MOTIVES FROM A CATEGORICAL POINT OF VIEW

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This article is based on the second overview lecture given at the Workshop. The principal aim of these survey lectures was to provide a bird's eye view of the theory of motives vis-a-vis some of the longer courses and special lectures that were to follow. Needless to say, such a sweeping overview involves compressing a vast area, thereby necessitating omission of many details. In this article, we have largely retained the flavour of the lecture, introducing various concepts, themes and conjectures from the theory of motives.

Apart from the references in the bibliography, the interested reader is also referred to the various articles on this subject in the homepages of B. Kahn, M. Levine and J. Milne.

1 Introduction

Classical Galois theory relates finite groups to the study of polynomial equations over fields. The theory of Motivic Galois groups is a vast higher dimensional analogue, wherein 'motives' are related to finite dimensional representations of some groups, called the 'Motivic Galois groups'. The study of motives encompasses deep questions coming from such diverse areas as Hodge theory, algebraic cycles, arithmetic geometry and Galois representations.

The essential idea is the following. If G is any group, and F a field, then the category $\mathbf{Rep}_F(G)$ of finite dimensional F-representations of G has a rich structure, namely that of a 'Tannakian category'. Consider the association

$$\begin{array}{ccc} \text{Groups} & \longrightarrow & \text{Tannakian categories} \\ G & \mapsto & \mathbf{Rep}_F(G). \end{array}$$

If G is compact, then the classical theorem of Tannaka and Krein shows how the group G may be recovered from its category of representations via the obvious forgetful functor

$$\mathbf{Rep}_F(G) \longrightarrow \mathbf{Vec}_F$$

into the category $\mathbf{Vec}_F$ of finite dimensional F-vector spaces.

An analogous theorem for algebraic groups or group schemes will be discussed below. The idea is to first attach group schemes to various suitable Tannakian (sub)categories arising from the theory of motives, using realizations (Betti, Hodge, l-adic....). The group schemes associated to these categories have the property that their corresponding categories of representations are in fact equivalent to the original categories that we started with. Schematically,

$$\begin{array}{c} \text{Category } \mathcal{M} \text{ coming from motives} \\ \wr \\ \downarrow \\ G' \quad \text{pro} - \text{algebraic group schemes} \\ \wr \\ \downarrow \\ \mathbf{Rep}\ G' \longleftrightarrow \mathcal{M}, \end{array}$$

where the first vertical arrow denotes the association mentioned above and the last two sided arrow denotes equivalence. This is made more precise in the language of Tannakian categories.

2 Tannakian Categories

The main references for this section are [28], [5] and the article by L. Breen in [23], Part 1.

Definition 2.1. Let R be a commutative ring. An *R-linear category* is a category $\mathcal{C}$ such that for every pair of objects M, N in $\mathcal{C}$, the set of morphisms $\mathcal{C}(M,N)$ is an R-module, and the composition law is R-bilinear. In addition, we shall also need that finite sums exist in$\mathcal{C}$. An R-functor between two such categories is an R-linear functor.

We impose additional conditions on such an R-linear category in the definition below, referring the reader to any of the references mentioned above, for more details.

Definition 2.2. A *tensor category* over R is an R-linear category $\mathcal{C}$ with an R-bilinear tensor functor $\otimes$,

$$\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$$

which satisfies the commutativity and associativity constraints, and such that there exists a unit object $\mathbb{I}$ in $\mathcal{C}$.

In particular, given objects L, M and N there are the following functorial isomorphisms in a tensor category:

$$\begin{array}{ll} \alpha_{LMN}: & L\otimes(M\otimes N)\simeq(L\otimes M)\otimes N \\ c_{MN}: & M\otimes N\simeq N\otimes M, \text{ with } c_{MN}\circ c_{NM}=1_{M\otimes N} \\ u_M: & M\otimes\mathbb{I}\simeq M,\ u'_M:\mathbb{I}\otimes M\simeq M, \end{array}$$

such that various compatibilities are expressed by the natural commutative diagrams. We remark that other equivalent terminology for a tensor category is $\otimes$-category ACU (Saavedra-Rivano) or symmetric monoidal category.

Definition 2.3. The category $\mathcal{C}$ has an *internal hom* functor

$$\begin{array}{rcl} \hom:\mathcal{C}\times\mathcal{C} & \to & \mathcal{C} \\ (X,Y) & \mapsto & \hom(X,Y) \end{array}$$

if $\hom(X,Y)$ is the representing object for the functor

$$\begin{array}{rcl} \mathcal{C}^{\mathrm{op}} & \to & \mathbf{Sets} \\ M & \mapsto & \mathcal{C}(M\otimes X,Y). \end{array}$$

Suppose that the internal hom functor exists in $\mathcal{C}$. Then the *dual object* $M^\vee$ for every object M of $\mathcal{C}$ is defined by $M^\vee=\hom(M,\mathbb{I})$.

We thus have a duality functor

$$\begin{array}{rcl} \vee:\mathcal{C} & \longrightarrow & \mathcal{C}^{\mathrm{op}} \\ M & \mapsto & M^\vee \\ \{M\overset{f}{\mapsto}N\} & \mapsto & \{N^\vee\overset{{}^tf}{\to}M^\vee\} \end{array}$$

and evaluation maps for every $M\in\mathcal{C}$,

$$\text{(evaluation)}\quad \varepsilon:\ M\otimes M^\vee\ \to\ \mathbb{I}.$$

The category is said to be *rigid* if there are also *coevaluation maps* η for every object $M\in\mathcal{C}$,

$$\text{(coevaluation)}\quad \eta:\ \mathbb{I}\ \to\ M^\vee\otimes M$$

with the property that the composites below

$$M\overset{u_M^{-1}}{\simeq}M\otimes\mathbb{I}\overset{1_M\otimes\eta}{\longrightarrow}M\otimes M^\vee\otimes M\overset{\varepsilon\otimes1_M}{\longrightarrow}\mathbb{I}\otimes M\simeq M,$$

$$M^\vee \overset{(u'_{M^\vee})^{-1}}{\simeq} \mathbb{I} \otimes M^\vee \overset{\eta\otimes 1_M}{\longrightarrow} M^\vee \otimes M \otimes M^\vee \overset{1_M \otimes \varepsilon}{\longrightarrow} M^\vee \otimes \mathbb{I} \overset{u_{M^\vee}}{\simeq} M^\vee,$$

are respectively 1_M and $1_{M^\vee}$. Further, there are functorial isomorphisms

$$\hom(Z, \hom(X, Y)) \simeq \hom(Z \otimes X, Y), \ \ X^\vee \otimes Y \simeq \mathit{hom}(X, Y).$$

Definition 2.4. Given an R-rigid tensor category $\mathcal{C}$, every endomorphism $f \in \mathrm{End}(M)$ has a *trace*, denoted by $\mathrm{tr}(f)$, which is an element of the commutative R-algebra $\mathrm{End}(\mathbb{I})$. It is defined as the composite

$$\mathbb{I} \overset{\eta}{\longrightarrow} M^\vee \otimes M \overset{1_{M^\vee} \otimes f}{\longrightarrow} M^\vee \otimes M \overset{c_{M^\vee M}}{\longrightarrow} M \otimes M^\vee \overset{\varepsilon}{\longrightarrow} \mathbb{I}.$$

We thus get a map

$$\mathrm{tr} : \mathrm{End}(M) \to \mathrm{End}(\mathbb{I})$$

for every object M of $\mathcal{C}$. The *dimension* or *rank* of an object M in $\mathcal{C}$ is then defined as

$$\dim M := \mathrm{tr}(I_M)$$

Examples. (1) The prototype is $\mathcal{C} = \mathbf{Rep}_F(G)$ where F is a field and G any group. The usual tensor product of representations gives the tensor functor while $\mathbb{I}$ is the trivial representation and $\vee$ denotes the contragredient representation functor. The notions of trace and dimension are the usual ones. More generally, if R is a commutative ring, the category of R-modules is a rigid tensor category.

