

# IRREDUCIBILITY OF GENERALIZED HERMITE-LAGUERRE POLYNOMIALS

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## 1. INTRODUCTION

Let  $n$  and  $1 \leq \alpha < d$  be positive integers with  $\gcd(\alpha, d) = 1$ . Any positive rational  $q$  is of the form  $q = u + \frac{\alpha}{d}$  where  $u$  is a non-negative integer. For integers  $a_0, a_1, \dots, a_n$ , let

$$G(x) := G_q(x) = a_n x^n + a_{n-1}(\alpha + (n-1+u)d)x^{n-1} + \dots + a_1 \left( \prod_{i=1}^{n-1} (\alpha + (i+u)d) \right) x + a_0 \left( \prod_{i=0}^{n-1} (\alpha + (i+u)d) \right).$$

This is an extension of Hermite polynomials and generalized Laguerre polynomials. Therefore we call  $G(x)$  the generalized Hermite-Laguerre polynomial. We prove

**Theorem 1.** *Let  $k \geq 1$  and  $a_0 a_n \in \{\pm 3^t : t \geq 0, t \in \mathbb{Z}\}$ . Then  $G_{\frac{1}{3}}$  and  $G_{\frac{2}{3}}$  do not have a factor of degree  $k \in [2, n-2]$ .*

**Theorem 2.** *Let  $k \geq 1$ ,  $0 \leq u \leq k$  and  $a_0 a_n \in \{\pm 2^t : t \geq 0, t \in \mathbb{Z}\}$ . Then  $G_{u+\frac{1}{2}}$  does not have a factor of degree  $k$  except possibly when  $k \in \{1, n-1\}$ ,  $u = 1$ .*

Schur [Sch29] proved that  $G_{\frac{1}{2}}(x^2)$  with  $a_n = \pm 1$  and  $a_0 = \pm 1$  are irreducible and this implies the irreducibility of  $H_{2n}$  where  $H_m$  is the  $m$ -th Hermite polynomial. Schur [Sch73] also established the irreducibility of Hermite polynomials  $H_{2n+1}$  by showing that  $G_{\frac{3}{2}}(x^2)$  with  $a_n = \pm 1$  and  $a_0 = \pm 1$  is irreducible except for some explicitly given finitely many values of  $n$  where it can have a quadratic factor. Further Allen and Filaseta [AlFi04] showed that  $G_{\frac{1}{2}}$  with  $a_{2j+1} = 0$  is irreducible. Finch and Saradha [FiSa] showed that  $G_{u+\frac{1}{2}}$  with  $0 \leq u \leq 13$  have no factor of degree  $k \in [2, n-2]$  except for an explicitly given finite set of values of  $u$  where it may have a factor of degree 2.

From now onwards, we always assume  $d \in \{2, 3\}$ . For an integer  $\nu > 1$ , we denote by  $P(\nu)$  the the greatest prime factor of  $\nu$  and we put  $P(1) = 1$ . A new ingredient in the proofs of Theorems 1 and 2 is the following result which we shall prove in Section 3.

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2000 Mathematics Subject Classification: Primary 11A41, 11B25, 11N05, 11N13, 11C08, 11Z05.  
Keywords: Irreducibility, Hermite-Laguerre Polynomials, Arithmetic Progressions, Primes.

**Theorem 3.** *Let  $k \geq 2$  and  $d = 2, 3$ . Let  $m$  and  $k$  be positive integers such that  $d \nmid m$  and  $m > dk$ . Then*

$$(1) \quad P(m(m+d) \cdots (m+d(k-1))) > \begin{cases} 3.5k & \text{if } d = 2 \text{ and } m \leq 2.5k \\ 4k & \text{if } d = 2 \text{ and } m > 2.5k \\ 3k & \text{if } d = 3 \end{cases}$$

*unless  $(m, k) \in \{(5, 2), (7, 2), (25, 2), (243, 2), (9, 4), (13, 5), (17, 6), (15, 7), (21, 8), (19, 9)\}$  when  $d = 2$  and  $(m, k) = (125, 2)$  when  $d = 3$ .*

If  $d = 2, 3$  and  $m > dk$ , this is an improvement of [LaSh06a].

In Section 4, we shall combine Theorem 3 with the irreducibility criterion from [ShTi] (see Lemma 4.1) to derive Theorems 1 and 2. This criterion comes from Newton polygons. If  $p$  is a prime and  $m$  is a nonzero integer, we define  $\nu(m) = \nu_p(m)$  to be the nonnegative integer such that  $p^{\nu(m)} \mid m$  and  $p^{\nu(m)+1} \nmid m$ . We define  $\nu(0) = +\infty$ . Consider  $f(x) = \sum_{j=0}^n a_j x^j \in \mathbb{Z}[x]$  with  $a_0 a_n \neq 0$  and let  $p$  be a prime. Let  $S$  be the following set of points in the extended plane:

$$S = \{(0, \nu(a_n)), (1, \nu(a_{n-1})), (2, \nu(a_{n-2})), \dots, (n-1, \nu(a_1)), (n, \nu(a_0))\}$$

Consider the lower edges along the convex hull of these points. The left-most endpoint is  $(0, \nu(a_n))$  and the right-most endpoint is  $(n, \nu(a_0))$ . The endpoints of each edge belong to  $S$ , and the slopes of the edges increase from left to right. When referring to the edges of a Newton polygon, we shall not allow two different edges to have the same slope. The polygonal path formed by these edges is called the Newton polygon of  $f(x)$  with respect to the prime  $p$ . For the proof of Theorems 1 and 2, we use [ShTi, Lemma 10.1] which is derived from an extension of result due to Dumas [Duma06].

**Lemma 1.1.** *Let  $g(x)$  and  $h(x)$  be in  $\mathbb{Z}[x]$  with  $g(0)h(0) \neq 0$ , and let  $p$  be a prime. Let  $k$  be a non-negative integer such that  $p^k$  divides the leading coefficient of  $g(x)h(x)$  but  $p^{k+1}$  does not. Then the edges of the Newton polygon for  $g(x)h(x)$  with respect to  $p$  can be formed by constructing a polygonal path beginning at  $(0, k)$  and using translates of the edges in the Newton polygon for  $g(x)$  and  $h(x)$  with respect to the prime  $p$  (using exactly one translate for each edge). Necessarily, the translated edges are translated in such a way as to form a polygonal path with the slopes of the edges increasing.*

A part of this work was done when the authors were visiting Max-Planck Institute for Mathematics in Bonn during August-October, 2008 and February-April, 2009, respectively. We would like to thank the MPIM for the invitation and the hospitality. We also thank Pieter Moree for his comments on a draft of this paper. The authors are indebted to the referee for his suggestions and remarks which improved the exposition of the paper.

