On the transcendence of infinite sums of values of rational functions

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Abstract

We investigate convergent sums $T = \sum_{n=1}^{\infty} \frac{P(n)}{Q(n)}$ and $U = \sum_{n=1}^{\infty} (-1)^n \frac{P(n)}{Q(n)}$ where $P(X), Q(X) \in \mathbb{Q}[X]$, and $Q(X)$ has only simple rational roots. Adhikari, Shorey and the authors have shown that T and U are either rational or transcendental. In the present paper simple necessary and sufficient conditions are formulated for the transcendence of T and U if the degree of Q is 3 and 2, respectively.

1 Introduction.

Throughout the paper we take $q > 1$ to be an integer and $f(x)$ a number theoretic function which is periodic mod q with $\sum_{i=1}^{q} f(i) = 0$. We denote by $\mathbb{Z}, \mathbb{Q}, \overline{\mathbb{Q}}$, the ring of rational integers, the field of rationals and the field of algebraic numbers. The first general result on the non-vanishing of

$$S = \sum_{n=1}^{\infty} \frac{f(n)}{n}$$

is probably due to Dirichlet who showed in 1839 by his class number formulas that $L(1, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n} \neq 0$ if χ is a non-principal Dirichlet character. Around 1970, Chowla and Siegel obtained some non-vanishing results on periodic functions f assuming only values $1, -1$ and 0, see [4]. Chowla [4] and Erdős (see [5]) formulated some related conjectures, one of which was proved by Baker, Birch and Wirsing [3] in 1973. They used Baker's theory on linear forms in logarithms to establish that $S \neq 0$ if f is a non-vanishing function defined on the integers with algebraic values and period q such that

(i) $f(r) = 0$ if $1 < \gcd(r, q) < q$.

(ii) the cyclotomic polynomial Φ_q is irreducible over $\mathbb{Q}(f(1), \ldots, f(q))$.

If q is prime, then (i) is vacuous and if f is rational valued then (ii) holds trivially. Baker, Birch and Wirsing further showed that their result would be false if (i) or (ii) is omitted. Indeed, let $q = p^2$ where p is a prime and f be defined by

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = (1 - p^{1-s})^2 \zeta(s)$$

where $\zeta(s)$ denotes the Riemann zeta function. For $p = 2$, this yields

$$1 - \frac{3}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{3}{6} + \frac{1}{7} + \frac{1}{8} + \cdots = 0$$

with period 4. Thus (i) is necessary. To see that (ii) cannot be omitted, they considered the quadratic characters $\chi,\ \chi'$ mod 12 with conductors 3 and 4, respectively, and $f = 2\chi - \sqrt{3}\chi'$. The vanishing of S follows immediately from the values $L(1, \chi) = \frac{\pi}{2\sqrt{3}}$ and $L(1, \chi') = \frac{\pi}{3}$. They also characterised all odd algebraic valued functions f for which $S = 0$.

In 1982 Okada [6] published a result which provides a description of all functions for which (ii) holds and $S = 0$. The criterion is a system of $\varphi(q) + \omega(q)$ homogeneous linear equations in $f(1), \ldots, f(q)$ with rational coefficients where $\varphi(q)$ denotes the Euler's totient function and $\omega(q)$ is the number of distinct prime divisors of q. The precise result is stated in Section 2. Okada's proof depends on the basic result on the linear independence of the logarithms of algebraic numbers and on the non-vanishing of $L(1, \chi)$ if χ is a non-principal Dirichlet character. Okada's result was used by Tijdeman [7] to prove that $S \neq 0$ if $f : \mathbb{Z} \to \overline{\mathbb{Q}}$ is completely multiplicative, and if f is multiplicative such that $|f(p^k)| < p - 1$ for every prime divisor p of q and every positive integer k (see Section 2).

In 2001, Adhikari, Saradha, Shorey and Tijdeman [1] proved that if $S \neq 0$, then S is transcendental. They used this result to prove that if $P(X) \in \overline{\mathbb{Q}}[X]$ and $Q(X) \in \mathbb{Q}[X]$ where $Q(X)$ is a polynomial with simple rational roots, then

$$T = \sum_{n=1}^{\infty} \frac{P(n)}{Q(n)}$$

is transcendental provided T converges and is irrational. They proved many related results. Thus the question of deciding whether $S = 0$ or not has gained importance. For more information on the developments sketched above we refer to [2] and [7].

In the present paper, we rephrase Okada's theorem so that it becomes a decomposition lemma (Lemma 1) and use it to derive that $S = 0$ implies $\sum_{n=1}^{\infty} \frac{f(an)}{n} = 0$ for any positive integer a coprime to q (Corollary 1). This is an intermediate result in [3, p.231, formula (7)].

