DIOPHANTINE APPROXIMATIONS, DIOPHANTINE EQUATIONS, TRANSCENDENCE AND APPLICATIONS

T. N. SHOREY

This article centres around the contributions of the author and therefore, it is confined to topics where the author has worked. Between these topics there are connections and we explain them by a result of Liouville in 1844 that for an algebraic number α of degree $n \geq 2$, there exists $c > 0$ depending only on α such that

$$| \alpha - \frac{p}{q} | > \frac{c}{q^n}$$

for all rational numbers $\frac{p}{q}$ with $q > 0$. This inequality is from diophantine approximations. Any non-trivial improvement of this inequality shows that certain class of diophantine equations, known as Thue equations, has only finitely many integral solutions. Also, the above inequality can be applied to establish the transcendence of numbers like $\sum_{n=1}^{\infty} \frac{1}{2^{n!}}$. For an other example on connection between these topics, we refer to an account on equation (2) in this article.

Both general and special diophantine equations have been considered. The most famous example of diophantine equations is the equation of Fermat but there are several others with a long history. The present article contains an account of some of these like equations of Catalan, Erdős and Selfridge and its extensions, Goormaghtigh, Nagell-Ljunggren, Pillai, Ramanujan-Nagell, Thue and Thue-Mahler and it may be of interest to a general reader. We prove that certain equations have only finitely many solutions and sometimes we determine all the solutions in which case computational ideas are also required and they have been carried out on a computer. Combinatorial arguments and elementary number theory form a basis for several contributions and very little prerequisites are required. Thus the work may be read by the ones not necessarily experts and it may be of interest to non number-theorists. The literature [6A,0B,86B,95B] serve excellent introduction to the material covered in this article. We also point out the methods and ideas required for the proofs and suggest open problems. The papers [86B,92B] have been written with similar intentions to some extent. We have partitioned the references in two lists A and B. The literature in the List A is ordered according to the names of the authors. The List A is followed by List B. The List B is the list of publications of the author and it is arranged according to the year of publication. If a reference appears in List B, it is either author's or his joint publication. The articles in List A are referred in the text as [1A],[2A], [3A], $\cdots$ and in List B as [0B], [1B], [2B], [3B], $\cdots$.

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For an integer ν with $|\nu| > 1$, we denote by $P(\nu)$ the greatest prime factor of ν and $\omega(\nu)$ the number of distinct prime divisors of ν, respectively. Further we put $P(1) = P(-1) = 1$ and $\omega(1) = \omega(-1) = 0$. All the constants $C, C_1, C_2 \cdots$ appearing in this article are effectively computable. This means that they can be determined explicitly in terms of the various parameters under consideration. Furthermore, unless otherwise specified, all the results contained in this article are effective. For non-zero algebraic numbers $\alpha_1, \cdots, \alpha_n$, let log $\alpha_1, \cdots,$ log α_n be arbitrary but fixed values of logarithms. Gelfond [36A] and Schneider [81A] in 1935 solved, independently, Hilbert's seventh problem that the linear independence of log α_1 and log α_2 over rationals implies the linear independence of log α_1 and log α_2 over algebraic numbers. Baker [2A] in 1966 extended the theorem of Gelfond and Schneider by proving that log $\alpha_1, \cdots,$ log α_n are linearly independent over algebraic numbers whenever they are linearly independent over rational numbers. Many improvements and variations of the theorem of Baker have been established as they are essential for several problems and these constitute the theory of linear forms in logarithms. Let $\beta_1, \cdots, \beta_n$ denote algebraic numbers of heights not exceeding $B \geq 3$. Suppose that the heights of $\alpha_1, \cdots, \alpha_n$ do not exceed $A_1, \cdots, A_n$, respectively, where $A_j \geq 3$ for $1 \leq j \leq n$ and we write

$$\begin{aligned} d &= [\mathbf{Q}(\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n) : \mathbf{Q}],\ A = \max_{1 \leq j \leq n} A_j, \\ \Omega &= \prod_{j=1}^{n} \log A_j,\ E = (\log \Omega + \log\log B). \end{aligned}$$

Shorey [12B] proved in 1976 the following result: Given $\epsilon > 0$ there exists a number C depending only on ϵ such that

$$|\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n|$$

exceeds

$$\exp\ (-(nd)^{Cn}\Omega(\log \Omega)^2(\log(\Omega B))^2 E^{2n+2+\epsilon})$$

provided that the above linear form in logarithms does not vanish. This sharpens a result of Stark [83A] and the improvement depends on a new idea on the size of inductive steps. Apart from the constant C, it was best known with respect to its dependence on n until Matveev [60A] replaced n^{Cn} by C^n in 2000. This has several applications. Let $f(X)$ be a polynomial with integer coefficients and at least two distinct roots. For a sufficiently large integer x, the estimate given above for linear forms in logarithms yields that $\omega(f(x))$ is at least constant times $\log\log x/\log\log\log x$ whenever $\log P(f(x)) \leq (\log\log x)^2$. This implies that $P(f(x))$ at integer x with $|x| \geq C_1$ exceeds $C_2 \log\log |x|$ for some numbers C_1 and $C_2 > 0$ depending only on f. Infact we obtain lower bounds for

$$\max_{1 \leq i \leq y} P(f(x+i))$$

for $\log y \leq (\log\log x)^{C_3}$ where C_3 is any absolute constant, see Shorey and Tijdeman [14B]. By applying a p-adic analogue (see [99A]) of the above result on linear forms in logarithms, Shorey, van der Poorten, Tijdeman and Schinzel [17B] extended the result on a lower bound for $P(f(x))$ to all binary forms with at least three pairwise non-proportional linear factors in their factorisations over $\mathbf{C}$.

If $n = 2$, the above estimate on linear forms in logarithms includes an earlier estimate of Shorey [6B] where the proof depends on integrating the function on a circle of very large radius. This idea gives the best possible bound in its dependence on A for linear forms in logarithms with α_i's very close to 1 and it is crucial for several applications. Sylvester [88A] proved in 1892 that a product of k consecutive positive integers greater than k is divisible by a prime exceeding k. By combining Jutila's result [46A] which depends on estimates for exponential sums and best possible (with respect to A) estimates on linear forms in logarithms with α_i's close to 1, Shorey [5B] proved that it suffices to take constant times $k(\text{loglog}k)/\text{log}k$ consecutive integers in place of k consecutive integers in the above result of Sylvester. This improves on the results of Tijdeman [92A], Ramachandra and Shorey [4B] and we refer to [86B] for earlier results. These special linear forms also find an application in a problem of Grimm [38A]. He conjectured that if $x, x+1, \cdots, x+k-1$ are all composite integers, then the number of distinct prime factors of $x(x+1)\cdots(x+k-1)$ is at least k. This conjecture, according to Erdős, implies that $p_{n+1} - p_n \leq C_4 p_n^{\frac{1}{2}-\epsilon}$ where $\epsilon > 0$, C_4 depends only on ϵ and $2 = p_1 < p_2 < \cdots$ is the sequence of all prime numbers. This consequence is stronger than what has been derived from Riemann hypothesis. Infact, the above difference between consecutive primes is at most $(\text{log}p_n)^2$ when n exceeds an absolute constant according to a conjecture of Cramer. Further Ramachandra, Shorey and Tijdeman [16B] confirmed Grimm's conjecture when $(\text{log}x)/(\text{log}k)^2$ exceeds certain absolute constant. Infact the assumption $x, x+1, \cdots x+k-1$ all composites is not required in the preceding result. A stronger question of Grimm is to choose pairwise distinct k prime divisors one from each of k consecutive positive integers $x, x+1, \cdots, x+k-1$ whenever $x, x+1, \cdots, x+k-1$ are all composites. Laishram and Shorey [113B] confirmed this conjecture whenever $x \leq 1.9(10)^{10}$. Ramachandra, Shorey and Tijdeman [9B] showed that this is possible whenever k does not exceed constant times $(\text{log}x/\text{loglog}x)^3$. We do not need $x, x+1, \cdots, x+k-1$ all composites in the preceding result. Further we observe that Cramer's conjecture implies stronger version of Grimm's conjecture for sufficiently large x. These applications of linear forms in logarithms with α_i's close to 1 continue to be best known. The studies of linear forms in logarithms with α_i's close to 1 were continued by Waldschmidt [96A] and they led to a remarkable estimate of Laurent, Mignotte and Nesterenko [49A] on linear forms in two logarithms with α_i's close to 1. It has several important applications. For example, it has been applied by Bennett [8A] to establish a striking theorem that for a positive integer a, the equation

$$(a+1)x^n - ay^n = 1 \text{ in integers } x \geq 1, y \geq 1, n \geq 3 \tag{1}$$

has no non-trivial solution i.e. has no solution other than given by $x = y = 1$. More applications of linear forms in logarithms with α_i's close to 1 will be mentioned at several places in this article. We refer to Baker [5A] and Wüstholz [98A] for surveys on linear forms in logarithms. Further there is an extensive p-adic theory for which we refer to Yu [99A]. Also see Sinnou David and Noriko Hirata-Kohno [25A] for an account of the theory of linear forms in elliptic logarithms and Győry [42A] for a survey of solving diophantine equations by linear forms in logarithms and its p-adic and elliptic analogues.

