ON THE μ-INVARIANT IN IWASAWA THEORY

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Introduction

The aim of this expository article is to discuss the μ-invariant associated to finitely generated modules over Iwasawa algebras. This is an important invariant which was first discovered by Iwasawa in the 1960's and occurs in his seminal work on cyclotomic fields [9], [10]. In fact, Iwasawa conjectured that his μ-invariant was always zero for the p-primary subgroup of the ideal class group of the field obtained by adjoining all p-power roots of unity to a given finite extension F of $\mathbb{Q}$, p being a fixed prime number. This was subsequently proven by Ferrero and Washington when F is an abelian extension of $\mathbb{Q}$, but remains open in general. Mazur found the first simple examples of a positive μ-invariant in the case of the Iwasawa theory of elliptic curves. For example, let A denote the 5-primary subgroup of the Tate-Shafarevich group of the elliptic curve

$$X_0(11) : y^2 + y = x^3 - x^2 - 10x - 20$$

over the field obtained by adjoining all 5-power roots of unity to $\mathbb{Q}$. Then Mazur showed that the μ-invariant of A is equal to 2; in fact, in this case, $5.A = 0$, but A is not a finitely generated abelian group since it has positive μ-invariant.

In recent years, the progress made in non-commutative Iwasawa theory has led to a revival of interest in this subtle invariant, and the open problems associated with it. As we shall see below, even though the definition of the μ-invariant is algebraic, all known deep results about its calculation are by purely analytic means! The article is organised as follows: Section 1 is preliminary in nature, where we also fix notation. Section 2 introduces the μ-invariant and we recall the important classical results in this set up. In Section 3, we consider the μ-invariant for elliptic curves over number fields. Section 4 discusses the relatively recent definition of the μ-invariant for modules over Iwasawa algebras of p-adic Lie groups which are not necessarily pro-p. Finally in section 5, we mention one of the main open problems related to this invariant, which seems quite difficult to attack. This problem is especially interesting in the context of the non-commutative Iwasawa theory and the formulation of the main conjecture.

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1. Notation and Preliminaries

Throughout, p will denote an odd prime and $\mathbb{Z}_p$ (respectively $\mathbb{Q}_p$), will denote the ring of p-adic integers (resp. the field of p-adic numbers). Given any number field F, we shall denote by F^{cyc} the cyclotomic $\mathbb{Z}_p$-extension of F, which we recall, is the unique subfield of $F(\mu_{p^\infty})$ with Galois group over F isomorphic to $\mathbb{Z}_p$. Recall that an extension $\mathcal{L}$ of F is said to be a p-adic Lie extension if the Galois group $\text{Gal}(\mathcal{L}/\mathbb{Q})$ is a p-adic Lie group [11]. The cyclotomic $\mathbb{Z}_p$-extension F^{cyc} is a basic example of a p-adic Lie extension of F, and has been studied extensively beginning with Iwasawa's seminal works [9], [10]. For any finite set S of primes of F, F_S will denote the maximal extension of F unramified outside S. We shall always assume that S contains the set S_p of primes in F that lie above p, and the archimedean primes. Given a profinite group G, the p-cohomological dimension of G will be denoted by $\text{cd}_p(G)$, which we recall is the least integer n such that $H^{n+i}(G, A) = 0$ for all $i \geq 1$, and any discrete p-primary torsion module A. For a compact p-adic Lie group with no elements of order p, it is a result of Serre and Lazard that $\text{cd}_p(G) = d$, where d is the dimension of G as a Lie group.If G is a profinite group, the Iwasawa algebra of G is denoted by $\Lambda(G)$, and is defined as

$$\Lambda(G) = \varprojlim_{U} \mathbb{Z}_p[G/U],$$

where U runs over all open normal subgroups of G, with the inverse limit being taken with respect to the natural surjections. For any number field F, we shall denote the Galois group $\text{Gal}(F^{\text{cyc}}/F)$ by Γ. Given a discrete module M over the Iwasawa algebra $\Lambda(G)$, we denote its compact Pontryagin dual by $M^\vee = \text{Hom}\,(M, \mathbb{Q}_p/\mathbb{Z}_p)$. For any module M over the Iwasawa algebra $\Lambda(G)$, $M(p)$ denotes the p-primary torsion submodule of M.

