Products of Fibonacci numbers with indices in an interval and at most four omitted being a power

Florian Luca
Instituto de Matemáticas
Universidad Nacional Autónoma de México
C.P. 58089, Morelia, Michoacán, México
fluca@matmor.unam.mx

T.N. Shorey
School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road
Mumbai, 400005, India
shorey@math.tifr.res.in

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Abstract

Here, we find all instances in which a product of Fibonacci numbers with indices in an interval of length k and at most four of them omitted is a perfect power.

1 Introduction

Let $(F_n)_{n\geq 0}$ be the sequence of Fibonacci numbers given by $F_0 = 0,\ F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. In the previous paper [5], we showed that the equation

$$F_n F_{n+1} \cdots F_{n+k-1} = y^m \tag{1}$$

in integers $n \geq 1,\ k \geq 2,\ m \geq 2,\ y \geq 1$ has only the solution $F_1 F_2 = 1$. With $k = 1$, the only solutions are $F_1 = F_2 = 1,\ F_6 = 2^3$ and $F_{12} = 12^2$ (see [2]). In our sequel [6] to [5], we proved that for large values of k, we can even remove some of the Fibonacci numbers from the product appearing on the left of (1) and the resulting Diophantine equation has only finitely many integer solutions (n, k, y, m). More precisely, we showed that there exist effectively computable absolute constants c_0 and c_1 such that if $k \geq 3$ and $n_1 < \cdots < n_t \in [n, n+k-1]$ are positive integers with $t > k - c_0 k \log\log k / \log k$ such that

$$F_{n_1} F_{n_2} \cdots F_{n_t} = y^m, \tag{2}$$

again with some integers y and $m \geq 2$, then $\max\{n, k, y, m\} \leq c_1$.

In this paper, we look at a specific instance of the Diophantine equation (2), namely when $t \in \{k-4, \ldots, k-1\}$. The result is the following.

Theorem 1. *Let $n \geq 1,\ k \geq 5$ and $\mathcal{I} \subseteq \{n, n+1, \ldots, n+k-1\}$ such that $|\mathcal{I}| \geq k-4$ and*

$$\prod_{i \in \mathcal{I}} F_i = y^m$$

holds with some integers y and $m \geq 2$. Then

$$\begin{aligned} I \ \in\ & \{\{1\}, \{2\}, \{6\}, \{12\}, \{1,2\}, \{1,6\}, \{2,6\}, \{3,6\}, \\ & \{1,2,6\}, \{1,3,6\}, \{2,3,6\}, \{1,2,3,6\}\}. \end{aligned}$$

The exceptions in Theorem 1 are necessary.

2 The proof of Theorem 1

We write our Diophantine equation as

$$\prod_{i \in \mathcal{I}} F_i = y^m, \quad \mathcal{I} \subset \{n, n+k, \ldots, n+k-1\}, \quad |\mathcal{I}| \in \{k-4, \ldots, k-1\}. \tag{3}$$

We start by showing that any counterexample to Theorem 1 must be very large. Later we shall prove that such large examples do not exist.

Recall that given a positive integer ℓ, its *order of apparition* in the Fibonacci sequence denoted by $z(\ell)$ is the smallest positive integer such that $\ell \mid F_{z(\ell)}$. It always exists and it has the property that $\ell \mid F_m$ if and only if $z(\ell) \mid m$. In particular, prime factors p of F_m have the property that

$z(p) \mid m$. If $z(p) = m$, then p is called a *primitive divisor* of F_m. In other words, a primitive divisor p of F_m is a prime factor of F_m that does not divide F_n for any positive integer $n < m$. Such primes always exist except if $m \in \{1, 2, 6, 12\}$ by a result of Carmichael [1], and they always satisfy $p \equiv \pm 1 \pmod{m}$ except if $m = 5$ when p is also 5.

A conjecture of Wall asserts that $p \| F_{z(p)}$. This was recently verified for all $p < 10^{14}$ by McIntosh and Roettger [7]. In particular, from their calculation we deduce that for all $m \notin \{1, 2, 6, 12\}$ and $m < 10^{14}$, the number F_m has a prime factor $p \| F_m$ such that $p \nmid F_n$ for any positive integer $n < m$.