For positive integers x and $k \geq 2$, we write

$$\triangle_0 = \triangle_0(x,k) = x(x+1)\cdots(x+k-1)$$

and give lower bounds for $P(\triangle_0)$ and $\omega(\triangle_0)$. As stated in the preceding paragraph, Sylvester [88A] proved that

$$P(\triangle_0(x,k)) > k \text{ if } x > k.$$

The assumption $x > k$ can not be removed since $P(\triangle_0(1,k)) \leq k$. Improving on the results of Sylvester [88A] and Hanson [44A], Laishram and Shorey [111B] proved that $P(\triangle_0) > 1.95k$ if $x > k$ except for an explicitly given finite set of exceptions. Here we observe that 1.95 can not be replaced by 2 since there are arbitrary long chains of composite positive integers. There is no exception when $k > 270$ or $x > k+11$. Now we turn to lower bounds for $\omega(\triangle_0)$. We see that $k!$ divides $\triangle_0(x,k)$ and therefore, Sylvester's theorem can be reformulated as

$$\omega(\triangle_0) > \pi(k) \text{ if } x > k.$$

A well-known conjecture states that $2^p - 1$ is prime for infinitely many primes. Thus $\omega(\triangle_0) = 2$ for infinitely many primes p when $x = 2^p - 1, k = 2$ according to the above conjecture. Now we assume that $k \geq 3$. Then Saradha and Shorey [99B] improved Sylvester's theorem to

$$\omega(\triangle_0) \geq \pi(k) + [\frac{1}{3}\pi(k)] + 2 \text{ if } x > k$$

except for an explicitly given finite set of possibilities. The above estimate is best known for $k \leq 18$. For $k \geq 19$, Laishram and Shorey [105B] sharpened to

$$\omega(\triangle_0) \geq \pi(k) + [\frac{3}{4}\pi(k)] - 1 \text{ if } x > k$$

except for explicitly given finitely many possibilities. We refer to [99B] and [105B] for the set of exceptions to the above estimates. These exceptions satisfy $\omega(\triangle_0) \geq \pi(2k) - 1$. Thus

$$\omega(\triangle_0) \geq \min(\pi(k) + [\frac{3}{4}\pi(k)] - 1, \pi(2k) - 1) \text{ if } x > k.$$

The proof depends on explicit estimates for $\pi(x)$ due to Dusart [26A,27A,28A]. The above inequality is valid for small values of k and it can be sharpened if k is sufficiently large. It has been shown in [105B] that for $\epsilon > 0$, there exists $k_0 = k_0(\epsilon)$ depending only on ϵ such that

$$\omega(\triangle_0) \geq (1-\epsilon)\pi(2k) \text{ for } k \geq k_0.$$

On the other hand, we observe that $\omega(\triangle_0(k+1,k)) = \pi(2k)$ and there are infinitely many k such that $\omega(\triangle_0) = \pi(2k) - 1$. Further we observe that $\omega(\triangle_0(74,57))$ $= \pi(2k) - 2, \omega(\triangle_0(3936,3879)) = \pi(2k) - 3, \omega(\triangle_0(1304,1239)) = \pi(2k) - 4$ and $\omega(\triangle_0(3932,3880)) = \pi(2k) - 5$. We refer to [105B] for more pairs but we do not know whether there are infinitely many such pairs. Let $r = r(k)$ be the largest integer such that $\omega(\triangle_0) = \pi(2k) - r$ holds for infinitely many values of k. A study of the function $r(k)$ is of interest.

Now we consider an analogue of Sylvester's theorem and its sharpenings for a product of terms in arithmetic progression. For relatively prime positive integers $x, d \geq 2$ and $k \geq 3$, we put

$$\triangle = \triangle(x,d,k) = x(x+d)\cdots(x+(k-1)d),\ \chi = x+(k-1)d$$

and we give lower bounds for $P(\triangle)$ and $\omega(\triangle)$. Let $W(\triangle)$ denote the number of terms in $\triangle$ divisible by at least one prime exceeding k. We observe that

$$\omega(\triangle) \geq W(\triangle) + \pi_d(k)$$

where $\pi_d(k)$ denotes the number of primes $\leq k$ and coprime to d. Thus lower bounds for $\omega(\triangle)$ follow from those of $W(\triangle)$. We observe that $P(\triangle(x,d,2)) = 2$ if and only if $x = 1$ and $d+1$ is a power of 2. Therefore we assume that $k \geq 3$. Sylvester [88A] proved that $P(\triangle) > k$ if $x \geq d+k$. Langevin [48A] replaced the assumption $x \geq d+k$ by $x > k$. Further Shorey and Tijdeman [52B] showed that

$$P(\triangle) > k \ \text{ unless } \ (x,d,k) = (2,7,3).$$

Laishram and Shorey [112B] proved that

$$P(\triangle(x,d,k) > 2k \ \text{ for } \ d > 2$$

unless $k = 3, (x,d) = (1,4),(1,7),(2,3),(2,7),(2,23),(2,79),(3,61),(4,23),$ $(5,11),(18,7); k = 4,(x,d) = (1,3),(1,13),(3,11); k = 10,(x,d) = (1,3)$. Further we check that $P(\triangle) < 2k$ for these exceptions and there is no loss of generality in assuming that $d > 2$ since the case $d = 2$ is similar to that of $d = 1$ already considered. A conjecture states that

$$P(\triangle) > ak \ \text{ for } \ d > a$$

where a is a positive integer. Thus the conjecture has been confirmed for $a = 1, 2$ according to the above inequalities. It has been proved in [50B,60B] that

$$P(\triangle) > C_5 k \text{loglog}\chi \ \text{ if } \chi > k(\text{log}k)^{\epsilon}$$

where $\epsilon > 0$ and $C_5 > 0$ depends only on ϵ. An assumption of the latter type is necessary in the preceding result. By observing $\chi \geq P$, we see that the assumption $\chi > k(\text{log}k)^{\epsilon}$ can not be replaced by $\chi > k(\text{loglog}k)^{\epsilon}$ with $0 < \epsilon < 1$.

Now we turn to giving lower bounds for $\omega(\triangle)$. It has been proved in [59B] that $\omega(\triangle) \geq [k\ \log(\chi/k)/\text{log}\chi]$. This is not far from the best possible when $\chi < k^{C_6}$ with C_6 close to 1 as is clear from the following result from [59B]: For every k and prime d, there exists $x < d$ with $\gcd(x,d) = 1$ and

$$\omega(\triangle(x,k,d)) \leq k\text{log}\frac{\text{log}\chi}{\text{log}k} + C_7\frac{k}{\text{log}k}$$

where C_7 is some absolute constant. On the other hand, the above lower bound can be improved when χ is quite large as compared with k. It has been shown in [59B] that $\omega(\triangle) \geq k-1$ when

$$k \geq C_8, \quad \log\chi \geq k^{4/3}(\text{log}k)^{C_8} \tag{2}$$

for some absolute constant C_8 and $\omega(\triangle) \geq k$ if (2) holds together with $a \geq k$.

Next we give lower bounds for $\omega(\triangle)$ and $W(\triangle)$ in terms of $\pi(k)$. Shorey and Tijdeman [46B] proved that $\omega(\triangle) \geq \pi(k)$ and Moree [64A] showed that $\omega(\triangle) > \pi(k)$ for

$k \geq 4$ and $(x, d, k) \neq (1, 2, 5)$. A well-known conjecture, known as Schinzel's Hypothesis H, implies that there are infinitely many d such that $1 + d, 1 + 2d, 1 + 3d, 1 + 4d$ are all primes. Thus Hypothesis H implies that the estimate of Moree is best possible for $k = 4, 5$. For $k \geq 6$, Saradha, Shorey and Tijdeman [94B] sharpened and extended the preceding inequality by showing $W(\triangle) \geq \frac{6}{5}\pi(k) - \pi_d(k) + 1$ for $k \geq 6$ if and only if $k = 6, (x, d) = (1, 2), (1, 3); k = 7, (x, d) = (1, 2), (1, 3), (1, 4), (2, 3), (2, 5), (3, 2); k = 8, (x, d) = (1, 2); k = 11, (x, d) = (1, 2), (1, 3); k = 13, (x, d) = (1, 2), (3, 2); k = 14, (x, d) = (1, 2)$. Denoting $\rho = 1$ if $d = 2, x \leq k$ and 0 otherwise, the above estimate for $W(\triangle)$ is best known for $k = 6, 7, 8$ but for $k \geq 9$, it has been improved to

$$W(\triangle) \geq \pi(2k) - \pi_d(k) - \rho$$

unless $x = 1, d = 3, k = 9, 10, 11, 12, 19, 22, 24, 31;\ x = 2, d = 3, k = 12; x = 4, d = 3, k = 9, 10; x = 2, d = 5, k = 9, 10; x = 1, d = 7, k = 10$. This implies that

$$\omega(\triangle) \geq \pi(2k) - 1 \text{ unless } (x, d, k) = (1, 3, 10)$$

confirming a conjecture of Moree [64A]. This is best possible when $d = 2$ by considering $\omega(\triangle(k + 1, 2, k)) = \pi(2k) - 1$. The proofs depend on explicit estimates for the number of primes in arithmetic progression due to Ramaré and Rumely [70A].

On the other hand, upper bounds for $W(\triangle)$ have been given in [77B] and [94B]. It has been shown in [94B] by Prime Number Theorem in arithmetic progression with error term that

$$W(\triangle) \leq \frac{k}{\log k} d' + C_9 \frac{k \log\log k}{(\log k)^2} d'$$

where $d' = \log(2d) + 5.2\log\log(2d) + 5.2$ and C_9 is an absolute constant. Finally it has recently been proved by Green and Tao that there are arbitrary long chains of primes in arithmetic progressions.

Baker's well-known sharpenings of bounds for linear forms in logarithms led to the study of exponential diophantine equations. It was an immediate consequence of his estimates that for non-zero integers A, B, k, m, x and y with $\max(|x|, |y|) > 1$, the equation $Ax^m + By^m = k$ implies that m is bounded by a number depending only on A, B and k. Schinzel [79A] applied these sharpenings to settle an old problem on primitive divisors of $A^n - B^n$ in algebraic number fields and Tijdeman [94A] showed that the Catalan equation

$$x^m - y^n = 1 \text{ in integers } x > 1, y > 1, m > 1, n > 1$$

has only finitely many solutions and explicit bounds for the magnitude of the solutions can be given. Catalan conjectured in 1844 that 9 and 8 are the only powers that differ by 1. This has recently been settled by Mihailescu [62A]. For integers $a > 0, b > 0$ and $k \neq 0$, we have more general equation than the equation of Catalan, namely, the equation of Pillai

(3) $$ax^m - by^n = k \text{ in integers } x > 1, y > 1, m > 1, n > 1 \text{ with } mn \geq 6.$$