Let us briefly discuss how the μ-invariant was discovered by Iwasawa in his study of class numbers. Let F be a number field and suppose that

$$F = F_0 \subset F_1 \subset \cdots \subset F_n \cdots \subset F_\infty = \bigcup F_n,$$

is a sequence of number fields with $\text{Gal}(F_n/F_0) \simeq \mathbb{Z}/p^n\mathbb{Z}$. Then the Galois group

$$\text{Gal}(F_\infty/F_0) = \varprojlim\,(\mathbb{Z}/p^n\mathbb{Z}) = \mathbb{Z}_p,$$

so that the extension F_∞/F_0 is a $\mathbb{Z}_p$-extension. The most natural example of such an extension is of course the cyclotomic $\mathbb{Z}_p$-extension F^{cyc}. Let p^{e_n} be the exact order of p that divides the order of the class number of F_n. Iwasawa proved that there exist non-negative integers λ, μ, ν such that

$$e_n = \lambda n + \mu p^n + \nu \tag{1}$$

for all sufficiently large n. The invariants λ and μ are respectively the λ and μ- invariants, which we denote by $\lambda(F_\infty/F)$ and $\mu(F_\infty/F)$ respectively. If F_∞ is the cyclotomic $\mathbb{Z}_p$-extension F^{cyc} of F, and the base field F is clear from the context, we shall simply denote them by λ and μ respectively. In fact, for $F = \mathbb{Q}$ and $F_\infty = \mathbb{Q}^{\text{cyc}}$, it is easy to prove (as Iwasawa did) that $\lambda = \mu = \nu = 0$. The argument breaks down as soon as one replaces $\mathbb{Q}$ by a finite extension, even a quadratic extension, if p splits in the extension. Iwasawa's celebrated "$\mu = 0$ conjecture" asserts that the invariant μ in (1) is zero for any number field F and its cyclotomic $\mathbb{Z}_p$-extension F^{cyc}. A deep theorem of Ferrero and Washington proves this conjecture for abelian extensions F of $\mathbb{Q}$. Iwasawa himself constructed examples of non-cyclotomic $\mathbb{Z}_p$-extensions with $\mu > 0$, (see Washington's book [20] for more details). In the next sections, we shall see a more algebraic definition of the μ-invariant.

2. The μ-invariant in Iwasawa theory

In this section, we give two other definitions of the μ-invariant in the more general context of finitely generated modules over Iwasawa algebras. Let therefore G be a pro-p p-adic Lie group of dimension d, which has no elements of order p. For example, if G is isomorphic to $\mathbb{Z}_p^d$, then it is well-known that there is an isomorphism

$$\Lambda(G) \simeq \mathbb{Z}_p[[T_1, \cdots, T_d]]$$

which sends a (fixed) independent set of topological generators $e_1, \cdots, e_d$ of G to $(1 + T_1), \cdots (1 + T_d)$ respectively. Thus $\Lambda(G)$ is a commutative, regular local ring of Krull dimension $d+1$, in this case. For a general non-commutative p-adic Lie group G of dimension d as above, the ring $\Lambda(G)$ is a non-commutative, left and right Noetherian local domain. It has additional nice properties, for instance it is Auslander regular (see [19]) of global dimension $d+1$, and hence affords a good dimension theory for finitely generated modules. Recall that a torsion module over $\Lambda(G)$ is said to be *pseudo-null* if it has dimension less than or equal to $d-1$. A homomorphism $f : M \to N$ between two finitely generated $\Lambda(G)$-modules is a *pseudo-isomorphism* if both its kernel and cokernel are pseudo-null. Let M be a finitely generated torsion module over $\Lambda(G)$ and $M(p)$ be its p-primary torsion submodule The structure theorem for finitely generated torsion $\Lambda(G)$-modules, was first proved by Iwasawa and then generalised by Serre, (cf. [2, Chap. VII, §4]) in the commutative case, and extended to general Iwasawa algberas $\Lambda(G)$, and finitely generated modules over $\Lambda(G)$ that are killed by a power of p by Susan Howson ([7]) and Venjakob [19] independently :-

Theorem 2.1. *Let M be a finitely generated torsion module over $\Lambda(G)$, where G is a pro-p p-adic Lie group with no elements of order p. Then there is a pseudo-isomorphism*

$$f : M(p) \to \bigoplus_{i=1}^{k} \Lambda(G)/p_i^{n_i}.$$

The set of integers n_i is unique upto ordering.

The importance of this theorem lies in the fact that it enables us to define a key invariant for finitely generated $\Lambda(G)$-modules. Let M be a finitely generated $\Lambda(G)$-module. Then the μ-*invariant* of M is defined as

$$\mu_G(M) = \sum_j n_j,$$

where the integers n_j are as in the structure theorem for $M(p)$. If $G \simeq \mathbb{Z}_p$, then by the general structure theorem of Serre-Iwasawa for finitely generated torsion $\Lambda(G)$-modules M, it follows that $M/M(p)$ is a finitely generated $\mathbb{Z}_p$-module. In this case, the λ-*invariant* of M, denoted $\lambda(M)$, is defined to be the $\mathbb{Z}_p$-rank of $M/M(p)$.

