PILLAI'S PROBLEM ON CONSECUTIVE INTEGERS

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ABSTRACT. For integers $m \geq 2$ and $d \geq 1$, we study the set S_m of m consecutive integers which satisfies the property that for each $x \in S_m$ there exists $y \in S_m$ such that $\gcd(x, y) > d$. This problem was first posed and studied by S. S. Pillai for the case $d = 1$. In this article, we elaborate on an argument of T. Vijayaraghavan for $d = 1$ and of Y. Caro for $d \geq 1$.

§1. Introduction

For an integer $m \geq 2$, let S_m be a set of m consecutive integers. Let $d \geq 1$ be an integer. We say that the set S_m has property P_d if there exists an element $x \in S_m$ such that $\gcd(x, y) \leq d$ for all $y \in S_m$ with $y \neq x$. In this case, we also say that the element x has *property* P_d . When no such element exists, we say that S_m does not have property P_d . Thus, if $d = 1$, S_m has property P_1 means that there exists $x \in S_m$ which is co-prime to all other elements in S_m .

In 1940, S. S. Pillai (in [12]) and, independently, Szekeres (see for instance, [9]) first studied the problem of finding sets S_m having property P_1 . Pillai was motivated by this problem while trying to solve a folklore conjecture that

a product of two or more consecutive integers is never a perfect power.

This remarkable result was proved by P. Erdős and J. L. Selfridge (in [6]) in 1975. Pillai (in [12]) showed that S_m has property P_1 for $m < 17$. Thus, any set of consecutive integers having less than 17 elements has property P_1 . Further, Pillai succeeded in proving that for $17 \leq m \leq 430$, there exist infinitely many sets S_m for which property P_1 does not hold. To prove this result, he used sieving techniques and introduced numbers known as *gap numbers*. Then he extended this result to $m \leq 12335$ (in [14]). Also, William Scott (in [17]) further extended this to $m \leq 2491906561$ in a private letter to Pillai. Finally, this result was completely solved for all $m \geq 17$ by A. T. Brauer (in [1]) in 1941. Later many authors including Pillai (in [15]), Erdős (in [5]), Evans (in [7]), Harborth (in [10] and [11]) and Gassko (in [9]) gave different proofs of this result. Indeed, Gassko

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characterized the sets S_m in terms of coverings of finite sequences of natural numbers by arithmetic progressions with prime differences. Evans extended the problem to blocks in arithmetic progression (see [8]). Since Pillai's problem for arithmetic progression poses no new difficulty, we restrict only to the case of consecutive integers. To summarize, we have

Theorem 1. *Any set S_m with $1 \leq m \leq 16$, has property P_1 . For every $m \geq 17$, there exist infinitely many sets S_m for which property P_1 does not hold.*

In a private communication to Pillai, around 1940, T. Vijayaraghavan gave an argument to prove the following theorem.

Theorem 2. *There exist infinitely many sets S_m which do not have property P_1 whenever m is sufficiently large.*

The second author came across this communication of Vijayaraghavan during the preparation of the collected works of Pillai in which he is jointly involved with R. Balasubramanian. Though Theorem 2 is much weaker than Theorem 1, the ideas proposed by Vijayaraghavan were illuminating and have some similarities with the works of Y. Caro (in [2]) for $d > 1$. In Section 2, we elaborate and give a complete proof of Theorem 2 following Vijayaraghavan's ideas from that letter.

Next let $d > 1$. Developing on Evan's proof of Theorem 1, in 1979 Caro (in [2]) proved the following stronger result.

Theorem 3. *Let $d > 1$. There exist infinitely many sets S_m which do not have property P_d whenever m exceeds an effectively computable number $G(d)$.*

Thus, for $m > G(d)$, there are infinitely many sets S_m such that for every $x \in S_m$, there exists $y \in S_m$ with $y \neq x$ such that $\gcd(x, y) > d$. For $m \leq G(d)$, it is still possible that there exists some set S_m for which P_d does not hold. Denote by $g(d)$ the smallest positive integer such that there exists a set $S_{g(d)}$ for which property P_d does not hold. By Theorem 1, $G(1) = g(1) = 17$. For example, the set $S_{17} = 2184, 2185, \dots, 2200$ does not have property P_1 . Since

$$30030 = 2 \times 3 \times 5 \times 7 \times 11 \times 13$$

and each term of this sequence $2184, 2185, \dots, 2200$ is divisible by one of the primes $p \leq 13$, we conclude that the infinite sets of 17 elements $\{2184 + 30030k, 2185 + 30030k, \dots, 2200 + 30030k\}$ for $k \in \mathbb{Z}$ do not have property P_1 .