Pillai [69A] conjectured that (3) has only finitely many solutions. This is a consequence of $a\ b\ c$ conjecture which states as follows: Let $\epsilon > 0$ and a, b, c be relatively prime positive integers such that $a+b=c$. Let N be the product of all prime divisors of abc. Then

$$N \leq C_{10} N^{1+\epsilon}$$

where C_{10} depends only on ϵ. Let $x_1, x_2, \cdots, x_\mu$ be relatively prime non-zero integers satisfying $x_1 + x_2 + \cdots + x_\mu = 0$ and no proper subsum of $x_1 + x_2 + \cdots + x_\mu$ vanishes. Then a generalised a b c conjecture states that

$$\max_{1 \leq i \leq \mu} \mid x_i \mid \leq N_1^{C_{11}}$$

where N_1 is the product of all prime divisors of $x_1 \cdots x_\mu$ and C_{11} depends only on μ. Now we give a more general version of Pillai's conjecture. Let $f(X)$ be a polynomial with rational coefficients such that it has at least two distinct roots and $f(0) \neq 0$. Let L be the number of non-zero coefficients of f and the maximum of the heights of the coefficients of f does not exceed H. Then Shorey [87B] conjectured that for integers $m \geq 2, x$ and y with $\mid y \mid > 1$ satisfying

$$f(x) = y^m, \tag{4}$$

there exists a number C_{12} depending only on L and H such that either $m \leq C_{12}$ or $y^m - f(x)$ has a proper subsum which vanishes. The dependence of C_{12} on L and H is necessary. This is also the case with the assumptions stated in the above conjecture. It is clear that it implies the conjecture of Pillai. Shorey [87B] showed that it is a consequence of generalised a b c conjecture stated above. The only case in which the conjecture has been confirmed is due to Schinzel and Tijdeman [80A] that m is bounded by a number depending only on f whenever (4) holds. This may be applied to show that (3) has only finitely many solutions if at least one of the four variables is fixed, see [15,0B:chapter 12]. Shorey [33B] showed that (3) with $k=1$ has at most 9 solutions in m and n and this is also the case in general if

$$\max(ax^m, by^n) > 953k^6.$$

The equation (3) with x or y fixed leads us to consider lower bounds for the distance between powers and integers composed of fixed primes, see [15B]. For example, it has been shown in [15B] that for x fixed and $n \geq 3$,

$$\mid k \mid \geq C_{13}(y^n)^{C_{14}}$$

where $C_{13} > 0$ and $C_{14} > 0$ are numbers depending only on a and b. Slightly weaker bounds for $n = 2, 3$ were already proved by Schinzel [78A]. For given integers $m > 1$ and $n > 1$ with $mn \geq 6$, an old result of Mahler [56A] states that $P(ax^m - by^n)$ tends to infinity as max $(\mid x \mid, \mid y \mid) \to \infty$ with $\gcd(x, y) = 1$. The proof of Mahler is non-effective but an effective version follows from the theory of linear forms in logarithms. Infact Shorey, van der Poorten, Tijdeman and Schinzel [17B] applied this theory to prove that $P(ax^m - by^n)$ tends to infinity with m uniformly in integers x, y with $\mid x \mid > 1$ and gcd $(x, y) = 1$. The proof depends on the result mentioned in the beginning on the greatest prime factor of a binary form. Shorey [18B] made the proof independent of this result and it led him to give a quantitative version $P(ax^m - by^n) \geq C_{15}$ $((\log m)$ $(\log\log m))^{1/2}$ which has been improved by Bugeaud [13A] to $C_{16}\log m$ where $C_{15} > 0$ and $C_{16} > 0$ depend only on a, b and n.

The equations of Catalan and Pillai are examples of equations involving four variables and we turn to giving more. For integers $A \neq 0, B \neq 0, C$ and D, Shorey [21B,27B] showed that

$$Ax^m + By^m = Cx^n + Dy^n$$

has only finitely many solutions in integers x, y, m, n with $\mid x \mid \neq \mid y \mid, 0 \leq n < m, m > 2, Ax^m \neq Cx^n, Ax^m + By^m \neq 0$ and $(m, n) \neq (4, 2)$. Further bounds for $\mid x \mid, \mid y \mid, m, n$ can be given in terms of A, B, C, D. It is easy to see that all the above assumptions are necessary. If m and n are fixed, more general equations than the preceding ones have been considered. Let $f(X, Y)$ and $g(X, Y)$ be binary forms with integer coefficients such that f is irreducible and $\deg(f) > \deg(g)$. Then it has been shown in [0B:corollary7.1] that there are only finitely many integers x and y satisfying

$$f(x, y) = g(x, y)$$

and this extends an effective version, due to Baker, of a theorem of Thue that f assumes a given non-zero value at only finitely many integral points. Extensions of the above result have been obtained by Evertse, Győry, Shorey and Tijdeman [40B]. The above equation with g a fixed non-zero integer or with g a non-zero integer composed of primes from a given finite set is called Thue equation or Thue-Mahler equation, respectively. Thus the estimate already stated on the greatest prime factor of the values of a binary form at integral values can be viewed as a contribution on Thue- Mahler equation.

Now we consider

$$(5) \qquad y^m = \frac{x^n - 1}{x - 1} \text{ in integers } x > 1, y > 1, m > 1, n > 2.$$

This equation has solutions given by

$$(x, y, n, m) = (3, 11, 5, 2), (7, 20, 4, 2), (18, 7, 3, 3).$$

We call these as exceptional solutions of (5). By a solution of (5), we shall always mean a non-exceptional solution. This equation asks for powers with all the digits equal to 1 in their x-adic expansions. This is called Nagell-Ljunggren equation as Nagell [65A] and Ljunggren [54A] made the initial contributions that (5) is not possible whenever 4 divides n or $m = 2$, respectively. Let $n = 2N$ with $N > 1$ odd and we re-write (5) as

$$y^m = \frac{x^N - 1}{x - 1}(x^N + 1).$$

We observe that the terms on the right hand side are relatively prime. Therefore $x^N + 1$ is an m-th power which is not possible by the result of Mihailescu on Catalan equation. Thus we assume from now onwards that n is odd and further $m > 2$ is prime. Several questions on (5) have been formulated in [87B]. It has been conjectured that (5) has only finitely many solutions. This follows from $a\ b\ c$ conjecture, see [86B]. For a proof that (5) has only finitely many solutions, it suffices to restrict to $\omega(n) = 1$, see [87B]. Shorey [31B,87B] showed that (5) has no solution when $\omega(n) > m - 2$. A stronger conjecture on (5) states that (5) has no solution.

Shorey [31B] showed that (5) has only finitely many solutions when n is divisible by a prime congruent to 1 mod m. The result of Bennett already stated on (1) implies that (5) does not hold whenever n is congruent to 1 mod m which has been used in the result stated above when $\omega(n) > m-2$. Shorey [32B] showed that (5) has only finitely many solutions when x is an m-th power. Infact (5) does not hold when x is an m-th power by Le [50A]. This result implies that

$$y^m + 1 = \frac{x^n - 1}{x - 1} \quad \text{in integers } x > 1, y > 1, m > 1, n > 2$$

has no solution. Here we may assume that $n > 3$ since a product of two consecutive positive integers is not a power. By subtracting 1 on both the sides of the above equation, we see that

$$x\frac{x^{n-1} - 1}{x - 1} = y^m$$

implying that x is m-th power and the assertion follows from the result of Le stated above.

Shorey and Tijdeman [15B] showed that (5) has only finitely many solutions whenever x is fixed. By using p-adic analogue of linear forms in logarithms with α_i's close to 1, Bugeaud [14A] solved (5) completely for several values of x. In particular, Bugeaud and Mignotte [16A] settled a problem, due to Inkeri, that there is no m-th power > 1 with digits identically equal to 1 in its decimal expansion. The preceding result with $m \le 19$ was already derived by Shorey and Tijdeman [15B] from Baker's irrationality measures [1A]. Saradha and Shorey [82B] showed that (5) is not possible if $x = z^2$ such that z runs through all integers > 31 and $z \in \{2, 3, 4, 8, 9, 16, 27\}$. Further Bugeaud, Mignotte, Roy and Shorey [83B] covered the remaining cases. Hence (5) is not possible if x is a square. This was also proved, independently, by Bennett [9A] who derived it from his general result on (1). An analogous result when x is a cube or a higher power was proved by Hirata - Kohno and Shorey [78B]: For a prime $\mu \ge 3$ satisfying $m > 2(\mu-1)(2\mu-3)$, equation (5) with $x = z^\mu$ implies that $\max(x, y, m, n)$ is bounded by an absolute constant. Thus (5) with $x = z^3$ and $m \notin \{5, 7, 11\}$ implies that $\max(x, y, m, n)$ is bounded by an absolute constant. The proofs of these results depend on estimates from the theory of linear forms in logarithms with α_i's close to 1. Further Saradha and Shorey [82B] showed that (5) implies that x is divisible by a prime congruent to 1 mod m whenever max (x, y, m, n) exceeds a sufficiently large absolute constant. Further they [82B] gave an infinite set S of positive integers containing all integers in the interval $(1, 20]$ other than 11 such that (5) has only finitely many solutions whenever x is a power of an element from S and this has been applied to show that certain numbers considered by Mahler [57A] are irrational. For an explicit construction of the set S, we refer to [82B,p.18].

For introducing these numbers of Mahler, we give some notation. Let $g \ge 2$ and $h \ge 2$ be integers. For any integer $n \ge 1$, we write $n = a_1 h^{r-1} + \cdots + a_r$ for some integers $r > 0$ and $0 \le a_i < h$ for $1 \le i \le r$ with $a_1 \ne 0$. We define $(n)_h = a_1 \cdots a_r$ i.e. the sequence of digits of n written in $h-$ary notation. For a sequence $\{n_i\}_{i=1}^\infty$ of non-negative integers, we put

$$a_h(g) = 0.(g^{n_1})_h (g^{n_2})_h \cdots .$$

Mahler [57A] proved that $a_{10}(g)$ is irrational for $\{n_i\}_{i=1}^\infty = \{i-1\}_{i=1}^\infty$. It is known that $a_h(g)$ is irrational for any unbounded sequence $\{n_i\}_{i=1}^\infty$ of non-negative integers, see [74A]. Shorey and Tijdeman [79B] extended this result by showing that

$$0.(g_1^{n_1})_h(g_2^{n_2})_h\cdots$$

is irrational for any sequence $\{g_i\}_{i=1}^{\infty}$ of non-negative integers and for any unbounded sequence $\{n_i\}_{i=1}^{\infty}$ of non-negative integers. Now we consider the irrationality of $a_h(g)$ when n_i is a bounded sequence. If an element occurs in a sequence infinitely many times, it is called a limit point of the sequence. If it has only one limit point, it is ultimately periodic and hence $a_h(g)$ is rational. We suppose that it has exactly two limit points $N_1 < N_2$ such that

$$g^{N_2-N_1} \neq h+1 \text{ whenever } g^{N_1} < h \tag{6}$$

and it is not ultimately periodic. Then the last stated result of Saradha and Shorey [82B] on (5) in the previous paragraph implies that for integers $g \geq 2$ and $h \in S$, if $a_h(g)$ is rational then N_2 is bounded by an absolute constant. For given g and h, we may assume (6) otherwise the assertion follows. The connection between (5) and the irrationality of these numbers is due to Sander [74A]. The corresponding irrationality problem for bounded sequences with more than two limit points remains open.

