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Irreducibility of generalized Hermite–Laguerre Polynomials II

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ABSTRACT

In this paper, we show that for each $n \geqslant 1$, the generalised Hermite–Laguerre Polynomials $G_{\frac{1}{4}}$ and $G_{\frac{3}{4}}$ are either irreducible or linear polynomial times an irreducible polynomial of degree $n-1$.

1. INTRODUCTION

Let n and $1 \leqslant \alpha < d$ be positive integers with $\gcd(\alpha, d) = 1$. Let $q = \frac{\alpha}{d}$ and let

$$(\alpha)_j = \alpha(\alpha+d)\cdots\big(\alpha+(j-1)d\big)$$

for non-negative integer j. We define

$$F(x) := F_q(x) = a_n \frac{d^n x^n}{(\alpha)_n} + a_{n-1}\frac{d^{n-1}x^{n-1}}{(\alpha)_{n-1}} + \cdots + a_1 \frac{dx}{(\alpha)_1} + a_0,$$

where $a_0, a_1, \ldots, a_n \in \mathbb{Z}$ and $P(|a_0 a_n|) \leqslant 2$. Here $P(\nu)$ is the maximum prime divisor for $|\nu| > 1$ and $P(1) = P(-1) = 1$. We put

$$G(x) := G_q(x) = (\alpha)_n F_q\left(\frac{x}{d}\right)$$
$$= a_n x^n + a_{n-1}\big(\alpha+(n-1)d\big)x^{n-1}$$

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$$+\cdots+a_1\left(\prod_{i=1}^{n-1}(\alpha+id)\right)x+a_0\left(\prod_{i=0}^{n-1}(\alpha+id)\right).$$

Schur [9] proved that $G_{\frac{1}{2}}$ with $|a_0|=|a_n|=1$ is irreducible. Laishram and Shorey [4] showed that $G_{\frac{1}{3}}$ and $G_{\frac{2}{3}}$ are either irreducible or linear polynomial times an irreducible polynomial of degree $n-1$ whenever $|a_0|=|a_n|=1$. For an account of earlier results, we refer to [8] and [2]. We prove the following theorem.

Theorem 1. *For each n, the polynomials $G_{\frac{1}{4}}$ and $G_{\frac{3}{4}}$ are either irreducible or linear polynomial times an irreducible polynomial of degree $n-1$.*

For Theorem 1, we prove the following lemma in Section 2.

Lemma 1. *Let $1\leqslant k\leqslant \frac{n}{2}$. Suppose there is a prime p satisfying*

$$p>d,\quad p\geqslant \min\big(2k,d(d-1)\big)$$

and

$$p\,\Big|\,\prod_{j=1}^{k}\big(\alpha+(n-j)d\big),\qquad p\nmid\prod_{j=1}^{k}\big(\alpha+(j-1)d\big). \tag{1}$$

Then $G(x)$ has no factor of degree k.

We compare Lemma 1 with [8, Lemma 10.1]. The assumption on p in [8, Lemma 10.1] has been relaxed. For any integer $\nu>1$, we denote by $\omega(\nu)$ the number of distinct prime factors of ν and $\omega(1)=0$. In Section 3, we give an upper bound for m when $\omega(\prod_{i=0}^{k-1}(m+id))\leqslant t$ for some t. In Section 4, we give preliminaries for the proof of Theorem 1. In Section 5, we complete the proof.

2. PROOF OF LEMMA 1

Let

$$\Delta_j=\alpha(\alpha+d)\cdots\big(\alpha+(j-1)d\big).$$

For each $1\leqslant l<d$ and $\gcd(l,d)=1$, we observe that $q|\Delta_k$ for all primes $q\equiv l^{-1}\alpha(\bmod d)$ and $q\leqslant \frac{kd}{l}$. Since $p>\alpha$ and $p\nmid\Delta_k$, we have $p>\frac{kd}{d-1}$. Let j_0 be the minimum j such that $p|(\alpha+(j-1)d)$ and we write $\alpha+(j_0-1)d=pl_0$. Then $j_0>k$ since $p\nmid\Delta_k$ and we observe that $1\leqslant l_0<d$ by the minimality of j_0. As in the proof of [8, Corollary 2.1], it suffices to show that

$$\phi_j=\frac{\mathrm{ord}_p(\Delta_j)}{j}<\frac{1}{k}\quad\text{for } 1\leqslant j\leqslant n.$$

