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Almost perfect powers in consecutive integers (II)

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ABSTRACT

Let $k \geqslant 4$ be an integer. We find all integers of the form by^l where $l \geqslant 2$ and the greatest prime factor of b is at most k (i.e. nearly a perfect power) such that they are also products of k consecutive integers with two terms omitted.

1. INTRODUCTION

Let $n \geqslant 0, k \geqslant 4,\ 0 \leqslant t \leqslant k-2$ and $0 \leqslant d_1 < d_2 < \cdots < d_{k-t} < k$ be integers. We put

$$\Delta_t = (n+d_1)\cdots(n+d_{k-t}).$$

When $t=0,\ d_i = i-1$ for $1 \leqslant i \leqslant k$ and we have

$$\Delta_0 = n(n+1)\cdots(n+k-1).$$

Thus Δ_t is a product taken from Δ_0 by omitting t terms. For an integer $v > 1$, we denote by $P(v)$ and $\omega(v)$ the greatest prime factor of v and the number of distinct prime factors of v and we put $P(1)=1$ and $\omega(1)=0$. We consider the equation

$$\Delta_t = (n+d_1)\cdots(n+d_{k-t}) = by^l \tag{1.1}$$

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in integers $b \geqslant 0, k, l \geqslant 2, n, t, y \geqslant 1, d_1, \ldots, d_{k-t}$ with $P(b) \leqslant k$. While dealing with equation (1.1), we assume that

$$(1.2) \qquad P(\Delta_t) > k$$

otherwise one can find infinitely many solutions. For instance, when $\Delta_t = (k-t)!$ we can take $y = 1$ and $b = (k-t)!$ and this holds for any k, l and for any $t < k$. Also note that there is no loss of generality in assuming that l is prime which we suppose from now on.

Let $t = 0$. Erdős and Selfridge [4] proved a remarkable result that (1.1) with (1.2) and $P(b) < k$ has no solution. The result of Erdős and Selfridge was extended to the case $P(b) \leqslant k$ by Saradha [9] for $k \geqslant 4$ and by Győry [5] for $k = 2, 3$.

Let $t = 1$. This is the case of one term being omitted from Δ_0. Saradha [9] showed that (1.1) with (1.2) implies that $k \leqslant 24$ if $l = 2$ and $k \leqslant 8$ if $l \geqslant 3$. In [10, Corollary 1], Saradha and Shorey showed that the only solutions of (1.1) with (1.2), $k \geqslant 4$ and $l = 2$ are given by

$$(n, k) \in \{(24, 4), (47, 4), (48, 4)\}.$$

Next, Hanrot, Saradha and Shorey [6, Theorem] proved the impossibility of (1.1) under (1.2) when $l \geqslant 3$ and $k \in \{6, 7, 8\}$. From the results of Saradha and Shorey in [10] and [11], it follows that the only solutions of (1.1) with $k \geqslant 4$ and $b = 1$ are given by

$$\frac{6!}{5} = 12^2; \qquad \frac{10!}{7} = 720^2; \qquad \frac{4!}{3} = 2^3.$$

This solves a conjecture of Erdős and Selfridge [4, p. 300]. Here the condition (1.2) is not assumed. Finally, Bennett [1] found all solutions of (1.1) with $k = 3$, $l \geqslant 3$ and $k \in \{4, 5\}$, $l \geqslant 2$, without the condition (1.2). There are 30 solutions.

In this paper we consider (1.1) with $t = 2$. First we take $l = 2$ and $k \geqslant 4$. When $k = 4$, (1.1) gives rise to Pell's equations which are known to have infinitely many solutions. For $k \geqslant 5$ it follows from Mukhopadhyaya and Shorey [7, Theorem 2] that the only solution of (1.1) with (1.2) is

$$k = 5, \qquad n \in \{45, 46, 47, 48, 96, 239, 240, 241, 242, 359, 360\}$$

and

$$k = 6, \qquad n \in \{45, 240\}.$$

Thus we need only to consider $l \geqslant 3$. Then we prove the following theorem.