For positive integers a and b with $ab > 1$ and gcd $(a, b) = 1$, a more general equation than (2), namely,

$$by^m = a\frac{x^n-1}{x-1} \text{ in integers } x > 1, y > 1, m > 1, n > 1 \tag{7}$$

has been considered. There is no loss of generality in assuming that m is prime in (7). The results on (5) have been extended to this equation by Inkeri [45A], Shorey and Tijdeman [15B] and Shorey [84B]. Infact (7) turns out to be easier than (5) and better results are available for (7). For elaborating this point, we give two results proved in [84B] for (7) but they are not yet available for (5). The equation (7) has only finitely many solutions whenever x is composed of primes from a given finite set. Further, for given x and m, the number of solutions of (7) in y and n is bounded by a number depending only on a and b. A weaker estimate, namely $m + C_{17}$ with C_{17} absolute constant, has been proved in [32] for the number of solutions of (5) in y and n. Under the assumption $\gcd(n, ab\phi(ab)) = 1$ which is satisfied if $a = b = 1$, Shorey [84B] proved that (7) with $x = z^2$ is not possible and (7) with $x = z^3, m \notin \{5, 7, 11\}$ implies that $\max(x, y, m, n)$ is bounded by a number depending only on a and b.

Next we turn to an equation of Goormaghtigh:

$$\frac{y^m-1}{y-1} = \frac{x^n-1}{x-1} \text{ in integers } x > 1, y > 1, m > 2, n > 2, m > n. \tag{8}$$

We observe that $x > y$ and (8) asks for positive integers whose all the digits are equal to one with respect to two distinct bases. Goormaghtigh [37A] observed that

$$31 = \frac{2^5-1}{2-1} = \frac{5^3-1}{5-1},\ 8191 = \frac{2^{13}-1}{2-1} = \frac{90^3-1}{90-1}$$

and it has been conjectured that these are the only solutions of (8). It follows from a b c conjecture that (8) has only finitely many solutions, see [86B:p.473]. Shorey [45B] showed that 31 and 8191 are the only primes N with $\omega(N-1) \leq 5$ such that all the digits of N are equal to one with respect to two distinct bases. For positive integers $A, B, x > 1$ and $y > 1$ with $x \neq y$, Shorey [33B] showed that there are at most 24 integers with all the digits equal to A in their $x-$adic expansions and all the digits equal to B in their $y-$adic expansions. If $AB = 1$, Bugeaud and Shorey [93B] replaced 24 by 2 and further, by 1 if x exceeds 10^{11} or $\gcd(x, y) > 1$. Balasubramanian and Shorey [19B] proved that (8) implies that max (x, y, m, n) is bounded by a number depending only on the greatest prime factor of x and y. Davenport Lewis and Schinzel [24A] showed that (8) has only finitely many solutions if m and n are fixed. They showed that the underlying polynomial for (8)

$$\frac{X^n - 1}{X - 1} - \frac{Y^m - 1}{Y - 1}$$

is irreducible over $\mathbf{C}$ and it has positive genus. Then the assertion follows from a well-known theorem of Siegel [82A] on integer solutions of polynomial equations in two variables and therefore, it is non-effective. On the other hand, they showed that it is effective when $\gcd(m - 1, n - 1) > 1$.

Nesterenko and Shorey [80B] extended the latter effective result when m and n are allowed to run through an explicitly given infinite set. More precisely, they proved the following: Let $d \geq 2, r \geq 1$ and $s \geq 1$ be integers with gcd $(r, s) = 1$. Assume that $m - 1 = dr$ and $n - 1 = ds$. Then (8) implies that max (x, y, m, n) is bounded by a number depending only on r and s. The proof depends on combining an elementary method of Runge with the theory of linear forms in logarithms. The result is effective and it is the first of the type where there is no restriction on bases x , y and m , n extend over an infinite set. This infinite set has been enlarged by Bugeaud and Shorey [93B] by replacing the dependence on r and s by the dependence on the ratio $(m - 1)/(n - 1)$: For $\alpha > 1$, (8) with $(m - 1)/(n - 1) \leq \alpha$ and gcd $(m - 1, n - 1) \geq 4\alpha + 6 + 1/\alpha$ implies that max (x, y, m, n) is bounded by a number depending only on α. Let r and s be fixed and we take $\alpha = \frac{m-1}{n-1}$. We may suppose that $\gcd(m - 1, n - 1) \geq 4\alpha + 6 + \frac{1}{\alpha}$ otherwise m, n are bounded and therefore, $\max(\mid x \mid, \mid y \mid)$ is bounded by the result of Davenport, Lewis and Schinzel stated above. Therefore the result of Bugeaud and Shorey includes the result of Nesterenko and Shorey stated in this paragraph.

Now we consider (8) with $n = 3$ which we assume without reference in this paragraph. If m is odd, Pingzhi Yuan [100A] proved recently that (8) has no solution other than the ones given by 31 and 8191. The preceding result with $m \leq 23$ is due to Nesterenko and Shorey [80B]. It remains to consider (8) only when m is even. Then $\gcd(x, y) > 1$, $y \nmid x$ by Le [52A] and the number of solutions of (8) with $y > 2$ in x and m is at most $2^{\omega(y)-1} - 1$ by Bugeaud and Shorey [88B] including the case $\omega(d) = 1$ of Le [52A]. The proofs depend on the theorem of Bilu, Hanrot and Voutier [12A] listing all the terms of Lucas and Lehmer sequences having no primitive divisor and we refer to the next paragraph for definitions of primitive divisors of terms of Lucas and Lehmer sequences.

We observe that $\frac{x^n-1}{x-1} = u_n$ for $n \geq 0$ where

$$u_n = (x + 1)u_{n-1} - xu_{n-2} \quad \text{for } n \geq 2$$

is a binary recursive sequence and we turn to introducing general binary recursive sequences. Let $u_0, u_1, \cdots$ be a sequence of integers such that

$$u_n = ru_{n-1} + su_{n-2} \quad \text{for} \quad n \geq 2$$

where r and $s \neq 0$ are integers with $r^2 + 4s \neq 0$. The sequence $\{u_n\}$ is called a binary recursive sequence. Further u_0, u_1 are the initial terms and $x^2 - rx - s$ is the companion polynomial of the sequence with α, β as its roots. We observe that $\alpha\beta \neq 0$ and $\alpha \neq \beta$. Further $u_n = a\alpha^n + b\beta^n$ for $n \geq 0$ where $a = \frac{u_0\beta - u_1}{\beta - \alpha}, b = \frac{u_1 - u_0\alpha}{\beta - \alpha}$. The sequence $\{u_n\}$ is called non-degenerate if $ab \neq 0$ and the quotient of the roots of its companion polynomial is not a root of unity. We shall always restrict to non -degenerate binary recursive sequences. A binary sequence with initial terms $u_0 = 0, u_1 = 1$ is called a Lucas sequence. The Fibonacci sequence given by $r = s = 1$ is a well-known example of Lucas sequence. For relatively prime non-zero integers r and s, a term u_n of a Lucas sequence has primitive divisor if there exists a prime p dividing u_n and p does not divide $(\alpha - \beta)^2 u_1 \cdots u_{n-1}$. Lehmer considered more general sequence corresponding to the polynomial $x^2 - \sqrt{r}x - s$ and these include Lucas sequences. These are known as Lehmer sequences and we refer to [20B,0B:chapter3] for analogous definitions.

Now we turn to applications of linear forms in logarithms to recursive sequences. Mahler [55A] proved, ineffectively, that $P(u_n)$ tends to infinity with n and an effective version is due to Schinzel [78A]. For $n > m > 0$ with $u_n u_m \neq 0$, Shorey [27B] proved that

$$P\left(\frac{u_n}{\gcd(u_n, u_m)}\right) \geq C_{18}\left(\frac{n}{\log n}\right)^{1/(d_1+1)} \tag{9}$$

where $d_1 = [\mathbf{Q}(\alpha, \beta) : \mathbf{Q}]$ and $C_{18} > 0$ depends only on α and β. Since $u_n \neq 0$ whenever n exceeds a number depending only on the sequence $\{u_n\}$, the inequality (9) includes a result of Stewart [85A] that

$$P(u_n) \geq C_{19}\left(\frac{n}{\log n}\right)^{1/(d_1+1)}$$

where $C_{19} > 0$ depends only on the sequence $\{u_n\}$. Further we derive from (9) that $u_n \mid u_m$ with $n > m$ implies that n is bounded by a number depending only on the sequence $\{u_n\}$. This includes a result of Parnami and Shorey [22B] that the members of a binary recursive sequence are distinct after a certain stage which can be determined explicitly in terms of the sequence $\{u_n\}$. This was extended by Shorey [27B] to all recursive sequences whose companion polynomials have at most two roots of maximal absolute value and by Mignotte, Shorey and Tijdeman [30B] to all recursive sequences of order 3. Furthermore they [30B] proved that the members of a recursive sequence of order 4 are non-zero after a certain stage which can be determined effectively in terms of the sequence. These results continue to be the best known effective results on recursive sequences, though more general ineffective versions are available. Finally Shorey [23B] gave lower bounds for the greatest square free factor $Q(u_n)$ of the members of a binary recursive sequence $\{u_n\}$

$$\log Q(u_n) \geq C_{20}(\log n)^2/\log\log n$$

where $C_{20} > 0$ depends only on the sequence. The above inequality for Lucas and Lehmer sequences was proved by Stewart [87A]. An analogue of Grimm's problem for binary recursive sequences have been considered in [26B]. If the coefficients of the companion polynomial of the binary recursive sequence are relatively prime, Shorey [26B] showed that there exist numbers C_{21} and C_{22} depending only on the sequence $\{u_n\}$ such that for $n \geq C_{21}$, it is possible to choose pairwise distinct primes $p_1, \cdots, p_g$ with $g = [C_{22}n^{1/2}]$ and $p_i \mid u_{n+i}$ for $1 \leq i \leq g$.