We may restrict to those j such that $\alpha+(j-1)d=pl$ for some l. Then $(j-j_0)d=p(l-l_0)$ implying $d|(l-l_0)$. Writing $l=l_0+sd$, we get $j=j_0+ps$. Note that if

$p|(\alpha+(i-1)d)$, then $\alpha+(i-1)d=p(l_0+rd)$ for some $r\geqslant 0$. Hence we have

$$
\begin{aligned}
(2)\qquad \operatorname{ord}_p(\Delta_j) &= \operatorname{ord}_p\big((pl_0)\big(p(l_0+d)\big)\cdots\big(p(l_0+sd)\big)\big)\\
&= s+1+\operatorname{ord}_p\big(l_0(l_0+d)\cdots(l_0+sd)\big)
\end{aligned}
$$

for some integer $s\geqslant 0$. Further we may suppose that $s>0$ otherwise the assertion follows since $p>d>l_0$. Let r_0 be such that $\operatorname{ord}_p(l_0+r_0d)$ is maximal. We consider two cases.

Case I. Assume that $s<p$. Then p divides at most one term of $\{l_0+id\colon 0\leqslant i\leqslant s\}$ and we obtain from (2) and $l_0+sd<(s+1)d<p^2$ that $\phi_j\leqslant\frac{s+2}{j_o+ps}$. Thus $\phi_j<\frac{1}{k}$ if $s(p-k)\geqslant k$ since $j_0-k+s(p-k)-k\geqslant 1+s(p-k)-k$. If $p\geqslant 2k$, then $s(p-k)\geqslant k$. Thus we may suppose that $p<2k$. Then $p\geqslant d(d-1)$. Since $p>\frac{kd}{d-1}$, we obtain $s(p-k)\geqslant k$ if $s\geqslant d-1$. We may suppose $s\leqslant d-2$. Then $l_0+sd\leqslant d-1+(d-2)d<p$ and therefore $\phi_j=\frac{s+1}{j_0+ps}\leqslant\frac{s+1}{k+1+(k+1)s}<\frac{1}{k}$.

Case II. Let $s\geqslant p$. Then

$$
\begin{aligned}
\operatorname{ord}_p(\Delta_j) &\leqslant s+1+\operatorname{ord}_p(l_0+r_0d)+\operatorname{ord}_p(s!)\\
&\leqslant s+1+\frac{\log(l_0+sd)}{\log p}+\frac{s}{p-1}.
\end{aligned}
$$

We have $p\geqslant d+1$. This with $l_0\leqslant d-1<p\leqslant s$ imply $\log(l_0+sd)\leqslant\log s(d+1)=\log s+\log(d+1)\leqslant\log s+\log p$. Hence

$$\operatorname{ord}_p(\Delta_j)\leqslant s+1+\frac{s}{p-1}+\frac{\log s}{\log p}+1.$$

Since $\frac{j}{k}=\frac{j_0+ps}{k}>1+\frac{p}{k}s$, it is enough to show that

$$\frac{p}{k}\geqslant 1+\frac{1}{p-1}+\frac{1}{s}+\frac{\log s}{s\log p}.$$

Since $s\geqslant p$, the right-hand side of the above inequality is at most $1+\frac{1}{p-1}+\frac{2}{p}$ and therefore it suffices to show

$$(3)\qquad 1+\frac{1}{p-1}+\frac{2}{p}\leqslant\frac{p}{k}.$$

Let $p\geqslant 2k$. Then $p\geqslant 2k+1\geqslant k+2$ and the left-hand side of (3) is at most

$$1+\frac{1}{2k}+\frac{2}{2k+1}\leqslant 1+\frac{2}{k}=\frac{k+2}{k}\leqslant\frac{p}{k}.$$

Thus we may assume that $p<2k$. Then $p>d(d-1)$ since $p\nmid d$. Further $d\geqslant 3$ since $p\geqslant\frac{kd}{d-1}$. Therefore the left-hand side of (3) is at most

$$1+\frac{3}{d(d-1)}\leqslant 1+\frac{1}{d-1}=\frac{d}{d-1}\leqslant\frac{p}{k}.$$

Hence the proof.

