

Extensions of Schur's irreducibility results

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ABSTRACT

We prove that the generalised Laguerre polynomials $L_n^{(\alpha)}(x)$ with $0 \leq \alpha \leq 50$ are irreducible except for finitely many pairs (n, α) and these exceptions are necessary. In fact we consider a more general polynomials $f_a(x)$ of degree n . Except for explicitly given finitely many pairs (n, a) , we show that $f_a(x)$ with $0 \leq a \leq 40$ are either irreducible or linear polynomial times an irreducible polynomial of degree $n - 1$. This is derived from a more general result proved in the paper.

1. Introduction

Let $n \geq 1, a \geq 0$ and a_0, a_1, \dots, a_n be integers. For $a \geq 0$, we consider

$$(1) \quad f(x) := f_a(x) = a_n \frac{x^n}{(n+a)!} + a_{n-1} \frac{x^n}{(n-1+a)!} + \dots + a_1 \frac{x}{(1+a)!} + a_0 \frac{1}{a!}.$$

Schur [Sch29] proved that $f(x)$ with $a = 0$ and $|a_0| = |a_n| = 1$ is irreducible. He also proved in [Sch73] that $f(x)$ with $a = 1$ and $|a_0| = |a_n| = 1$ is irreducible unless $n + 1 = 2^r$ for some r where it may have a linear factor or $n = 8$ where it may have a quadratic factor. Also for $a = 2$ and many other values of a the polynomial $f(x)$ may have a linear factor. If $f(x)$ is reducible, then $f(x)$ has a factor of degree k with $1 \leq k \leq \frac{n}{2}$. Shorey and Tijdeman [ShTi] proved that $f(x)$ with $2 \leq k \leq \frac{n}{2}$, $0 \leq a \leq \frac{3}{2}k$ and $|a_0| = |a_n| = 1$ has no factor of degree k except when

$$(2) \quad (n, k, a) \in \{(6, 2, 3), (7, 2, 2), (7, 2, 3), (7, 3, 3), (8, 2, 1), (8, 3, 2), \\ (12, 3, 4), (13, 2, 3), (22, 2, 3), (46, 3, 4), (78, 2, 3)\}.$$

Furthermore all the exceptions in (2) are necessary. They also showed that for $f(x)$ with $3 \leq k \leq \frac{n}{2}$, $|a_0| = |a_n| = 1$ and $0 \leq a \leq 10$ when $k = 3, 4$ or $0 \leq a \leq 30$ when $k \geq 5$ has no factor of degree k except when

$$(3) \quad (n, k, a) \in \{(7, 3, 3), (8, 3, 2), (12, 3, 4), (18, 4, 9), (18, 4, 10), (46, 3, 4), \\ (56, 4, 10), (17, 5, 11), (19, 5, 9), (40, 5, 12)\}.$$

We extend the validity of their results as follows.

THEOREM 1. *Let $2 \leq k \leq \frac{n}{2}$, $0 \leq a \leq 5k$ and $|a_0| = |a_n| = 1$. Then $f_a(x)$ has no factor of degree k except possibly when (n, k, a) is given by (2) or (3) or*

$$(4) \quad \begin{aligned} k = 2, & \quad (n, a) \in \{(4, 5), (6, 4), (8, 8), (12, 4), (17, 8), (21, 4), (22, 6), (23, 5), \\ & \quad (23, 10), (24, 9), (36, 9), (43, 6), (44, 5), (46, 9), (58, 6), (59, 5), \\ & \quad (72, 9), (73, 8), (77, 4), (91, 9), (112, 9), (233, 10), (234, 9)\}; \\ k = 3, & \quad (n, a) \in \{(14, 12), (17, 11), (53, 12)\}; \\ k = 4, & \quad (n, a) \in \{(16, 12), (17, 11), (38, 13), (39, 18)\}. \end{aligned}$$

THEOREM 2. Let $2 \leq k \leq \frac{n}{2}$, $|a_0| = |a_n| = 1$ and $0 \leq a \leq 40$ if $k = 2$ and $0 \leq a \leq 50$ if $k \geq 3$. Then $f_a(x)$ has no factor of degree k except possibly when (n, k, a) is given by (2) or (3) or (4) or $n + a \leq 100$ or $a \in \{13, 14, 19, 33\}$, $n + a \in \{126, 225, 2401, 4375\}$ or

a	$n + a$	a	$n + a$	a	$n + a$
12	169, 729	15, 16	289	17	513
18	361, 513, 1216	19, 20	243	21	529
21, 22	121, 576	24	325, 625, 676	27	784
28	145	29	961	31	243
32	243, 289, 1089	33	136, 256, 289, 5832	36	1369
38	325, 625, 676	39	1025, 6561	40	288

in the case $k = 2$ or

$$k = 3, (n, a) \in \{(14, 12), (17, 11), (53, 12)\};$$

$$k = 4, (n, a) \in \{(16, 12), (17, 11), (38, 13), (39, 18)\}.$$

The following assertion for $a \leq 12$ follows from Theorem 2.

THEOREM 3. The polynomial $f_a(x)$ with $a_0 a_n = \pm 1$, $a_1 = a_2 = \dots = a_{n-1} = 1$ and $a \leq 12$ has at most one linear factor.

We shall use the results of [ShTi] stated above without reference in this paper. Thus we always suppose that $a > 3$ if $k = 2$, $a > 10$ if $k = 3, 4$ and $a > 30$ if $k \geq 5$. Further we observe that Theorem 2 with $k \geq 10$ follows from Theorem 1.

Schur polynomials are generalisations of a class of orthogonal polynomials called Laguerre polynomials. For $n \in \mathbb{Z}$, $\alpha \in \mathbb{R}$ with $n \geq 1$, we define the generalised Laguerre polynomials of degree n as

$$L_n^{(\alpha)}(x) = \sum_{j=0}^n \frac{(n + \alpha)(n - 1 + \alpha) \cdots (j + 1 + \alpha)(-x)^j}{(n - j)!j!}.$$

It is easy to see that by taking

$$a = \alpha \text{ and } a_j = (-1)^j \binom{n}{j} \text{ for } 0 \leq j \leq n,$$

we obtain $(n + \alpha)!f_\alpha(x) = n!L_n^{(\alpha)}(x)$. There is an extensive literature on Laguerre polynomials. In particular, the irreducibility of these polynomials has been well studied. The irreducibility of $L_n^{(-2n-1)}$ proved by Filaseta and Trifonov [FiTr02] is equivalent to that all Bessel polynomials are irreducible. Also Laguerre polynomials provide examples of polynomials of degree n with associated Galois group A_n and the irreducibility of $L_n^{(n)}$ proved by Filaseta, Kidd and Trifonov [FiKiTr] has been used to settle explicitly the *Inverse Galois problem* that for every $n > 1$ there exists an explicit Laguerre polynomial of degree n with the associated Galois group A_n . As a consequence of Theorems 1 and 2, we obtain

THEOREM 4. Let $0 \leq \alpha \leq 50$. Then $L_n^{(\alpha)}(x)$ irreducible except when $n = 2, \alpha \in \{2, 7, 14, 23, 34, 47\}$ and $n = 4, \alpha \in \{5, 23\}$ where it has a linear factor.

For the exceptions, we have

$$\begin{aligned}
 L_2^{(2)}(x) &= \frac{1}{2}(x-2)(x-6); & L_2^{(7)}(x) &= \frac{1}{2}(x-6)(x-12); \\
 L_2^{(14)}(x) &= \frac{1}{2}(x-12)(x-20); & L_2^{(23)}(x) &= \frac{1}{2}(x-20)(x-30); \\
 L_2^{(34)}(x) &= \frac{1}{2}(x-30)(x-42); & L_2^{(47)}(x) &= \frac{1}{2}(x-42)(x-56); \\
 L_4^{(5)}(x) &= \frac{1}{24}(x-6)(x^3-30x^2+252x-504); \\
 L_4^{(23)}(x) &= \frac{1}{24}(x-30)(x^3-78x^2+1872x-14040).
 \end{aligned}$$

Theorem 4 is an extension of a result of Filaseta, Finch and Leidy [FiFiLe] where they proved that $L_n^{(\alpha)}(x)$ is irreducible for all n and $0 \leq \alpha \leq 10$ except when $(n, \alpha) \in \{(2, 2), (4, 5), (2, 7)\}$. Therefore we shall always assume that $\alpha > 10$ in the proof of Theorem 4. Analogous to Theorem 1, we also have

THEOREM 5. *Let $1 \leq k \leq \frac{n}{2}$ and $0 \leq \alpha \leq 5k$. Then $L_n^{(\alpha)}(x)$ has no factor of degree k except when $k = 1, (n, \alpha) \in \{(2, 2), (4, 5)\}$ where it has a linear factor.*

For the exceptions, we have

$$L_2^{(2)}(x) = \frac{1}{2}(x-2)(x-6) \text{ and } L_4^{(5)}(x) = \frac{1}{24}(x-6)(x^3-30x^2+252x-504).$$

Theorem 5 follows immediately from Theorem 4 for $k \leq 8$ and Theorem 1 for $k > 8$.

The new ingredients in the proofs of our theorems is the following Irreducibility Lemma and sharper lower estimates for the greatest prime factor of $\Delta(m, k)$ where

$$(5) \quad \Delta(m, k) = m(m+1) \cdots (m+k-1).$$

LEMMA 1. *Let $a > 0, 1 \leq k \leq \frac{n}{2}$ and $u_0 = \frac{a}{k}$.*

(A) *Assume that there is prime $p \geq k+2$ with*

$$(6) \quad p \mid \prod_{i=1}^k (a+n-k+i), \quad p \nmid a_0 a_n$$

and

$$(7) \quad p \nmid \prod_{i=1}^k (a+i).$$

Suppose

$$(8) \quad p \geq \min(2u_0, k+u_0)$$

or

$$(9) \quad p > 2k \text{ and } p^2 - p \geq a.$$

Then $f_a(x)$ has no factor of degree k .

