

**SQUARES IN PRODUCTS IN ARITHMETIC
PROGRESSION WITH AT MOST ONE TERM OMITTED
AND COMMON DIFFERENCE A PRIME POWER**

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ABSTRACT. It is shown that a product of $k-1$ terms out of $k \geq 7$ terms in arithmetic progression with common difference a prime power > 1 is not a square. In fact it is not of the form by^2 where the greatest prime factor of b is less than or equal to k . Also, we show that product of 11 or more terms in an arithmetic progression with common difference a prime power > 1 is not of the form by^2 where the greatest prime factor of b is less than or equal to $p_{\pi(k)+2}$.

1. INTRODUCTION

For an integer $x > 1$, we denote by $P(x)$ and $\omega(x)$ the greatest prime factor of x and the number of distinct prime divisors of x , respectively. Further we put $P(1) = 1$ and $\omega(1) = 0$. Let p_i be the i -th prime number. Let $k \geq 4, t \geq k-2$ and $\gamma_1 < \gamma_2 < \dots < \gamma_t$ be integers with $0 \leq \gamma_i < k$ for $1 \leq i \leq t$. Thus $t \in \{k, k-1, k-2\}$, $\gamma_t \geq k-3$ and $\gamma_i = i-1$ for $1 \leq i \leq t$ if $t = k$. We put $\psi = k-t$. Let b be a positive squarefree integer and we shall assume, unless otherwise specified, that $P(b) \leq k$. We consider the equation

$$(1.1) \quad \Delta = \Delta(n, d, k) = (n + \gamma_1 d) \cdots (n + \gamma_t d) = by^2$$

in positive integers n, d, k, b, y, t . It has been proved (see [SaSh03a] and [MuSh04a]) that (1.1) with $\psi = 1, k \geq 9, d \nmid n, P(b) < k$ and $\omega(d) = 1$ does not hold. Further it has been shown in [TSH06] that the assertion continues to be valid for $6 \leq k \leq 8$ provided $b = 1$. We show

Theorem 1. *Let $\psi = 1, k \geq 7$ and $d \nmid n$. Then (1.1) with $\omega(d) = 1$ does not hold.*

Thus the assumption $P(b) < k$ and $k \geq 9$ (in [SaSh03a] and [MuSh04a]) has been relaxed to $P(b) \leq k$ and $k \geq 7$, respectively, in Theorem 1. As an immediate consequence of Theorem 1, we see that (1.1) with $\psi = 0, k \geq 7, d \nmid n, P(b) \leq p_{\pi(k)+1}$ and $\omega(d) = 1$ is not possible. If $k \geq 11$, we relax the assumption $P(b) \leq p_{\pi(k)+1}$ to $P(b) \leq p_{\pi(k)+2}$ in the next result.

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Theorem 2. *Let $\psi = 0, k \geq 11$ and $d \nmid n$. Assume that $P(b) \leq p_{\pi(k)+2}$. Then (1.1) with $\omega(d) = 1$ does not hold.*

For related results on (1.1), we refer to [LaSh07].

2. NOTATIONS AND PRELIMINARIES

We assume (1.1) with $\gcd(n, d) = 1$ in this section. Then we have

$$(2.1) \quad n + \gamma_i d = a_{\gamma_i} x_{\gamma_i}^2 \text{ for } 1 \leq i \leq t$$

with a_{γ_i} squarefree such that $P(a_{\gamma_i}) \leq \max(k-1, P(b))$. Thus (1.1) with b as the squarefree part of $a_{\gamma_1} \cdots a_{\gamma_t}$ is determined by the t -tuple $(a_{\gamma_1}, \dots, a_{\gamma_t})$. Further we write

$$b_i = a_{\gamma_i}, \quad y_i = x_{\gamma_i}.$$

Since $\gcd(n, d) = 1$, we see from (2.1) that

$$(2.2) \quad (b_i, d) = (y_i, d) = 1 \text{ for } 1 \leq i \leq t.$$

Let

$$R = \{b_i : 1 \leq i \leq t\}.$$

Lemma 2.1. ([LaSh07])

Equation (1.1) with $\omega(d) = 1$ and $k \geq 9$ implies that $t - |R| \leq 1$.

Lemma 2.2. *Let $\psi = 0, k \geq 4$ and $d \nmid n$. Then (1.1) with $\omega(d) = 1$ implies $(n, d, k, b) = (75, 23, 4, 6)$.*

This is proved in [SaSh03a] and [MuSh03] unless $k = 5, P(b) = 5$ and then it is a particular case of a result of Tengely [Sz07].

Lemma 2.3. ([SaSh03a, Theorem 4] and [MuSh04a])

Let $\psi = 1, k \geq 9$ and $d \nmid n$. Assume that $P(b) < k$. Then (1.1) with $\omega(d) = 1$ does not hold.

Lemma 2.4. ([LaSh07])

Let $\psi = 2, k \geq 15$ and $d \nmid n$. Then (1.1) with $\omega(d) = 1$ does not hold.

Lemma 2.5. *Let $\psi = 1, k = 7$ and $d \nmid n$. Assume that (1.1) holds. Then (a_0, a_1, \dots, a_6) is different from the ones given by the following tuples or their mirror images.*

$$(2.3) \quad \begin{aligned} k = 7 : & (1, 2, 3, *, 5, 6, 7), (2, 1, 6, *, 10, 3, 14), (2, 1, 14, 3, 10, *, 6), \\ & (*, 3, 1, 5, 6, 7, 2), (3, 1, 5, 6, 7, 2, *), (3, *, 5, 6, 7, 2, 1), \\ & (1, 5, 6, 7, 2, *, 10), (*, 5, 6, 7, 2, 1, 10), (5, 6, 7, 2, 1, 10, *), \\ & (6, 7, 2, 1, 10, *, 3), (10, 3, 14, 1, 2, 5, *), \\ & (*, 10, 3, 14, 1, 2, 5), (5, 2, 1, 14, 3, 10, *), (*, 5, 2, 1, 14, 3, 10). \end{aligned}$$

THE EQUATION $\Delta(n, d, k) = by^2$ WITH $\omega(d) = 1$ AND AT MOST ONE TERM OMITTED
Further (a_1, \dots, a_6) is different from $(1, 2, 3, *, 5, 6)$, $(2, 1, 6, *, 10, 3)$ and their mirror images.

The proof of Lemma 2.5 is given in Section 3.

The following result is contained in [BBGH06, Lemma 4.1].

Lemma 2.6. *There are no coprime positive integers n', d' satisfying the diophantine equations*

$$\prod(0, 1, 2, 3) = by^2, \quad b \in \{1, 2, 3, 5, 15\}$$

$$\prod(0, 1, 3, 4) = by^2, \quad b \in \{1, 2, 3, 6, 30\}$$

where $\prod(0, i, j, l) = n'(n' + id')(n' + jd')(n' + ld')$.

Lemma 2.7. *Equation (1.1) with $\psi = 1, k = 7$ is not possible if*

- (i) $a_1 = a_4 = 1, a_6 = 6$ and either $a_3 = 3$ or $a_2 = 2$
- (ii) $a_1 = a_6 = 1$ and at least two of $a_2 = 2, a_4 = 6, a_5 = 5$ hold.
- (iii) $a_0 = a_6 = 2, a_5 = 3$ and either $a_2 = 6$ or $a_4 = 1$
- (iv) $a_0 = a_5 = 1$ and at least two of $a_1 = 5, a_2 = 6, a_4 = 2$ hold.
- (v) $a_3 = a_6 = 1, a_1 = 6$ and $a_2 = 5$
- (vi) $a_0 = a_4 = 1, a_3 = 3$ and $a_6 = 2$
- (vii) $a_0 = a_5 = 1$ and at least two of $a_1 = 2, a_3 = 6, a_6 = 3$ hold.

Proof. The proof of Lemma 2.7 uses **MAGMA** to compute integral points on quartic curves. For this we first make a quartic curve and find a integral point on it. Then we compute all integral points on the curve by using **MAGMA** command *IntegralQuarticPoints* and we exclude them.