Theorem 1. *Assume* (1.1) *with* $k \geqslant 4, t = 2, l \geqslant 3$ *and* (1.2). *Then we have*

$$l = 3, \qquad k = 4, \qquad n = 125 \quad \text{or} \quad k = 7, \quad P(b) = k.$$

It will be clear from the proof of Theorem 1 that for excluding the case $k = 7$, $P(b) = k, l \geqslant 3$, we need to solve the equations

$$\left|2^{\alpha_1}3^{\beta_1}7^{\gamma}x^l - 2^{\alpha_2}3^{\beta_2}y^l\right| = 1,\ 2,\ 3,\ 4,$$

where $l \geqslant 3, x > 0, y > 0, 0 \leqslant \alpha_1, \beta_1, \alpha_2, \beta_2 < l$ and $0 < \gamma < l$. The theorem shows that 16000 ($=2 \cdot 20^3 = 125 \cdot 128$) is the only integer of the form by^l with $l \geqslant 3$, $P(b) < k$, that can be expressed as a product of k ($\geqslant 4$) consecutive integers with two terms omitted. Let $b = 1$. In this case a complete result can be given without the assumption (1.2). Let $l = 2$. Then it was shown in [7, Corollary 3] that (1.1) implies that

$$(n, k) \in \{(1, 4), (2, 5), (1, 6), (2, 6), (3, 6), (5, 6), (1, 7), (3, 7), (4, 7), (3, 8), (14, 8), (2, 9), (1, 10), (2, 10), (5, 10), (1, 11), (4, 11)\}.$$

So it is enough to consider the case of $l \geqslant 3$. Then we prove the following corollary.

Corollary 2. *Let $k \geqslant 4, t = 2, l \geqslant 3$ and $b = 1$. Then* (1.1) *is not possible unless*

$$l = 3, (n, k) \in \{(1, 4), (2, 4), (1, 5), (4, 6)\}.$$

The corollary shows that 8 ($=2^3$) and 1728 ($=12^3$) are the only two perfect powers which are not squares and which can be expressed as product of k ($\geqslant 4$) consecutive integers with two terms omitted. It was shown in [9] that (1.1) with $t = 2$ and (1.2) implies that $k \leqslant 11380$ if $l \geqslant 3$. Hence for the proof of Theorem 1 we need only to consider $l \geqslant 3$ and $k \leqslant 11380$. The values of $k \geqslant 16$ are covered by elementary and combinatorial arguments of Erdős and Selfridge [4]. As in [10], [6] and [1] the values of $k \leqslant 15$ are covered using results based on modular methods and by solving several Thue equations using PARI. For a survey of (1.1) with $t \geqslant 2$, we refer to [13]. We thank the referee for his remarks on an earlier version of the paper.

2. NOTATION AND PRELIMINARIES

We assume from now on that (1.1) holds with $l \geqslant 3$. We write

$$n + d_i = a_i x_i^l, \qquad P(a_i) \leqslant k, \quad a_i \text{ is } l\text{th power free for } 1 \leqslant i \leqslant k - t$$

and put $H = \{a_1, \ldots, a_{k-t}\}$. Also we write

$$n + d_i = A_i X_i^l, \qquad P(A_i) \leqslant k, \qquad \gcd\left(X_i, \prod_{p \leqslant k} p\right) = 1 \quad \text{for } 1 \leqslant i \leqslant k - t.$$

Let $m_1 \geqslant 1,\ m_2 \geqslant 0$ and $m_3 \geqslant 0$ be integers such that $m_1 + m_2 + m_3 = \pi(k)$. Let

$$p_1 < p_2 < \cdots$$

be the sequence of all primes. For given m_1 and m_2, let $H(k) = H(k, m_1, m_2)$ denote the number of i such that $a_i \in H$ is composed only of $p_1, \ldots, p_{m_1}$ and divisible by at most one of the primes $p_{m_1+1}, \dots, p_{m_1+m_2}$ which divides a_i at most to the first power. In particular, when $m_2 = 0$, $H(k)$ represents the number of i such that $a_i \in H$ is composed of only $p_1, \dots, p_{m_1}$. Thus

$$H(k) \geqslant k - t - \sum_{\mu=m_1+m_2+1}^{m_1+m_2+m_3} \left\lceil \frac{k}{p_\mu} \right\rceil - \sum_{m_1+1 \leqslant \mu, \nu \leqslant m_1+m_2} \left\lceil \frac{k}{p_\mu p_\nu} \right\rceil =: H_0(k).$$

In particular, when $m_2 = 0$,

$$H(k) \geqslant k - t - \sum_{\mu=m_1+1}^{m_1+m_3} \left\lceil \frac{k}{p_\mu} \right\rceil = H_0(k).$$

Here for any real $x > 0$, $\lceil x \rceil$ denotes the smallest integer $\geqslant x$. Now we define a function $G(k) = G(k, m_1, m_2)$ for $4 \leqslant k \leqslant 11380$ as follows.

