SIMULTANEOUS RATIONAL APPROXIMATION VIA RICKERT’S INTEGRALS

N. SARADHA AND DIVYUM SHARMA

Abstract. Using Rickert’s contour integrals, we give effective lower bounds for simultaneous rational approximations to numbers in the sets
\[ \left\{ \left( 1 - \frac{a}{N} \right)^{\nu}, \left( 1 + \frac{a}{N} \right)^{\nu} \right\} \quad \text{and} \quad \left\{ \left( 1 + \frac{a}{N} \right)^{\nu}, \left( 1 + \frac{2a}{N} \right)^{\nu} \right\}. \]
Here \( N > a \geq 1 \) are integers, \( 0 < \nu < 1 \) is a rational number and at least one of the radicals is irrational in each set. The result is valid for all \( q \geq 1 \) where \( q \) denotes the denominator of the approximating rational number.

1. Introduction

The method of Padé approximation has been widely used to obtain effective lower bounds for simultaneous rational approximations to algebraic numbers. See for example [9], [2], [7], [8], [10] and [5]. In these papers hypergeometric polynomials were used as Padé approximants to transcendental functions of the form
\[ (1 + ax)^\nu \text{ with } a \in \mathbb{Z} \setminus \{0\} \text{ and } \nu \in \mathbb{Q}^+ \setminus \mathbb{N}. \]
Rickert [11] used contour integral representation for the Padé approximants to these transcendental functions to show the following result.

Let
\[ \theta_1 = \sqrt{(1 - 1/N)}, \theta_2 = \sqrt{(1 + 1/N)} \text{ with } N \geq 2. \]
Then
\[ \max \left\{ \left| \theta_1 - \frac{p_1}{q} \right|, \left| \theta_2 - \frac{p_2}{q} \right| \right\} > \frac{c}{q^\lambda} \]
for all integers \( p_1, p_2, q \) with \( q > 0 \) where
\[ \lambda = 1 + \frac{\log(12N\sqrt{3} + 24)}{\log(27(N^2 - 1)/32)} \text{ and } c = \frac{1}{271N}. \]

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This result leads to bounds for solutions to certain simultaneous Pell-type equations as soon as $\lambda < 2$. See [11] for details. By taking $N = 49$ in the above result, he derives that

\begin{equation}
\max \left\{ \left| \sqrt{2} - \frac{p_1}{q} \right|, \left| \sqrt{3} - \frac{p_2}{q} \right| \right\} > \frac{c_0}{q^{\lambda_0}}
\end{equation}

with $(c_0, \lambda_0) = (10^{-7}, 1.913)$ for all integers $p_1, p_2, q$ with $q > 0$. In [4], Bennett improved the value of $\lambda$ in (2) asymptotically, i.e. for $q \geq q_0$ where $q_0$ is effectively computable but not explicitly given. In fact, this is a special case of a more general result ([4, Theorem 1.1]) proved by elaborating on the ideas of Chudnovsky [5] which included estimating more precisely the common denominators of the coefficients of the approximants. As a result, he showed that (3) holds with $(c_0, \lambda_0) = (1, 1.79155)$ for all integers $p_1, p_2, q$ with $q \geq q_0$, where $q_0$ is a large inexplicit constant. In [3], he made this result explicit with $q_0 = 1$ and $(c_0, \lambda_0) = (10^{-10}, 1.8161)$. This is a particular case of the following result [3, Theorem 4.1].

Let $\theta_1, \theta_2$ be given by (1) and $N \geq 13$. Then

\[ \max \left\{ \left| \theta_1 - \frac{p_1}{q} \right|, \left| \theta_2 - \frac{p_2}{q} \right| \right\} > \frac{1}{1.7 \times 10^6 N q^\lambda} \]

for all integers $p_1, p_2, q$ with $q > 0$ where

\[ \lambda = \lambda(N) = 1 + \frac{\log(8N\sqrt{3} + 16)}{\log(81(N^2 - 1)/64)}. \]

Let $a, N$ be positive integers and let $0 < \nu < 1$ be a rational number. Write $\nu = s/n$ with $\gcd(s, n) = 1$. Our aim in this paper is to extend the above result of Bennett for numbers $(\theta_1, \theta_2)$ where

\[ (\theta_1, \theta_2) \in \left\{ \left( \left( 1 - \frac{a}{N} \right)^\nu, \left( 1 + \frac{a}{N} \right)^\nu \right), \left( \left( 1 + \frac{a}{N} \right)^\nu, \left( 1 + \frac{2a}{N} \right)^\nu \right) \right\}. \]

To state our results, we put

\[ e_0 = \prod_{p \mid n} p^{\max\{\text{ord}_p(s/n) + \frac{1}{p-1}, 0\}} \]

and

\begin{equation}
(e, e', e'') = \begin{cases} (2, 4/3, 168) & \text{if } n = 2 \\ (4, 3, 679) & \text{if } n = 4 \\ (e_0, 3e_0/2, 1) & \text{otherwise.} \end{cases}
\end{equation}
Theorem 1.1. Let \( \theta_1 = \left(1 - \frac{a}{N}\right)^\nu \) and \( \theta_2 = \left(1 + \frac{a}{N}\right)^\nu \) with \( N \geq 6.794e^2a^3 \). Assume that at least one of \( \theta_1 \) and \( \theta_2 \) is irrational. Then (2) is valid with
\[
\lambda = 1 + \frac{\log(6\sqrt{3}eN + 12ae')}{\log(27(N^2 - a^2)/(16a^3e'))}
\]
and
\[
c = 0.005 \frac{ae'e''N}{e''N} \left(1 - \frac{a}{N}\right)^\nu \left(1 + \frac{2a}{N\sqrt{3}}\right)^{-1-\nu} \left(\frac{9Na^''\nu(1-\nu^2)}{2(2N-a)\nu^2(N-2a)}\right)^{-\lambda+1}
\]

Remark 1.2. Taking \( a = 1 \) and \( \nu = \frac{1}{2}, \frac{1}{4} \) in Theorem 1.1, we get Theorems 4.1 and 4.3 of [3]. Note that \( \lambda < 2 \) whenever \( N \geq 13 \) if \( \nu = 1/2 \) and \( N \geq 62 \) if \( \nu = 1/4 \).