We illustrate this with an example. Consider (ii). Then from $x_6^2 - x_1^2 = n + 6d - (n + d) = 5d$ and $\gcd(x_6 - x_1, x_6 + x_1) = 1$, we get either

$$(2.4) \quad x_6 - x_1 = 5, x_6 + x_1 = d$$

or

$$(2.5) \quad x_6 - x_1 = 1, x_6 + x_1 = 5d.$$

Assume (2.4). Then $d = 2x_1 + 5$. This with $n + d = x_1^2$ gives

$$2x_2^2 = n + 2d = n + d + d = x_1^2 + 2x_1 + 5 = (x_1 + 1)^2 + 4 \text{ if } a_2 = 2$$

$$6x_4^2 = n + 4d = n + d + 3d = x_1^2 + 6x_1 + 15 = (x_1 + 3)^2 + 6 \text{ if } a_4 = 6$$

$$5x_5^2 = n + 5d = n + d + 4d = x_1^2 + 8x_1 + 20 = (x_1 + 4)^2 + 4 \text{ if } a_5 = 5.$$

When $a_2 = 2, a_4 = 6$, by putting $X = x_1 + 1, Y = 6x_2x_4$, we get the quartic curve $Y^2 = 3(X^2 + 4)((X + 2)^2 + 6) = 3X^4 + 12X^3 + 42X^2 + 48X + 120$ in

positive integers X and Y with $X = x_1 + 1 \geq 2$. Observing that $(X, Y) = (1, 15)$ is an integral point on this curve, we obtain by **MAGMA** command

$$\text{IntegralQuarticPoints}([3, 12, 42, 48, 120], [1, 15]);$$

that all integral points on the curve are given by

$$(X, Y) \in \{(1, \pm 15), (-2, \pm 12), (-14, \pm 300), (-29, \pm 1365)\}.$$

Since none of the points (X, Y) satisfy $X \geq 2$, we exclude the case $a_2 = 2, a_4 = 6$. Further when $a_2 = 2, a_5 = 5$, by putting $X = x_1 + 1$ and $Y = 10x_2x_5$, we get the curve $Y^2 = 10(X^2 + 4)((X + 3)^2 + 4) = 10X^4 + 60X^3 + 170X^2 + 240X + 520$ on which $(X, Y) = (-1, 20)$ is an integral point. It follows by **MAGMA** that all the integral points on the curve satisfy $X \leq 1$, and also this case is excluded. When $a_4 = 6, a_5 = 5$, by putting $X = x_1 + 3$ and $Y = 30x_4x_5$, we get the curve $Y^2 = 30(X^2 + 6)((X + 1)^2 + 4) = 30X^4 + 60X^3 + 330X^2 + 360X + 900$ on which $(X, Y) = (0, 30)$ is an integral point. It follows by **MAGMA** that all the integral points on the curve other than $(X, Y) = (11, 500)$ satisfy $X \leq 1$. Since $X > 1, 30|Y$ and $30 \nmid 500$, also this case is excluded. When (2.5) holds, we get $5d = 2x_1 + 1$ and this with $n + d = x_1^2$ implies

$$\begin{aligned} 2(5x_2)^2 &= 25(n + d) + 25d = 25x_1^2 + 10x_1 + 5 = (5x_1 + 1)^2 + 4 \text{ if } a_2 = 2 \\ 6(5x_4)^2 &= 25(n + d) + 75d = 25x_1^2 + 30x_1 + 15 = (5x_1 + 3)^2 + 6 \text{ if } a_4 = 6 \\ 5(5x_5)^2 &= 25(n + d) + 100d = 25x_1^2 + 40x_1 + 20 = (5x_1 + 4)^2 + 4 \text{ if } a_5 = 5. \end{aligned}$$

As in the case (2.4), these gives rise to the same quartic curves $Y^2 = 3X^4 + 12X^3 + 42X^2 + 48X + 120$; $Y^2 = 10X^4 + 60X^3 + 170X^2 + 240X + 520$ and $Y^2 = 30X^4 + 60X^3 + 330X^2 + 360X + 900$ when $a_2 = 2, a_3 = 6$; $a_2 = 2, a_5 = 5$ and $a_4 = 6, a_5 = 5$, respectively. This is not possible.

Similarly all the other cases are excluded. In the case (iii), we have $n = 2x_0^2$ and obtain either $d = 2x_0 + 3$ or $3d = 2x_0 + 1$. Then we use $2a_i x_i^2 = 2(n + id) = (2x_0)^2 + 2i(2x_0 + 3) = (2x_0 + i)^2 + 6i - i^2$ if $d = 2x_0 + 3$ and $2a_i(3x_i)^2 = 18(n + id) = (6x_0)^2 + 6i(2x_0 + 1) = (6x_0 + i)^2 + 6i - i^2$ if $3d = 2x_0 + 1$ to get quartic equations. In the case (vi), we obtain the quartic equation $Y^2 = 6X^4 + 36X^3 + 108X - 54 = 6(X^4 + 6X^3 + 18X - 9)$. For any integral point (X, Y) on this curve, we obtain $3|(X^4 + 6X^3 + 18X - 9)$ giving $3|X$. Then $\text{ord}_3(X^4 + 6X^3 + 18X - 9) = 2$ giving $\text{ord}_3(Y^2) = \text{ord}_3(6) + 2 = 3$, a contradiction. \square

3. PROOF OF LEMMA 2.5

For the proof of Lemma 2.5, we use the so-called elliptic Chabauty's method (see [NB02],[NB03]). Bruin's routines related to elliptic Chabauty's method are contained in **MAGMA** [MAGMA], so here we indicate the main

THE EQUATION $\Delta(n, d, k) = by^2$ WITH $\omega(d) = 1$ AND AT MOST ONE TERM OMITTED steps only and a MAGMA routine which can be used to verify the computations is available from the third author.

First consider the tuple $(6, 7, 2, 1, 10, *, 3)$. Using that $n = -2(n + 3d) + 3(n + 2d) = -2x_3^2 + 6x_2^2$ and $d = (n + 3d) - (n + 2d) = x_3^2 - 2x_2^2$ we obtain the following system of equations

$$\begin{aligned} -x_3^2 + 3x_2^2 &= 3x_0^2, \\ -x_3^2 + 4x_2^2 &= 7x_1^2, \\ x_3^2 - x_2^2 &= 5x_4^2, \\ 4x_3^2 - 6x_2^2 &= 3x_6^2. \end{aligned}$$

The first equation implies that x_3 is divisible by 3, that is there exists a $z \in \mathbb{Z}$ such that $x_3 = 3z$. By standard factorization argument we get that

$$(\sqrt{3}z + x_2)(3z + x_2)(12z^2 - 2x_2^2) = \delta\Delta,$$

where $\delta \in \{\pm 2 + \sqrt{3}, \pm 10 + 5\sqrt{3}\}$. Thus putting $X = z/x_2$ it is sufficient to find all points (X, Y) on the curves

$$(3.1) \quad C_\delta : \delta(\sqrt{3}X + 1)(3X + 1)(12X^2 - 2) = Y^2,$$

for which $X \in \mathbb{Q}$ and $Y \in \mathbb{Q}(\sqrt{3})$. For all possible values of δ the point $(X, Y) = (-1/3, 0)$ is on the curves, therefore we can transform them to elliptic curves. We note that $X = z/x_2 = -1/3$ does not yield appropriate arithmetic progressions.

I. $\delta = 2 + \sqrt{3}$. In this case $C_{2+\sqrt{3}}$ is isomorphic to the elliptic curve

$$E_{2+\sqrt{3}} : y^2 = x^3 + (-\sqrt{3} - 1)x^2 + (6\sqrt{3} - 9)x + (11\sqrt{3} - 19).$$

Using MAGMA, we get that the rank of $E_{2+\sqrt{3}}$ is 0 and the only point on $C_{2+\sqrt{3}}$ for which $X \in \mathbb{Q}$ is $(X, Y) = (-\frac{1}{3}, 0)$.

II. $\delta = -2 + \sqrt{3}$. Applying elliptic Chabauty with $p = 7$, we get that $z/x_2 \in \{-1/2, -1/3, -33/74, 0\}$. Among these values $z/x_2 = -1/2$ gives $n = 6, d = 1$.

III. $\delta = 10 + 5\sqrt{3}$. Applying again elliptic Chabauty with $p = 23$, we get that $z/x_2 \in \{1/2, -1/3\}$. Here $z/x_2 = 1/2$ corresponds to $n = 6, d = 1$.

IV. $\delta = -10 + 5\sqrt{3}$. The elliptic curve $E_{-10+5\sqrt{3}}$ is of rank 0 and the the only point on $C_{-10+5\sqrt{3}}$ for which $X \in \mathbb{Q}$ is $(X, Y) = (-\frac{1}{3}, 0)$.

We have proved that there is no arithmetic progression for which $(a_0, a_1, \dots, a_6) = (6, 7, 2, 1, 10, *, 3)$ and $d \nmid n$.

Now consider the tuple $(1, 5, 6, 7, 2, *, 10)$. The system of equation we use is

$$\begin{aligned} x_6^2 - 3x_1^2 &= -2 \left(\frac{x_0}{2} \right)^2, \\ x_6^2 + 2x_1^2 &= 3x_2^2, \\ 4x_6^2 + 3x_1^2 &= 7x_3^2, \\ 3x_6^2 + x_1^2 &= x_4^2. \end{aligned}$$

We factor the first equation over $\mathbb{Q}(\sqrt{3})$ and the fourth over $\mathbb{Q}(\sqrt{-3})$. We obtain

$$\begin{aligned} x_6 + \sqrt{3}x_1 &= \delta_1 \square, \\ \frac{\sqrt{-3}x_6 + x_1}{2} &= \delta_2 \square, \end{aligned}$$

where $\delta_1 \in \{\pm 1 + \sqrt{3}, \pm 1 - \sqrt{3}, \pm 5 + 3\sqrt{3}, \pm 5 - 3\sqrt{3}\}$ and $\delta_2 \in \{\pm 1, \frac{\pm 1 + \sqrt{-3}}{2}, \frac{\pm 1 - \sqrt{-3}}{2}\}$. The curves for which we apply elliptic Chabauty's method are

$$C_\delta : 3\delta(X + \sqrt{3})(\sqrt{-3}X + 1)(X^2 + 2) = Y^2,$$

defined over $\mathbb{Q}(\alpha)$, where $\alpha^4 + 36 = 0$. It turns out that there is no arithmetic progression with $(a_0, a_1, \dots, a_6) = (1, 5, 6, 7, 2, *, 10)$ and $d \nmid n$.