(B) *If there is a prime $p \geq k+2$ with*

$$(10) \quad p \mid \prod_{i=1}^k (n-k+i)(a+n-k+i)$$

and (7) and satisfying (8) or (9), then $L_n^{(a)}(x)$ has no factor of degree k .

In view of applications in further work, we have stated Lemma 1 and some of the subsequent lemmas in a more general way than required for the proof of our theorems. We prove Lemma 1

in Section 2. In Section 3, we give a refinement of argument of Erdős and Sylvester. In Sections 5–10, we prove Theorems 1-5 by combining Lemma 1 with the refinement of argument of Erdős and Sylvester in Section 3, results on Grimm’s conjecture (see Lemma 4.4) and estimates from prime number theory. The Section 4 contain preliminaries required for the proof of our theorems. For any real $u > 0$, let $\lfloor u \rfloor$ and $\lceil u \rceil$ be floor function of u and ceiling function of u , respectively. Thus $\lfloor u \rfloor$ is the greatest integer less than or equal to u and $\lceil u \rceil$ is the least integer exceeding u .

2. Proof of Lemma 1

We will use the notations introduced in this section throughout the paper. We write

$$\Delta_j = \Delta(a + 1, j) = (a + 1)(a + 2) \cdots (a + j).$$

We observe that $q|\Delta_k$ for all primes $k < q \leq \frac{a+k}{\lfloor u_0 \rfloor}$ since $a \leq k\lceil u_0 \rceil < q\lceil u_0 \rceil \leq a + k$. Suppose there is a prime p satisfying the condition of the lemma. Then $p > \frac{a+k}{\lfloor u_0 \rfloor}$. As in the proof of [ShTi, Lemma 4.2], it suffices to show that

$$(11) \quad \phi_j := \phi_j(p) := \frac{\text{ord}_p(\Delta_j)}{j} < \frac{1}{k} \quad \text{for } 1 \leq j \leq n$$

for showing that $f_a(x)$ has no factor of degree k . Also as in the proof of [FiFiLe, Lemma 2.4], for showing $L_n^{(a)}(x)$ has no factor of degree k , it suffices to show

$$(12) \quad \phi'_j := \phi'_j(p) := \frac{\text{ord}_p\left(\frac{\Delta_j}{\binom{n}{j}}\right)}{j} < \frac{1}{k} \quad \text{for } 1 \leq j \leq n.$$

Since $\phi'_j \leq \phi_j$, we show that (11) holds for all j .

Let j_0 be the minimum j such that $p|(a + j)$ and write $a + j_0 = pl_0$ for some l_0 . Then $j_0 \leq p$ and $j_0 > k$ since $p \nmid \Delta_k$. Also we see that $l_0 \leq \lceil u_0 \rceil$ which we shall use in the proof without reference.

We may restrict to those j such that $a + j = pl$ for some l . Then $j - j_0 = p(l - l_0)$. Writing $l = l_0 + s$, we get $j = j_0 + ps$. Note that if $p|(a + j)$, then $a + j = p(l_0 + r)$ for some r . Hence we have

$$(13) \quad \text{ord}_p(\Delta_j) = \text{ord}_p((pl_0)(p(l_0 + 1)) \cdots (p(l_0 + s))) = s + 1 + \text{ord}_p(l_0(l_0 + 1) \cdots (l_0 + s)).$$

Let r_0 be such that $\text{ord}_p(l_0 + r_0)$ is maximal. We consider two cases.

Case I: Assume that $l_0 + s < p^2$. If $p \nmid (l_0 + i)$ for $0 \leq i \leq s$, then $\phi_j = \frac{s+1}{j_0+ps} < \frac{s+1}{k+ks} = \frac{1}{k}$. Hence we may suppose that $p|(l_0 + i)$ for some $0 \leq i \leq s$ and further $l_0 + s = pl_1$ for some $1 \leq l_1 < p$. Assume $s = 0$. Then $p|l_0$ which together with $l_0 < p^2$ implies $\text{ord}_p(\Delta_j) = \text{ord}_p(a + j_0) = 2$. Therefore $a + p \geq a + j_0 \geq p^2$ implying $a \geq p^2 - p$. If (8) holds, then $a \leq \max(k(p - k), \frac{pk}{2}) < p(p - 1)$ which is not possible. Thus (9) holds and hence $p \geq 2k + 1$ and $a = p^2 - p$ implying $j_0 = p$. Therefore $\phi_j = \frac{2}{j_0} = \frac{2}{p} < \frac{1}{k}$. Thus we have $s \neq 0$ and we obtain from (13) that $\text{ord}_p(\Delta_j) = s + 1 + l_1$ implying $\phi_j \leq \frac{s+1+l_1}{j_0+ps}$. Hence $\phi_j < \frac{1}{k}$ if $(p - k)\frac{s}{l_1} \geq k$ since $\frac{j_0+sp}{k} > 1 + s\frac{p}{k}$.

Suppose p satisfy (9). Then we may assume that $s < l_1$. Since $l_1 < p$, we have $s < p$ implying $\text{ord}_p(\Delta_j) \leq s + 2$ giving $\phi_j < \frac{s+2}{k+ps} \leq \frac{1}{k}$ since $s > 0$.

Thus we assume that p satisfy (8). Since $p \geq k + 2$, $s = pl_1 - l_0$ and $l_0 \leq \lceil u_0 \rceil$, we have $(p - k)\frac{s}{l_1} - k \geq 2(p - \frac{l_0}{l_1}) - k \geq 2p - k - 2\lceil u_0 \rceil$. Hence it suffices to show $2p - k \geq 2\lceil u_0 \rceil$. Since

$p \geq \min(2u_0, k + u_0)$, we have

$$2p - k = p + p - k \geq \begin{cases} 2u_0 + 2 \geq 2\lceil u_0 \rceil & \text{if } p \geq 2u_0 \\ 2(k + \lceil u_0 \rceil) - k \geq 2\lceil u_0 \rceil & \text{if } p \geq k + u_0, \end{cases}$$

noting that $p \geq k + u_0$ implies $p \geq k + \lceil u_0 \rceil$.

Case II: Let $l_0 + s \geq p^2$. Then we get from (13) that

$$\text{ord}_p(\Delta_j) \leq s + 1 + \text{ord}_p(l_0 + r_0) + \text{ord}_p(s!) \leq s + 1 + \frac{\log(l_0 + s)}{\log p} + \frac{s}{p-1}.$$

Since $\frac{j}{k} = \frac{j_0 + ps}{k} > 1 + \frac{p}{k}s$, it is enough to show that

$$\frac{p}{k} \geq 1 + \frac{1}{p-1} + \frac{\log(l_0 + s)}{s \log p}.$$

Observe that $\frac{\log(l_0 + s)}{s \log p}$ is a decreasing function of s . Since $s \geq p^2 - l_0$, it suffices to show

$$\frac{p}{k} \geq 1 + \frac{1}{p-1} + \frac{2}{p^2 - l_0}.$$

Suppose p satisfy (8). Then from $l_0 \leq \lceil u_0 \rceil \leq p$ and $p \geq k + 2$, we have $p^2 - l_0 \geq (k + 2)^2 - (k + 2) \geq 2(k + 1)$ implying

$$1 + \frac{1}{p-1} + \frac{2}{p^2 - l_0} \leq 1 + \frac{1}{k+1} + \frac{2}{2(k+1)} < 1 + \frac{2}{k} \leq \frac{p}{k}.$$

Suppose p satisfy (9). Then from $l_0 \leq \lceil u_0 \rceil \leq a$ and $p > 2k$, we obtain $p^2 - l_0 \geq p^2 - a \geq p > 2k$ implying

$$1 + \frac{1}{p-1} + \frac{2}{p^2 - l_0} \leq 1 + \frac{1}{2k} + \frac{2}{2k} < 1 + \frac{2}{k} \leq \frac{p}{k}.$$

Hence the assertion. \square

COROLLARY 2.1. Let k, p and $\mathfrak{A}_{k,p}$ be given by

$$\begin{aligned} k = 1, p = 3, \mathfrak{A}_{1,3} &= \{3r, 3r + 1 : 0 \leq r \leq 16\} \setminus \{7, 16, 24, 25, 34, 43\} \\ k = 1, p = 5, \mathfrak{A}_{1,5} &= \{5r, 5r + 1, 5r + 2, 5r + 3 : 0 \leq r \leq 9\} \cup \{50\} \setminus \{23, 48\} \\ k = 1, p = 7, \mathfrak{A}_{1,7} &= [0, 50] \cap \mathbb{Z} \setminus \{6, 13, 20, 27, 34, 41, 47, 48\} \\ k = 2, p = 5, \mathfrak{A}_{2,5} &= \{5r, 5r + 1, 5r + 2 : 0 \leq r \leq 8\} \cup \{45, 50\} \setminus \{21, 22\} \\ k = 2, p = 7, \mathfrak{A}_{2,7} &= [0, 50] \cap \mathbb{Z} \setminus (\{7r - 1, 7r - 2 : 1 \leq r \leq 7\} \cup \{45, 46\}) \\ k = 3, p = 5, \mathfrak{A}_{3,5} &= \{0, 1, 5, 6, 10, 11, 15, 25, 26, 30, 31, 35, 36, 40, 50\} \\ k = 3, p = 7, \mathfrak{A}_{3,7} &= \{7r, 7r + 1, 7r + 2, 7r + 3 : 0 \leq r \leq 5\} \cup \{42, 49, 50\} \\ k = 4, p = 7, \mathfrak{A}_{4,7} &= \{7r, 7r + 1, 7r + 2 : 0 \leq r \leq 4\} \cup \{35, 36, 49, 50\} \\ k = 5, p = 7, \mathfrak{A}_{5,7} &= \{0, 1, 7, 8, 14, 15, 21, 22, 28, 49, 50\}. \end{aligned}$$

Suppose $n \geq 2k$ and p satisfies (6). Then $f_a(x)$ has no factor of degree k for $a \in \mathfrak{A}_{k,p}$. Further if p satisfy (10), then $L_n^{(a)}(x)$ has no factor of degree k for $a \in \mathfrak{A}_{k,p}$.