$16 \leqslant k \leqslant 106$:	$(m_1, m_2) = (3, 0)$,	$G(k) = 7$;
$107 \leqslant k \leqslant 312$:	$(m_1, m_2) = (4, 0)$,	$G(k) = 13$;
$313 \leqslant k \leqslant 642$:	$(m_1, m_2) = (5, 0)$,	$G(k) = 22$;
$643 \leqslant k \leqslant 1162$:	$(m_1, m_2) = (6, 0)$,	$G(k) = 38$;
$1163 \leqslant k \leqslant 6479$:	$(m_1, m_2) = (4, 11)$,	$G(k) = 112$;
$6480 \leqslant k \leqslant 7120$:	$(m_1, m_2) = (4, 12)$,	$G(k) = 121$;
$7121 \leqslant k \leqslant 11380$:	$(m_1, m_2) = (5, 11)$,	$G(k) = 195$.

Then for any given k with $16 \leqslant k \leqslant 11380$ and $t = 2$ we find that

(2.1) $\qquad H_0(k) \geqslant G(k).$

We now turn our attention to the condition (1.2). An old result of Sylvester states that

$$P(\Delta_0) > k \quad \text{if } n > k.$$

Since Δ_0 is divisible by $k!$, all primes $\leqslant k$ divide Δ_0. Thus the above result is equivalent to

$$\omega(\Delta_0) > \pi(k) \quad \text{if } n > k.$$

This was sharpened by Saradha and Shorey [11] as

(2.2) $\qquad \omega(\Delta_0) > \pi(k) + \left[\frac{\pi(k)}{3} \right] + 2 \quad \text{if } n > k \geqslant 3$

except when (n, k) belongs to the set S given below:

$$S = \begin{cases} n \in \{4, 6, 7, 8, 16\}, & k = 3; \\ n \in \{6\}, & k = 4; \\ n \in \{6, 7, 8, 9, 12, 14, 15, 16, 23, 24\}, & k = 5; \\ n \in \{7, 8, 15\}, & k = 6; \\ n \in \{8, 9, 10, 12, 14, 15, 24\}, & k = 7; \\ n \in \{9, 14\}, & k = 8; \\ n \in \{14, 15, 16, 18, 20, 21, 24\}, & k = 13; \\ n \in \{15, 20\}, & k = 14; \\ n = 20, & k = 17. \end{cases}$$

By (2.2), we see that Δ_0 is divisible by at least $[\frac{\pi(k)}{3}] + 2$ primes $> k$. Thus

$$(2.3) \qquad P(\Delta_2) > k \quad \text{for } n > k \geqslant 5 \text{ except when } (n, k) \in S.$$

Note that when (1.1) holds with (1.2), we have

$$n + k - 1 \geqslant (k + 1)^l > k^l + lk^{l-1}.$$

Thus

$$(2.4) \qquad n > k^l.$$

3. APPLICATION OF THE METHOD OF ERDŐS AND SELFRIDGE

The first lemma is a consequence of Lemmas 4 and 6 of [10].

Lemma 3. *Let* $1 \leqslant l' \leqslant l - 1$. *Suppose* (1.1) *holds with* (1.2). *Then for no distinct* l'*-tuples* $(i_1, \ldots, i_{l'})$ *and* $(j_1, \ldots, j_{l'})$ *with* $i_1 \leqslant \cdots \leqslant i_{l'}$ *and* $j_1 \leqslant \cdots \leqslant j_{l'}$, *the ratio of the two products* $a_{i_1} \cdots a_{i_{l'}}$ *and* $a_{j_1} \cdots a_{j_{l'}}$ *is an* l*th power of a rational number. Further*

$$(3.1) \qquad \binom{H(k) + l' - 1}{l'} \leqslant l^{m_1} \binom{l' + m_2}{m_2}$$

where the left-hand side is 0 *if* $H(k) < 1$.

As an application of the above lemma, we show that the following lemma holds.

