A generalization of the Enright-Varadarajan Modules

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For a semisimple Lie group admitting discrete series Enright and Varadarajan have constructed a class of modules. Denoted $D_{P,\lambda}$ (cf. [3]). Their infinitesimal description based on the theory of Verma modules parallels that of finite dimensional irreducible modules. The introduction of the modules $D_{P,\lambda}$ in [3] was primarily to give an infinitesimal characterization of discrete series but we feel that [3] may well be a starting point for a fresh approach towards dealing with the problem of classification of irreducible representations of a general semisimple Lie algebra.

In order to give more momentum to such an approach we first construct modules which broadly generalize those in [3]. We briefly describe them now.

Let $g_0$ be any real semisimple Lie algebra, $g_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ a Cartan decomposition and $\theta$ the associated Cartan involution. Let $g = \mathfrak{k} + \mathfrak{p}$ be the complexification. Let $U(g), U(\mathfrak{k})$ be the enveloping algebras of $g, k$ respectively and let $U_k$ be the centralizer of $k$ in $U(g)$. For each $\theta$ stable parabolic subalgebra $q$ of $g$ we associate in this paper a class of irreducible $k$ finite $U(g)$ modules having the following property: Like finite dimensional irreducible modules and like the Enright-Varadarajan modules $D_{P,\lambda}$, any member of this class comes with a special irreducible $k$-type occurring in it with multiplicity one, with an explicit description of the action of $U_k$ on the corresponding isotypical $k$-type. We obtain these modules by extending the techniques in [3].

To see in what way these modules are related to the $\theta$ invariant parabolic subalgebra $q$ we refer the reader to §2.

When our parabolic subalgebra $q$ is minimal in $g$ and when rank of $g = \operatorname{rank} \mathfrak{k}$, the classes of $U(g)$ modules which we associate to this $q$ coincides with the class of modules $D_{P,\lambda}$ of [3] (with a slight difference)
in parametrization). On the other hand when \( q = g \) is the maximal parabolic subalgebra, the class we obtain is just the class of all finite dimensional irreducible representations of \( g \).

If \( k \) has trivial center, the trivial one dimensional \( U(g) \) module is not equivalent to any of the modules \( D_{P,\lambda} \) of [3]. This gap is bridged by the introduction of our class of \( U(g) \) modules for every intermediate \( \theta \) invariant parabolic subalgebra \( q \) between \( q = g \) and \( q = a \) \( \theta \) invariant Borel subalgebra of \( g \).

We have to point out that the knowledge of [3] is a necessary prerequisite to read this paper. If an argument or construction needed at some stage of this paper is parallel to that in [3] then instead of repeating them, we simply refer to [3].

§1. \( \theta \)-stable parabolic subalgebras

As in the introduction, \( g = \mathfrak{k} + \mathfrak{p} \) is the complexified Cartan decomposition arising from a real one \( g_0 = \mathfrak{k}_0 + \mathfrak{p}_0 \). Let \( \theta \) be the Cartan involution. Let \( \mathfrak{b} \) the complexification of a fixed Cartan subalgebra \( \mathfrak{b}_0 \) of \( \mathfrak{k}_0 \). Then the centralizer of \( \mathfrak{b} \) in \( g \) is a \( \theta \) stable Cartan subalgebra \( \mathfrak{h} \), of \( g \). We can write

\[
\mathfrak{h} = \mathfrak{b} + \mathfrak{a}
\]

where \( \mathfrak{a} = \mathfrak{p} \cap \mathfrak{h} \). Let \( \mathfrak{a}_0 = \mathfrak{a} \cap \mathfrak{g}_0 \) and \( \mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{g}_0 \). Let \( \Delta \) be set of roots of \( (g, \mathfrak{h}) \). For \( \alpha \) in \( \Delta \), denote by \( g^\alpha \) the corresponding root space.

\[\theta(\mathfrak{h}_0) = \mathfrak{h}_0 \] so \( \mathfrak{h}_0 \in \mathfrak{b} \).

\[
\Delta(\mathfrak{q}) = \{ \alpha \in \Delta \mid \alpha(H_\mu') \geq 0 \}
\]

\[
\theta(H'_\mu) = H'_\mu \text{ so } H'_\mu \in \mathfrak{b}.
\]

\[\Delta(\mathfrak{q}) = \{ \alpha \in \Delta \mid \alpha(H'_\mu) \geq 0 \}.
\]
Then one can see that

\[(1.5) \quad q = \mathfrak{h} + \sum_{\alpha \in \Delta(q)} \mathfrak{g}^\alpha.\]

Let \( C_\mathfrak{t} \) be the open Weyl chamber in \( i\mathfrak{b}_0 \) for \((\mathfrak{t}, \mathfrak{b})\) defined by the Borel subalgebra \( r_\mathfrak{t} \). Since we assumed that \( r_\mathfrak{t} \subseteq q \), it follows from 1.5 that

\[(1.6) \quad H'_\mu \subseteq \overline{C_\mathfrak{t}} = \text{the closure of } C_\mathfrak{t}.\]

Let \( \alpha \) be in \( \Delta \). If \( \alpha \) is identically zero on \( \mathfrak{b} \), it would follow that \( \mathfrak{b} \) is not maximal abelian in \( \mathfrak{t} \). Hence \( \alpha \) is not identically zero on \( \mathfrak{b} \). Let \( C'_\mathfrak{t} \) be the open subset of \( C_\mathfrak{t} \) got by deleting points of \( C_\mathfrak{t} \) where some \( \alpha \) belonging to \( \Delta \) vanishes. Then \( C'_\mathfrak{t} \) is the disjoint union

\[(1.7) \quad C'_\mathfrak{t} = \bigcup_{i=1}^{N} C'_{\mathfrak{t},j}\]

of its connected components and one has

\[(1.8) \quad \overline{C_\mathfrak{t}} = \bigcup_{i=1}^{N} \overline{C'_{\mathfrak{t},j}}.\]

Choose an index \( M \) between 1 and \( N \) such that

\[(1.9) \quad H'_\mu \subseteq \overline{C'_{\mathfrak{t},M}}.\]

Now choose an element \( X_j \) in \( C'_{\mathfrak{t},j} \) and consider the weight space decomposition of \( \mathfrak{g} \) with respect to \( \text{ad}(X_j) \). We now define a Borel subalgebra \( \mathfrak{r} \) of \( \mathfrak{g} \) by

\[(1.10) \quad \mathfrak{r}^j = \text{the sum of the eigen spaces for } \text{ad}(X_j) \]

with nonnegative eigenvalues.

If we define

\[(1.11) \quad P^j = \{ \alpha \in \Delta \mid \alpha(X_j) > 0 \}\]

then clearly \( P^j \) is a positive system of roots in \( \Delta \) and \( \mathfrak{r}^j = \mathfrak{h} + \sum_{\alpha \in P^j} \mathfrak{g}^\alpha \). Since \( X_j \) belongs to \( \mathfrak{t} \) clearly both \( \mathfrak{r}^j \) and \( P^j \) are \( \theta \) stable. 1.9 implies that for every \( \alpha \) in \( P^M \), \( \alpha(H'_\mu) \) is nonnegative. Hence from 1.4 and 1.5

\[(1.12) \quad \mathfrak{r}^M \subseteq q.\]
Also since $X_M$ belongs to $C_t$, (1.10) implies that

(1.13) $\tau_t$ is contained in $\tau^M$.

(q.e.d.)

(1.14) Corollary Let $\tau_t$ be as in Lemma 1.2. Let $\tau$ be a $\theta$ stable Borel subalgebra of $g$ containing $\tau_t$. Then $\tau$ equals one of the $N$ Borel subalgebras $\tau^j$ of (1.10).

Proof Since $\tau$ contains $b$, $\tau$ contains a Cartan subalgebra of $g$ containing $b$. $h$ is the unique Cartan subalgebra of $g$ containing $b$. Hence $\tau$ contains $h$. In the proof of Lemma 1.2 take $q = \tau$. Then it is seen $\tau = \tau^M$. (q.e.d.)

Rather than starting with a Borel subalgebra $\tau_t$ of $k_t$ containing $b$, we want to start with an arbitrary $\theta$ invariant parabolic subalgebra of $g$ and recover the set up in Lemma 1.2. For this we prove the following lemma.

(1.15) Lemma Let $q$ be an arbitrary $\theta$ stable parabolic subalgebra of $g$. Then $q$ contains a Borel subalgebra of $k$.

