

On nonvanishing infinite sums

N. Saradha

*Dedicated to Professor K. Ramachandra
on the occasion of his 70th birthday*

1 Introduction

Let $1 < q = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ with $p_1 < \cdots < p_r, p_i$ prime and $\alpha_i \geq 1$ for $1 \leq i \leq r$. We denote by $\varphi(q)$ and $\omega(q)$, the Euler totient function of q and the number of distinct prime divisors of q , respectively. Suppose f is an arithmetic function, periodic mod q defined by

$$f(n) = \begin{cases} \pm 1 & \text{if } n \text{ is not a multiple of } q \\ 0 & \text{otherwise.} \end{cases}$$

Then Erdős (see [2]), posed the problem whether the infinite series

$$(1) \quad S = \sum_{n=1}^{\infty} \frac{f(n)}{n}$$

can vanish whenever convergent? A necessary and sufficient condition for the series to converge is that

$$\sum_{n=1}^q f(n) = \sum_{n=1}^{q-1} f(n) = 0.$$

Hence if q is even, the series S does not converge. Thus we restrict to q odd. In [5], Okada showed that the series S does not vanish provided

$$(2) \quad 2\varphi(q) \geq q.$$

This condition is satisfied for all q with $\omega(q) \leq 2$. In particular, when q is a prime then S does not vanish. By a result of Adhikari, Saradha, Shorey and Tijdeman [1], the non-vanishing of S implies that S is transcendental. Their result was a consequence of Baker's theory on linear forms in logarithms and a theorem of Baker, Birch and Wirsing [4] in which they derived a necessary and sufficient condition for the nonvanishing of the series

$$T = \sum_{n=1}^{\infty} \frac{g(n)}{n}$$

where g is an algebraic valued, periodic function with period q . Okada [5] translated the necessary and sufficient conditions into a criterion consisting of a set of $\varphi(q) + \omega(q)$ homogeneous linear equations to be satisfied by the values of g . Using Okada's criterion Tijdeman [7] showed that T does not vanish if g is completely multiplicative or if g is multiplicative such that $|g(p^k)| < p - 1$ for every prime divisor p of q and every positive integer k . The criterion of Okada was reformulated by Saradha and Tijdeman ([6], see Lemma 1). Using this, they gave explicit conditions under which some special infinite sums do not vanish. In this paper we shall use the reformulated criterion of [6] to study the nonvanishing of S when $\omega(q) = 3, 4$. We show

Theorem 1 (i) *Let $\omega(q) = 3$. Then S does not vanish except possibly when*

$$(p_1, p_2, p_3) \in \{(3, 5, 7), (3, 5, 11), (3, 5, 13)\}$$

and $q \notin A$ where A is given by

$$(3) \quad A = \{3 \cdot 5 \cdot 7, 3^2 \cdot 5 \cdot 7, 3 \cdot 5 \cdot 11, 3 \cdot 5 \cdot 13, 3 \cdot 5^2 \cdot 11, 3 \cdot 5^2 \cdot 13, 3^2 \cdot 5 \cdot 11, \\ 3^2 \cdot 5 \cdot 13, 3^2 \cdot 5^2 \cdot 11, 3^2 \cdot 5^2 \cdot 13, 3^3 \cdot 5 \cdot 11, 3^3 \cdot 5 \cdot 13, 3^3 \cdot 5^2 \cdot 11, 3^3 \cdot 5^2 \cdot 13\}.$$

(ii) *Let $\omega(q) = 4$. Then S does not vanish except possibly when*

$(p_1, p_2, p_3, p_4) = (3, 5, 7, p_4); (3, 5, 11, p_4); (3, 5, 13, p_4); (3, 5, 17, p_4)$ with $p_4 \leq 251$;
 $(3, 5, 19, p_4)$ with $p_4 \leq 89$; $(3, 5, 23, p_4)$ with $p_4 \leq 47$; $(3, 5, 29, p_4)$ with $p_4 \leq 31$;
 $(3, 7, 11, p_4)$ with $p_4 \leq 23$; $(3, 7, 13, p_4)$ with $p_4 \leq 19$

and q not listed in Table 1.

As a consequence of Theorem 1, we get

Corollary *Let $q \notin \{525, 735, 945\}$. Then S does not vanish for all $q \leq 1154$.*

We derive Theorem 1 from the following result.