Lemma 1 and Corollary 1 are used in Section 3 to prove that

$$\sum_{n=0}^{\infty} \frac{(-1)^n(an+b)}{(qn+s_1)(qn+s_2)} \tag{1}$$

with $a, b, s_1, s_2 \in \mathbb{Z}, s_1 \neq s_2, |a| + |b| > 0$ and $-\frac{s_1}{q}, -\frac{s_2}{q}$ never a non-negative integer, is transcendental except when $s_1 \equiv s_2 \pmod q$ and $a = 0$. In the exceptional case the sum is rational. In Section 4, we use Lemma 1 to prove that

$$\sum_{n=0}^{\infty} \frac{an+b}{(qn+s_1)(qn+s_2)(qn+s_3)} \tag{2}$$

with $a, b, s_1, s_2, s_3 \in \mathbb{Z}, s_1, s_2, s_3$ distinct, $a^2 + b^2 \neq 0$ and $-s_1/q, -s_2/q, -s_3/q$ never a non-negative integer, is transcendental except when $s_1 \equiv s_2 \equiv s_3 (\bmod q)$ or $s_1 \equiv s_2 \pmod q$ and $as_3 = bq$ or $s_1 \equiv s_3 \pmod q$ and $as_2 = bq$ or $s_2 \equiv s_3 \pmod q$ and $as_1 = bq$. In the exceptional cases the sum is rational. On the other hand, the examples

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(3n+1)(3n+3)(3n+5)} = \frac{1}{16}; \ \sum_{n=0}^{\infty} \frac{16n^2+12n-1}{(4n+1)(4n+2)(4n+3)(4n+4)} = 0$$

show that the corresponding results for (1) and (2) when the denominators are replaced with 3 and 4 factors, respectively, are not valid.

2 Decomposition Lemma.

We first introduce some notation and state Okada's main result. We denote by P the set of all primes dividing q. For $p \in P$ and $n \in \mathbb{Z}$, we denote by $v_p(n)$ the exponent to which p divides n. We write

$$\begin{aligned} J &= \{a \in \mathbb{Z} \mid 1 \leq a \leq q, \ \ \text{gcd } (a,q) = 1\}, \\ L &= \{r \in \mathbb{Z} \mid 1 \leq r \leq q, \ \ 1 < \text{gcd } (r,q) < q\} \end{aligned}$$

and

$$L' = L \cup \{q\}.$$

We define for $r \in L'$ and $p \in P$,

$$P(r) = \{p \in P \mid v_p(r) \geq v_p(q)\}$$

and

$$\varepsilon(r,p) = \begin{cases} v_p(q) + \frac{1}{p-1} \text{ if } \ p \in P(r) \\ v_p(r) \text{ otherwise.} \end{cases}$$

We further define for $r \in L'$ and $a \in J$,

$$A(r,a) = \frac{1}{\gcd(r,q)} \prod_{p \in P(r)} (1 - \frac{1}{p^{\varphi(q)}})^{-1} \sum_{n \in S(r)} \frac{\delta(r,a,n)}{n}$$

where

$$S(r) = \left\{ \prod_{p \in P(r)} p^{\alpha(p)} \mid 0 \leq \alpha(p) < \varphi(q) \right\}$$

and

$$\delta(r,a,n) = \begin{cases} 1 \text{ if } \ r \equiv an \gcd(r,q) \pmod{q} \\ 0 \text{ otherwise} \end{cases}$$

Theorem A. (Okada.) *If Φ_q is irreducible over $\mathbb{Q}(f(1),\ldots,f(q))$, then $S = 0$ if and only if*

$$f(a) + \sum_{r \in L} f(r)A(r,a) + \frac{f(q)}{\varphi(q)} = 0 \text{ for } a \in J \tag{3}$$

and

$$\sum_{r \in L'} f(r)\varepsilon(r,p) = 0 \text{ for } p \in P. \tag{4}$$

We observe that (3) and (4) form a system of $\varphi(q)+\omega(q)$ homogeneous linear equations in $f(1),\ldots,f(q)$ with non-negative rational coefficients. Suppose $f \not\equiv 0$ and $S = 0$. By (4), $f(q) = 0$ if $f(r) = 0$ for each $r \in L$. Hence, by (3), there exists at least one $r \in L$ for which $f(r) \neq 0$. So Theorem A implies the result of Baker, Birch and Wirsing mentioned in Section 1. In particular we find that if q is prime, then $S \neq 0$ in accordance with Chowla's conjecture.

Now we give an equivalent version of Theorem A.

Lemma 1. (Decomposition lemma.) *Let Φ_q be irreducible over $\mathbb{Q}(f(1),\ldots,f(q))$. Let M be the set of positive integers which are composed of prime factors of q and let*

$$\varepsilon(r,p) = \begin{cases} v_p(q) + \frac{1}{p-1} \text{ if } v_p(r) \geq v_p(q) \\ v_p(r) \ \textit{otherwise} \end{cases}$$

Then $S = 0$ *if and only if*

$$\sum_{m\in M}\frac{f(am)}{m} = 0 \textit{ for every } a \textit{ with } 0 < a < q,\ \gcd(a,q) = 1 \tag{5}$$

and

$$\sum_{\substack{r=1\\ \gcd(r,q)>1}}^{q} f(r)\varepsilon(r,p) = 0 \textit{ for every prime divisor } p \textit{ of } q. \tag{6}$$