Shorey and Stewart [25B] as well as Pethő [68A], independently, settled an old question by showing that there are only finitely many powers in a binary recursive sequence. As an application of this result, Shorey and Stewart [25B] showed that for non-zero integers A_1, A_2, A_3, B with $A_2^2 - 4A_1A_3 \neq 0$ and integers x, y, t with $\mid x \mid > 1$ and $t > 1$, the equation $A_1x^{2t} + A_2x^ty + A_3y^2 = B$ implies that max $(\mid x \mid, \mid y \mid, t)$ is bounded by a number depending only A_1A_2, A_3 and B. Thus the equation $A_1x^2 + A_2xy + A_3y^2 = B$ may have infinitely many solutions in integers x, y but there are only finitely many when x is a power. An inhomogeneous analogue of the preceding result was given by Shorey and Stewart [38B] which also contains lower bounds for distance between members of a binary recursive sequence and powers. For example, the distance from

$$u_n = (2 + \sqrt{7})^n + (2 - \sqrt{7})^n$$

to the closest power tends to infinity with n and we refer to [38B] for precise formulation of the results in this direction. Finally Luca and Shorey [108B] showed that a product of two or more terms of Fibonacci sequence is never a power.

For relatively prime positive integers A and B with $A > B$, it has been conjectured that $P(A^n - B^n)/n$ intends to infinity with n The first result is due to Birkhoff and Vandiver [7A] that $P(A^n - B^n) > n$ for $n > 6$. This has been improved to $P(A^n - B^n) > 2n - 1$ by Schinzel [77A] if AB is a square or twice a square unless $n \neq 4, 6, 12$ when $(A, B) = (2, 1)$. Stewart [84A] confirmed the conjecture for all n with $\omega(n) \leq K$ where $0 < K < 1/\log 2$ which is satisfied for almost all n. Further Erdős and Shorey [13B] gave lower bounds for $P(A^n - B^n)/n$ by applying the result on linear forms in logarithms stated in the beginning of this article. In particular

$$P(2^p - 1) > C_{23}p \log p$$

where $C_{23} > 0$ is an absolute constant. Further Erdős and Shorey [13B] combined the theory of linear forms in logarithms with Brun's Sieve to show that

$$P(2^p - 1) > p\ (\log p)^2/(\log\log p)^3$$

for almost all primes p. For a sufficiently large integer n, the proof depends on comparing a lower bound and an upper bound for the number of primes $(2^{p_1} - 1) \cdots (2^{p_s} - 1)$ where $p_1, \cdots, p_s$ are all the primes between n and $2n$ for which the above inequality is not satisfied. Here lower bound is given by the theory of linear forms in logarithms whereas upper bound depends on Brun's Sieve. Stewart [85A] extended his result stated above to all Lucas and Lehmer sequences whose companion polynomials have real roots. Further Shorey and Stewart [20B] proved the analogous result for Lucas and Lehmer sequences with companion polynomials having non-real roots.

An important example of Lucas sequence is defined by Ramanujan function $r(n)$ satisfying

$$\sum_{n=1}^{\infty} \tau(n)q^n = q \prod_{m=1}^{\infty} (1-q^m)^{24}.$$

Let p be a prime such that $\tau(p) \neq 0$. We have a binary recursive relation

$$\tau(p^{m+1}) = \tau(p)\tau(p^m) - p^{11}\tau(p^{m-1}) \quad \text{for} \quad m \geq 1$$

giving a Lucas sequence $\{u_m\}$

$$u_0 = 0, u_m = \tau(p^{m-1}) = \frac{\alpha_p^m - \beta_p^m}{\alpha_p - \beta_p} \quad \text{for} \quad m \geq 1$$

where α_p and β_p are roots of $X^2 - \tau(p)X + p^{11}$. Shorey [36B] applied the theory of linear forms in logarithms via the above relations to obtain estimates for $\mid \tau(p^m) - \tau(p^n) \mid$ and $P(\tau(p^m))$ by our study of Lucas sequences as already described. It has been shown in [36B] that for $m > n$ and $m \geq C_{25}$,

$$\mid \tau(p^m) - \tau(p^n) \mid > p^{\frac{11}{2}m} \exp(-C_{24}\Lambda)$$

where $\Lambda = \log m \, \log(n+2) \, (\log p)^2 \log\log(p+1)$ and C_{24}, C_{25} are absolute constants. In particular $\tau(p^m) \neq \tau(p^n)$ whenever $m > n$ and $m \geq C_{25}$. Infact this is the case for all $m > n > 0$ when $p \geq C_{25}$ by [86A] , [36B] and when $p \geq 2$ by [12A]. There is no loss of generality in assuming that $p \not| \tau(p)$ in the proofs of [86A] and [12A] for the above result. Kumar Murty, Ram Murty and Shorey [39B] showed that for non-zero odd integer a, the equation

$$\tau(n) = a$$

implies that $\log \ n \leq (2|a|)^{C_{26}}$ where C_{26} is an absolute constant. In particular, the above equation has only finitely many solutions in integers $n \geq 1$.

Ramanujan conjectured and Nagell [66A] proved that equation, now known as Ramanujan-Nagell equation,

$$x^2 + 7 = 2^n \quad \text{in integers} \quad x \geq 1, n \geq 1$$

has only solutions $(x, n) = (1, 3), (3, 4), (5, 5), (11, 7), (181, 15)$. Let $y \geq 2, D_1$ and D_2 be positive integers such that $\gcd(D_1, D_2) = 1, D = D_1D_2$ and $\lambda \in \{2^{1/2}, 2\}$. We consider the generalised Ramanujan-Nagell equation

$$D_1x^2 + D_2 = \lambda^2 y^n \tag{10}$$

in integers $x \geq 1$ and $n \geq 1$. We denote by $\mathcal{N}(\lambda, D_1, D_2, y)$ the number of solutions (x, n) of (10) and we write p for a prime. Le [51A,52A] proved that $\mathcal{N}(\lambda, D_1, D_2, p) \leq 2$ unless $\mathcal{N}(2, 1, 7, 2) = 5, \mathcal{N}(2, 3, 5, 2) = \mathcal{N}(2, 1, 11, 3) = \mathcal{N}(2, 1, 19, 5) = \mathcal{N}(2, 1, 7, 2) = \mathcal{N}(1, 2, 1, 3) = 3$. There are three infinite families of triples (D_1, D_2, y) for which $\mathcal{N}(\lambda, D_1, D_2, y) \geq 2$. If (D_1, D_2, p) does not belong to any of these three infinite families, then Bugeaud and Shorey [88B] showed that $\mathcal{N}(\lambda, D_1, D_2, p) \leq 1$ unless $\mathcal{N}(2, 13, 3, 2) = \mathcal{N}(\sqrt{2}, 7, 11, 3) = \mathcal{N}(1, 2, 1, 3) = \mathcal{N}(2, 7, 1, 2) = \mathcal{N}(\sqrt{2}, 1, 1, 5) = \mathcal{N}(\sqrt{2}, 1, 1, 13) = \mathcal{N}(2, 1, 3, 7) = 2$. If (D_1, D_2, p) belongs to any of these three infinite families, then Bugeaud and Shorey [88B] showed that $\mathcal{N}(\lambda, D_1, D_2, p) = 2$ unless $\mathcal{N}(2,1,7,2) = 5, \mathcal{N}(\sqrt{2},3,5,2) = \mathcal{N}(2,1,11,3) = \mathcal{N}(2,1,19,5) = 3, \mathcal{N}(1,2,1,3) = 3$. This settles an old question and the proof depends on a theorem of Bilu, Hanrot and Voutier already referred. Thus Ramanujan-Nagell equation has the maximal number of solutions. Infact, Bugeaud and Shorey [88B] extended their result to more general equation (10) where y is not necessarily a prime. The solutions of this equation (10) can be put into $2^{\omega(y)-1}$ classes. Apart from the equations corresponding to elements in three infinite families and explicitly given finitely many equations (10), there is at most one solution of (10) in any class such that

$$n < \frac{4\sqrt{D}}{\pi}\log(2e\sqrt{D}).$$

This generalisation led to determining in [88B] all the solutions of the generalised Ramanujan-Nagell equation in positive integers x, y and n for several values of D_1 and D_2. For example, the equation $x^2+D_2 = y^p$ with $p \geq 3$ prime has been completely solved in [88B] with $D_2 = 2,5,6,10,13,14,17,21,22,30,33,34,37,41,42,46,57,58,62,$ $65,66,69,70,73,77,78,82,85,93,94,97$. Further it has been shown in [88B] that $x^2 + 7 = 4y^n$ has no solution in positive integers $x \geq 1, y \geq 2, n \geq 2$ other than $(x,y,n) = (3,2,2),(5,2,3),(11,2,5),(181,2,13)$. More difficult equation $x^2+7 = y^n$ and many others have been completely solved recently by Bugeaud, Mignotte and Siksek [19A]. Now all the equations $x^2 + D_2 = y^n$ with $1 \leq D_2 \leq 100$ are completely solved, see [20A] and [19A] for a survey.