Theorem 2 *Let $h = \max_{1 \leq i \leq r} (p_i^{\alpha_i})$. Assume that*

$$(4) \quad 2\varphi(q) \geq q\left(1 - \frac{1}{h}\right).$$

Then S does not vanish.

The values of q in (3) and Table 1 for which S does not vanish are not covered by the result of Okada. The nonvanishing of S when $q = 525, 735, 945$ is not known. For a survey on related problems, refer to [2], [3] and [7].

I wish to thank Professor T.N. Shorey for his suggestions on an earlier draft.

2. Lemmas We begin with the reformulated criterion of [6, Lemma 1].

Lemma 1 *Let g be an algebraic valued, periodic function with period q . Let M be the set of integers composed only of p_1, \dots, p_r . Suppose for any integer $n > 0$, $\nu_p(n)$ denotes the order of the prime p in n . Define*

$$\epsilon(r, p) = \begin{cases} \nu_p(r) & \text{if } \nu_p(r) < \nu_p(q) \\ \nu_p(q) + \frac{1}{p-1} & \text{if } \nu_p(r) \geq \nu_p(q) \end{cases}$$

Then

$$T = \sum_{n=1}^{\infty} \frac{g(n)}{n}$$

vanishes if and only if the following two conditions hold.

$$(i) \quad \sum_{n \in M} \frac{g(an)}{n} = 0 \text{ for every } a \text{ with } 1 \leq a < q \text{ and } \gcd(a, q) = 1$$

$$(ii) \quad \sum_{\substack{t=1 \\ \gcd(t, q) > 1}}^q g(t) \epsilon(t, p_i) = 0 \text{ for } 1 \leq i \leq r.$$

Lemma 2 Suppose S vanishes. Let a, b be two positive integers with $\gcd(a, q) = \gcd(b, q) = 1$ and $a \equiv b \pmod{\frac{q}{p_i^{\alpha_i}}}$ for some i with $1 \leq i \leq r$. Suppose

$$(5) \quad \frac{q}{\varphi(q)} \frac{p_i^{\alpha_i} - 1}{p_i^{\alpha_i}} < 2.$$

Then $f(a) = f(b)$.

Proof We apply Lemma 1 with $T = S$. Then (i) and (ii) are valid with $g = f$. From (i), we have

$$(6) \quad f(a') = - \sum_{\substack{m \in M \\ m > 1}} \frac{f(a'm)}{m} \text{ for any } a' \text{ with } \gcd(a', q) = 1.$$

Since $a \equiv b \pmod{\frac{q}{p_i^{\alpha_i}}}$, we have $f(am) = f(bm)$ whenever $p_i^{\alpha_i} \mid m$. Thus it follows from (6) that

$$\begin{aligned} f(a) - f(b) &= \sum_{\substack{m \in M \\ m > 1}} \frac{f(bm) - f(am)}{m} \\ &= \sum_{\substack{m \in M, m > 1 \\ p_i \nmid m}} \frac{f(bm) - f(am)}{m} + \sum_{\substack{m \in M, m > 1 \\ 0 < \nu_{p_i}(m) < \alpha_i}} \frac{f(bm) - f(am)}{m}. \end{aligned}$$

Hence

$$|f(a) - f(b)| \leq 2 \left(\sum_{\substack{m \in M, m > 1 \\ p_i \nmid m}} \frac{1}{m} + \sum_{\substack{m \in M, m > 1 \\ 0 < \nu_{p_i}(m) < \alpha_i}} \frac{1}{m} \right).$$

Suppose

$$P_i = \left(1 - \frac{1}{p_1}\right)^{-1} \cdots \left(1 - \frac{1}{p_r}\right)^{-1} \left(1 - \frac{1}{p_i}\right) = \frac{q}{\varphi(q)} \frac{p_i - 1}{p_i}.$$

Since M is the set of integers composed only of p_1, \dots, p_r , we see that

$$\sum_{\substack{m \in M, m > 1 \\ p_i \nmid m}} \frac{1}{m} = \frac{\left(1 + \frac{1}{p_1} + \cdots\right) \cdots \left(1 + \frac{1}{p_r} + \cdots\right)}{\left(1 + \frac{1}{p_i} + \cdots\right)} - 1 = P_i - 1$$

and

$$\sum_{\substack{m \in M, m > 1 \\ 0 < \nu_{p_i}(m) < \alpha_i}} \frac{1}{m} = \frac{\left(1 + \frac{1}{p_1} + \cdots\right) \cdots \left(1 + \frac{1}{p_r} + \cdots\right)}{\left(1 + \frac{1}{p_i} + \cdots\right)} \left(\frac{1}{p_i} + \cdots + \frac{1}{p_i^{\alpha_i - 1}}\right) = \frac{P_i}{p_i - 1} \left(1 - \frac{1}{p_i^{\alpha_i - 1}}\right).$$