Caro showed that

$$g(d) < 45d \log d, \quad G(d) < 54d \log d. \quad (1.1)$$

In Section 3, we first sketch the argument of Y. Caro and then give better bounds for $G(d)$ and $g(d)$ than the above.

The exact values of $g(d)$ and $G(d)$ for any $d \geq 2$ are not known. It will be interesting to characterize those sets for which P_d holds with $d \geq 2$.

§2. An Argument of Vijayaraghavan

We split the argument of Vijayaraghavan into several lemmas to bring out its essence. Let t and T denote the smallest and largest integer in S_m , respectively. Thus, $T - t = m$. Let $p(x)$ denote the least prime divisor of x . Then

Lemma 1. *A necessary and sufficient condition for $x \in S_m$ not to have property P_1 is that*

$$p(x) \leq \max \{x - t, T - x\}. \quad (2.1)$$

Proof. We observe that $x \in S_m$ does not have property P_1 if and only if it has a common factor with some other element, say, y of S_m . Then

$$p(x) \leq |y - x| \leq \max \{x - t, T - x\}.$$

The assertion of the lemma follows. $\square$

Let $X \geq 2$ be any real number and $\pi(X)$ denote the number of primes $\leq X$. This counting function of primes plays an important role in many areas of Number Theory. Extensive work on the estimates for $\pi(X)$ was done by Rosser and Sch6nenfeld (in [16]). Some of these estimates were improved by P. Dusart (see [3] and [4]) in 1998. For the purpose of this paper, we shall use the estimates from [16] to get

$$\frac{X}{\log X} \leq \pi(X) \leq \frac{X}{\log X} + \frac{1.5X}{\log^2 X}.$$

Hence,

$$\frac{X}{\log X} < \pi(x) \leq 1.5 \frac{X}{\log X} \text{ for all } X \geq 21. \quad (2.2)$$

Let $2 = p_1, p_2, \dots$ be the sequence of all prime numbers. Then we have

$$n \log n < p_n < n(\log n + \log \log n) \text{ for } n \geq 6 \quad (2.3)$$

(see [12, p.69]). We apply (2.3) to get

Lemma 2. *For any $\alpha \geq 2.5$, we have*

$$p_\ell \leq 2\alpha X \text{ for all } X \geq 2\alpha,$$

where $\ell = \left\lceil \frac{\alpha X}{\log X} \right\rceil$.

Proof. From (2.3), we get

$$\begin{aligned} p_\ell &\leq \ell(\log \ell + \log \log \ell) \\ &\leq \frac{\alpha X}{\log X} (\log \alpha + \log X - \log \log X + \log(\log \alpha X)) \\ &\leq \frac{\alpha X}{\log X} \left(\log \alpha + \log X + \log \left(1 + \frac{\log \alpha}{\log X} \right) \right) \\ &\leq \frac{\alpha X}{\log X} (\log 2\alpha + \log X) \leq 2\alpha X, \end{aligned}$$

as $X \geq 2\alpha$. □

Put

$$A = A_X = p_1 p_2 \cdots p_{\pi(X)}$$

and let

$$1 = a_0 < a_1 < \cdots < a_n < \cdots \quad (2.4)$$

be the sequence of all integers co-prime to A . For example, when $X = 2$, the sequence in (2.4) consists of all odd integers. Observe that $a_1 = 1$ and a_2 is the first prime exceeding X . Thus, $a_2 = p_{\pi(X)+1}$. Also, the sequence (2.4) is periodic with period $\phi(A)$ (where $\phi(\cdot)$ denotes the Euler's phi-function) in the following sense

$$A + a_i = a_{i+\phi(A)} \text{ and } a_{\phi(A)} = A - 1. \quad (2.5)$$

Thus, to know all the values of this sequence, it is enough to compute the values of a_i for $i = 1, 2, \dots, \phi(A) - 1$.