(2) Let F be a field, and $\mathcal{C} := \mathbf{VecGr}_F$ be the category of $\mathbb{Z}$-graded F-vector spaces (V_n) such that $\underset{n}{\oplus} V_n$ has finite dimension. We shall mainly consider this category, but with the Koszul rule for the commutativity constraint. In other words, consider the isomorphisms

$$\overset{\star}{\phi} : V \otimes W \simeq W \otimes V,$$

with $\overset{\star}{\phi} = \underset{r,s}{\oplus} (-1)^{rs} \phi^{r,s}$, where

$$\phi^{r,s} : V^r \otimes W^s \to W^s \otimes V^r$$

is the usual isomorphism in $\mathcal{C}$. With this latter definintion, if $V = (V_n)$ is an object of $\mathcal{C}$, then the rank of V is the 'super-dimension' $\dim V^+ - \dim V^-$, where $V^+ = \oplus V^{2k}$ and $V^- = \oplus V^{2k+1}$. With the usual tensor functor $\otimes_F$, the category $\mathcal{C}$ is a rigid tensor category.

(3) The category of vector bundles over a variety X/F is a F-rigid tensor category.

A tensor functor $\Phi : \mathcal{C} \to \mathcal{C}'$ between tensor categories is a functor preserving the tensor structure, i.e. there exist functorial isomorphisms

$$\kappa_{M,N} : \Phi(M) \otimes \Phi(N) \simeq \Phi(M \otimes N)$$

which are compatible with the associativity and commutativity constraints and such that the identity object of $\mathcal{C}$ is mapped to that of $\mathcal{C}'$. If further, $\mathcal{C}$ and $\mathcal{C}'$ are rigid, then there are functorial isomorphisms

$$\Phi(M^\vee) \simeq \Phi(M)^\vee.$$

There is an obvious notion of tensor equivalence between tensor categories. Further, we have $\mathrm{tr}(\Phi(f)) = \Phi(\mathrm{tr}(f))$ and $\dim(\Phi(M)) = \Phi(\dim(M))$.

If Φ and Φ' are tensor functors, then $\hom^{\otimes}(\Phi, \Phi')$ is the set of morphisms (i.e. natural transformations) of tensor functors. Further, if the categories $\mathcal{C}$ and $\mathcal{C}'$ are rigid, then any morphism of tensor functors is an isomorphism.

We now outline how the set $\hom^{\otimes}$ is given an additional structure. For any field F and an F-algebra R, there is a canonical $\otimes$-functor,

$$\begin{array}{rcl} \Phi_R : \mathbf{Vec}_F & \to & \underline{\mathrm{Mod}}_R \\ V & \mapsto & V \otimes_F R. \end{array}$$

If Ψ and Λ are tensor functors from $\mathcal{C} \to \mathbf{Vec}_F$, then we define $\hom^{\otimes}(\Psi, \Lambda)$ to be the functor from the category of F-algebras to the category of sets such that

$$\hom^{\otimes}(\Psi, \Lambda)(R) = \hom^{\otimes}(\Phi_R \circ \Psi, \Phi_R \circ \Lambda).$$

Definition 2.5. An *additive (resp. abelian) tensor category* is a tensor category $\mathcal{C}$ over R such that $\mathcal{C}$ is additive (resp. abelian) and the tensor functor is biadditive. If in addition, we have $R = \mathrm{End}(\mathbb{I})$, then such a category is said to be an *additive* (resp. *abelian*) *tensorial* category.

There is the notion of tensor subcategories generated by subsets of objects; briefly this is the smallest tensor subcategory containing the generating set of objects.

We now come to the important notion of fibre functors which is crucial to define Tannakian categories.

Definition 2.6. Let $R = F$ be a field and $\mathcal{C}$ a rigid abelian tensorial category so that $\mathrm{End}(\mathbb{I}) = F$. A *fibre functor* on $\mathcal{C}$ is a faithful, exact, tensor functor

$$\omega : \mathcal{C} \to \mathbf{Vec}_{F'}$$

into the rigid category of F'-vector spaces of finite dimension over F', where F' is an unspecified algebraic field extension of F.

We say that $\mathcal{C}$ is *Tannakian* if $\mathcal{C}$ has a fibre functor; $\mathcal{C}$ is *neutral* if $\mathcal{C}$ has a fibre functor into $\mathbf{Vec}_F$ and $\mathcal{C}$ is *neutralized* if a fibre functor into $\mathbf{Vec}_F$ has been specified.

Given a fibre functor $\omega : \mathcal{C} \to \mathbf{Vec}_F$, one can define the affine group scheme $\mathbf{G}(\omega)$ over F by

$$\mathbf{G}(\omega) = \mathrm{Aut}^{\otimes} \omega$$

where the latter is viewed as a scheme via its 'functor of points' on F-algebras.

The following deep theorem is the centrepiece of Tannakian formalism.

Theorem 2.7. *Let $(\mathcal{C}, \omega)$ be a neutralized Tannakian category over a field F, and let $\mathbf{G}(\omega) = \mathrm{Aut}^{\otimes} \omega$ be the associated group scheme over F. Then $\mathbf{G}(\omega)$ is an affine, flat F-group scheme and the functor*

$$\rho : \mathcal{C} \to \mathbf{Rep}_F(\mathbf{G}(\omega))$$

is an equivalence of categories.

In the simplest case of $\mathcal{C} = \mathbf{Rep}_F(G)$ for an algebraic group G, the group can thus be recovered as the automorphism group of the canonical fibre functor, given by the forgetful functor on representations.

Examples. (i) Let $\mathcal{C} = \mathbf{VecGr}_K$ as before. An object is thus a collection $V = (V_n)_{n \in \mathbb{Z}}$ such that $\oplus V_n$ is finite dimensional over the field K. Consider the fibre functor

$$\begin{aligned} \omega : \mathcal{C} &\to \mathbf{Vec}_K \\ V &\mapsto \oplus V_n. \end{aligned}$$

Then $\mathbf{G}(w) = \mathbb{G}_m$.

(ii) **Hodge structures:** Let $\mathcal{C} = \mathbf{Hod}_{\mathbb{R}}$, the catgory of real Hodge structures. If V is an object of $\mathcal{C}$, then recall that V is a real vector space of finite dimension such that there exists an isomorphism

$$V_{\mathbb{C}} := V \otimes \mathbb{C} \simeq \oplus V^{p,q}, \quad \text{with} \quad V^{p,q} = \overline{V^{q,p}},$$

where $^{-}$ denotes complex conjugation. Then $\mathcal{C}$ is a Tannakian category and we have the natural fibre functor

$$\begin{array}{rcl}\omega : \mathcal{C} & \rightarrow & \mathbf{Vec}_{\mathbb{R}} \\ V & \mapsto & V.\end{array}$$

In this case, the group $\mathbf{G}(\omega) = \mathbb{S}$ which is the torus defined by $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$, where $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}$ denotes Weil restriction of scalars.

Now define the *weight cocharacter* $\omega : \mathbb{G}_m \rightarrow \mathbb{S}$ given on points by the natural inclusion $\mathbb{R}^* = \mathbb{G}_{m,\mathbb{R}}(\mathbb{R}) \rightarrow \mathbb{S}(\mathbb{R}) = \mathbb{C}^*$. An arbitrary $\mathbb{Q}$-Hodge structure is a mixed Hodge structure (see [6, 2.3.1, 2.3.8], [7]), which is described by a vector space $V/\mathbb{Q}$ of finite dimension along with a homomorphism $h : \mathbb{S} \rightarrow \mathrm{GL}(V)_{\mathbb{R}}$ such that

$$h \circ \omega : \mathbb{G}_{m,\mathbb{R}} \rightarrow \mathbb{S}$$

is defined over $\mathbb{Q}$. In other words, the weight decomposition of $V_{\mathbb{C}}$ comes from a decomposition over $\mathbb{Q}$.

We now turn to some properties of the affine group scheme.