Lemma 4. *Suppose* (1.1) *with* $t = 2$ *and* (1.2) *holds. Then* $k \leqslant 9$ *or* $k = 11, 13$. *Further we have* $l = 3$ *if* $k = 11, 13$.

Proof. By Lemma 3, we see that a_i's are distinct and (3.1) is valid for any $l' < l$. Let $16 \leqslant k \leqslant 11380$. For every k in this range we use (2.1) to find that

$$(3.2) \qquad H(k) \geqslant H_0(k) \geqslant G(k).$$

We take $l' = l - 1$ in Lemma 3. We find that (3.1) does not hold with $l = 3$. We see by induction that if (3.1) does not hold for some odd $l = l_1$, then it does not hold for $l = l_1 + 2$ provided that $H(k)$ satisfies

$$(H(k) + l_1)(H(k) + l_1 - 1) > (1 + 2/l_1)^{m_1}(l_1 + 1 + m_2)(l_1 + m_2).$$

By (3.2) it is enough to check that the above inequality is valid with $H(k)$ replaced by $G(k)$ and this is true by the choice of $G(k)$. Thus (3.1) does not hold for any $l \geqslant 3$. It follows that $k < 16$.

Let $k = 15$. Then the number of a_i's divisible by the primes 13, 11 and 7 does not exceed 2, 2 and 3, respectively. Also we note that the number of a_i's divisible by either 13 or 7 cannot exceed 4. Hence $H(15) = H(15, 3, 0) \geqslant 7$ which gives the necessary contradiction as in the previous paragraph. Let $k = 12, 14$. Then $H(k) = H(k, 2, 0) \geqslant 3$. When $k = 14$, the primes 5, 11 and 13 can divide only at most 6 terms and when $k = 12$, the primes 5 and 11 can divide at most 4 terms giving $H(k) = H(k, 2, 0) \geqslant 4$. This inequality is also true when $k = 10$. Thus we get the necessary contradiction as earlier for the cases $k = 10, 12, 14$.

We also observe that (3.1) is not valid if

$$H(k) = H(k, 3, 0) \geqslant 6 \quad \text{with } l \geqslant 5 \text{ and } l' = l - 1.$$

Now the lemma follows since $H(k) = H(k, 3, 0) \geqslant 6$ for $k = 11, 13$. □

4. APPLICATION OF MODULAR METHOD AND LINEAR FORMS IN LOGARITHMS

In this section, we present some results on generalized Fermat equations which are solved using modular methods. These are applied to form certain Thue equations for the values of k and l given by Lemma 4. These equations are used in the proofs of Theorem 1 and Corollary 2. The first lemma is the main result of Bennett [1].

Lemma 5. *If m, h, α, β, y and l are non-negative integers with $l \geqslant 3$ and $y \geqslant 1$, then the only solutions to the equation*

$$m(m + 2^h) = 2^\alpha 3^\beta y^l$$

are those with

$$m \in \{2^h, 2^{h\pm1}, 3 \cdot 2^h, 2^{h\pm3}\}.$$

The following result is (15) of Proposition 3.1 of Bennett, Bruin, Győry and Hajdu [3].

Lemma 6. *The equation*
$x^l + 2^\alpha y^l = 3^\beta z^2$, $l \geqslant 7$ prime, α, β non-negative integers with $\alpha \neq 1$ has no solution in non-zero coprime integers (x, y, z) with $xy \neq \pm 1$.

The next result is part of Proposition 5.1 in [2].

Lemma 7. *The only solutions to the equation*

$$x^l - 2y^l = 3$$

in integers x, y *and* $l \geqslant 3$ *are given by* $(x, y, l) = (1, -1, l)$ *for odd* l *and by* $(x, y, l) = (-5, -4, 3)$.

We apply Lemmas 5 and 6 to get the following one.

Lemma 8. *Let* $4 \leqslant k \leqslant 9$ *or* $k = 11, 13$. *Suppose* (1.1) *holds with* $t = 2$ *and* (1.2). *Assume also that* $P(b) < k$ *for* $k = 7$. *Then there exist integers* u, v *with* $1 \leqslant u < v \leqslant k - 2$ *such that* $d_v - d_u = 3$ *and* $(A_u, A_v) \in \{(1, 2^\alpha), (2^\alpha, 1)\}$, *with* $\alpha = 1$ *whenever* $l \geqslant 7$.