Thus we find that

$$\begin{aligned} |f(a) - f(b)| &\leq 2 \left(P_i - 1 + P_i \frac{1 - \frac{1}{p_i^{\alpha_i - 1}}}{p_i - 1} \right) \\ &\leq 2 \left(\frac{q}{\varphi(q)} \frac{p_i^{\alpha_i} - 1}{p_i^{\alpha_i}} - 1 \right) < 2 \end{aligned}$$

by our supposition. Thus $f(a) = f(b)$. \square

Lemma 3. *Assume that S vanishes. Suppose (5) holds for every i with $1 \leq i \leq r$. Then*

$$f(a) = f(b)$$

for integers a, b with $\gcd(a, q) = \gcd(b, q) = 1$.

Proof. It is enough to prove the result for a, b with $1 \leq a, b < q$ and $\gcd(a, q) = \gcd(b, q) = 1$. There are $\varphi(q)$ number of such a 's. For every i with $1 \leq i \leq r$, we partition the a 's into $\varphi\left(\frac{q}{p_i^{\alpha_i}}\right)$ classes, say

$$C_1^{(i)}, \dots, C_{\varphi(q/p_i^{\alpha_i})}^{(i)},$$

each class consisting of $\varphi(p_i^{\alpha_i})$ elements such that if $a, b \in C_h^{(i)}$ for some h with $1 \leq h \leq \varphi(q/p_i^{\alpha_i})$, then $a \equiv b \pmod{q/p_i^{\alpha_i}}$. Hence by Lemma 2, we have

$$(7) \quad f(a) = f(b) \text{ if } a, b \in C_h^{(i)} \text{ with } 1 \leq h \leq \varphi(q/p_i^{\alpha_i}).$$

We set

$$X_i = \{C_1^{(i)}, \dots, C_{\varphi(q/p_i^{\alpha_i})}^{(i)}\} \text{ for } 1 \leq i \leq r.$$

Suppose $a, b \in C_{h_1}^{(i)}$ for some $0 \leq h_1 \leq \varphi(q/p_i^{\alpha_i})$ and $a \in C_{h_2}^{(j)}$ for some $0 \leq h_2 \leq \varphi(q/p_j^{\alpha_j})$ with $0 \leq i < j \leq r$. Then we claim that $b \notin C_{h_2}^{(j)}$. For, otherwise, we have $a \equiv b \pmod{q/p_j^{\alpha_j}}$. Also $a \equiv b \pmod{p_j^{\alpha_j}}$ since $a, b \in C_{h_1}^{(i)}$. Hence $a \equiv b \pmod{q}$, a contradiction which proves the claim. Hence $\varphi(p_i^{\alpha_i})$ elements in $C_{h_1}^{(i)}$ fall into different classes of X_j . (Note that $\varphi(p_i^{\alpha_i}) \leq \varphi(q/p_j^{\alpha_j})$). Thus there are

$$\varphi(p_i^{\alpha_i})\varphi(p_j^{\alpha_j})$$

elements coprime to q for which f takes the same value. Starting from $p_1^{\alpha_1}$ and moving up to $p_r^{\alpha_r}$, we find that there are

$$\varphi(p_1^{\alpha_1}) \cdots \varphi(p_r^{\alpha_r}) = \varphi(q)$$

elements, coprime to q for which f takes the same value. □

Lemma 4. *Suppose that S vanishes. Assume that (5) holds for every i with $1 \leq i \leq r$. Then condition (ii) of Lemma 1 does not hold for any $p = p_i$.*

Proof Suppose S vanishes and (5) holds for every i with $1 \leq i \leq r$. Then Lemmas 1 and 3 are valid. Since $\sum_{h=1}^q f(h) = 0$, we find that

$$(8) \quad \sum_{\substack{a=1 \\ \gcd(a,q)=1}}^q f(a) = - \sum_{\substack{h=1 \\ \gcd(h,q)>1}}^q f(h).$$