Now suppose that $n + k - 1 < 10^{14}$ and let $i_0 \in \mathcal{I}$ be the maximal element in $\mathcal{I}$. If $i_0 \notin \{1, 2, 6, 12\}$, then, we get that there exists a prime $p_0 \| F_{i_0}$ such that $p_0 \nmid F_i$ for any positive integer $i < i_0$. Thus,

$$p_0 \| \prod_{i \in \mathcal{I}} F_i, \tag{4}$$

and this certainly shows that the Diophantine equation (3) cannot hold. Thus, $i_0 \in \{1, 2, 6, 12\}$ and, in particular, $\mathcal{I} \subset \{1, \ldots, 12\}$. If $|\mathcal{I}| \geq 2$, then let i_1 be the largest element in $\mathcal{I} \backslash \{1, 2, 3, 4, 6, 12\}$ assuming that this last set is nonempty. Then every primitive prime factor $p \| F_{i_1}$ satisfies $p > 3$, so it will have the property that p does not divide F_i for any $i \neq i_1$ in $\mathcal{I}$, which leads to a contradicton. Thus, $\mathcal{I} \subseteq \{1, 2, 3, 4, 6, 12\}$. Now if $4 \in \mathcal{I}$, then $3 = F_4$, so $\mathcal{I}$ must contain another multiple of 4. The only such possibility is 12. Hence, $12 \in \mathcal{I}$, but this is impossible since then $\mathcal{I}$ will have a gap of at least 6. Thus, $\mathcal{I} \subset \{1, 2, 3, 6, 12\}$ assuming that $n + k - 1 < 10^{14}$, from which the desired conclusion is easily derived.

Thus, indeed $\mathcal{I} \subseteq \{1, 2, 3, 6, 12\}$ assuming that $n + k - 1 < 10^{14}$.

Next we shall show that there are no exceptions to our Theorem 1 with $n + k - 1 \geq 10^{14}$. We suppose that this last inequality holds as well as the Diophantine equation (3) and in order to get a contradiction we distinguish various cases according to the size of n versus k.

Case 2. *The case when $n \geq k^2$ and $k \geq 19$.*

We write $P(m)$ for the largest prime factor of the positive integer m. Then the inequality

$$p_0 := P(\prod_{i \in \mathcal{I}} i) > k \tag{5}$$

holds provided that $|\mathcal{I}| \geq k - \pi(2k) + \pi(k)$ (see [4] inequalities (8)–(10)). Here and in what follows, for a positive real number x we use $\pi(x)$ for the number of primes $p \leq x$. Since $|\mathcal{I}| \geq k - 4$, it follows that it suffices that $\pi(2k) - \pi(k) \geq 4$. It is easy to see that this last inequality holds when $k \geq 19$. Indeed, since the inequalities

$$\frac{x}{\log x - 0.5} < \pi(x) < \frac{x}{\log x - 1.5} \qquad \text{hold for all} \qquad x > 67 \tag{6}$$

(see [8]), it follows that it is enough to check that

$$\frac{2k}{\log(2k) - 0.5} - \frac{k}{\log k - 1.5} \geq 4 \qquad \text{holds for all} \qquad k > 134.$$

The last inequality holds for all $k \geq 135$. For the values $k \in [19, 134]$, one checks directly that indeed $\pi(2k) - \pi(k) \geq 4$. Hence, inequality (5) holds. Let i_0 for the unique index in $\mathcal{I}$ such that $p_0 \mid i_0$, and further write $i_0 = p_0^a j_0$, where $p_0 \nmid j_0$. Rewrite equation (3) as

$$F_{p_0^a} \left(\frac{F_{i_0}}{F_{p_0^a}} \right) \prod_{\substack{i \in \mathcal{I} \\ i \neq i_0}} F_i = y^m. \tag{7}$$

Recall that the relation

$$\gcd(F_u, F_v) = F_{\gcd(u,v)} \tag{8}$$

holds for all positive integers u and v. Since p_0 does not divide i for all $i \neq i_0$ in $\mathcal{I}$, relation (8) tells us that $F_{p_0^a}$ is coprime to $\prod_{\substack{i \in \mathcal{I} \\ i \neq i_0}} F_i$. Next we show that $F_{p_0^a}$ is also coprime to $F_{i_0}/F_{p_0^a}$. Indeed, it is known that if q is a prime factor dividing both $F_{p_0^a}$ and $F_{i_0}/F_{p_0^a}$, then q must divide $i_0/p_0^a = j_0$, so, in particular, $q \leq P(j_0) < p_0$. However, every prime factor q of $F_{p_0^a}$ has the property that $z(q)$ is a divisor of p_0^a. Since $p_0 > k \geq 5$, it follows that $q \equiv \pm 1 \pmod{p_0}$, therefore $q \geq 2p_0 - 1 > p_0 > P(j_0)$, again a contradiction. Hence, indeed $F_{p_0^a}$ and $F_{i_0}/F_{p_0^a}$ are also coprime. Now equation (7) shows that $F_{p_0^a}$ is a perfect power, so $p_0^a \in \{1, 2, 6, 12\}$, and this is impossible since $p_0 > 5$ is prime.

