DIVISIBILITY PROPERTIES OF HYPERGEOMETRIC POLYNOMIALS

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Abstract. In this paper we give effective upper bounds for the degree k of divisors (over $\mathbb{Q}$) of hypergeometric polynomials defined by

$$\sum_{j=0}^{n} a_j \frac{(a)_j}{(b)_j (c)_j} x^j,$$

where $(m)_j = m(m+1)\cdots(m+j-1)$ denotes the Pochhammer symbol and $a_0,\dots,a_n$ are integers with $|a_0| = |a_n| = 1, a = -n-r, b = \alpha+1, c \geq 1$ and $\alpha = -tn-s-1, tn+s$ for integers $r \geq 0, t \geq 1, s, c$ bounded in terms of k. These results generalize on earlier results of the authors and others on generalized Laguerre polynomials.

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1. Introduction and results

For rational numbers a, b, c the hypergeometric polynomials are defined by

$$g_{a,b,c}(x) = \sum_{j=0}^{n} \frac{(a)_j}{(b)_j (c)_j} x^j,$$

where $(m)_j = m(m+1)\cdots(m+j-1)$ denotes the Pochhammer symbol. We mention that such polynomials appear by truncating the infinite series given by generalized hypergeometric functions of type ${}_2F_2(a,1;b,c;x)$ (with the usual notation for such functions). For $a = -n, b = \alpha+1, c = 1$ one gets

$$\begin{aligned} g_{-n,\alpha+1,1}(x) &= \frac{n!}{(\alpha+1)\cdots(\alpha+n)} \sum_{j=0}^{n} \frac{(\alpha+n)\cdots(\alpha+j+1)}{(n-j)!j!}(-x)^j \\ &= \frac{n!}{(\alpha+1)\cdots(\alpha+n)} L_n^{(\alpha)}(x) \end{aligned}$$

the generalized Laguerre polynomials (up to a constant).

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Let n, s and t be integers with $n \geq 2$ and $|s| \leq n$ and

$$\alpha = -tn - s - 1 \text{ with } t \geq 2 \tag{1}$$

or

$$\alpha = tn + s \text{ with } t \geq 1. \tag{2}$$

Such polynomials have been studied extensively, especially the case $L_n^{(\alpha)}(x)$, starting with work by Schur [10, 11], Coleman [1] and Filaseta and others (see e.g. [3]). We also mention the papers [2] and [5] since we shall be using their arguments. Now we are additionally assuming that

$$a = -n - r, \quad b = \alpha + 1, \quad c \geq 1 \tag{3}$$

for integers r, c with $r \geq 0$. Let α satisfy (1). Then

$$g_{a,b,c}(x) = \frac{(n+r)!(c-1)!}{((t-1)n+s+1)\cdots(tn+s)(n+r+c-1)!} \sum_{j=0}^{n} c_j x^{n-j}$$

with

$$c_j = \binom{n+r+c-1}{r+j}((t-1)n+s+1)\cdots((t-1)n+s+j) \tag{4}$$

and therefore we have for $m \in \{0, \ldots, n\}$ that

$$\frac{c_n}{c_{n-m}} = \frac{(tn+s)!}{(tn+s-m)!} \frac{(n+r-m)!}{(n+r)!} \frac{(m+c-1)!}{(c-1)!}. \tag{5}$$

Now let α satisfy (2). Then we have

$$g_{a,b,c}(x) = \frac{(-1)^n (c-1)!(n+r)!}{(tn+s+1)\cdots((t+1)n+s)(n+c+r-1)!} \sum_{j=0}^{n} c'_j x^{n-j}$$

with

$$c'_j = (-1)^j \binom{n+r+c-1}{r+j}((t+1)n+s-j+1)\cdots((t+1)n+s) \tag{6}$$

and therefore we have

$$\frac{c'_n}{c'_{n-m}} = (-1)^m \frac{(tn+s+m)!}{(tn+s)!} \frac{(n+r-m)!}{(n+r)!} \frac{(m+c-1)!}{(c-1)!} \tag{7}$$

for $m \in \{0, \ldots, n\}$. For $0 \leq j \leq n$, we write $d_j = c_j$ or c'_j according as α satisfies (1) or (2). Moreover, we set

$$f(x) = \sum_{j=0}^{n} d_j x^{n-j}$$

and

$$F(x) = \sum_{j=0}^{n} a_j d_j x^{n-j}$$

for integers $a_0, \dots, a_n$. Here we notice that $F(x)$ is the polynomial stated in the abstract.