Proof. Note that $\varepsilon(r,p) \neq 0$ implies $p|r$, hence $r \in L'$. So, by Theorem A, it suffices to show that

$$f(a) + \sum_{r\in L} f(r)A(r,a) + \frac{f(q)}{\varphi(q)} = \sum_{m\in M}\frac{f(am)}{m}. \tag{7}$$

For any integer $r \geq 1$, we denote by $M(r)$ the set of positive integers which are composed of prime factors from $P(r)$. Thus $M(r) = M(\gcd(r,q))$ and $M = M(q)$. We consider

$$\begin{aligned}\sum_{m\in M}\frac{f(am)}{m} &= f(a) + \sum_{r\in L}\sum_{\substack{m\in M\\ am\equiv r \bmod q}}\frac{f(am)}{m} + \sum_{\substack{m\in M\\ q|m}}\frac{f(am)}{m}\\ &= f(a) + \sum_{r\in L}\frac{f(r)}{\gcd(r,q)}\sum_{n\in M(r)}\frac{\delta(r,a,n)}{n} + \frac{f(q)}{q}\sum_{n\in M(q)}\frac{1}{n} =: V.\end{aligned}$$

We have

$$\sum_{n\in M(q)}\frac{1}{n} = \prod_{p|q}\left(1+\frac{1}{p}+\frac{1}{p^2}+\cdots\right) = \prod_{p|q}\frac{1}{1-\frac{1}{p}} = \frac{q}{\varphi(q)}. \tag{8}$$

Let $n \in M(r)$ and $\delta(r,a,n) = 1$. We have $v_p(n) = 0$ if $p \notin P(r)$. If $p \in P(r)$, then $\gcd(p, \frac{q}{\gcd(r,q)}) = 1$ whence $p^{\varphi(\frac{q}{\gcd(r,q)})} \equiv 1 (\mathrm{mod} \frac{q}{\gcd(r,q)})$. Since $\varphi(\frac{q}{\gcd(r,q)}) \mid \varphi(q)$, we obtain $p^{\varphi(q)}\gcd(r,q) \equiv \gcd(r,q) (\mathrm{mod}\ q)$ whence $\delta(r,a,np^{\varphi(q)}) = \delta(r,a,n)$. Thus

$$\sum_{n\in M(r)}\frac{\delta(r,a,n)}{n} = \sum_{n\in S(r)}\frac{\delta(r,a,n)}{n}\prod_{p\in P(r)}\left(1+\frac{1}{p^{\varphi(q)}}+\frac{1}{p^{2\varphi(q)}}+\cdots\right). \tag{9}$$

We substitute (8) and (9) in the expression V to obtain (7).

$\square$

As a consequence of Lemma 1 we derive the following corollary which is formula (7) of [3].

Corollary 1. *Let Φ_q be irreducible over $\mathbb{Q}(f(1), \cdots, f(q))$. Suppose $S = 0$. Then*

$$\sum_{n=1}^{\infty} \frac{f(kn)}{n} = 0 \textit{ for every } k \textit{ with } \gcd(k,q) = 1.$$

Proof. By Lemma 1, we find that (5) and (6) hold. Thus, by (5),

$$\sum_{m \in M} \frac{f(akm)}{m} = \sum_{m \in M} \frac{f(a'm)}{m} = 0$$

for every a with $1 \leq a < q$, $\gcd(a,q) = 1$ where $a' \equiv ak \pmod{q}$. If $kr \equiv r' (\bmod q)$, then $\varepsilon(r,p) = \varepsilon(r',p)$. Hence

$$\sum_{r \in L'} f(kr)\varepsilon(r,p) = \sum_{r' \in L'} f(r')\varepsilon(r,'p) = 0$$

by (6). Now the assertion follows from the converse part of Lemma 1.

□

We write formally

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = \sum_{\gcd(a,q)=1} \frac{1}{a} \left(\sum_{m \in M(q)} \frac{f(am)}{m} \right).$$

It follows from Lemma 1 that if the series on the left hand side vanishes, so does the expression within brackets for every a coprime to q. The converse is not true in general. For instance, if $q = 2$ and $f(n) = (-1)^n$, $n \in \mathbb{Z}$, then

$$\sum_{m \in M(2)} \frac{f(am)}{m} = f(a) + \frac{f(2a)}{2} + \frac{f(4a)}{4} + \cdots = 0$$

for every odd a, but

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -\log 2 \neq 0.$$

In this example we have $\sum_{r \in L'} f(r)\varepsilon(r,p) = 2f(2) \neq 0$ and hence (6) is not satisfied.

As another consequence of Lemma 1, we derive Theorems 9 and 10 of [7].

