PILLAI'S PROBLEM ON CONSECUTIVE INTEGERS

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ABSTRACT. For integers $m \geq 2$ and $d \geq 1$, we study the set S_m of m consecutive integers which satisfies the property that for each $x \in S_m$ there exists $y \in S_m$ such that $\gcd(x, y) > d$. This problem was first posed and studied by S. S. Pillai for the case $d = 1$. In this article, we elaborate on an argument of T. Vijayaraghavan for $d = 1$ and of Y. Caro for $d \geq 1$.

§1. Introduction

For an integer $m \geq 2$, let S_m be a set of m consecutive integers. Let $d \geq 1$ be an integer. We say that the set S_m has property P_d if there exists an element $x \in S_m$ such that $\gcd(x, y) \leq d$ for all $y \in S_m$ with $y \neq x$. In this case, we also say that the element x has *property* P_d. When no such element exists, we say that S_m does not have property P_d. Thus, if $d = 1, S_m$ has property P_1 means that there exists $x \in S_m$ which is co-prime to all other elements in S_m.

In 1940, S. S. Pillai (in [12]) and, independently, Szekeres (see for instance, [9]) first studied the problem of finding sets S_m having property P_1. Pillai was motivated by this problem while trying to solve a folklore conjecture that

a product of two or more consecutive integers is never a perfect power.

This remarkable result was proved by P. Erdős and J. L. Selfridge (in [6]) in 1975. Pillai (in [12]) showed that S_m has property P_1 for $m < 17$. Thus, any set of consecutive integers having less than 17 elements has property P_1. Further, Pillai succeeded in proving that for $17 \leq m \leq 430$, there exist infinitely many sets S_m for which property P_1 does not hold. To prove this result, he used sieving techniques and introduced numbers known as *gap numbers.* Then he extended this result to $m \leq 12335$ (in [14])). Also, William Scott (in [17]) further extended this to $m \leq 2491906561$ in a private letter to Pillai. Finally, this result was completely solved for all $m \geq 17$ by A. T. Brauer (in [1]) in 1941. Later many authors including Pillai (in [15]), Erdős (in [5]), Evans (in [7]), Harborth (in [10] and [11]) and Gassko (in [9]) gave different proofs of this result. Indeed, Gassko

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characterized the sets S_m in terms of coverings of finite sequences of natural numbers by arithmetic progressions with prime differences. Evans extended the problem to blocks in arithmetic progression (see [8]). Since Pillai's problem for arithmetic progression poses no new difficulty, we restrict only to the case of consecutive integers. To summarize, we have

Theorem 1. *Any set* S_m *with* $1 \leq m \leq 16$, *has property* P_1. *For every* $m \geq 17$, *there exist infinitely many sets* S_m *for which property* P_1 *does not hold.*

In a private communication to Pillai, around 1940, T. Vijayaraghavan gave an argument to prove the following theorem.

Theorem 2. *There exist infinitely many sets* S_m *which do not have property* P_1 *whenever* m *is sufficiently large.*

The second author came across this communication of Vijayaraghavan during the preparation of the collected works of Pillai in which he is jointly involved with R. Balasubramanian. Though Theorem 2 is much weaker than Theorem 1, the ideas proposed by Vijayaraghavan were illuminating and have some similarities with the works of Y. Caro (in [2]) for $d > 1$. In Section 2, we elaborate and give a complete proof of Theorem 2 following Vijayaraghavan's ideas from that letter.

Next let $d > 1$. Developing on Evan's proof of Theorem 1, in 1979 Caro (in [2]) proved the following stronger result.

Theorem 3. *Let* $d > 1$. *There exist infinitely many sets* S_m *which do not have property* P_d *whenever* m *exceeds an effectively computable number* $G(d)$.

Thus, for $m > G(d)$, there are infinitely many sets S_m such that for every $x \in S_m$, there exists $y \in S_m$ with $y \neq x$ such that $\gcd(x, y) > d$. For $m \leq G(d)$, it is still possible that there exists some set S_m for which P_d does not hold. Denote by $g(d)$ the smallest positive integer such that there exists a set $S_{g(d)}$ for which property P_d does not hold. By Theorem 1, $G(1) = g(1) = 17$. For example, the set $S_{17} = 2184, 2185, \cdots, 2200$ does not have property P_1. Since

$$30030 = 2 \times 3 \times 5 \times 7 \times 11 \times 13$$

and each term of this sequence $2184, 2185, \cdots, 2200$ is divisible by one of the primes $p \leq 13$, we conclude that the infinite sets of 17 elements $\{2184 + 30030k, 2185 + 30030k, \cdots, 2200 + 30030k\}$ for $k \in \mathbb{Z}$ do not have property P_1.