Let F be a number field and F^{cyc} be the cyclotomic $\mathbb{Z}_p$-extension of F, with $\Gamma = \text{Gal}(F^{\text{cyc}}/F) \simeq \mathbb{Z}_p$. We denote by $M(F^{\text{cyc}})$ the maximal abelian, p-extension of F^{cyc} which is unramified everywhere. In other words, $M(F^{\text{cyc}})$ is the p-Hilbert class field of F^{cyc}. Let $X^{\text{cyc}} = X^{\text{cyc}}(F)$ denote the Galois group $\text{Gal}(M(F^{\text{cyc}})/F^{\text{cyc}})$. As X^{cyc} is abelian, it has a natural structure of a Γ-module. Indeed, given an element γ in Γ and an element x in X^{cyc}, the action is defined by

$$\gamma.x = \tilde{\gamma}x\tilde{\gamma}^{-1},$$

where $\tilde{\gamma}$ is any lift of γ to $\text{Gal}(M(F^{\text{cyc}})/F)$. It is easily checked that the action is independent of the lift since X^{cyc} is abelian. Further, as any compact $\mathbb{Z}_p$-module with a continuous Γ-action has a natural structure of a $\mathbb{Z}_p[[\Gamma]]$-module, it is plain that X^{cyc} is a compact module over the Iwasawa algebra $\Lambda(\Gamma)$. The following theorem is due to Iwasawa [9]:-

Theorem 2.2. *(Iwasawa) X^{cyc} is a finitely generated torsion $\Lambda(\Gamma)$-module.*

Iwasawa proved that the invariants λ and μ occurring in (1) exist by combining this structure theorem with an arithmetic argument, and indeed the μ and λ invariants appearing in (1) are precisely the μ-invariant and λ- invariant that arise from the structure theorem for the finitely generated torsion module X^{cyc}.

There is another equivalent definition for the μ-invariant. Let G be as above and $\Lambda(G)$ be the Iwasawa algebra. Suppose that M is a finitely generated module over $\Lambda(G)$. Then the Euler characteristic $\chi(G, M(p))$ is defined as

$$\chi(G, M(p)) = \prod_i \# \left(H_i(G, M(p))\right)^{(-1)^i}, \tag{2}$$

where $H_i(G, M(p))$ are the homology groups $\text{Tor}_i^{\Lambda(G)}(\mathbb{Z}_p, M(p))$. Note that as $\Lambda(G)$ is Auslander regular, the homology groups vanish for $i > (d+1)$ and hence the Euler characteristic is well-defined. We then have (see [8], [19])

$$\mu_G(M) = \text{ord}_p(\chi(G, M(p))). \tag{3}$$

Thus the μ-invariant can be interpreted also as an Euler characteristic. It is obvious from the above discussion that a finitely generated p-primary torsion module over $\Lambda(G)$

is pseudo-null if and only if it has μ-invariant zero. Further, the μ-invariant is additive along exact sequences of torsion $\Lambda(G)$-modules.

Again, suppose G is a pro-p, p-adic Lie group of dimension d with no elements of order p. Let $\Omega(G)$ denote the Iwasawa algebra over $\mathbb{F}_p$, i.e.

$$(4) \qquad \Omega(G) := \varprojlim_{U} \mathbb{F}_p[G/U],$$

where the inverse limit is taken over open, normal subgroups as before, with respect to the natural maps. The ring $\Omega(G)$ is again a left and right Noetherian local domain which is Auslander regular (cf. [19]) with finite global dimension equal to d. The rank of a finitely generated module over $\Omega(G)$ is well-defined since $\Omega(G)$ is a domain. For any finitely generated $\Lambda(G)$-module M, there exists an integer $r(M)$ such that the p-primary torsion submodule $M(p)$ is annnihilated by $p^{r(M)}$. Let $\mathrm{gr}_p\, M(p)$ denote the p-graded module of $M(p)$, i.e.

$$(5) \qquad \mathrm{gr}_p\, M(p) := \bigoplus_{k=1}^{r(M)} M_{p^k}/M_{p^{k-1}},$$

where M_{p^t} denotes the submodule of p^t-torsion elements of M. Then each graded piece is an $\Omega(G)$-module, and $\mu(M)$ is easily seen to be the sum of the ranks of the graded pieces.

Let S be a finite set of primes of F as in §1 and F_S be the maximal extension of F unramified outside S. It is plain that F^{cyc} is contained in F_S. Let $F_S(p)$ denote the subfield of F_S containing F such that the Galois group $G_S(p, F) := \mathrm{Gal}(F_S(p)/F)$ is the maximal pro-p quotient of $\mathrm{Gal}(F_S/F)$. It was proven by Iwasawa that for any number field F, we have

$$(6) \qquad H_2(\mathrm{Gal}(F_S/F^{\mathrm{cyc}}), \mathbb{Z}_p) = 0.$$

Recall that a pro-p group G is *free* if $\mathrm{cd}_p(G) \leq 1$. Let $G_S(p, F^{\mathrm{cyc}}) = \mathrm{Gal}(F_S(p)/F^{\mathrm{cyc}})$. Using (6), it can be shown that $G_S(p, F^{\mathrm{cyc}})$ is free if and only if $\mu(F^{\mathrm{cyc}}/F) = 0$ (see [13, Chap. XI, §3] and [18] for more details). This provides an interesting connection between the classical Iwasawa μ-invariant for a number field F and the freeness of the pro-p Galois group $G_S(p, F^{\mathrm{cyc}})$.