Properties: Let $\mathbf{G} = \mathbf{G}(\omega)$ be obtained by the above formalism on some Tannakian category $\mathcal{C}$. Then:

- $\mathbf{G}$ is finite if and only if there is an object M of $\mathbf{Rep}_F(\mathbf{G}) \simeq \mathcal{C}$ such that every object of $\mathbf{Rep}_F\,\mathbf{G}$ is isomorphic to a subquotient of $M^n, \quad n \geq 0$.
- $\mathbf{G}$ is algebraic if and only if there exists an object M of $\mathbf{Rep}_F\,\mathbf{G}$ which is a tensor generator of $\mathbf{Rep}_F\,\mathbf{G}$.
- Assume that the field F has characteristic zero. Then $\mathbf{G}$ is connected if and only if for any nontrivial representation M of $\mathbf{G}$, the strictly full subcategory of $\mathbf{Rep}_F\,\mathbf{G}$ whose objects are isomorphic to subquotients of $M^n, \quad n \geq 0$, is not stable under the tensor product.
- Assume that F has characteristic zero and that $\mathbf{G}$ is connected. Then $\mathbf{G}$ is pro-reductive (i.e. projective limit of reductive groups) if and only if $\mathbf{Rep}_F\,\mathbf{G}$ is semisimple.

3 Category of motives

The references for this section are [11], [19], [20] and the articles by S. Kleiman and A. Scholl in [23].

In this section, k will denote a fixed base field. We consider the category $\mathcal{V}_k$ of smooth projective varieties over k and an adequate equivalence relation $\sim$ on the groups

$Z^i(X)$, (see any of the references above for definition of adequate equivalence) where $Z^i(X)$ denotes the group of algebraic cycles of codimension i on X, such that given a morphism $f : X \to Y$ in $\mathcal{V}_k$, the pushforward f_*, pull-back f^* and intersection of cycles are well-defined modulo the equivalence relation $\sim$. Examples of adequate relations are:

$$\begin{array}{ll} \sim_{\mathrm{rat}} : \text{rational equivalence} & \sim_{\mathrm{alg}} : \text{algebraic equivalence} \\ \sim_{\mathrm{num}} : \text{numerical equivalence} & \sim_{\mathrm{hom}} : \text{homological equivalence} \end{array}$$

We set

$$\mathcal{A}^i_{\sim}(X) := Z^i(X)/\sim .$$

The idea is to construct a (Tannakian) category of pure motives whose most basic constituents come from smooth projective varieties. This category is obtained in three steps, starting from $\mathcal{V}_k$:

Linearization $\rightsquigarrow$ pseudoabelianization $\rightsquigarrow$ Inversion.

Step 1 : Linearization

A different set of morphisms on $\mathcal{V}_k$ using algebraic cycles is defined which enables one to give an additive structure on the set of morphisms. More precisely, we define a category $\mathbf{Corr}_{\sim}(k)$, the category of correspondences over k, whose objects are the same as those of $\mathcal{V}_k$, but denoted by $h(X)$ for X in $\mathcal{V}_k$. The morphims are given by

$$\mathbf{Corr}_{\sim}(k)(X,Y) = \bigoplus_i \mathcal{A}^{\dim X_i}_{\sim}(X_i \times Y),$$

where X_i are the irreducible components of X, and $\mathcal{A}^*_{\sim}$ denotes the group of cycles modulo the adequate equivalence relation $\sim$. The composition of two morphisms α in $\mathbf{Corr}_{\sim}(k)(X,Y)$ and β in $\mathbf{Corr}_{\sim}(k)(Y,Z)$ (which we assume to be simple, in the sense that there is a single element in the sum) is given by $p_{XZ*}(p^*_{XY}(\alpha) \cdot p^*_{YZ}(\beta))$. Here $\cdot$ denotes the intersection product in $X \times Y \times Z$, and p_{XY}, p_{YZ} and p_{XZ} are the projections onto the corresponding products of the factors. Given any morphism $f : X \to Y$ of smooth projective varieties, the associated graph Γ_f is a correspondence and we therefore obtain a *contravariant* functor

$$h : \mathcal{V}_k \to \mathbf{Corr}_{\sim}(k).$$

Note that here we are following the classical convention of Grothendieck. The other convention is to define the morphisms using correspondences with codimension Y_j, as Y_j

varies over the irreducible components of Y, so as to make the functor covariant, as is followed in Fulton's book or by Voevodsky (see the article [19] for more on the philosophy behind these conventions).

The category $\mathbf{Corr}_\sim(k)$ has a tensor structure given by $h(X) \otimes h(Y) = h(X \times Y)$, and also an identity object given by $\mathbb{I}_k = h(\operatorname{Spec} k)$. Further, if A is any commutative ring, then $\mathbf{Corr}_\sim(k, A)$ defined by tensoring morphisms with A makes it into an A-linear, additive tensor category.

Step 2: Pseudoabelianization

This is a fairly general construction on categories, and a partial step towards obtaining abelian categories from additive ones. It consists of formally adjoining the kernels to idempotent endomorphisms or projectors in the given additive category (recall that idempotent endomorphisms p in the category are those satisfying the property $p^2 = p$). Let $\mathbf{Mot}^{\mathrm{eff}}_\sim(k, A)$ denote the pseudoabelianization (also called idempotent completion, or the Karoubi envelope) of $\mathbf{Corr}_\sim(k, A)$. The objects here are now pairs $(h(X), p)$ with $p^2 = p$ in $\operatorname{End}(h(X))$ and morphisms are given by

$$\mathbf{Mot}^{\mathrm{eff}}_\sim(k, A)((h(X), p), (h(Y), q)) = q\, \mathbf{Corr}_\sim(h(X), h(Y))p.$$

The tensor structure is given by

$$(h(X), p) \otimes (h(Y), q) = (h(X \times Y),\ p \times q),$$

and we thus get a pseudoabelian A-linear additive tensor category, $\mathbf{Mot}^{\mathrm{eff}}_\sim(k, A)$ called the category of *effective pure motives.* An effective pure motive is essentially a direct factor of the motive of a smooth projective variety and we have a natural tensor functor

$$\begin{array}{ccc} \mathbf{Corr}_\sim(k, A) & \to & \mathbf{Mot}^{\mathrm{eff}}_\sim(k, A) \\ h(X) & \mapsto & (h(X), 1_X). \end{array}$$

In $\mathbf{Mot}^{\mathrm{eff}}_\sim(k, A)$, the motive of $\mathbb{P}^1_k$ decomposes as

$$h(\mathbb{P}^1_k) = \mathbb{I}_k \oplus \mathbb{L},$$

where $\mathbb{L}$ is the *Lefschetz motive.* This splitting arises from a choice of a k-point which gives the factor $\mathbb{I}_k$.

Step 3: Inversion

This is again a general construction with tensor categories and consists of formally inverting an object in the given tensor category, thereby forming a new category. Applying this construction to $\mathbf{Mot}^{\mathrm{eff}}_{\sim}(k, A)$ and formally inverting the Lefschetz motive, we get a category which is denoted by $\mathbf{Mot}_{\sim}(k, A)$. In this specific case, this step brings us closer to obtaining a rigid category. The objects of $\mathbf{Mot}_{\sim}(k, A)$ consists of pairs (M, i) where M is an object of $\mathbf{Mot}^{\mathrm{eff}}_{\sim}(k, A)$ and $i \in \mathbb{Z}$. The morphisms are given by

$$\mathbf{Mot}_{\sim}(k, A)((M, i), (N, j)) = \lim_{\rightarrow n} \mathbf{Mot}^{\mathrm{eff}}_{\sim}(M \otimes \mathbb{L}^{n-i}, N \otimes \mathbb{L}^{n-j}) \otimes_{\mathbb{Z}} A,$$

where $\mathbb{L}^m := \mathbb{L}^{\otimes m}$.