Proof. By Lemma 3, (3.1) holds. Let k be given. Suppose the number of i such that A_i is composed of only 2 and 3 is $\geqslant 4$. Then $H(k) = H(k, 2, 0)$ is also $\geqslant 4$. In this case as in Lemma 4, we take $l' = l - 1$ in Lemma 3 and check that (3.1) is not valid for $l = 3$ and also for any odd $l > 3$, by induction. This is a contradiction. Hence we may assume that the number of such A_i's is $\leqslant 3$. When there are exactly three such A_i's, we say that *property* P_3 is satisfied. We observe that there are at least two A_i's composed of 2 and 3 since $k \geqslant 4$. Thus there exist at least two integers $0 \leqslant d_u < d_v < k$ with $P(A_u A_v) \leqslant 3$ and we may write

$$n + d_u = 2^{\alpha_u} 3^{\beta_u} X_u^l; \qquad n + d_v = 2^{\alpha_v} 3^{\beta_v} X_v^l$$

and

$$(n + d_u)(n + d_v) = 2^\alpha 3^\beta y_1^l. \tag{4.1}$$

Suppose $d_v - d_u = 2^h$. Then by taking $m = n + d_u$, the above equation becomes an equation considered in Lemma 5. Hence we get by (2.4), that

$$n + d_u \leqslant 8 \cdot 2^h < 8k < k^3 < n,$$

a contradiction. Thus we may assume that

$$d_v - d_u \neq 2^h \quad \text{for } h \geqslant 0. \tag{4.2}$$

Suppose $d_v - d_u = 3$. Then one of $n + d_u, n + d_v$ is even and the other is odd. We discuss the case when $n + d_u$ is even. The case $n + d_v$ even can be treated similarly. We get

$$3^{\beta_v} X_v^l - 2^{\alpha_u} 3^{\beta_u} X_u^l = 3.$$

Note that $\min(\beta_u, \beta_v) = 1$ or $\beta_u = \beta_v = 0$. In both cases after canceling 3, if necessary, we get an equation of the form

$$3^{\beta_v - 1} X_v^l - 2^{\alpha_u} X_u^l = 1 \quad \text{or} \quad X_v^l - 2^{\alpha_u} 3^{\beta_u - 1} X_u^l = 1 \quad \text{or} \quad X_v^l - 2^{\alpha_u} X_u^l = 3. \tag{4.3}$$

Let us consider the first equation in (4.3). By taking $m = 2^{\alpha_u} X_u^l$, we see that $m + 1 = 3^{\beta_v - 1} X_v^l$ and $m(m+1) = 2^{\alpha_u} 3^{\beta_v - 1} X^l$ where $X = X_u X_v$. This is an equation considered in Lemma 5 with $h = 0$. Hence

$$n + d_u \leqslant 3m \leqslant 24 < k^3 < n$$

which is not possible. Similarly the second equation in (4.3) is also not possible. For the last equation, we apply Lemma 6 to conclude that $\alpha_u = 1$ if $l \geqslant 7$. This gives the assertion of the lemma. Hence it remains to consider that $d_v - d_u$ does not equal 3 or 2^h. Thus

(4.4) $\quad d_v - d_u \geqslant 5.$

This implies that $k \geqslant 6$. Suppose there exists another term $n + d_w$ with $P(A_w) \leqslant 3$ and $u < v < w$. Then by repeating the above argument with $n + d_v$ and $n + d_w$, we may assume that $d_w - d_v \geqslant 5$ which gives

(4.5) $\quad d_w - d_u \geqslant 10.$

Thus if property P_3 is satisfied, then $k \geqslant 11$. Let $k = 6$. By (4.4), we may take $(d_u, d_v) = (0, 5)$. This means 5 divides only one A_i. Hence property P_3 is satisfied which is not possible. Let $k = 7$. Since $P(b) < 7$, we see that if 7 divides Δ_2, then it divides at most one term and to an lth power. Hence there are at least three A_i's composed of only 2, 3 and 7 which occurs to an lth power. So we can form equations as in (4.1) and (4.3) and conclude that (4.5) holds which is not possible.

Let $k = 8$. We may take $(d_u, d_v) \in \{(0,5), (0,6), (0,7), (1,6), (1,7), (2,7)\}$. Since P_3 does not hold we have

$$7 \text{ divides } A_0, A_7; \qquad 5 \text{ divides } A_1, A_6.$$

Hence choices taken for (d_u, d_v) are not possible.