Corollary 2. *Let f be completely multiplicative, or multiplicative with $|f(p^k)| < p - 1$ for every prime divisor p of q and every positive integer k. Also let Φ_q be irreducible over $\mathbb{Q}(f(1), \ldots, f(q))$. Then $S \neq 0$.*

Proof. Suppose $S = 0$. Since f is multiplicative, (5) with $a = 1$ implies that

$$\sum_{m \in M} \frac{f(m)}{m} = 0.$$

Note that the series is absolutely convergent because of the periodicity of f. Since

$$\sum_{m \in M} \frac{f(m)}{m} = \prod_{p|q} \left(\sum_{j=0}^{\infty} \frac{f(p^j)}{p^j} \right),$$

we get

$$\sum_{j=0}^{\infty} \frac{f(p^j)}{p^j} = 0 \ \text{ for some prime divisor } p \text{ of } q \tag{10.}$$

If f is completely multiplicative, then

$$\sum_{j=0}^{\infty} \left(\frac{f(p)}{p} \right)^j = 0 \text{ for some prime divisor } p \text{ of } q$$

which is not possible. If f is multiplicative with $|f(p^k)| < p-1$ for every prime divisor p of q and every positive integer k, then

$$| \sum_{j=0}^{\infty} \frac{f(p^j)}{p^j} | \ > \ 1 - \frac{p-1}{p}(1 + \frac{1}{p} + \cdots) = 0$$

contradicting (10).

□

We remark here that in the statement of Theorem 9 in [7], the condition $|f(p^k)| \le p-1$ should be replaced by $|f(p^k)| < p-1$. The example $f(0) = 2, f(1) = f(5) = 1, f(2) = f(4) = -1, f(3) = -2$ with period 6 shows that the statement is false under the former condition. The application to Erdős' problem in [7] is not affected by this correction.

3 Alternating Series.

In this section, we apply Lemma 1 and Corollary 1 to investigate sums

$$T = \sum_{n=0}^{\infty} \frac{(-1)^n(\alpha n + \beta)}{(qn + s_1)(qn + s_2)} \tag{11}$$

with $\alpha, \ \beta \ \in \overline{\mathbb{Q}}, \ s_1, \ s_2 \ \in \ \mathbb{Z}$. We prove

Theorem 1. *Suppose T is given by* (11) *with* $|\alpha| + |\beta| \ > 0$. *Let* Φ_{2q} *be irreducible over* $\mathbb{Q}(\alpha, \beta)$ *and* s_1, s_2 *distinct integers such that* $qn+s_1,\ qn+s_2$ *do not vanish for* $n \geq 0$. *Assume that* $\alpha \neq 0$ *if* $s_1 \equiv s_2 \pmod{q}$. *Then T is transcendental.*

We derive Theorem 1 from the following lemma.

Lemma 2. *Let*

$$T' = \sum_{n=0}^{\infty} \frac{(-1)^n(\alpha' n + \beta')}{(qn + r_1)(qn + r_2)}$$

with $0 < r_1 \leq q,\ 0 < r_2 \leq q,\ r_1 \neq r_2,\ r_1,\ r_2 \in \mathbb{Z},\ \alpha',\ \beta' \in \overline{\mathbb{Q}}$. *Suppose* $|\alpha'| + |\beta'| \ > 0$ *and* Φ_{2q} *is irreducible over* $\mathbb{Q}(\alpha', \beta')$. *Then T' is transcendental.*

Proof. First we show that $T' \neq 0$. Suppose $T' = 0$. We may assume without loss of generality that $\gcd(q, r_1, r_2) = 1$. By partial fractions, we see that

$$T' = \sum_{n=0}^{\infty} (-1)^n \left\{ \frac{A'}{qn + r_1} + \frac{B'}{qn + r_2} \right\}$$

where $A' = -\frac{q\beta' - \alpha' r_1}{q(r_1 - r_2)}$ and $B' = \frac{q\beta' - \alpha' r_2}{q(r_1 - r_2)}$. Suppose $A' = 0$. Then $B' \neq 0$ and

$$T' = B' \sum_{n=0}^{\infty} \left\{ \frac{1}{2qn + r_2} - \frac{1}{2qn + q + r_2} \right\}.$$

Hence $T' \neq 0$. Similarly if $A' \neq 0$ and $B' = 0$, we have $T' \neq 0$. Thus we may suppose that $A' \neq 0,\ B' \neq 0$. We define $f(n)$ for $n \geq 0$ as

$$f(n) = \begin{cases} A' \text{ if } n \equiv r_1 (\text{mod } 2q) \\ B' \text{ if } n \equiv r_2 (\text{mod } 2q) \\ -A' \text{ if } n \equiv q + r_1 (\text{mod } 2q) \\ -B' \text{ if } n \equiv q + r_2 (\text{mod } 2q) \\ 0 \text{ otherwise.} \end{cases}$$

Thus f is a periodic function with period $2q$ and

$$(12) \qquad T' = \sum_{n=1}^{\infty} \frac{f(n)}{n} = \sum_{n=0}^{\infty} \left\{ \frac{f(r_1)}{2qn + r_1} + \frac{f(r_2)}{2qn + r_2} - \frac{f(r_1)}{2qn + q + r_1} - \frac{f(r_2)}{2qn + q + r_2} \right\} = 0.$$

Hence (5) and (6) are valid by Lemma 1.