We now make some observations here. If

$$(3.2) \quad u(n + id) + v(n + jd) = w(n + ld)$$

holds with $0 \leq i, j, l \leq k - 1$ and integers u, v, w , then

$$u + v = w \quad \text{and} \quad ui + vj = wl.$$

Therefore

$$u(n + (k - 1 - i)d) + v(n + (k - 1 - j)d) = w(n + (k - 1 - l)d)$$

holds and vice versa. Therefore any tuple $(a_0, a_1, a_2, \dots, a_6)$ and its mirror tuple (a_6, \dots, a_1, a_0) give rise to same set of equations. Hence it suffices to exclude any one of them. Also it suffices to exclude any one of $(*, a_1, a_2, \dots, a_6)$ and $(a_0, a_1, a_2, \dots, a_5, *)$.

Further if we define $a'_i = \frac{a_i}{2}$ if a_i is even and $a'_i = 2a_i$ if a_i is odd, then $(a'_0, a'_1, \dots, a'_6)$ and (a_0, a_1, \dots, a_6) give rise to same set of equations. Let i, j, l satisfy (3.2). If $n + id = a_i x_i^2, n + jd = a_j x_j^2, n + ld = a_l x_l^2$ is the one given by (a_0, a_1, \dots, a_6) and $n + id = a'_i x_i'^2, n + jd = a'_j x_j'^2, n + ld = a'_l x_l'^2$ given by $(a'_0, a'_1, \dots, a'_6)$, then we get from (3.2) that

$$(3.3) \quad ua_i x_i^2 + va_j x_j^2 = wa_l x_l^2$$

and

$$(3.4) \quad ua'_i x_i'^2 + va'_j x_j'^2 = wa'_l x_l'^2,$$

THE EQUATION $\Delta(n, d, k) = by^2$ WITH $\omega(d) = 1$ AND AT MOST ONE TERM OMITTED

respectively. Since $2a'_i x_i'^2 = a_i y_i^2$ for some y_i , multiplying by 2 to (3.4), we obtain an equation exactly similar to (3.3). Hence if we exclude one of $(a'_0, a'_1, \dots, a'_6)$, or (a_0, a_1, \dots, a_6) , the other tuple is excluded.

In view of the above observations and since $(a_0, a_1, \dots, a_6) = (1, 2, 3, *, 5, 6, 7)$ is excluded if $(a_1, a_2, \dots, a_6) = (1, 2, 3, *, 5, 6)$ is excluded, it suffices to consider the tuples

$$(a_0, a_1, \dots, a_6) \in \{(*, 3, 1, 5, 6, 7, 2), (3, *, 5, 6, 7, 2, 1), (1, 5, 6, 7, 2, *, 10), \\ (*, 5, 6, 7, 2, 1, 10), (6, 7, 2, 1, 10, *, 3), (*, 1, 2, 3, *, 5, 6)\}.$$

Already $(a_0, a_1, \dots, a_6) \in \{(1, 5, 6, 7, 2, *, 10), (6, 7, 2, 1, 10, *, 3)\}$ are excluded. In the table below, we indicate the relevant quartic polynomials for the remaining tuples as follows:

tuple	polynomial
$(1, 2, 3, *, 5, 6)$	$2\delta_{A1}(X + \sqrt{-1})(X + 3\sqrt{-1})(5X^2 - 3)$
$(*, 3, 1, 5, 6, 7, 2)$	$\delta_{A2}(X + \sqrt{-1})(2X + \sqrt{-1})(5X^2 - 1)$
$(3, *, 5, 6, 7, 2, 1)$	$5\delta_{A3}(2X + 3\sqrt{-1})(X + \sqrt{-1})(12X^2 - 3)$
$(*, 5, 6, 7, 2, 1, 10)$	$\delta_{A4}(X + \sqrt{-2})(2\sqrt{-2}X + 1)(3X^2 + 1)$

□

4. PROOF OF THEOREM 1

Suppose that the assumptions of Theorem 1 are satisfied and assume (1.1) with $\omega(d) = 1$. Let $k \geq 9$. By Lemma 2.3, we may suppose that $P(b) = k$ implying k is a prime. After deleting the term divisible by k on the left hand side of (1.1) and using Lemma 2.4, the assertion for $k \geq 15$ follows. Thus it suffices to prove the assertion for $k \in \{7, 8, 11, 13\}$ with $P(b) \leq k$ for $k \in \{7, 8\}$ and $P(b) = k$ for $k \in \{11, 13\}$.

Therefore we always restrict to $k \in \{7, 8, 11, 13\}$ and with $P(b) \leq k$ for $k \in \{7, 8\}$ and $P(b) = k$ for $k \in \{11, 13\}$. In view of Lemma 2.1, we arrive at a contradiction by showing $t - |R| \geq 2$ when $k \in \{11, 13\}$. Further Lemma 2.1 also implies that $p \nmid d$ for $p \leq k$ whenever $k \in \{11, 13\}$.

For a prime $p \leq k$ and $p \nmid d$, let i_p be such that $0 \leq i_p < p$ and $p|n + i_p d$. For any subset $\mathcal{I} \subseteq [0, k) \cap \mathbb{Z}$ and primes p_1, p_2 with $p_i \leq k$ and $p_i \nmid d$, $i = 1, 2$, we define

$$\mathcal{I}_1 = \{i \in \mathcal{I} : \left(\frac{i - i_{p_1}}{p_1}\right) = \left(\frac{i - i_{p_2}}{p_2}\right)\} \text{ and } \mathcal{I}_2 = \{i \in \mathcal{I} : \left(\frac{i - i_{p_1}}{p_1}\right) \neq \left(\frac{i - i_{p_2}}{p_2}\right)\}.$$

Then from $\left(\frac{a_i}{p}\right) = \left(\frac{i - i_p}{p}\right) \left(\frac{d}{p}\right)$, we see that either

$$(4.1) \quad \left(\frac{a_i}{p_1}\right) \neq \left(\frac{a_i}{p_2}\right) \text{ for all } i \in \mathcal{I}_1 \text{ and } \left(\frac{a_i}{p_1}\right) = \left(\frac{a_i}{p_2}\right) \text{ for all } i \in \mathcal{I}_2$$

or

$$(4.2) \quad \left(\frac{a_i}{p_1}\right) \neq \left(\frac{a_i}{p_2}\right) \text{ for all } i \in \mathcal{I}_2 \text{ and } \left(\frac{a_i}{p_1}\right) = \left(\frac{a_i}{p_2}\right) \text{ for all } i \in \mathcal{I}_1.$$

We define $(\mathcal{M}, \mathcal{B}) = (\mathcal{I}_1, \mathcal{I}_2)$ in the case (4.1) and $(\mathcal{M}, \mathcal{B}) = (\mathcal{I}_2, \mathcal{I}_1)$ in the case (4.2). We call $(\mathcal{I}_1, \mathcal{I}_2, \mathcal{M}, \mathcal{B}) = (\mathcal{I}_1^k, \mathcal{I}_2^k, \mathcal{M}^k, \mathcal{B}^k)$ when $\mathcal{I} = [0, k) \cap \mathbb{Z}$. Then for any $\mathcal{I} \subseteq [0, k) \cap \mathbb{Z}$, we have

$$\mathcal{I}_1 \subseteq \mathcal{I}_1^k, \mathcal{I}_2 \subseteq \mathcal{I}_2^k, \mathcal{M} \subseteq \mathcal{M}^k, \mathcal{B} \subseteq \mathcal{B}^k$$

and

$$(4.3) \quad |\mathcal{M}| \geq |\mathcal{M}^k| - (k - |\mathcal{I}|), \quad |\mathcal{B}| \geq |\mathcal{B}^k| - (k - |\mathcal{I}|).$$

By taking $m = n + \gamma_t d$ and $\gamma'_i = \gamma_t - \gamma_{t-i+1}$, we re-write (1.1) as

$$(4.4) \quad (m - \gamma'_1 d) \cdots (m - \gamma'_t d) = by^2.$$

The equation (4.4) is called the mirror image of (1.1). The corresponding t -tuple $(a_{\gamma'_1}, a_{\gamma'_2}, \dots, a_{\gamma'_t})$ is called the mirror image of $(a_{\gamma_1}, \dots, a_{\gamma_t})$.