3. AN UPPER BOUND FOR m WHEN $\omega(\Delta(m,d,k)) \leqslant t$

Let m and k be positive integers with $m > kd$ and $\gcd(m,d) = 1$. We write

$$\Delta(m,d,k) = m(m+d)\cdots\big(m+(k-1)d\big).$$

Assume that

$$\omega\big(\Delta(m,d,k)\big) \leqslant t \tag{4}$$

for some integer t. For every prime p dividing Δ, we delete a term $m+i_p d$ such that $\mathrm{ord}_p(m+i_p d)$ is maximal. Then we have a set T of terms in $\Delta(m,k)$ with

$$|T| = k - t := t_0.$$

We arrange the elements of T as $m+i_1 d < m+i_2 d < \cdots < m+i_{t_0} d$. Let

$$\mathfrak{P} := \prod_{\nu=1}^{t_0} (m+i_\nu d) \geqslant m^{t_0}. \tag{5}$$

Now we deduce an upper bound for $\mathfrak{P}$. For a prime p, let r be the highest power of p such that $p^r \leqslant k-1$. Let $w_l = \#\{m+id\colon\ p^l|(m+i), m+i \in T\}$ for $1 \leqslant l \leqslant r$. By Sylvester and Erdős argument, we have $w_l \leqslant [\frac{i_0}{p^l}] + [\frac{k-1-i_o}{p^l}] \leqslant [\frac{k-1}{p^l}]$. Let $h_p > 0$ be such that $[\frac{k-1}{p^{h_p+1}}] \leqslant t_0 < [\frac{k-1}{p^{h_p}}]$. Then $|\{m+id \in T\colon\ \mathrm{ord}_p(n+id) \leqslant h_p\}| \leqslant t_0 - w_{h_p+1}$. Hence

$$\begin{aligned}
\mathrm{ord}_p(\mathfrak{P}) &\leqslant r w_r + \sum_{u=h_p+1}^{r-1} u(w_u - w_{u+1}) + h_p(t_0 - w_{h_p+1}) \\
&= w_r + w_{r-1} + \cdots + w_{h_p+1} + h_p t_0 \\
&\leqslant \sum_{u=1}^{r} \left\lfloor \frac{k-1}{p^u} \right\rfloor + h_p t_0 - \sum_{u=1}^{h_p} \left\lfloor \frac{k-1}{p^u} \right\rfloor \\
&= \mathrm{ord}_p\big((k-1)!\big) + h_p t_0 - \sum_{u=1}^{h_p} \left\lfloor \frac{k-1}{p^u} \right\rfloor.
\end{aligned}$$

It is also easy to see that $\mathrm{ord}_p(\mathfrak{P}) \leqslant \mathrm{ord}_p(k-1)!)$ if $p \nmid d$ and $\mathrm{ord}_p(\mathfrak{P}) = 0$ if $p|d$. Therefore

$$m^{t_0} \leqslant \mathfrak{P} \leqslant (k-1)! \prod_{p \leqslant k} p^{L_0(p)},$$

where

$$L_0(p) = \begin{cases} \min\big(0, h_p t_0 - \sum_{u=1}^{h_p} \lfloor \frac{k-1}{p^u} \rfloor\big) & \text{if } p \nmid d, \\ -\mathrm{ord}_p\big((k-1)!\big) & \text{if } p|d. \end{cases}$$

Observe that

$$(6)\qquad m^{t_0} \leqslant (k-1)! \prod_{p\mid d} p^{-\operatorname{ord}_p((k-1)!)}.$$

We also note that $L_0(p) \leqslant 0$ for any prime p. Hence for any $l \geqslant 1$, we have from (5) that

$$(7)\qquad m \leqslant (\mathfrak{P})^{\frac{1}{t_0}} \leqslant \Big((k-1)! \prod_{p\leqslant p_l} p^{L_0(p)}\Big)^{\frac{1}{t_0}} =: L(k,l).$$

4. PRELIMINARIES FOR THEOREMS 1

Let m and k be positive integers with $m > kd$ and $\gcd(m,d) = 1$. We write

$$\Delta(m,d,k) = m(m+d)\cdots\big(m+(k-1)d\big).$$

For positive integers ν, μ and $1 \leqslant l < \mu$ with $\gcd(l,\mu) = 1$, we write

$$\pi(\nu,\mu,l) = \sum_{\substack{p\leqslant \nu\\ p\equiv l \pmod{\mu}}} 1, \quad \pi(\nu) = \pi(\nu,1,1),$$

$$\theta(\nu,\mu,l) = \sum_{\substack{p\leqslant \nu\\ p\equiv l \pmod{\mu}}} \log p.$$

Let $p_{i,\mu,l}$ denote the ith prime congruent to l modulo μ. Let $\delta_\mu(i,l) = p_{i+1,\mu,l} - p_{i,\mu,l}$ and $W_\mu(i,l) = (p_{i,\mu,l}, p_{i+1,\mu,l})$. We recall some well-known estimates from prime number theory.