## 2. PRELIMINARIES FOR THEOREM 3

Let  $m$  and  $k$  be positive integers with  $m > kd$  and  $\gcd(m, d) = 1$ . We write

$$\Delta(m, d, k) = m(m + d) \cdots (m + (k - 1)d).$$

For positive integers  $\nu, \mu$  and  $1 \leq l < \mu$  with  $\gcd(l, \mu) = 1$ , we write

$$\begin{aligned} \pi(\nu, \mu, l) &= \sum_{\substack{p \leq \nu \\ p \equiv l \pmod{\mu}}} 1, \quad \pi(\nu) = \pi(\nu, 1, 1) \\ \theta(\nu, \mu, l) &= \sum_{\substack{p \leq \nu \\ p \equiv l \pmod{\mu}}} \log p. \end{aligned}$$

Let  $p_{i,\mu,l}$  denote the  $i$ th prime congruent to  $l$  modulo  $\mu$ . Let  $\delta_\mu(i, l) = p_{i+1,\mu,l} - p_{i,\mu,l}$  and  $W_\mu(i, l) = (p_{i,\mu,l}, p_{i+1,\mu,l})$ . Let  $M_0 = 1.92367 \times 10^{10}$ .

We recall some well-known estimates on prime number theory.

**Lemma 2.1.** *We have*

- (i)  $\pi(\nu) \leq \frac{\nu}{\log \nu} \left( 1 + \frac{1.2762}{\log \nu} \right)$  for  $\nu > 1$
- (ii)  $\nu \left( 1 - \frac{3.965}{\log^2 \nu} \right) \leq \theta(\nu) < 1.00008\nu$  for  $\nu > 1$
- (iii)  $\sqrt{2\pi k} e^{-k} k^k e^{\frac{1}{12k+1}} < k! < \sqrt{2\pi k} e^{-k} k^k e^{\frac{1}{12k}}$  for  $k > 1$
- (iv)  $\text{ord}_p(k!) \geq \frac{k-p}{p-1} - \frac{\log(k-1)}{\log p}$  for  $k > 1$  and  $p < k$ .

The estimates (i), (ii) are due to Dusart [Dus98, p.14], [Dus99]. The estimate (iii) is [Rob55, Theorem 6]. For a proof of (v), see [LaSh04b, Lemma 2(i)].  $\square$

The following lemma is due to Ramaré and Rumely [RaRu96, Theorems 1, 2].

**Lemma 2.2.** *Let  $l \in \{1, 2\}$ . For  $\nu_0 \leq 10^{10}$ , we have*

$$(2) \quad \theta(\nu, 3, l) \geq \begin{cases} \frac{\nu}{2}(1 - 0.002238) & \text{for } \nu \geq 10^{10} \\ \frac{\nu}{2} \left( 1 - \frac{2 \times 1.798158}{\sqrt{\nu_0}} \right) & \text{for } 10^{10} > \nu \geq \nu_0 \end{cases}$$

and

$$(3) \quad \theta(\nu, 3, l) \leq \begin{cases} \frac{\nu}{2}(1 + 0.002238) & \text{for } \nu \geq 10^{10} \\ \frac{\nu}{2} \left( 1 + \frac{2 \times 1.798158}{\sqrt{\nu_0}} \right) & \text{for } 10^{10} > \nu \geq \nu_0 \end{cases}.$$

We derive from Lemmas 2.1 and 2.2 the following result.

**Corollary 2.3.** *Let  $M_0 < m \leq 131 \times 2k$  if  $d = 2$  and  $6450 \leq m \leq 10.6 \times 3k$  if  $d = 3$ . Then  $P(\Delta(m, d, k)) \geq m$ .*

*Proof.* Let  $M_0 < m \leq 131 \times 2k$  if  $d = 2$  and  $6450 \leq m \leq 10.6 \times 3k$  if  $d = 3$ . Then  $k \geq k_1$  where  $k_1 = 7.34 \times 10^7, 203$  when  $d = 2, 3$ , respectively. Let  $1 \leq l < d$  and assume  $m \equiv l \pmod{d}$ . We observe that  $P(\Delta(m, d, k) \geq m)$  holds if

$$\theta(m + d(k-1), d, l) - \theta(m-1, d, l) = \sum_{\substack{m \leq p \leq m+(k-1)d \\ p \equiv l \pmod{d}}} \log p > 0.$$

Now from Lemmas 2.1 and 2.2, we have

$$\frac{\theta(m-1, d, l)}{\frac{m-1}{\phi(d)}} < \theta_1 := \begin{cases} 1.00008 & \text{if } d = 2 \\ 1 + \frac{2 \times 1.798158}{\sqrt{6450}} & \text{if } d = 3 \end{cases}$$

and

$$\frac{\theta(m + (k-1)d, d, l)}{\frac{m+(k-1)d}{\phi(d)}} > \theta_2 := \begin{cases} 1 - \frac{3.965}{\log^2(10^{10})} & \text{if } d = 2 \\ 1 - \frac{2 \times 1.798158}{\sqrt{6450}} & \text{if } d = 3. \end{cases}$$

Thus  $P(\Delta(m, d, k) \geq m)$  holds if

$$\theta_2(m + d(k-1)) > \theta_1 m$$

i.e., if

$$\frac{d(k-1)}{m} > \frac{\theta_1}{\theta_2} - 1.$$

This is true since for  $k \geq k_1$ , we have

$$\frac{dk(1 - \frac{1}{k})}{\frac{\theta_1}{\theta_2} - 1} \geq \frac{dk(1 - \frac{1}{k_1})}{\frac{\theta_1}{\theta_2} - 1} > (dk) \begin{cases} 131.3 & \text{if } d = 2 \\ 10.6 & \text{if } d = 3 \end{cases}$$

and  $m$  is less than the last expression. Hence the assertion.  $\square$

Now we give some results for  $d = 2$ . The next result follows from Lemma 2.1 (ii).