Case 1. Let q be odd. Then one of $r_i,\ q+r_i$ is even and the other is odd for $i=1,2$. Further $\varepsilon(r_i,2)=0$ if r_i is odd and $\varepsilon(r_i,2)=2$ if r_i is even. We apply (6) for $p=2$ to obtain $|f(r_1)| = |f(r_2)|$. We re-write (12) as

$$T' = qf(r_1)\sum_{n=0}^{\infty}\left\{\frac{1}{(2qn+r_1)(2qn+q+r_1)} \pm \frac{1}{(2qn+r_2)(2qn+q+r_2)}\right\}.$$

We see that all terms within the curly brackets have the same sign whence $T'\neq 0$, which is a contradiction. Thus q is not odd.

Case 2. Let q be even. Then $r_i,\ q+r_i$ are both odd or both even for $i=1,2$. By (5) and (6), there exists an $r\in L$ for which $f(r)\neq 0$. Let $r=r_1$ and $\tilde{p}$ be a prime factor of q dividing r_1. Then it follows from (6) that there exists $s\leq 2q$ with $\tilde{p}\mid s$ and $s\neq r_1$ such that $f(s)\neq 0$. Now $s\neq r_2,\ s\neq q+r_2$, since $\gcd(q,r_1,r_2)=1$. Hence $s=q+r_1$.

Assume $\gcd(r_2,2q)=1$. Then $\gcd(q+r_2,2q)=1$. Suppose $v_2(r_1)\geq v_2(q)$. Then either $v_2(r_1)>v_2(q), v_2(q+r_1)=v_2(q)$ or $v_2(r_1)=v_2(q), v_2(q+r_1)>v_2(q)$. On applying (6) with $p=2$, we obtain $\varepsilon(r_1,2)=\varepsilon(q+r_1,2)$. Since $v_2(r_1)\neq v_2(q+r_1)$, this is possible only when $v_2(r_1)\geq v_2(2q)$ and $v_2(q+r_1)\geq v_2(2q)$ which is false. Thus we get

$$v_2(r_1)=v_2(q+r_1)<v_2(q). \tag{13}$$

Next we show that

$$r_2, r_2+\frac{q}{\tilde{p}},\cdots,r_2+(\tilde{p}-1)\frac{q}{\tilde{p}}$$

are all coprime to $2q$. Suppose there exists a prime p dividing both $r_2+\frac{iq}{\tilde{p}}$ and $2q$ for some i with $0<i<\tilde{p}$. Then $p=\tilde{p},\ v_{\tilde{p}}(q)=1$ and there exists exactly one such i, say i_0. Now we apply (5) with $a=r_2$ and $r_2+i\frac{q}{\tilde{p}}$ to see that

$$f(r_2)=f(r_2+i\frac{q}{\tilde{p}}) \text{ for } 0<i<\tilde{p},\ i\neq i_0,$$

since all the terms in (5) for $m>1$ are equal because $f(am)\neq 0$ for $m>1, m\in M$ only when $am\equiv r_1$ or $q+r_1 \pmod{2q}$ and $\tilde{p}$ divides r_1 and q. Since r_2 and $q+r_2$ are the only integers k mod $2q$ coprime to q with $f(k)\neq 0$, we conclude that $\tilde{p}=2$. Thus $r_1=R_1 2^{\alpha_1},\ q+r_1=R_2 2^{\alpha_1}$ with $\gcd(R_1,2q)=\gcd(R_2,2q)=1$ and $0<\alpha_1<v_2(q)$ by (13). We apply (5) with $a=R_1,\ R_2$ to get

$$f(R_1)+\frac{f(R_1 2^{\alpha_1})}{2^{\alpha_1}}=0,\ f(R_2)+\frac{f(R_2 2^{\alpha_1})}{2^{\alpha_1}}=0.$$

Hence $f(R_1) \neq 0,\ f(R_2) \neq 0$. Thus $\{R_1, R_2\} = \{r_2, q + r_2\}$ which gives $q = |R_2 - R_1|$ implying $\alpha_1 = 0$, which is a contradiction. This proves that $\gcd(r_i, 2q) > 1$ for $i = 1, 2$.

Note that r_1 or r_2 is odd. Assume r_1 is odd. (The case r_2 is odd is similar.) Put $d = \gcd(r_1, 2q) > 1$. Hence d is odd. We put $a = \frac{r_1}{d},\ b = \frac{q+r_1}{d}$. Then $\gcd(a, \frac{2q}{d}) = \gcd(b, \frac{2q}{d}) = 1$. Now we show that it is possible to choose an integer m such that

$$m \text{ is prime},\ m > 2q,\ m \not\equiv q + 1 \ (\mathrm{mod}\ 2q), m \equiv a^{-1}b \ (\mathrm{mod} \frac{2q}{d}). \tag{14}$$

If $a^{-1}b \not\equiv q + 1$ (mod $\frac{2q}{d}$), then we choose any prime $> 2q$ which is $\equiv a^{-1}b$ (mod$\frac{2q}{d}$) and (14) is satisfied. So we suppose that $a^{-1}b \equiv q + 1$ (mod $\frac{2q}{d}$). There are $\varphi(2q)/\varphi(\frac{2q}{d})$ primitive residue classes mod $2q$ in the residue class $a^{-1}b$ (mod $\frac{2q}{d}$). Now

$$\frac{\varphi(2q)}{\varphi(\frac{2q}{d})} \geq d \prod_{\substack{p|d \\ p \text{ prime}}} (1 - \frac{1}{p}) \geq 2$$

since d is odd. Hence we may take a primitive residue class $\not\equiv q + 1$ (mod $2q$) and $\equiv a^{-1}b$ (mod $\frac{2q}{d}$). By Dirichlet's theorem, we can find a prime m satisfying (14). Note that

$$m(r_1 + q) = mad + mq \equiv bd + mq = r_1 + q + mq \equiv r_1 (\mathrm{mod}\ 2q). \tag{15}$$