The category $\mathbf{Mot}_{\sim}(k, A)$ is a rigid pseudoabelian tensor category in which the Lefschetz motive $\mathbb{L}$ is invertible. We remark in passing that though not at all evident, it is not difficult to prove that the tensor structure on $\mathbf{Mot}^{\mathrm{eff}}_{\sim}(k, A)$ passes over to $\mathbf{Mot}_{\sim}(k, A)$. There is a natural fully faithful functor of tensor categories

$$\begin{array}{ccc} \mathbf{Mot}^{\mathrm{eff}}_{\sim}(k, A) & \rightarrow & \mathbf{Mot}_{\sim}(k, A) \\ M & \mapsto & (M, 0) \end{array}$$

and $\mathbf{Mot}_{\sim}(k, A)$ satisfies the universal property of the Lefschetz motive being invertible. In the category $\mathbf{Mot}^{\mathrm{eff}}_{\sim}$, the objects are given by triples $(h(X), p, n)$, where X is a smooth projective variety, p an indempotent in $\mathrm{End}(h(X))$ and n an integer. Summarizing, we have the diagram

$$\mathcal{V}_k \overset{h}{\rightsquigarrow} \mathbf{Corr}_{\sim}(k, A) \overset{\sharp}{\rightsquigarrow} \mathbf{Mot}^{\mathrm{eff}}_{\sim}(k, A) \overset{\mathbb{L}^{-1}}{\rightsquigarrow} \mathbf{Mot}_{\sim}(k, A),$$

where $\sharp$ denotes pseudoabelianization. The object $(\mathbb{I}, 1, -1)$ in $\mathbf{Mot}_{\sim}(k, A)$ is also denoted by $\mathbb{L}^{-1}$ or $\mathbb{T}$ and is called the *Tate motive*. For any object M of $\mathbf{Mot}_{\sim}(k, A)$, we set $M(1) := M \otimes \mathbb{L}^{-1}$. For simplicity, we shall use the notation $h(X)$ to denote the image of a smooth, irreducible, projective variety X in $\mathbf{Mot}_{\sim}(k, A)$. Further, the category $\mathbf{Mot}_{\sim}(k, A)$ is rigid, the dual $h(X)^{\vee}$ being given by $h(X)(d)$, where d is the dimension of X.

The category $\mathbf{Mot}_{\mathrm{rat}}(k, A)$ is the category of *pure Chow motives*, which was originally considered by Grothendieck.

4 Motives and Tannakian categories

The references for this section are [1], [2], [5] and [17].

From now on, the coefficient ring A is assumed to be a field F of characteristic zero. We next study the question of how and when one obtains Tannakian categories from the category $\mathbf{Mot}_{\sim}(k, F)$. The following result is due to Deligne:

Theorem 4.1. *(Deligne) Let $\mathcal{C}$ be an abelian F-linear, rigid tensorial category. Then $\mathcal{C}$ is Tannakian if and only if the dimension of any object of $\mathcal{C}$ is a natural number.*

Recall that an object in an abelian category $\mathcal{C}$ is *simple* if it does not possess any proper non-zero subobject. The abelian category $\mathcal{C}$ is *semisimple* if every object of $\mathcal{C}$ is a direct sum of simple objects. The following striking result of Jannsen gives sufficient conditions for the category $\mathbf{Mot}_{\sim}(k, F)$ to be abelian semisimple.

Theorem 4.2. *(Jannsen) The following assertions are equivalent:*
(a) $\mathbf{Mot}_{\sim}(k, F)$ is a semisimple abelian category.
(b) The group of algebraic cycles $Z^{\dim \mathrm{X}}(X \times X) \otimes F$ on $X \times X$ of dimension equal to that of X is a finite dimensional semisimple F-algebra for every object X of $\mathbf{Mot}_{\sim}(k, F)$.
(c) The relation $\sim$ is numerical equivalence.

Corollary 4.3. *The category $\mathbf{Mot}_{\mathrm{num}}(k, F)$ is a semisimple F-linear, rigid tensorial category.*

Given this result, a natural question is the following: When is $\mathbf{Mot}_{\mathrm{num}}(k, F)$ Tannakian. Note that all that is needed for $\mathbf{Mot}_{\mathrm{num}}(k, F)$ to be Tannakian is that there exist a fibre functor with values in $\mathbf{Vec}_{F'}$ for some finite extension F' of $F = \mathrm{End}(\mathbb{I})$. In this case, it is easily seen that the dimension of $h(X)$ is a certain Euler characteristic (cf. Example (2) in §2) and hence need not always be a positive integer. However, if a fibre functor exists, its dimension in the category of vector spaces will be a natural number. To reconcile this dilemma, Deligne considers the category $\mathbf{Mot}_{\mathrm{AHS}}(k, \mathbb{Q})$ of motives with respect to *Absolute Hodge Cycles*, which in fact has a structure of a graded category (i.e. objects are $\mathbb{Z}$-graded). The commutativity constraint is then modified by using the grading. This modification though subtle, is fundamental. While it leaves the objects, morphisms and the tensor structure unchanged, the commutativity constraint is modified. Under certain strong conditions, which are always conjectured to be true, this construction can also be imitated in the category $\mathbf{Mot}_{\mathrm{num}}(k, A)$, and we denote the new modified category thus obtained by $\dot{\mathbf{Mot}}_{\mathrm{num}}(k, F)$. This is stated formally in the following theorem due to Jannsen. The hypotheses of the theorem is elaborated upon in the two interludes that follow the statement.

Theorem 4.4. *(Jannsen) The F-linear tensorial category $\mathbf{Mot}_{\mathrm{num}}(k, F)$ is semisimple. If the Künneth components of the diagonal (with respect to some fixed Weil cohomology theory) are algebraic for every X in $\mathcal{V}_k$, then the modified category $\dot{\mathbf{Mot}}_{\mathrm{num}}(k, F)$ is a semisimple Tannakian category.*

Interlude on Weil cohomology

A *Weil cohomology theory with coefficients in a field F* is a functor

$$H^* : \mathcal{V}_k^{\mathrm{op}} \to \mathbf{VecGr}_F$$

such that the following hold:
• $\dim\, H^2(\mathbb{P}^1) = 1$
• Künneth fomula: $H^*(X \times Y) \simeq H^*(X) \otimes H^*(Y)$.
Observing that $H^2(\mathbb{P}^1)$ is invertible in $\mathbf{VecGr}_F$, since $\dim_F H^2(\mathbb{P}^1) = 1$, we note that for any integer r, the *Tate twist*

$$V \to V \otimes H^2(\mathbb{P}^1)^{\otimes(-r)} =: V(r),$$

is an operation in $\mathbf{VecGr}_F$.
• Multiplicative trace map: There is a *trace map*

$$\mathrm{tr} : H^{2d}(X)(d) \to F$$

where $d = \dim\, X$ which induces a 'Poincaré duality'.
• Given a homomorphism $A \to F$, there are multiplicative, contravariant and normalized cycle class maps

$$\mathrm{cl_n} : Z^n(X) \otimes A \to H^{2n}(X)(n).$$

We shall need the following compatibility condition, as was pointed out by the referee. For $n = 1$ and $X = \mathbb{P}^1$, the above cycle map cl_1 gives a map from $\mathbb{Z}$ (on identifying the Chow group of zero cycles on the projective line with $\mathbb{Z}$, via the degree map) to F. We shall require that this map takes the identity element to the identity element. With this assumption, the datum of a Weil cohomology is almost equivalent to that of a $\otimes$-functor from the category of Chow motives to the category of graded vector spaces.

Recall that a cycle is homologically equivalent to zero with respect to H if and only if it maps to zero under the cycle class maps. The classical cohomology theories given by Betti cohomology, étale cohomology (with $\mathbb{Q}_l$-coefficients for $l \neq \mathrm{char}\, k$) and de Rham cohomology are all Weil cohomology theories. If $\mathrm{char}\, k = p$, and F is the quotient field of the ring of Witt vectors $W(k)$ of k, then crystalline cohomology $H_{\mathrm{cris}}(X)$ is a Weil cohomology theory. Any algebraic correspondence in $\mathbf{Corr}_k(X, Y)$ induces a map between $H^*(X) \to H^*(Y)$. More generally, the axioms of the Weil cohomology theory yield canonical isomorphisms (we warn the reader that we are neglecting Tate twists in this paragraph and the next),

$$H^*(X \times Y) \simeq H^*(X) \otimes H^*(Y) \simeq \mathrm{Hom}(H^*(X), H^*(Y)).$$

Thus an element u of $H^*(X \times Y)$ may be viewed as an operator from $H^*(X)$ to $H^*(Y)$, and is referred to as a *cohomological correspondence*. If u lies in the $\mathbb{Q}$-vector subspace of $H^*(X \times X)$ generated by the images of the cycle map on $X \times X$, then u is said to be *algebraic*.

We now explain the hypothesis that the diagonal components are algebraic, remarking that it is conjectured to be true. Let $\Delta_X \subset X \times X$ denote the diagonal and consider its image in $H^{2d}(X \times X)$ under the cycle class map with respect to a Weil cohomology theory, where $d = \dim X$. By Künneth decomposition, we have

$$H^{2d}(X \times X) \simeq \bigoplus_j H^{2d-j}(X) \otimes H^j(X),$$

and the corresponding components of Δ_X considered in $H^{2d}(X \times X)$ are the Künneth components. The hypothesis of algebraicity is the statement that each component in fact is algebraic (see Kleiman's article in Volume 1 of [23]).