Proof. For k, p and $a \in \mathfrak{A}_{k,p}$ given in the statement of Corollary 2.1, we check that $p \nmid \Delta_k$ and $\frac{\text{ord}_p(\Delta_j)}{j} < \frac{1}{k}$ for $j \leq 50$. As in the proof of Lemma 1, it suffices to check that $\frac{\text{ord}_p(\Delta_j)}{j} < \frac{1}{k}$ for all $j \geq 1$. Since $\text{ord}_p(s!) \leq \frac{s}{p-1}$, we have for $j > 50$ that

$$\frac{\text{ord}_p(\Delta_j)}{j} = \frac{\text{ord}_p((a+j)!) - \text{ord}_p(a!)}{j} \leq \frac{\frac{a+j}{p-1} - \text{ord}_p(a!)}{j} \leq \frac{1}{p-1} + \frac{\frac{a}{p-1} - \text{ord}_p(a!)}{51} < \frac{1}{k}.$$

Thus $\frac{\text{ord}_p(\Delta_j)}{j} < \frac{1}{k}$ for all $j \geq 1$. □

COROLLARY 2.2. *Let $a > 0$ and $1 \leq k \leq \frac{n}{2}$.*

(i) *If there is a prime $p > a + k$ satisfying (6), then $f_a(x)$ has no factor of degree k .*

(ii) *Let $p \geq k + 2$ be a prime satisfying (6) and let*

$$\mathcal{A}_p = \bigcup_{i=1}^{r_p} ((ip - k, ip - 1] \cap \mathbb{Z}_{>0}) \cup \{j > pr_p, j \in \mathbb{Z}\}$$

where

$$r_p = \lfloor \frac{k}{2} \rfloor \text{ if } p < 2k \text{ and } p - 1 \text{ if } p \geq 2k.$$

Then $f_a(x)$ has no factor of degree k for $a \notin \mathcal{A}_p$.

(iii) *Let $P_1 > P_2 > \dots > P_s \geq k + 2$ be primes satisfying (6). For a subset $\{Q_1, Q_2, \dots, Q_g\} \subseteq \{P_1, P_2, \dots, P_s\}$, let*

$$\mathcal{B}\{Q_1, \dots, Q_g\} = \bigcap_{l=1}^g \mathcal{A}_{Q_l}.$$

Then $f_a(x)$ has no factor of degree k for $a \notin \mathcal{B}\{Q_1, \dots, Q_g\}$.

In earlier results, Corollary 2.2 (i) has been used. This is possible only if there is a $p > k + a$ satisfying (6). But it is possible to apply Lemma 1 even when $p \leq k + a$ for all p satisfying (6). For example, take $n = 15, a = 13, k = 3$. Here $p < k + a$ for all p satisfying (6). However (6), (7) and (9) are satisfied with $p = 13$ and hence $f_{13}(x)$ has no factor of degree 3 by Lemma 1.

Proof. (i) is immediate from Lemma 1. Consider (ii). We may assume that $p \leq k + a$ by (i). Let $a \notin \mathcal{A}_p$. Then $a \leq pr_p$ implying $a \leq p^2 - p$ if $p \geq 2k$ and $2u_0 = \frac{2a}{k} \leq \frac{2pr_p}{k} \leq p$ if $p < 2k$ satisfying either (9) or (8). Since $a \notin \mathcal{A}_p$, there is some i for which $ip - 1 < a < (i + 1)p - k$ implying $ip < a + 1 < a + k < (i + 1)p$. Therefore $p \nmid \prod_{j=1}^k (a + j)$ which together with (6) and $p \geq k + 2$ satisfy the conditions of Lemma 1. Now the assertion follows by Lemma 1. The assertion (iii) follows from (ii). □

3. An upper bound for m when $\omega(\Delta(m, k)) \leq t$

Let m, k and t be positive integers such that

$$(14) \quad \omega(\Delta(m, k)) \leq t$$

and we write

$$v = \frac{m}{k}.$$

For every prime p dividing $\Delta(m, k)$, we delete a term $m + i_p$ in $\Delta(m, k)$ such that $\text{ord}_p(m + i_p)$ is maximal. Then we have a set T of terms in $\Delta(m, k)$ with

$$|T| = k - t := t_0.$$

We arrange the elements of T as $m + i_1 < m + i_2 < \dots < m + i_{t_0}$. Let

$$(15) \quad \mathfrak{P} := \prod_{\nu=1}^{t_0} (m + i_\nu) \geq (vk)^{t_0}.$$

Now we obtain an upper bound for \mathfrak{P} . For a prime p , let r be the highest power of p such that $p^r \leq k-1$. Let $w_l = \#\{m+i : p^l(m+i), m+i \in T\}$ for $1 \leq l \leq r$. By Sylvester and Erdős argument, we have $w_l \leq \lfloor \frac{iq}{p^l} \rfloor + \lfloor \frac{k-1-i_0}{p^l} \rfloor \leq \lfloor \frac{k-1}{p^l} \rfloor$. Let $h_p > 0$ be such that $\lfloor \frac{k-1}{p^{h_p+1}} \rfloor \leq t_0 < \lfloor \frac{k-1}{p^{h_p}} \rfloor$. Then there are at most $t_0 - w_{h_p+1}$ terms in T exactly divisible by p^l with $l \leq h_p$. Hence

$$\begin{aligned} \text{ord}_p(\mathfrak{P}) &\leq r w_r + \sum_{u=h_p+1}^{r-1} u(w_u - w_{u+1}) + h_p(t_0 - w_{h_p+1}) \\ &= w_r + w_{r-1} + \cdots + w_{h_p+1} + h_p t_0 \\ &\leq \sum_{u=1}^r \lfloor \frac{k-1}{p^u} \rfloor + h_p t_0 - \sum_{u=1}^{h_p} \lfloor \frac{k-1}{p^u} \rfloor = \text{ord}_p((k-1)!) + h_p t_0 - \sum_{u=1}^{h_p} \lfloor \frac{k-1}{p^u} \rfloor. \end{aligned}$$

It is also easy to see that $\text{ord}_p(\mathfrak{P}) \leq \text{ord}_p(k-1)!$. Let $L_0(p) = \min(0, h_p t_0 - \sum_{u=1}^{h_p} \lfloor \frac{k-1}{p^u} \rfloor)$. For any $l \geq 1$, we have from (15) that

$$(16) \quad m = vk \leq (\mathfrak{P})^{\frac{1}{t_0}} \leq \left((k-1)! \prod_{p \leq p_i} p^{L_0(p)} \right)^{\frac{1}{t_0}} =: L(k, l).$$

Observe that

$$(17) \quad (vk)^{t_0} \leq (L(k, l))^{t_0} \leq (k-1)!.$$

4. Preliminaries for Theorems 1-5

For a positive integer $\nu > 1$, we denote by $\omega(\nu)$ and $P(\nu)$ the number of distinct prime factors and the greatest prime factor of ν , respectively, and we put $\omega(1) = 0, P(1) = 1$. For positive integers ν , we write

$$\begin{aligned} \pi(\nu) &= \sum_{p \leq \nu} 1, \\ \theta(\nu) &= \sum_{p \leq \nu} \log p. \end{aligned}$$

Let p_i denote the i -th prime.

We begin with some results on prime number theory.

LEMMA 4.1. *Let $k \in \mathbb{Z}$ and $\nu \in \mathbb{R}$. We have*

- (i) $\frac{\nu}{\log \nu - 1} \leq \pi(\nu) \leq \frac{\nu}{\log \nu} \left(1 + \frac{1.2762}{\log \nu} \right)$.
- (ii) $\pi(\nu_1 + \nu_2) \leq \pi(\nu_1) + \pi(\nu_2)$ for $2 \leq \nu_1 < \nu_2 \leq \frac{7}{5}\nu_1(\log \nu_1)(\log \log \nu_1)$.
- (iii) $\nu(1 - \frac{3.965}{\log^2 \nu}) \leq \theta(\nu) < 1.00008\nu$ for $\nu > 1$.
- (iv) $p_k \geq k \log k$ for $k \geq 1$.
- (v) $\text{ord}_p(k-1)! \geq \frac{k-p}{p-1} - \frac{\log(k-1)}{\log p}$ for $k \geq 2$.
- (vi) $\sqrt{2\pi k} e^{-k} k^k e^{\frac{1}{12k+1}} < k! < \sqrt{2\pi k} e^{-k} k^k e^{\frac{1}{12k}}$.

The estimates (i), (iii) and (ii) are due to Dusart ([Dus99] and [Dus02], respectively). The estimate (iv) is due to Rosser [Ros38] and estimate (vi) is due to Robbins [Rob55, Theorem 6]. For a proof of (v), see [LaSh04, Lemma 2(i)]. \square

We derive from Lemma 4.1 the following results.