Our intention is to generalize the results of [6] to this extended setting. It was proved there that for integers $a_0, \dots, a_n$ with $|a_0| = |a_n| = 1$ there exist effectively computable absolute constants η_0 and ε such that for all $\eta_0 < k \le \frac{n}{2}$ and for all α with $t < \varepsilon \log k, 0 \le s < \varepsilon k \log k$ the polynomial $F(x)$ does not have a factor of degree k. We also mention that for $2 \le k \le \frac{n}{2}$, it was proved in [9, Theorem 1.3] that if for given $\varepsilon > 0$ the hypergeometric polynomial $g_{-n-r,\alpha+1,c}$ with $0 \le \alpha \le k$ and $r+c < (1/3-\varepsilon)k$ has a divisor of degree k, then k is bounded by an effectively computable constant depending only on ε, and in [9, Theorem 1.4] that $g_{-n,\alpha+1,1}$ with $\alpha = -n-s-1$ and $0 \le s \le 0.95k$ has no factor of degree k at all.

In the sequel we will denote by $\eta_1, \eta_2, \dots$ effectively computable absolute positive real constants.

Theorem 1. *Let $a_0, \dots, a_n$ be any integers with $|a_0| = |a_n| = 1$. Then there exist constants $\varepsilon > 0$ and η_1 such that for all $\eta_1 < k \le \frac{n}{2}$ and for all α satisfying* (1) *with $t \ge 4$ or* (2) *with $t \ge 3$ and for*

$$t < \varepsilon \log k, \quad \max\{r, c\} < k, \quad |s| < \varepsilon k \vartheta,$$

where $\vartheta = \log k$, the polynomial $F(x)$ does not have factor of degree k.

Moreover, under the abc-conjecture, the statement holds true with $\vartheta = \log n$.

For small values of t in both the cases for α we also get results, but under slightly stronger restrictions. We state them separately in the following theorem.

Theorem 2. *The statement of Theorem 1 holds true for*

$$\max\{r, c, |s|\} < k,\ r + c + |s| < n^{6/11+\varepsilon} \quad \text{if } \alpha \text{ satisfies (1) with } t = 2,$$
$$\max\{r, c, |s|\} < k,\ r + c < n^{6/11+\varepsilon} \quad \text{if } \alpha \text{ satisfies (2) with } t = 1,$$

and for

$$\max\{r, c\} < k, \quad |s| < \varepsilon k \vartheta, \quad r + c < n^{6/11+\varepsilon}$$

if α satisfies (1) *with $t = 3$ or* (2) *with $t = 2$.*

The two theorems imply [6, Theorem 1,2] apart from the values of ε. Further we observe that we cover the negative values of s in contrast to the situation in [6].

In the proof we will again use the p-adic Newton polygon, where the prime p satisfies certain properties. Let us write v_p for the p-adic valuation and $v_p(0)=\infty$. Then we use the following lemma, which we take from [2]:

Lemma 1. *Let k and l be integers with $k>l\geq 0$ and $k\leq \frac{n}{2}$. Suppose that*

$$g(x)=\sum_{j=0}^{n} b_j x^{n-j}\in\mathbb{Z}[x]$$

and p is a prime such that $p\nmid b_0, p|b_j$ for all $j\in\{l+1,\ldots,n\}$ and the slope of the right-most edge of the Newton polygon for $f(x)$

$$\max_{1\leq m\leq n}\left\{\frac{v(b_n)-v(b_{n-m})}{m}\right\}$$

is $<1/k$. Then for any integers $a_0,\ldots,a_n$ with $|a_0|=|a_n|=1$, the polynomial

$$G(x)=\sum_{j=0}^{n} a_j b_j x^{n-j}$$

cannot have a factor with degree in the interval $[l+1,k]$.

The existence of such primes is the main challenge (and also the most significant difference to our results in [6]) and this will be guaranteed by tools from analytic number theory. The result on primes that we are needing is the following lemma:

Lemma 2. *There exists a constant η_2 such that for all $x>\eta_2$ and for all $\frac{6}{11}<\theta\leq 1$ we have*

$$0.969\frac{y}{\log x}\leq \pi(x)-\pi(x-y)$$

for $y=x^\theta$, where $\pi(x)$ is the prime counting function.

This result is taken from [7]. Moreover, for the conditional result in Theorem 1 we recall the abc-conjecture that we will use.