4.1. The case $k = 7, 8$. We may assume that $k = 7$ since the case $k = 8$ follows from that of $k = 7$.

In this subsection, we take $d \in \{2^\alpha, p^\alpha, 2p^\alpha\}$ where p is any odd prime and α is a positive integer. In fact, we prove

Lemma 4.1. *Let $\psi = 1, k = 7$ and $d \nmid n$. Then (1.1) with $d \in \{2^\alpha, p^\alpha, 2p^\alpha\}$ does not hold.*

First we check that (1.1) does not hold for $d \leq 23$ and $n + 5d \leq 324$. Thus we assume that either $d > 23$ or $n + 5d > 324$. Hence $n + id > 24i$ since $n > 208$ if $d \leq 23$. Then (1.1) with $\psi = 0, k \geq 4$ and $\omega(d) = 1$ has no solution by Lemma 2.2. Let $d = 2$ or $d = 4$. Suppose $a_i = a_j$ with $i > j$. Then $x_i - x_j = r_1$ and $x_i + x_j = r_2$ with r_1, r_2 even and $\gcd(r_1, r_2) = 2$. Now from $a_i x_i^2 = n + id > 24i \geq 4i^2$, we get

$$i - j \geq \frac{a_i(x_i + x_j)}{2} \geq \frac{(a_i x_i^2)^{\frac{1}{2}} + (a_j a_j^2)^{\frac{1}{2}}}{2} > \frac{2i + 2j}{2} \geq i,$$

a contradiction. Therefore $a_i \neq a_j$ whenever $i \neq j$ giving $|R| = k - 1$. But $|\{a_i : P(a_i) \leq 5\}| \leq 4$ implying $|R| \leq 4 + 1 < k - 1$, a contradiction. Let $8|d$. From (2.1), we get $\left(\frac{a_i}{8}\right) = \left(\frac{n+id}{8}\right) = \left(\frac{n}{8}\right)$ implying a_i 's belong to exactly one distinct residue class modulo 8. Therefore $|\{a_i : P(a_i) \leq 5\}| \leq 1$ and which together with $|\{j : a_j = a_i\}| \leq 2$ for $a_i \in R$ implies $|\{i : P(a_i) \leq 5\}| \leq 2$. This is a contradiction since $|\{i : P(a_i) \leq 5\}| \geq 7 - 2 = 5$. Thus $d \neq 2^\alpha$. Let $t - |R| \geq 2$. Then we observe from [LaSh06a, Lemma 3.10] that $d_2 = d < 24$ and $n + 5d < 324$. This is not possible.

Therefore $t - |R| \leq 1$ implying $|R| \geq k - 2 = 5$. If $7|d$, then we get a contradiction since $7 \nmid a_i$ for any i and $|\{a_i : P(a_i) \leq 5\}| \leq 4$ implying

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$|R| \leq 4 < k - 2$. If $3|d$ or $5|d$, then also we obtain a contradiction since $|\{a_i : P(a_i) \leq 5\}| \leq 2$ implying $|R| \leq 2 + 1 < k - 2$.

Thus $\gcd(p, d) = 1$ for each prime $p \leq 7$. Therefore $5|n + i_5d$ and $7|n + i_7d$ with $0 \leq i_5 < 5$ and $0 \leq i_7 < 7$. By taking the mirror image (4.4) of (1.1), we may suppose that $0 \leq i_7 \leq 3$.

Let $p_1 = 5$, $p_2 = 7$ and $\mathcal{I} = \{\gamma_1, \gamma_2, \dots, \gamma_6\}$. We observe that $P(a_i) \leq 3$ for $i \in \mathcal{M} \cup \mathcal{B}$. Since $\binom{2}{5} \neq \binom{2}{7}$ but $\binom{3}{5} = \binom{3}{7}$, we observe that $a_i \in \{2, 6\}$ whenever $i \in \mathcal{M}$ and $a_i \in \{1, 3\}$ whenever $i \in \mathcal{B}$.

We now define four sets

$$\begin{aligned}\mathcal{I}_{++}^k &= \{i : 0 \leq i < k, \binom{i - i_{p_1}}{p_1} = \binom{i - i_{p_2}}{p_2} = 1\}, \\ \mathcal{I}_{--}^k &= \{i : 0 \leq i < k, \binom{i - i_{p_1}}{p_1} = \binom{i - i_{p_2}}{p_2} = -1\}, \\ \mathcal{I}_{+-}^k &= \{i : 0 \leq i < k, \binom{i - i_{p_1}}{p_1} = 1, \binom{i - i_{p_2}}{p_2} = -1\}, \\ \mathcal{I}_{-+}^k &= \{i : 0 \leq i < k, \binom{i - i_{p_1}}{p_1} = -1, \binom{i - i_{p_2}}{p_2} = 1\}.\end{aligned}$$

and let $\mathcal{I}_{++} = \mathcal{I}_{++}^k \cap \mathcal{I}$, $\mathcal{I}_{--} = \mathcal{I}_{--}^k \cap \mathcal{I}$, $\mathcal{I}_{+-} = \mathcal{I}_{+-}^k \cap \mathcal{I}$, $\mathcal{I}_{-+} = \mathcal{I}_{-+}^k \cap \mathcal{I}$. We observe here that $\mathcal{I}_1 = \mathcal{I}_{++} \cup \mathcal{I}_{--}$ and $\mathcal{I}_2 = \mathcal{I}_{+-} \cup \mathcal{I}_{-+}$. Since $a_i \in \{1, 2, 3, 6\}$ for $i \in \mathcal{I}_1 \cup \mathcal{I}_2$ and $\binom{a_i}{p} = \binom{i - i_p}{p} \binom{d}{p}$, we obtain four possibilities *I, II, III* and *IV* according as $\binom{d}{5} = \binom{d}{7} = 1$; $\binom{d}{5} = \binom{d}{7} = -1$; $\binom{d}{5} = 1, \binom{d}{7} = -1$; $\binom{d}{5} = -1, \binom{d}{7} = 1$, respectively.

	$\{a_i : i \in \mathcal{I}_{++}\}$	$\{a_i : i \in \mathcal{I}_{--}\}$	$\{a_i : i \in \mathcal{I}_{+-}\}$	$\{a_i : i \in \mathcal{I}_{-+}\}$
<i>I</i>	$\{1\}$	$\{3\}$	$\{6\}$	$\{2\}$
<i>II</i>	$\{3\}$	$\{1\}$	$\{2\}$	$\{6\}$
<i>III</i>	$\{2\}$	$\{6\}$	$\{3\}$	$\{1\}$
<i>IV</i>	$\{6\}$	$\{2\}$	$\{1\}$	$\{3\}$

In the case *I*, we have $\binom{a_i}{p} = \binom{i - i_p}{p}$ for $p \in \{5, 7\}$ which together with $\binom{a_i}{5} = 1$ for $a_i \in \{1, 6\}$, $\binom{a_i}{5} = -1$ for $a_i \in \{2, 3\}$, $\binom{a_i}{7} = 1$ for $a_i \in \{1, 2\}$ and $\binom{a_i}{7} = -1$ for $a_i \in \{3, 6\}$ implies the assertion. The assertion for the cases *II, III* and *IV* follows similarly. For simplicity, we write $\mathcal{A}_7 = (a_0, a_1, a_2, a_3, a_4, a_5, a_6)$.

For each possibility $0 \leq i_5 < 5$ and $0 \leq i_7 \leq 3$, we compute $\mathcal{I}_{++}^k, \mathcal{I}_{--}^k, \mathcal{I}_{+-}^k, \mathcal{I}_{-+}^k$ and restrict to those pairs (i_5, i_7) for which $\max(|\mathcal{I}_1^k|, |\mathcal{I}_2^k|) \leq 4$. Then we check for the possibilities *I, II, III* or *IV*.

Suppose $d = 2p^\alpha$. Then $b_i \in \{1, 3\}$ whenever $P(b_i) \leq 3$. If $i_5 \neq 0, 1$, then $|R| \leq 2 + 2 = 4$ giving $t - |R| \geq 7 - 1 - 4 = 2$, a contradiction. Thus $i_5 \in \{0, 1\}$. Further $\mathcal{M} = \emptyset$ and $a_i \in \{1, 3\}$ for $i \in \mathcal{B}$. Therefore either $|\mathcal{I}_1^k| \leq 1$ or $|\mathcal{I}_1^k| \leq 2$. We find that this is the case only when $(i_5, i_7) \in \{(0, 1), (1, 2)\}$. Let $(i_5, i_7) = (0, 1)$. We get $\mathcal{I}_{++}^k = \mathcal{I}_{--}^k = \emptyset$,

$\mathcal{I}_{+-}^k = \{4, 6\}$ and $\mathcal{I}_{-+}^k = \{2, 3\}$. It suffices to consider the cases *III* and *IV* since $b_i \in \{1, 3\}$ whenever $P(b_i) \leq 3$. Suppose *III* holds. Then by modulo 3, we obtain $4 \notin \mathcal{I}$, $a_6 = 3$ and $a_2 = a_3 = 1$. By modulo 3 again, we get $a_1 \notin \{1, 7, 3\}$ which is not possible since $5 \nmid a_1$. Suppose *IV* holds. Then by modulo 3, we obtain $2 \notin \mathcal{I}$, $a_4 = a_6 = 1$ and $a_3 = 3$. We now get $a_1 \in \{1, 7\}$ and by $t - |R| \leq 1$, we get $a_1 = 7$. This is not possible since $-1 = \binom{a_1 a_4}{5} = \binom{(1-0)(4-0)}{5} = 1$. Similarly $(i_5, i_7) = (1, 2)$ is excluded. Hence $d = p^\alpha$ from now on.