**Corollary 2.4.** *Let  $d = 2, k > 1$  and  $2k < m < 4k$ . Then*

$$(4) \quad P(\Delta(m, d, k)) > \begin{cases} 3.5k & \text{if } m \leq 2.5k \\ 4k & \text{if } m > 2.5k \end{cases}$$

unless  $(m, k) \in \{(5, 2), (7, 2), (9, 4), (13, 5), (17, 6), (15, 7), (21, 8), (19, 9)\}$ .

*Proof.* We observe that the set  $\{m, m+2, \dots, m+2(k-1)\}$  contains all primes between  $3.5k$  and  $4k$  if  $m \leq 2.5k$  and all primes between  $4k$  and  $4.5k$  if  $2.5k < m < 4k$ . Therefore (4) holds if

$$\theta(4k) > \theta(3.5k) \quad \text{and} \quad \theta(4.5k) > \theta(4k).$$

Let  $(r, s) = (3.5, 4)$  or  $(4, 4.5)$ . Then from Lemma 2.1, we see that  $\theta(sk) > \theta(rk)$  if

$$sk \left(1 - \frac{3.965}{\log^2(sk)}\right) > 1.00008 \times rk$$

or

$$\frac{s - 1.00008r}{1.00008r} > \frac{s}{1.00008r} \frac{3.965}{\log^2(sk)}$$

or

$$k > \frac{1}{s} \exp \left( \sqrt{\frac{3.965s}{s - 1.00008r}} \right).$$

This is true for  $k \geq 88$ . Thus  $k \leq 87$ . For  $10 \leq k \leq 87$ , we check that there is always a prime in the intervals  $(3.5k, 4k)$  and  $(4k, 4.5k)$  and hence (4) follows in this case. For  $2 \leq k \leq 9$ , the assertion follows by computing  $P(\Delta(m, 2, k))$  for each  $2k < m < 4k$ .  $\square$

The following result concerns Grimm's Conjecture, [LaSh06b, Theorem 1].

**Lemma 2.5.** *Let  $m \leq M_0$  and  $l$  be such that  $m+1, m+2, \dots, m+l$  are all composite numbers. Then there are distinct primes  $P_i$  such that  $P_i | (m+i)$  for each  $1 \leq i \leq l$ .*

As a consequence, we have

**Corollary 2.6.** *Let  $4k < m \leq M_0$ . Then either  $P(\Delta(m, 2, k)) > 4k$  or  $P(\Delta(m, 2, k)) \geq p_{k+1}$ .*

*Proof.* If  $m+2i$  is prime for some  $i$  with  $0 \leq i < k$ , then the assertion holds clearly since  $P(\Delta(m, 2, k)) \geq m+2i > 4k$ . Thus we suppose that  $m+2i$  is composite for all  $0 \leq i < k$ . Since  $m$  is odd, we obtain that  $m+2i+1$  with  $0 \leq i < k$  are all even and hence composite. Therefore  $m, m+1, m+2, \dots, m+2k-1$  are all composite and hence, by Lemma 2.5, there are distinct primes  $P_j$  with  $P_j | (m-1+j)$  for each  $1 \leq j \leq 2k$ . Therefore  $\omega(\Delta(m, 2, k)) \geq k$  implying  $P(\Delta(m, 2, k)) \geq p_{k+1}$ .  $\square$

**Corollary 2.7.** *Let  $d = 2$  and  $4k < m \leq M_0$ . Then  $P(\Delta(m, 2, k)) > 4k$  for  $k \geq 30$ .*

*Proof.* By Corollary 2.6, we may assume that  $P(\Delta(m, 2, k)) \geq p_{k+1}$ . By Lemma 2.1, we get  $p_{k+1} \geq k \log k$  which is  $> 4k$  for  $k \geq 60$ . For  $30 \leq k < 60$ , we check that  $p_{k+1} > 4k$ . Hence the assertion follows.  $\square$

The following result follows from [Leh64, Tables IIA, IIIA].

**Lemma 2.8.** *Let  $d = 2$ ,  $m > 4k$  and  $2 \leq k \leq 37, k \neq 35$ . Then  $P(\Delta(m, 2, k)) > 4k$ .*

*Proof.* The case  $k = 2$  is immediate from [Leh64, Table IIA]. Let  $k \geq 3$  and  $m \geq 4k$ . For  $m$  and  $1 \leq i < k$  such that  $m+2i = N$  with  $N$  given in [Leh64, Tables IIA, IIIA], we check that  $P(\Delta(m, 2, k)) > 4k$ . Hence assume that  $m+2i$  with  $1 \leq i < k$  is different from those  $N$  given in [Leh64, Tables IIA, IIIA].

For every prime  $31 < p \leq 4k$ , we delete a term in  $\{m, m+2, \dots, m+2(k-1)\}$  divisible by  $p$ . Let  $i_1 < i_2 < \dots < i_l$  be such that  $m+2i_j$  is in the remaining set where  $l \geq k - (\pi(4k) - \pi(31))$ . From [Leh64, Tables IIA, IIIA], we observe that  $i_{j+1} - i_j \geq 3$

implying  $k - 1 \geq i_l - i_1 \geq 3(l - 1) \geq 3(k - \pi(4k) + 10)$ . However we find that the inequality  $k - 1 \geq 3(k - \pi(4k) + 10)$  is not valid except when  $k = 28, 29$ . Hence the assertion of the Lemma is valid except possibly for  $k = 28, 29$ .

Therefore we may assume that  $k = 28, 29$ . Further we suppose that  $l = k - (\pi(4k) - \pi(31)) = 10$  otherwise  $3(l - 1) \geq 30 > k - 1$ , a contradiction. Thus we have either  $i_{10} - i_1 = 27$  implying  $i_1 = 0, i_{j+1} = i_j + 3 = 3j$  for  $1 \leq j \leq 9$  or  $i_1 = 1, i_{j+1} = i_j + 3 = 3j + 1$  for  $1 \leq j \leq 9$  or  $i_{10} - i_1 = 28$  implying  $i_1 = 0, i_{j+1} = \begin{cases} 3j & \text{if } 1 \leq j \leq r \\ 3j + 1 & \text{if } r < j \leq 9 \end{cases}$  for some  $r \geq 1$ . Let  $X = m + 2i_1 - 6$ . Note that  $X$  is odd since  $m$  is odd. Also  $X \geq 4k + 1 - 6 \geq 107$ . We have either

$$(5) \quad P((X + 6) \cdots (X + 54)(X + 60)) \leq 31$$

or there is some  $r \geq 1$  for which

$$(6) \quad P((X + 6) \cdots (X + 6r)(X + 6(r + 1) + 2) \cdots (X + 60 + 2)) \leq 31.$$

Note that (5) is the only possibility when  $k = 28$ . Now we consider (5). Suppose  $3|X$ . Then putting  $Y = \frac{X}{3}$ , we get  $P((Y+2) \cdots (Y+18)(Y+20)) \leq 31$  which implies  $Y+2 < 20$  by Corollary 2.4 and Lemma 2.8 with  $k = 10$ . Since  $X + 6 \geq m \geq 113$ , we get a contradiction. Hence we may assume that  $3 \nmid X$ . Then  $3 \nmid (X+6) \cdots (X+54)(X+60)$ . After deleting terms  $X + 6i$  divisible by primes  $11 \leq p \leq 31$ , we are left with three terms divisible by primes 5 and 7 and hence  $m \leq X + 6 \leq 35$  which is again a contradiction. Therefore (5) is not possible.