Since $\gcd(m, 2q) = 1$, we may apply Corollary 1 with $k = m$ to obtain

$$\sum_{n=0}^{\infty} \left\{ \frac{f(r_1)}{2qn + r_1^*} + \frac{f(r_2)}{2qn + r_2^*} - \frac{f(r_1)}{2qn + r_3^*} - \frac{f(r_2)}{2qn + r_4^*} \right\} = 0 \tag{16}$$

where $r_i^* \equiv m^{-1}r_i$ (mod $2q$), $0 < r_i^* \leq 2q$ for $1 \leq i \leq 4$ with $r_3 = q + r_1,\ r_4 = q + r_2$. We have

$$r_3^* \equiv m^{-1}r_3 = m^{-1}(q + r_1) \equiv ab^{-1}bd = ad = r_1 (\mathrm{mod}\ 2q) \tag{17}$$

and by (15),

$$r_1^* \equiv m^{-1}r_1 \equiv m^{-1}(m(q + r_1)) \equiv r_3 \ (\mathrm{mod}\ 2q). \tag{18}$$

Thus $r_3^* = r_1, r_1^* = r_3 = q + r_1$. From (15), it follows that $(m - 1)r_1 \equiv 0$ (mod q). Suppose $r_2^* \equiv r_2$ (mod q). Then $(m-1)r_2 \equiv 0$ (mod q). Hence $m \equiv 1$ (mod q) since $\gcd(q, r_1, r_2) = 1$. If $m \equiv 1$ (mod $2q$), then $a \equiv b$(mod $\frac{2q}{d}$) which is not possible since $b - a = \frac{q}{d}$. On the other hand, $m \not\equiv q + 1$ (mod $2q$) by (14). Thus $r_2^* \not\equiv r_2$ (mod q). Now we add (12) with (16) and use (17), (18) to get

$$f(r_2) \sum_{n=0}^{\infty} \left\{ \Big(\frac{1}{2qn + r_2} - \frac{1}{2qn + q + r_2} \Big) + \Big(\frac{1}{2qn + r_2^*} - \frac{1}{2qn + r_4^*} \Big) \right\} = 0.$$

Note that $r_4^* \equiv m^{-1}r_4 = m^{-1}(q + r_2) \equiv q + r_2^* \pmod{2q}$ so that $r_4^* = r_2^* \pm q$. By the monotonicity of $\frac{1}{2qn+k} - \frac{1}{2qn+k+q}$ with respect to k, the sign of the expression between the curly brackets in the infinite sum above depends only on r_2 and r_2^*. Hence all these expressions have the same sign. Since $r_2^* \not\equiv r_2 \pmod{q}$, the sum does not vanish. Hence $f(r_2) = 0$, which is a contradiction.

Thus $T' \neq 0$. Now we apply [1, Theorem 1] to see that T' is transcendental.

□

Proof of Theorem 1. We assume without loss of generality that $\gcd(q, s_1, s_2) = 1$. By using partial fractions, we have

$$T = \sum_{n=0}^{\infty} (-1)^n \left\{ \frac{A}{qn + s_1} + \frac{B}{qn + s_2} \right\}$$

where $A = -\frac{q\beta - \alpha s_1}{q(s_1 - s_2)}$ and $B = \frac{q\beta - \alpha s_2}{q(s_1 - s_2)}$. Let $s_1 \equiv r_1 \pmod{q}$ and $s_2 \equiv r_2 \pmod{q}$ with $0 < r_1 \leq q, 0 < r_2 \leq q$. Then

$$T = \gamma \pm \sum_{n=0}^{\infty} (-1)^n \left\{ \frac{A}{qn + r_1} + \frac{B}{qn + r_2} \right\} \tag{19}$$

where $\gamma \in \mathbb{Q}(\alpha,\ \beta)$. If $r_1 = r_2$, then

$$T = \gamma \pm \frac{\alpha}{q} \sum_{n=0}^{\infty} \frac{(-1)^n}{qn + r_1} = \gamma \pm \frac{\alpha}{q} \sum_{n=0}^{\infty} \left\{ \frac{1}{2qn + r_1} - \frac{1}{2qn + q + r_1} \right\}. \tag{20}$$

We see that the infinite sum in the latter expression for T does not vanish. Now we apply [1, Theorem 1] with

$$f(n) = \begin{cases} 1 \text{ if } n \equiv r_1 (\text{mod } 2q) \\ -1 \text{ if } n \equiv q + r_1 \pmod{2q} \\ 0 \text{ otherwise} \end{cases}$$

to conclude that the infinite series and hence T is transcendental. Thus we may assume that $r_1 \neq r_2$. Now we apply Lemma 2 to (19) to see that T is transcendental.