Lemma 3 (abc-Conjecture). *For every $\epsilon>0$ there exists a constant $\gamma=\gamma(\epsilon)$ depending only on ϵ such that for all coprime nonzero integers a,b,c with $a+b=c$ the inequality*

$$\max\{|a|,|b|,|c|\}<\gamma N(abc)^{1+\epsilon}$$

holds, where $N(m)$ *denotes the product over all different prime divisors of* m.

Now we have everything ready to give the proof of Theorem 1 and Theorem 2 that will be done simultaneously in the next section.

2. Proof of Theorem 1 and 2

For the proof we assume that $F(x)$ has a factor of degree k such that $k \leq \frac{n}{2}$ and k exceeds a sufficiently large constant η_1. Let $\vartheta = \log n$ if the abc-conjecture holds and $\vartheta = \log k$ otherwise. Moreover, we put $\delta = 1/4$.

By Lemma 2 there exists ℓ with

$$n^{13/22} \leq \ell < ((t+1)n+s)^{13/22}$$

such that $(t-1)n+s+\ell$ or $(t+1)n+s-\ell+1$ is a prime p according as (1) or (2) holds, respectively. Then it follows from (4) and (6), respectively, that $p \| d_j$ for $j \in \{\ell, \dots, n\}$ (here we use, as usual, $d \| d_j$ for $d | d_j$ and $d^2 \nmid d_j$). Next we show that $p > n+c+r$. For this we have to take special care of the small values of t. We have

$$\begin{aligned}
(t-1)n+s+\ell \geq n-|s|+n^{6/11+\varepsilon} &> n+c+r \qquad \text{if (1) with } t=2,\\
(t-1)n+s+\ell \geq n+n^{6/11+\varepsilon} &> n+c+r \qquad \text{if (1) with } t=3,\\
(t+1)n+s-\ell+1 > n+(n/2-n^{7/11})+1 &\\
> n+n^{6/11+\varepsilon} &> n+c+r \qquad \text{if (2) with } t=1,\\
(t+1)n+s-\ell+1 > n+(n-n^{7/11}) &\\
> n+n^{6/11+\varepsilon} &> n+c+r \qquad \text{if (2) with } t=2
\end{aligned}$$

and finally $p > 2n \geq n+c+r$ in all other cases. This implies $p \nmid d_0$. Therefore, the right-most edge of the p-adic Newton polygon for $f(x)$ has slope $< 1/k$. By Lemma 1 we conclude that $k \leq \ell \leq (3\varepsilon n \log n)^{13/22} \leq n^{7/11}$.

Now we will first consider the case (1), i.e. that $\alpha = -tn-s-1$. We write $z = 6\varepsilon k\vartheta$. Observe that every prime $p > z \geq k$ that divides $((t-1)n+s+1)\cdots((t-1)n+s+k)$ divides exactly one of the factors, so $p | (t-1)n+s+1+\ell$ for some $0 \leq \ell \leq k-1$. We shall show that a prime with this property exists. For this purpose we use the following lemma (cf. [4, Lemma 6] and [6, Lemma 5]).

Lemma 4. *Let z be a positive real number. For each prime $p \leq z$, let $d_p \in \{n, n-1, \dots, n-k+1\}$ with $v_p(d_p)$ maximal. Define*

$$Q_z = Q_z(n,k) = \prod_{p>z} p^{v_p(A)}$$

with $A = n(n-1)\cdots(n-k+1)$. Then

$$Q_z \geq \frac{n(n-1)\cdots(n-k+1)}{(k-1)!\prod_{p\leq z} p^{v_p(d_p)}} \geq \frac{(n-k+1)^{k-\pi(z)}}{(k-1)!},$$

where $\pi(z)$ denotes the number of primes $\leq z$.

By the above lemma we get for $\vartheta = \log k$ that

$$\begin{aligned} Q_z((t-1)n+s+k,k) &\geq \frac{((t-1)n+s+1)^k}{(k-1)!((t-1)n+s)^{\pi(z)}} \geq n^{k-2\pi(z)-7k/11} \\ &\geq n^{(4/11-12\varepsilon(1+\delta)^2)k} > 1, \end{aligned}$$

where we have used the inequality $(k-1)! \leq k^k \leq n^{7k/11}$ and the estimate

$$\pi(z) \leq \frac{(1+\delta)6\varepsilon k\vartheta}{\log(6\varepsilon k\vartheta)} \leq 6\varepsilon(1+\delta)^2 k,$$