COROLLARY 4.2. *Let $10^{10} < m \leq 123k$. Then there are primes p, q with $m \leq p < m + k$ and $\frac{m}{2} \leq q < \frac{m+k}{2}$.*

Proof. Let $10^{10} < m \leq 123 \times k$. We observe that the assertion holds if

$$\theta\left(\frac{m+k-1}{s}\right) - \theta\left(\frac{m-1}{s}\right) = \sum_{\frac{m-1}{s} < p \leq \frac{m+k-1}{s}} \log p > 0$$

for $s = 1, 2$. Now from Lemmas 4.1 and since $m > 10^{10}$, it suffices to show

$$\theta\left(\frac{m+k-1}{s}\right) - \theta\left(\frac{m-1}{s}\right) > \frac{m+k-1}{s} \left(1 - \frac{3.965}{\log^2(5 \cdot 10^9)}\right) - 1.00008 \frac{m-1}{s} > 0$$

or

$$k\left(1 - \frac{3.965}{\log^2(5 \cdot 10^9)}\right) > (m-1)\left(\frac{8}{10^5} + \frac{3.965}{\log^2(5 \cdot 10^9)}\right).$$

This is true since $m \leq 123k$ and

$$\frac{1 - \frac{3.965}{\log^2(5 \cdot 10^9)}}{\frac{8}{10^5} + \frac{3.965}{\log^2(5 \cdot 10^9)}} > 123.$$

□

COROLLARY 4.3. *We have*

$$(18) \quad \pi(k) + \pi\left(\frac{k}{2}\right) + \pi\left(\frac{k}{3}\right) + \pi\left(\frac{k}{4}\right) + \pi\left(\frac{6k}{5}\right) \leq \begin{cases} k-2 & \text{for } k \geq 61 \\ \pi(4k) & \text{for } k \geq 8000. \end{cases}$$

Proof. Let $k \geq 30000$. We have from $\frac{\log y}{\log x} = 1 + \frac{\log y/x}{\log x}$ and Lemma 4.1 (i) that

$$\begin{aligned} & (\log 4k) \left(\pi(4k) - \pi\left(\frac{6k}{5}\right) - \pi(k) - \pi\left(\frac{k}{2}\right) - \pi\left(\frac{k}{3}\right) - \pi\left(\frac{k}{4}\right) \right) \\ & \geq \frac{4k}{\log 4k - 1} + \\ & k \left(4 - \frac{6}{5} \left(1 + \frac{\log \frac{10}{3}}{\log \frac{6k}{5}} \right) \left(1 + \frac{1.2762}{\log \frac{6k}{5}} \right) - \sum_{j=1}^4 \frac{1}{j} \left(1 + \frac{\log 4j}{\log \frac{k}{j}} \right) \left(1 + \frac{1.2762}{\log \frac{k}{j}} \right) \right). \end{aligned}$$

The right hand side of the above inequality is an increasing function of k and it is positive at $k = 30000$. Therefore the left hand side of (18) is at most $\pi(4k)$ for $k \geq 30000$. By using exact values, we find that it is valid for $k \geq 8000$.

Also $\pi(4k) \leq \frac{4k}{\log 4k} \left(1 + \frac{1.2762}{\log 4k} \right) \leq k-2$ is true for $k \geq 8000$. Therefore the left hand side of (18) is at most $k-2$ for $k \geq 8000$. Finally we check using exact values of π function that the left hand side of (18) is at most $k-2$ for $61 \leq k < 8000$. □

The following result is on Grimm's Conjecture, [LaSh06b, Theorem 1].

LEMMA 4.4. *Let $m \leq 1.9 \times 10^{10}$ and $l \geq 1$ be such that $m+1, m+2, \dots, m+l$ are all composite numbers. Then there are distinct primes P_i such that $P_i | (m+i)$ for each $1 \leq i \leq l$.*

The following result follows from [SaSh03, Lemma 3].

LEMMA 4.5. *Let $m+k-1 < k^{\frac{3}{2}}$. Let $|\{i : P(m+i) \leq k\}| = \mu$. Then*

$$\binom{m+k-1}{k} \leq (2.83)^{k+\sqrt{m+k-1}} (m+k-1)^{k-\mu}.$$

5. Prelude to the proof of Theorems 1-3

Let $k \geq 2$, $n \geq 2k$, $a \geq 0$, $m = n + a - k + 1$ and $|a_0 a_n| = 1$. Then $m > k + a$. We consider the polynomials $f_a(x)$ with $3 < a \leq 40$ when $k = 2$; $10 < a \leq 50$ when $k \in \{3, 4\}$ and $\max(30, 1.5k) < a \leq \max(50, 5k)$ when $k \geq 5$. Let $P_1 > P_2 > \dots > P_s \geq k + 2$ be primes dividing $\Delta(m, k)$ and we write $P_{m,k} = \{P_1, P_2, \dots, P_s\}$. We use Corollaries 2.2 and 2.1 to apply the following procedure which we refer to as *Procedure I*.

Procedure I: Let k be fixed. For all a with $3 < a \leq 40$ if $k = 2$; $10 < a \leq 50$ if $k \in \{3, 4\}$ and $\max(30, 1.5k) < a \leq \max(50, 5k)$ if $k \geq 5$, it suffices to consider all (m, k, a) with $P_1 \leq k + a$ by Corollary 2.2 (i). We restrict to triplets (m, k, a) with $P_1 \leq k + a$. By Corollary 2.2 (iii), we have $a \in \mathfrak{B}_0(m, k) := \mathfrak{B}\{P_1, P_2, \dots, P_s\}$. Therefore we further restrict to (m, k, a) with $a \in \mathfrak{B}_0(m, k)$. Further for $k \in \{2, 3, 4, 5\}$ and $p = 5 \in P_{m,k}$ if $k = 2$; $p = 5 \in P_{m,k}$ or $p = 7 \in P_{m,k}$ if $k = 3$ and $p = 7 \in P_{m,k}$ if $k \in \{4, 5\}$, we restrict to those (m, k, a) with $a \notin \mathfrak{A}_{k,p}$ by using Corollary 2.1 and recalling $n = m + k - 1 - a$. Every (m, k, a) gives rise to the triplet (n, k, a) .

We try to exclude the triplets (n, k, a) given by *Procedure I* to prove our theorems.

Let

$$\omega_0(a) = \begin{cases} \pi(a+k) - \pi(k+1) + \pi(k+1) & \text{if } a \leq k+1 \\ \sum_{j=1}^2 \left(\pi\left(\frac{a+k}{j}\right) - \pi(\max(k+1, \frac{a}{j})) \right) + \pi(k+1) & \text{if } k+1 < a \leq 2k+2 \\ \sum_{j=1}^3 \left(\pi\left(\frac{a+k}{j}\right) - \pi(\max(k+1, \frac{a}{j})) \right) + \pi(k+1) & \text{if } 2k+2 < a \leq 3k+3 \\ \sum_{j=1}^4 \left(\pi\left(\frac{a+k}{j}\right) - \pi(\max(k+1, \frac{a}{j})) \right) + \pi(k+1) & \text{if } 3k+3 < a \leq 4k+4 \\ \sum_{j=1}^5 \left(\pi\left(\frac{a+k}{j}\right) - \pi(\max(k+1, \frac{a}{j})) \right) + \pi(k+1) & \text{if } 4k+4 < a \leq 5k \end{cases}$$

and ω_1 be the maximum of $\omega_0(a)$ for $1.5k < a \leq 5k$. Then $\omega(\Delta(a+1, k)) \leq \omega_1$.

Let $k \geq 10$. Assume that $\omega(\Delta(m, k)) > \omega_1$. Then there is a prime $p \geq k+2$ with $p | \Delta(m, k)$ such that $p \nmid \Delta(a+1, k)$ and $p \nmid a_0 a_n$. Further $p \geq 13 > 2u_0$ since $u_0 \leq 5$. Hence $f(x)$ has no factor of degree k by Lemma 1. Therefore we may suppose that

$$(19) \quad \omega(\Delta(m, k)) \leq \omega_1 \text{ for } k \geq 10.$$

Let $k \geq 100$. Let $(i-1)(k+1) < a \leq i(k+1)$ with $1 \leq i \leq 5$. For $1 \leq j < i$, we have $\frac{a}{j} > \frac{k}{j} \geq \frac{100}{4}$ implying $\frac{\frac{a}{j}}{\frac{k}{j}} = \frac{a}{k} \leq 5 \leq \frac{7}{5} \log(25) \log \log(25) \leq \frac{7}{5} \log\left(\frac{k}{j}\right) \log \log\left(\frac{k}{j}\right)$. Hence $\pi\left(\frac{a+k}{j}\right) - \pi\left(\frac{a}{j}\right) \leq \pi\left(\frac{k}{j}\right)$ for $1 \leq j < i$ by Lemma 4.1 (ii). Therefore

$$\omega_0(a) \leq \begin{cases} \pi(k+k+1) & \text{if } a \leq k+1 \\ \pi(k) + \pi\left(\frac{k}{2} + k+1\right) & \text{if } k+1 < a \leq 2k+2 \\ \pi(k) + \pi\left(\frac{k}{2}\right) + \pi\left(\frac{k}{3} + k+1\right) & \text{if } 2k+2 < a \leq 3k+3 \\ \pi(k) + \pi\left(\frac{k}{2}\right) + \pi\left(\frac{k}{3}\right) + \pi\left(\frac{k}{4} + k+1\right) & \text{if } 3k+3 < a \leq 4k+4 \\ \pi(k) + \pi\left(\frac{k}{2}\right) + \pi\left(\frac{k}{3}\right) + \pi\left(\frac{k}{4}\right) + \pi\left(\frac{k}{5} + k\right) & \text{if } 4k+4 < a \leq 5k \end{cases}$$

which, again by Lemma 4.1 (ii), implies

$$(20) \quad \omega_1 \leq \pi(k) + \pi\left(\frac{k}{2}\right) + \pi\left(\frac{k}{3}\right) + \pi\left(\frac{k}{4}\right) + \pi\left(\frac{6k}{5}\right) =: \omega_2 \text{ for } k \geq 100.$$