Proof Let $Ad(g)$ be the adjoint group of $g$ and $Q$ the parabolic subgroup with Lie algebra $q$. Let $G^u$ be the compact form of $Ad(g)$ with Lie algebra $t_0 + ip_0$. Note that $G^u$ is $\theta$- stable. It is well known that $G^u \cap Q$ is a compact form of a reductive Levi factor of $Q$ (cf.[8, § 1.2]). But $G^u \cap Q$ is $\theta$ stable since $G^u$ and $Q$ are $\theta$ stable. Thus, going to the Lie algebra level, $q$ has a reductive Levi supplement which is $\theta$ stable. In this reductive Levi supplement we can surely find some $\theta$ stable Cartan subalgebra $h'$ of $g$. Then, as in the proof of Lemma 1.2, we can find an element $H'_\mu$ in $h'$ such that $\theta(H'_\mu) = H'_\mu$ and such that $q$ is the sum of the nonnegative eigenspaces of $ad(H'_\mu)$. Since $H'_\mu$ lies in $h' \cap k$, clearly it follows that $q$ contains a Borel subalgebra of $k$. (q.e.d)

(1.16) Corollary Let $\tau$ be any $\theta$ stable Borel subalgebra of $g$. Then $\tau \cap k$ is a Borel subalgebra of $k$.  

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§2. The objects \( r, r', PP' \) and the choice of \( P'' \) associated with a \( \theta \) stable parabolic subalgebra \( q \)

Now let \( q \) be a \( \theta \) stable parabolic subalgebra of \( g \). By (1.15) we can find a Borel subalgebra \( r_\Omega \) of \( \mathfrak{t} \) contained in \( q \). We fix a Cartan subalgebra \( b_0 \) of \( \mathfrak{t}_0 \) contained in \( r_\Omega \). Let \( a_0 \) be the centralizer of \( b_0 \) in \( p_0 \). Then \( h_0 = b_0 + a_0 \) is a \( \theta \) stable Cartan subalgebra of \( g_0 \). Let \( h = b + a \) be its complexification. Note that \( h \subseteq q \). By (1.12), we can find a \( \theta \) stable Borel subalgebra \( r \) of \( g \) such that \( r \subseteq r_\Omega \) and \( r \subseteq q \). One has then \( h \subseteq r \). There is a unique Borel subalgebra \( r' \) of \( g \) contained in \( q \) such that

\[
(2.1) \quad r \cap r' = h + u \text{ where } u \text{ is the unipotent radical of } q.
\]

Since \( \theta(r') \) has the same property, we have \( \theta(r') = r' \). Let \( r'_k \subseteq r' \cap \mathfrak{t} \). Then by (1.16), \( r'_k \) is a Borel subalgebra of \( \mathfrak{r} \). We observe that \( r'_k \) is the unique Borel subalgebra of \( \mathfrak{r} \) such that

\[
(2.2) \quad r_k \cap r'_k = b + u_k \text{ where } u_k \text{ is the unipotent radical of } q_k (= q \cap \mathfrak{t}).
\]

We denote by \( W_\mathfrak{t} \) the Weyl group of \((\mathfrak{t}, b)\) and by \( W_\mathfrak{g} \) the Weyl group of \((\mathfrak{g}, \mathfrak{h})\). \( W_\mathfrak{t} \) is naturally embedded in \( W_\mathfrak{g} \) as follows. If \( s \) belongs to \( W_\mathfrak{t} \) then \( s \) normalizes \( b \), hence also normalizes the centralizer of \( b \) in \( g \) which is precisely \( h \). Thus \( s \) belongs to \( W_\mathfrak{g} \).

We will now define two distinguished elements of the Weyl group \( W_\mathfrak{t} \). Let \( t \) be the unique element of \( W_\mathfrak{t} \) such that \( t(P_\mathfrak{t}) = -P_\mathfrak{t} \). Next we denote by \( \tau \) the unique element of the Weyl group \( W_\mathfrak{t} \) such that \( \tau(P_\mathfrak{t}) = P'_\mathfrak{t} \). The class of \( U(\mathfrak{g}) \) modules associated to \( q \) will be parametrized by some subsets of \( h^X \). We now prepare to describe these. Let \( \Delta_\mathfrak{t} \) be the set of roots for \((\mathfrak{t}, b)\). Whenever possible we will denote elements of \( \Delta_\mathfrak{t} \) by \( \varphi \) while elements of \( \Delta(= \text{the roots of } (\mathfrak{g}, \mathfrak{h})) \) will be denoted by \( \alpha \). For a root \( \varphi \) in \( \Delta_\mathfrak{t} \), denote by \( X_\varphi \) a nonzero root vector in \( \mathfrak{t} \) of weight \( \varphi \). For \( \alpha \) in \( \Delta \), we denote by \( E_\alpha \) a nonzero root vector in \( \mathfrak{g} \) of weight \( \alpha \). Let \( P \) and \( P' \) be the sets of positive roots in \( \Delta \) defined respectively by \( \mathfrak{t} \) and \( \mathfrak{r}' \). Next let \( P_\mathfrak{t} \) and \( P'_\mathfrak{t} \) be the sets of positive roots in \( \Delta_\mathfrak{t} \) defined respectively by \( \mathfrak{t} \) and \( \mathfrak{t}' \). Let \( \delta \) and \( \delta' \) denote half the sum of the roots in \( P \) and \( P' \) respectively and let \( \delta_\mathfrak{t} \) and \( \delta'_\mathfrak{t} \) denote half the sum of the roots in \( P_\mathfrak{t} \) and \( P'_\mathfrak{t} \) respectively.
Let $P''$ be a $\theta$ stable positive system of roots in $\triangle$ such that if $r''$ is the corresponding $\theta$ stable Borel subalgebra of $g$ then

\[(2.3) \quad r'' \supseteq r'_t \quad \text{and} \quad P'' \supseteq P' \cap -P.\]

\[(2.4) \quad r'' \supseteq r'_k \quad \text{and} \quad P'' \supseteq P' \cap -P.\]

\[\textbf{(2.5) Remark} \quad \text{If one takes } P'' = P' \text{ then (2.3) and (2.4) are clearly satisfied. If } q \text{ is a Borel subalgebra then } P' = P \text{ and } P'' \text{ which satisfies (2.3) also satisfies (2.4). If } q = g, \text{ then } P' = -P; \text{ the only candidate which satisfies (2.3) and (2.4) is } P''.\]

We can now describe the modules that we want to construct. As usual for $\alpha$ in $P$ denote by $H_\alpha$ the element of $i b_0 + a_0$ such that $\lambda(H_\alpha) = 2(\lambda, \alpha)/(\alpha, \alpha)$ for every $\lambda$ in $bX$. Similarly for $\phi$ in $P_k$, denote by $H^k_\phi$ the element of $i b_0$ such that $\lambda(H^k_\phi) = 2(\lambda, \phi)/(\phi, \phi)$ for every $\lambda$ in $bX$. (Note: The Killing form of $g$ induces a nondegenerate bilinear form on $b$ which in turn induces one on $bX$).

Let $F(P'' : q, r)$ be the set of all elements $\mu$ in $hX$ with the following properties:

\[(2.6) \quad \mu(H_\alpha) \text{ is a nonnegative integer for every } \alpha \text{ in } P''.\]

\[(2.7) \quad \mu(H^k_\phi) \text{ is nonzero for every } \phi \text{ in } P_k \text{ and } \mu(H_\phi) \text{ is nonzero for every } \alpha \text{ in } P \cap -P'.\]

\[\textbf{Example} \quad \text{Suppose } \mu \text{ belonging to } hX \text{ is such that } \mu(H_\alpha) \text{ is a positive integer for every } \alpha \text{ in } P''. \text{ Then one can show that } \mu \text{ belongs to } F(P'' : q, r). \text{ The method of showing that } \mu(H^k_\phi) \text{ is nonzero for every } \phi \text{ in } P_k \text{ can be found in the proof of (3.6).}\]

We now use some definitions and notations from [3, §§ 2, 5] (cf. also §§ 3, 5 here). Let $U^t$ be the centralizer of $t$ in $U(g)$. Let $\mu \in F(P'' : q, r)$. Our aim is to construct a $t$-finite irreducible $U(g)$ module, denoted $D_{P'' : q, r}(\mu)$ in which the irreducible $t$ type with highest weight $-t(\tau\mu + \tau\delta - \tau\delta_t - \delta_t)$ (cf. 3.7) occurs with multiplicity one and such that on the corresponding isotypical $U(t)$ submodule, elements of $U^t$ act by scalars given by the homomorphism $\chi_{P'' : q, r}$ (cf. § 5).

\[(2.8) \quad \text{Remark} \quad \text{Fix } q \text{ and } r. \text{ For any compatible choice of } P'' \text{ and for any element } \mu \text{ in } F(P'' : q, r), \text{ we will show (cf.3.6) that (i) } -\mu - \delta(H_\alpha) \text{ is a nonnegative integer for every } \alpha \text{ in } P \cap -P' \text{ and (ii) } \tau\mu + \tau\delta - \tau\delta_t - \delta_t(H^k_\phi) \text{ is a nonnegative integer for every } \phi \text{ in } P_t. \text{ Now define } F(q, r) \text{ to consist of all } \mu \text{ in } hX \text{ satisfying (i) and (ii) above. In general } F(q, r) \text{ properly contains } \cup_{P''} F(P'' : q : r). \text{ Our constructions}\]

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and proofs in §§ 3, 4, 5 go through perfectly well for any \( \mu \) in \( \mathcal{F}(q, r) \) and so we do have a \( \mathfrak{t} \)-finite irreducible \( U(\mathfrak{g}) \) module in which the irreducible \( \mathfrak{t} \) type with highest weight \(-t(\tau \mu + \tau \delta - \tau \delta_\mathfrak{t})\) occurs with multiplicity one and such that on the corresponding isotypical \( U(\mathfrak{t}) \) submodule elements of \( U^\mathfrak{t} \) act by scalars given by \( \chi_{P_{\mathfrak{t},-\mu-\delta}} \). We have restricted ourselves to the subsets \( F(P' : q, r) \) rather than all of \( \mathcal{F}(q, r) \) only because condition (ii) is the definition of \( \mathcal{F}(q, r) \) is quite incomprehensible.

\[ \section{3} \]

Choose and fix an element \( \mu \) in \( \mathcal{F}(P'' : q, r) \) as in \( \section{2} \) (cf. (2.6) and (2.7)). For facts about Verma modules that we will be using we refer to \([1, 2, 5, 6]\).