Caro showed that

$$g(d) \ < \ 45d \log d, \quad G(d) \ < \ 54d \log d. \tag{1.1}$$

In Section 3, we first sketch the argument of Y. Caro and then give better bounds for $G(d)$ and $g(d)$ than the above.

The exact values of $g(d)$ and $G(d)$ for any $d \geq 2$ are not known. It will be interesting to characterize those sets for which P_d holds with $d \geq 2$.

§2. An Argument of Vijayaraghavan

We split the argument of Vijayaraghavan into several lemmas to bring out its essence. Let t and T denote the smallest and largest integer in S_m, respectively. Thus, $T - t = m$. Let $p(x)$ denote the least prime divisor of x. Then

Lemma 1. *A necessary and sufficient condition for $x \in S_m$ not to have property P_1 is that*

$$p(x) \leq \max\{x - t, T - x\}. \tag{2.1}$$

Proof. We observe that $x \in S_m$ does not have property P_1 if and only if it has a common factor with some other element, say, y of S_m. Then

$$p(x) \leq |y - x| \leq \max\{x - t, T - x\}.$$

The assertion of the lemma follows. □

Let $X \geq 2$ be any real number and $\pi(X)$ denote the number of primes $\leq X$. This counting function of primes plays an important role in many areas of Number Theory. Extensive work on the estimates for $\pi(X)$ was done by Rosser and Schöenfeld (in [16]). Some of these estimates were improved by P. Dusart (see [3] and [4]) in 1998. For the purpose of this paper, we shall use the estimates from [16] to get

$$\frac{X}{\log X} \leq \pi(X) \leq \frac{X}{\log X} + \frac{1.5X}{\log^2 X}.$$

Hence,

$$\frac{X}{\log X} \ < \ \pi(x) \leq 1.5\frac{X}{\log X} \text{ for all } X \geq 21. \tag{2.2}$$

Let $2 = p_1, p_2, \cdots$ be the sequence of all prime numbers. Then we have

$$n \log n < p_n < n(\log n + \log\log n) \text{ for } n \geq 6 \tag{2.3}$$

(see [12, p.69]). We apply (2.3) to get

Lemma 2. *For any $\alpha \geq 2.5$, we have*

$$p_\ell \leq 2\alpha X \text{ for all } X \geq 2\alpha,$$

where $\ell = \left[\dfrac{\alpha X}{\log\ X}\right]$.

Proof. From (2.3), we get

$$\begin{aligned}
p_\ell &\leq \ell(\log \ell + \log\log \ell)\\
&\leq \frac{\alpha X}{\log X}(\log\alpha + \log X - \log\log X + \log(\log \alpha X))\\
&\leq \frac{\alpha X}{\log X}\left(\log\alpha + \log X + \log\left(1 + \frac{\log\alpha}{\log X}\right)\right)\\
&\leq \frac{\alpha X}{\log X}(\log 2\alpha + \log X) \leq 2\alpha X,
\end{aligned}$$

as $X \geq 2\alpha$. □

Put

$$A = A_X = p_1 p_2 \cdots p_{\pi(X)}$$

and let

$$1 = a_0 < a_1 < \cdots < a_n < \cdots \tag{2.4}$$

be the sequence of all integers co-prime to A. For example, when $X = 2$, the sequence in (2.4) consists of all odd integers. Observe that $a_1 = 1$ and a_2 is the first prime exceeding X. Thus, $a_2 = p_{\pi(X)+1}$. Also, the sequence (2.4) is periodic with period $\phi(A)$ (where $\phi(.)$ denotes the Euler's phi-function) in the following sense

$$A + a_i = a_{i+\phi(A)} \text{ and } a_{\phi(A)} = A - 1. \tag{2.5}$$

Thus, to know all the values of this sequence, it is enough to compute the values of a_i for $i = 1, 2, \cdots, \phi(A) - 1$.