This takes care of the case when $n \geq k^2$ and $k \geq 19$.

From now on, we work under the assumption that $n < k^2$ for $k \geq 19$.

Case 2. *The case when $k < n < k^2$.*

Observe that $k^2 + k - 1 > n + k - 1 > 10^{14}$ in this case, so k is very large. Thus, the inequality

$$|\mathcal{I}| \geq k - \lfloor \pi(k)/3 \rfloor - 1$$

holds. Theorem 3 in [9] shows that if we put $P(m)$ for the largest prime factor of the positive integer m, then the inequality

$$P(\prod_{i \in \mathcal{I}} i) > k \qquad (9)$$

holds with finitely many exceptions in n and k, which are explicitly given in the statement of Theorem 3 in [9] together with the Remark at the end of that paper. All these exceptions have $k \leq 17$, which is not the case for us. Now the arguments from Case 1 show that if we put p for the number appearing in the left hand side of equation (9) and a for the exact power at which p^a divides $\prod_{i \in \mathcal{I}} i$, then $p^a \in \{1, 2, 6, 12\}$, which is a contradiction.

This takes care of the case when $k < n < k^2$.

Case 3. *The case when $n \leq k$.*

Observe that if $n \leq k - 1$, we have $(n + k - 1)/2 \geq n$. Thus, putting $m := n+k-1$, we have that the primes in the interval $(m/2, m]$ are contained in $[n, n + k - 1]$. When $n = k$, the interval $[n, n + k - 1]$ is $[k, 2k - 1]$ and since $k \geq 5$, it follows that if we put $m := 2k$, then again the primes in the interval $(m/2, m]$ are contained in $[n, n + k - 1]$. We shall next show that in our range of variables, we have $\pi(m) - \pi(m/2) \geq 5$. Indeed, observe that $2k > n + k - 1 > 10^{14}$, therefore $m > 10^{14}$. Using again the inequalities (6), it suffices to check that

$$\pi(m) - \pi(m/2) > \frac{m}{\log m - 0.5} - \frac{m}{2\log(m/2) - 3} > 5,$$

and the inequality above holds for all $m > 125$, in particular for our range for m. Since $\mathcal{I}$ misses at most 4 integers from the interval $[n, n + k - 1]$, it follows that $\mathcal{I}$ contains a prime exceeding $m/2 \geq (n + k - 1)/2$. Let i_0 be this prime. Now the argument from Case 1 shows that F_{i_0} is a perfect power and i_0 is prime, therefore $i_0 = 2$. However, this is impossible since $i_0 \geq m/2 > 10^{13}$.

This takes care of the case when $n \leq k$.

Case 1. *The case when $k < 19$.*

Assume first that $|\mathcal{I}| = 1$. Then the main result from [2] shows that $I \subset \{1, 2, 6, 12\}$, which is a contradiction since $\mathcal{I}$ must consist of one large element. Assume next that $|\mathcal{I}| \geq 2$. Let $i_1 < i_2$ be the smallest two elements in $\mathcal{I}$. Then $i_2 - i_1 \in \{1, 2, 3, 4, 5\}$. Let $d := \gcd(i_1, i_2)$ and put $x := i_2/d$, $y := i_1/d$. Then $d \in \{1, 2, 3, 4\}$, x and y are coprime and $x - y \in \{1, 2, 3, 4\}$. Theorem 6.3 in [10] shows that if $P(xy) \leq 13$, then $x < 10^{11}$, therefore $n < dx < 4 \cdot 10^{11}$, so $n + k - 1 < 4 \cdot 10^{11} + 18 < 10^{14}$, which is false. Thus, $P(xy) \geq 17$. In particular,

$$p = P(\prod_{i \in \mathcal{I}} i) \geq 17.$$

If there exists exactly one value of $i \in \mathcal{I}$ such that $p \mid i$, then the arguments from Case 1 lead to a contradiction. Hence, assume that there exist two indices $i_1 < i_2$ in $\mathcal{I}$ such that $p \mid i_1$ and $p \mid i_2$. Observe that since $k < 19$, this is possible only when $k = 18$, $p = 17$, i_1 is the smallest element in $\mathcal{I}$ and i_2 is the largest element in $\mathcal{I}$. Now $\mathcal{I}$ must have at least 14 elements, of which only two of them are multiples of 17. Hence, there must exist two consecutive elements in $\mathcal{I}$ (say the second one and the third one), which are coprime to 17. Let them be $i_1 < i_2$. Then $i_2 - i_1 \in \{1, 2, 3, 4\}$ and $P(i_1 i_2) \leq 13$, which by the result from [10] shows again that $n + k - 1 < 10^{14}$, which is a contradiction.

This takes care of this last case and completes the proof of the theorem.

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