If we assume Grothendieck's standard conjectures in $\operatorname{char} k = 0$ (see [11]), then $\mathbf{Mot}_{\mathrm{num}}(k, F) = \mathbf{Mot}_{\mathrm{hom}}(k, F)$ and the category $\overset{\bullet}{\mathbf{Mot}}_{\mathrm{num}}(k, F)$ is semisimple Tannakian. Of course, the standard conjectures lie very deep and we only briefly touch upon this subject, especially as this is well treated in the literature. Finally, we remark that the theorems of Jannsen, when they were first proved, were greeted with surprise as the semisimplicity with respect to numerical equivalence did not require assuming the standard conjectures.

Interlude on Standard Conjectures

Inspired by Serre's work on Kählerian varieties and his letter to Weil, Grothendieck formulated the standard conjectures on algebraic cycles in the 1960's and showed (as had Bombieri, independently) that these imply the Weil conjectures. We briefly recall these conjectures below, remarking that unlike the Weil conjectures, they remain largely open to this day. Let $X \in \mathcal{V}_k$ be a smooth projective algebraic variety of dimension d and let H^* denote a fixed Weil cohomology theory. Let $D \in \operatorname{Pic}(X)$ be the class of an ample divisor on X and let η be its image in $H^2(X)(1)$ under the cycle class map; η is then called a *polarisation* of X. The *Lefschetz operator* $L = L_\eta$ is the cup-product by η on H^* and there are maps

$$L^{d-i} : H^i(X)(r) \to H^{2d-i}(X)(d - i + r)$$

for all $i \geq 0$. The strong and weak Lefschetz theorems are assertions about this operator. The strong Lefschetz theorem asserts that it is an isomorphism for all natural integers $i \leq d$ and any integer r. The weak Lefschetz theorem states that for $L = L_{\eta_Y}$, corresponding to the class η_Y of a smooth hyperplane section $i : Y \hookrightarrow X$, the induced map $H^i(X) \to H^i(Y)$ is an isomorphism for every $i \leq d-2$ and is injective for $i = d-1$. The Lefschetz involution $*_L = *_{L,X}$ is defined as the operator L^{d-i} on $\bigoplus_{i,r} H^i(X)(r)$ if $i \leq d$ and its inverse for $i > d$. There is also the Hodge involution $*_H = *_{H,X}$ which we do not describe precisely here except for stating that it is the Lefschetz involution upto a sign on a certain 'primitive' decomposition of $H^*(X)$ (see the references at the beginning of §4 for more details). We shall also need the fact that $*_H$ induces a $\mathbb{Q}$-valued quadratic form $q_H(X)$ on $\mathcal{A}^*_{\text{hom}}(X)\otimes\mathbb{Q}$, the $\mathbb{Q}$-vector space of algebraic cycles modulo homological equivalence. The following are the assertions of the Standard Conjectures:

I. C(X) (Standard conjecture of Künneth type): The Künneth components of the diagonal Δ_X are algebraic.

We have already commented on the diagonal components with respect to the Künneth decomposition. This conjecture is known to be true for all X if k is a finite field and H^* is any of the classical Weil cohomologies. It is also known to hold if X is an abelian variety over an arbitrary base field k and H^* is an arbitrary Weil cohomology theory.

II. B(X) (Standard conjecture of Lefschetz type): The Lefschetz involution $*_{L,X}$ is algebraic (with $\mathbb{Q}$-coefficients).

In other words, this conjecture asserts that the map between the cohomology groups defined by the Lefschetz involution, viewed as a correspondence, is algebraic in the sense mentioned before. This conjecture is known to be true in dimensions at most 2 (and at most 4 in characteristic zero) and for abelian varieties.

III. I(X) (Standard conjecture of Hodge Type): The quadratic form $q_H(X)$ with values in $\mathbb{Q}$ is positive definite.

It can be verified that for the classical Weil cohomology theories, this conjecture is true for any field k once it is known to be true for all finite fields. If k is of characteristic zero and H^* is one of the classical cohomology theories, this conjecture reduces to the Hodge index theorem. In any characteristic, it is known to hold for cycles modulo homological equivalence over $\mathbb{Q}$ in dimensions $0\,,1\,,d-1$ and d. For abelian varieties over finite fields, there are some partial results.

IV. D(X) (Numerical and Homological equivalence): Homological equivalence coincides over $\mathbb{Q}$ with numerical equivalence, i.e. $\mathcal{A}^*_{\text{num}}(X) \otimes \mathbb{Q} = \mathcal{A}^*_{\text{hom}}(X) \otimes \mathbb{Q}$.

This conjecture is trivially true for cycles of codimension 0 and d in any characteristic. If the characteristic of k is zero, it is also true in codimensions 1 and $d-1$. In the case of characteristic zero, it is known to hold also for abelian varieties.

We remark that the conjectures B(X) and I(X) together imply D(X). In characteristic zero, B(X) implies all the standard conjectures.

5 Motivic Galois groups

The references for this section are [1], [21] and the articles by N. Schappacher and J.-P. Serre in [23].

In this section, we shall assume that the field k has characteristic zero. In addition, we make the very strong assumption that Grothendieck's standard conjectures are true. Under these hypotheses, note that $\overset{\bullet}{\mathbf{Mot}}_{\mathrm{num}}(k,\mathbb{Q})$ is semisimple Tannakian, by Theorem 4.4. The classical cohomology theories, in particular, H^*_σ, which is the Betti realization for an embedding $\sigma : k \hookrightarrow \mathbb{C}$, gives a fibre functor

$$H^*_\sigma : \overset{\bullet}{\mathbf{Mot}}_{\mathrm{num}}(k,\mathbb{Q}) \to \mathbf{Vec}_{\mathbb{Q}}.$$

Hence $\overset{\bullet}{\mathbf{Mot}}_{\mathrm{num}}(k,\mathbb{Q})$ is neutral and by the Tannakian formalism outlined in §2, we obtain the corresponding pro-algebraic, affine $\mathbb{Q}$-group scheme $\mathbf{G}_{\mathbf{Mot},k} = \mathbf{G}_{\mathbf{Mot},B} := \mathrm{Aut}^{\otimes}(H^*_\sigma)$. This group of course depends on the chosen embedding σ but we shall tacitly ignore this dependence as it is largely irrelevant for our purposes.

Given a smooth projective variety X in $\overset{\bullet}{\mathbf{Mot}}_{\mathrm{num}}(k,\mathbb{Q})$, consider the Tannakian subcategory $\mathcal{M}_X$ generated by $h(X)$. By restricting the fibre functor above to this subcategory, we then obtain the corresponding group $\mathbf{G}_{\mathbf{Mot},B}(X)$ over $\mathbb{Q}$, called the *motivic Galois group* of X. More generally, given an object E in $\overset{\bullet}{\mathbf{Mot}}_{\mathrm{num}}(k,\mathbb{Q})$, we can analogously define the Tannakian subcategory $\mathcal{M}_E$ and the associated motivic Galois group $\mathbf{G}_{\mathbf{Mot},B}(E)$. Let $h(X)$, $h(X')$ be objects in $\overset{\bullet}{\mathbf{Mot}}_{\mathrm{num}}(k,\mathbb{Q})$. We introduce an ordering $M \prec M'$ in $\overset{\bullet}{\mathbf{Mot}}_{\mathrm{num}}(k,\mathbb{Q})$ if M belongs to $\mathcal{M}_{M'}$. Then there are natural *surjective* transition maps $\mathbf{G}_{mot,B}(X') \to \mathbf{G}_{\mathbf{Mot},B}(X)$ with respect to this ordering, and we define

$$\mathbf{G}_{\mathbf{Mot},B} := \varprojlim \mathbf{G}_{\mathbf{Mot},B}(X)$$

where the projective limit is taken with respect to these surjective transition maps.