Let $N_1(p) = \{N : P(N(N-1)) \leq p\}$ and $N_2(p) = \{N : P(N(N-2)) \leq p, N \text{ odd}\}$. Then N_1 and N_2 are given by [Leh64, Table IA] for $p \leq 41$ and [Leh64, Table IIA] for $p \leq 31$, respectively

and we shall use them without reference. For given k, N and j with $1 \leq j < k$, we put

$$M_j(N, k) = \prod_{i=0}^{k-1} (N - j + i).$$

Let

$$\mathcal{N}_j(k) := \{N \in N_1(41) : P(M_j(N, k)) \leq 59\}.$$

By observing that

$$M_1(N, k+1) = M_1(N, k)(N-1+k), \quad M_k(N, k+1) = (N-k)M_{k-1}(N, k)$$

and

$$M_j(N, k+1) = M_j(N, k)(N-j+k) = (N-j)M_{j-1}(N, k) \quad \text{for } 1 < j < k,$$

we can compute $\mathcal{N}_j(k)$ recursively as follows. Recall that $P(N(N-1)) \leq 41$ for $N \in N_1(41)$. Hence we have

$$\mathcal{N}_1(3) := \{N \in N_1(41) : P(N+1) \leq 59\}, \quad \mathcal{N}_2(3) := \{N \in N_1(41) : P(N-2) \leq 59\}.$$

For $k \geq 3$ and $1 \leq j \leq k$, we obtain $\mathcal{N}_j(k+1)$ recursively by

$$\mathcal{N}_1(k+1) := \{N \in \mathcal{N}_1(k) : P(N-1+k) \leq 59\}, \quad \mathcal{N}_k(k+1) := \{N \in \mathcal{N}_{k-1}(k) : P(N-k) \leq 59\}$$

and

$$\mathcal{N}_j(k+1) := \{N \in \mathcal{N}_j(k) : P(N-j+k) \leq 59\} \cup \{N \in \mathcal{N}_{j-1}(k) : P(N-j) \leq 59\} \quad \text{for } 1 < j < k.$$

6. Proof of Theorems 1 and 2 for $k < 10$

Let $k = 2$. Then $a \leq 40$. By Corollary 2.2 (i), we first restrict to those m for which $P(m(m+1)) \leq 41$. They are given by $m = N-1$ with $N \in N_1(41)$. By *Procedure I*, we obtain the tuples $(n, 2, a)$ given by the following table.

a	$n+a$	a	$n+a$	a	$n+a$
4, 5	9	4	10	5, 6	28, 49, 64
4, 8, 9	16, 25, 81	9	33, 45, 55, 100, 121, 243	10	33, 243
12	27, 28, 49, 64, 91, 169, 729	13	21, 25, 28, 36, 50, 64	14	25
13, 14 19, 33	81, 126, 225, 2401, 4375	15, 16	289	17	513
18	25, 76, 81, 96, 361, 513, 1216	19	25, 28, 36, 49, 50, 64, 243	20	28, 33, 49, 64, 243
21	25, 33, 45, 55, 529	21, 22	46, 81, 100, 121, 576	23	81
24	40, 81, 65, 325, 625, 676	26	49, 64	27	49, 64, 784
28	81, 145	29	81, 125, 961	31	243
32	243, 289, 1089	33	49, 50, 51, 64, 85, 136, 256, 289, 5832	34	49, 50, 64, 81
36	1369	38	65, 81, 325, 625, 676	39	81, 82, 1025, 6561
40	49, 64, 82, 288				

Let $3 \leq k \leq 9$. Then $10 < a \leq 50$ if $k = 3, 4$ and $30 < a \leq 50$ if $5 \leq k \leq 9$. Thus we may assume that $P(\Delta(m, k)) \leq 59$ by Corollary 2.2 (i).

Let $m \leq 10000$. We need to consider $[k, 59] \cup \mathcal{M}(k)$ where $\mathcal{M}(k) = \{60 \leq m \leq 10000 : P(\Delta(m, k)) \leq 59\}$. We compute $\mathcal{M}(3)$ and further from the identity $\Delta(m, k+1) = (m+k)\Delta(m, k)$,

we obtain $\mathcal{M}(k+1) = \{m \in \mathcal{M}(k) : P(m+k) \leq 59\}$ for $k \geq 3$ recursively. In fact we obtain

$$\mathcal{M}(6) = \{90, 91, 116, 184, 185, 285, 340\}, \quad \mathcal{M}(7) = \{90, 184\}$$

and $\mathcal{M}(8) = \mathcal{M}(9) = \emptyset$. We now apply Procedure *I* on $m \in [k, 59] \cup \mathcal{M}(k)$. We obtain

a	$n+a$	a	$n+a$
11	28	12	26, 27, 28, 65
19, 20	56, 100	20	46, 162
21	46	32	51, 56, 100, 121
33	51	38, 39	82
41, 43	56, 100	43, 44, 45	162

or $a \in \{12, 13, 18, 19, 20, 27, 32, 33, 34, 39, 41, 43, 44\}$, $n+a = 50$ if $k = 3$ and

a	$n+a$	a	$n+a$	a	$n+a$	a	$n+a$
11, 12	27, 28	13, 31, 32, 33	51	18	57	10	66

if $k = 4$.

Thus $m > 10000$. Suppose that $m+j = N \in N_1(41)$ for some $1 \leq j < k$. Then $\Delta(m, k) = M_j(N, k)$ which imply $N \in \mathcal{N}_j(k)$ since $P(\Delta(m, k)) \leq 59$. Let $\mathcal{N}'_j(k) = \mathcal{N}_j(k) \cap [10000, \infty)$. We find that

$$\mathcal{N}'_1(3) = \{13311, 13455, 17576, 17577, 19551, 29601, 32799, 212381\}$$

$$\mathcal{N}'_2(3) = \{10881, 11662, 13312, 13456, 13690, 16170, 17577, 23375, 27456, 31213, 134850, 212382, 1205646\}$$

$$\mathcal{N}'_1(4) = \{17576\}, \quad \mathcal{N}'_2(4) = \{17577\}, \quad \mathcal{N}'_3(4) = \{10881\}$$

and $\mathcal{N}'_j(k) = \emptyset$ for $k \geq 5$ and $1 \leq j < k$. We now take $m = N - j$ with $N \in \mathcal{N}_j(k)$ for $1 \leq j < k$ and apply Procedure *I* to find that there are no triplets (n, k, a) .

Thus we may suppose that $m+j \notin N_1(41)$ for all $1 \leq j < k$. Then $P((m+i)(m+i+1)) > 41$ for each $0 \leq i < k-1$. By Corollary 2.2 (i), we may suppose that $P(\Delta(m, k)) \leq 53$ for $k \leq 8$ and $P(\Delta(m, k)) \leq 59$ for $k = 9$. Taking $V(m, k) = \{P((m+2i)(m+2i+1)) : 0 \leq i < \frac{k}{2}\}$, we have $V(m, k) \subseteq \{43, 47, 53\}$ for $4 \leq k \leq 7$ and $V(m, k) = \{43, 47, 53, 59\}$ if $k = 8, 9$. Then $k \neq 8$ and computing $\{a \leq 50 : a \in \mathfrak{B}\{Q_1, Q_2\}\}$ for $(Q_1, Q_2) \in \{(47, 43), (53, 43), (53, 53)\}$ if $k = 4, 5$; $(Q_1, Q_2) = (53, 43)$ if $k = 6, 7, 9$, we find that the set is \emptyset except when $k = 5$, $(Q_1, Q_2) = (43, 47)$ where it is $\{42\}$. Thus we may assume that $k = 5$ and $a = 42$. Further $P(\Delta(m, k)) = 47$ and $43 \mid \Delta(m, k)$. If $p \mid \Delta(m, k)$ with $13 \leq p \leq 41$, then $42 \notin \mathfrak{B}\{47, p\}$ by Corollary 2.2 (iii). Thus we may further suppose that $p \mid \Delta(m, k)$ implies $p \leq 11$ or $p \in \{43, 47\}$. Also $P(m) \leq 41$ otherwise each of $P(m), P((m+1)(m+2)), P((m+3)(m+4))$ is > 41 which is not possible. Again $P(m+2) \leq 41$ otherwise each of $P(m(m+1)), P(m+2), P((m+3)(m+4))$ is > 41 . Therefore $P(m(m+2)) \leq 41$ implying $P(m(m+2)) \leq 11$. If m is odd, then $m = N - 2$ for $N \in N_2(11)$ and we check that there is a prime $p > 11, p \notin \{43, 47\}$ with $P \mid \Delta(m, k)$ which is a contradiction. Thus m is even and we have $P(\frac{m}{2}(\frac{m}{2}+1)) \leq 11$ implying $m = 2N - 2$ with $N \in N_1(11)$. This is again not possible as above.