Lemma 4.1. *Let $k \in \mathbb{Z}$ and $\nu \in \mathbb{R}$ be positive. We have:*

(i) $\pi(\nu) \leqslant (1 + \frac{1.2762}{\log \nu})$ *for* $\nu > 1$*;*
(ii) $\operatorname{ord}_p(k-1)! \geqslant \frac{k-p}{p-1} - \frac{\log(k-1)}{\log p}$ *for* $k \geqslant 2$*;*
(iii) $\sqrt{2\pi k}\, e^{-k} k^k e^{\frac{1}{12k+1}} < k! < \sqrt{2\pi k}\, e^{-k} k^k e^{\frac{1}{12k}}$.

The estimates (i) is due to Dusart [1]. The estimate (iii) is due to Robbins [7, Theorem 6]. For a proof of (ii), see [3, Lemma 2(i)].

The following lemma is due to Ramaré and Rumely [6, Theorems 1, 2].

Lemma 4.2. *Let $d = 4$ and $l \in \{1,3\}$. For $\nu_0 \leqslant 10^{10}$, we have*

$$(8)\qquad \theta(\nu,d,l) \geqslant \begin{cases} \frac{\nu}{2}(1-0.002238) & \textit{for } \nu \geqslant 10^{10}, \\ \frac{\nu}{2}\big(1 - \frac{2\times 1.798158}{\sqrt{\nu_0}}\big) & \textit{for } 10^{10} > \nu \geqslant \nu_0 \end{cases}$$

and

$$\theta(\nu, d, l) \leqslant \begin{cases} \frac{\nu}{2}(1+0.002238) & \textit{for } \nu \geqslant 10^{10}, \\ \frac{\nu}{2}\Big(1+\frac{2\times 1.798158}{\sqrt{\nu_0}}\Big) & \textit{for } 10^{10} > \nu \geqslant \nu_0. \end{cases} \tag{9}$$

We derive from Lemmas 4.1 and 4.2 the following result.

Corollary 4.3. *Let* $10^6 < m \leqslant 138 \times 4k$. *Then* $P(\Delta(m, 4, k)) \geqslant m$.

Proof. Let $d = 4$ and $10^6 \leqslant m \leqslant 138 \times dk$. Let $l \in \{1, 3\}$ and assume $m \equiv l \pmod d$. We observe that $P(\Delta(m, d, k) \geqslant m$ holds if

$$\theta\big(m + d(k-1), d, l\big) - \theta(m - d, d, l) = \sum_{\substack{m < p \leqslant m+(k-1)d \\ p \equiv l(d)}} \log p > 0.$$

From Lemmas 4.1 and 4.2, we have

$$\frac{\theta(m-d, d, l)}{\frac{m-d}{\phi(d)}} < 1 + \frac{2 \times 1.798158}{\sqrt{10^6}}$$

and

$$\frac{\theta(m+(k-1)d, d, l)}{\frac{m-d+dk}{\phi(d)}} > 1 - \frac{2 \times 1.798158}{\sqrt{10^6}}.$$

Thus $P(\Delta(m, d, k) \geqslant m$ holds if

$$\left(1 - \frac{2 \times 1.798158}{10^3}\right) dk > \frac{4 \times 1.798158}{10^3}(m-d)$$

which is true since

$$\frac{m}{dk} \leqslant 138 < \frac{10^3}{4 \times 1.798158} - \frac{1}{2}.$$

Hence the assertion. □

The following lemma is a computational result.

Lemma 4.4. *Let* $l \in \{1, 3\}$. *Then* $\delta_4(i, l) \leqslant 24, 32, 60, 200$ *according as* $p_{i,4,l} \leqslant 120, 250, 2400, 10^6$, *respectively.*

As a consequence, we obtain the following corollary.

Corollary 4.5. *Let* $d = 4$, $k \geqslant 6$ *and* m *be such that* $m \leqslant 120, 250, 2400, 10^6$ *when* $6 \leqslant k < 8$, $8 \leqslant k < 15$, $15 \leqslant k < 50$ *and* $k \geqslant 50$, *respectively. Then* $P(\Delta(m, d, k)) \geqslant m$.