By Lemma 3, we get

$$(9) \quad \left| \sum_{\substack{a=1 \\ \gcd(a,q)=1}}^q f(a) \right| = \varphi(q).$$

We write the set of integers $\leq q$ and divisible by p_i as

$$S^{(i)} = \cup_{j=1}^{\alpha_i} S_j^{(i)}$$

where $S_j^{(i)}$ for $1 \leq j < \alpha_i$ denotes those integers $\leq q$ which are exactly divisible by p_i^j . Further, $S_{\alpha_i}^{(i)}$ denotes all integers $\leq q$ which are divisible by $p_i^{\alpha_i}$. Then condition (ii) of Lemma 1 gives

$$(10) \quad \sum_{h \in S_1^{(i)}} f(h) + 2 \sum_{h \in S_2^{(i)}} f(h) + \cdots + (\alpha_i - 1) \sum_{h \in S_{\alpha_i-1}^{(i)}} f(h) + \left(\alpha_i + \frac{1}{p_i - 1}\right) \sum_{h \in S_{\alpha_i}^{(i)}} f(h) = 0 \text{ for } 1 \leq i \leq r.$$

If further, p_i exactly divides q , then

$$(11) \quad \sum_{h \in S^{(i)}} f(h) = 0.$$

Now we consider the right hand side of (8) and apply (10) and (11) to get

$$\sum_{\substack{h=1 \\ \gcd(h,q)>1}}^q f(h) = - \sum_{\substack{h=1 \\ \gcd(h,q)>1, h \notin S^{(i)}}}^q f(h) - \sum_{\substack{h=1 \\ \gcd(h,q)>1, h \in S^{(i)}}}^q f(h)$$

$$\begin{aligned}
&= - \sum_{\substack{h=1 \\ \gcd(h,q)>1, h \notin S^{(i)}}}^q f(h) + \left(\sum_{h \in S_2^{(i)}} f(h) + \cdots + (\alpha_i - 2) \sum_{h \in S_{\alpha_i-1}^{(i)}} f(h) + \right. \\
&\quad \left. (\alpha_i - 1 + \frac{1}{p_i - 1}) \sum_{h \in S_{\alpha_i}^{(i)}} f(h) \right) \delta_i
\end{aligned}$$

where

$$\delta_i = \begin{cases} 0 & \text{if } p_i \parallel q \\ 1 & \text{otherwise.} \end{cases}$$

Thus

$$\begin{aligned}
\left| - \sum_{\substack{h=1 \\ \gcd(h,q)>1}}^q f(h) \right| &\leq q - \varphi(q) - \frac{q}{p_i} + \delta_i \left(\frac{q}{p_i^2} - \frac{q}{p_i^3} + 2 \left(\frac{q}{p_i^3} - \frac{q}{p_i^4} \right) + \cdots \right. \\
&\quad \left. + (\alpha_i - 2) \left(\frac{q}{p_i^{\alpha_i-1}} - \frac{q}{p_i^{\alpha_i}} \right) + (\alpha_i - 1 + \frac{1}{p_i - 1}) \frac{q}{p_i^{\alpha_i}} \right) \\
&= q - \varphi(q) - \frac{q}{p_i} + \delta_i \left(\frac{q}{p_i^2} + \frac{q}{p_i^3} + \cdots + \frac{q}{p_i^{\alpha_i}} + \frac{1}{p_i - 1} \frac{q}{p_i^{\alpha_i}} \right) \\
&= q - \varphi(q) - \frac{q}{p_i} + \delta_i \left(\frac{q(p_i^{\alpha_i-1} - 1)}{p_i^{\alpha_i}(p_i - 1)} + \frac{q}{p_i^{\alpha_i}(p_i - 1)} \right) \\
&= q - \varphi(q) - \frac{q}{p_i} + \delta_i \frac{q}{p_i(p_i - 1)}
\end{aligned}$$

which by (8) and (9) gives

$$(12) \quad 2\varphi(q) \leq q \left(1 - \frac{1}{p_i} + \frac{\delta_i}{p_i(p_i - 1)} \right).$$