Now we consider (6) which is possible only when  $k = 29$ . Since  $X + 6 = m > 4k = 116$ , we have  $X > 110$ . Suppose  $r = 1, 9$ . Then we have  $P((X + 12 + 2) \cdots (X + 54 + 2)(X + 60 + 2)) \leq 31$  if  $r = 1$  and  $P((X + 6) \cdots (X + 54)) \leq 31$  if  $r = 9$ . Putting  $Y = X + 8$  in the first case and  $Y = X$  in the latter, we get  $P((Y+6) \cdots (Y+54)) \leq 31$ . Suppose  $3|Y$ . Then putting  $Z = \frac{Y}{3}$ , we get  $P((Z+2) \cdots (Z+18)) \leq 31$  which implies  $Z + 2 \leq 18$  by Corollary 2.4 and Lemma 2.8 with  $k = 9$ . Since  $Z + 2 \geq \frac{X}{3} > \frac{110}{3}$ , we get a contradiction. Hence we may assume that  $3 \nmid Y$ . Then  $3 \nmid (Y + 6) \cdots (Y + 54)$ . After deleting terms  $Y + 6i$  divisible by primes  $11 \leq p \leq 31$ , we are left with two terms divisible by primes 5 and 7 only. Let  $Y + 6i = 5^{a_1} 7^{b_1}$  and  $Y + 6j = 5^{a_2} 7^{b_2}$  where  $b_1 \leq 1 < b_2$  and  $a_2 \leq 1 < a_1$ . Since  $|i - j| \leq 8$ , the equality  $6(i - j) = 5^{a_1} 7^{b_1} - 5^{a_2} 7^{b_2}$  implies  $5^a - 7^b = \pm 6, \pm 12, \pm 18, \pm 24, \pm 36, \pm 48$ . By taking modulo 6, we get  $(-1)^a \equiv 1$  modulo 6 implying  $a$  is even. Taking modulo 8 again, we get either

$$b \text{ is even, } 5^a - 7^b = (5^{\frac{a}{2}} - 7^{\frac{b}{2}})(5^{\frac{a}{2}} + 7^{\frac{b}{2}}) = \pm 24, \pm 48$$

giving

$$(7) \quad 5^a = 25, 7^b = 49$$

or

$$b \text{ is odd, } 5^a - 7^b = -6, 18.$$

Let  $5^a - 7^b = -6$ . Considering modulo 5, we get  $2^b \equiv 1$  implying  $4|b$ , a contradiction. Let  $5^a - 7^b = 18$ . By considering modulo 7 and modulo 9 and since  $a$  is even, we get  $3|(a - 2)$  and  $3|(b - 1)$  implying  $(5^{\frac{a+1}{3}})^3 + 35(-7^{\frac{b-1}{3}})^3 = 90$ . Solving the

Thue equation  $x^3 + 35y^3 = 90$  gives  $x = 5, y = -1$  or  $25 - 7 = 18$  is the only solution. Hence  $6 \cdot 3 = 25 - 7 = X + 6i - (X + 6j)$ . Also the solution (7) implies  $-6 \cdot 4 = 25 - 49 = X + 6i - (X + 6j)$ . Thus  $X \leq 25$  which is not possible.

Assume now that  $2 \leq r \leq 8$ . Then  $P((X + 6)(X + 12)(X + 56)(X + 62)) \leq 31$ . Suppose  $3|X(X + 2)$ . Putting  $Y = \frac{X+6}{3}$  if  $3|X$  and  $Y = \frac{X+56}{3}$  if  $3|(X + 2)$ , we get either  $P(Y(Y + 2)(3Y + 50)(6Y + 56)) \leq 31$  or  $P(Y(Y + 2)(3Y - 50)(3Y - 44)) \leq 31$ . In particular  $P(Y(Y + 2)) \leq 31$ . For  $Y = N - 2$  given by [Leh64, Table IIA] such that  $P(Y(Y + 2)) \leq 31$ , we check that  $P((3Y + 50)(3Y + 56)) > 31$  and  $P((3Y - 50)(3Y - 44)) > 31$  except when  $Y \in \{55, 145, 297, 1573\}$ . This gives  $m = X + 6 = 3Y - 50$  and then we further check that  $P(\Delta(m, 2, k)) > 116$ . Hence we suppose  $3 \nmid X(X + 2)$ . Then  $3 \nmid (X + 6) \cdots (X + 6r)(X + 6(r + 1) + 2) \cdots (X + 60 + 2)$ . If a prime power  $p^a$  divides two terms of the product, then  $p^a|(X + 6j), p^a|(X + 6i)$  or  $p^a|(X + 6j + 2), p^a|(X + 6i + 2)$  or  $p^a|(X + 6j), p^a|(X + 6i + 2)$  for some  $i, j$ . Hence  $p^a|6(i - j)$  or  $p^a|6(i - j) + 2$ . Since  $1 \leq j < i \leq 10$ , we get  $p^a \in \{5, 7, 11, 13, 19, 25\}$ . After deleting terms divisible by primes  $5 \leq p \leq 31$  to their highest powers, we are left with two terms such that their product divides  $25 \cdot 7 \cdot 11 \cdot 13 \cdot 19$  and hence  $X + 6 \leq \sqrt{25 \cdot 7 \cdot 11 \cdot 13 \cdot 19}$  or  $X + 6 \leq 689$ . We check that  $P((X + 6)(X + 12)(X + 56)(X + 62)) > 31$  for  $110 \leq X \leq 683$  except when  $X \in \{113, 379\}$ . Further we check that  $P(\Delta(m, 2, k)) > 116$  for  $m = X + 6$ . Hence the result.  $\square$

The remaining results in this section deal with the case  $d = 3$ . The first one is a computational result.