that follows at once from the prime number theorem. It remains to show that we also have a prime $p > \eta_3 k \log n > z$ for some η_3 and for ε small enough, dividing $((t-1)n+s+1)\cdots((t-1)n+s+k)$, if we assume the abc-conjecture to be true. For this we just have to follow the arguments of [8, Theorem 1]. We give the proof for the readers convenience (and since the statement that is proved there, at first sight, does not seem to be connected to what we need). For a prime p dividing two different factors of this product of k consecutive terms we have $p \leq k$. Thus

$$\prod_{i=1}^{k} N((t-1)n+s+i) \leq \left(\prod_{p\leq P} p\right)\prod_{p\leq k} p^{\lfloor k/p\rfloor} \leq \eta_4 \exp\left(\eta_5(P + k\log k)\right),$$

where P denotes the largest prime divisor of $((t-1)n+s+1)\cdots((t-1)n+s+k)$ and $N(m)$ the product over all primes dividing m. Now let j_1, j_2 with $N((t-1)n+s+j_1) \leq N((t-1)n+s+j_2)$ be the smallest two values in the set $\{N((t-1)n+s+j); 1 \leq j \leq k\}$. It follows

$$\begin{aligned} N((t-1)n+s+j_2) &\leq \left(\prod_{i=1}^{k} N((t-1)n+s+i)\right)^{1/(k-1)} \\ &= \exp\left(\eta_6(P/k + \log k)\right). \end{aligned}$$

We apply Lemma 3 with $\epsilon = 1$ to the equation

$$\frac{(t-1)n+s+j_1}{d} - \frac{(t-1)n+s+j_2}{d} = \frac{j_1-j_2}{d}$$

and get

$$\begin{aligned} n &\le \eta_7 \left(N\left(\frac{(t-1)n+s+j_1}{d}\right) N\left(\frac{(t-1)n+s+j_2}{d}\right) \frac{|j_1-j_2|}{d} \right)^2 \\ &\le \exp\left(\eta_8(P/k+\log k)\right), \end{aligned}$$

where d denotes the greatest common divisor of $(t-1)n+s+j_1$ and $(t-1)n+s+j_2$. Finally, this implies $P > \eta_9 k \log n$.

Thus there is a prime $p > z$ dividing $((t-1)n+s+1)\cdots((t-1)n+s+k)$, say p divides $(t-1)n+s+1+\ell$ with $0 \le \ell \le k-1$. We may assume that $p \nmid n+c+i$ for every $0 \le i \le r-1$, since assuming the contrary we have $p|n+c+i$, which implies that p divides $|(t-1)n+s+1+\ell-(t-1)(n+c+i)| \le |s|+1+\ell+tc+tr \le \varepsilon k\vartheta+k+2\varepsilon k\log k \le 4\varepsilon k\vartheta \le z$, a contradiction. It follows that p satisfies $p|c_j$ for $\ell+1 \le j \le n$ and $p \nmid c_0$. Define $m = m(p) \in \{1,\dots,n\}$ such that

$$\frac{v_p(c_n) - v_p(c_{n-m(p)})}{m(p)} = \max_{1\le m\le n}\left\{\frac{v_p(c_n)-v_p(c_{n-m})}{m}\right\}$$

is the slope of the right most edge of the p-adic Newton polygon for $f(x)$ with respect to p. Then by Lemma 1 and (5) we conclude

$$\begin{aligned} \frac{1}{k} &\le \frac{v_p(c_n)-v_p(c_{n-m})}{m} \\ &\le \frac{1}{m}\left[v_p\left(\frac{(tn+s)!}{(tn+s-m)!}\right) - v_p\left(\binom{n+r}{m}\right) + v_p\left(\binom{m+c-1}{c-1}\right)\right]. \end{aligned}$$

For estimating the third summand we may assume that $m > 5\varepsilon k\vartheta$, since otherwise $m+c-1 \le 6\varepsilon k\vartheta < p$ and so this summand is zero, and therefore we get

$$\begin{aligned} \frac{1}{m} v_p\left(\binom{m+c-1}{c-1}\right) &\le \frac{1}{m} v_p((m+c-1)!) \le \frac{m+c-1}{m(p-1)} \\ &= \frac{1}{p-1} + \frac{c-1}{m(p-1)} < \frac{1}{5\varepsilon k\vartheta} + \frac{k}{5\varepsilon k\vartheta 5\varepsilon k\vartheta} \le \frac{1}{4k}. \end{aligned}$$

If p does not divide $(tn+s-m+1)\cdots(tn+s)$ then we immediately get a contradiction. Thus we may assume that p divides $tn+s-i$ with $0 \le i \le m-1$. But then it also divides $t((t-1)n+s+\ell+1)-(t-1)(tn+s-i) = t(\ell+1)+s+(t-1)i$ and therefore $p \le 2\varepsilon k\theta + \varepsilon m \log k \le 2\varepsilon k\vartheta + 2\varepsilon m\log k$.