Let \( M \) be any \( U(\mathfrak{g}) \) module. Let \( Q \) be a subset of \( \Delta_\mathfrak{k} \). An element \( v \) of \( M \) is said to be \( Q \) extreme if \( X_\varphi \cdot v = 0 \) for every \( \varphi \) in \( Q \). For \( \lambda \) in \( \mathfrak{h}^X \), \( v \) is called a weight vector of weight \( \lambda \) with respect to \( \mathfrak{b} \) if \( H \cdot v = \lambda(H) \cdot v \) for all \( H \) in \( \mathfrak{b} \). By \( J(M) \) we denote the set of all \( \lambda \) in \( \mathfrak{b}^X \) for which there exists a nonzero weight vector of weight \( \lambda \) in \( M \), which is \( P_\mathfrak{t} \) extreme where \( P_\mathfrak{t} \) is the positive system of roots in \( \Delta_\mathfrak{t} \) defined in \( \section{2} \). For \( \varphi \) in \( \Delta_\mathfrak{t} \), \( M \) is said to be \( X_\varphi \) free if \( X_\varphi \cdot v = 0 \) implies \( v = 0 \). For a subalgebra \( \mathfrak{s} \) of \( \mathfrak{g} \), \( M \) is said to be \( \mathfrak{s} \)-finite if every vector of \( M \) lies in a finite dimensional \( \mathfrak{s} \) submodule of \( M \). For any \( \eta \) in \( \pi_\mathfrak{r} \) let \( m(\eta) \) denote the subalgebra of \( \mathfrak{g} \) spanned by the elements \( X_\eta, X_{-\eta} \) and \( H^k_\eta \). For the notion of \( U(\mathfrak{t}) \) module of \( \text{‘type } P_\mathfrak{t} \) we refer to \([3, \section{2}]\).

Let \( P_0 \) be a positive system of roots of \( \Delta \) and let \( \lambda \) in \( \mathfrak{b}^X \). The Verma module \( V_{\mathfrak{g},P_0,\lambda} \) of \( U(\mathfrak{g}) \) is defined as follows: It is the quotient of \( U(\mathfrak{g}) \) by the left ideal generated by the elements \( H - \lambda(H), (H \in \mathfrak{h}) \) and \( E_\alpha (\alpha \in P_0) \). The Verma modules of \( U(\mathfrak{t}) \) are defined similarly. We will suppress \( \mathfrak{g} \) and write \( V_{P_0,\lambda} \) for the Verma module \( V_{\mathfrak{g},P_0,\lambda} \).

We have the inclusions \( \mathfrak{h} \subseteq \mathfrak{r} \subseteq \mathfrak{q} \) (cf. \( \section{2} \)). Let \( \pi \) be the set of simple roots for \( P \). The parabolic subalgebras of \( \mathfrak{g} \) containing \( \mathfrak{r} \) are in one to one correspondence with subsets of \( \pi \). The subset of \( \pi \) corresponding to \( \mathfrak{q} \) is got as follows: Let \( \sigma \) in \( \mathfrak{h}^X \) be defined by \( \sigma(H) = \text{trace } (\text{ad}H) \vert u \). Then

\[ \pi(\mathfrak{q}) = \{ \alpha \in \pi \mid (\sigma, \alpha) = 0 \} \]

From standard facts about parabolic subalgebras (cf. \([8, \section{1.2}]\)) we know that elements of \( P \cap -P' \) are of the form \( \sum m_i \alpha_i \) where \( m_i \) are nonnegative integers and \( \alpha_i \) are in \( \pi(\mathfrak{q}) \). For \( \alpha \) in \( \Delta \) the element \( s_\alpha \) of
$W_g$ is the reflection corresponding to $\alpha$. It is given by $s_\alpha(\lambda) = \lambda - 2(\lambda, \alpha)/(\alpha, \alpha) \cdot \alpha$. We now define a $U(\mathfrak{g})$ module $W_1$ by

\begin{equation}
W_1 = V_{P, -\mu - \delta}
\end{equation}

considered as a $U(\mathfrak{t})$ module it has some nice properties.

(3.4) Lemma $W_1$ considered as a module for $U(\mathfrak{t})$ is a weight module with respect to $\mathfrak{b}$; i.e. $W_1$ is the sum of the weight spaces with respect to $\mathfrak{b}$. Denoting also $-\mu - \delta$ the restriction of $-\mu - \delta$ to $\mathfrak{b}$, all the weights are of the form $-\mu - \delta - \sum n_i \varphi_i$ where $\varphi_i$ are elements of $P$ and $n_i$ are positive integers. Finally the weight spaces are finite dimensional and the weight space corresponding to $-\mu - \delta$ is one dimensional.

Proof Since as a $U(\mathfrak{g})$ module $W_1$ is the sum of weight spaces with respect to $\mathfrak{h} = \mathfrak{b} + \mathfrak{a}$, the first statement is clear. Since no root $\alpha$ in $\triangle$ is identically zero on $\mathfrak{b}$, we can pick up an element $H$ in $\mathfrak{b}$ such that for every $\alpha$ in $P$, $\alpha(H)$ is real and positive. As a $U(\mathfrak{g})$ module, the weights of $W_1$ with respect to $\mathfrak{h}$ are of the form $-\mu - \delta - \sum m_i \alpha_i$ ($\alpha_i \in P, m_i$ nonnegative integers). By considering the action of $H$ it is clear that weight spaces of $W_1$ with respect to $\mathfrak{b}$ are finite dimensional and the weight space of $\mathfrak{b}$ with weight $-\mu - \delta$ is one dimensional. Finally since $P$ is $\theta$ stable the restriction to $\mathfrak{b}$ of the weights with respect to $\mathfrak{h}$ are of the form $-\mu - \delta - \sum n_i \varphi_i$ where $\varphi_i$ are in $P$ and $n_i$ nonnegative integers.

(q.e.d)

(3.5) Corollary The $U(k)$ submodule of $W_1$ generated by the unique weight vector in $W_1$ of weight $-\mu - \delta$ is isomorphic to the $U(\mathfrak{t})$ Verma module $V_{\mathfrak{t}, P, -\mu - \delta}$. $W_1$ is $X_{-\varphi}$ free for every $\varphi$ in $P_t$.

Proof Let $v_1$ be the nonzero weight vector in $W_1$ of weight $-\mu - \delta$. $v_1$ is killed by every element of $[\mathfrak{r}, \mathfrak{r}]$ hence in particular by every element of $[\mathfrak{r}, \mathfrak{r}]$. On the other hand let $\mathfrak{r}$ be the unique Borel subalgebra of $\mathfrak{g}$ such that $\mathfrak{r} \cap \mathfrak{r} = \mathfrak{h}$ and let $\mathfrak{n}(\mathfrak{r})$ be the unipotent radical of $\mathfrak{r}$. If $\mathfrak{r}_t = \mathfrak{r} \cap \mathfrak{t}$, then $\mathfrak{r}_t$ is the unique Borel subalgebra of $\mathfrak{t}$ such that $\mathfrak{r}_t \cap \mathfrak{r}_t = \mathfrak{b}$. Let $U(\mathfrak{n}(\mathfrak{r}))$ and $U(\mathfrak{n}(\mathfrak{r}_t))$ denote the corresponding enveloping algebras considered as subalgebra of $U(\mathfrak{g})$. One knows that $W_1$ is $U(\mathfrak{n}(\mathfrak{r}_t))$ free, [2]. Hence in particular it is $U(\mathfrak{n}(\mathfrak{r}_t))$ free. The corollary now follows from [2,7.1.8].

(q.e.d.)
There is an ascending chain of $U(\mathfrak{k})$ Verma modules containing $V_{t,P_{\mu}-\mu-\delta}$. This chain will give rise to a chain of $U(\mathfrak{g})$ modules, which is fundamental in the work [3].

Recall the two distinguished elements $t$ and $\tau$ of $W_{\mathfrak{t}}$ from § 2. The highest weight of the special irreducible representation of $\mathfrak{t}$ which the $U(\mathfrak{g})$ module $D_{\alpha^\vee,\mu}(\mu)$ will contain is described in the corollary to the lemma below.