3. The μ-invariant for elliptic curves

The discussion in the previous section may be viewed as the theory of the μ-invariant for the trivial Tate motive $\mathbb{Z}_p$. We shall consider the motive associated to an elliptic curve and describe the μ-invariants that arise in this context. Let E be an elliptic curve defined over a number field F such that E has good ordinary reduction at all primes of F that lie above p. We first describe the arithmetic module that corresponds to the module X^{cyc} considered in the previous section. The set S is now any finite set of primes of F that contains the primes above p and all the primes of bad reduction for E. Put $G_F = \mathrm{Gal}(\bar{F}/F)$ for the absolute Galois group of F. Let

$$E_{p^\infty} := \bigcup_{n \geq 0} E_{p^n}$$

where $E_{p^n} := E_{p^n}(\bar{F})$ is the discrete G_F-module of p^n-torsion points on $E(\bar{F})$. The Tate module of E, denoted $T_p(E)$, is defined by

$$T_p(E) = \varprojlim E_{p^n},$$

and $V_p(E) = T_p(E) \otimes \mathbb{Q}_p$ is the corresponding two dimensional $\mathbb{Q}_p$-vector space with a continuous action of G_F. For the purposes of this article, we consider the G_F-module $V_p(E)$ as the motive of the elliptic curve E.

Let F^{cyc} be the cyclotomic $\mathbb{Z}_p$-extension of F. Classical descent theory for elliptic curves suggests that the $\Lambda(\Gamma)$-module to be considered in this context is the Pontryagin dual of the *Selmer group* over the cyclotomic extension. Recall that for a finite Galois extension L of F, the Selmer group $S(E/L)$ of E over L is a discrete $\mathrm{Gal}(L/F)$-module and is defined by

$$S(E/L) = \mathrm{Ker}\left(H^1(\mathrm{Gal}(F_S/L), E_{p^\infty})) \to \underset{v \in S}{\oplus} J_v(E/L)\right), \tag{7}$$

where

$$J_v(E/L) = \underset{w|v}{\oplus} H^1(\mathrm{Gal}(\bar{L}_w/L_w), E)(p).$$

The Selmer group of E over F^{cyc} is defined as the direct limit of $S(E/L)$ as L varies over finite extensions of F in F^{cyc}. Its Pontryagin dual is denoted by $\mathfrak{X}(E/F^{\mathrm{cyc}})$ and is easily seen to be a finitely generated module over $\Lambda(\Gamma)$. The Selmer group and its dual are defined analogously for any infinite extension L_∞ of F, and denoted by $S(E/L_\infty)$ and $\mathfrak{X}(E/L_\infty)$ respectively. It is a deep conjecture due to Mazur that since E has good ordinary reduction at the primes above p, the module $\mathfrak{X}(E/F^{\mathrm{cyc}})$ is $\Lambda(\Gamma)$-torsion. There are plenty of numerical examples known where this conjecture holds. There are also the important theoretical result due to Kato who proves Mazur's conjecture for $\mathfrak{X}(E/F^{\mathrm{cyc}})$, for number fields F that are abelian over $\mathbb{Q}$, provided E is defined over $\mathbb{Q}$. One is thus led naturally to wonder if $\mathfrak{X}(E/F^{\mathrm{cyc}})$ is a finitely generated $\mathbb{Z}_p$-module, which would be the analogue of Iwasawa's $\mu = 0$ conjecture. However, Mazur already gave examples where this is not true, and the fact that the μ-invariant of $\mathfrak{X}(E/F^{\mathrm{cyc}})$ is not always zero causes endless technical difficulties in the Iwasawa theory of elliptic curves! An explicit example is the elliptic curve $E/\mathbb{Q}$ of conductor 11, defined by

$$E : y^2 + y = x^3 - x^2 - 10x - 20.$$

For the prime $p = 5$, Mazur showed that $\mathfrak{X}(E/\mathbb{Q}^{\mathrm{cyc}})$ is not a finitely generated $\mathbb{Z}_5$-module, although it is $\Lambda(\Gamma)$-torsion.