Let $k = 3$. Then $P(\Delta(m, k)) \leq 53$ by Corollary 2.2 (i). Also $P_1 > 41$ since $m+j \notin N_1(41)$ for $1 \leq j < k$. Further $P((m+1)(m+2)) > 41$ if $P(m) > 41$ and $P(m(m+1)) > 41$ if $P(m+2) > 41$ which are excluded by Corollary 2.2 (iii) as above. Thus we may suppose that $P_1 = P(m+1) > 41$ and $P(m(m+2)) \leq 41$. If m is even, then $m = 2N - 2$ for $N \in N_1(41)$ and we check that either $P_1 > 53$ or $a > 50$ for $a \in \mathfrak{B}\{P_1, P_2, \dots\}$. Thus m is odd. If $P(m(m+2)) \leq 31$, then $m = N - 2$ with $N \in N_2(31)$ and we check that either $P_1 > 53$ or $a > 50$ for $a \in \mathfrak{B}\{P_1, P_2, \dots\}$ which is excluded. Thus $P_2 = P(m(m+2)) \in \{37, 41\}$ which together with $41 < P_1 \leq 53$ imply $a > 50$ for $a \in \mathfrak{B}\{P_1, P_2\}$ except when $P_1 = 43, P_2 = 41$ where $a = 40 \in \mathfrak{B}\{P_1, P_2\}$. Thus

$a = 40$, $P(m+1) = 43$ and $P(m(m+2)) = 41$. Further by Corollary 2.2 (iii), we may assume $p \in \{2, 3, 7, 41, 43\}$ for $p|\Delta(m, 3)$ and $2 \cdot 43|(m+1)$. By looking at the possible prime factorisations of $m, m+1, m+2$ and taking $(m+2) - m$ or $m - (m+2)$, we have the following possibilities.

$$\begin{aligned} m+1 &= 2^r \cdot 7^y \cdot 43^t, & 3^x - 41^z &= \pm 2; \\ m+1 &= 2^r \cdot 3^x \cdot 43^t, & 7^y - 41^z &= \pm 2; \\ m+1 &= 2^r \cdot 43^t, & 3^x - 41^z &= \pm 2; \\ m+1 &= 2^r \cdot 43^t, & 3^x \cdot 7^y - 41^z &= \pm 2; \\ m+1 &= 2^r \cdot 43^t, & 3^x - 7^y \cdot 41^z &= \pm 2; \\ m+1 &= 2^r \cdot 43^t, & 7^y - 3^x \cdot 41^z &= \pm 2; \end{aligned}$$

where r, x, y, z, t are positive integers. The second and fourth equations are excluded by using modulo 7. Using modulo 8 for the remaining possibilities, we get the following four simultaneous equations

$C1$	$3^x - 41^z = 2,$	$3^x - 2^r \cdot 7^y \cdot 43^t = 1,$	$2^r \cdot 7^y \cdot 43^t - 41^z = 1,$	x odd
$C2$	$3^x - 41^z = 2,$	$3^x - 2^r \cdot 43^t = 1,$	$2^r \cdot 43^t - 41^z = 1,$	x odd
$C3$	$3^x - 7^y \cdot 41^z = 2,$	$3^x - 2^r \cdot 43^t = 1,$	$2^r \cdot 43^t - 7^y \cdot 41^z = 1$	
$C4$	$3^x \cdot 41^z - 7^y = 2,$	$3^x \cdot 41^z - 2^r \cdot 43^t = 1,$	$2^r \cdot 43^t - 7^y = 1$	

with x odd in $C1$ and $C2$. If $4|2^r$ in $C2$, we get a contradiction modulo 4 since x is odd. Thus $2^r = 2$ in $C2$. Taking modulo 7 in all the possibilities, we find that $C1$ is excluded since x is odd. Further $6|(x-1)$ in $C2$; $6|(x-2)$, $3|r$ in $C3$ and $3|r$ in $C4$. Note that $x \geq 2$. Taking modulo 9 again, we find that $3|(z+1)$ in $C2$; $3|t$ in $C3$ and $3|t$, $3|(y-1)$ in $C4$. Thus we have $(-41^{\frac{z+1}{3}})^3 + 3 \cdot 41(3^{\frac{x-1}{3}})^3 = 2 \cdot 41$ in $C2$, $(-2^{\frac{r}{3}} \cdot 43^{\frac{t}{3}})^3 + 9(3^{\frac{x-2}{3}})^3 = 1$ in $C3$ and $(2^{\frac{r}{3}} \cdot 43^{\frac{t}{3}})^3 + 7(-7^{\frac{y-1}{3}})^3 = 1$ in $C4$. We solve the Thue equations $X^3 + 123Y^3 = 82$, $X^3 + 9Y^3 = 1$ and $X^3 + 7Y^3 = 1$ with X, Y integers in *Pari* to find that it is not possible.

We recall that Theorem 2 follows from Theorem 1 when $k \geq 10$. Therefore we prove Theorem 1 with $k \geq 10$ in Sections 7, 8 and this will complete the proofs of Theorems 1 and 2.

7. Proof of Theorem 1 for $k \geq 10$

We may suppose by Corollary 2.2 (i) that $P(\Delta(m, k)) \leq a + k \leq 6k$. Let $k \leq 17$. We may suppose that $\max(30, 1.5k) < a \leq 5k$. First assume that $m + j \notin N_1(41)$ for any $1 \leq j < k$. Let

$$\mathfrak{L}_i(k, a) := \left\{ p : \max\left(41, \frac{a}{i}\right) < p \leq \frac{a+k}{i} \right\} \text{ for } 1 \leq i \leq 5$$

and $\ell(k) := \max_{1.5k < a \leq 5k} |\cup_{i=1}^5 \mathfrak{L}_i(k, a)|$. There are at most $\ell(k)$ primes > 41 dividing $\Delta(a+1, k)$ and we delete terms in $\{m, m+1, \dots, m+k-1\}$ divisible by those primes. We are left with at least $k - \ell(k)$ terms. We observe that the prime factors of each of these terms are at most 41 otherwise the assertion follows by Lemma 1. Call by U the maximum of these terms. From [Leh64, Tables IA], we may assume that each of these terms is at least at a distance 2 from the preceding one. Thus $m+k-1 \geq U \geq m+2(k-\ell(k)-1)$. Hence we have a contradiction if $k-2\ell(k)-1 > 0$. This is the case since $\ell(k) = 2, 3, 4, 5$ when $k = 10, k \in \{11, 12\}, k \in \{13, 14\}, k \in \{15, 16, 17\}$, respectively. Therefore we suppose that $m+j_0 = N \in N_1(41)$ for some $1 \leq j_0 \leq k-1$. Then $\Delta(m, k) = M_{j_0}(N, k)$. We check that $P(M_j(N, 7)) > 102$ for $1 \leq j < 7$ when $N > 10000$ and $N \in N_1(41)$. Thus $m < N \leq 10000$. For each $m < 10000$, we check that $P(\Delta(m, 10)) > 102$ for $m \geq 118$. Therefore $P(\Delta(m, k)) > 6k$ when $m \geq 118$. Further we find that $p_{i+1} - p_i \leq 10$ for $p_i < 118$. Hence for $m < 118$, $P(\Delta(m, k)) \geq m$ since $k \geq 10$. Therefore we have $P(\Delta(m, k)) \geq \min(m, 6k+1) > k+a$ for all m . Now the assertion follows by Corollary 2.2 (i).

Thus $k \geq 18$. First we check that $\omega_1 < k$ for $k \leq 100$ with together with (20) and Corollary

4.3 imply $\omega_1 < k$ for all k . Suppose $m \leq 10^{10}$. If at least one of $m, m+1, \dots, m+k-1$ is a prime, then $P(\Delta(m, k)) \geq m > k+a$ and hence the assertion follows from Corollary 2.2 (i). Hence we may suppose that each of $m, m+1, \dots, m+k-1$ is composite. By Lemma 4.4, we obtain $\omega(\Delta(m, k)) \geq k > \omega_1$ which contradicts (19). Therefore we have $m > 10^{10}$ which imply $k > 500$ by (19) and (16) with $t = \omega_1$.

By (19) and (20), we have

$$(21) \quad \omega(\Delta(m, k)) \leq \omega_2.$$

Taking $t = \omega_2$ in (21), we obtain from (17) and Lemma 4.1 (vi) and $k > 500$ that

$$(22) \quad m^{k-\omega_2} < (k-1)! = \frac{k!}{k} \leq \frac{\sqrt{2\pi k}}{k} \left(\frac{k}{e}\right)^k \exp\left(\frac{1}{12k}\right) < \left(\frac{k}{e}\right)^k.$$

Since $m \geq 10^{10}$, we get

$$\log k - 1 > \frac{(k - \omega_2) \log m}{k} \geq 10(\log 10) \left(1 - \frac{\omega_2}{k}\right).$$

By using estimates of $\pi(\nu)$ from Lemma 4.1 (i), we obtain

$$k > \exp \left(1 + 10(\log 10) \left(1 - \frac{\frac{6}{5}}{\log \frac{6k}{5}} \left(1 + \frac{1.2762}{\log \frac{6k}{5}} \right) - \sum_{j=1}^4 \frac{1}{j \log \frac{k}{j}} \left(1 + \frac{1.2762}{\log \frac{k}{j}} \right) \right) \right) =: J(k)$$

Since $J(k)$ is an increasing function of k and $k > 500$, we have $k > J(500) \geq 4581$. Further $k > J(4581) \geq 578802$ and hence $k > J(578802) > 4.5 \times 10^7$. Let $m \leq 123k$. Then, by Corollary 4.2, there is a prime $P_1 \geq m$ such that $P_1 | \Delta(m, k)$. Since $m > a+k$, the assertion follows by Corollary 2.2 (i). Therefore we may suppose that $m > 123k$.