(3.6) Lemma (i) $-\mu - \delta(H_{\alpha})$ is a nonnegative integer for every $\alpha$ in $P \cap -P'$ and (ii) $\tau\mu + \tau\delta - \tau\delta(H_{\phi})$ is a nonnegative integer for every $\phi$ in $P_\mathfrak{k}$.

Proof By (2.4), (2.7) and (2.8), one sees that $-\mu(H_{\alpha})$ is a positive integer for every $\alpha$ in $P \cap -P'$. The elements of $P \cap -P'$ are nonnegative integral linear combination of elements of $\pi(q)$. Since $\delta(H_{\alpha}) = 1$ for every $\alpha$ in $\pi(q)$ it now follows that $-\mu - \delta(H_{\alpha})$ is a nonnegative integer for every $\alpha$ in $P \cap -P'$.

To prove (ii) first suppose $\phi$ lies in $P_{\phi}' \cap P_{\mathfrak{k}}$. We will show that $\tau\mu - \delta(H_{\phi})$ and $\tau\delta - \tau\delta(H_{\phi})$ are both nonnegative integers. For this it is enough to show that $\tau\mu(H_{\phi})$ is a positive integer for every $\phi$ in $P_{\mathfrak{k}}$ and that $\tau\mu(H_{\phi})$ is a positive integer for every $\phi$ in $\tau P_{\mathfrak{k}}$. By (2.6) there exists a finite dimensional representation of $\mathfrak{g}$ having a weight vector $\nu$ of weight $\delta$ with respect to the Cartan subalgebra $\mathfrak{h}$ and such that $\nu$ is annihilated by $[\mathfrak{r}', \mathfrak{r}']$ (cf. (2.3)). Since $\mathfrak{r}' \subseteq \mathfrak{r}'', \nu$ is in particular annihilated by $[\mathfrak{r}', \mathfrak{r}']$. It is clear from this that $\mu(H_{\phi})$ is a nonnegative integer for every $\phi$ in $P_{\phi}'$. In view of (2.7), $\mu(H_{\phi})$ is then a positive integer for every $\phi$ in $P_{\phi}'$. Note that $\tau P_{\phi}' = P_{\mathfrak{k}}$. Hence $\tau\mu(H_{\phi})$ is a positive integer for every $\phi$ in $P_{\mathfrak{k}}$. It remains to show that $\tau\delta(H_{\phi})$ is a positive integer for every $\phi$ in $P_{\mathfrak{k}}$. For this consider the representation $\rho$ of $\mathfrak{g}$ having a weight vector $\nu$ of weight $\delta$ with respect to the Cartan subalgebra $\mathfrak{h}$ and annihilated by $[\mathfrak{r}, \mathfrak{r}]$. Clearly then $\nu$ is annihilated by $[\mathfrak{r}, \mathfrak{r}]$, hence $\delta(H_{\phi})$ is a positive integer for every $\phi$ in $P_{\mathfrak{k}}$. To show that $\delta(H_{\phi})$ is nonzero we give the following reason: one can easily see that the stabilizer of $\nu$ in $\mathfrak{g}$ is exactly $\mathfrak{r}$. If $\delta(H_{\phi})$ is zero for some $\phi$ in $P_{\mathfrak{k}}$, then $X_{-\phi}$ would stabilize $\nu$. But $X_{-\phi}$ does not belong to $\mathfrak{r}$. Hence $\delta(H_{\phi})$ is a positive integer for every $\phi$ in $P_{\mathfrak{k}}$, so that $\tau\delta(H_{\phi})$ is a positive integer for every $\phi$ in $\tau P_{\mathfrak{k}}$.

Now suppose $\phi$ lies in $P_{\phi'} \cap -P_{\phi}'$. Let $\mathfrak{r}(q)$ be the maximal reductive subalgebra of $\mathfrak{q}$ defined by $\mathfrak{r}(q) = \mathfrak{h} + \sum_{\alpha \in \mathfrak{q} \cap -\mathfrak{p}'} (\mathfrak{g}^\alpha + \mathfrak{g}^{-\alpha})$. By (ii) $-\mu - \delta(H_{\alpha})$ is a nonnegative integer for every $\alpha$ in $P \cap -P'$. Hence, if $n_{\mathfrak{q}}(q) = \sum_{\alpha \in \mathfrak{q} \cap -\mathfrak{p}'} \mathfrak{g}^\alpha$, there exists a finite dimensional representation of $\mathfrak{r}(q)$ and a weight vector for $\mathfrak{h}$ of weight $-\mu - \delta$ annihilated by all of
$\mathfrak{n}_{\mathfrak{t}(q)}$, hence in particular by $\mathfrak{t} \cap \mathfrak{n}_{\mathfrak{t}(q)}$. Observe that $P_t \cap -P_t'$ is precisely the set of roots in $P_t$, whose corresponding root spaces span $\mathfrak{t} \cap \mathfrak{n}_{\mathfrak{t}(q)}$. Thus there exists a finite dimensional representation of $b + \sum_{\varphi \in P_t \cap -P_t'} (C \cdot X_{\varphi} + C \cdot X_{\varphi})$ with a weight vector for $b$ of weight $-\mu - \delta$ annihilated by $X_{\varphi}$ for every $\varphi$ in $P_t \cap -P_t'$. Hence we conclude that $-\mu - \delta(H^\ell_{i})$ is a nonnegative integer for every $\varphi$ in $P_t \cap -P_t'$. Since $-\tau(P_t \cap -P_t') = P_t \cap P_t'$, $\tau(\mu + \delta)(H^\ell_{i})$ is a nonnegative integer for every $\varphi$ in $P_t \cap -P_t'$. On the other hand $\tau\delta = \delta^t = \text{half the sum of}$ the roots in $P_t'$, while $\delta_t + \delta^t(H^\ell_{i}) = 0$ for every $\varphi$ in $P_t \cap -P_t'$. Thus $\tau\mu + \tau\delta - \tau\delta_t - \delta_t(H^\ell_{i})$ is a nonnegative integer for every $\varphi$ in $P_t \cap -P_t'$.

This completes the proof of (3.6). (q.e.d.)

**Corollary** \(-t(\tau\mu + \tau\delta - \tau\delta_t - \delta_t)(H^\ell_{i})\) is a nonnegative integer for every $\varphi$ in $P_t$.

**Proof** Clear since $-tP_t = P_t$. (q.e.d.)

Let $\pi_t$ be the set of simple roots of $P_t$. For $\varphi$ in $P_t$, let $s_{\varphi}$ be the reflection $s_{\varphi}(\lambda) = \lambda - \lambda(H_{\varphi})\varphi$ of $b^X$. If $\varphi$ lies in $\pi_t$, $s_{\varphi}$ is called a simple reflection. For $w$ in $W_t$, the length $N(w)$ of $w$ is the smallest integer $N$ such that $w$ is a product of $N$ simple reflections. A reduced word for $w$ is an expression of $w$ as a product of $N(w)$ simple reflections. Choose any reduced word for the element $\tau t$ of $W_t$. Following the notation in [5, §4.15], we write it as

$$\tau t = s_1 s_2 \cdots s_m$$

where $s_i = s_{\eta_i}, \eta_i = \varphi_{j_i}, \varphi_{j_i} \in \pi_t$. For $\lambda$ in $b^X$ and $w$ in $W_t$ write $w'(\lambda) = w(\lambda + \delta_t) - \delta_t$.

Having chosen the element $\mu$ in $F(P'' : q, r)$ we now define elements $\mu_i$ of $b^X$ as follows:

$$\mu_{m+1} = -t(\tau\mu + \tau\delta - \tau\delta_t - \delta_t)$$

and

$$\mu_i = (s_is_{i+1} \cdots s_m)'\mu_{m+1} (i = 1, \cdots, m)$$

**Note** that $\mu_1 = (\tau t)'\mu_{m+1} = -\mu - \delta$ and that $\mu_1$ and $\mu_{m+1}$ are independent of the reduced expression (3.8). We now define the positive integers $e_i$ by

$$e_i = \mu_{i+1} + \delta_t(H^\ell_{i}) \cdot (i = 1, \cdots, m).$$
With $\mu_i$ defined as above, the following inclusion relations between Verma modules are well known [2.6]:

\[(3.11) \quad V_{t,P,\mu_1} \subseteq V_{t,P,\mu_2} \subseteq \cdots \subseteq V_{t,P,\mu_{m+1}}.\]

Define elements $v_1, v_2, \ldots, v_{m+1}$ of $V_{t,P,\mu_{m+1}}$ as follows: $\mu_{m+1}$ is the unique nonzero weight vector of $V_{t,P,\mu_{m+1}}$ of weight $\mu_{m+1}$. For $i = 1, 2, \ldots, m$, $v_i = X^{e_i} \cdot v_{i+1}$. Then one knows that $v_i$ is of weight $\mu_i$ and that $V_{t,P,\mu_i} = U(\mathfrak{g})v_i$. Associated to the reduced word (3.8) and $\mu$ in $F(P'': q, r)$ is a fundamental chain of $U(\mathfrak{g})$ modules: $W_1 \subseteq W_2 \subseteq \cdots \subseteq W_{m+1}$. It will turn out that $W_1$ and $W_{m+1}$ are independent of the reduced expression (3.8). They are defined as follows: $W_1$ is defined to be $V_{\mathfrak{p},-\mu-\delta}$ as in (3.3). Then $W_{m+1}$ is given by the following lemma.