Another interesting case is that of elliptic curves with complex multiplication. Let E be an elliptic curve with complex multiplication by the ring of integers $\mathcal{O}_K$ of an imaginary quadratic field K of class number one. Assume that E is defined over K and that the odd prime p splits in K, say $p\mathcal{O}_K = \mathfrak{p}\mathfrak{p}^*$. Assume also that E has good reduction at $\mathfrak{p}$ and $\mathfrak{p}^*$. Let $K(E_{\mathfrak{p}^n})$ denote the field extension of K obtained by adjoining the co-ordinates of the $\mathfrak{p}^n$-division points of E and consider the infinite Galois extension

$$K_\infty := K(E_{\mathfrak{p}^\infty}) = \bigcup_{n\geq 0} K(E_{\mathfrak{p}^n}).$$

Then the Galois group $G = \mathrm{Gal}(K(E_{\mathfrak{p}^\infty})/K)$ is isomorphic to $\mathbb{Z}_p^\times$ and K_∞ is a $\mathbb{Z}_p$-extension of $K(E_\mathfrak{p})$, and we write $\Lambda(G)$ for the associated Iwasawa algebra. Let X_∞ be the maximal abelian p-extension of K_∞ that is unramified outside of the set of primes above $\mathfrak{p}$. It is known, thanks to results of Coates and Wiles [5], [14] that X_∞ is a finitely generated torsion $\Lambda(G)$-module. Let $S^\mathfrak{p}(E/K_\infty)$ denote the $\mathfrak{p}$-Selmer group of E over K_∞. To be precise, this is defined by taking $E_{\mathfrak{p}^\infty}$ instead of E_{p^∞} in (7). Let $\mathfrak{X}^\mathfrak{p}(E/K_\infty)$ be the compact dual of the Selmer group. We then have [14]

$$S^\mathfrak{p}(E/K_\infty) = \mathrm{Hom}\,(X_\infty, E_{\mathfrak{p}^\infty}),$$

and hence it follows that $\mathfrak{X}^\mathfrak{p}(E/K_\infty)$ is a finitely generated $\Lambda(G)$-torsion module. Put $\Gamma = \mathrm{Gal}(K_\infty/K(E_p))$, which is isomorphic to $\mathbb{Z}_p$. The following theorem was proven by Gillard and Schneps independently ([6], [16]) using analytic methods:-

Theorem 3.1. *Viewing $\mathfrak{X}^\mathfrak{p}(E/K_\infty)$ as a $\Lambda(\Gamma)$=module, we have $\mu(\mathfrak{X}^\mathfrak{p}(E/K_\infty)) = 0$.*

Let K'_∞ be the Galois extension of K obtained by adjoining the co-ordinates of the p-power division points of E, i.e.

$$K'_\infty := K(E_{p^\infty}) = \bigcup_{n\geq 0} K(E_{p^n}).$$

Then K'_∞ is a Galois extension of $K(E_p)$, whose Galois group G' is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. Let $\Lambda(G')$ be its Iwasawa algebra, so that we have

$$\Lambda(G') \simeq \mathbb{Z}_p[[T_1, T_2]].$$

The Pontryagin dual $\mathfrak{X}^\mathfrak{p}(E/K'_\infty)$ of the discrete Selmer group $S^\mathfrak{p}(E/K'_\infty)$ is a compact, finitely generated torsion $\Lambda(G')$-module, and as the extension K'_∞/K_∞ is pro-p, the following result is an easy consequence of a generalisation of Theorem 3.1, where $\mu_{G'}$ denotes the μ-invariant for $\Lambda(G')$-modules.

Proposition 3.2. *We have $\mu_{G'}(\mathfrak{X}^\mathfrak{p}(E/K'_\infty)) = 0$.*

For a general elliptic curve E over a number field F, and a p-adic Lie extension L_∞/F with pro-p Galois group $\mathrm{Gal}(L_\infty/F) =: G$ having no element of order p, the μ-invariant $\mu_G(\mathfrak{X}(E/L_\infty))$ is a big mystery and we discuss some aspects of this in §5.

4. The μ-invariant for p-adic Lie groups

In the previous sections, the μ-invariant was defined for modules over Iwasawa algebras of pro-p, p-adic Lie groups G with the additional hypothesis that G has no elements of order p. It is clear from the discussion in §2, that the μ-invariant is well-defined on Grothendieck groups. Let $\mathcal{C}$ be a small abelian category. Recall that its Grothendieck group $K_0(\mathcal{C})$, is an abelian group with generators $[N]$ where N runs over all the objects of $\mathcal{C}$, and relations $[N] = [N'] + [N'']$ for any short exact sequence

$$0 \to N' \to N \to N'' \to 0$$

of objects in $\mathcal{C}$. Now consider the abelian categories $\mathcal{M}(G)$ and $\mathcal{P}(G)$ which are respectively the categories of finitely generated (resp. finitely generated, projective) modules over $\Omega(G)$. The additional hypothesis above on G ensures that the Grothendieck groups $K_0(\mathcal{M}(G))$ and $K_0(\mathcal{P}(G))$ are naturally isomorphic. Further, it is plain that the μ-invariant gives a homomorphism from $K_0(\mathcal{C})$ into $\mathbb{Z}$. where $\mathcal{C}$ denotes either of the two categories above. As noted before, $\mu([N])$ is zero if and only if N is pseudo-null.