Let $\gamma = 0.577215665\ldots$ denote the Euler constant. Thus, $1.78 < e^\gamma < 1.8$. Set

$$M = \left[\frac{15X}{\log X}\right].$$

Then the following lemma explores the gaps between the elements of the sequence (2.4).

Lemma 3. *There are infinitely many pairs* $\{c_t, c_{t+M}\}$ *with* c_t, c_{t+M} *elements of the sequence in* (2.4) *such that*

$$c_{t+M} - c_t > 25X \text{ for } t \geq 0. \tag{2.6}$$

Proof. It is well-known that the density of the sequence (2.4) is

$$\frac{\phi(p_1 p_2 \cdots p_{\pi(X)})}{p_1 p_2 \cdots p_{\pi(X)}}.$$

By Mertens' estimate, we know that

$$\frac{\phi(p_1p_2\cdots p_{\pi(X)})}{p_1p_2\cdots p_{\pi(X)}} = \prod_{p\le X}\left(1-\frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log X}.$$

Consider the sequence

$$a_0, a_M, a_{2M}, \cdots a_{tM}, \cdots$$

Since $\left[\frac{15X}{\log X}\right] > \frac{(14.5)X}{\log X}$, we see that the density of this sequence is

$$\frac{1}{M}\frac{e^{-\gamma}}{\log X} > \frac{1}{14.5e^{\gamma}X},$$

since $M < 15X/\log X$. Thus, there exists h such that

$$a_{(h+1)M} - a_{hM} > (14.5)e^{\gamma}X > 25X.$$

Hence, by (2.5), we see that for any $t \ge 0$,

$$a_{(h+1)M+t\phi(A)} - a_{hM+t\phi(A)} = a_{(h+1)M} - a_{hM} > 25X.$$

Now put $c_t = a_{hM+t\phi(A)}$ to get the assertion of the lemma. □

Remark 1. When $X = 2$, the above lemma is clear since in this case $M = 43, a_i = 2i+1, a_{43+i} = 87+2i$, giving $a_{43+i} - a_i = 86 > 25X$ for any $i \ge 0$. Hence, we take $h = 0$. Thus, all the pairs $(a_0, a_{43}), (a_1, a_{44}), \cdots$ satisfy the assertion of the lemma.

Remark 2. Lemma 3 says that for any $X \ge 2$, the gap $a_j - a_i$ between any two elements $a_i < a_j$ in (2.4) is as large as X provided $j - i = M$. When X is large, it is natural to expect large gap $a_j - a_i$, even if $j - i = 1$. Indeed, this phenomenon is true. To see this, let

$$H(X) = \max_{i\ge 1}(a_{i+1} - a_i).$$

In fact, it is clear from (2.5) that

$$H(X) = \max_{1\le i\le \phi(A_X)-1}(a_{i+1} - a_i).$$

As observed earlier, $a_1 = 1$ and $a_2 = p_{\pi(X)+1}$. Hence, by (2.3), we have

$$H(X) \ge a_2 - a_1 \ge p_{\pi(X)+1} - 1 > (\pi(X)+1)\log(\pi(X)+1) - 1.$$

Now we use (2.2) to get

$$H(X) \ge .6X \text{ for } X \ge 20.$$

During his investigation on gaps between consecutive primes, P. Erdős (in [5]) showed that

there exists a positive constant c such that we can find

$$c\ p_{\pi(X)} \log p_{\pi(X)} (\log\log p_{\pi(X)})^{-2}$$

consecutive integers so that no one of them is relatively prime to A_X.

This result was based on Brun's sieve and several other intricate arguments. From this result, it follows that

$$H(X) \geq c_1 \frac{X \log\ X}{(\log\log\ X)^2}$$

for some positive constant c_1, whenever X is large.