(3.12) Lemma There exists a $U(\mathfrak{g})$ module $W_{m+1} = U(\mathfrak{g}) \cdot v_{m+1}$ such that (a) $W_1$ is a $U(\mathfrak{g})$ submodule of $W_{m+1}$, (b) $v_1$ belongs to $U(\mathfrak{g})v_{m+1}$, (c) $v_{m+1}$ is a $P_t$ extreme weight vector (with respect to $\mathfrak{h}$) of weight $\mu_{m+1}$ (d) $W_{m+1}$ is $X_\varphi$ free for all $\varphi$ in $P_t$ and (e) $W_{m+1}$ is a sum of $U(\mathfrak{g})$ submodules of type $P_t$.

Proof Start with the conclusion of $V_{t,P,\mu_1}$ in $W_1$ given by Corollary 3.5 and the inclusion of $V_{t,P,\mu_i}$ in $V_{t,P,\mu_{m+1}}$ given by 3.11. By 3.5 we know that $W_1$ is $X_\varphi$ free for every $\varphi$ in $P_t$. Now [3, Lemma 4] gives us the module $W_{m+1}$ with the properties required in the lemma. (One easily sees that the results of [3 § 2] do not depend on the assumption there that rank of $\mathfrak{g} = \text{rank of } \mathfrak{t}$).

(q.e.d)

(3.13) Remark If $V$ and $\overline{V}$ are Verma modules for, say, $U(\mathfrak{t})$ then the space of $U(\mathfrak{t})$ homomorphisms of $V$ into $\overline{V}$ has dimension equal to zero or one. Thus the inclusion of $V_{t,P,\mu_{m+1}}$ given by (3.11) is independent of the reduced expression (3.8) for $\tau t$. Hence also the $U(\mathfrak{g})$ module $W_{m+1}$ and the inclusion of $W_1$ in $W_{m+1}$ with the properties listed in Lemma 3.12 can be chosen to be independent of the reduced expression (3.8).

Having defined $W_1$ and $W_{m+1}$ as above, now for any given reduced word for $\tau t$ such as (3.8), we define submodules $W_2, W_3, \ldots, W_{m}$ of $W_{m+1}$ by

\[(3.14) \quad W_i = U(\mathfrak{g})v_i\]

where $v_i$ are the elements of $W_{m+1}$ defined after (3.11). We have
$W_1 \subseteq W_2 \subseteq \cdots \subseteq W_{m+1}$ because $v_i$ belongs to $U(\mathfrak{t})v_{i+1}, (i = 1, \cdots , m)$. The properties of this chain of $U(\mathfrak{g})$ modules are summarized below from the work of [3, § 3]:

(3.15) $W_1 = V_{\mu - \delta}$ and each $W_i$ is the sum of its weight spaces with respect to $\mathfrak{b}$. Moreover as a $U(\mathfrak{t})$ module $W_i$ is the sum of $U(\mathfrak{t})$ submodules of type $P_k$.

(3.16) Each $W_i$ is a cyclic $U(\mathfrak{g})$ module with a cyclic vector $v_i$, which is a $P_k$ extreme weight vector of weight $\mu_i$ with respect to $\mathfrak{b}$, $i = 1, \cdots , m+1$.

(3.17) The $P_k$ extreme vectors of weight $\mu_i$ in $W_i$ are scalar multiples of $v_i$; for $i = 1, \cdots , m+1$, the vector $v_i$ does not belong to $W_{i-1}$.

(3.18) Each $W_i$ is $\mathfrak{X}_{-\varphi}$ free for every $\varphi$ in $P_k$ and $W_{i+1}/W_i$ is $m(\eta_i)$ finite ($i = 1, \cdots , m$).

(3.19) $v_i = \mathfrak{X}_{\eta_i}v_{i+1}(i = 1, \cdots , m)$.

(3.20) Let $w$ be in $W_k$. Let $i = 1, \cdots , m$. Suppose $w'(\mu_{m+1})$ belongs to $J(W_i)$. Then $N(w)$ equals at least $m + 1 - i$.

We will not prove the properties (3.15) to (3.20) here since they are essentially proved in [3, Lemma 5]. Though (3.20) has the same form as [3, Lemma 5, vi] its proof is different in our case. It is important to first know the case $i = 1$ of (3.20) to carry over the inductive arguments of [3, § 3] to our situation. To this end we prove the following lemma. Before that we make the following remark.

(3.21) Remark Let $H'_q$ be the element of $\mathfrak{h}$ defined by $(H'_q, H) = \text{trace } (\text{ad } H | u)$, for every $H$ belonging to $\mathfrak{h}$, where $u$ is the unipotent radical of $\mathfrak{q}$. Since $\mathfrak{q}$ and $\mathfrak{h}$ are $\theta$ invariant $\theta(H'_q) = H'_q^\ast$; hence $H'_q$ belongs to $\mathfrak{b}$. One can easily prove the following: For every $\alpha$ in $P \cap -P'$, $\alpha(H'_q)$ is a positive real number; and for every $\varphi$ in $P_\ell \cap -P'_\ell$, $\varphi(H'_q)$ equals zero while for every $\varphi$ in $P_\ell \cap P'_\ell$, $\varphi(H'_q)$ is a positive real number. (Observe that any $\varphi$ in $P_\ell \cap -P'_\ell$ is the restriction to $\mathfrak{b}$ of some $\alpha$ in $P \cap -P'$).

Now we come to the lemma which is basic to carry over the inductive arguments of [3, § 3].
\textbf{(3.22) Lemma} \textit{Let $w$ be in $W_t$. Suppose $w'(\mu_{m+1})$ belongs to $J(W_1)$. Then $N(w)$ is greater than or equal to $m$.}

\textbf{Proof} Since $w'(\mu_{m+1})$ belongs to $J(W_1)$ it is in particular a weight of $W_1$ of for $b$. Hence by (3.4), $w'(\mu_{m+1})$ is of the form $\mu_1 - \sum n_i \alpha_i | b$, where $n_i$ are nonnegative integers and $\alpha_i$ are in $P$. That is $w(\mu_{m+1} + \delta_t) = \mu_1 - \sum n_i \alpha_i | b = \tau t(\mu_{m+1} + \delta_t) - \delta_t - \sum n_i \alpha_i b$. Thus

$$\tau t(\mu_{m+1} + \delta_t) - w(\mu_{m+1} + \delta_t) = \sum n_i \alpha_i | b.$$ 

Write $\mu'_{m+1} = -t\mu_{m+1}$. Hence

(3.23) 

$$-\tau(\mu'_{m+1} + \delta_t) + wt(\mu'_{m+1} + \delta_t) = \sum n_i \alpha_i | b$$

where $n_i$ are nonnegative integers and $\alpha_i$ are in $P$. The left side of the equality in (3.23) is the sum of $wt(\mu'_{m+1} + \delta_t) - (\mu'_{m+1} + \delta_t)$ and $(\mu'_{m+1} + \delta_t) - \tau(\mu'_{m+1} + \delta_t)$. We claim that (3.23) implies

(3.24) 

$$P_t \cap -wt P_t$$

is contained in $P_t \cap -\tau P_t$.

To see this enumerate the elements of $P_t \cap -wt P_t$ in a sequence $(\epsilon_1, \epsilon_2, \ldots, \epsilon_k)$ such that $\epsilon_1$ is a simple root of $P_t$ and $\epsilon_{i+1}$ is a simple root of $s_{\epsilon_i} \cdots s_{\epsilon_1} P_t (i = 1, \ldots, k - 1)$. Then $wt = s_{\epsilon_k} \cdots s_{\epsilon_1}$ (cf. (5, 4.15.10) and [7, 8.9.13]). By induction on $i$ one can show that $(\mu'_{m+1} + \delta_t) - s_{\epsilon_i} \cdots s_{\epsilon_1} (\mu'_{m+1} + \delta_t)$ can be written as $\sum d_{j,i} \epsilon_j$ where $d_{j,i}$ are positive integers. Thus $(\mu'_{m+1} + \delta_t) - wt(\mu'_{m+1} + \delta_t)$ can be written as $d_{1,\epsilon_1} + d_{2,\epsilon_2} + \cdots + d_{k,\epsilon_k}$ where $d_j$ are positive integers. Similarly $(\mu'_{m+1} + \delta_t) - \tau(\mu'_{m+1} + \delta_t)$ can be written as $d'_1 \epsilon'_1 + d'_2 \epsilon'_2 + \cdots + d'_h \epsilon'_h$ where $d'_i$ are positive integers and $(\epsilon'_1, \ldots, \epsilon'_h)$ is an enumeration of $P_t \cap -\tau P_t$. With these observations we can write

(3.25) 

$$-\tau(\mu'_{m+1} + \delta_t) + wt(\mu'_{m+1} + \delta_t)$$

$$= (d'_1 \epsilon'_1 + \cdots + d'_h \epsilon'_h) - (d_1 \epsilon_1 + \cdots + d_k \epsilon_k)$$

where $d'_1, \ldots, d'_h, d_1, \ldots, d_k$ are positive integers. Let $H'_{q'}$ be the element of $\mathfrak{h}$ defined by $(H'_{q'}, H) = \text{trace} \left( \text{ad} \ H \ | \ u \right)$, where $u$ is the unipotent radical of $q$. Then $H'_{q'}$ belongs to $b$. We can apply remark (3.21) to (3.25) and conclude that $[-\tau(\mu'_{m+1} + \delta_t) + wt(\mu'_{m+1} + \delta_t)](H'_{q'})$ is a strictly negative real number unless (3.24) holds. But by looking at the right hand side of (3.23) and applying remark (3.21), we see that $[-\tau(\mu'_{m+1} + \delta_t) + wt(\mu'_{m+1} + \delta_t)](H'_{q'})$ is a nonnegative real number.
Thus we have proved the validity of (3.24). Now (3.24) implies that $N(wt)$ is less than or equal to $N(\tau)$. But note that $N(wt) = N(t) - N(w)$, while $N(\tau) = N(t) - N(\tau t) = N(t) - m$. Hence $N(w)$ is greater than or equal to $m$. (q.e.d.)