□

We note that if $s_1 \equiv s_2 \pmod{q}$ and $\alpha = 0$, then by (20), we have T algebraic.

4 Series with terms having three products in the denominator.

In this section we consider series of the form

$$U=\sum_{n=0}^{\infty}\frac{\alpha n+\beta}{(qn+s_1)(qn+s_2)(qn+s_3)} \tag{21}$$

with $\alpha,\ \beta\ \in\overline{\mathbb{Q}},\ s_1,\ s_2,\ s_3\ \in\ \mathbb{Z}$. We prove

Theorem 2. *Suppose U is given by (21) with $|\alpha|+|\beta|\ >0$. Let Φ_q be irreducible over $\mathbb{Q}(\alpha,\beta)$ and $s_1,\ s_2,\ s_3$ be distinct integers such that $qn+s_1,\ qn+s_2,\ qn+s_3$ do not vanish for $n\geq 0$. Assume that $s_1,\ s_2,\ s_3$ are not in the same residue class* mod q. *Further let*

$$s_1\not\equiv s_2 (\mathrm{mod}\ q)\ if\ \alpha s_3=\beta q; s_2\not\equiv s_3(\mathrm{mod}\ q)\ if\ \alpha s_1=\beta q; s_3\not\equiv s_1(\mathrm{mod}\ q)\ if\ \alpha s_2=\beta q. \tag{22}$$

Then U is transcendental.

We derive Theorem 2 from the following lemma.

Lemma 3. *Let*

$$U'=\sum_{n=0}^{\infty}\frac{\alpha' n+\beta'}{(qn+r_1)(qn+r_2)(qn+r_3)}$$

with $r_1,\ r_2,\ r_3$ distinct positive integers $\leq q$ and $\alpha',\beta'\ \in\overline{\mathbb{Q}}$. *Suppose $|\alpha'|+|\beta'|\ >0$ and Φ_q is irreducible over $\mathbb{Q}(\alpha',\beta')$. Then U' is transcendental.*

Proof. First we show that $U'\neq 0$. Suppose $U'=0$. We may assume without loss of generality that $\gcd(q,r_1,\ r_2,\ r_3)=1$. Using partial fractions, we get

$$U'=\sum_{n=0}^{\infty}\left\{\frac{A'}{qn+r_1}+\frac{B'}{qn+r_2}+\frac{C'}{qn+r_3}\right\}$$

where

$$A'=\frac{q\beta'-\alpha' r_1}{q(r_1-r_2)(r_1-r_3)},\ B'=\frac{q\beta'-\alpha' r_2}{q(r_2-r_1)(r_2-r_3)},\ C'=\frac{q\beta'-\alpha' r_3}{q(r_3-r_1)(r_3-r_2)}.$$

Observe that $A'+B'+C'=0$. Hence if any of $A',\ B',\ C'$ vanishes then the series U' reduces to some series in which all the expressions between curly brackets have the same sign. Then $U'\neq 0$. Thus we may assume that none of $A',\ B',C'$ vanishes. We define $f(n)$ for $n\geq 0$ as

$$f(n) = \begin{cases} A' \text{ if } n \equiv r_1 \pmod{q} \\ B' \text{ if } n \equiv r_2 \pmod{q} \\ C' \text{ if } n \equiv r_3 (\text{mod } q) \\ 0 \text{ otherwise.} \end{cases}$$

Thus f is a periodic function with period q taking only three non-zero values $f(r_1),\ f(r_2),\ f(r_3)$ with

$$f(r_1) + f(r_2) + f(r_3) = 0 \text{ and } U' = \sum_{n=1}^{\infty} \frac{f(n)}{n} = 0. \tag{23}$$

Hence (5) and (6) are valid by Lemma 1. Thus there exist $r, s \in \{r_1, r_2, r_3\}$ with $r \neq s$ and a prime p_1 dividing $\gcd(q, r, s)$. Without loss of generality, we may take $r = r_1,\ s = r_2$. If $p_1 \parallel q$, by (6) with $p = p_1$, we have $f(r_1) + f(r_2) = 0$ which gives $f(r_3) = 0$ by (23), which is a contradiction. Thus $p_1^2 | q$. Now suppose $\gcd(r_3, q) = 1$. Let $r_3 + \frac{iq}{p_1} \equiv a_i \pmod{q}$ where $0 < a_i < q$ for $0 \leq i < p_1$. We note that a_i 's are all distinct and coprime to q. When we apply (5) to the numbers a_i, then all non-zero terms corresponding to $m > 1$ are the same and we find

$$f(a_i) \ = \ f(r_3) \ \neq \ 0 \quad \text{for } i = 0, 1, ..., p_1 - 1.$$

Since $p_1 \geq 2$, this is a contradiction. Thus $\gcd(r_3, q) > 1$. Further we may suppose that $p_1 \not| r_3$ since $\gcd(q, r_1, r_2, r_3) = 1$. Let q_1 be a prime with $q_1 \mid \gcd(r_3, q)$. By applying (6) with $p = q_1$, we may assume that $q_1 \mid r_2$. Then $q_1 \not| \, r_1$. Thus we have