Let $\gamma = 0.577215665\dots$ denote the Euler constant. Thus, $1.78 < e^\gamma < 1.8$. Set

$$M = \left\lceil \frac{15X}{\log X} \right\rceil.$$

Then the following lemma explores the gaps between the elements of the sequence (2.4).

Lemma 3. *There are infinitely many pairs $\{c_t, c_{t+M}\}$ with c_t, c_{t+M} elements of the sequence in (2.4) such that*

$$c_{t+M} - c_t > 25X \text{ for } t \geq 0. \quad (2.6)$$

Proof. It is well-known that the density of the sequence (2.4) is

$$\frac{\phi(p_1 p_2 \cdots p_{\pi(X)})}{p_1 p_2 \cdots p_{\pi(X)}}.$$

By Mertens' estimate, we know that

$$\frac{\phi(p_1 p_2 \cdots p_{\pi(X)})}{p_1 p_2 \cdots p_{\pi(X)}} = \prod_{p \leq X} \left(1 - \frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log X}.$$

Consider the sequence

$$a_0, a_M, a_{2M}, \cdots a_{tM}, \cdots$$

Since $\left\lfloor \frac{15X}{\log X} \right\rfloor > \frac{(14.5)X}{\log X}$, we see that the density of this sequence is

$$\frac{1}{M} \frac{e^{-\gamma}}{\log X} > \frac{1}{14.5e^\gamma X},$$

since $M < 15X/\log X$. Thus, there exists h such that

$$a_{(h+1)M} - a_{hM} > (14.5)e^\gamma X > 25X.$$

Hence, by (2.5), we see that for any $t \geq 0$,

$$a_{(h+1)M+t\phi(A)} - a_{hM+t\phi(A)} = a_{(h+1)M} - a_{hM} > 25X.$$

Now put $c_t = a_{hM+t\phi(A)}$ to get the assertion of the lemma. $\square$

Remark 1. When $X = 2$, the above lemma is clear since in this case $M = 43, a_i = 2i + 1, a_{43+i} = 87 + 2i$, giving $a_{43+i} - a_i = 86 > 25X$ for any $i \geq 0$. Hence, we take $h = 0$. Thus, all the pairs $(a_0, a_{43}), (a_1, a_{44}), \cdots$ satisfy the assertion of the lemma.

Remark 2. Lemma 3 says that for any $X \geq 2$, the gap $a_j - a_i$ between any two elements $a_i < a_j$ in (2.4) is as large as X provided $j - i = M$. When X is large, it is natural to expect large gap $a_j - a_i$, even if $j - i = 1$. Indeed, this phenomenon is true. To see this, let

$$H(X) = \max_{i \geq 1} (a_{i+1} - a_i).$$

In fact, it is clear from (2.5) that

$$H(X) = \max_{1 \leq i \leq \phi(A_X)-1} (a_{i+1} - a_i).$$

As observed earlier, $a_1 = 1$ and $a_2 = p_{\pi(X)+1}$. Hence, by (2.3), we have

$$H(X) \geq a_2 - a_1 \geq p_{\pi(X)+1} - 1 > (\pi(X) + 1) \log(\pi(X) + 1) - 1.$$

Now we use (2.2) to get

$$H(X) \geq .6X \text{ for } X \geq 20.$$

During his investigation on gaps between consecutive primes, P. Erdős (in [5]) showed that

there exists a positive constant c such that we can find

$$c p_{\pi(X)} \log p_{\pi(X)} (\log \log p_{\pi(X)})^{-2}$$

consecutive integers so that no one of them is relatively prime to A_X .

This result was based on Brun's sieve and several other intricate arguments. From this result, it follows that

$$H(X) \geq c_1 \frac{X \log X}{(\log \log X)^2}$$

for some positive constant c_1 , whenever X is large.