Proof. We may assume that $p_{i,d,l} < m < m + (k-1)d < p_{i+1,d,l}$ for some i otherwise the assertion follows. Thus $p_{i+1,d,l} \geqslant d+m+(k-1)d$ and $p_{i,d,l} \leqslant m-d$. Therefore $\delta_d(i,l) = p_{i+1,d,l} - p_{i,d,l} \geqslant d+m+(k-1)d-(m-d) = d(k+1) > dk$. Now the assertion follows from Lemma 4.4. □

5. PROOF OF THEOREM 1

Let $2 \leqslant k \leqslant \frac{n}{2}$ and assume that $G(x)$ has a factor of degree k. We take $m = \alpha + 4(n-k)$. Since $n \geqslant 2k$, we have $m > 4k$. We may assume that $P(\Delta(m,4,k)) \leqslant 4k$ otherwise the assertion follows from Lemma 1 since $\alpha + 4(k-1) < 4k$. Thus $P(\Delta(m,4,k)) \leqslant 4k < m$.

Let $k \leqslant 6$. Then $P(\Delta(m,4,k)) \leqslant 4k \leqslant 23$ implying $P(m(m+4)) \leqslant 24$. Then $m+4 = N$ where N is given by [5, Table IIA] for $p \leqslant 23$. For each such N and for each $2 \leqslant k \leqslant 6$, we first restrict to those $m = N-4 > 4k$ such that $P(\Delta(m,4,k)) \leqslant 4k$. They are given by $k=2$, $m \in \{21, 45\}$. Here $P(m(m+4)) = 7$ and since $m \equiv 1$ modulo 4, the assertion follows by taking $p = 7$ in Lemma 1.

Therefore $k \geqslant 7$. Let $\omega_1(k) := \max_{\alpha\in\{1,3\}} \omega(\Delta(\alpha,4,k))$. If $\omega(\Delta(m,4,k)) > \omega_1$, then there is a prime p satisfying (1) implying $p > k \geqslant 7$. Observe that $11|\Delta(3,4,k)$ and $11|\Delta(1,4,k)$ for $k \geqslant 9$. For $k \in \{7,8\}$, if $\omega(\Delta(m,4,k)) > \omega_1$, then there are two primes $p > k$ dividing $\Delta(m,4,k)$ but $p \nmid \Delta(1,4,k)$ and hence there is a prime $p > 11$ satisfying (1). Therefore by Lemma 1, we may assume that $\omega(\Delta(m,4,k)) \leqslant \omega_1$. Taking $t = \omega_1$, we obtain from (7) with $p_l = 7$ that $m \leqslant 104, 245, 2353$ according as $k \leqslant 10, 20, 400$, respectively. This is not possible by Corollary 4.5.

Hence $k > 400$ and further $m > 10^6$ by Corollary 4.5. By Corollary 4.3, we may further suppose that $m \geqslant v_0 \cdot 4k$ where $v_0 := 138$. Since $P(\Delta(m,d,k)) \leqslant 4k$, we have $\omega(\Delta(m,d,k)) \leqslant \pi(4k) - 1$. Taking $t = \pi(4k) - 1$ in (4), we obtain from (6) that

$$(v_0 \cdot 4k)^{k-\pi(4k)+1} \leqslant (k-1)! 2^{-\operatorname{ord}_2((k-1)!)} = \frac{k!}{k} 2^{-\operatorname{ord}_2((k-1)!)}.$$

By using estimates of $\operatorname{ord}_p((k-1)!)$ and $k!$ from Lemma 4.1, we obtain

$$(v_0 \cdot 4k)^{k-\pi(4k)} < \frac{1}{k(v_0 \cdot 4k)} \left(\frac{k}{e}\right)^k \left((2\pi k)^{\frac{1}{2}} \exp\left(\frac{1}{12k}\right)\right) \left(2^{-k+2}(k-1)\right)$$

or

$$(v_0 \cdot 4 \cdot e \cdot 2)^k < (v_0 \cdot 4k)^{\pi(4k)} \frac{((2\pi)^{\frac{1}{2}} \exp(\frac{1}{12k}))}{v_0 \cdot \sqrt{k}} < (v_0 \cdot 4k)^{\pi(4k)}$$

since $k > 400$. By using estimates of $\pi(4k)$ from Lemma 4.1, we get

$$\log(v_0 \cdot 8 \cdot e) < \frac{4\log(v_0 \cdot 4k)}{\log(4k)} \left(1 + \frac{1.2762}{\log(4k)}\right).$$

The right-hand side of the above expression is a decreasing function of k and the inequality does not hold at $k = 401$. This is a contradiction.

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