Let $(i_5, i_7) = (0, 0)$. We obtain $\mathcal{I}_{++}^k = \{1, 4\}$, $\mathcal{I}_{--}^k = \{3\}$, $\mathcal{I}_{+-}^k = \{6\}$ and $\mathcal{I}_{-+}^k = \{2\}$. We may assume that $1 \in \mathcal{I}$ otherwise $P(a_2 a_3 a_4 a_5 a_6) \leq 5$ and this is excluded by Lemma 2.2 with $k = 5$. Further $i \notin \mathcal{I}$ for exactly one of $i \in \{2, 3, 4\}$ otherwise $P(a_1 a_2 a_3 a_4) \leq 3$ and this is not possible by Lemma 2.2 with $k = 4$ since $d > 23$. Consider the possibilities *II* and *IV*. By modulo 3, we obtain $2 \notin \mathcal{I}$, $3 \mid a_1 a_4$ and $a_3 a_6 = 2$. This is not possible by modulo 3 since $-1 = \binom{a_3 a_6}{3} = \binom{(3-1)(6-1)}{3} = 1$, a contradiction. Suppose *I* holds. Then $a_1 = 1$ and $a_6 = 6$. If $4 \in \mathcal{I}$, then $a_1 = a_4 = 1$ and at least one of $a_3 = 3, a_2 = 2$ holds and this is excluded by Lemma 2.7 (i). Assume that $4 \notin \mathcal{I}$. Then $a_1 = 1, a_2 = 2, a_3 = 3, a_6 = 6$ giving $a_5 = 5$ by modulo 2 and 3. Thus we have $(a_1, \dots, a_5, a_6) = (1, 2, 3, *, 5, 6)$. This is not possible by Lemma 2.5. Suppose *III* holds. Then $4 \notin \mathcal{I}$, $a_1 = 2, a_2 = 1, a_3 = 6, a_6 = 3$ giving $a_5 = 10$ by modulo 2 and 3. Thus $(a_1, \dots, a_5, a_6) = (2, 1, 6, *, 10, 3)$ which is also excluded by Lemma 2.5.

Let $(i_5, i_7) = (0, 1)$. We obtain $\mathcal{I}_{++}^k = \mathcal{I}_{--}^k = \emptyset$, $\mathcal{I}_{+-}^k = \{4, 6\}$ and $\mathcal{I}_{-+}^k = \{2, 3\}$. The possibility *I* is excluded by parity and modulo 3. The possibility *II* implies that $3 \notin \mathcal{I}$, $a_4 = a_6 = 2$ and $a_2 = 3$. This is not possible by modulo 3. Suppose *III* holds. Then $a_2 = a_3 = 1$ and either $4 \notin \mathcal{I}, a_6 = 3$ or $6 \notin \mathcal{I}, a_4 = 3$. By modulo 3, we obtain $4 \notin \mathcal{I}, a_6 = 3$ and $\binom{a_5}{3} = \binom{a_2}{3} = 1$. This gives $a_5 \in \{1, 10\}$ which together with $t - |R| \leq 1$ implies $a_5 = 10$. But this is not possible by Lemma 2.6 with $n' = n + 2d, d' = d$ and $(i, j, l) = (1, 3, 4)$. Hence *III* is excluded. Suppose *IV* holds. Then $a_4 = a_6 = 1$ and $2 \notin \mathcal{I}, a_3 = 3$ by modulo 3. By modulo 3, we get $a_5 \in \{2, 5\}$ and we may take $a_5 = 5$ otherwise we get a contradiction from $d > 23$ and Lemma 2.2 with $k = 4$ applied to $(n + 3d)(n + 4d)(n + 5d)(n + 6d)$. This is again not possible by Lemma 2.6 with $n' = n + 3d, d' = d$ and $(i, j, l) = (1, 2, 3)$.

Let $(i_5, i_7) = (0, 3)$. We obtain $\mathcal{I}_{++}^k = \{4\}$, $\mathcal{I}_{--}^k = \{2\}$, $\mathcal{I}_{+-}^k = \{1, 6\}$ and $\mathcal{I}_{-+}^k = \emptyset$. By modulo 3, we observe that the possibilities *I* and *III* are excluded. Suppose *II* happens. Then $a_2 = 1, a_4 = 3$ and either $a_6 = 2, 1 \notin \mathcal{I}$ or $a_1 = 2, 6 \notin \mathcal{I}$. If $a_6 = 2, 1 \notin \mathcal{I}$, then $a_5 \in \{1, 5\}$ which gives $a_5 = 1$ by modulo 3. This is not possible by modulo 7 since $-1 = \binom{a_4 a_5}{7} =$

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$\left(\frac{(4-3)(5-3)}{7}\right) = 1$. Thus $a_1 = 2, 6 \notin \mathcal{I}$. Then $a_0 = 5, a_5 = 10, a_3 = 14$ by modulo 3 giving $(a_0, a_1, \dots, a_5, a_6) = (5, 2, 1, 14, 3, 10, *)$. Suppose *IV* happens. Let $1, 6 \in \mathcal{I}$. Then $a_1 = a_6 = 1$ and either $a_2 = 2$ or $a_4 = 6$. By Lemma 2.7 (ii), we may assume that either $2 \notin \mathcal{I}$ or $4 \notin \mathcal{I}$. If $2 \notin \mathcal{I}$, then $a_4 = 6, a_3 = 7$ and $a_5 = 5$ which is excluded by Lemma 2.7 (ii). Thus $4 \notin \mathcal{I}, a_2 = 2$ and $a_5 = 5$ since $3 \nmid a_5$. This is also excluded by Lemma 2.7 (ii). Therefore $a_2 = 2, a_4 = 6$ and either $6 \notin \mathcal{I}, a_1 = 1$ or $1 \notin \mathcal{I}, a_6 = 1$. Now $7|a_3$ otherwise $P(a_1a_2 \cdots a_5) \leq 5$ if $1 \in \mathcal{I}$ or $P(a_2a_3 \cdots a_6) \leq 5$ if $6 \in \mathcal{I}$ and this is excluded by Lemma 2.2 with $k = 5$. Further by modulo 3, we get $a_3 = 7, a_0 = 10$ and $a_5 = 5$. Hence we obtain $\mathcal{A}_7 = (10, *, 2, 7, 6, 5, 1)$ or $\mathcal{A}_7 = (10, 1, 2, 7, 6, 5, *)$.

Let $(i_5, i_7) = (1, 0)$. We obtain $\mathcal{I}_{++}^k = \{2\}, \mathcal{I}_{--}^k = \{3\}, \mathcal{I}_{+-}^k = \{5\}$ and $\mathcal{I}_{-+}^k = \{4\}$. We consider the possibility *I*. By parity argument, we have either $5 \notin \mathcal{I}$ or $4 \notin \mathcal{I}$. Again by modulo 3, either $3 \notin \mathcal{I}$ or $5 \notin \mathcal{I}$. Thus $5 \notin \mathcal{I}$ giving $a_2 = 1, a_3 = 3, a_4 = 2$. Now $5|a_1$ otherwise we get a contradiction from $P(a_1a_2a_3a_4) \leq 3$, Lemma 2.2 with $k = 4$ and $d > 23$. Hence $a_1 = 5$. This is a again a contradiction since $-1 = \left(\frac{a_1a_2}{7}\right) = \left(\frac{(1-0)(2-0)}{7}\right) = 1$. Thus the possibility *I* is excluded. If the possibility *III* holds, then $3 \notin \mathcal{I}, a_2 = 2, a_5 = 3, a_4 = 1$ giving $a_1 \in \{1, 5\}$ and $a_6 = 5$. By modulo 3, we get $a_1 = 1$. But this is not possible by Lemma 2.6 with $n' = n + 2d, d' = d$ and $(i, j, l) = (1, 3, 4)$. Similarly, the possibilities *II* and *IV* are also excluded. If *II* holds, then $4 \notin \mathcal{I}, a_2 = 3, a_3 = 1, a_5 = 2$. Now $a_6 \in \{1, 5\}$ and further by modulo 3, we get $a_6 = 1$. This is not possible by Lemma 2.6 with $n' = n + 2d, d' = d$ and $(i, j, l) = (1, 3, 4)$. If *IV* holds, then $2 \notin \mathcal{I}, a_3 = 2, a_5 = 1, a_4 = 3$. Then $a_6 \in \{1, 5\}$ giving $a_6 = 5$ by modulo 3. This is not possible modulo 7.

Let $(i_5, i_7) = (1, 1)$. We obtain $\mathcal{I}_{++}^k = \{2, 5\}, \mathcal{I}_{--}^k = \{4\}, \mathcal{I}_{+-}^k = \{0\}$ and $\mathcal{I}_{-+}^k = \{3\}$. We consider the possibilities *III* and *IV*. By parity, we obtain $5 \notin \mathcal{I}$. But then we get a contradiction modulo 3 since $a_4 = 6, a_0 = 3$ if *III* holds and $a_2 = 6, a_3 = 3$ if *IV* holds are not possible. Next we consider the possibility *I*. Then $0 \notin \mathcal{I}$ by modulo 2 and 3 and we get $P(a_2a_3 \cdots a_6) \leq 5$ and this is excluded by Lemma 2.2 with $k = 5$. Let *II* hold. Then $3 \notin \mathcal{I}$ by modulo 2 and 3 and $a_2 = a_5 = 3, a_4 = 1, a_0 = 2$. Further $a_6 \in \{5, 10\}$ which together with modulo 3 gives $a_6 = 5$. Now we get a contradiction modulo 7 from $a_5 = 3, a_6 = 5$.