Take any pair $\{c_t, c_{t+M}\}$ as in Lemma 3. Let $e_t = \frac{c_t + c_{t+M}}{2}$. Re-arrange the elements $c_t, c_{t+1}, \dots, c_{t+M-1}, c_{t+M}$ as

$$e_t + y_0, e_t + y_1, \dots, e_t + y_M \text{ with } |y_0| \leq |y_1| \leq \dots \leq |y_M|. \quad (2.7)$$

Let N be the largest integer such that

$$|y_N| \leq 2X. \quad (2.8)$$

In the case $X = 2$, take $(a_0, a_{43}) = (c_t, c_{t+M})$. Then $e_t = \frac{a_0 + a_{43}}{2} = 44$. Thus, (2.7) becomes

$$43, 45; 41, 47; 39, 49; 37, 51; 35, 53; 33, 55; 31, 57; 29, 59; 27, 61; \dots; 1, 87.$$

Hence, $N = 2$.

Let N be very small. This means there are only very few y_i 's with their absolute values small. Thus, there are very few elements of (2.7) which are near e_t . Hence, in this case we may expect to have an interval around e_t in which property P_1 does not hold. We make this precise in the following lemma.

Lemma 4. *Let $X \geq 33$ be any integer. Suppose $N \leq \frac{14X}{\log X}$. Then there exists a λ such that no integer in the interval*

$$I_1 = (e_\lambda - 31X, e_\lambda + 31X)$$

has property P_1 .

Proof. By the Chinese Remainder Theorem, we can choose an integer x such that

$$\begin{aligned} x &\equiv e_0 \pmod{A} \\ x &\equiv -y_1 \pmod{p_{\pi(X)+1}} \\ \dots &\quad \dots \\ x &\equiv -y_M \pmod{p_{\pi(X)+M}}. \end{aligned}$$

Thus, $x = e_0 + \lambda A = e_\lambda$ for some λ . Further,

$$p_{\pi(X)+i} \mid (e_\lambda + y_i) \text{ for } 1 \leq i \leq M. \quad (2.9)$$

Now consider $x \in I_1$. Suppose that $x \neq e_\lambda + y_i$ for any i . Then, as x is not co-prime to A ,

$$p(x) \leq X. \quad (2.10)$$

If $x = e_\lambda + y_i$, then by (2.7) and Lemma 2, we have

$$p(x) \leq p_{\pi(X)+i} \leq p_{\pi(X)+M} \leq 33X, \quad (2.11)$$

since $X \geq 33 = 2\alpha$ with $\alpha = 16.5$.

If $x = e_\lambda + y_i$ with $i \leq N$, then, using the estimates for p_n , we have

$$p(x) \leq p_{\pi(X)+N} \leq 31X. \quad (2.12)$$

For a given $x \in I_1$, let

$$L(x) = \max(x - e_\lambda + 31X, e_\lambda + 31X - x).$$

We now show that

$$p(x) \leq L(x) \text{ for every } x \in I_1. \quad (2.13)$$

By (2.9) and (2.12), we need to consider the case

$$x = e_\lambda + y_i \text{ where } i > N.$$

By the definition of N , we have

$$|y_i| = |x - e_\lambda| > 2X.$$

Thus,

$$L(x) > 33X \geq p(X),$$

by (2.11). Hence, (2.13) follows and hence the lemma. $\square$

Next we look at the case when N is large. Then there are few values of $x = e_\lambda + y_i$ with y_i 's large and these values of x are near c_λ . Hence, it is likely that integers in a suitable interval $[c_\lambda, c_\lambda + \rho]$ do not have property P_1 . Thus, we have the following lemma.

Lemma 5. *Let $X \geq 21$ be any integer. Suppose $N > \frac{14X}{\log X}$. Then there exists a λ such that no integer in the interval*

$$I_2 = [c_\lambda, c_\lambda + 10X]$$

has property P_1 .

Proof. For any fixed λ , in the interval $(c_\lambda, c_{\lambda+M})$ there are M elements of the form $e_\lambda + y_i$. Out of these, there are N elements with

$$|y_i| \leq 2X.$$

Thus, there are at most

$$M - N \leq \frac{X}{\log X} := L$$

such elements which lie outside the interval

$$(e_\lambda - 2X, e_\lambda + 2X).$$

We observe by (2.5) that

$$e_\lambda - 2X = \frac{c_\lambda + c_{\lambda+M}}{2} - 2X > \frac{2c_\lambda + 25X}{2} - 2X > c_\lambda + 10X.$$

Thus, there are at most L elements $e_\lambda - z_i$ lying in the interval

$$I_2 = (c_\lambda, c_\lambda + 10X),$$

where $z_i = -y_i$ and $z_i > 2X$.