The K-theoretic context is important from an other point of view as well. The centrepiece of Iwasawa theory is the formulation of (and attempts to prove!) the *Main Conjecture*, which we shall not discuss in detail here. However, we do mention that the formulation of the main conjecture in non-commutative Iwasawa theory involves the characteristic element of a certain class of $\Lambda(G)$-modules, which is an element of a localisation of the Iwasawa alegbra $\Lambda(G)$ (see [3] for details). From this perspective, it is then natural to wonder if the theory, and in particular, the μ-invariant can be extended to more general compact p-adic Lie groups. Let G be a p-adic Lie group with no element of order p and suppose N is a finitely generated $\Omega(G)$-module. We can also view N as a module over $\Lambda(G)$ that is killed by p. The characteristic element of N, denoted ξ_N, is an element of $K_1(\Lambda(G)_T)$ where T is the *central Ore set* in $\Lambda(G)$ defined by $T = \{1, p, p^2, \cdots\}$, and $\Lambda(G)_T$ denotes the localisation of $\Lambda(G)$ at T. For a ring R, let $R^\times$ denote the multiplicative group of units of R. There is a natural map

$$\theta \,:\, (\Lambda(G))_T^\times \to K_1(\Lambda(G)_T). \tag{8}$$

We have the following

Theorem 4.1. *Let G be a pro-p, p-adic Lie group with no elements of order p, and suppose that N is a finitely generated $\Omega(G)$- module. Then*

$$\xi_N = \theta\left(p^{\mu(N)}\right).$$

Moreover, the following assertions are equivalent:-
(i) N is pseudo-null (ii) $\chi(G, N) = 1$ (iii) $\xi_N = 1$.

Ardakov and Wadsley [1] study the characteristic elements for modules over $\Omega(G)$ where they allow G to be an arbitrary compact p-adic Lie group which has no elements of order

p. The Iwasawa algebra $\Lambda(G)$ has finitely many simple modules up to isomorphism; if G is pro-p with no elements of order p, then of course, there is a unique simple module, namely $\mathbb{F}_p$. Let $V_1, \cdots, V_s$ be the finitely many simple modules, each V_i being a finite dimensional $\mathbb{F}_p$-vector space. Assume that G has no elements of order p (but is not necessarily pro-p). Then the Euler characteristic definition of the μ-invariant, given in §2 is valid and allows us to define the μ-invariant $\mu(N)$ for any finitely generated $\Lambda(G)$-module N killed by a power of p. Ardakov and Wadsley also introduce *twisted μ-invariants*, denoted $\mu_i(N)$, for $1 \leq i \leq s$, defined by the formula

$$\mu_i(N) = \frac{\log_p \chi(G, \mathrm{gr}_p N \otimes_{\mathbb{F}_p} V_i^*)}{\dim_{\mathbb{F}_p} \mathrm{End}_{\Omega(G)}(V_i)}.$$

Here V_i^* is the dual vector space V_i and $\mathrm{gr}_p(N)$ is the graded module defined by (5).

Theorem 4.2. *(Ardakov-Wadsley) Assume that G is a compact p-adic analytic group with no elements of order p and let N be a finitely generated p-primary $\Lambda(G)$-module. Then the twisted μ-invariants $\mu_i(N)$ $(1 \leq i \leq s)$ are integers and*

$$\xi_N = \theta\left(\prod_{i=1}^{s} f_i^{\mu_i(N)}\right),$$

where $f_i = 1 + (p-1)e_i$, and e_i is an idempotent in $\Lambda(G)$ such that V_i is the unique simple quotient module of $e_i\Lambda(G)$.

For a general p-adic Lie group G which has no element of order p, but is not necessarily pro-p, even the integrality of the Euler characteristic $\chi(G, N)$ is unclear for a finitely generated p-primary $\Lambda(G)$-module. In [1, 9.6], there is an explicit example of a dimension one p-adic analytic group G and a module N over $\Omega(G)$ for which the Euler characteristic $\chi(G, N)$ is not integral. It is also known that in this generality, it is no longer true that N is pseudo-null if and only $\xi_N = 1$. In [1], there is a characterisation of those groups for which this assertion holds.

Theorem 4.3. *(Ardakov-Wadsley) Let G be a compact p-adic analytic group with no elements of order p. Then the following assertions are equivalent:-*
(i) $\xi_N = 1$ for all finitely generated p-primary pseudo-null $\Lambda(G)$-modules N (ii) $\chi(G, N) \in \mathbb{Z}$ for all finitely generated p-primary $\Lambda(G)$-modules N (iii) G is a semidirect product of a finite group of order prime to p with a pro-p group.