Since $p > z = 6\varepsilon k\vartheta$, this implies that

$$\frac{2k\vartheta}{\log k} < m. \tag{8}$$

Moreover we get

$$\frac{3}{4k} \leq \frac{1}{m} v_p\left(\frac{(tn+s)!}{(tn+s-m)!}\right) \leq \frac{1}{m}\sum_{j=1}^{\infty}\left(\left\lfloor\frac{tn+s}{p^j}\right\rfloor - \left\lfloor\frac{tn+s-m}{p^j}\right\rfloor\right)$$
$$\leq \frac{1}{m}\sum_{j=1}^{J}\left(\frac{m}{p^j}+1\right) \leq \frac{1}{p-1}+\frac{J}{m} \leq \frac{1}{12k}+\frac{J}{m},$$

where

$$J := \left\lfloor\frac{\log(tn+s)}{\log p}\right\rfloor.$$

This gives $m \leq 3kJ/2$ and thus

$$m \leq \frac{3k}{2}\frac{\log(tn+s)}{\log p} < \frac{3(1+\delta)k\log n}{2\log k}. \tag{9}$$

If $\vartheta = \log n$, then it follows from (8) and (9) that $2\log n = 2\vartheta < \frac{3}{2}(1+\delta)\log n$. This contradiction proves the result if we assume that the abc-conjecture is true. Now we give the proof for the case $\vartheta = \log k$. In fact all the above statements are true for every prime $p > z$ dividing $(t-1)n+s+1+\ell$ for some $0 \leq \ell \leq k-1$. Especially this is the case for the inequalities (8) and (9). Next we will prove the existence of such a prime with even stronger assumptions.

Let U be the set of numbers $(t-1)n+s+1+j$ with $0 \leq j \leq k-1$, where for every prime $q \leq z$ we have removed those numbers $d_q \in \{(t-1)n+s+1,\ldots,(t-1)n+s+k-1\}$ with $v_q(d_q)$ maximal. We mention that all elements of U are $\geq n$ if $t > 2$ and $\geq n/2$ if $t = 2$. Now let Ω be the set of all primes $q > z$ with $v_q(u) > 0$ for some $u \in U$ and $q^{v_q(u)} \leq (2k+m)\varepsilon\log k$ for all $u \in U$. Observe that all such q divide exactly one $u \in U$, since $q > z \geq k$. Thus we have

$$\log\left(\prod_{u\in U}\prod_{q\in\Omega} q^{v_q(u)}\right) \leq \log\left(\prod_{z<q\leq(2k+m)\varepsilon\log k}(2k+m)\varepsilon\log k\right)$$
$$\leq \pi((2k+m)\varepsilon\log k)\log((2k+m)\varepsilon\log k) \leq (1+\delta)\varepsilon(2k+m)\log k$$
$$\leq \varepsilon(1+\delta)(2k\log k+3k\log n) \leq 5\varepsilon(1+\delta)k\log n,$$

where for the second summand (9) was used. It follows that

$$\log\left(\prod_{u\in U}\prod_{q\le z} q^{v_q(u)}\right)+\log\left(\prod_{u\in U}\prod_{q\in\Omega} q^{v_q(u)}\right)$$
$$\le \frac{(1+\delta)6\varepsilon k\log k}{\log(6\varepsilon k\log k)}\log k+5\varepsilon(1+\delta)k\log n\le\frac{2}{3}k\log n,$$

since $k\le n^{7/11}$. On the other side we have

$$\log\left(\prod_{u\in U}u\right)\ge\log(((t-1)n+s+1)\cdots((t-1)n+s+k-\pi(z)))$$
$$\ge(k-\pi(z))\log\frac{n}{2}>\frac{2}{3}k\log n.$$

By comparing the lower and upper bound just obtained we conclude that there is a prime $q>z$ that divides some element $u\in U$ with the additional property that $q^{v_q(u)}>(2k+m)\varepsilon\log k$. We write $u=(t-1)n+s+\ell+1, 0\le \ell\le k-1$ and define f by $q^{f-1}\le(2k+m)\varepsilon\log k<q^f$ and such that $q^f|u$. Observe that $1\le f\le v_q(u)$.