By the Chinese Remainder theorem, we can choose an integer x such that

$$\begin{aligned} x &\equiv e_0 \pmod{A} \\ x &\equiv z_1 \pmod{p_{\pi(X)+1}} \\ &\dots \quad \dots \\ x &\equiv z_L \pmod{p_{\pi(X)+L}}. \end{aligned}$$

Then, $x = e_\lambda = e_0 + \lambda A$ for some λ and

$$p_{\pi(X)+j} \mid (e_\lambda - z_j) \text{ for } 1 \leq j \leq L.$$

If $x \in I_2$ and $x \neq e_\lambda - z_j$ for any j , then

$$p(x) \leq X. \tag{2.14}$$

If $x \in I_2$ and $x = e_\lambda - z_j$ for some j , then

$$p(x) \leq p_{\pi(X)+j} \leq p_{\pi(X)+L} \leq 5X, \tag{2.15}$$

since $X \geq 13$. For any $x \in I_2$, we let

$$M(x) = \max(x - c_\lambda, c_\lambda + 10X - x).$$

We show that

$$p(x) \leq M(x) \text{ for any } x \in I_2.$$

By (2.14) and (2.15), we may assume that $x = e_\lambda - z_j$ for some j and $x - c_\lambda \geq 5X$. Then $10X + c_\lambda - x \geq 5X$. Hence, $p(x) \leq M(x)$. $\square$

Proof of Theorem 2. We combine Lemmas 4 and 5 to observe that when X is sufficiently large there exist infinitely many integers μ and an absolute constant c such that the interval

$$[\mu, \mu + cX]$$

does not have property P_1 . Thus, the property P_1 does not hold for infinitely many sets S_m with $m = cX$ and X sufficiently large. $\square$

§3. Bounds for $g(d)$ and $G(d)$.

Caro (in [2]) extended Pillai's problem for sets S_m with property P_d with $d \geq 1$. We give a description of his construction. Let $d \geq 1$ be fixed. For any interval J , we denote by $L(J)$, the length of the interval. Let $N(d)$ denote a number such that there are at least $4d - 1$ primes between

$X/2$ and $3X/4$ for all $X > N(d)$. The existence of such a number $N(d)$ follows from the Prime Number Theorem and the estimates for $\pi(X)$. Let $X > N(d)$ be fixed. Further, let

$$d+1 < p_{\pi(d+1)+1} < \cdots < p_t < X/2 \leq p_{t+1} < \cdots < p_{t+4d-1} < \cdots \leq 3X/4.$$

Also let

$$q_1 < q_2 < \cdots < q_k$$

be all the primes $\leq d+1$ and co-prime to d . Put

$$R = d^2 \prod_{i=1}^k q_i^{e_i},$$

where e_i is the smallest integer with $q_i^{e_i} > d$. By the Chinese Remainder Theorem, we can choose infinitely many x satisfying the following congruence:

$$\begin{aligned} x &\equiv -d \pmod{p_{t+1}} \\ &\equiv -d+1 \pmod{p_{t+2}} \\ &\dots\dots\dots \\ &\equiv -1 \pmod{p_{t+d}} \\ &\equiv 1 \pmod{p_{t+d+1}} \\ &\dots\dots\dots \\ &\equiv d \pmod{p_{t+2d}} \\ &\equiv -p_{t+1} \pmod{p_{t+2d+1}} \\ &\dots\dots\dots \\ &\equiv -p_{t+2d-1} \pmod{p_{t+4d-1}} \\ &\equiv 0 \pmod{R \prod_{i=\pi(d+1)+1}^t p_i}. \end{aligned}$$