Examples. (1) Let $\mathcal{M}^0$ be the Tannakian category of Artin motives (corresponding to Artin representations realized over $\mathbb{Q}$). Recall that this is the subcategory of $\overset{\bullet}{\mathbf{Mot}}_{\mathrm{num}}(k,\mathbb{Q})$ generated by $h(\operatorname{Spec} E)$ where E runs over all finite extensions of k. Then the corresponding group $\mathbf{G}_{\mathbf{Mot},\mathcal{M}^0} = \mathrm{Gal}(\bar{k}/k)$. Thus the Galois group of $\Gamma_k = \mathrm{Gal}(\bar{k}/k)$, where $\bar{k}$ is a separable closure of k, is a natural quotient of the motivic Galois group. There is a natural exact sequence

$$0 \to \mathbf{G}^0_{\mathbf{Mot},B} \to \mathbf{G}_{\mathbf{Mot},B} \to \Gamma_k \to 0,$$

which is functorial with respect to algebraic extensions of the base field k. Conjecturally, $\mathbf{G}^0_{\mathbf{Mot},B} = \mathbf{G}_{\mathbf{Mot},\bar{k}}$.

(2) Let $\mathcal{M}_{\mathbb{T}}$ denote the Tannakian subcategory of Tate motives, which is the subcategory generated by the Tate or Lefschetz motive. Then $\mathbf{G}_{\mathbf{Mot},\mathcal{M}_{\mathbb{T}}} = \mathbb{G}_m$.

(3) Let E be an elliptic curve over $\mathbb{Q}$. Then

$$\mathbf{G}_{\mathbf{Mot},B}(E) = \begin{cases} \mathrm{GL}_2 & \text{if E has no complex multiplication} \\ \mathrm{T} & \text{if E has complex multiplication,} \end{cases}$$

where T is either a maximal torus in GL_2 or a normaliser of a maximal torus. This is, of course, related to the l-adic realizations of the motive associated to E. Further, the inclusion $\mathcal{M}_{\mathbb{T}} \to \mathcal{M}_E$ corresponds to the morphism $\mathbf{G}_{\mathbf{Mot},B}(E) \to \mathbb{G}_m$ given by the determinant of matrices.

(4) Consider the (semisimple Tannakian) category of pure motives $\overset{\bullet}{\mathbf{Mot}}_{\mathrm{AHS}}(k, \mathbb{Q})$ with respect to Absolute Hodge cycles, à la Deligne. Let $\mathcal{CM}_k$ be the smallest full Tannakian subcategory generated by the motives $h^1(A)$ for all abelian varieties A defined over k, such that A is potentially CM. With respect to the fibre functor defined by the Betti realization, we obtain a pro-algebraic group scheme over $\mathbb{Q}$, which is the Taniyama group, as considered by Langlands. This will be treated in greater detail later in the lectures of Clozel and Fargues (see [21]).

(5) Let V be a $\mathbb{Q}$-Hodge structure, given by the homomorphism $h : \mathbb{S} \to \mathrm{GL(V)}_{\mathbb{R}}$ (see §2). The Mumford-Tate group, denoted $\mathbf{MT}(V)$ is the smallest algebraic subgroup $\mathbf{M} \subset \mathrm{GL(V)}$, defined over $\mathbb{Q}$, such that h factors through $\mathbf{M}_{\mathbb{R}}$. In particular, consider an abelian variety A over k with a fixed embedding of k into $\mathbb{C}$, and the associated natural Hodge structure on $V := H^1(A(\mathbb{C}), \mathbb{Q})$. Let $\mathcal{M}_V$ be the Tannakian subcategory generated by V and consider its associated motivic Galois group $\mathbf{G}_{\mathbf{Mot},B}(V)$. Assume that the Hodge conjecture holds for all products A^n, $n \geq 1$. Then the Mumford-Tate group $\mathbf{MT}(V)$ is the connected component of $\mathbf{G}_{\mathbf{Mot},B}(V)$. For a prime l, let ρ_l be the associated l-adic Galois representation

$$\rho_l : \Gamma_k \to \mathrm{GL}(V_l),$$

where $V_l \simeq V \otimes \mathbb{Q}_l$. Assume now that k is finitely generated over $\mathbb{Q}$. Let $\mathcal{G}_l(V)$ denote the image of ρ_l. Conjectures of Grothendieck and Mumford-Tate assert that the Zariski closure of $\mathcal{G}_l(V)$ is $\mathbf{G}_{\mathbf{Mot},B}(V)_{\mathbb{Q}_l}$.

Realizations and Conjectures

Let $E \in \overset{\bullet}{\mathbf{Mot}}_{\mathrm{num}}(k, \mathbb{Q})$, where k is of finite type over $\mathbb{Q}$. Fix a prime l and let P be a set of prime numbers. Considering the l-adic realization, (cf. the talk of M. Kim), we obtain a $\mathbb{Q}_l$-vector space $V_l(E)$ on which the Galois group $\Gamma_k := \mathrm{Gal}(\bar{k}/k)$ acts. There is a Galois representation

$$\rho_{l,E} : \Gamma_k \to \mathrm{GL}(V_l(E)).$$

Let $G_{l,E}$ be the image of this representation. Again, conjecturally, the Zariski closure of $G_{l,E}$ is $\mathbf{G}_{\mathbf{Mot},B}(E)_{BbbQ_l}$. As l varies the different ρ_l form a strictly compatible system of Galois representations (see [29]), and we have a homomorphism

$$\rho_E : \Gamma_k \to \prod_{l \in P} G_{l,E}.$$

We next explain how the classical Tate conjecture and Hodge conjecture can be reformulated in a categorical framework (see [1]).

Let X be a smooth, projective algebraic variety over k and fix an embedding $\sigma : k \hookrightarrow \mathbb{C}$. The étale cohomology groups $H^{2j}_{ét}(\bar{X}, \mathbb{Q}_l)(j)$ with the Tate twists, have an action of the Galois group Γ_k. Let $\mathcal{A}^j(X) \subset H^{2j}_{ét}(\bar{X}, \mathbb{Q}_l)(j)$ be the $\mathbb{Q}$-span of the image of the cycle class map. Tate's conjecture is the assertion that

$$\mathbb{Q}_l \otimes_{\mathbb{Q}} \mathcal{A}^j(X) = (H^{2j}_{ét}(\bar{X}, \mathbb{Q}_l(j))^{\Gamma_k}.$$

In order to reformulate this, we consider the functor

$$\mathcal{R}_{\mathrm{Tate}} : \overset{\bullet}{\mathbf{Mot}}_{\mathrm{num}}(k, \mathbb{Q}_l) \to \mathbf{Rep}_{\mathbb{Q}_l}(\Gamma_k)$$

given by the l-adic Galois representation associated to a motive, as explained in the paragraph above. An equivalent formulation of Tate's conjecture then is the assertion that $\mathcal{R}_{\mathrm{Tate}}$ is fully faithful.

In a similar vein, there is also a reformulation of the Hodge conjecture. We now assume that the base field k is of characteristic zero and fix an embedding $k \hookrightarrow \mathbb{C}$. Recall that the Hodge conjecture is the assertion that the rational (p, p) classes in the Hodge decomposition are in fact algebraic; here p is the dimension of $X_{\mathbb{C}}$. Let $\mathbb{Q}\mathbf{HS}$ be the category of pure Hodge structures over $\mathbb{Q}$. There is a functor

$$\mathcal{R}_{\mathrm{Hodge}} : \overset{\bullet}{\mathbf{Mot}}_{\mathrm{num}}(k, \mathbb{Q}) \to \mathbb{Q}\mathbf{HS}$$

and the Hodge conjecture is equivalent to the assertion that $\mathcal{R}_{\text{Hodge}}$ is fully faithful.

We end this section with the following remarks. The first one is the rather vague statement that conjecturally, Shimura varieties parametrize families of motives, and provide a bridge to the theory of automorphic forms via algebraic representations on adélic groups (see [21] and the lectures of Clozel, Fargues on this subject). Secondly, as is clear from the discussion in this section, the definition of the motivic Galois groups rely very strongly on the standard conjectures. Deligne weakens this dependence with his theory of Absolute Hodge cycles. André and André-Kahn study other methods of obtaining the unconditional existence of motivic Galois groups.