**Lemma 2.9.** *Let  $l \in \{1, 2\}$ . If  $p_{i,3,l} \leq 6450$ , then  $\delta_3(i, l) \leq 60$ .*

As a consequence, we obtain

**Corollary 2.10.** *Let  $d = 3$  and  $3k < m \leq 6450$  with  $\gcd(m, 3) = 1$ . Then (1) holds unless  $(m, k) = (125, 2)$ .*

*Proof.* For  $k \leq 20$ , it follows by direct computation. For  $k > 20$ , (1) follows as  $3(k - 1) \geq 60$  and, by Lemma 2.9, the set  $\{m + 3i : 0 \leq i < k\}$  contains a prime.  $\square$

We shall also need the following result of Nagell [Nag58](see [Cao99]) on diophantine equations.

**Lemma 2.11.** *Let  $a, b, c \in \{2, 3, 5\}$  and  $a < b$ . Then the solutions of*

$$a^x + b^y = c^z \text{ in integers } x > 0, y > 0, z > 0$$

*are given by*

$$(a^x, b^y, c^z) \in \{(2, 3, 5), (2^4, 3^2, 5^2), (2, 5^2, 3^3), (2^2, 5, 3^2), (3, 5, 2^3), (3^3, 5, 2^5), (3, 5^3, 2^7)\}.$$

As a corollary, we have

**Corollary 2.12.** *Let  $X > 80, 3 \nmid X$  and  $1 \leq i \leq 7$ . Then the solutions of*

$$P(X(X + 3i)) = 5 \quad \text{and} \quad 2|X(X + 3i)$$

*are given by*

$$(i, X) \in \{(1, 125), (2, 250), (4, 500), (5, 625)\}.$$

*Proof.* Let  $1 \leq i \leq 7$ . We observe that  $2|X, 2|(X + 3i)$  only if  $X$  and  $i$  are both even and  $5|X, 5|(X + 3i)$  only if  $i = 5$ . Let the positive integers  $r, s$  and  $\delta = \text{ord}_2(i) \in \{0, 1, 2\}$  be given by

$$(8) \quad X = 2^{r+\delta}, \quad X + 3i = 2^\delta 5^s \quad \text{or} \quad X = 2^\delta 5^s, \quad X + 3i = 2^{r+\delta} \quad \text{if } i \neq 5$$

and

$$(9) \quad X = 5^{s+1}, \quad X + 3i = 5 \times 2^r \quad \text{or} \quad X = 5 \times 2^r, \quad X + 3i = 5^{s+1} \quad \text{if } i = 5,$$

where  $r + 2 \geq r + \delta \geq 7$  and  $s \geq 2$  since  $X > 80$ . Hence we have

$$(10) \quad 2^r - 5^s = \pm \left( \frac{X + 3i}{2^{\text{ord}_2(i)} \cdot 5^{\text{ord}_5(i)}} - \frac{X}{2^{\text{ord}_2(i)} \cdot 5^{\text{ord}_5(i)}} \right) = \pm 3 \times \frac{i}{2^{\text{ord}_5(i)} \cdot 5^{\text{ord}_5(i)}}.$$

Let  $i \in \{1, 2, 4, 5\}$ . Then  $2^r - 5^s = \pm 3$ . By Lemma 2.11, we have  $2^r = 2^7, 5^s = 5^3$  and  $2^7 - 5^3 = 3$  implying  $X = 2^{\text{ord}_2(i)} \cdot 5^{3+\text{ord}_5(i)}$  and  $X + 3i = 2^{7+\delta} \cdot 5^{\text{ord}_5(i)}$ . These give the solutions stated in the Corollary.

Let  $i \in \{3, 6\}$ . Then  $2^r - 5^s = \pm 9 = \pm 3^2$ . Since  $\min(2^r, 5^s) > 16$ , we observe from Lemma 2.11 that there is no solution.

Let  $i = 7$ . Then  $2^r - 5^s = \pm 21$ . Let  $s$  be even. Since  $2^r > 16$ , taking modulo 8, we find that  $-1 \equiv \pm 21 \pmod{8}$  which is not possible. Hence  $s$  is odd. Then  $2^r - 5^s \equiv 2^r + 2^s \equiv 0 \pmod{7}$ . Since  $2^r, 2^s \equiv 1, 2, 4 \pmod{7}$ , we get a contradiction.  $\square$

### 3. PROOF OF THEOREM 3

Let  $D = 4, 3$  according as  $d = 2, 3$ , respectively. Let  $v = \frac{m}{dk}$ . Assume that

$$(11) \quad P(\Delta(m, d, k)) = P(m(m + d) \cdots (m + (k - 1)d)) < Dk.$$

Then

$$(12) \quad \omega(\Delta(m, d, k)) \leq \pi(Dk) - 1.$$

For every prime  $p \leq Dk$  dividing  $\Delta$ , we delete a term  $m + i_p d$  such that  $\text{ord}_p(m + i_p d)$  is maximal. Note that  $p|(m + id)$  for at most one  $i$  if  $p \geq k$ . Then we are left with a set  $T$  with  $1 + t := |T| \geq k - \pi(Dk) + 1 := 1 + t_0$ . Let  $t_0 \geq 0$  which we assume in this section to ensure that  $T$  is non-empty. We arrange the elements of  $T$  as  $m + i'_0 d < m + i'_1 d < \cdots < m + i'_{t_0} d < \cdots < m + i'_t d$ . Let

$$(13) \quad \mathfrak{P} := \prod_{\nu=0}^{t_0} (m + i'_\nu d) \geq d^{k-\pi(Dk)+1} \prod_{i=0}^{k-\pi(Dk)} (vk + i).$$



We now apply [LaSh04b, Lemma 2.1, (14)] to get

$$\mathfrak{P} < (k-1)!d^{-\text{ord}_d(k-1)!}.$$

Comparing the upper and lower bounds of  $\mathfrak{P}$ , we have

$$d^{\pi(Dk)} > \frac{d^{k+1} \prod_{i=0}^{k-\pi(Dk)} (vk+i)}{(k-1)!d^{-\text{ord}_d(k-1)!}}$$

which imply

$$(14) \quad d^{\pi(Dk)} > \frac{d^{k+1} d^{\text{ord}_d(k-1)!} (vk)^{k+1-\pi(Dk)}}{(k-1)!}.$$