Now consider the integers in the interval

$$J_1 = \left[x - \frac{X}{4}, x + p_{t+2d} - 1 \right].$$

- (i) If $r \in J_1$ and $r = x + j$ with $1 \leq j \leq d$, then choose $s = x + j + p_{t+d+1-j} \in J_1$, to find $\gcd(r, s) = p_{t+d+1-j} \geq d$.
- (ii) If $r \in J_1$ and $r = x - j$ with $1 \leq j \leq d$, then choose $s = x - j + p_{t+d+j} \in J_1$, to find $\gcd(r, s) = p_{t+d+j} \geq d$.
- (iii) If $r \in J_1$ and $r = x + p_{t+j}$ with $1 \leq j \leq 2d-1$, then choose $s = x + p_{t+j} - p_{t+2d+j}$ which is in J_1 since $p_{t+2d+j} - p_{t+j} < X/4$. Moreover, $\gcd(r, s) = p_{t+2d+j} \geq d$.
- (iv) If $r \in J_1$ and $r = x \pm j$ with $j \notin [1, d]$ and $j \neq p_{t+l}$ with $1 \leq l \leq 2d-1$, then choose $s = x$ to find $\gcd(r, s) \geq d$.

Thus, the integers in the interval J_1 do not have property P_d .

Now

$$L(J_1) \geq p_{t+2d} - 1 + \frac{X}{4} \geq \frac{3X}{4}.$$

Caro (in [2]) noticed that this interval can be enlarged as

$$J_2 = [x - p_{t+1} + 1, x + p_{t+2d} - 1],$$

and still the integers in the interval do not have property P_d . Note that

$$L(J_2) - L(J_1) \geq p_{t+1} - 1 - \frac{X}{4} \geq \frac{X}{4}.$$

Thus, starting with J_1 and extending on the left up to a length of $\frac{X}{4}$, we get increasing blocks of consecutive integers for which P_d does not hold. Next we choose $X_1 > X$ such that

$$p_{t+1} < \frac{X_1}{2} \leq p_{t+2}.$$

Then the integers in the interval

$$J_3 = \left[x - \frac{X_1}{4}, x + p_{t+2d+1} - 1 \right]$$

do not have property P_d . As before, we can enlarge this interval to

$$J_4 = [x - p_{t+2} + 1, x + p_{t+2d+1} - 1].$$

Now it is easy to see that $J_4 \supseteq J_2$. Proceeding iteratively, we find that for every integer $m > G(d)$ there exist infinitely many blocks S_m which do not have property P_d . Note that

$$L(J_2) = p_{t+2d} + p_{t+1} - 2 \leq X.$$

Thus,

$$G(d) \leq X, \tag{3.1}$$

where X is chosen such that there are $4d - 1$ primes between $X/2$ and $3X/4$. On the other hand, if X is chosen such that there are $4d - 1$ primes between $X/2$ and X , then we take J'_1 as

$$J'_1 = \left[x - \frac{X}{2}, x + p_{t+2d} - 1 \right]$$

and the integers in J'_1 do not have property P_d . Also

$$L(J'_1) = p_{t+2d} - 1 + \frac{X}{2} \leq \frac{3X}{2}.$$

Here it may not be possible to enlarge the set to

$$J'_2 = [x - p_{t+1} + 1, x + p_{t+2d} - 1],$$

for instance when $p_{t+1} - 1 = X/2$. Thus,

$$g(d) \leq \frac{3X}{2}, \quad (3.2)$$

where X is chosen such that there are $4d - 1$ primes between $X/2$ and X .

An Example. Let $d = 2$ and $X = 60$. There are 7 primes between 30 and 60. Here $k = 1, q_1 = 3$,

$$d + 1 < p_3 < \cdots < p_{10} < 30 < p_{11} < \cdots < p_{17} < 60.$$

We choose x such that

$$\begin{aligned} x &\equiv -2 \pmod{31} \\ &\equiv -1 \pmod{37} \\ &\equiv 1 \pmod{41} \\ &\equiv 2 \pmod{43} \\ &\equiv -31 \pmod{47} \\ &\equiv -37 \pmod{53} \\ &\equiv -41 \pmod{59} \\ &\equiv 0 \pmod{4 \cdot 3 \cdot 5 \cdot 7 \cdots 29}. \end{aligned}$$

Consider the interval

$$J = [x - 30, x + 42].$$

We see that $\gcd(x - 1, x + 40) = 41, \gcd(x - 2, x + 41) = 43, \gcd(x + 1, x + 38) = 37, \gcd(x + 2, x + 33) = 31, \gcd(x + 31, x - 16) = 47, \gcd(x + 37, x - 16) = 53, \gcd(x + 41, x - 18) = 59$. For all other $x + j$, $\gcd(x + j, x) \geq 3$. Thus, the integers in J do not have property P_2 . We also see that we cannot enlarge the set since $\gcd(x - 31, n)$ or $\gcd(x + 43, n)$ for $n \in J$ is not known.