Take any pair $\{c_t, c_{t+M}\}$ as in Lemma 3. Let $e_t = \dfrac{c_t + c_{t+M}}{2}$. Re-arrange the elements $c_t, c_{t+1}, \cdots, c_{t+M-1}, c_{t+M}$ as

$$e_t + y_0, e_t + y_1, \cdots, e_t + y_M \text{ with } |y_0| \leq |y_1| \leq \cdots \leq |y_M|. \tag{2.7}$$

Let N be the largest integer such that

$$|y_N| \leq 2X. \tag{2.8}$$

In the case $X = 2$, take $(a_0, a_{43}) = (c_t, c_{t+M})$. Then $e_t = \dfrac{a_0 + a_{43}}{2} = 44$. Thus, (2.7) becomes

$$43, 45; 41, 47; 39, 49; 37, 51; 35, 53; 33, 55; 31, 57; 29, 59; 27, 61; \cdots; 1, 87.$$

Hence, $N = 2$.

Let N be very small. This means there are only very few y_i's with their absolute values small. Thus, there are very few elements of (2.7) which are near e_t. Hence, in this case we may expect to have an interval around e_t in which property P_1 does not hold. We make this precise in the following lemma.

Lemma 4. *Let $X \geq 33$ be any integer. Suppose $N \leq \frac{14X}{\log X}$. Then there exists a λ such that no integer in the interval*

$$I_1 = (e_\lambda - 31X, e_\lambda + 31X)$$

has property P_1.

Proof. By the Chinese Remainder Theorem, we can choose an integer x such that

$$\begin{array}{rcl} x & \equiv & e_0 \pmod{A} \\ x & \equiv & -y_1 \pmod{p_{\pi(X)+1}} \\ \cdots & & \cdots \\ x & \equiv & -y_M \pmod{p_{\pi(X)+M}}. \end{array}$$

Thus, $x = e_0 + \lambda A = e_\lambda$ for some λ. Further,

$$p_{\pi(X)+i} \mid (e_\lambda + y_i) \text{ for } 1 \leq i \leq M. \tag{2.9}$$

Now consider $x \in I_1$. Suppose that $x \neq e_\lambda + y_i$ for any i. Then, as x is not co-prime to A,

$$p(x) \leq X. \tag{2.10}$$

If $x = e_\lambda + y_i$, then by (2.7) and Lemma 2, we have

$$p(x) \leq p_{\pi(X)+i} \leq p_{\pi(X)+M} \leq 33X, \tag{2.11}$$

since $X \geq 33 = 2\alpha$ with $\alpha = 16.5$.

If $x = e_\lambda + y_i$ with $i \leq N$, then, using the estimates for p_n, we have

$$p(x) \leq p_{\pi(X)+N} \leq 31X. \tag{2.12}$$

For a given $x \in I_1$, let

$$L(x) = \max\left(x - e_\lambda + 31X, e_\lambda + 31X - x\right).$$

We now show that

$$p(x) \leq L(x) \text{ for every } x \in I_1. \tag{2.13}$$

By (2.9) and (2.12), we need to consider the case

$$x = e_\lambda + y_i \text{ where } i > N.$$

By the definition of N, we have

$$|y_i| = |x - e_\lambda| > 2X.$$

Thus,

$$L(x) > 33X \geq p(X),$$

by (2.11). Hence, (2.13) follows and hence the lemma. □

Next we look at the case when N is large. Then there are few values of $x = e_\lambda + y_i$ with y_i's large and these values of x are near c_λ. Hence, it is likely that integers in a suitable interval $[c_\lambda, c_\lambda + \rho]$ do not have property P_1. Thus, we have the following lemma.

Lemma 5. *Let $X \geq 21$ be any integer. Suppose $N > \dfrac{14X}{\log X}$. Then there exists a λ such that no integer in the interval*

$$I_2 = [c_\lambda, c_\lambda + 10X]$$

has property P_1.

Proof. For any fixed λ, in the interval $(c_\lambda, c_{\lambda+M})$ there are M elements of the form $e_\lambda + y_i$. Out of these, there are N elements with

$$|y_i| \leq 2X.$$

Thus, there are at most

$$M - N \leq \frac{X}{\log X} := L$$

such elements which lie outside the interval

$$\left(e_\lambda - 2X, e_\lambda + 2X\right).$$

We observe by (2.5) that

$$e_\lambda - 2X = \frac{c_\lambda + c_{\lambda+M}}{2} - 2X > \frac{2c_\lambda + 25X}{2} - 2X > c_\lambda + 10X.$$

Thus, there are at most L elements $e_\lambda - z_i$ lying in the interval

$$I_2 = (c_\lambda, c_\lambda + 10X),$$

where $z_i = -y_i$ and $z_i > 2X$.