Now if q^f divides $tn+s-i$ for some $0\le i\le m-1$, then it also divides $|t((t-1)n+s+\ell+1)-(t-1)(tn+s-i)|\le t\ell+t+|s|+(t-1)i<3\varepsilon k\log k$ which contradicts the fact that $q^f>(2k+m)\varepsilon\log k$ by (8). Thus q^f does not divide $tn+s-i$ for any $0\le i\le m-1$ and we conclude

$$\frac{3}{4k}\le\frac{1}{m}v_q\left(\frac{(tn+s)!}{(tn+s-m)!}\right)\le\frac{1}{m}\sum_{j=1}^{f-1}\left(\frac{m}{q^j}+1\right)\le\frac{1}{q-1}+\frac{f-1}{m}.$$

For $f=1$ we immediately get a contradiction. For $f\ge2$ we get

$$2(2k+m)\varepsilon\log k\ge q^{f-1}+(2k+m)\varepsilon\log k\log k\ge(f-1)6\varepsilon k\log k+4\varepsilon k\log k,$$

where we have used (8), and therefore $3(f-1)k<m$, which gives

$$\frac{3}{4k}\le\frac{1}{q-1}+\frac{f-1}{m}<\frac{5}{12k}+\frac{1}{3k}=\frac{3}{4k},$$

a contradiction again. This completes the proof in this case.

Now we come to the case (2), i.e. that $\alpha=tn+s$. Here we can argue in almost the same way. We have to consider primes $p>z=6\varepsilon k\vartheta$ that divide $((t+1)n+s-k+1)\cdots((t+1)n+s)$ and we show by following the arguments from above that such a prime exists. As before we may assume that a prime dividing $(t+1)n+s-\ell$ for some $0\le\ell\le k-1$ does not divide any of $n+c+i$ for $0\le i\le r-1$, since otherwise we have that p divides $|(t+1)n+s-\ell-(t+1)(n+c+i)|\le |s|+\ell+(t+1)(c+r)\le \varepsilon k\vartheta+k+4\varepsilon k\log k\le 6\varepsilon k\vartheta=z$. Therefore it follows that such a p satisfies $p|c'_j$ for $\ell+1\le j\le n$ and $p\nmid c'_0$. Proceeding as in the previous case we conclude from Lemma 1 and (7) that

$$\frac{1}{k}\le \frac{v_p(c'_n)-v_p(c'_{n-m})}{m}$$
$$\le \frac{1}{m}\left[v_p\left(\frac{(tn+s+m)!}{(tn+s)!}\right)-v_p\left(\binom{n+r}{m}\right)+v_p\left(\binom{m+c-1}{c-1}\right)\right].$$

In the same way as before we can estimate the third summand and we may assume that p divides $tn+s+m-i$ with $0\le i\le m-1$ and therefore $6\varepsilon k\vartheta=z<p\le |t((t+1)n+s-\ell)-(t+1)(tn+s+m-i)|\le t\ell+|s|+(t+1)m\le 2\varepsilon k\vartheta+2m\varepsilon\log k$, which again implies $2k\vartheta/\log k<m$. On the other hand, one shows $m\le 3kJ/2$ with

$$J:=\left\lfloor\frac{\log((t+1)n+s)}{\log p}\right\rfloor,$$

which gives $m<3(1+\delta)k\log n/(2\log k)$. For $\vartheta=\log n$ we conclude the proof by comparing the lower and upper bound for m. Thus we may assume that $\vartheta=\log k$. By arguing as above we get a prime $q>z$ that divides exactly one element of the form $u=(t+1)n+s-\ell, 0\le\ell\le k-1$ and with $q^{v_q(u)}>(2k+m)\varepsilon\log k$ (observe that now all such elements u are $\ge n$). By defining f as before we conclude that q^f does not divide $tn+s+1+i$ for any $0\le i\le m-1$, since otherwise it would divide $t((t+1)n+s-\ell)-(t+1)(tn+s+1+i)=-t\ell-ts-(t+1)(1+i)$ that contradicts $q^f>z$ being large. Similar as in the case (1) the proof can be finished.

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