When $\delta_i = 0$, by (5) and (12), we have

$$2 \leq \frac{q}{\varphi(q)} \left(1 - \frac{1}{p_i} \right) < 2,$$

a contradiction. Let $\delta_i = 1$. Then $\alpha_i > 1$ and

$$2 \leq \frac{2p_i^{\alpha_i}}{(p_i^{\alpha_i} - 1)} \left(1 - \frac{1}{p_i} + \frac{1}{p_i(p_i - 1)} \right)$$

implying

$$p_i^{\alpha_i-1} - 1 \leq \frac{p_i^{\alpha_i-1}}{p_i - 1}$$

which is impossible. □

Suppose we have

$$(13) \quad \alpha \leq \frac{\varphi(q)}{q} \leq \beta.$$

We define

$$s_0 = s'_0 = 1, s_i = \prod_{j=1}^i \left(1 - \frac{1}{p_j}\right) \text{ for } i \geq 1.$$

Since

$$(14) \quad \frac{\varphi(q)}{q} = \left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_i}\right) \cdots \left(1 - \frac{1}{p_r}\right),$$

we see that

$$\left(1 - \frac{1}{p_i}\right)^{r-i+1} s_{i-1} \leq \beta.$$

Thus we get

$$(15) \quad p_i < \frac{1}{1 - \left(\frac{\beta}{s_{i-1}}\right)^{1/(r-i+1)}} \text{ for } 1 \leq i \leq r.$$

The following lemma is a consequence of (13) with $\beta = 1/2$.

Lemma 5 *Suppose $2\varphi(q) \leq q$. Then*

(i) *If $\omega(q) = 3$, then $(p_1, p_2, p_3) = (3, 5, 7)$ or $(3, 5, 11)$ or $(3, 5, 13)$.*

(ii) *If $\omega(q) = 4$, then*

$(p_1, p_2, p_3, p_4) = (3, 5, 7, p_4); (3, 5, 11, p_4); (3, 5, 13, p_4); (3, 5, 17, p_4)$ with $p_4 \leq 251$;

$(3, 5, 19, p_4)$ with $p_4 \leq 89$; $(3, 5, 23, p_4)$ with $p_4 \leq 47$; $(3, 5, 29, p_4)$ with $p_4 \leq 31$;

$(3, 7, 11, p_4)$ with $p_4 \leq 23$; $(3, 7, 13, p_4)$ with $p_4 \leq 19$.

Proof (i) Suppose $\omega(q) = 3$. Then we have , by (14) and (13) with $\beta = \frac{1}{2}$ that

$$p_1 < \frac{1}{1 - (1/2)^{1/3}}$$

implying $p_1 = 3$. Next,

$$p_2 < \frac{1}{1 - (3/4)^{1/2}}$$

giving $p_2 = 5, 7$. When $p_2 = 7$, we check that $2\varphi(q) > q$. Thus $p_2 = 5$. Lastly, we have

$$p_3 < \frac{1}{1 - (15/16)}$$

giving $p_3 < 16$ i.e $p_3 = 7, 11, 13$, which proves the assertion.

Proof of (ii) is similar. □

3. Proofs of Theorems 1, 2 and Corollary

Proof of Theorem 2 Suppose S vanishes. Then Lemma 1 holds. Hence assertion (ii) of Lemma 1 is valid. On the other hand since (4) holds, we see that (5) is valid for every i with $1 \leq i \leq r$. In that case, by Lemma 4, assertion (ii) of Lemma 1 does not hold, a contradiction. □

Proof of Theorem 1 Let $\omega(q) = 3$. By the result of Okada, we assume that $2\varphi(q) \leq q$. We consider those q for which

$$\frac{1}{2}\left(1 - \frac{1}{h}\right) \leq \frac{\varphi(q)}{q} \leq \frac{1}{2}.$$

We first apply Lemma 5 (i) to get

$$(p_1, p_2, p_3) \in \{(3, 5, 7), (3, 5, 11), (3, 5, 13)\}.$$

Now we apply (13) with $\alpha = \frac{1}{2}\left(1 - \frac{1}{h}\right)$. We explain with an example. Let $(p_1, p_2, p_3) = (3, 5, 7)$. Then from (13) with $\alpha = \frac{1}{2}\left(1 - \frac{1}{h}\right)$ and (14) we have