By the Chinese Remainder theorem, we can choose an integer x such that

$$\begin{array}{rcll} x & \equiv & e_0 & (\text{mod } A) \\ x & \equiv & z_1 & (\text{mod } p_{\pi(X)+1}) \\ \cdots & & \cdots & \\ x & \equiv & z_L & (\text{mod } p_{\pi(X)+L}). \end{array}$$

Then, $x = e_\lambda = e_0 + \lambda A$ for some λ and

$$p_{\pi(X)+j} \mid (e_\lambda - z_j) \text{ for } 1 \le j \le L.$$

If $x \in I_2$ and $x \ne e_\lambda - z_j$ for any j, then

$$p(x) \le X. \tag{2.14}$$

If $x \in I_2$ and $x = e_\lambda - z_j$ for some j, then

$$p(x) \le p_{\pi(X)+j} \le p_{\pi(X)+L} \le 5X, \tag{2.15}$$

since $X \ge 13$. For any $x \in I_2$, we let

$$M(x) = \max(x - c_\lambda, c_\lambda + 10X - x).$$

We show that

$$p(x) \le M(x) \text{ for any } x \in I_2.$$

By (2.14) and (2.15), we may assume that $x = e_\lambda - z_j$ for some j and $x - c_\lambda \ge 5X$. Then $10X + c_\lambda - x \ge 5X$. Hence, $p(x) \le M(x)$. □

Proof of Theorem 2. We combine Lemmas 4 and 5 to observe that when X is sufficiently large there exist infinitely many integers μ and an absolute constant c such that the interval

$$[\mu, \mu + cX]$$

does not have property P_1. Thus, the property P_1 does not hold for infinitely many sets S_m with $m = cX$ and X sufficiently large. □

§3. Bounds for $g(d)$ and $G(d)$.

Caro (in [2]) extended Pillai's problem for sets S_m with property P_d with $d \ge 1$. We give a description of his construction. Let $d \ge 1$ be fixed. For any interval J, we denote by $L(J)$, the length of the interval. Let $N(d)$ denote a number such that there are at least $4d-1$ primes between

$X/2$ and $3X/4$ for all $X > N(d)$. The existence of such a number $N(d)$ follows from the Prime Number Theorem and the estimates for $\pi(X)$. Let $X > N(d)$ be fixed. Further, let

$$d+1 < p_{\pi(d+1)+1} < \cdots < p_t < X/2 \le p_{t+1} < \cdots < p_{t+4d-1} < \cdots \le 3X/4.$$

Also let

$$q_1 < q_2 < \cdots < q_k$$

be all the primes $\le d+1$ and co-prime to d. Put

$$R = d^2 \prod_{i=1}^{k} q_i^{e_i},$$

where e_i is the smallest integer with $q_i^{e_i} > d$. By the Chinese Remainder Theorem, we can choose infinitely many x satisfying the following congruence:

$$\begin{aligned} x &\equiv -d \pmod{p_{t+1}} \\ &\equiv -d+1 \pmod{p_{t+2}} \\ &\cdots\cdots \\ &\equiv -1 \pmod{p_{t+d}} \\ &\equiv 1 \pmod{p_{t+d+1}} \\ &\cdots\cdots \\ &\equiv d \pmod{p_{t+2d}} \\ &\equiv -p_{t+1} \pmod{p_{t+2d+1}} \\ &\cdots\cdots \\ &\equiv -p_{t+2d-1} \pmod{p_{t+4d-1}} \\ &\equiv 0 \pmod{R \prod_{i=\pi(d+1)+1}^{t} p_i}. \end{aligned}$$

Now consider the integers in the interval

$$J_1 = \left[x - \frac{X}{4}, x + p_{t+2d} - 1 \right].$$

(i) If $r \in J_1$ and $r = x + j$ with $1 \le j \le d$, then choose $s = x + j + p_{t+d+1-j} \in J_1$, to find $\gcd(r,s) = p_{t+d+1-j} \ge d$.

(ii) If $r \in J_1$ and $r = x - j$ with $1 \le j \le d$, then choose $s = x - j + p_{t+d+j} \in J_1$, to find $\gcd(r,s) = p_{t+d+j} \ge d$.