By using the estimates for  $\text{ord}_d((k-1)!)$  and  $(k-1)!$  given in Lemma 2.1, we obtain

$$\begin{aligned} (vdk)^{\pi(Dk)} &> \frac{(vdk)^{k+1} d^{(k-d)/(d-1)} (k-1)^{-1}}{\sqrt{2(k-1)} \pi \left(\frac{k-1}{e}\right)^{k-1} \exp\left(\frac{1}{12(k-1)}\right)} \\ &= \left( evd^{\frac{d}{d-1}} \frac{k}{k-1} \right)^k \frac{v\sqrt{k}}{ed^{1/(d-1)} \sqrt{2\pi}} \sqrt{\frac{k}{k-1}} \exp\left(-\frac{1}{12(k-1)}\right) \end{aligned}$$

implying

$$(15) \quad \frac{\pi(Dk)}{k} > \frac{\log( evd^{\frac{d}{d-1}} ) + \left(1 + \frac{1}{2k}\right) \log\left(\frac{k}{k-1}\right) - \frac{1}{12k(k-1)} + \frac{1}{2k} \log \frac{v^2 k}{2\pi e^2 d^{\frac{2}{d-1}}}}{\log(vdk)}.$$

Again by using the estimates for  $\pi(\nu)$  given in Lemma 2.1 and  $\frac{\log(vdk)}{\log(Dk)} = 1 + \frac{\log \frac{vd}{D}}{\log(Dk)}$ , we derive

$$(16) \quad \begin{aligned} 0 &> \log( evd^{\frac{d}{d-1}} ) - \frac{1}{2k} \left( \frac{1}{6(k-1)} - \log \frac{v^2 k}{2\pi e^2 d^{\frac{2}{d-1}}} \right) \\ &\quad - D \left( 1 + \frac{\log \frac{vd}{D}}{\log(Dk)} \right) \left( 1 + \frac{1.2762}{\log(Dk)} \right) \end{aligned}$$

Note that

$$\frac{1}{2k} \left( \frac{1}{6(k-1)} - \log \frac{v^2 k}{2\pi e^2 d^{\frac{2}{d-1}}} \right) = \left( \frac{1}{2k} \right) \left( -\log \frac{v^2}{2\pi e^2 d^{\frac{2}{d-1}}} - \log k + \frac{1}{6(k-1)} \right)$$

is a decreasing function for  $k > 1$ . Also for a fixed  $v$  with  $vd \geq D$ , the term  $D \left( 1 + \frac{\log \frac{vd}{D}}{\log(Dk)} \right) \left( 1 + \frac{1.2762}{\log(Dk)} \right)$  is a decreasing function of  $k$ . Thus for a fixed  $v$  with  $vd \geq D$ , we observe that the right hand side of (16) is an increasing function for  $k > 1$ . For a fixed  $k$ , if (16) is not valid at some  $v = v_0$ , then (14) is also not valid at  $v = v_0$ . Observe that for a fixed  $k$ , if (14) is not valid at some  $v = v_0$ , then (14) is also not valid when  $v \geq v_0$ . Therefore whenever (16) is not valid for  $k = k_0 > 1$  at some  $v = v_0$  with  $v_0 d \geq D$ , then (14) is not valid for all  $k \geq k_0$  and  $v \geq v_0$ .

3(A). PROOF OF THEOREM 3 FOR THE CASE  $d = 3$ 

Let  $d = 3$  and let the assumptions of Theorem 3 be satisfied. Let  $2 \leq k \leq 11$  and  $m > 3k$ . Observe that  $k - \pi(3k) + 1 = 0$  for  $k \leq 8$  and  $k - \pi(3k) + 1 = 1$  for  $9 \leq k \leq 11$ . If  $T \neq \phi$ , then  $m \leq 2^3 \times 5 \times 7 = 280$ .

By Corollary 2.10, we may assume that  $2 \leq k \leq 8$ ,  $m \geq 6450$  and  $T = \phi$ . Further  $i_p$  exists for each prime  $p \leq 3k$ ,  $p \neq 3$  and  $i_p \neq i_q$  for  $p \neq q$  otherwise  $|T| \geq k - \pi(3k) + 1 + 1 > 0$ . Also  $pq \nmid (m + id)$  for any  $i$  whenever  $p, q \geq k$  otherwise  $T \neq \phi$ . Thus  $P((m + 3i_2)(m + 3i_5)) = 5$  if  $k < 8$ . For  $k = 8$ , we get  $P((m + 3i_2)(m + 3i_5)) \leq 7$  with  $P((m + 3i_2)(m + 3i_5)) = 7$  only if  $7|m$  and  $\{i_2, i_5\} \cap \{0, 7\} \neq \phi$ .

Let  $k \leq 7$  or  $k = 8$  with  $P((m + 3i_2)(m + 3i_5)) = 5$ . Let  $j_0 = \min(i_2, i_5)$ ,  $X = n + 3j_0$  and  $i = |i_2 - i_5|$ . Then  $X \geq 6450$  and this is excluded by Corollary 2.12.

Let  $k = 8$  and  $P((m + 3i_2)(m + 3i_5)) = 7$ . Then  $7|m$  and  $\{i_2, i_5\} \cap \{0, 7\} \neq \phi$ . Hence  $i_7 = 0$  or  $7$  and  $7 \in \{i_2, i_5\}$  if  $i_7 = 0$  and  $0 \in \{i_2, i_5\}$  if  $i_7 = 7$ . If  $5 \nmid m(m + 21)$ , then  $\{i_2, i_7\} = \{0, 7\}$  and either

$$m = 7 \times 2^r, m + 21 = 7^{1+s} \quad \text{or} \quad m = 7^{1+s}, m + 21 = 7 \times 2^r$$

implying  $2^r - 7^s = \pm 3$ . Since  $2^r \geq \frac{m}{7} > 40$ , we get by taking modulo 8 that  $(-1)^{s+1} \equiv \pm 3$  which is a contradiction. Thus  $5|m(m+21)$  implying  $2 \times 5 \times 7 | m(m+21)$ . By taking the prime factorization, we obtain

$$m = 2^{a_0} 5^{b_0} 7^{c_0}, m + 21 = 2^{a_1} 5^{b_1} 7^{c_1}$$

with  $\min(a_0, a_1) = \min(b_0, b_1) = 0$ ,  $\min(c_0, c_1) = 1$  and further  $b_0 + b_1 = 1$  if  $i_2 \in \{0, 7\}$  and  $a_0 + a_1 \leq 2$  if  $i_5 \in \{0, 7\}$ . From the identity  $\frac{m+21}{7} - \frac{m}{7} = 3$ , we obtain one of