Erdős and Selfridge [33A] proved in 1975 that a product of two or more consecutive positive integers is never a power. Further Saradha and Shorey [89B] showed that there is no power other than

$$\frac{6!}{5} = (12)^2, \frac{10!}{7} = (720)^2, 1.2.4 = 2.4 = 2^3$$

which is a product of $k-1$ distinct integers out of $k \geq 3$ consecutive positive integers $x, x+1, \cdots, x+k-1$. This settles a conjecture of Erdős and Selfridge [33A]. The proof depends on combining the elementary method of Erdős and Selfridge with the method of Wiles [97A] on Fermat equation. Further Mukhopadhyay and Shorey [103B] showed that a product of $k-2$ distinct integers out of $k \geq 4$ consecutive positive integers is a square only when

$$\frac{6!}{5} = \frac{7!}{5\cdot 7} = 12^2,\ \frac{10!}{1\cdot 7} = \frac{11!}{7\cdot 11} = 20^2,\ \frac{4!}{2.3} = 2^2,$$
$$\frac{6!}{4.5} = 6^2,\ \frac{8!}{2.5.7} = 24^2,\ \frac{10!}{2.3.4.6.7} = 60^2,$$
$$\frac{9!}{2.5.7} = 72^2,\ \frac{10!}{2.3.6.7} = 120^2,\ \frac{10!}{2.7.8} = 180^2,\ \frac{10!}{4.7} = 360^2,$$
$$\frac{21!}{13!.17.19} = 5040^2,\ \frac{14!}{2.3.4.11.13} = 5040^2,\ \frac{14!}{2.3.11.13} = 10080^2.$$

Let $m > 2$ be a prime, $k \geq 2$ and $x > k^m$. Erdős and Selfridge [33A] showed that a product $x(x+1)\cdots(x+k-1)$ with $k \geq 3$ is not of the form by^m with $P(b) < k$. Further the assumption $P(b) < k$ has been relaxed to $P(b) \leq k$ by Saradha [75A] for

$k \geq 4$ and Győry [40A] for $k = 3$. The particular case $b = k!$ of the result of Saradha and Győry was already settled by Erdős [30A] for $k \geq 4$ and Győry [39A] for $k = 3$. Also the case $k = 2$ of their result is not valid since Pell's equation $x^2 + 1 = 2y^2$ has infinitely many integer solutions. We assume now that $k \geq 3$. Hanrot, Saradha and Shorey [90B] showed that the product in the result of Saradha and Shorey in the preceding paragraph is not of the form by^m with $P(b) < k$ unless $k = 4$ and it is not of the form by^m with $P(b) \leq k$ unless $k \in \{3, 4, 5\}$ which cases are covered by Bennett [9A]. The assumption $x > k^m$ in the results stated above can not be removed but it can be replaced by the necessary assumption that the product of integers appearing in the results of this paragraph is divisible by a prime greater than k. Hence a product divisible by a prime exceeding k of $k-1$ distinct integers out of k consecutive positive integers is not of the form by^m with $P(b) \leq k$. The proof depends on the elementary method of Erdős and Selfridge, Baker's method on linear forms in logarithms of finding all integral solutions of Thue equations and the contributions of Wiles, Ribet and others on Fermat equation. For applying Baker's method, we need to keep a control on the degree as well as the coefficients of Thue equations. The contributions on Fermat equation keep a check on the coefficients whereas a check on the degree is given by the elementary method of Erdős and Selfridge. The theory of linear forms in logarithms has also been combined with the ideas coming out of Fermat equation for considering analogous problems for products of terms in arithmetic progression which we shall consider in this article. This is also the case with the results of Bugeaud, Mignotte and Siksek [19A] on the equation $x^2 + 7 = y^n$ and their result [18A] on finding all powers in the Fibonacci sequence. They call this approach as combining classical and modular methods. The analogous result for $m = 2$ is given in Saradha and Shorey [99B] where it has been proved that a product of $k - 1$ distinct integers out of $x, x+1, \cdots, x+k-1$ with $x > k^2$ and $k \geq 4$ is of the form by^2 with $P(b) \leq k$ only when $(x, k) = (24, 4), (47, 4), (48, 4)$. Here the assumption $k \geq 4$ is necessary since Pell's equations have infinitely many integer solutions. Further Mukhopadhyay and Shorey [103B] showed that a product of at least $k - 7$ distinct integers out of k consecutive positive integers with $x > k^2$ and $k \geq 10$ is a square only if it is given by

$$240.243.245 = 3780^2,\ 242.245.250 = 3850^2,\ 240.242.243.250 = 59400^2$$

and we refer to [103B] for more general formulation.

We consider an extension of the above results. Let $m \geq 2$ be an integer. We always write b for a positive integers such that $P(b) \leq k$ and let $d_1, \cdots, d_t$ be distinct integers in $[0, k)$. We consider the equation

$$(x + d_1) \cdots (x + d_t) = by^m. \tag{11}$$

We assume that the left hand side of (11) is divisible by a prime exceeding k. Let $m > 2$. We put

$$\nu_m = \frac{1}{2}(1 + \frac{4m^2 - 8m + 7}{2(m-1)(2m^2 - 5m + 4)}).$$

We observe that $\nu_3 = \frac{47}{56}, \nu_4 = \frac{45}{64}$ and $\nu_m < \frac{2}{3}$ for $m \geq 5$. Shorey [31B,35B] proved that (11) with $t \geq \nu_m k$ implies that k is bounded by an absolute constant. This is a considerable improvement of a result of Erdős [31A] with $1 - (1-\epsilon)\frac{\log\log k}{\log k}$ in place of

ν_m. Furthermore the assumption on t is relaxed to $t \geq km^{-1/11} + \pi(k) + 2$ in [31B] whenever m is sufficiently large. Infact $\frac{1}{11}$ has been replaced by $\frac{1}{3}$, see [72B]. The proofs depend on an estimate of Shorey [31B] on linear forms in logarithms with α_i's close to 1, the estimates of Baker [1A,3A] on irrationality measures proved by hypergeometric method and the estimates of Roth and Halberstam on difference between consecutive m free integers. While proving these results, Shorey [31B] showed for the first time that sharp estimates on linear forms in logarithms with α_i's close to 1 combine well with the irrationality measures of Baker [1A,3A] proved by hypergeometric method. This approach has led to several results. This is the case with the results of Hirata-Kohno and Shorey [78B] on (5), Bugeaud and Shorey on (8) stated above, Bugeaud and Dujella [15A] on diophantine tuples for higher powers and Bennett and de Weger [11A] leading to a theorem of Bennett already stated on (1). This has also been followed to confirm a conjecture of Erdős on (12) with x fixed and $m \geq 7$, see an account on (12). For fixed $m \geq 7$, Shorey and Nesterenko [72B] showed that we can take

$$\nu_m = \begin{cases} \frac{4}{m}(1 - \frac{1}{(.875)m}) & \text{if } m \equiv 1 \pmod 2 \\ \frac{4}{m}(1 - \frac{1}{(1.412)m}) & \text{if } m \equiv 0 \pmod 2 \end{cases}$$

in the above result of Shorey [31B,35B] on (11). Here

$$\nu_7 \leq .4832,\ \nu_8 \leq .4556,\ \nu_9 \leq .3878, \nu_{10} \leq .3664,$$
$$\nu_{11} \leq .3243, \nu_{12} \leq .3076, \nu_{13} \leq .2787,\ \nu_{14} \leq .2655.$$

Let $m = 2$. For $\epsilon > 0$, Shorey [35B] proved that (11) with $t \geq k - (1-\epsilon)k^{\frac{\log\ \log\ k}{\log\ k}}$ implies that k is bounded by a number depending only on ϵ. This answers a question of Erdős [31A] and sharpens his result with $t \geq k - C_{27}\frac{k}{\log k}$ for some absolute constant C_{27}. The proof depends on a theorem of Baker [4A] that hyper-elliptic equation (4), under necessary assumptions, has only finitely many integral solutions. Furthermore Balasubramanian and Shorey [63B] relaxed slightly the assumption on t and this turns out to be not far from the best possible. Next Mukhopadhyay and Shorey [103B] found all the solutions of (11) with $P(b) \leq k, t \geq k - 7$ and $k \geq 10$. We see that there is no solution if $k \geq 14$ and for $k = 10, 11, 12, 13$, we find that x is less than 5040. Here the assumption $k \geq 10$ is necessary since $x^2 - 2y^2 = -1$ has infinitely many integer solutions.

For relatively prime positive integers x, d and positive integer b with $P(b) \leq k$, we consider the equation

$$(12) \qquad x(x+d)\cdots(x+(k-1)d) = by^m \ \text{ in integers } x > 0, y > 0, k \geq 3, m \geq 2.$$

We assume that $d \geq 2$ as the case $d = 1$ has already been considered. Now we always suppose in (12) that $(x, d, k) \neq (2, 7, 3)$ so that, as already stated, the left hand side of (12) is divisible by a prime exceeding k. There is no loss of generality in assuming that m is prime in (12) which we suppose in this paragraph. We assume from now onwards in this paragraph that (12) holds and k exceeds a sufficiently large absolute constant. Erdős conjectured that k is bounded by an absolute constant. Marszalek [59A] confirmed the conjecture for fixed d. Further Shorey and Tijdeman [48B,57B] showed that

$$d \geq k^{C_{28}\ \log\ \log\ k}$$

where $C_{28} > 0$ is an absolute constant and Shorey[86B:p.490] applied this inequality to derive the conjecture of Erdős from $a\ b\ c$ conjecture if $m > 3$. Further Granville (unpublished) showed that $a\ b\ c$ conjecture implies the conjecture of Erdős with $m = 2, 3$. For a proof, see Laishram [47A]. Shorey [71B,44B] applied linear forms in logarithms with $\alpha_i's$ close to 1 and the irrationality measures of Baker [3A] by hypergeometric method to show that

$$x \geq k^{C_{29} \log\ \log\ k} \text{ for } m \geq 7$$

where $C_{29} > 0$ is an absolute constant. Thus k is bounded by a number depending only on x whenever $m \geq 7$. If $m \geq 3$, Shorey [43B] applied the theory of linear forms in logarithms for proving that k is bounded by a number depending only on the greatest prime factor of d. Let d_1 be the maximal divisor of d such that all the prime divisors of d_1 are congruent to 1 mod m. Then Shorey [43B] showed that $d_1 > 1$ which implies that we need to verify the preceding assertion for only finitely many m and the proof depends on estimates for the magnitude of solutions of Thue-Mahler equations. Infact for a given $m \geq 2$, Shorey and Tijdeman [48B] proved that k is bounded by a number depending only on $\omega(d)$. Next Shorey and Tijdeman [57B] proved that if $\epsilon > 0, \eta = (1-\epsilon)/\log 2$ for $m > 30$ and $\eta = (1-\epsilon)/\log m$ for $m \leq 30$, we have $2P(d_1) \geq \eta\ m\ \log k\ \log\log k$ for $m \geq 7$ and $P(d) \geq \eta\ m\ \log k\ \log\log k$ for $m = 2, 3, 5$. Further the least prime factor of d_1 is greater than constant times k whenever $m > 4\omega(d_1) + 2$. An extension of (12) analogous to (11) was also considered by Shorey and Tijdeman [58B] and Shorey [86B:p.489] used this to show that the assumption gcd $(x, d) = 1$ can be replaced by $d \nmid x$ in the result stated above that k is bounded by a number depending only on m and $\omega(d)$.