$\frac{1}{2}(1 - \frac{1}{h}) \leq \frac{48}{105}$ implying $h \leq 11$. Thus the only possible values of q are $3 \cdot 5 \cdot 7$ or $3^2 \cdot 5 \cdot 7$. By similar argument for the other values of (p_1, p_2, p_3) we obtain that

$$(15) \quad q \in \{3 \cdot 5 \cdot 7, 3^2 \cdot 5 \cdot 7, 3 \cdot 5 \cdot 11, 3 \cdot 5 \cdot 13, 3 \cdot 5^2 \cdot 11, 3 \cdot 5^2 \cdot 13, 3^2 \cdot 5 \cdot 11, 3^2 \cdot 5 \cdot 13, 3^2 \cdot 5^2 \cdot 11, 3^2 \cdot 5^2 \cdot 13, 3^3 \cdot 5 \cdot 11, 3^3 \cdot 5 \cdot 13, 3^3 \cdot 5^2 \cdot 11, 3^3 \cdot 5^2 \cdot 13\}.$$

Thus for the values of q in (15) we find by Theorem 2 that S does not vanish. This proves the assertion of Theorem 1 for $\omega(q) = 3$.

Let $\omega(q) = 4$. We use Lemma 5 (ii) and (13) with $\alpha = \frac{1}{2}(1 - \frac{1}{h})$ to get an upper bound for α_i for every p_i . The upper bounds for α'_i s with $1 \leq i \leq 3$ are shown in the Table 1 for various values of p_1, p_2, p_3, p_4 . Table 1 is read as follows. We always take $\alpha_4 = 1$. Suppose $(p_1, p_2, p_3, p_4) = (3, 5, 17, 251)$, then $\alpha_1 \leq 8, \alpha_2 \leq 5$ and $\alpha_3 \leq 3$. Thus $3^8 \cdot 5^5 \cdot 17^3 \cdot 251$ is a value of q from the table. We see that S does not vanish for each of these q 's by Theorem 2. \square

Proof of Corollary. Let $q \leq 1154$ and $q \notin \{525, 735, 945\}$. The only values of q for which (2) is not satisfied are $3 \cdot 5 \cdot 7, 3 \cdot 5 \cdot 11, 3 \cdot 5 \cdot 13, 3^2 \cdot 5 \cdot 7, 3^2 \cdot 5 \cdot 11, 3^2 \cdot 5 \cdot 13, 3 \cdot 5^2 \cdot 11, 3 \cdot 5^2 \cdot 13$ and S does not vanish for these values of q by Theorem 1(i). \square

(p_1, p_2, p_3)	p_4	α_1	α_2	α_3	(p_1, p_2, p_3)	p_4	α_1	α_2	α_3
(3, 5, 17)	19	2	1	1	(3, 5, 19)	71 – 73	5	3	1
	23	2	2	1		79 – 83	5	3	2
	29 – 61	3	2	1		89	6	4	2
	67 – 83	4	2	1	(3, 5, 23)	29 – 31	3	2	1
	89 – 113	4	3	1		37 – 41	4	3	1
	127 – 131	5	3	1		43	5	3	1
	137 – 181	5	3	2		47	5	4	2
	191 – 229	6	4	2	(3, 5, 29)	31	5	3	1
	233	7	4	2	(3, 7, 11)	13	2	1	1
	239 – 241	7	5	2		17	3	1	1
251	8	5	3	19		3	2	1	
(3, 5, 19)	23 – 43	3	2	1	(3, 7, 13)	23	4	2	2
	47 – 53	4	2	1		17	4	2	1
	59 – 67	4	3	1		19	6	3	2

Table 1

REFERENCES

- [1] S.D. Adhikari, N. Saradha, T.N. Shorey and R. Tijdeman, *Transcendental infinite sums*, Indag. Math. (N.S) 12 (2001), 1-14.
- [2] S.D. Adhikari, *Transcendental Infinite sums and related questions*, Number Theory and discrete mathematics. Proceedings of conference, Chandigarh, 2000 (Hindustan Book Agency, 2002), 169-178.
- [3] S.D. Adhikari and N. Saradha, *Arithmetic nature of sums of certain convergent series*, Resonance, Vol 7, No.11 (2002), 35-46.

- [4] A. Baker, B.J. Birch and E.A. Wirsing, *On a problem of Chowla*, J. Number Theory 5 (1973), 224-236.
- [5] T. Okada, *On certain infinite sums for periodic arithmetical functions*, Acta Arith. 40 (1982), 143-153.
- [6] N. Saradha and R. Tijdeman, *On the transcendence of infinite sums of values of rational functions*, J. London Math. Soc. 67 (2003), 1-13.
- [7] R. Tijdeman, *Some applications of diophantine approximations*, Number theory for the millennium, Proceedings of conference, Urbana, IL, 2000-Vol. III (A.K. Peters, Natick, MA, 2002), 261-284.