Assume that $m+k-1 \geq k^{\frac{3}{2}}$. Then $m > \frac{k^{\frac{3}{2}}}{e}$ and we get from (22) and Corollary 4.3 that

$$k^k > \left(k^{\frac{3}{2}}\right)^{k-\pi(4k)}$$

which together with estimate of $\pi(\nu)$ from Lemma 4.1 imply

$$0 > \frac{k - 3\pi(4k)}{k} \geq 1 - \frac{12}{\log 4k} \left(1 + \frac{1.2762}{\log 4k} \right).$$

The right hand expression is an increasing function of k and the inequality does not hold at $k = 10^6$. Therefore $m+k-1 < k^{\frac{3}{2}}$. By Lemma 4.5, we get

$$\binom{m+k-1}{k} \leq (2.83)^{k+k^{\frac{3}{4}}} k^{\frac{3}{2}(\pi(4k)-\pi(k))}$$

since $|\{i : P(m+i) \leq k\}| \geq k - (\pi(4k) - \pi(k))$ by (21) and Corollary 4.3. On the other hand, we have $m > 123k$ implying

$$\begin{aligned} \binom{m+k-1}{k} &\geq \binom{124k}{k} = \frac{(124k)!}{k!(123k)!} > \frac{\sqrt{2\pi(124k)} \left(\frac{124k}{e}\right)^{124k}}{\sqrt{2\pi k} \left(\frac{k}{e}\right)^k \exp \frac{1}{12k} \sqrt{2\pi(123k)} \left(\frac{123k}{e}\right)^{123k} \exp \frac{1}{123k}} \\ &> \frac{0.4}{\sqrt{k}} e^{-\frac{1}{8k}} (335.7)^k \end{aligned}$$

using estimates of $\nu!$ from Lemma 4.1. Comparing the upper and lower bounds, we obtain

$$0 > \log(0.4) - \frac{1}{8k} - .5 \log k + k \log\left(\frac{335.7}{2.83}\right) - k^{\frac{3}{4}} \log(2.83) - \frac{3}{2}(\pi(4k) - \pi(k)) \log k.$$

By using estimates of $\pi(\nu)$ from Lemma 4.1, we obtain

$$\begin{aligned} \frac{(\pi(4k) - \pi(k)) \log k}{k} &\leq \frac{4 \log k}{\log 4k} \left(1 + \frac{1.2762}{\log 4k}\right) - \frac{\log k}{\log k - 1} \\ &\leq 4 \left(1 - \frac{\log 4}{\log 4k}\right) \left(1 + \frac{1.2762}{\log 4k}\right) - 1 \\ &\leq 4 \left(1 - \frac{\log 4 - 1.2762}{\log 4k}\right) - 1 \leq 3. \end{aligned}$$

Therefore we have

$$0 > \frac{\log(0.4) - \frac{1}{8k} - .5 \log k}{k} + \log\left(\frac{335.7}{2.83}\right) - k^{-\frac{1}{4}} \log(2.83) - 4.5.$$

The right hand side of the above inequality is an increasing function of k and the inequality is not valid at $k = 10^6$. This is a contradiction. \square

8. Proof of Theorems 3

By Theorem 2, we restrict to those triplets (n, a, k) given in the statement of Theorem 2 with $a \leq 12$. We now factorise $f_a(x)$ with $a_0 a_n = \pm 1, a_1 = a_2 = \dots = a_{n-1} = 1$ to find that these $f_a(x)$ are irreducible. Hence the assertion follows. \square

9. Proof of Theorem 4

For the proof of Theorem 4, we put $\alpha = a$ through out this section. As remarked in Section 1 after the statement of Theorem 4, we may assume that $10 < a \leq 40$. For $n \leq 18$ and $n \in \{24, 25, 27, 30, 32, 36, 45, 48, 54, 60, 64, 72, 75, 80, 90, 112, 120\}$, we find that $L_n^{(a)}(x)$ is irreducible except for (n, a) listed in Theorem 4. Thus we assume $n > 18, n \notin \{24, 25, 27, 30, 32, 36, 45, 48, 54, 60, 64, 72, 75, 80, 90, 112, 120\}$. Assume that $L_n^{(\alpha)}(x)$ is reducible. Then $L_n^{(\alpha)}(x)$ has a factor of degree k with $1 \leq k \leq \frac{n}{2}$. First we prove the following lemma.

LEMMA 9.1. *Let $k \geq 2$. Then $L_n^{(a)}(x)$ has no factor of degree k .*

Proof. Let $k \geq 2$ and $a \leq 40$ if $k = 2$. We may restrict to those (n, k, a) given in the exception of Theorem 2. For each of these triplets (n, k, a) , we first check if there is a prime $p \geq k + 2$ with (10) such that either (8) or (9) is satisfied and they are excluded by Lemma 1. We are now left with triples (n, k, a) given by $k = 2, (n, a) \in \{(100, 21), (40, 24), (256, 33), (42, 40)\}$. For these (n, a) , we check that $L_n^{(a)}(x)$ are irreducible.

Let $k = 2$ and $40 < a \leq 50$. Suppose $n \notin N_1(23)$ and $n+a \notin N_1(23)$. Then $P_1 = P(n(n-1)) > 23$ and $P_2 = P((n+a)(n+a-1)) > 23$. Further either $P_1 \nmid (a+1)(a+2)$ or $P_2 \nmid (a+1)(a+2)$ and the assertion follows by Lemma 1. Therefore we may assume that either $n = N \in N_1(23)$ or $n+a = N \in N_1(23)$. Further we may also suppose that $P(n(n-1)(n+a)(n+a-1)) \leq P((a+1)(a+2))$ otherwise assertion follows by Lemma 1. For $N \in N_1(23)$ and $N > 10000$, we check that $P((N-a)(N-a-1)) > P((a+1)(a+2))$ and $P((N+a)(N+a-1)) > P((a+1)(a+2))$ except when $(a, N) = (45, 10648), (46, 12168)$ where $P(N(N-1)) = 13, 23$, respectively. Observe that $N(N-1) | n(n-1)(n+a)(n+a-1)$. By taking $p = P(N(N-1))$, the assertion follows from Lemma 1. We now consider $n \leq 10000$. Let a be given. By Lemma 1, we first restrict to those n for which $P(n(n-1)(n+a)(n+a-1)) \leq P((a+1)(a+2))$. Further we check that there is a prime $p | n(n-1)(n+a)(n+a-1), p > 7$ and $p \nmid (a+1)(a+2)$. Lemma 1 implies the assertion now. \square

By Lemma 9.1, we need to consider $k = 1$. If there is a prime $p | n(n+a), p \nmid (a+1)$ with either

$p \geq 11$ or $p = 7, a \neq 47$ or $p = 5, a \notin \{23, 48\}$ or $p = 3, a \notin \{16, 24, 25, 34, 43\} =: S_1$, then the assertion follows by Lemma 1 and Corollary 2.1. Let $P_a = \{2\} \cup \{p : p|(a+1)\}$ if $a \notin S_1 \cup \{23, 47, 48\}$, $P_a = \{2, 3\} \cup \{p : p|(a+1)\}$ if $a \in S_1$, $P_a = \{2, 3, 5\}$ if $a = 23$, $P_a = \{2, 3, 7\}$ if $a = 47$ and $P_a = \{2, 5, 7\}$ if $a = 48$. Thus for a given a , we may assume that $p|n(n+a)$ implies $p \in P_a$.

Let a be given. Let $p|n$ with $p > 2$. Then $p \in P_a$. As in the proof of Lemma 1, if we have $\phi'_j < 1$ for all $1 \leq j \leq n$, then $L_n^{(\alpha)}(x)$ does not have a linear factor and we are done. Let $1 \leq j \leq 50$. We compute ϕ_j to find that $\phi_j < 1$ for $j > 1$ except when $(p, a) \in T_1 := \{(3, 16), (3, 17), (5, 23), (5, 24), (3, 34), (3, 35), (3, 43), (3, 44), (5, 48), (5, 49), (7, 47), (7, 48)\}$ where $\phi_j < 1$ for $j > 2$ and except when $23 \leq a \leq 26, p = 3$ where $\phi_j < 1$ for $j > 4$. Let $j > 50$. By using $\text{ord}_p(s!) \leq \frac{s}{p-1}$, we find that

$$\phi_j = \frac{\text{ord}_p((a+j)!) - \text{ord}_p(a!)}{j} \leq \frac{\frac{a+j}{p-1} - \text{ord}_p(a!)}{j} \leq \frac{1}{p-1} + \frac{\frac{a}{p-1} - \text{ord}_p(a!)}{51} < 1.$$

It suffices to show that $\phi'_1 < 1$ except when $(p, a) \in T_1$ for which we need to show $\phi'_j < 1, 1 \leq j \leq 2$ and except when $23 \leq a \leq 26, p = 3$ for which we need to show $\phi'_j < 1$ for $1 \leq j \leq 4$. Let $\phi'_0 = \max\{\phi'_i\}$ for $1 \leq i \leq 4$. It suffices to show $\phi'_0 < 1$ is always valid. This is the case except when $a \in \{24, 49\}, p = 5; a \in \{17, 24, 25, 26, 35, 44\}, p = 3$ and $a = 48, p = 7$. Further $\text{ord}_5(n) \leq 1$ when $a \in \{24, 49\}$, $\text{ord}_7(n) \leq 1$ when $a = 48$, $\text{ord}_3(n) \leq 1$ when $a \in \{17, 24, 25, 35, 44\}$ and $\text{ord}_3(n) \leq 2$ when $a = 26$ otherwise $\phi'_0 < 1$. Let $a \in \{17, 26, 35\}$ and $3||n$ or $9||n$. Then from $n(n+a) = 2^\alpha 3^{\beta_3}$ and $\text{gcd}(n, n+a) \leq 2$, we obtain $n \in \{3, 6, 9, 18\}$ which is not possible. Let $a = 49$ and $5||n$. Then from $n(n+a) = 2^\alpha 5^{\beta_5}$ and $\text{gcd}(n, n+a) = 1$, we obtain $n = 5$ which is again not possible.