$$(i) 2^a - 5 \cdot 7^c = \pm 3 \quad \text{or} \quad (ii) 5 \cdot 2^a - 7^c = \pm 3 \\ \text{or} \quad (iii) 5^b - 2^\delta \cdot 7^c = \pm 3 \quad \text{or} \quad (iv) 2^\delta \cdot 5^b - 7^c = \pm 3$$

with  $\delta \in \{1, 2\}$ . Further from  $m \geq 6450$ , we obtain  $c \geq 3$  and

$$(17) \quad a \geq 9, a \geq 7, b \geq 4, b \geq 3$$

according as (i), (ii), (iii), (iv) hold, respectively. These equations give rise to a Thue equation

$$(18) \quad X^3 + AY^3 = B$$

with integers  $X, Y, A > 0, B > 0$  given by

	$c$ (mod 3)	Equation	$A$	$B$	$X$	$Y$
(i)	0, 1	$2^a - 5 \cdot 7^c = \pm 3$	$5 \cdot 2^{a'} \cdot 7^{c'}$	$3 \cdot 2^{a'}$	$\pm 2^{\frac{a+a'}{3}}$	$\pm 7^{\frac{c-c'}{3}}$
(ii)	0, 1	$5 \cdot 2^a - 7^c = \pm 3$	$25 \cdot 2^{a'} \cdot 7^{c'}$	$75 \cdot 2^{a'}$	$\pm 5 \cdot 2^{\frac{a+a'}{3}}$	$\pm 7^{\frac{c-c'}{3}}$
(iii)	0, 1	$5^b - 2^\delta \cdot 7^c = \pm 3$	$2^\delta \cdot 5^{b'} \cdot 7^{c'}$	$3 \cdot 5^{b'}$	$\pm 5^{\frac{b+b'}{3}}$	$\pm 7^{\frac{c-c'}{3}}$
(iv)	0, 1	$2^\delta \cdot 5^b - 7^c = \pm 3$	$2^{3-\delta} \cdot 5^{b'} \cdot 7^{c'}$	$2^{3-\delta} \cdot 5^{b'} \cdot 3$	$\pm 2 \cdot 5^{\frac{b+b'}{3}}$	$\pm 7^{\frac{c-c'}{3}}$
(v)	2	$2^a - 5 \cdot 7^c = \pm 3$	$175 \cdot 2^{a'}$	525	$\pm 5 \cdot 7^{\frac{c+1}{3}}$	$\pm 2^{\frac{a-a'}{3}}$
(vi)	2	$5 \cdot 2^a - 7^c = \pm 3$	$35 \cdot 2^{a'}$	21	$\pm 7^{\frac{c+1}{3}}$	$\pm 2^{\frac{a-a'}{3}}$
(vii)	2	$5^b - 2^\delta \cdot 7^c = \pm 3$	$2^{3-\delta} \cdot 5^{b'} \cdot 7$	$21 \cdot 2^{3-\delta}$	$\pm 2 \cdot 7^{\frac{c+1}{3}}$	$\pm 5^{\frac{b-b'}{3}}$
(viii)	2	$2^\delta \cdot 5^b - 7^c = \pm 3$	$2^\delta \cdot 5^{b'} \cdot 7$	21	$\pm 7^{\frac{c+1}{3}}$	$\pm 5^{\frac{b-b'}{3}}$

where  $0 \leq a', b' < 3$  are such that  $X, Y$  are integers and  $c' = 0, 1$  according as  $c \pmod{3} = 0, 1$ , respectively. For example,  $2^a - 5 \cdot 7^c = \pm 3$  with  $c \equiv 0, 1 \pmod{3}$  implies  $(\pm 2^{\frac{a+a'}{3}})^3 + 5 \cdot 2^{a'} 7^{c'} (\pm 7^{\frac{c-c'}{3}})^3 = 3 \cdot 2^{a'}$  where  $a'$  is such that  $3|(a+a')$ . This give a Thue equation (18) with  $A = 5 \cdot 2^{a'} 7^{c'}$  and  $B = 3 \cdot 2^{a'}$ .

By using (17), we see that at least two of

$$(19) \quad \text{ord}_2(XY) \geq 2 \text{ or } \text{ord}_5(XY) \geq 1 \text{ or } \text{ord}_7(XY) \geq 1$$

hold except for (vi) and (viii) where  $\text{ord}_2(XY) \geq 1$ ,  $\text{ord}_7(XY) \geq 1$  in case of (vi) and  $\text{ord}_2(XY) = 0$ ,  $\text{ord}_7(XY) \geq 1$  in case of (viii). Using the command

$$T := \text{Thue}(X^3 + A); \text{Solutions}(T, B);$$

in *Kash*, we compute all the solutions in integers  $X, Y$  of the above Thue equations. We find that none of solutions of Thue equations satisfy (19).

Hence we have  $k \geq 12$ . For the proof of Theorem 3, we may suppose from Corollaries 2.10 and 2.3 that

$$(20) \quad m \geq \max(6450, 10.6 \times 3k).$$

Let  $12 \leq k \leq 19$ . Since  $t_0 \geq 1, 2$  for  $12 \leq k \leq 16$  and  $17 \leq k \leq 19$ , respectively, we have

$$m \leq \sqrt{\mathfrak{P}} \leq \sqrt{4 \times 8 \times 5^2 \times 7^2 \times 11 \times 13} < 6450 \quad \text{if } 12 \leq k \leq 16$$

$$m \leq \sqrt[3]{\mathfrak{P}} \leq \sqrt[3]{4 \times 8 \times 16 \times 5^3 \times 7^2 \times 11 \times 13 \times 17} < 6450 \quad \text{if } 17 \leq k \leq 19.$$

This is not possible by (20).