6 Motives over finite fields

The main reference for this short section is the article by Milne in [23]. We now suppose that $k = \mathbb{F}_q$ is a finite field of characteristic p. In this case, the hypothesis of algebraicity of the Künneth components of the diagonal is satisfied and Jannsen's theorem is therefore valid. Thus $\dot{\mathbf{Mot}}_{\text{num}}(k, \mathbb{Q})$ is a semisimple $\mathbb{Q}$-linear Tannakian category (see Theorem 4.4). However, if $k \supset \mathbb{F}_{p^2}$, then, as was pointed out by Serre, the category $\dot{\mathbf{Mot}}_{\text{num}}(k, F)$ is *not* neutral. In other words, there is no F-valued fibre functor if $F \subset \mathbb{Q}$ or $\mathbb{Q}_p$. In such cases, when there is a Tannakian category with a fibre functor over an extension of the field of coefficients F, the associated object $\text{Aut}^{\otimes}(\omega)$ for a fibre functor ω into the category $\mathbf{Vec}_{F'}$, with $[F' : F] > 1$, is a *gerbe* or *groupoid* and is related to non-abelian cohomology [10].

We now describe the simple objects in the semisimple category $\dot{\mathbf{Mot}}_{\text{num}}(k, F)$. For each motive M, there is a Frobenius endomorphism π_M in $\text{End}(M)$. When M is the Tate motive, the action of π_M is just multiplication by q^{-1}, and for any pure motive M, the $\mathbb{Q}$-algebra $\mathbb{Q}[\pi_M] \subset \text{End}(M)$ is a product of fields. By the algebraicity of the Künneth components, for a smooth projective variety X, we have $h(X) = h^0(X) + h^1(X) + \cdots + h^{2d}(X)$, where d is the dimension of X. A consequence of the Weil conjectures, proved by Deligne, is the fact that for every homomorphism $\phi : \mathbb{Q}[\pi_{h^i(X)}] \to \mathbb{C}$, we have $\left|\phi(\pi_{h^i(X)})\right| = q^{i/2}$.

Recall that an algebraic number κ is said to be a *Weil q-number of weight m* if for every embedding $\sigma : \mathbb{Q}[\kappa] \hookrightarrow \mathbb{C}$, we have $|\sigma(\kappa)| = q^{m/2}$, and for some k, $q^k\kappa$ is an algebraic integer. Let $\mathcal{W}(q)$ denote the set of Weil q-numbers. Then the Galois group $\text{Gal}(\bar{k}/k)$ acts on $\mathcal{W}(q)$. It can be shown that there is a bijection between the set of isomorphism classes of simple objects in $\dot{\mathbf{Mot}}_{\text{num}}(k, F)$ and the set of Galois orbit classes of Weil q-numbers.

7 Mixed Motives and Algebraic K-theory

This section is a prelude to the lectures of Marc Levine and those of André, Kahn, Riou and Ivorra. The main references are [4], [18], [15], [16], [12], [13], [14], [22], [24], [25] and the articles by B. Kahn and M. Levine in [9].

Recall that the category of pure motives is constructed from smooth *projective* varieties. In a nutshell, the (conjectural) category of mixed motives is supposed to take into account *all* smooth varieties. The conjectural description of the existence of such an abelian tensor category of mixed motives is due to Deligne and independently, Beilinson (see [3], [27] and the articles of Deligne and Beilinson in [23]). The Beilinson conjectures are formulated in the framework of this category. Furthermore, the category $\mathbf{Mot}_{\mathrm{num}}$ of pure numerical motives is expected to be contained in this larger abelian tensor category as the full subcategory consisting of semisimple objects. Let k be a field and $\mathcal{MM}_k$ denote the (conjectural abelian) category of mixed motives. For integers q, there are the *Tate objects* $\mathbb{Z}(q)$ in $\mathcal{MM}_k$, and in keeping with this notation (but changing the convention from our §3), we shall denote the Tate motive by $\mathbb{Z}(1)$. For an object $M \in \mathcal{MM}_k$ and an integer n, the Tate twist $M(n)$ is defined as $M \otimes \mathbb{Z}(n)$.

Recall that if X is a smooth variety over k, there are defined the algebraic K-groups due to Quillen, denoted $K_i(X)$, for integers $i \geq 0$ [26]. The weight p-subspaces of $K_i(X)_{\mathbb{Q}} := K_i(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ are defined by

$$K_i(X)^{(p)} := \{x \in K_i(X)_{\mathbb{Q}} \mid \psi_k(x) = k^p.x \,\forall\, k \geq 2\},$$

where ψ_k is a certain operator on the K-groups called the k-th Adams operator.

The conjectures of Beilinson relating values of L-functions to algebraic K-theory [27] are via the *motivic cohomology groups*, which are bigraded and are defined as certain Ext groups in the derived category $D^b(\mathcal{MM}_k)$. Specifically, if X a smooth variety, then there is an object $h(X)$ in the derived category $D^b(\mathcal{MM}_k)$, and the bigraded 'motivic cohomology groups', denoted $H^p_{\mathcal{M}}(X, \mathbb{Z}(q))$, for integers p, q, are defined as

$$H^p_{\mathcal{M}}(X, \mathbb{Z}(q)) := \mathrm{Ext}^p_{\mathcal{MM}_k}(\mathbb{I}, h(X)(q)).$$

Here $\mathbb{I} := \mathbb{Z}(0)$ is the identity object of the tensor category $\mathcal{MM}_k$. Just as the classical Atiyah-Hirzebruch spectral sequence relates singular cohomology groups to topological K-theory, it is conjectured that the motivic cohomology groups are related to algebraic K-theory by a spectral sequence. Another important component of these theories is Bloch's higher Chow groups.

There is the notion of a Bloch-Ogus cohomology theory, which associates (as a contravariant functor) bigraded abelian groups to smooth varieties. Roughly speaking, for the theory of mixed motives, the Bloch-Ogus theory plays a role similar to that of Weil cohomology for pure motives. All the classical Weil cohomology theories in fact turn out to extend as Bloch-Ogus theories satisfying the Künneth formula. Further, motivic cohomology is also a Bloch-Ogus cohomology theory, and is even supposed to be *universal*, in the sense that any other Bloch-Ogus theory factors through motivic cohomology. We now give a quick description of the conjectural properties of the theory of mixed motives. Denote the category of smooth varieties over k by $\mathbf{Sm}_k$. Let $\mathbf{Ab}$ denote the category of abelian groups and $\mathbf{D}(\mathbf{Ab})$ be its derived category. If Γ is a Bloch-Ogus cohomology theory, then conjecturally, there is a contravariant functor from $\mathbf{Sm}_k$ to $\mathbf{D}^b(\mathcal{MM}_k)$ and a realization functor

$$\mathcal{R}_\Gamma : \mathcal{MM}_k \to \mathbf{D}(\mathrm{Ab}).$$

We now state the conjectural properties expected of the category of mixed motives (see Levine's articles cited above).

Conjecture: (1) Let k be a field. There is a rigid tensor category $\mathcal{MM}_k$, containing the 'Tate objects' $\mathbb{Z}(n)$, $n \in \mathbb{Z}$, and a functor

$$h : \mathbf{Sm}_k^{\mathrm{op}} \to \mathbf{D}^b(\mathcal{MM}_k)$$

such that the functor $X \mapsto \underset{p,q}{\oplus} H^p_{\mathcal{M}}(X, \mathbb{Z}(q))$ is the universal Bloch-Ogus cohomology theory on $\mathbf{Sm}_k$. For each $X \in \mathbf{Sm}_k$, the object

$$h^i(X)(q) := H^i(h(X)) \otimes \mathbb{Q}(q)$$

is in $\mathcal{MM}_k$.

(2) In $\mathcal{MM}_k \otimes \mathbb{Q}$, the full subcategory of semisimple objects is equivalent to the category $\mathbf{Mot}_{\mathrm{num}}(k, \mathbb{Q})$ of pure motives.