Let $(i_5, i_7) = (3, 1)$. We obtain $\mathcal{I}_{++}^k = \{2\}, \mathcal{I}_{--}^k = \{0, 6\}, \mathcal{I}_{+-}^k = \{4\}$ and $\mathcal{I}_{-+}^k = \{5\}$. We may assume that $i \notin \mathcal{I}$ for exactly one of $i \in \{0, 2, 4, 6\}$ otherwise n is even, $P(a_0a_2a_4a_6) \leq 3$ and this is excluded by $k = 4$ of Lemma 2.2 applied to $\frac{n}{2}(\frac{n}{2} + d)(\frac{n}{2} + 2d)(\frac{n}{2} + 3d)$. We consider the possibilities *I* and *III*. By modulo 3, we get $4 \notin \mathcal{I}, a_0 = a_6, 3|a_0$ and $a_2a_5 = 2$. This is not

possible by modulo 3. Next we consider the possibility *II*. Then $4 \notin \mathcal{I}$ by parity argument. Further $a_0 = a_6 = 1$, $a_2 = 3$, $a_5 = 6$. This is not possible since $8|x_6^2 - x_0^2 = n + 6d - n = 6d$ and d is odd. Finally we consider the possibility *IV*. If $2 \notin \mathcal{I}$ or $4 \notin \mathcal{I}$, then $a_0 = a_6 = 2$, $a_5 = 3$ and one of $a_2 = 6$ and $a_4 = 1$. This is excluded by Lemma 2.7 (iii). Thus $a_2 = 6$, $a_4 = 1$, $a_5 = 3$ and either $a_0 = 2$, $6 \notin \mathcal{I}$ or $a_6 = 2$, $0 \notin \mathcal{I}$. Then $a_1 = 7$, $a_3 = 5$ by parity and modulo 3. Hence $\mathcal{A}_7 = (2, 7, 6, 5, 1, 3, *)$ or $\mathcal{A}_7 = (*, 7, 6, 5, 1, 3, 2)$.

All the other pairs are excluded similarly. For $(i_5, i_7) = (0, 2)$, we obtain either $\mathcal{A}_7 = (1, 2, 3, *, 5, 6)$ or $(5, 6, 7, 2, 1, 10, *)$ or $(10, 3, 14, 1, 2, 5, *)$ which are all excluded by Lemma 2.5. For $(i_5, i_7) = (1, 3)$, we obtain $\mathcal{A}_7 = (1, 5, 6, 7, 2, *, 10)$, $(*, 5, 6, 7, 2, 1, 10)$ or $(*, 10, 3, 14, 1, 2, 5)$ which is not possible by Lemma 2.5, or $a_0 = a_5 = 1$ and at least two of $a_1 = 5$, $a_2 = 6$, $a_4 = 2$ hold which is again excluded by Lemma 2.7 (iv). For $(i_5, i_7) = (2, 0)$, we obtain $\mathcal{A}_7 = (14, 3, 10, *, 6, 1, 2)$, $(7, 6, 5, *, 3, 2, 1)$ or $a_3 = a_6 = 1$, $a_0 = 7$, $a_1 = 6$, $a_2 = 5$, $a_4 = 3$ or $a_5 = 2$. These are impossible by Lemma 2.7 (v). For $(i_5, i_7) = (2, 1)$, we obtain $a_0 = a_4 = 1$, $a_3 = 3$, $a_6 = 2$ which is not possible by Lemma 2.7 (vi). For $(i_5, i_7) = (4, 1)$, we obtain $\mathcal{A}_7 = (6, 7, 2, 1, 10, *, 3)$ which is also excluded. For $(i_5, i_7) = (4, 2)$, we obtain $\mathcal{A}_7 = (2, 1, 14, 3, 10, *, 6)$, $(1, 2, 7, 6, 5, *, 3)$, $(*, 2, 7, 6, 5, 1, 3)$ or $a_0 = a_5 = 1$ and at least two of $a_1 = 2$, $a_3 = 6$, $a_6 = 3$ hold. The previous possibility is excluded by Lemma 2.5 and the latter by Lemma 2.7 (vii).

4.2. The case $k = 11$. We may assume that $11|a_i$ for some $i \in \{4, 5, 6\}$ whenever $i \notin \mathcal{I}$ otherwise the lemma follows from Lemma 4.1.

Let $p_1 = 5$, $p_2 = 11$ and $\mathcal{I} = \{\gamma_1, \gamma_2, \dots, \gamma_t\}$. We observe that $P(a_i) \leq 7$ for $i \in \mathcal{M} \cup \mathcal{B}$. Since $\binom{3}{5} \neq \binom{3}{11}$ but $\binom{q}{5} = \binom{q}{11}$ for a prime $q < k$ other than 3, 5, 11, we observe that $3|a_i$ whenever $i \in \mathcal{M}$. Since $\sigma_3 \leq 4$ and $|\mathcal{I}| = k - 1$, we obtain from (4.3) that $|\mathcal{M}^k| \leq 5$ and $3|a_i$ for at least $|\mathcal{M}^k| - 1$ i 's with $i \in \mathcal{M}^k$. Further $a_i \in \{1, 2, 7, 14\}$ for $i \in \mathcal{B}$ giving $|\mathcal{B}| \leq 5$ otherwise $t - |R| \geq 2$. Hence $|\mathcal{B}^k| \leq 6$ by (4.3).

By taking the mirror image (4.4) of (1.1), we may suppose that $4 \leq i_{11} \leq 5$. For each possibility $0 \leq i_5 < 5$ and $4 \leq i_{11} \leq 5$, we compute $|\mathcal{I}_1^k|$, $|\mathcal{I}_2^k|$ and restrict to those pairs (i_5, i_{11}) for which $\max(|\mathcal{I}_1^k|, |\mathcal{I}_2^k|) \leq 6$. Further we restrict to those pairs (i_5, i_{11}) for which either

$$(4.5) \quad 3|a_i \text{ for at least } |\mathcal{I}_1^k| - 1 \text{ elements } i \in \mathcal{I}_1^k$$

or

$$(4.6) \quad 3|a_i \text{ for at least } |\mathcal{I}_2^k| - 1 \text{ elements } i \in \mathcal{I}_2^k.$$

We find that exactly one of (4.5) or (4.6) happens. We have $\mathcal{M}^k = \mathcal{I}_1^k$, $\mathcal{B}^k = \mathcal{I}_2^k$ when (4.5) holds and $\mathcal{M}^k = \mathcal{I}_2^k$, $\mathcal{B}^k = \mathcal{I}_1^k$ when (4.6) holds. If $3|a_i$ for exactly $|\mathcal{M}^k| - 1$ elements $i \in \mathcal{M}^k$, then $\mathcal{B} = \mathcal{B}^k$ and we restrict to such

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pairs (i_5, i_{11}) for which there are at most 3 elements $i \in \mathcal{B}^k$ with $P(a_i) \leq 2$ otherwise $t - |R| \geq 2$. Now all the pairs (i_5, i_{11}) are excluded other than

$$(4.7) \quad (0, 4), (1, 5), (4, 5).$$

For these pairs, we find that $|\mathcal{B}^k| \geq 5$. Hence we may suppose that $7|a_i$ for some $i \in \mathcal{B}$ otherwise $a_i \in \{1, 2\}$ for $i \in \mathcal{B}$ which together with $|\mathcal{B}| \geq 4$ gives $t - |R| \geq 2$. Further if $|\mathcal{B}^k| = 6$, then we may assume that $7|a_i, 7|a_{i+7}$ for some $0 \leq i \leq 3$.

Let $(i_5, i_{11}) = (0, 4)$. Then $\mathcal{M}^k = \{3, 9\}$ and $\mathcal{B}^k = \{1, 2, 6, 7, 8\}$ giving $i_3 = 0$. If $7|a_6a_7$, then $|\mathcal{B}| = |\mathcal{B}^k| - 1$ and $a_i \in \{3, 6\}$ for $i \in \mathcal{M} = \mathcal{M}^k$ but $\left(\frac{a_3a_9}{7}\right) = \left(\frac{(3-i_7)(9-i_7)}{7}\right) = -1$ for $i_7 = 6, 7$, a contradiction. If $7|a_2$, then $a_i \in \{5, 10\}$ for $i \in \{5, 10\} \subseteq \mathcal{I}$ but $\left(\frac{a_5a_{10}}{7}\right) = \left(\frac{(5-2)(10-2)}{7}\right) = -1$, a contradiction again. Thus $7|a_1a_8$ and $a_i \in \{1, 2\}$ for $\{2, 6, 7\} \cap \mathcal{B}^k$. From $\left(\frac{a_i}{7}\right) = \left(\frac{i-1}{7}\right) \left(\frac{d}{7}\right)$, $\left(\frac{6-1}{7}\right) = \left(\frac{7-1}{7}\right) = -1$ and $\left(\frac{2-1}{7}\right) = 1$, we find that $2 \notin \mathcal{I}$. This is not possible.