(iii) If $r \in J_1$ and $r = x + p_{t+j}$ with $1 \le j \le 2d-1$, then choose $s = x + p_{t+j} - p_{t+2d+j}$ which is in J_1 since $p_{t+2d+j} - p_{t+j} < X/4$. Moreover, $\gcd(r,s) = p_{t+2d+j} \ge d$.

(iv) If $r \in J_1$ and $r = x \pm j$ with $j \notin [1,d]$ and $j \ne p_{t+l}$ with $1 \le l \le 2d-1$, then choose $s = x$ to find $\gcd(r,s) \ge d$.

Thus, the integers in the interval J_1 do not have property P_d.

Now

$$L(J_1) \geq p_{t+2d} - 1 + \frac{X}{4} \geq \frac{3X}{4}.$$

Caro (in [2]) noticed that this interval can be enlarged as

$$J_2 = [x - p_{t+1} + 1, x + p_{t+2d} - 1],$$

and still the integers in the interval do not have property P_d. Note that

$$L(J_2) - L(J_1) \geq p_{t+1} - 1 - \frac{X}{4} \geq \frac{X}{4}.$$

Thus, starting with J_1 and extending on the left up to a length of $\frac{X}{4}$, we get increasing blocks of consecutive integers for which P_d does not hold. Next we choose $X_1 > X$ such that

$$p_{t+1} \ < \ \frac{X_1}{2} \leq p_{t+2}.$$

Then the integers in the interval

$$J_3 = \left[x - \frac{X_1}{4}, x + p_{t+2d+1} - 1\right]$$

do not have property P_d. As before, we can enlarge this interval to

$$J_4 = [x - p_{t+2} + 1, x + p_{t+2d+1} - 1].$$

Now it is easy to see that $J_4 \supseteq J_2$. Proceeding iteratively, we find that for every integer $m > G(d)$ there exist infinitely many blocks S_m which do not have property P_d. Note that

$$L(J_2) = p_{t+2d} + p_{t+1} - 2 \leq X.$$

Thus,

$$G(d) \leq X, \tag{3.1}$$

where X is chosen such that there are $4d-1$ primes between $X/2$ and $3X/4$. On the other hand, if X is chosen such that there are $4d-1$ primes between $X/2$ and X, then we take J_1' as

$$J_1' = \left[x - \frac{X}{2}, x + p_{t+2d} - 1\right]$$

and the integers in J_1' do not have property P_d. Also

$$L(J_1') = p_{t+2d} - 1 + \frac{X}{2} \leq \frac{3X}{2}.$$

Here it may not be possible to enlarge the set to

$$J_2' = [x - p_{t+1} + 1, x + p_{t+2d} - 1],$$

for instance when $p_{t+1} - 1 = X/2$. Thus,

$$g(d) \le \frac{3X}{2}, \tag{3.2}$$

where X is chosen such that there are $4d - 1$ primes between $X/2$ and X.

An Example. Let $d = 2$ and $X = 60$. There are 7 primes between 30 and 60. Here $k = 1, q_1 = 3$,

$$d + 1 < p_3 < \cdots < p_{10} < 30 < p_{11} < \cdots < p_{17} < 60.$$

We choose x such that

$$\begin{aligned} x &\equiv -2 \pmod{31} \\ &\equiv -1 \pmod{37} \\ &\equiv 1 \pmod{41} \\ &\equiv 2 \pmod{43} \\ &\equiv -31 \pmod{47} \\ &\equiv -37 \pmod{53} \\ &\equiv -41 \pmod{59} \\ &\equiv 0 \pmod{4 \cdot 3 \cdot 5 \cdot 7 \cdots 29}. \end{aligned}$$

Consider the interval

$$J = [x - 30, x + 42].$$

We see that $\gcd(x-1, x+40) = 41, \gcd(x-2, x+41) = 43, \gcd(x+1, x+38) = 37, \gcd(x+2, x+33) = 31, \gcd(x+31, x-16) = 47, \gcd(x+37, x-16) = 53, \gcd(x+41, x-18) = 59$. For all other $x + j$, gcd $(x+j, x) \ge 3$. Thus, the integers in J do not have property P_2. We also see that we cannot enlarge the set since $\gcd(x-31, n)$ or $\gcd(x+43, n)$ for $n \in J$ is not known.