5. Open Problems

In this final section, we describe some open problems related to the μ-invariant. As these arise naturally in the formulation of the main conjecture in non-commutative Iwasawa theory (see [3]), we begin by discussing the background. For brevity, we shall restrict ourselves to the case of elliptic curves, though it is obvious that the problems can be posed in the wider context of more general motives. Let therefore E be an eliptic curve over a number field F without complex multiplication, such that E has good ordinary reduction at the primes of F that lie above p. Recall that this means that the endomorphism ring $\mathrm{End}(E)$ of the elliptic curve is isomorphic to $\mathbb{Z}$, and by a deep result of Serre, [17], the image of the Galois representation

$$\rho_E \,:\, \mathrm{Gal}(\bar{\mathbb{Q}}/F) \to GL_2(V),$$

where $V = V_p(E)$ is as in §3, is an open subgroup of $GL_2(\mathbb{Z}_p)$. Let G denote this image. Then G is the Galois group F of the extension

$$F_\infty := F(E_{p^\infty})$$

obtained by attaching the co-ordinates of all the p-power division points of E to F. Henceforth assume that $p \geq 5$, as this guarantees that G has no elements of order p. We shall further assume that F contains the p-division points of E, i.e. $F = F(E_p)$, so that $G = \mathrm{Gal}(F_\infty/F)$ is a pro-p, p-adic Lie group. Let $\Lambda(G)$ be the $\mathbb{Z}_p$-Iwasawa algebra of G. As before, we consider the compact Pontryagin dual of the Selmer group $\mathfrak{X}(E/F_\infty)$. It is not difficult to prove that $\mathfrak{X}(E/F_\infty)$ is a finitely generated module over the Iwasawa algebra $\Lambda(G)$. By the Weil pairing, the extension F_∞ contains the cyclotomic $\mathbb{Z}_p$-extension F^{cyc} and we denote by H the Galois group $\mathrm{Gal}(F_\infty/F^{\mathrm{cyc}})$, and by Γ the quotient $G/H = \mathrm{Gal}(F^{\mathrm{cyc}}/F)$. We now make the following definition.

Definition 5.1. The category $\mathfrak{M}_H(G)$ consists of all finitely generated compact modules X over $\Lambda(G)$ such that $X/X(p)$ is finitely generated over $\Lambda(H)$.

Put

$$\mathfrak{Y}(E/F_\infty) = \mathfrak{X}(E/F_\infty)/\mathfrak{X}(E/F_\infty)(p). \tag{9}$$

A key hypothesis that is essential for the formulation of the non-commutative main conjecture is the following [3]:-

Conjecture 5.2. *($\mathfrak{M}_H(G)$ Conjecture) The $\Lambda(G)$-module $\mathfrak{X}(E/F_\infty)$ is in $\mathfrak{M}_H(G)$, i.e. the module $\mathfrak{Y}(E/F_\infty)$ is a finitely generated $\Lambda(H)$-module.*

This conjecture is essential in order to define the characteristic element ξ_E of $\mathfrak{X}(E/F_\infty)$. In fact, it is shown in [3] that there is a surjection

$$\delta : K_1((\Lambda(G)_{S^*})) \to K_0(\mathfrak{M}_H(G)),$$

where S^* is a certain canonical Ore set in $\Lambda(G)$ defined in [3]. The kernel of δ is the natural image of $K_1(\Lambda(G)$ in $K_1(\Lambda(G)_{S^*})$. For any element $[M]$ in $K_0(\mathfrak{M}_H(G))$, the characteristic element ξ_M is defined to be the class of a lift of M under the map δ above. It is well-defined modulo the natural image of $K_1(\Lambda(G)$ in $K_1(\Lambda(G)_{S^*})$.

Let us now explain the relevance of the μ-invariant for this conjecture. As before, $\mathfrak{X}(E/F^{\mathrm{cyc}})$ denotes the dual Selmer group of E over the cyclotomic extension F^{cyc} of F.

An easy application of Nakayama's lemma shows that if $\mathfrak{X}(E/F^{\text{cyc}})$ is a finitely generated $\mathbb{Z}_p$-module, i.e. $\mathfrak{X}(E/F^{\text{cyc}})$ has μ-invariant zero, then Conjecture 5.2 holds for E. Indeed, one proves that $\mathfrak{X}(E/F_\infty)$ is itself a finitely generated $\Lambda(H)$-module. More generally, if the elliptic curve E is isogenous over any subfield of F_∞, to a curve whose dual Selmer group over the cyclotomic $\mathbb{Z}_p$-extension of this field has μ-invariant zero, then Conjecture 5.2 is true for E. Thus there are plenty of numerical examples known where the conjecture is valid. For example, consider the three isogenous elliptic curves over $\mathbb{Q}$ of conductor 11, given by the following equations:-

$$\begin{aligned} y^2 + y &= x^3 - x^2, \\ y^2 + y &= x^3 - x^2 - 10x - 20, \\ y^2 + y &= x^3 - x^2 - 7820x - 263580. \end{aligned}$$

The field F_∞ obtained by attaching the co-ordinates of the p-power division points for any prime p, is the same for the three curves as they are isogenous. Take $p = 5$, and $L = \mathbb{Q}(\mu_5)$. Then $G_L = \text{Gal}(F_\infty/F)$ is pro-5 and $\mathfrak{X}(E/F_\infty)$ is in $\mathfrak{M}_H(G)$ for these three curves, because $\mathfrak{X}(E/L^{\text{cyc}}) = 0$ for the first curve $E := y^2 + y = x^3 - x^2$, see [4]. However it is unknown whether $\mathfrak{X}(E/F_\infty)$ is in $\mathfrak{M}_H(G)$ for any prime of good ordinary reduction greater than 5.

