which restricts non-trivially to $S_H$ and if the centraliser of $H(\mathbb{R})$ in $G(\mathbb{R})$ is strictly larger than the centre of $G$, then the $G(\mathbb{A}_f)$-module generated by the cycle class $[S_H(\Gamma)]$ is infinite dimensional.

In particular, the existence of holomorphic automorphic representations $A_q$, implies the automorphy of $A_q$ with $A_q$ of Hodge type $(p, p)$.

Examples: (1) If $G = U(p, q)$ and $H = \prod_{1 \leq i \leq r} U(p_i, q_i)$ with $\sum p_i = p$ or $\sum q_i = q$.

(2) $G = Sp_g$ and $H = Sp_{g_1} \times \cdots \times Sp_{g_r}$, with $\sum g_i = g$.

In contrast, if these equalities are not satisfied (i.e. $\sum p_i < p$ and $\sum q_i < q$ and $\sum g_i < g$, then the cycle class $[S_H(\Gamma)]$ generates the trivial $G(\mathbb{A}_f)$-module.

These and similar computations raise the possibility that the following may have a positive answer.

**Question 1.** Given a simple Lie group $G$ defined over $\mathbb{Q}$ and a semi-simple $\mathbb{Q}$-subgroup $H$ such that the centralizer of $H$ in $G$ is non-compact, is it always the case that $[S_H(\Gamma)]$ lies in $H^*(\tilde{X}_G)$ (i.e. generates the trivial $G(\mathbb{A}_f)$-module)?

### 1.3. Mumford-Tate Groups

The conjectures of Langlands on the zeta functions on Shimura Varieties (and their extension to the non-tempered case by Kottwitz and Arthur) predict that in low degrees, the Galois group (of the number field over which a Shimura variety is defined) acts by a “small” group. In particular, for very low degrees of cohomology, the Galois action is potentially abelian. This is equivalent to saying (modulo the Mumford-Tate conjecture on the relation between
the Galois group and the Mumford-Tate group) that the Mumford-Tate group of the $\mathbb{Q}$-Hodge structure associated to low degree cohomology is abelian. This implication can be proved for several arithmetic quotients of classical Hermitian domains. (see [Bla-Rog], [Mu-Ra2] for related results).

**Theorem 6.** (1) If $G(\mathbb{R}) = Sp_g$ and $g \geq 2$ then the Mumford-Tate group of the $\mathbb{Q}$-Hodge structure of $H^g(S_G)$ is abelian.

(2) If $G(\mathbb{R}) = SU(p,q)$ and $2 \leq p \leq q$, then the Mumford-Tate group of the $\mathbb{Q}$-Hodge structure associated to $H^p(S_G)$ is abelian.

Here is a sketch of the proof. We use the criteria of restriction to deduce that the cohomology restricts injectively to a product of Shimura Curves in $S_G$, and then use the fact that the Hodge types of cohomological representations in low degrees are highly restricted (Vogan-Zuckerman). Then the following Lemma completes the proof.

**Lemma 7.** Suppose that $W$ is an irreducible pure $\mathbb{Q}$-Hodge structure whose Hodge types are holomorphic or anti-holomorphic: $W \otimes \mathbb{C} = W^{m,0} \oplus W^{0,m}$, with $m \geq 0$. Suppose that $W$ is contained in a tensor product of two irreducible $\mathbb{Q}$-Hodge structures $A$ and $B$, such that $A \otimes \mathbb{C} = \oplus_{p,q \geq 0, p+q=a} A^{p,q}$ and $B \otimes \mathbb{C} = \oplus_{p,q \geq 0, p+q=b} B^{p,q}$. Then the Mumford-Tate groups of $W$, $A$ and $B$ are all Abelian.

2. The Action of the cohomology of the Compact Dual

Under the Matsushima decomposition

$$H^*(S_G) = \oplus m(\pi)H^*(g, K_{\infty}, \pi_{\infty}) \otimes \pi_f,$$

the part which corresponds to the trivial representation $\pi$ is the cohomology of the compact dual $H^*(\hat{X}_G)$. Therefore, it acts on the cohomology of $S_G$ by cup product. If $X$ is Hermitian symmetric, it is possible to split the Hodge structure $H^*(S_G)$ into smaller pieces according to this action.

Suppose $\pi_{\infty} = A_q$ and $\pi_{\infty}' = A_{q'}$ are two cohomological representations which have strongly primitive cohomology in degree $i$. Suppose that $L$ and $L'$ are respectively the Levi subgroups of the parabolic subgroups $Q$ and $Q'$ corresponding to the $\theta$-stable parabolic subalgebras $q$ and $q'$. We consider the restriction maps $r_L : H^*(\hat{X}_G) \to H^*(\hat{X}_L)$ and $r_{L'} : H^*(\hat{X}_G) \to H^*(\hat{X}_{L'})$. Denote by $Hod^i(A_q)$ the smallest $\mathbb{Q}$-Hodge structure whose complex points contain all the strongly primitive cohomology classes in degree $i$ of type $A_q$. Define similarly, $Hod^i(A_{q'})$.