A stronger version of the conjecture of Erdős, referred as ES, states that if (12) holds, then

$$(k, m) \in (3, 2), (4, 2), (3, 3).$$

In each of the above three cases, we can find b such that (12) has infinitely many solutions. This conjecture has been formulated in [107B] but it is already available in [95A] for the finite version. Let $m > 2$ and $k \geq 4$. Saradha and Shorey [89B] showed that Shorey's inequality $d_1 > 1$ for sufficiently large k is valid for all k whenever (12) holds. Thus (12) implies that d is divisible by a prime congruent to 1 mod m. Consequently there are infinitely many $d = 2^a 3^b 5^c > 1$ for which (12) never holds. Thus conjecture ES is confirmed for infinitely many many d. Further Saradha and Shorey [107B] confirmed conjecture ES for a large number of values of d. They proved that (12) implies that $d > d_1$ where d_1 equals 30 if $m = 3$; 750 if $m = 4$; 5.10^4 if $m = 5, 6$; 10^8 if $m = 7, 8, 9, 10$ and 10^{15} if $m \geq 11$. Here we observe from d tends to infinity with k and the result of Schinzel and Tijdeman already stated on (4) that d tends to infinity with m. The number C_{28} in the lower bound for d stated above turns out to be small giving trivial estimate for small values of k and Saradha and Shorey [89B] gave non trivial estimates for small values of k. Let $\theta = 1$ if $m \nmid d$ and $1/m$ of $m \mid d$. Then it has been proved in [89B] that (12) with m prime implies that

$$d \geq d_1 > \alpha\theta k^{\beta}$$

where (α, β) is defined as $(1.59, m/2 - 3 - 5/(2m))$ if $m \geq 17; (1.1, 43/13)$ if $m = 13; (.93, 25/11)$ if $m = 11; (.73, 9/7)$ if $m = 7; (.6, 7/5)$ if $m = 5, 5 \mid d; (.65, 1/5)$ if

$m = 5, 5 \not| d$ and $(.41, 1/3)$ if $m = 3$. Let $m = 2$ and $k \geq 4$. Then conjecture ES has been confirmed when $d \leq d_0$ where $d_0 = 104, 30, 22$ by Saradha and Shorey [98B], Saradha [76A], Filakovszky and Hajdu [34A], respectively. As already stated, equation (12) implies that k is bounded by a number depending only on $\omega(d)$. If $\omega(d) = 1$ i.e. d is a prime power, Saradha and Shorey [98B] showed that a product of four or more terms in arithmetic progression is never a square. The case $k = 3$ of the preceding result remains open and it is likely that (12) with $b = 1, k = 3$ and $\omega(d) = 1$ has infinitely many solutions. Further it follows from the works of [98B] and [100B] that (12) with $\omega(d) = 1$ and $(x, d, k, b, y) \neq (75, 23, 4, 6, 4620)$ implies that $k = 5$ and $P(b) = 5$ which case we have not been able to resolve. Thus (12) has infinitely many solutions if $k = 4$ but only one $(x, d, k, b, y) = (75, 23, 4, 6, 4620)$ if d is restricted to prime powers. This follows from the method of Baker and Davenport on solving a pair of Pell's equations with one common variable. Finally Laishram and Shorey [115B] confirmed conjecture ES when $b = 1$ and $\omega(d) = 2, 3, 4$. Further the assumption $b = 1$ has been relaxed to $P(b) < k$ in [115B] if we exclude the cases stated in the next paragraph when $a_0, a_1, \cdots a_{k-1}$ are consecutive positive integers. This may be applied to sharpen the value of d_0 considerably. Baker and Davenport applied their method on solving a pair of Pell's equations referred above to show that the product of any two out of $1, 3, 8, x$ with $x \neq 120$ added by 1 is not a square and we refer to [15A] for this method and its development.

So far we have been considering (12) with k as variable and with some restrictions on d. Now we consider (12) with k fixed and without any restriction on d. We consider the case of squares i.e. $m = 2$. The first result is due to Fermat that there are no four squares in an arithmetic progression. This is also the case when $k = 5$ by Obläth [67A] and $6 \leq k \leq 110$ by Hirata-Kohno, Laishram, Shorey and Tijdeman [114B]. The cases $6 \leq k \leq 11$ are covered independently by Bennett, Bruin, Győry and Hajdu [10A]. Next we consider (12) with $P(b) < k$. This equation, as already pointed out, has infinitely many solutions for $k = 4$ but no solution when $k = 5$ due to Mukhopadhyay and Shorey [100B]. This is also the case when $k = 6$ by Bennett, Bruin, Győry, Hajdu [10A] and the proof depends on Chabauty method. Further Hirata-Kohno, Laishram, Shorey and Tijdeman [114B] showed that (12) with $7 \leq k \leq 100$ does not hold unless $(a_0, a_1, \cdots, a_{k-1}) = (2, 3, 1, 5, 6, 7, 2, 1), (3, 1, 5, 6, 7, 2, 1, 10)$ if $k = 8; (2, 3, 1, 5, 6, 7, 2, 1, 10)$ if $k = 9; (3, 1, 5, 6, 7, 2, 1, 10, 11, 3, 13, 14, 15, 1)$ if $k = 14; (5, 6, 7, 2, 1, 10, 11, 3, 13, 14, 15, 1, 17, 2, 19, 5, 21, 22, 23, 6, 1, 26, 3, 7)$ if $k = 24$ or their mirror images. Here $n + id = a_i x_i^2$ for $0 \leq i < k$ such that a_i are squarefree. When $6 \leq k \leq 11$, it has been proved in [10A] that (12) with $P(b) \leq 5$ is not possible and this is a particular case of the preceding result. Let $m > 2$ be prime. The first result is due to Darmon and Merel [23A] that $n, n+d, n+2d$ are all not m-th powers. Győry [41A] showed that (12) with $k = 3$ and $P(b) < k$ is not possible. This is also the case when $k = 4, 5, b = 1$ by Győry, Hajdu, Saradha [43A] (see also [10A] for the case $m = 3$.) and $6 \leq k \leq 11, b = 1$ by Bennett, Bruin, Győry, Hajdu [10A].

Now we consider an equation obtained from (12) by omitting a term on the left hand side. For $k \geq 4$ and $0 < i < k - 1$, we consider

$$x(x + d) \cdots (x + (i - 1)d)(x + (i + 1)d) \cdots (x + (k - 1)d) = by^m. \tag{13}$$

The case $d = 1$ has already been considered and therefore, we suppose that $d > 1$. Then it has been shown in [98B] and [104B] that (13) with $m = 2, \omega(d) = 1$ and $P(b) < k$ does not hold when $k \geq 9$ and in [110B], the cases $5 \leq k \leq 8$ are covered when $b = 1$. Therefore (13) with $m = 2, \omega(d) = 1, b = 1$ and $k \geq 5$ is not possible. Let $m > 2$ prime. It is shown in [89B] that the above equation with $k \geq 9$ and $P(b) \leq k$ implies that d is divisible by a prime congruent to 1 mod m and when $b = 1$, the remaining cases $k \leq 8$ are covered in [109B]. Thus (13) with $b = 1$ implies that d is divisible by a prime congruent to 1 mod m. Consequently (13) with $b = 1$ is not possible whenever d is composed of $2, 3$ and 5 only.

Next we consider (11) with $b = 1, m = 2, t = k$ and the left hand side replaced by two blocks of consecutive integers. More precisely, we consider the equation

$$X(X+1)\cdots(X+K-1)Y(Y+1)\cdots(Y+K'-1) = Z^2$$

where $K \geq 3, K' \geq 3$ and $X \geq Y + K'$. It has been conjectured by Erdős and Graham that this equation has only finitely many solutions in all the integral variables $X > 0, Y > 0, Z > 0, K$ and K'. This conjecture implies that

$$x(x+1)\cdots(x+k-1) = y(y+1)\cdots(y+k+l-1)$$

has only finitely many solutions in integers $x > 0, y > 0, k \geq 3$ and $l \geq 0$ satisfying $x \geq y+k+l$. More generally, for positive integers A and B, Erdős conjectured in 1975 that there are only finitely many integers $x > 0, y > 0, k \geq 3, l \geq 0$ with $x \geq y+k+l$ satisfying

$$\text{(14)} \qquad Ax(x+1)\cdots(x+k-1) = By(y+1)\cdots(y+k+l-1).$$

The first result is due to Mordell [63A] that (14) with $A = B = 1$ and $k = 2, l = 1$ has no solution in integers $x > 0$ and $y > 0$ and we refer to [85B] for earlier results. Beukers, Shorey and Tijdeman [85B] applied a well-known theorem of Siegel [82A] on integral points on curves to confirm the conjecture if k and l are fixed. The work involves establishing irreducibility and computing genus of the curve under consideration so that the assumptions of the theorem of Siegel [82A] are satisfied, see [85B:Theorems 2.1,2.2]. Because of the ineffective nature of Siegel's result, it has not been possible to give an explicit estimate for the magnitude of the solutions. If $l = 0$, Shorey [29B] confirmed the conjecture of Erdős when x and y are composed of fixed primes and Saradha and Shorey [49B] extended this result to all $l \geq 0$. This is rather a restrictive result but the proof depends on several applications of linear forms in logarithms. Further they [49B] showed that (14) implies that $x - y \geq C_{30}x^{2/3}$ where $C_{30} > 0$ depends only on A and B.

It is a difficult problem to confirm the preceding conjecture of Erdős in general. We consider (14) with $A = B = 1$ and $k + l$ an integral multiple of k. In this case, for an integer $m \geq 2$, we re-write (14) as

$$\text{(15)} \qquad x(x+1)\cdots(x+k-1) = y(y+1)\cdots(y+mk-1)$$

in integers $x > 0, y > 0, k \geq 2$.