(3) Suppose that k embeds into $\mathbb{C}$ and let $\mathcal{R}_B$ denote the realization functor with respect to the Betti cohomology, which is a Bloch-Ogus cohomology theory. Then the functor

$$H^0 \circ \mathcal{R}_B : \mathcal{MM}_k \otimes \mathbb{Q} \to \mathbf{Vec}_{\mathbb{Q}}$$

is a fibre functor, making $\mathcal{MM}_k \otimes \mathbb{Q}$ a neutral Tannakian category over $\mathbb{Q}$. Similarly, if $\mathcal{R}_l$ is the l-adic realization functor, then

$$H^0 \circ \mathcal{R}_l : \mathcal{MM}_k \otimes \mathbb{Q}_l \to \mathbf{Vec}_{\mathbb{Q}_l}$$

is a fibre functor and $\mathcal{MM}_k \otimes \mathbb{Q}_l$ is a neutral Tannakian category over $\mathbb{Q}_l$.
(4) For each object M in $\mathcal{MM}_k$, there is a natural finite weight filtration

$$0 = W_{n-1}(M) \subset W_n(M) \subset \cdots \subset W_m(M) = M$$

such that the graded quotients $\mathrm{gr}_W^* M$, after tensoring with $\mathbb{Q}$, are in $\mathbf{Mot}_{\mathrm{num}}(k, \mathbb{Q})$. For $M = h^i(X)$, the weight filtration is sent to the weight filtration of singular cohomology, respectively étale cohomology under the corresponding realization functors.
(5) For X smooth projective over k of dimension d, $h^i(X)$ is pure ("of weight i") and there is a decomposition (not necessarily unique) in $\mathbf{D}^b(\mathcal{MM}_k \otimes \mathbb{Q})$ such that

$$h(X)_{\mathbb{Q}} = \bigoplus_{i=0}^{2d} h^i(X)[-i].$$

(This statement can in fact be derived from the earlier statements).
(6) There are natural isomorphisms

$$H^p_{\mathcal{M}}(X, \mathbb{Z}(q)) \otimes \mathbb{Q} \simeq K_{2q-p}(X)^{(q)}$$

which should arise from an Atiyah-Hirzebruch type spectral sequence

$$E_2^{p,q} = H^{p-q}_{\mathcal{M}}(X, \mathbb{Z}(-q)) \Longrightarrow K_{-q-p}(X).$$

Further, this spectral sequence degenerates at E_2 after tensoring with $\mathbb{Q}$.

To date, there is no satisfactory construction of an abelian category of mixed motives that satisfies all the conditions in the above conjecture. There are the analogues of 'absolute Hodge cycles' à la Deligne and constructions by Deligne and Jannsen of Tannakian categories of mixed motives. For other constructions using realizations, we refer to the works of Beilinson, Huber and Nori.

Observe however, that the conjectured motivic cohomology groups themselves require only the *derived* category $D^b(\mathcal{MM}_k)$. Voevodsky (see [8]) constructed (in the 1990's) a tensor triangulated category of motives, $\mathbf{DM}_{\mathrm{gm}}(k)$, which has all the expected structural properties of $D^b(\mathcal{MM}_k)$. There are also other constructions of such triangulated categories of motives, due to Hanamura and M. Levine. Assuming resolution of singularities for k, the triangulated categories of Levine and Voevodsky are known to be equivalent (even with integer coefficients, see [22]). For a general perfect field k, one needs to tensor with $\mathbb{Q}$ to get a comparison theorem, as F. Ivorra has shown. In addition, it is known that the $\mathbb{Q}$-motivic cohomology groups obtained by Hanamura's construction are also the same. In particular, there is now a very good candidate for the motivic cohomology groups. There are also several constructions for the analogue of the Atiyah-Hirzebruch spectral sequence, though a solution to the foundational problem of constructing an abelian category of mixed motives still remains elusive.

Mixed Tate Motives

We denote by $\mathbf{TM}_k$ the category of mixed Tate motives, which is the full abelian Tannakian subcategory of $(\mathcal{MM}_K \otimes \mathbb{Q})$ generated by the Tate objects $\mathbb{Q}(n)$, $n \in \mathbb{Z}$, and closed under extensions. There is also a triangulated version of this construction, which we denote by $\mathbf{DTM}(k)$ and which is defined as the triangulated subcategory of $\mathbf{DM}_{\mathrm{gm}}(k)$ generated by the Tate objects $\mathbb{Z}(n)$. It comes equipped with a duality functor on $\mathbf{DTM}(k)$. A natural question is whether the conjecture above on mixed motives holds for mixed Tate motives. But even here, there is an obstruction which is called the *Beilinson-Soulé vanishing conjecture* and is the following conjecture:

Conjecture: (Beilinson-Soulé vanishing conjecture): Let F be a field. Then $K_p(F)^{(q)} = 0$ if $2q \leq p$ and $p > 0$.

An equivalent formulation is the following.

Conjecture: Let F be a field. Then

$$H^p(F, \mathbb{Q}(q)) := \mathrm{Hom}_{\mathbf{DTM}(k)}(\mathbb{Q}, \mathbb{Q}(q)[p]) = 0 \text{ if } p \leq 0 \text{ and } q > 0.$$

The conjecture is known to be true for global fields.

It can be shown that if $\mathbf{DTM}(k)$ is the bounded derived category of a rigid abelian tensor category $\mathbf{T}_k$ with 'good' properties (i.e. those expected from the Beilinson-Deligne theory), then the vanishing conjecture above would hold. We also have the following partial converse due to Levine.

Theorem 7.1. *(Levine) Let k be a field and assume that the Beilinson-Soulé vanishing conjecture holds for k. Then there is a t-structure on $\mathbf{DTM}(k)$ with heart $\mathbf{TM}_k$ satisfying:*
(a) $\mathbf{TM}_k$ contains all the Tate objects $\mathbb{Q}(n)$ and these generate $\mathbf{TM}_k$ as an abelian category, which is closed under extensions in $\mathbf{DTM}(k)$.
(b) The tensor operation and duality on $\mathbf{DTM}(k)$ restrict to $\mathbf{TM}_k$, making $\mathbf{TM}_k$ a rigid tensor category.

We close this paragraph by remarking that various mathematicians (Deligne, Goncharov, Manin, Terasoma...) have shown that there is a relation between the category of mixed Tate motives and (multi)zeta values and polylogarithms.

Mixed Motivic Galois groups

Recall that conjecturally the category of pure motives $\mathbf{Mot}_{\mathrm{num}}(k,\mathbb{Q})$ embeds as a semisimple full subcategory of the conjectured category of mixed motives $\mathcal{MM}_k$. Assume now that k admits an embedding into $\mathbb{C}$ and fix such an embedding. Then by the conjecture stated above, there is fibre functor $H^0 \circ \mathcal{R}_B$ over $\mathbb{Q}$ whch makes $\mathcal{MM}_k \otimes \mathbb{Q}$ a neutral Tannakian category. We denote the associated motivic Galois group over $\mathbb{Q}$ by $\mathbf{MG}_{\mathbf{Mot},B}$ and clearly there is a natural surjection

$$\mathbf{MG}_{\mathbf{Mot},B} \to \mathbf{G}_{\mathbf{Mot},B}.$$

On the other hand, there is an exact tensor functor $\mathcal{MM}_k \otimes \mathbb{Q} \to \mathbf{Mot}_{\mathrm{num}}(k,\mathbb{Q})$, splitting the inclusion, which via the Tannakian formalism, corresponds to taking the associated graded for the weight filtration. This gives a splitting to the above surjection and an exact sequence

$$1 \to \mathbf{U}_k \to \mathbf{MG}_{\mathbf{Mot},B} \to \mathbf{G}_{\mathbf{Mot},B} \to 1$$

with $\mathbf{U}_k$ a connected pro-unipotent algebraic group scheme over $\mathbb{Q}$. We now restrict ourselves to Tate motives. Then the category of mixed Tate motives, $\mathbf{TM}_k$ contains the abelian full subcategory $\mathcal{M}_{\mathbb{T}}$ of the pure Tate motives considered earlier. On the other hand, as explained above, there is an exact tensor functor $\mathbf{TM}_k \to \mathcal{M}_{\mathbb{T}}$ splitting the inclusion. Let $\mathbf{G}_{\mathbf{TM},k}$ denote the motivic Galois group associated to $\mathbf{TM}_k$. We thus get a split surjection

$$\mathbf{G}_{\mathbf{TM},k} \to \mathbb{G}_m \to 1,$$

whose kernel is a pro-unipotent group with an action of $\mathbb{G}_m$.

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