Therefore n is a power of 2 except when $a = 24$ where $\text{ord}_3(n) \leq 1$ or $\text{ord}_5(n) \leq 1; a = 25$ where $\text{ord}_3(n) \leq 1; a = 44$ where $\text{ord}_3(n) \leq 1$ and $a = 48$ where $\text{ord}_7(n) \leq 1$. From the definition of P_a , we observe that $n(n+a)$ has at most two odd prime factors except when $a = 34$ where it has at most three odd prime factors. Hence we always have $n, n+a$ of the form

$$(23) \quad \begin{aligned} n = 2^{\alpha+\delta}, \quad \frac{n+a}{2^\delta} &= p^{\beta_p} && \text{if } P_a = \{2, p\} \\ n = 2^{\alpha+\delta}, \quad \frac{n+a}{2^\delta} &\in \{p_1^{\beta_{p_1}}, p_2^{\beta_{p_2}}, p_1^{\beta_{p_1}} p_2^{\beta_{p_2}}\} && \text{if } P_a = \{2, p_1, p_2\} \\ n = 2^{\alpha+\delta}, \quad \frac{n+a}{2^\delta} &\in \{p_1^{\beta_{p_1}}, p_2^{\beta_{p_2}}, p_3^{\beta_{p_3}}, p_1^{\beta_{p_1}} p_2^{\beta_{p_2}}, p_1^{\beta_{p_1}} p_3^{\beta_{p_3}}, \\ & \quad p_2^{\beta_{p_2}} p_3^{\beta_{p_3}}, p_1^{\beta_{p_1}} p_2^{\beta_{p_2}} p_3^{\beta_{p_3}}\} && \text{if } P_a = \{2, p_1, p_2, p_3\}. \end{aligned}$$

where $2^\delta || a$ and in addition $n, n+a$ is of the form

$$(24) \quad \begin{aligned} n = 15 \cdot 2^{\alpha+3}, \quad n+a &= 8 \cdot 3^{\beta_3+1} && \text{if } a = 24 \\ \text{or } n = 3 \cdot 2^{\alpha+3}, \quad n+a &\in \{8 \cdot 3^{\beta_3+1}, 8 \cdot 3^{\beta_3+1} 5^{\beta_5}\} && \text{if } a = 24 \\ n = 3 \cdot 2^\alpha, \quad n+a &= 13^{\beta_{13}} && \text{if } a = 25 \\ n = 3 \cdot 2^{\alpha+2}, \quad n+a &= 4 \cdot 5^{\beta_5} && \text{if } a = 44 \\ n = 7 \cdot 2^{\alpha+4}, \quad n+a &= 16 \cdot 5^{\beta_5} && \text{if } a = 48. \end{aligned}$$

Here all the exponents of odd prime powers appearing in (23) and (24) are positive. For $n < 512$ and n of the form given by (23) or (24) which are given by $n \in \{96, 128, 192, 224, 240, 256, 384, 448, 480\}$, we check that there is a prime $p|(n+a), p \notin P_a$ except when $(n, a) \in \{(256, 14), (128, 16), (256, 16), (96, 24), (192, 24), (256, 32), (256, 33), (128, 34)\}$. We find that for each of these (n, a) , the polynomial $L_n^{(a)}(x)$ is irreducible. Therefore we have $n \geq 512$.

From the equality $\frac{n+a}{2^\delta} - \frac{n}{2^\delta} = \frac{a}{2^\delta}$, we obtain an equation of the form

$$p^{\beta_p} - 2^\alpha = \frac{a}{2^\delta} \quad \text{or} \quad p_1^{\beta_{p_1}} p_2^{\beta_{p_2}} - 2^\alpha = \frac{a}{2^\delta}$$

or further $3^{\beta_3} 5^{\beta_5} 7^{\beta_7} - 2^\alpha = 17$ (only when $a = 34$) or $3^{\beta_3} - 5 \cdot 2^\alpha = 1$ (only when $a = 24$) or $13^{\beta_{13}} - 3 \cdot 2^\alpha = 25$ (only when $a = 25$) or $5^{\beta_5} - 3 \cdot 2^\alpha = 11$ (only when $a = 44$) or $5^{\beta_5} - 7 \cdot 2^\alpha = 3$

(only when $a = 48$). In each of the equations thus obtained, we note that $8|2^\alpha$ since $n \geq 512$. Out of all the equations, we need to consider only those which are valid modulo 8 and hence we restrict to those equations. Here we use $p^{\beta_p} \equiv 1$ or p modulo 8 according as β_p is even or odd, respectively. They are now expressed as the Thue equation

$$X^3 + AY^3 = B$$

and we solve them in **PariGp**. For instance, let $a = 32$. Then we obtain equations of the form $3^{\beta_3} - 2^\alpha = 1$, $11^{\beta_{11}} - 2^\alpha = 1$, $3^{\beta_3}11^{\beta_{11}} - 2^\alpha = 1$. By modulo 8, we find that $\beta_3, \beta_{11}, \beta_3 + \beta_{11}$ are even for the first, second and third equation, respectively. This imply $3^{\frac{\beta_3}{2}} - 1 = 2, 3^{\frac{\beta_3}{2}} + 1 = 2^{\alpha-1}$ giving $3^{\beta_3} = 9, 2^\alpha = 8$ for the first equation and $11^{\frac{\beta_{11}}{2}} - 1 = 2, 11^{\frac{\beta_{11}}{2}} + 1 = 2^{\alpha-1}$ giving a contradiction for the second equation. Observe that $2^\alpha > 8$ since $n \geq 512$. Thus we are left with $3^{\beta_3}11^{\beta_{11}} - 2^\alpha = 1$. For some $0 \leq r, s, t \leq 2$, we have $\alpha + r, \beta_3 - s, \beta_{11} - t$ all multiples of 3 and from $-2^{\alpha+r} + 2^r3^s11^t3^{\beta_3-s}11^{\beta_{11}-t} = 2^r$, we obtain the Thue equations $X^3 + AY^3 = B$ with $B = 2^r, A = 2^r3^s11^t, 0 \leq r, s, t \leq 2$ and X a power of 2 and $33|AY$. There are 27 possibilities of pairs (A, B) . If $A = 1$, then $B = 1$ and we factorise $X^3 + Y^3$ to get a contradiction and the case $A = 1$ is excluded. For all other values of (A, B) than those given by $t = 2$, we check in **PariGp** that none of solutions (X, Y) of Thue equations thus obtained satisfy the condition X a power of 2 and $33|AY$ except when $A = 66, B = 2$ where $X = -4$ and $Y = 1$ from which we obtain $n = 1024$. When $t = 2$, from $3^{\beta_3-s+3}11^{\beta_{11}-2+3} - 2^{3-r}3^{3-s} \cdot 11 \cdot 2^{\alpha+r-3} = 3^{3-s} \cdot 11$, we obtain the Thue equations $X^3 + AY^3 = B$ with $B = 3^{3-s} \cdot 11, A = 2^{3-r}3^{3-s} \cdot 11, 0 \leq r, s \leq 2$ and $33|X$ and Y is a power of 2. We check again in **PariGp** that none of solutions (X, Y) of these Thue equations thus satisfy the condition $33|X$ and Y is a power of 2. Hence we need to consider $n = 1024$ when $a = 32$. For another example, let $a = 48$. We obtain the equations of the form $5^{\beta_5} - 2^\alpha = 3, 7^{\beta_7} - 2^\alpha = 3, 5^{\beta_5} - 7 \cdot 2^\alpha = 3$ and $5^{\beta_5}7^{\beta_7} - 2^\alpha = 3$. First three equations are excluded modulo 8 and for the last equation, we find that β_5, β_7 are both odd. Modulo 7 imply $3|(\alpha - 2)$ or $3|(\alpha + 1)$ and hence from the equation $-2^{\alpha+1} + 2 \cdot 5^{\beta_5}7^{\beta_7} = 6$, we obtain the Thue equations $X^3 + AY^3 = B$ with $B = 6, A = 2 \cdot 5^s7^t, 0 \leq s, t \leq 2$ and X a power of 2 and $70|AY$. When $t = 2$, from $5^{\beta_5-s+3}7^{\beta_7+1} - 4 \cdot 5^{3-s} \cdot 7 \cdot 2^{\alpha-2} = 3 \cdot 5^{3-s} \cdot 7$, we obtain the Thue equations $X^3 + AY^3 = B$ with $B = 21 \cdot 5^{3-s}, A = 28 \cdot 5^{3-s}, 0 \leq s \leq 2$ and $35|X$ and Y is a power of 2. We check in **PariGp** that all the solutions (X, Y) of these Thue equations are excluded except when $(A, B) = (70, 6)$ where $X = -4, Y = -1$ and we obtain $n = 512$. Hence we need to consider $n = 512$ when $a = 48$. Similarly, all other a 's are excluded except when $a \in \{20, 24\}$ where we obtain $(n, a) \in \{(4096, 20), (1920, 24)\}$.

Thus we now exclude the cases $(n, a) \in \{(4096, 20), (1920, 24), (1024, 32), (512, 48)\}$. We take $p = 2$ and show that $\phi'_j < 1$ for all $1 \leq j \leq n$. This is shown by checking $\text{ord}_2(\Delta_j) - \text{ord}_2\left(\binom{n}{j}\right) < j$ for j such that $\text{ord}_2(\Delta_j) \geq j$ for these pairs (n, a) . Hence they are all excluded. \square

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