The first results are due to MacLeod and Barrodale and we refer again to [85B] for an account of earlier results. Saradha and Shorey [54B] by extending an old elementary effective method of Runge [73A] to exponential diophantine equations, proved that

(15) implies that $\max(x, y, k)$ is bounded by a number depending only m. Saradha and Shorey [53B] and Mignotte and Shorey [73B] showed that (15) with $2 \leq m \leq 6$ implies that $x = 8, y = 1, k = 3, m = 2$ and it has been conjectured in [69B] that (15) with $m > 6$ has no solution. For positive integers l, m, d_1 and d_2 with $l < m$ and gcd $(l, m) = 1$, we consider a more general equation than (15), namely,

$$x(x + d_1) \cdots (x + (lk - 1)d_1) = y(y + d_2) \cdots (y + (mk - 1)d_2) \quad (16)$$

in integers $x > 0, y > 0, k \geq 2$.

By developing the elementary method of Runge as referred above, Saradha and Shorey [55B,64B] and Saradha, Shorey and Tijdeman [65B] showed that (16) implies that either max (x, y, k) is bounded by a number depending only on m, d_1, d_2 or $m = 2, k = 2, d_1 = 2d_2^2,\ x = y^2 + 3d_2y$. On the other hand, (16) with $m = 2$ is satisfied whenever the latter possibilities hold. Balasubramanian and Shorey [62B] and Saradha, Shorey and Tijdeman [70B] gave an analogue of the above results for a more general equation than (16), namely,

$$f(x)f(x + d_1) \cdots f(x + (lk - 1)d_1) = f(y)f(y + d_2) \cdots f(y + (mk - 1)d_2)$$

where $f(X)$ is a monic polynomial of degree $\nu > 0$ with rational numbers as coefficients such that it is a power of an irreducible polynomial. It has been proved in [70B] that there exists a number C_{31} depending only on d_1, d_2, m and f such that the above equation with

$$f(x + jd_1) \neq 0 \quad \text{for} \quad 0 \leq j \leq lk - 1$$

implies that max $(x, y, k) \leq C_{31}$ unless $l = 1, m = k = 2, d_1 = 2d_2^2, f(X) = (X + r)^\nu$ where r is an integer such that $(x + r) = (y + r)(y + r + 3d_2)$. Infact the assumption that f is a power of an irreducible polynomial is not required to bound k. On the other hand, it remains open to bound x and y for a fixed k when f has at least two distinct irreducible factors.

Let $l = m = 1$ in (16). It is clear that (16) with $k = 2$ has infinitely many solutions. Further Gabovich [35A] gave an infinite class of solutions of (16) with $k = 3, 4$. Some infinite classes of solutions of (16) with $k = 5$ were given by Szymiczek [89A] and Choudhry [21A] where he also provided an infinite class of solutions of (16) with arbitrary k. Next we take d_1 and d_2 fixed. There is no loss generality in assuming that $x > y$ and gcd $(x, y, d_1, d_2) = 1$. Then $d_1 < d_2$. Saradha, Shorey and Tijdeman [67B] proved that either max (x, y, k) is bounded by a number depending only on d_2 or $x = k + 1, y = 2, d_1 = 1, d_2 = 4$. The latter possibilities can not be excluded by $(k + 1) \cdots (2k) = 2.6 \cdots (4k - 2)$ due to Makowski [58A]. The proof depends on Prime Number Theorem for arithmetic progression. Further they [68B] obtained sharp bounds for the magnitude of solutions of (16) with $d_1 = 1$ and $l = 1, m = 1, 2$. These bounds have been applied by Saradha, Shorey and Tijdeman [68B] to show that all the solutions of (16) with $l = m = d_1 = 1$ and $d_2 = 2, 3, 5, 6, 7, 9, 10$ are given by $2.3 = 1.6, 7.8.9 = 4.9.14, 8.9 = 6.12, 5.6 = 3.10, 4.5.6 = 1.8.15, 15.16.17 = 10.17.24, 9.10 = 6.15, 7.8 = 4.14, 24.25 = 20.30$ and $32.33.34 = 24.34.44$. Further they [68B] proved that (16) with $l = d_1 = 1, m = 2$ and $d_2 = 5, 6$ is possible only when $32.33 = 1.6.11.16$ and $207.208 = 8.13.18.23$.

Erdős and Woods states that if for positive integers $x, y, k \geq 3$ and that for every i with $0 \leq i < k$ the set of prime divisors of $x+i$ coincides with the set of prime divisors of $y+i$, then $x = y$. This conjecture has applications in logic. Balasubramanian, Shorey and Waldschmidt [47B] showed that $\log k \leq C_{32}(\log x)^{1/2}\log\log x$ for some absolute constant C_{32} whenever $x > y > 0$ and the set of prime divisors of $x+i$ coincides with the set of prime divisors of $y+i$ for every i with $0 \leq i < k$. Further they [74B] jointly with M. Langevin considered the above problem for sequences of integers in arithmetic progressions and solved it assuming $a \;\; b \;\; c$ conjecture.

Let α be an irrational real number with $[a_0, a_1, \cdots]$ as its simple continued fraction expansion. Let p_n/q_n and $\alpha_n = [a_n, a_{n+1}, \cdots]$ be the n-th convergent and the n-th complete quotient in the simple continued fraction expansion of α, respectively. If α is algebraic of degree ≥ 3 and d_{α_n} denotes the denominator of α_n, then Győry and Shorey [41B] showed that $d_{\alpha_n} \geq C_{33}C_{34}^n,\; Q(d_{\alpha_n}) \geq n^{C_{35}},\; P(d_{\alpha_n}) \geq C_{36}\log n$ where $n > 1$ and $C_{33}, C_{34} > 1, C_{35}, C_{36}$ are positive numbers depending only on α. As an application of the estimate on linear forms in logarithms stated in the beginning of this article, Shorey [10B] derived that $P(p_nq_n) \geq C_{37}$ log log q_n if α is algebraic. Here $C_{37} > 0$ depends only on α. This is an improved and an effective version of a result of Mahler that $P(p_nq_n)$ tends to infinity with n. Erdős and Mahler [32A] conjectured that if $P(p_nq_n)$ is bounded for infinitely many n, then α has to be a Liouville number. If α is a non-Liouville number such that $P(p_{n_k}q_{n_k})$ is bounded for $k \geq 1$ and $n_1 < n_2 < \cdots$, then Shorey [24B] showed that

$$\lim_{k\to\infty} \frac{\log\log n_k}{\log k} = \infty.$$

Further Shorey [24B] sharpened the results of Erdős and Mahler [32A] on lower bounds for $P(q_{n-1}q_nq_{n+1})$. For example, it has been shown in [24B] that $P(q_{n-1}q_nq_{n+1})$ tends to infinity with n if $\log\log a_{n+1} = o(\log q_n)$ which is an assumption not far from best possible. The analogous results are also valid for $P(p_{n-1}p_np_{n+1})$. For $\delta > 0$, Shorey and Srinivasan [37B] gave a lower bound $q_n(\log q_n)^{-1-\delta}$ for the greatest square free factor $Q(p_n)$ of p_n and $Q(q_n)$ of q_n for almost all α with $0 < \alpha < 1$ and all sufficiently large n and this is close to the best possible. Thus q_n is never a power for almost all α with $0 < \alpha < 1$ and all sufficiently large n. Further, for almost all $0 < \alpha < 1$, they [37B] derived that $Q(q_m) = Q(q_n)$ with $m \neq n$ has only finitely many solutions in m and n.

A result of Siegel and Schneider (rediscovered by Lang and Ramachandra) states that

$$(17) \qquad \mid 2^{\pi} - \alpha_1 \mid + \mid 2^{\pi^2} - \alpha_2 \mid + \mid 2^{\pi^3} - \alpha_3 \mid$$

is positive where α_1, α_2 and α_3 are algebraic numbers. The question whether at least one of the numbers 2^{π} and 2^{π^2} is transcendental remains open and, if stated in generality, it is the well-known four exponential conjecture. Shorey [8B] gave a positive lower bound for (17) in terms of the heights and degrees of α_1, α_2 and α_3. A theorem of Baker [2A] that a linear form in logarithms of algebraic numbers with algebraic coefficients is either zero or transcendental has been applied to prove the transcendence of certain infinite series. For example, it has been shown in [91B] that

$L(1,\chi)$ with χ a non-principal character and $\sum_{n=1}^{\infty} \frac{F_n}{n2^n}$ with F_n the Fibonacci sequence are transcendental. Infact, these are examples of more general series which have been proved to be transcendental in [91B].

Shorey [2B] proved a p-adic analogue of a result of Tijdeman [91A] on a bound for the number of zeros of a general exponential polynomial in a disk and he applied it to give p-adic analogues of the results of Tijdeman [90A] on algebraic independence of certain numbers connected with the exponential function. For a prime p, let T_p be the completion of the algebraic closure of $\mathbf{Q}_p$ and we denote the p-adic valuation on T_p by $||_p$. Let $\alpha_i \in T_p$ with $\mid \alpha_i \mid_p < p^{-1/p-1}$ be linearly independent over $\mathbf{Q}$ and $\eta_j \in T_p$ with $\mid \eta_j \mid_p \leq 1$ be linearly independent over $\mathbf{Q}$. Then Shorey [2B] proved that at least two of the numbers from each of the following sets

$$\alpha_i, exp(\alpha_i\eta_j) \;\text{ with }\; i = 0,1,2,3, j = 0,1,2,$$
$$\alpha_i, \eta_j, exp(\alpha_i\eta_j) \;\text{ with }\; i = 0,1,2, j = 0,1$$

and

$$\alpha_i, exp(\alpha_i\eta_j) \;\text{ with }\; i = 0,1,2, j = 0,1,2,3$$

are algebraically independent over $\mathbf{Q}$. By taking $\alpha_i = pe^{ip}$ with $i = 0,1,2$ and $\eta_j = e^{jp}$ with $j = 0,1$ we derive that for $p > 2$, at least two of the numbers

$$e^p, e^{pe^p}, e^{pe^{2p}}, e^{pe^{3p}}$$

are algebraically independent. This implies that at least one of the last three numbers is transcendental. This is weaker than available in the complex case, namely, that at least one of e^e, e^{e^2} is transcendental due to Brownawell and Waldschmidt . The p-adic analogue of a result of Tijdeman [93A] for small values of a general exponential polynomial was given in [3B] where it has also been applied to give a p-adic analogue of a lower bound for (17).

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School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India

E-mail address: shorey@math.tifr.res.in