$$r_1 = R_1 p_1^{\alpha_1} \cdots, \ r_2 = R_2 p_1^{\beta_1} q_1^{\gamma_1} \cdots, \ r_3 = R_3 q_1^{\delta_1} \cdots$$

where the dots represent other prime factors of q and $\gcd(R_i, q) = 1$ for $i = 1, 2, 3$. Applying (6) with $p = p_1$ and $p = q_1$, we find that $\frac{f(r_1)}{f(r_2)}$ and $\frac{f(r_2)}{f(r_3)}$ are negative and, by (23), that $|f(r_2)| = |f(r_1) + f(r_3)|$. Hence $|f(r_2)| > |f(r_1)|$ and $|f(r_2)| > |f(r_3)|$. Again using (6) we get $\alpha_1 > \beta_1,\ \delta_1 > \gamma_1$ and $v_{p_1}(r_2) < v_{p_1}(q)$, $v_{q_1}(r_2) < v_{q_1}(q)$. It follows that $\gcd(r_1, r_3, q) = 1$ and that $v_p(r_2) < v_p(q)$ for every prime divisor p of q. Hence we may write

$$r_1 = R_1 p_1^{\alpha_1} \cdots p_t^{\alpha_t},\ r_2 = R_2 p_1^{\beta_1} \cdots p_t^{\beta_t} q_1^{\gamma_1} \cdots q_s^{\gamma_s},\ r_3 = R_3 q_1^{\delta_1} \cdots q_s^{\delta_s}$$

where p_i 's and q_i 's are distinct sets of primes. By permuting r_1 and r_3 if necessary, we may assume that all the p_i are odd. We select a number a_0 coprime to q as follows. Suppose

$$R_1 \equiv R_2 (\text{mod } \frac{q}{p_1^{\beta_1} \cdots p_t^{\beta_t} q_1^{\gamma_1} \cdots q_s^{\gamma_s}}).$$

Then both the numbers $R_1 \pm \frac{q}{p_1^{\beta_1+1}}$ are not congruent to $R_2(\text{mod}\frac{q}{p_1^{\beta_1}\cdots p_t^{\beta_t}q_1^{\gamma_1}\cdots q_s^{\gamma_s}})$ and one of them is coprime to q. We take a_0 to be one of $R_1 \pm \frac{q}{p_1^{\beta_1+1}}$ which is coprime to q. If $R_1 \not\equiv R_2$ $(\text{mod } \frac{q}{p_1^{\beta_1}\cdots p_t^{\beta_t}q_1^{\gamma_1}\cdots q_s^{\gamma_s}})$, then we take $a_0 = R_1$. Hence in (5) with $a = a_0$, no term occurs involving $f(r_2)$. Thus we get

$$0 = Af(r_1) + Bf(r_3)$$

where A and B represent certain non-vanishing series of positive terms. Thus $\frac{f(r_1)}{f(r_3)} < 0$ which is a contradiction with $\frac{f(r_1)}{f(r_2)} < 0$, $\frac{f(r_2)}{f(r_3)} < 0$.

□

Proof of Theorem 2. We assume without loss of generality that gcd $(q, s_1, s_2, s_3) = 1$. By using partial fractions, we have

$$U = \sum_{n=0}^{\infty}\{\frac{A}{qn+s_1} + \frac{B}{qn+s_2} + \frac{C}{qn+s_3}\}$$

where

$$A = \frac{q\beta - \alpha s_1}{q(s_1-s_2)(s_1-s_3)}, B = \frac{q\beta - \alpha s_2}{q(s_2-s_1)(s_2-s_3)}, C = \frac{q\beta - \alpha s_3}{q(s_3-s_1)(s_3-s_2)}.$$

We observe that

$$A + B + C = 0. \tag{24}$$

Let $s_i \equiv r_i(\text{mod } q)$ with $0 < r_i \leq q$ for $i = 1, 2, 3$. Then we re-write

$$U = \gamma + \sum_{n=0}^{\infty}\{\frac{A}{qn+r_1} + \frac{B}{qn+r_2} + \frac{C}{qn+r_3}\} \tag{25}$$

where $\gamma \in \mathbf{Q}(\alpha, \beta)$. Suppose $r_1 = r_2$. Then $r_1 \neq r_3$ and by (24), we have

$$U = \gamma + C(r_1 - r_3)\sum_{n=0}^{\infty}\frac{1}{(qn+r_1)(qn+r_3)}. \tag{26}$$

By our assumption $C \neq 0$. Hence the infinite sum in (26) does not vanish. Now we apply [1, Theorem 1] to conclude that U is transcendental. Similarly U is transcendental whenever $r_1 = r_3$ or $r_2 = r_3$. Thus we may assume that r_1, r_2, r_3 are all distinct. Now we apply Lemma 3 to conclude that the infinite sum in (25) and hence U is trasnscendental. □

We observe from (22),(23),(24) that in the cases when s_1, s_2, s_3 are all in the same residue class (mod q) or when (22) is not valid, then U is algebraic.

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