**Theorem 8.** If the kernels of the maps $r_L$ and $r_{L'}$ are distinct, then the $\mathbb{Q}$ Hodge Structures $Hod^i(A_q)$ and $Hod^i(A_{q'})$ are disjoint (their intersection is the zero vector space).
As an example, consider $G$ such that $G(\mathbb{R}) = SU(2, 2)$ up to compact factors. Then, in degree $i = 2$, there are three parabolic subalgebras $q$ whose $A_q$ have cohomology in degree 2. Two of them (say $q_1$ and $q_2$ are holomorphic (of Hodge type $(2, 0)$) and the other (say, $q_3$) is of Hodge type $(1, 1)$. It can be verified from the criterion of Theorem 8 that the associated $\mathbb{Q}$-Hodge structures are all disjoint.

By the Lefschetz $(1, 1)$-theorem, the Hodge structure associated to $q_3$ consists of algebraic cycles. This can be shown to yield the following.

**Corollary 3.** If $G(\mathbb{R}) = U(2, 2)$ up to compact factors, then all Tate classes in $H^2(S_G)$ are algebraic.

**Remark 1.** The Tate conjecture for $H^2$ for most Shimura varieties is known, in all dimensions at least five (by unpublished work of Blasius and Rogawski (a much earlier work of Harder-Langlands ([Har-Lan]), Murty-Ramakrishnan ([Mu-Ra]) and Klingenberg ([Kli]) treats the case of Hilbert modular surfaces). The above Corollary shows that for $U(2, 2)$ also, the Tate Conjecture holds. The main open case is then that of compact quotients of the two fold product of the upper half plane by cocompact irreducible lattices in $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$.

### 3. Non-Hermitian Case

To tackle the general (non-hermitian) case, M.Harris and J-S.Li devised an alternative approach ([H-L]). This is in terms of the “automorphic dual” of $G$ in the sense of Burger, Li and Sarnak ([Bu-Sa]). Recall that $G(\mathbb{R})$ is a real semi-simple Lie group and denote by $\hat{G}(\mathbb{R})$ the space of equivalence classes of irreducible unitary representations of $G(\mathbb{R})$ under the Fell topology (of uniform convergence of matrix coefficients of representations on compact subsets of $G(\mathbb{R})$). Denote by $\hat{G}(\mathbb{R})_{\text{Aut}}$ the closure of the union (over all congruence subgroups $\Gamma$) of the collections of irreducibles $\pi$ which occur weakly in $L^2(\Gamma \backslash G(\mathbb{R}))$.

The following conjecture is due to many people ([H-L], [Ber] and [Ber-Cl]).

**Conjecture 1.** (Harris-Li, Bergeron and Clozel) If $\pi$ is a cohomological representation, then $\pi$ is not a limit of complementary series representations $\sigma$ with $\sigma \in \hat{G}(\mathbb{R})_{\text{Aut}}$.

In particular, if $\pi$ is a non-tempered cohomological representation, then it is isolated in the automorphic dual od $G$.

A very special case of this is when $\pi$ is the trivial representation of $SL_2$ and the conjecture is equivalent to saying that the non-zero eigenvalues of the Laplacian on quotients of the upper half plane by congruence subgroups of $SL_2(\mathbb{Z})$ are bounded away from zero (Selberg’s “3/16” Theorem ([Sel])). For a general semi-simple group $G$ defined over $\mathbb{Q}$, Clozel has proved that conjecture 1 is true for the trivial representation ([Clo]). A result of Vogan ([Vog]) says that for most groups, the cohomological representations $A_q$ are isolated even in the unitary dual (the
only ones which are not isolated are those for which the Levi subgroup $L$ over $\mathbb{R}$ is a product of copies of $SO(m,1)$ or $SU(m,1)$).

Harris and Li showed that under the assumption of Conjecture 1, the question of the non-vanishing of the restriction of cohomology may be reduced purely to a question of the discrete occurrence of a suitable cohomological representation of the smaller group $H(\mathbb{R})$ in a cohomological representation of the larger group $G(\mathbb{R})$. In the special case that $G(\mathbb{R}) = SU(n,1)$ (up to compact factors), they proved that Shimura subvarieties of the complex hyperbolic manifold $S_G$ satisfy a Lefschetz property namely

$$Res : H^i(S_G) \rightarrow \prod_{g \in G(\Lambda_f)} H^*S_H,$$

is injective provided $i \leq \text{dim} S_H$ (they even proved this unconditionally in the case that $i \leq 2$).

Clozel and Bergeron have an analogue for the real hyperbolic manifolds under the assumption of Conjecture 1, namely the following theorem.

**Theorem 9.** (Clozel and Bergeron). Under the assumption of Conjecture 1, if $G(\mathbb{R}) = SO(n,1)$ and $H(\mathbb{R}) = SO(m,1)$ up to compact factors, then the restriction map

$$Res : H^i(S_G) \rightarrow \prod_{g \in G(\Lambda_f)} H^i(S_H),$$

is injective provided $i \leq \lfloor m/2 \rfloor$ (where $\lfloor x \rfloor$ is the integral part of $x$).

Clozel and Bergeron have shown that Conjecture 1 follows from well known conjectures of Arthur on the possible non-tempered automorphic representations which can arise. Because of recent progress on the Fundamental Lemma, and results of Arthur on consequences of the Fundamental Lemma, Conjecture 1 is close to being settled. The details will appear in a work by Bergeron and Clozel.

Theorem 9 has been proved unconditionally (only for $i = 1$) by [Ra-V], [lub] and [V4]).

In [Sp-V], Conjecture 1 for $SO(n,1)$ is reduced to the case when the cohomological representation is tempered.

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