Let $(i_5, i_{11}) = (1, 5)$. Then $\mathcal{M}^k = \{4, 10\}$ and $\mathcal{B}^k = \{0, 2, 3, 7, 8, 9\}$ giving $i_3 = 1$. Thus $\mathcal{M} = \mathcal{M}^k$, $a_i \in \{3, 6\}$ for $i \in \mathcal{M}$ and $|\mathcal{B}| = |\mathcal{B}^k| - 1$, $a_i \in \{1, 2, 7, 14\}$ for $i \in \mathcal{B}$. Further we have either $7|a_0a_7$ or $7|a_2a_9$. Taking $\left(\frac{a_i}{7}\right)$ for $i \in \{4, 10, 0, 2, 3, 7, 8, 9\}$, we find that $7|a_2a_9$ and $3 \notin \mathcal{B}$. This is not possible.

Let $(i_5, i_{11}) = (4, 5)$. Then $\mathcal{M}^k = \{0, 6\}$ and $\mathcal{B}^k = \{1, 2, 3, 7, 8, 10\}$ giving $\mathcal{M} = \mathcal{M}^k$ and $i_3 = 0$. Further $7|a_1a_8$ or $7|a_3a_{10}$. Taking $\left(\frac{a_i}{7}\right)$ for $i \in \mathcal{M} \cup \mathcal{B}^k$, we find that $7|a_1a_8$ and $\mathcal{B} = \mathcal{B}^k \setminus \{7\}$. This is not possible since $7 \in \mathcal{I}$.

4.3. The case $k = 13$. We may assume that $13|a_3a_4a_5a_6a_7a_8a_9$ otherwise the assertion follows from Theorem 1 with $k = 11$.

Let $p_1 = 11$, $p_2 = 13$ and $\mathcal{I} = \{\gamma_1, \gamma_2, \dots, \gamma_t\}$. Since $\left(\frac{5}{11}\right) \neq \left(\frac{5}{13}\right)$ but $\left(\frac{q}{11}\right) = \left(\frac{q}{13}\right)$ for $q = 2, 3, 7$, we observe that for $5|a_i$ for $i \in \mathcal{M}$ and $P(a_i) \leq 7$, $5 \nmid a_i$ for $i \in \mathcal{B}$. Since $\sigma_5 \leq 3$, we obtain $|\mathcal{M}^k| \leq 4$ and $5|a_i$ for at least $|\mathcal{M}^k| - 1$ i 's with $i \in \mathcal{M}^k$.

By taking the mirror image (4.4) of (1.1), we may suppose that $3 \leq i_{13} \leq 6$ and $0 \leq i_{11} \leq 10$. We may suppose that $i_{13} \geq 4, 5$ if $i_{11} = 0, 1$, respectively and $\max(i_{11}, i_{13}) \geq 6$ if $i_{11} \geq 2$ otherwise the assertion follows from Lemma 4.1.

Since $\max(|\mathcal{I}_1^k|, |\mathcal{I}_2^k|) \geq 5$ and $|\mathcal{M}^k| \leq 4$, we restrict to those pairs satisfying $\min(|\mathcal{I}_1^k|, |\mathcal{I}_2^k|) \leq 4$ and further \mathcal{M}^k is exactly one of \mathcal{I}_1^k or \mathcal{I}_2^k with minimum cardinality and hence \mathcal{B}^k is the other one. Now we restrict to those pairs (i_{11}, i_{13}) for which $5|a_i$ for at least $|\mathcal{M}^k| - 1$ elements $i \in \mathcal{M}^k$. If $5|a_i$ for exactly $|\mathcal{M}^k| - 1$ elements $i \in \mathcal{M}^k$, then $\mathcal{B} = \mathcal{B}^k$ and hence we may assume that $|\mathcal{B}| = |\mathcal{B}^k| \leq 7$ otherwise there are at least 6 elements $i \in \mathcal{B}$ for which $a_i \in \{1, 2, 3, 6\}$ giving $t - |R| \geq 2$. Therefore we now exclude

those pairs (i_{11}, i_{13}) for which $5|a_i$ for exactly $|\mathcal{M}^k| - 1$ elements $i \in \mathcal{M}^k$ and $|\mathcal{B}^k| > 7$. We find that all the pairs (i_{11}, i_{13}) are excluded other than

$$(4.8) \quad (1, 3), (2, 4), (3, 5), (4, 2), (5, 3), (6, 4).$$

From $i_{13} \geq 5$ if $i_{11} = 1$ and $\max(i_{11}, i_{13}) \geq 6$ if $i_{11} \geq 2$, we find that all these pairs are excluded other than $(6, 4)$.

Let $(i_{11}, i_{13}) = (6, 4)$. Then $\mathcal{M}^k = \{0, 2, 7, 12\}$ and $\mathcal{B}^k = \{1, 3, 5, 8, 9, 10, 11\}$ giving $i_5 = 1$, $\mathcal{M} = \{2, 7, 12\}$ and $0 \notin \mathcal{I}$. This is excluded by applying Lemma 4.1 to $\prod_{i=0}^5 (n + d + 2i)$. \square

5. PROOF OF THEOREM 2

By Lemma 2.2, we may suppose that $P(b) > k$. If $P(b) = p_{\pi(k)+1}$ or $P(b) = p_{\pi(k)+2}$ with $p_{\pi(k)+1} \nmid b$, then the assertion follows from Theorem 1. Thus we may suppose that $P(b) = p_{\pi(k)+2}$ and $p_{\pi(k)+1} | b$. Then we delete the terms divisible by $p_{\pi(k)+1}, p_{\pi(k)+2}$ on the left hand side of (1.1) and the assertion for $k \geq 15$ follows from Lemma 2.4. Thus $11 \leq k \leq 14$ and it suffices to prove the assertion for $k = 11$ and $k = 13$. After removing the i 's for which $p|a_i$ with $p \in \{13, 17\}$ when $k = 11$ and $p|a_i$ with $p \in \{17, 19\}$ when $k = 13$, we observe that from Lemma 2.1 that $k - |R| \leq 1$ and $p \nmid d$ for each $p \leq k$.

5.1. The case $k = 11$. Let $p_1 = 11$, $p_2 = 13$ and $\mathcal{I} = \{0, 1, 2, \dots, 10\}$. Since $\left(\frac{5}{11}\right) \neq \left(\frac{5}{13}\right)$, $\left(\frac{17}{11}\right) \neq \left(\frac{17}{13}\right)$ but $\left(\frac{q}{11}\right) = \left(\frac{q}{13}\right)$ for $q = 2, 3, 7$, we observe that either $5|a_i$ or $17|a_i$ for $i \in \mathcal{M}$ and either $5 \cdot 17|a_i$ or $P(a_i) \leq 7$ for $i \in \mathcal{B}$. Since $\sigma_5 \leq 3$, we obtain $|\mathcal{M}| \leq 4$.

By taking the mirror image (4.4) of (1.1), we may suppose that $0 \leq i_{13} \leq 5$ and $0 \leq i_{11} \leq 10$. If both i_{11}, i_{13} are odd, then we may suppose that i_{17} is even otherwise we get a contradiction from Lemma 4.1 applied to $\prod_{i=0}^5 a_{n+i(2d)}$. Also we may suppose that $\max(i_{11}, i_{13}) \geq 4$ otherwise we get a contradiction from Lemma 4.1 applied to $\prod_{i=0}^6 a_{n+4d+id}$. Further from Lemma 4.1, we may assume $i_{17} > 4$ if $\max(i_{11}, i_{13}) = 4$.

Since $\max(|\mathcal{I}_1^k|, |\mathcal{I}_2^k|) \geq 5$ and $|\mathcal{M}^k| \leq 4$, we restrict to those pairs satisfying $\min(|\mathcal{I}_1^k|, |\mathcal{I}_2^k|) \leq 4$ and further \mathcal{M}^k is exactly one of \mathcal{I}_1^k or \mathcal{I}_2^k with minimum cardinality and hence \mathcal{B}^k is the other one. Now we restrict to those pairs (i_{11}, i_{13}) for which either $5|a_i$ or $17|a_i$ whenever $i \in \mathcal{M}$. Let $\mathcal{B}' = \mathcal{B} \setminus \{i : 5 \cdot 17|a_i\}$. If $|\mathcal{B}'| \geq 8$, then there are at least 6 elements $i \in \mathcal{B}'$ such that $P(a_i) \leq 3$ giving $k - |R| \geq 2$. Thus we restrict to those pairs for which $|\mathcal{B}'| \leq 7$. Further we observe that $7|a_i$ and $7|a_{i+7}$ for some $i, i+7 \in \mathcal{B}'$ if $|\mathcal{B}'| = 7$.