Thus  $k \geq 20$ . Then  $m \geq 6450$  and  $v \geq 10.6$  by (20) satisfying  $v_0 d \geq D = d = 3$ . Now we check that (16) is not valid for  $k = 180$  at  $v = 10.6$ . Therefore (14) is not valid for  $k \geq 180$  and  $v \geq 10.6$ . Thus  $k < 180$ . Further we check that (15) is not valid for  $21 \leq k < 180$  at  $v = \frac{6450}{3k}$  except when  $k \in \{21, 25, 28, 37, 38\}$ . Hence (14) is not valid for  $20 \leq k < 180$  when  $v \geq \frac{6450}{3k}$  except when  $k \in \{21, 25, 28, 37, 38\}$ . Thus it suffices to consider  $k \in \{21, 25, 28, 37\}$  where we check that (14) is not valid

at  $v = \frac{6450}{3k}$  and hence it is not valid for all  $v \geq \frac{6450}{3k}$ . Finally we consider  $k = 38$  where we find that (14) is not valid at  $v = \frac{8000}{3k}$ . Thus  $m < 8000$ . For  $l \in \{1, 2\}$  and  $p_{i,3,l} \leq 8000$ , we find that  $\delta_3(i, 3, l) < 90$  implying the set  $\{m, m+3, \dots, m+3(38-1)\}$  contains a prime. Hence the assertion follows since  $m > 3k$ .  $\square$

### 3(B). PROOF OF THEOREM 3 FOR $d = 2$

Let  $d = 2$  and let the assumptions of Theorem 3 be satisfied. The assertion for Theorem 3 with  $k \geq 2$  and  $m \leq 4k$  follows from Corollary 2.4. Thus  $m > 4k$ . For  $2 \leq k \leq 37$ ,  $k \neq 35$ , Lemma 2.8 gives the result. Hence for the proof of Theorem 3, we may suppose that  $k = 35$  or  $k \geq 38$ . Further from Corollaries 2.3 and 2.7, we may assume that

$$(21) \quad m \geq \max(M_0, 131 \times 2k).$$

Let  $k = 35, 38$ . Then  $t_0 = 1, 2$  for  $k = 35, 38$ , respectively and we have

$$m \leq \sqrt{\mathfrak{P}} \leq \sqrt{27 \cdot 9 \cdot 25 \cdot 5 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19 \cdot 23 \cdot 29 \cdot 31} < 10^{10} \text{ if } k = 35$$

$$m \leq \sqrt[3]{\mathfrak{P}} \leq \sqrt[3]{27 \cdot 9^2 \cdot 25 \cdot 5^2 \cdot 7^3 \cdot 11^3 \cdot 13^2 \cdot 17^2 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37} < 10^{10} \text{ if } k = 38.$$

This is not possible by (21).

Thus we assume that  $k \geq 39$ . Now we check that (16) is not valid for  $k = 500$  at  $v = 131$ . Therefore (14) is not valid for  $k \geq 500$  when  $v \geq 131$ . Hence from (21), we get  $k < 500$ . Further  $v \geq \frac{M_0}{2 \times 500} \geq 10^7$ . We check that (16) is not valid for  $v = 10^7$  at  $k = 70$ . Therefore (14) is not valid for  $k \geq 70$  when  $v \geq 10^7$ . Thus  $k < 70$ . For each  $39 \leq k < 70$ , we find that (14) is not valid at  $v = \frac{M_0}{2k}$  and hence for all  $v \geq \frac{M_0}{2k}$ . This is a contradiction.  $\square$

## 4. PROOF OF THEOREMS 1 AND 2

Recall that  $q = u + \frac{\alpha}{d}$  with  $1 \leq \alpha < d$ . The following result is [ShTi, Lemma 10.1].

**Lemma 4.1.** *Let  $1 \leq k \leq \frac{n}{2}$  and*

$$d \leq 2\alpha + 2 \text{ if } (k, u) = (1, 0).$$

*If there is a prime  $p$  with*

$$p | (\alpha + (n + u - k)d) \cdots (\alpha + (n + u - 1)d), \quad p \nmid a_0 a_n.$$

*such that*

$$p \geq \begin{cases} (k + u - 1)d + \alpha + 1 & \text{if } u > 0 \\ (k + u - 1)d + \alpha + 2 & \text{if } u = 0 \end{cases}$$

*Then  $G(x)$  has no factor of degree  $k$ .*

Let  $d = 3$ . By putting  $m = \alpha + 3(n - k)$  and taking  $p = P(\Delta(m, 3, k))$ , we find from Lemma 4.1 and Theorem 3 that  $G_{\frac{1}{3}}$  and  $G_{\frac{2}{3}}$  does not have a factor of degree  $k \geq 2$  except possibly when  $k = 2, \alpha = 2, m = 2 + 3(n - 2) = 125$ . This gives  $n = 43$  and we use [ShTi, Lemma 2.12] with  $p = 2, r = 2$  to show that  $G_{\frac{2}{3}}$  do not have a factor of degree 2. Further except possibly for  $\alpha = 2$  and  $m = 2 + 3(n - 1) = 2^l$  for positive integers  $l$ ,  $G_{\frac{1}{3}}$  and  $G_{\frac{2}{3}}$  are irreducible. This proves Theorem 1.

Let  $d = 2$ . By [ShTi, Theorem 10.2], we may suppose that  $u > 0$ . Further we may assume that  $k \geq 2$ . For  $(m, k) \in \{(5, 2), (7, 2), (13, 5), (17, 6), (15, 7), (21, 8), (19, 9)\}$ , we check that  $P(\Delta(m, 2, k)) \geq m$ . For  $1 \leq u \leq k$ , by putting  $m = 1 + 2(n + u - k)$ , we find from  $n \geq 2k$  and Theorem 3 that

$$P(\Delta(m, 2, k)) > 2(k + u) = \begin{cases} \min(2(k + u), 3.5k) & \text{if } u \leq 0.5k \\ \min(2(k + u), 4k) & \text{if } 0.5k < u \leq k \end{cases}$$

except when  $k = 2, m \in \{25, 243\}$ . Now we take  $p = P(\Delta(m, 2, k))$  in Lemma 4.1 to obtain that  $G_{u+\frac{1}{2}}$  do not have a factor of degree  $k$  with  $k \geq 1$  except possibly when  $k = 2, u = 1, n \in \{13, 122\}$  or  $k = 2, u = 2, n \in \{12, 121\}$ . We use [ShTi, Lemma 2.12] with  $(p, r) = (3, 1), (7, 1)$  to show that  $G_{u+\frac{1}{2}}$  do not have a factor of degree 2 when  $(u, n) = (1, 13), (2, 12)$  and  $(u, n) = (2, 121)$ , respectively. The case  $k = 2, (u, n) = (1, 122)$  is excluded by Lemma 4.1 since  $P(243 \cdot 245) = 7$ .  $\square$

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