(3.22) Enables us to carry over the inductive arguments in [3, § 3] without any further change and obtain the properties (3.15) to (3.20).

§ 4. The ℓ-finite quotient $U(g)$ module of $W_{m+1}$

The difference between the special situation in [3] and our more general situation becomes more apparent in this section which parallels [3, § 4].

Start with an arbitrary reduced word (3.8) for $\tau t$ and let $W_1 \subseteq W_2 \subseteq \cdots \subseteq W_{m+1}$ be a fundamental chain of $U(g)$ modules satisfying (3.15) through (3.20). Recall $W_1 = V_{P,-\mu,-\delta}$. Recall the subset $\pi(q) \subseteq \pi$ corresponding to the parabolic subalgebra $q$. For $\alpha$ in $\pi$ and $\lambda$ in $h_X$ define $s^X_\alpha(\lambda) = s_\alpha(\lambda + \delta) - \delta$. By lemma 3.6, $-\mu - \delta(H_\alpha)$ is a nonnegative integer for every $\alpha$ in $P \cap -P'$, hence in particular for every $\alpha$ in $\pi(q)$. Thus one has the inclusion of the Verma modules $V_{P,s^X_\alpha(-\mu,-\delta)} \subseteq V_{P,-\mu,-\delta}$ for every $\alpha$ in $\pi(q)$. We now define a $U(g)$ submodule.

\begin{equation}
W_0 = \sum_{\alpha \in \pi(q)} V_{P,s^X_\alpha(-\mu,-\delta)} \text{ of } W_1.
\end{equation}

As is well known the Verma modules have unique proper maximal submodules. Let $I$ be the proper maximal $U(g)$ submodule of $V_{P,-\mu,-\delta}$.

Then each $V_{P,s^X_\alpha(-\mu,-\delta)} (\alpha \in \pi(q))$, is contained in $I$. Hence

\begin{equation}
v_1 \text{ does not belong to } W_0.
\end{equation}

Now fix some $i_i (i = 1, \cdots, m)$. Define a $U(g)$ submodule (relative to some reduced word (3.8) for $\tau t$) $\overline{W}_i$ of $W_{m+1}$ as follows: Let $W_{i,0}$ be the $U(g)$ submodule of all vectors in $W_{m+1}$ that are $m(\eta_i)$ finite mod $W_{i-1}$; once $W_{i,0}, \cdots, W_{i,p-1}$ are defined, $W_{i,p}$ is the $U(g)$ submodule of all vectors in $W_{m+1}$ that are $m(\eta_i, \cdots, \eta_{i+p})$ finite mod $W_{i,p-1}$, $p = 1, 2, \cdots, m - i$. We have $W_{i,0} \subseteq \cdots \subseteq W_{i,m-i}$. We then define $\overline{W}_i = W_{i,m-i}$. Define

\begin{equation}
\overline{W} = W_m + \overline{W}_1 + \overline{W}_2 + \cdots + \overline{W}_m.
\end{equation}
Thus for each reduced expression (3.8) for $\tau t$, we have defined a $U(g)$ submodule $W$ of $W_{m+1}$.

(4.4) **Proposition** For any reduced word (3.8) for $\tau t$ define the $U(g)$ submodule $W$ of $W_{m+1}$ as above. Then $v_{m+1}$ does not belong to $W$. If $\lambda \in \mathfrak{b}^X$ is such that $W_{m+1}$ has a nonzero $P_t$ extreme weight vector (with respect to $\mathfrak{b}$) of weight $\lambda$ which is nonzero mod $W$, then $(\tau t)'\lambda$ is a $P_t$ extreme weight of $W_1/W_0$.

**Proof** We refer to the proof of [3, Lemma 9].

Since we do not have a full chain of $U(g)$ modules corresponding to a reduced word for $t$ as in [3] but only a shorter chain corresponding to a reduced word for $\tau t$, we have to work more to obtain a $\mathfrak{t}$-finite quotient $U(g)$ module of $W_{m+1}$. We now define

(4.5) $W_X = \sum W$, the summation being over all reduced expressions (3.8) for $\tau$.

(4.6) **Lemma** $v_{m+1}$ does not belong to $W_X$. Let $\lambda \in \mathfrak{b}^X$ be such that there is a $P_t$ extreme vector in $W_{m+1}$ of weight $\lambda$ which is nonzero mod $W_X$. Then $(\tau t)'\lambda(H^t\phi)$ is a nonnegative integer for every $\phi$ in $P_t\cap -P_t$.

**Proof** $v_{m+1}$ is a $P_t$ extreme weight vector in $W_{m+1}$ of weight $\mu_{m+1}$. From (3.7) and the definition of $\mu_{m+1}$, we know that $\mu_{m+1}(H^t\phi)$ is a nonnegative integer for every $\phi$ in $P_t$. Now suppose $v_{m+1}$ belongs to $W_X$. Since $W_X = \sum W$, $W_X$ is a quotient of the abstract direct sum $\oplus W$, the summation being over all reduced words (3.8) for $\tau t$. We can then apply [3, Lemma 7] and conclude that for some reduced word (3.8) for $\tau t$, the corresponding $W$ has a nonzero $P_t$ extreme vector of weight $\mu_{m+1}$. This vector has to be a nonzero scalar multiple of $v_{m+1}$ in view of (3.17). Hence $v_{m+1}$ belongs to that $W$. But this contradicts (4.4). This proves the first assertion in (4.6).

Next let $\lambda$ be as in the lemma. Let $c$ be the reductive component of $\mathfrak{q}$ defined by $c = \mathfrak{h} + \sum_{\alpha \in \mathfrak{p} \cap -\mathfrak{p}'} (\mathfrak{g}^\alpha + \mathfrak{g}^{-\alpha})$. We claim that $W_1/W_0$ is $c$-finite. For this it is enough to show that the image $\overline{\sigma}_1$ in $W_1/W_0$ of $v_1$ is $c$-finite. For any $\alpha$ in $\pi(\mathfrak{q})$ the submodule $V_{g,\mathfrak{p},\sigma}^\alpha(\mu_1)$ of $W_1$ coincides with $U(\mathfrak{g})X^\alpha_{\sigma_1} X_{\sigma_1}^{\mu_1} \cdot v_1$ (cf. [2, 7.1.15]). Thus we have $W_0 = \sum_{\alpha \in \pi(\mathfrak{q})} U(\mathfrak{g})X^\alpha_{\sigma_1} X_{\sigma_1}^{\mu_1} \cdot v_1$. Hence the annihilator in $U(\mathfrak{g})$ of $\overline{\sigma}_1$ contains $U(\mathfrak{g})X^\alpha_{\sigma_1} X_{\sigma_1}^{\mu_1}$ for every $\alpha$ in $\pi(\mathfrak{q})$. This suffices in view of [2, 7.2.5] to conclude that $\overline{\sigma}_1$ is $c$-finite. Thus $W_1/W_0$ is $c$-finite.
Let \( c_\ell = c \cap \ell \). Then in particular \( W_1/W_0 \) is \( \ell \)-finite. But note that \( c_\ell = b + \sum_{\varphi \in P_\ell \cap -P_\ell} (\mathbb{C} \cdot X_\varphi + \mathbb{C} \cdot X_{-\varphi}) \).

Now choose some reduced word \((3.8)\) for \( \tau t \) and relative to it define \( \overline{W} \) as in (4.3). Note that \( \overline{W} \subseteq W_X \). For \( \lambda \) as in the lemma, choose a \( P_\ell \) extreme weight vector \( v \) in \( W_{m+1} \) which is nonzero mod \( W_X \) and is of weight \( \lambda \). Then \( v \) is in particular nonzero mod \( \overline{W} \). Hence from (4.4), \((\tau t)^' \lambda \) is a \( P_\ell \) extreme weight of \( W_1/W_0 \). Since \( W_1/W_0 \) is \( c_\ell \)-finite, it now follows that \((\tau t)^' \lambda (H_\varphi^\ell) \) is a nonnegative integer for every \( \varphi \) in \( P_\ell \cap -P'_\ell \). (q.e.d)

For our proof of the \( \ell \)-finiteness of \( W_{m+1}/W_\ell \), we need one more lemma.