Let $(i_{11}, i_{13}) = (2, 4)$. Then $\mathcal{M}^k = \{1, 6, 8\}$ and $\mathcal{B}^k = \{0, 3, 5, 7, 9, 10\}$ giving $i_5 = 1$, $17|a_8$ and $P(a_i) \leq 7$ for $i \in \mathcal{B}$. For each possibility $i_7 \in$

$\{0, 3, 4, 5\}$, and $i_{17} = 8$, we take $p_1 = 7, p_2 = 17, \mathcal{I} = \mathcal{B}^k$ and compute \mathcal{I}_1 and \mathcal{I}_2 . Since $\left(\frac{p}{7}\right) = \left(\frac{p}{17}\right)$ for $p \in \{2, 3\}$, we should have either $\mathcal{I}_1 = \emptyset$ or $\mathcal{I}_2 = \emptyset$. We find that $\min(|\mathcal{I}_1|, |\mathcal{I}_2|) > 0$ for each possibility $i_7 \in \{0, 3, 4, 5\}$. Hence $(i_{11}, i_{13}) = (2, 4)$ is excluded. Similarly all pairs (i_{11}, i_{13}) are excluded except $(i_{11}, i_{13}) \in \{(4, 2), (6, 4)\}$. When $(i_{11}, i_{13}) = (3, 5)$, we get $\mathcal{M}^k = \{2, 7, 9\}$ giving $5|a_2a_7, 17|a_9$ and hence it is excluded. When $(i_{11}, i_{13}) = (1, 4)$, we obtain $\mathcal{M}^k = \{5, 9\}$ and $\mathcal{B}^k = \{0, 2, 3, 6, 7, 8, 10\}$ giving either $5|a_5, 17|a_9$ or $17|a_5, 5|a_9$. Also $i_7 \in \{0, 3\}$. Thus we have $(i_7, i_{17}) \in \{(0, 5), (0, 9), (3, 5), (3, 9)\}$ and apply the procedure for each of these possibilities.

Let $(i_{11}, i_{13}) = (6, 4)$. Then $\mathcal{M}^k = \{0, 2, 7\}$ and $\mathcal{B}^k = \{1, 3, 5, 8, 9, 10\}$ giving $i_5 = 2, 17|a_0$ and $P(a_i) \leq 7$ for $i \in \mathcal{B}$. For each possibility $i_7 \in \{1, 3, 4, 5\}$, and $i_{17} = 0$, we take $p_1 = 7, p_2 = 17$ and $\mathcal{I} = \mathcal{B}^k$. Since $\left(\frac{p}{7}\right) = \left(\frac{p}{17}\right)$ for $p \in \{2, 3\}$, we observe that either $\mathcal{I}_1 = \emptyset$ or $\mathcal{I}_2 = \emptyset$. We find that this happens only when $i_7 = 3$ where we get $\mathcal{I}_1 = \emptyset$ and $\mathcal{I}_2 = \{1, 5, 8, 9\}$. By taking modulo 7, we get $a_i \in \{1, 2\}$ for $i \in \{1, 8, 9\}$ and $a_5 \in \{3, 6\}$. Further by modulo 5, we obtain $a_1 = a_8 = 1, a_9 = 2, a_5 = 3, a_4, a_{10} = 7$ and this is excluded by Runge's method as in [MuSh03]. When $(i_{11}, i_{13}) = (4, 2)$, we get $\mathcal{M}^k = \{0, 5, 10\}$ and $\mathcal{B}^k = \{1, 3, 6, 7, 8, 9\}$ giving $5|a_0a_5a_{10}$ and $i_{17} \in \{5, 10\}$. Here we obtain $i_{17} = 10, i_7 = 3$ where $\mathcal{I}_1 = \emptyset$ and $\mathcal{I}_2 = \{1, 6, 7, 8, 9\}$. This is not possible by Lemma 2.2 with $k = 4$ applied to $(n + 6d)(n + 6d + d)(n + 6d + 2d)(n + 6d + 3d)$.

5.2. The case $k = 13$. Let $p_1 = 11, p_2 = 13$ and $\mathcal{I} = \{0, 1, 2, \dots, 12\}$. Since $\left(\frac{5}{11}\right) \neq \left(\frac{5}{13}\right), \left(\frac{17}{11}\right) \neq \left(\frac{17}{13}\right)$ but $\left(\frac{q}{11}\right) = \left(\frac{q}{13}\right)$ for $q = 2, 3, 7$, we observe that either $5|a_i$ or $17|a_i$ for $i \in \mathcal{M}^k$ and either $5 \cdot 17|a_i$ or $19|a_i$ or $P(a_i) \leq 7$ for $i \in \mathcal{B}^k$. Since $\sigma_5 \leq 3$, we obtain $|\mathcal{M}^k| \leq 4$.

By taking the mirror image (4.4) of (1.1), we may suppose that $0 \leq i_{13} \leq 6$ and $0 \leq i_{11} \leq 10$. We may assume that $i_{11}, i_{13}, i_{17}, i_{19}$ are not all even otherwise $P(\prod_{i=0}^5 a_{2i+1}) \leq 7$ which is excluded by Lemma 4.1. Further exactly two of $i_{11}, i_{13}, i_{17}, i_{19}$ are even and other two odd otherwise this is excluded again by Lemma 4.1 applied to $\prod_{i=0}^6 (n + i(2d))$ if n is odd and $\prod_{i=0}^6 \left(\frac{n}{2} + id\right)$ if n is even. Also exactly two of $i_{11}, i_{13}, i_{17}, i_{19}$ lie in each set $\{2, 3, 4, 5, 6, 7, 8\}$ and $\{3, 4, 5, 6, 7, 8, 9\}$ otherwise this is excluded by Lemma 4.1.

Since $\max(|\mathcal{I}_1^k|, |\mathcal{I}_2^k|) \geq 5$ and $|\mathcal{M}^k| \leq 4$, we restrict to those pairs satisfying $\min(|\mathcal{I}_1^k|, |\mathcal{I}_2^k|) \leq 4$ and further \mathcal{M}^k is exactly one of \mathcal{I}_1^k or \mathcal{I}_2^k with minimum cardinality and hence \mathcal{B}^k is the other one. Now we restrict to those pairs (i_{11}, i_{13}) for which either $5|a_i$ or $17|a_i$ whenever $i \in \mathcal{M}$. Let $\mathcal{B}' = \mathcal{B}^k \setminus \{i : 5 \cdot 17|a_i\}$. If $|\mathcal{B}'| \geq 9$, then there are at least 6 elements $i \in \mathcal{B}'$ such that $P(a_i) \leq 3$ giving $k - |R| \geq 2$. Thus we restrict to those pairs for

which $|\mathcal{B}'| \leq 8$. For instance, let $(i_{11}, i_{13}) = (0, 0)$. We obtain $\mathcal{M}^k = \{5, 10\}$ and $\mathcal{B}^k = \{1, 2, 3, 4, 6, 7, 8, 9, 12\}$ giving $i_5 = 0$, $i_{17} \in \{5, 10\}$, $\mathcal{B}' = \mathcal{B}^k$ and $|\mathcal{B}^k| = 9$. This is excluded.

Let $(i_{11}, i_{13}) = (1, 1)$. Then $\mathcal{M}^k = \{0, 6, 11\}$ and $\mathcal{B}^k = \{2, 3, 4, 5, 7, 8, 9, 10\}$ giving $i_5 = 1$, $i_{17} = 0$. This is excluded. Similarly $(i_{11}, i_{13}) \in \{(1, 3), (2, 4), (3, 5), (4, 6), (6, 4), (7, 5)\}$, are excluded where we find that i_{17} is of the same parity as i_{11}, i_{13} .

Let $(i_{11}, i_{13}) = (4, 2)$. Then $\mathcal{M}^k = \{0, 5, 10\}$ and $\mathcal{B}^k = \{1, 3, 6, 7, 8, 9, 11, 12\}$ giving $5|a_0, 5|a_{10}$ and $i_{17} = 5$. Further for $i \in \mathcal{B}^k$, we have either $19|a_i$ or $P(a_i) \leq 7$. Also $7|a_1$ and $7|a_8$ otherwise $k - |R| \geq 2$. We now take $(i_7, i_{17}) = (1, 5)$, $p_1 = 7, p_2 = 17$, $\mathcal{I} = \mathcal{B}^k$ and compute \mathcal{I}_1 and \mathcal{I}_2 . Since $\left(\frac{p}{7}\right) = \left(\frac{p}{17}\right)$ for $p \in \{2, 3\}$, and $\left(\frac{19}{7}\right) = \left(\frac{19}{17}\right)$, we should have either $|\mathcal{I}_1| = 1$ or $|\mathcal{I}_2| = 1$. We find that $\mathcal{I}_1 = \{3, 9, 11\}$ $\mathcal{I}_2 = \{6, 7, 12\}$ which is a contradiction. Similarly $(i_{11}, i_{13}) \in \{(5, 3), (8, 4)\}$ are also excluded. When $(i_{11}, i_{13}) = (5, 3)$, we find that $i_{17} = 6$ and $i_7 \in \{0, 2\}$ and this is excluded. \square

6. A REMARK

We consider (1.1) with $\psi = 0, \omega(d) = 2$ and the assumption $\gcd(n, d) = 1$ replaced by $d \nmid n$ if $b > 1$. It is proved in [LaSh06a] that (1.1) with $\psi = 0, b = 1$ and $k \geq 8$ is not possible. We show that (1.1) with $\psi = 0, k \geq 6$ and $\omega(d) = 2$ is not possible. The case $k = 6$ has already been solved in [BBGH06]. Let $k \geq 7$. As in [LaSh06a] and since $d \nmid n$, the assertion follows if (1.1) with $\psi = 1, k \geq 7, \omega(d) = 1$ and $\gcd(n, d) = 1$ does not hold. This follows from Theorem 1.

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