Now we proceed to get an estimate for $g(d)$ and $G(d)$ using (3.1) and (3.2). We apply the estimates for $\pi(x)$ in (2.2) to show

Lemma 6 *(i) Suppose* $d \ge 20$. *Then*

$$g(d) \le 27 d \log\ d.$$

(ii) Suppose $d \ge 11$. *Then*

$$G(d) \le 44 d \log\ d.$$

Proof. (i) By the above description of Caro's method, we need to find X such that

$$\pi(X) - \pi(X/2) \ge 4d - 1.$$

By (2.2), it is enough to show that

$$\frac{X}{\log\ X} - \frac{X}{2(\log\ X - \log\ 2)} - \frac{.75X}{(\log\ X - \log\ 2)^2} \geq 4d - 1. \tag{3.3}$$

We observe that the left hand side is an increasing function of X. Thus, if this inequality is valid for some $X = X_0$, then it is valid for all $X > X_0$. Also we see that X_0 has to be chosen as a function of d to the order $d \log d$. We set

$$X_0 = 18d \log d.$$

Then left hand side of (3.3) becomes

$$18d\left(1 + \frac{\log\log d + \log 18}{\log d}\right)^{-1} - 18d\left(2\left(1 + \frac{\log\log d + \log 9}{\log d}\right)\right)^{-1} -$$

$$(0.75)18d\left(\log d\left(1 + \frac{\log\log d + \log 9}{\log d}\right)^2\right)^{-1}.$$

We see that this is an increasing function of d and this expression exceeds

$$(.2332)18d > 4d \quad \text{whenever } d \geq 1000.$$

Thus, for $d \geq 1000$, the assertion is true. For $20 \leq d < 1000$, we check by direct computation using *Mathematica* that

$$\pi(18d\log\ d) - \pi(9d\log\ d) \geq 4d - 1$$

holds. This completes the proof of first assertion.

(ii) Here we need to find X such that

$$\pi(3X/4) - \pi(X/2) \geq 4d - 1.$$

Now we follow the argument in (i) with $X_0 = 44d \log d$ to get the second assertion. □

Remark 3. The bounds for $g(d)$ and $G(d)$ given by Lemma 6 are better than the bounds given by Caro. It is clear from the proof of the lemma that for large d, it is possible to get better bounds using the estimates of $\pi(x)$ for large x.

Remark 4. We computed bounds for $g(d)$ with $1 \leq d \leq 19$ and $G(d)$ with $1 \leq d \leq 10$. For instance, let $d = 19$. Lemma 6 and estimates (1.1) suggest that $27d\log\ d < g(d) < 45d\log\ d$. A computer search gives $X_0 = 1021$. Thus, $g(19) \leq 1531$. In the Table below, we give the bounds obtained:

d	$g(d)$	$G(d)$	d	$g(d)$
1	25	50	11	763
2	79	134	12	898
3	151	239	13	928
4	208	335	14	1009
5	286	463	15	1114
6	361	578	16	1234
7	424	650	17	1315
8	529	799	18	1429
9	628	879	19	1531
10	664	1050	–	–

Table

Remark 5. Let $d = 1$. From the Table we have $g(d) \leq 25, G(d) \leq 50$. Thus, there are infinitely many sets of consecutive integers S_m for every $m \geq 50$ for which P_1 does not hold. Now let $m \leq 49$. Since $g(1) \leq 25$, there are at least three primes with $m/2 \leq p_{t+1} < p_{t+2} < p_{t+3} < \cdots < m$ for $m \geq 17$. By our construction, there exist x such that

$$x - [m/2], \cdots, x - 1, x, x + 1, \cdots x + (p_{t+2} - 1)$$

do not have property P_1. Hence, the set of integers

$$x - (m - p_{t+2}), \cdots, x - 1, x, x + 1, \cdots x + (p_{t+2} - 1)$$

does not have property P_1. This is the result due to Evans [7] which complements the result of Pillai. From the Table, we have $g(2) \leq 79$. We also know that $g(2) \geq 17$. It will be interesting to determine the exact value of $g(2)$.

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