**Lemma (4.7)** Let \( \eta \) be in \( b^X \). Suppose \( \eta (H_\varphi^\ell) \) is nonnegative for every \( \varphi \) in \( P_\ell \). Let \( s \) be in \( W_\ell \). Suppose \( \tau ts'^\ell \eta (H_\varphi^\ell) \) is nonnegative for every \( \varphi \) in \( P_\ell \cap -P'_\ell \). Then \( N(\tau t) = N(\tau ts') + N(s'^{-1}) \).

**Proof** \((\tau ts') \eta = \tau ts(\eta + \delta_t) - \delta_t \). Since \( \eta (H_\varphi^\ell) \) is nonnegative for every \( \varphi \) in \( P_\ell \), \( \tau ts(\eta + \delta_t)(H_\varphi^\ell) \) is negative for every \( \varphi \) in \(-\tau tsP_\ell \). Also \(-\delta_t (H_\varphi^\ell) \) is negative for every \( \varphi \) in \( P_\ell \). Hence \((\tau ts') \eta (H_\varphi^\ell) \) is negative for every \( \varphi \) in \((-\tau tsP_\ell) \cap P_\ell \). Hence the assumption implies

\[(4.8) \ P_\ell \cap -\tau tsP_\ell \subseteq \text{complement of } P_\ell \cap -P'_\ell \text{ in } P_\ell.\]

Note that \( TP_\ell = -P_\ell \) and \( \tau P_\ell = P'_\ell \). So, \(-P'_\ell = \tau tP_\ell \). So, the complement of \( P_\ell \cap -P'_\ell \) in \( P_\ell \) is \( P_\ell \cap -\tau tP_\ell \). Hence from (4.8) we have

\[(4.9) \ P_\ell \cap (-\tau tsP_\ell) \subseteq P_\ell \cap (-\tau tP_\ell).\]

Let \((\epsilon_1, \epsilon_2, \cdots, \epsilon_j, \epsilon_{j+1}, \cdots, \epsilon_m)\) be an enumeration of the elements of \((-\tau tP_\ell) \cap P_\ell \) such that \( \epsilon_1 \) is a simple root of \( P_\ell \), \( \epsilon_2 \) is a simple root of \( s_{\epsilon_1} P_\ell, \cdots, \epsilon_{j+1} \) is a simple root of \( s_{\epsilon_1} s_{\epsilon_{j-1}} \cdots s_{\epsilon_1} P_\ell (i = 1, \cdots, m - 1) \). Because of (4.9) we can further assume \((\epsilon_1, \cdots, \epsilon_j)\) is an enumeration of \((-\tau tsP_\ell) \cap P_\ell \). Let

\[\varphi'_1 = s_{\epsilon_1} \cdots s_{\epsilon_{i-1}} (\epsilon_i)(i = 1, \cdots, m)(\varphi'_1 = \epsilon_1).\]

Then \( \varphi'_1 \) belongs to \( \pi_\ell \). One can show that \( \tau t = s_{\epsilon_m} \cdots s_{\epsilon_1} \) and a reduced word for \( \tau t \) is

\[\tau t = s_{\varphi'_1} s_{\varphi'_2} \cdots s_{\varphi'_m}.\]
Similarly $\tau ts = s_\epsilon \ldots s_{\epsilon, 1}$ and a reduced word for $\tau ts$ is

(4.11) \[ \tau ts = s_{\varphi_1'} \ldots s_{\varphi_j'}. \]

Note that $N(\tau t) = m$ and $N(\tau ts) = j$. Now from (4.10) and (4.11) it is clear that $s^{-1} = s_{\varphi_{j+1}'} \ldots s_{\varphi_m'}$ is a reduced word for $s^{-1}$. These observations substantially prove the lemma. (q.e.d.)

(4.12) Remark With the data assumed in Lemma 4.7 we have actually proved more than what is asserted in (4.7): There exists a reduced word $\tau t = s_{\varphi_1'} \ldots s_{\varphi_j'} s_{\varphi_{j+1}'} \ldots s_{\varphi_m'}$ for $\tau t$ such that $s^{-1} = s_{\varphi_{j+1}'} \ldots s_{\varphi_m'}$.

The following proposition gives the $\mathfrak{t}$-finite $U(\mathfrak{g})$ module quotient of $W_{m+1}$.

(4.13) Proposition The $U(\mathfrak{g})$ module $W_{m+1}/W_X$ is $\mathfrak{t}$-finite.

Proof Let $\tau m+1$ be the image of $v_{m+1}$ in $W_{m+1}/W_X$. Since $U(\mathfrak{g})\tau m+1 = W_{m+1}/W_X$, it suffices to prove that $U(\mathfrak{t}) \cdot \tau m+1$ has finite dimension over $\mathbb{C}$. For this again, by well known facts [2, 7.2.5] it suffices to prove that the annihilator of $\tau m+1$ in $U(\mathfrak{t})$ contains $X^{e(\varphi)}$ for every $\varphi$ in $\pi_\mathfrak{t}$, where $e(\varphi) = \mu_{m+1}(H_\varphi^\mathfrak{t}) + 1$ (observe that in view of (3.7), $\mu_{m+1}(H_\varphi^\mathfrak{t})$ is a nonnegative integer for every $\varphi$ in $\pi_\mathfrak{t}$). Thus it suffices to show that for every $\varphi$ in $\pi_\mathfrak{t}$,

(4.14) \[ X^{e(\varphi)} \cdot v_{m+1} \text{ belongs to } W_X. \]

Suppose (4.14) is not true. Choose a $\varphi$ in $\pi_\mathfrak{t}$, such that $X^{e(\varphi)} v_{m+1}$ does not belong to $W_X$. Then $X^{e(\varphi)} v_{m+1}$ is a $P_\mathfrak{t}$ extreme vector of weight $s_{\varphi}(\mu_{m+1})$ in $W_{m+1}$ which is nonzero mod $W_X$. Hence by (4.6), $(\tau ts_{\varphi})' \mu_{m+1}(H_{\varphi'}^\mathfrak{t})$ is a nonnegative integer for every $\varphi'$ in $P_\mathfrak{t} \cap -P'_{\mathfrak{t}}$. We can now apply (4.7) and (4.12) and conclude that there exists a reduced word.

(4.15) \[ \tau t = s_{\varphi_1'} s_{\varphi_2'} \ldots s_{\varphi_{m-1}'} s_{\varphi_m'} (\varphi_i' \in \pi_\mathfrak{t}) \]

for $\tau t$ such that

(4.16) \[ \varphi_m' = \varphi. \]
Take the reduced word (4.15) for $\tau t$ in (3.8) and consider the corresponding modules $W_m$ and $W$. By definition $W_m \subseteq \overline{W}$. But in the fundamental chain $W_1 \subseteq \cdots \subseteq W_m \subseteq W_{m+1}$ associated to the reduced word (4.15) for $\tau t$, the module $W_m$ is simply $U(g) \cdot X_{-\varphi} v_{m+1}$. This is clear from the definitions (cf. (3.14) and the definition of $v_i$ after (3.11) and (4.16). Thus it follows that $X_{-\varphi} v_{m+1} \in \overline{W} \subseteq W_X$. But this is a contradiction to the hypothesis. Thus (4.14) is true and proved and with that also the $\mathfrak{e}$-finiteness of $W_{m+1}/W_X$. (q.e.d)

§ 5

Let $\mathfrak{b}$ be a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{h}$ its centralizer in $\mathfrak{g}$, so that $\mathfrak{h}$ is a $\theta$ stable Cartan subalgebra of $\mathfrak{g}$. Let $P$ be a system of positive roots for $(\mathfrak{g}, \mathfrak{h})$ such that $\theta(P) = P$. Let

$$n^+ = \sum_{\alpha \in P} g^\alpha$$

and

$$n^- = \sum_{\alpha \in P} g^{-\alpha}.$$

The following fact is standard if $\mathfrak{b} = \mathfrak{h}$, but it remains true in our general case.

(5.1) Lemma Let $U^\mathfrak{b}$ be the centralizer of $\mathfrak{b}$ in $U(\mathfrak{g})$. If the set $P$ of positive roots satisfies $\theta(P) = P$, we have a unique homomorphism

(5.2) $\beta_P : U^\mathfrak{b} \rightarrow U(\mathfrak{h})$

such that for any $y$ in $U^\mathfrak{b}$.

(5.3) $y \equiv \beta_P(y) \pmod{U(\mathfrak{g})n^+}$.