Next we take $k \in \{9, 11, 13\}$. In these cases P_3 holds. Hence by (4.5) $k \neq 9$. If $k = 11$, then $H(k) = H(k, 2, 0) \geqslant 3$. In fact, $H(k) = 3$ since property P_3 holds. This implies that 5 divides exactly three A_i's. Hence 5 divides A_0, A_5 and A_{10}. Hence $d_u \geqslant 1$ and $d_w \leqslant 9$ contradicting (4.5). When $k = 13$, then $H(k) = H(k, 2, 0) = 3$ implies that 11 divides exactly two A_i's and 5 divides three other A_i's. Hence we have either

$$11 \text{ divides } A_0, A_{11}; \qquad 5 \text{ divides } A_2, A_7, A_{12}$$

or

$$11 \text{ divides } A_1, A_{12}; \qquad 5 \text{ divides } A_0, A_5, A_{10}.$$

Thus $d_u \geqslant 1, d_w \leqslant 10$ or $d_u \geqslant 2, d_w \leqslant 11$, respectively, which contradicts (4.5). □

We apply Lemmas 4 and 8 for the proofs of Theorem 1 and Corollary 2.

Proof of Theorem 1. By Lemmas 4 and 8 we may assume that $4 \leqslant k \leqslant 9$ or $k = 11, 13$ and an equation of the form

$$x^l + 2^\alpha z^l = 3 \quad \text{with } \alpha = 1 \text{ for } l \geqslant 7$$

holds in integers x and z. Thus we have the following set of equations:

$$(4.6) \qquad x^l + 2z^l = 3; \qquad x^3 + 4z^3 = 3; \qquad x^5 + 2^\alpha y^5 = 3 \quad \text{with } \alpha \in \{2, 3, 4\}.$$

Thus we need to solve the above set of Thue equations. When the first equation in (4.6) holds we use Lemma 7 to conclude that $l = 3$ and two terms of the product are 125 and 128. By (1.1), we see that 124 and 129 cannot be terms of Δ_2 since $k \leqslant 13$ and $\operatorname{ord}_{31}(\Delta_2)$ and $\operatorname{ord}_{43}(\Delta_2)$ are not $\equiv 0 \pmod 3$. Hence this case leads only to the solution $(n, k) = (125, 4)$.

For the remaining equations in (4.6) we use the computer package PARI which utilizes the method of linear forms in logarithms for solving Thue equations. We find that there is no non-trivial solution. □

Proof of Corollary 2. Let $k = 4$. Then we have either

$$n(n+2) = y^l \quad \text{or} \quad (n+1)(n+3) = y^l \quad \text{or} \quad n(n+3) = y^l.$$

For the first two equations, we apply Lemma 5 to get $n \leqslant 16$ implying $(n, l) \in \{(1, 3), (2, 3)\}$. The third equation gives rise to an equation of the form

$$x^l + 3^\alpha z^l = 1 \quad \text{with } 0 < \alpha < l$$

in integers x and z. It follows from an old result of Serre [12] that the above equation has no non-trivial solution except perhaps when $l = 3, 5$ and 7. In these cases we check with PARI that the above Thue equation has no non-trivial solution. Thus we suppose from now on that $k \geqslant 5$.

Let $n > k$. By (2.3), we find that (1.2) is satisfied except when $(n, k) \in S$. Hence by Theorem 1, we may assume that

$$(n, k) \in S.$$

Now we check directly that Δ_2 is not a perfect power for these finitely many values of (n, k). Let $n \leqslant k$. Then we use the inequality from [8, p. 69] that

$$\pi(2x) - \pi(x) \geqslant \frac{3x}{5 \log x} \qquad \text{for } x \geqslant 20.5$$

to get for $n + k - 1 \geqslant 41$ that

$$\pi(n + k - 1) - \pi\left(\frac{n + k - 1}{2}\right) \geqslant 3.$$

This means that there exists a prime $p > \frac{n+k-1}{2}$ implying $p \geqslant n$ and $\mathrm{ord}_p(\Delta_2) = 1$. This contradicts (1.1) since $b = 1$ and $l > 1$. Thus we have $n + k - 1 \leqslant 40$. We check directly that for these finitely many values of n and k, (1.1) is not satisfied except for the values of n, k, l mentioned in Corollary 2. □

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