Now we proceed to get an estimate for $g(d)$ and $G(d)$ using (3.1) and (3.2). We apply the estimates for $\pi(x)$ in (2.2) to show

Lemma 6 (i) Suppose $d \geq 20$. Then

$$g(d) \leq 27d \log d.$$

(ii) Suppose $d \geq 11$. Then

$$G(d) \leq 44d \log d.$$

Proof. (i) By the above description of Caro's method, we need to find X such that

$$\pi(X) - \pi(X/2) \geq 4d - 1.$$

By (2.2), it is enough to show that

$$\frac{X}{\log X} - \frac{X}{2(\log X - \log 2)} - \frac{.75X}{(\log X - \log 2)^2} \geq 4d - 1. \quad (3.3)$$

We observe that the left hand side is an increasing function of X . Thus, if this inequality is valid for some $X = X_0$, then it is valid for all $X > X_0$. Also we see that X_0 has to be chosen as a function of d to the order $d \log d$. We set

$$X_0 = 18d \log d.$$

Then left hand side of (3.3) becomes

$$\begin{aligned} & 18d \left(1 + \frac{\log \log d + \log 18}{\log d} \right)^{-1} - 18d \left(2 \left(1 + \frac{\log \log d + \log 9}{\log d} \right) \right)^{-1} - \\ & (0.75)18d \left(\log d \left(1 + \frac{\log \log d + \log 9}{\log d} \right)^2 \right)^{-1}. \end{aligned}$$

We see that this is an increasing function of d and this expression exceeds

$$(.2332)18d > 4d \text{ whenever } d \geq 1000.$$

Thus, for $d \geq 1000$, the assertion is true. For $20 \leq d < 1000$, we check by direct computation using *Mathematica* that

$$\pi(18d \log d) - \pi(9d \log d) \geq 4d - 1$$

holds. This completes the proof of first assertion.

(ii) Here we need to find X such that

$$\pi(3X/4) - \pi(X/2) \geq 4d - 1.$$

Now we follow the argument in (i) with $X_0 = 44d \log d$ to get the second assertion. $\square$

Remark 3. The bounds for $g(d)$ and $G(d)$ given by Lemma 6 are better than the bounds given by Caro. It is clear from the proof of the lemma that for large d , it is possible to get better bounds using the estimates of $\pi(x)$ for large x .

Remark 4. We computed bounds for $g(d)$ with $1 \leq d \leq 19$ and $G(d)$ with $1 \leq d \leq 10$. For instance, let $d = 19$. Lemma 6 and estimates (1.1) suggest that $27d \log d < g(d) < 45d \log d$. A computer search gives $X_0 = 1021$. Thus, $g(19) \leq 1531$. In the Table below, we give the bounds obtained:

d	$g(d)$	$G(d)$	d	$g(d)$
1	25	50	11	763
2	79	134	12	898
3	151	239	13	928
4	208	335	14	1009
5	286	463	15	1114
6	361	578	16	1234
7	424	650	17	1315
8	529	799	18	1429
9	628	879	19	1531
10	664	1050	—	—

Table

Remark 5. Let $d = 1$. From the Table we have $g(d) \leq 25, G(d) \leq 50$. Thus, there are infinitely many sets of consecutive integers S_m for every $m \geq 50$ for which P_1 does not hold. Now let $m \leq 49$. Since $g(1) \leq 25$, there are at least three primes with $m/2 \leq p_{t+1} < p_{t+2} < p_{t+3} < \cdots < m$ for $m \geq 17$. By our construction, there exist x such that

$$x - [m/2], \dots, x - 1, x, x + 1, \dots, x + (p_{t+2} - 1)$$

do not have property P_1 . Hence, the set of integers

$$x - (m - p_{t+2}), \dots, x - 1, x, x + 1, \dots, x + (p_{t+2} - 1)$$

does not have property P_1 . This is the result due to Evans [7] which complements the result of Pillai. From the Table, we have $g(2) \leq 79$. We also know that $g(2) \geq 17$. It will be interesting to determine the exact value of $g(2)$.

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