For any finite extension L of F contained in F_∞, let

$$G_L = \text{Gal}(F_\infty/L),\ \Gamma_L = \text{Gal}(F_\infty/L^{\text{cyc}}).$$

Let $\mu_{G_L}(\mathfrak{X}(E/F_\infty))$ (resp. $\mu_{\Gamma_L}(\mathfrak{X}(E/F^{\text{cyc}})))$ denote the μ-invariants of the corresponding dual Selmer groups over the Iwasawa algebras $\Lambda(G_L)$, (resp. Λ_{Γ_L}). Let $H_L = \text{Gal}(F_\infty/L^{\text{cyc}})$. The following result is proved in [3, §5]:-

Proposition 5.3. *Let E/F be an elliptic curve as above and assume Mazur's conjecture holds for $\mathfrak{X}(E/L^{\text{cyc}})$ for all finite extensions L of F in F_∞. Suppose in addition, that there exists a finite extension L_1 of F in F_∞ such that $\mu_{\Gamma_{L_1}}(\mathfrak{X}(E/L_1^{\text{cyc}})) = \mu_{G_{L_1}}(\mathfrak{X}(E/F_\infty))$, and that $H_1(H_{L_1}, \mathfrak{Y}(E/F_\infty))$ is finite. Then $\mathfrak{X}(E/F_\infty)$ is in $\mathfrak{M}_H(G)$. Conversely, if $\mathfrak{X}(E/F_\infty)$ is in $\mathfrak{M}_H(G)$, then $\mu_{G_L}(\mathfrak{X}(E/F_\infty)) = \mu_{\Gamma_L}(\mathfrak{X}(E/L^{\text{cyc}}))$ and $H_1(H_L, \mathfrak{Y}(E/F_\infty))$ is finite for each finite extension L of F in F_∞.*

It is clear that Conjecture 5.2 can be suitably modified and posed in a wider context of *admissible p-adic Lie extensions*, i.e. infinite extensions of F with a pro-p, p-adic Galois group G that does not have elements of order p, and which contain F^{cyc}. Further, one could also consider more general motives and their associated Galois representations. We do not go further into the details and refer the interested reader to [12] for a striking result on these non-commutative main conjectures.

We end this paper with a discussion of the special case of elliptic curves with complex multiplication. As in §3, let K be an imaginary quadratic field of class number one, and E an elliptic curve defined over K with complex multiplication by the ring of integers $\mathcal{O}_K$

of K. Again, let p be a rational prime which splits in K, $p\mathcal{O}_K = \mathfrak{p}\mathfrak{p}^*$, and assume that E has good reductin at $\mathfrak{p}$ and $\mathfrak{p}^*$. Let

$$K_\infty = K(E_{\mathfrak{p}^\infty}),\ K'_\infty = K(E_{p^\infty}),$$

and $G' = \mathrm{Gal}(K'_\infty/K)$. As noted before, the $\mathfrak{p}$-dual Selmer group $\mathfrak{X}^{\mathfrak{p}}(E/K_\infty)$ is a finitely generated $\mathbb{Z}_p$-module. This in turn implies that $\mu_{G'}(\mathfrak{X}(E/K'_\infty)) = 0$. In fact one can show that $\mathfrak{X}(E/K'_\infty)(p) = 0$. Thus the analogue of Conjecture 5.2 in this case would be the assertion that $\mathfrak{X}(E/K'_\infty)$ is a finitely generated $\Lambda(H)$-module, where

$$H = \mathrm{Gal}(K'_\infty/F)$$

with $F = K(E_p)$. It is then easy to see that this latter assertion holds if and only if

$$\mathfrak{X}(E/F^{\mathrm{cyc}}) \text{ is a finitely generated } \mathbb{Z}_p - \text{module.} \tag{10}$$

Unfortunately, (10) remains unproven today. Note also that if E is defined over $\mathbb{Q}$, assertion (10) implies that $\mathfrak{X}(E/\mathbb{Q}^{\mathrm{cyc}})$ is always a finitely generated $\mathbb{Z}_p$-module. In other words, Mazur's examples of positive μ-invariant over $\mathbb{Q}^{\mathrm{cyc}}$ should never occur for elliptic curves with complex multiplication. But we stress that even this special consequence of (10) remains unknown. although all the numerical dats supports it. Of course, unlike the case of curves without complex multiplication, we here can formulate the main conjecture in Iwasawa theory for E/K_∞ without knowing the assertion (10), because in this case the Galois group G' is abelian. Moreover, this main conjecture has been proven by the work of Rubin and Yager [15], [21].

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