Proof We have

(5.4) $U(\mathfrak{g}) = U(n^- + \mathfrak{h}) \oplus U(\mathfrak{g})n^+$

and this decomposition is stable under $adH$ for every $H$ in $\mathfrak{h}$ i.e. $adH(U(n^- + \mathfrak{h})) \subseteq U(n^- + \mathfrak{h})$ and $adH(U(\mathfrak{g})n^+) \subseteq U(\mathfrak{g})n^+$. For $y$ in $U^\mathfrak{b}$, let $y = y_0 + y_1$ be its decomposition with respect to (5.4). Define
$\beta_P(y) = y_0$. We claim $\beta_P(y)$ belongs to the subalgebra $U(\mathfrak{h})$ of $U(n^- + \mathfrak{h})$. Since $y$ is in $U^b$, $y_0$ and $y_1$ are also in $U^b$. Let $S(n^- + \mathfrak{h})$ and $S(\mathfrak{h})$ denote the symmetric algebras and $\lambda$ the symmetrizer map of $S(n^- + \mathfrak{h})$ onto $U(n^- + \mathfrak{h})$. Then for $H$ in $\mathfrak{b}$, $\lambda^{-1}(y_0)$ is annihilated by $adH$ (extended as a derivation to $S(n^- + \mathfrak{h})$). It is enough to show that $\lambda^{-1}(y_0)$ belongs to $S(\mathfrak{h})$. Using (1.14), one can show that there exists an element $X_P$ in $\mathfrak{b}$ such that $\alpha(X_P)$ is a nonzero real number for every $\alpha$ in $\Delta (= \text{the roots of } (\mathfrak{g}, \mathfrak{h}))$ and such that $P$ consists of precisely those $\alpha$ in $\Delta$ such that $\alpha(\lambda^{-1}(y_0))$ is positive. It is then clear that in $S(n^- + \mathfrak{h})$, the null space for $adX_P$ is just $S(\mathfrak{h})$. Since $adX(\lambda^{-1}(y_0)) = 0$ for every $X$ in $\mathfrak{b}$, in particular $adX_P(\lambda^{-1}(y_0)) = 0$. Hence $\lambda^{-1}(y_0)$ belongs to $S(\mathfrak{h})$, so that $\beta_P(y)$ belongs to $U(\mathfrak{h})$.

Now suppose $y$ and $y'$ are in $U^b$. Let $y = y_0 + y_1$ and $y' = y_0' + y_1'$ be their decomposition with respect to (5.4), so that $\beta_P(y) = y_0$ and $\beta_P(y') = y_0'$. Then $yy' = y_0y_0' + y_0y_1' + y_1y_0' + y_1y_1'$. Clearly $y_0y_0'$ belongs to $U(\mathfrak{h})$ and $y_0y_1' + y_1y_0'$ belongs to $U(\mathfrak{g})n^+$. Also $y_1y_0' \in U(\mathfrak{g})n^+ \cdot U(\mathfrak{h}) \subseteq U(\mathfrak{g})U(\mathfrak{h})n^+$. Thus $y_0y_0'$ is the component of $yy'$ in $U(n^- + \mathfrak{h})$ with respect to (5.4). We already know that this component is in $U(\mathfrak{h})$. Thus $\beta_P$ is a homomorphism of algebras.

The centralizer $U^\mathfrak{t}$ of $\mathfrak{t}$ in $U(\mathfrak{g})$ is contained in $U^b$. As usual interpret elements of $S(\mathfrak{h})$ as polynomials on $\mathfrak{h}^\mathfrak{X}$. For any $\varphi$ in $\mathfrak{h}^\mathfrak{X}$, define a homomorphism $\chi_{P,\varphi}$ of $U^\mathfrak{t}$ into $\mathbb{C}$ as follows:

$$\chi_{P,\varphi}(y) = \beta_P(y)(\varphi) \quad (y \in U^\mathfrak{t}).$$

The main results of the previous sections can now be formulated. Let $\mathfrak{b}_0$ be a Cartan subalgebra of $\mathfrak{t}_0$ and $\mathfrak{b}$ its complexification. Let $\mathfrak{q}$ be a $\theta$ stable parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$. The centralizer $\mathfrak{h}$ of $\mathfrak{b}$ in $\mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{q}$ contains $\mathfrak{h}$. Let $\mathfrak{r}$ be a $\theta$ stable Borel subalgebra of $\mathfrak{g}$ contained in $\mathfrak{q}$ (cf. (1.15) and (1.2)). Let $P$ be the set of positive roots for $(\mathfrak{g}, \mathfrak{h})$ corresponding to $\mathfrak{r}$. Define the $\theta$ stable Borel subalgebra $\mathfrak{r}' \subseteq \mathfrak{q}$ by (2.1). Choose a $\theta$ stable positive system $P''$ of roots of $(\mathfrak{g}, \mathfrak{h})$ having properties (2.3) and (2.4). Denote by $F(P'': \mathfrak{q}, \mathfrak{r})$ the set of all elements $\mu$ in $\mathfrak{h}^\mathfrak{X}$ having properties (2.6) and (2.7). Now choose a $\mu$ in $F(P'': \mathfrak{q}, \mathfrak{r})$ and recall the objects associated to it in §§ 3, 4.

We can now state

**Theorem** Let $\mathfrak{q}$ be a $\theta$ stable parabolic subalgebra. Let $\mu \in F(P'': \mathfrak{q}, \mathfrak{r})$. Let $W_{P''(\mu)} = W_{m+1}/W_X$ (cf. (3.12) and (4.5)). Then $W_{P''(\mu)}$ is a $\mathfrak{t}$ finite $U(\mathfrak{g})$ module having the following properties:
(i) $W_{P': q, r}^{\tau}(\mu) = U(g)\overline{\pi}_{m+1}$, where $\overline{\pi}_{m+1}$ is the image of the vector $v_{m+1}$ of $W_{m+1}$. The irreducible finite dimensional representation of $\mathfrak{t}$ with highest weight $-t(\tau\mu + \tau\delta - \tau\delta_t - \delta_t)$ occurs with multiplicity one in $W_{P': q, r}^{\tau}(\mu)$. The corresponding isotypical $U(\mathfrak{k})$ submodule of $W_{P': q, r}^{\tau}$ is $U(\mathfrak{k})\overline{\pi}_{m+1}$; on this space elements of $U(\mathfrak{g})$ act by scalars given by the homomorphism $\chi_{P', -\mu - \delta}$.

(ii) If $\tau_\lambda$ is an irreducible finite dimensional representation of $\mathfrak{k}$ with highest weight $\lambda$ with respect to $P_\mathfrak{t}$, then the multiplicity of $\tau_\lambda$ in $W_{P': q, r}^{\tau}(\mu)$ is finite; it is zero if $\lambda$ is not of the form $\tau_\lambda'(-\mu - \delta - \sum_{\varphi \in P} m_\varphi \varphi) | b$ where $m_\varphi$ are nonnegative integers.

**Proof** By (4.13), we know that $W_{P': q, r}^{\tau}(\mu)$ is nonzero and $\mathfrak{t}$-finite. By (4.6) the vector $v_{m+1}$ of $W_{m+1}$ does not belong to $W_X$. The image of $v_{m+1}$ in $W_{P': q, r}^{\tau}(\mu)$ is $P_\mathfrak{t}$ extreme of weight $(\tau_\lambda')(-\mu - \delta) = -t(\tau\mu + \tau\delta - \tau\delta_t - \delta_t)$ (which is dominant by (3.7)) and this image generates an irreducible $\mathfrak{k}$-module with highest weight $-t(\tau\mu + \tau\delta + \tau\delta_t - \delta_t)$ with respect to $P_\mathfrak{t}$.

Based on the preceding sections one can complete the proof of the theorem in the same way as [3, Theorem 1].

It is easy to conclude from (5.6) that $W_{P': q, r}^{\tau}(\mu)$ has a unique proper maximal $U(\mathfrak{g})$ submodule and hence $W_{P': q, r}^{\tau}(\mu)$ has a unique nonzero quotient $U(\mathfrak{g})$ module which is irreducible. We denote this $U(\mathfrak{g})$ module by $D_{P': q, r}^{\tau}(\mu)$. The following theorem is now immediate from (5.6).

(5.7) **Theorem** Let $\mu \in F(P'' : q, r)$. Up to equivalence there exists a unique $\mathfrak{t}$-finite irreducible $U(\mathfrak{g})$ module $D_{P'' : q, r}^{\tau}(\mu)$ having the following property: The finite dimensional irreducible $U(\mathfrak{k})$ module with highest weight $-t(\tau\mu + \tau\delta - \tau\delta_t - \delta_t)$ (with respect to $P_\mathfrak{t}$) occurs with multiplicity one in $D_{P'' : q, r}^{\tau}(\mu)$ and the action of $U(\mathfrak{t})$ on the corresponding isotypical $U(\mathfrak{k})$ submodule is given by the homomorphism $\chi_{P', -\mu - \delta}$.

The uniqueness follows from the well known theorem of Harish Chandra [4]: An irreducible $\mathfrak{t}$- finite $U(\mathfrak{g})$ module $M$ is completely determined by a nonzero isotypical $U(\mathfrak{t})$ submodule of $M$ and the action of $U(\mathfrak{t})$ on it.
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