Example. By taking \( N = 18^3, a = 1 \) and \( \nu = 1/3 \), we obtain
\[
\lambda(\sqrt[3]{17}, \sqrt[3]{5833}) \leq 1.8264.
\]

Theorem 1.3. Let \( \theta_1 = \left(1 + \frac{a}{N}\right)^\nu \) and \( \theta_2 = \left(1 + \frac{2a}{N}\right)^\nu \) with \( N \geq 8.637e^2a^3 \). Assume that at least one of \( \theta_1 \) and \( \theta_2 \) is irrational. Then (2) is valid with
\[
\lambda = 1 + \frac{\log(6\sqrt{3}eN + 6ae'(2 + \sqrt{3}))}{\log(27(N-a)(N-2a)/(16a^3e'))}
\]
and
\[
c = 0.005 \frac{ae'e''N}{a^2e''N} \left(1 + \frac{(2 + \sqrt{3})a}{N\sqrt{3}}\right)^{-1-\nu} \left(\frac{9Na^''\nu(1-\nu^2)}{2(2N-a)\nu^2(N-2a)}\right)^{-\lambda+1}
\]

Remark 1.4. Let \( \theta_1 \) and \( \theta_2 \) be algebraic numbers such that \( 1, \theta_1, \theta_2 \) are linearly independent over \( \mathbb{Q} \). Then it is known by a celebrated theorem of Schmidt [12] that given \( \epsilon > 0 \) there exists \( c = c(\theta_1, \theta_2, \epsilon) > 0 \) such that
\[
\max\left\{\left|\theta_1 - \frac{p_1}{q}\right|, \left|\theta_2 - \frac{p_2}{q}\right|\right\} > \frac{c}{q^\lambda}
\]
with \( \lambda = 1.5 + \epsilon \), for all integers \( p_1, p_2, q \) with \( q > 0 \). By the classical theorem of Dirichlet on Diophantine approximation, it follows that such a result cannot hold for \( \lambda = 1.5 \). Thus Schmidt’s result is optimal, but it is ineffective in the sense that \( c \) cannot be computed. In Theorems 1.1 and 1.3, \( c \) is explicitly determined for the specified numbers \( \theta_1, \theta_2 \). Further, the exponent \( \lambda \) of \( q \) is found to be less than 2.
Let $\theta_1$ and $\theta_2$ be either as in Theorem 1.1 or Theorem 1.3, with $\nu = 1/2$. Take $N = N_0^2$ and write

$$\theta_i = \frac{s_i \sqrt{\alpha_i}}{N_0}$$

with $N_0$, $s_i$ positive integers and $\alpha_i$ square free for $i = 1, 2$. By the irrationality assumption on $\theta_1$ and $\theta_2$, at least one of the $\alpha_i$’s exceeds 1. We apply Theorems 1.1 and 1.3 with $q$ replaced by $N_0q$ and $p_i$ replaced by $s_i p_i$ to get

$$\max \left\{ \left| \frac{s_1 \sqrt{\alpha_1}}{N_0} - \frac{s_1 p_1}{N_0q} \right|, \left| \frac{s_2 \sqrt{\alpha_2}}{N_0} - \frac{s_2 p_2}{N_0q} \right| \right\} > \frac{c}{(N_0q)^\lambda}.$$  

This shows that

$$\max \left\{ \left| \sqrt{\alpha_1} - \frac{p_1}{q} \right|, \left| \sqrt{\alpha_2} - \frac{p_2}{q} \right| \right\} > \frac{c'}{q^\lambda}$$

with $c' = c/(N_0^{\lambda-1} \max(s_1, s_2))$. We give few values of $a$, $N_0$, the corresponding $\alpha_1$, $\alpha_2$ and $\lambda$ in Tables 1 and 2 below for Theorems 1.1 and 1.3, respectively. In all these cases $c'$ may be taken as $10^{-17}$. The values of $a$ and $N$ that we have chosen are taken from Table 1 of [4] in which an asymptotic value of $\lambda$ is given. The values of $\lambda$ given in the tables below are valid for all $q \geq 1$.

**Table 1. Examples for Theorem 1.1**

<table>
<thead>
<tr>
<th>$a$</th>
<th>$N_0$</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>61</td>
<td>22</td>
<td>19</td>
<td>1.8109</td>
</tr>
<tr>
<td>4</td>
<td>29</td>
<td>93</td>
<td>5</td>
<td>1.9815</td>
</tr>
<tr>
<td>5</td>
<td>169</td>
<td>59</td>
<td>6</td>
<td>1.8093</td>
</tr>
<tr>
<td>6</td>
<td>136</td>
<td>10</td>
<td>22</td>
<td>1.8583</td>
</tr>
</tbody>
</table>

**Table 2. Examples for Theorem 1.3**

<table>
<thead>
<tr>
<th>$a$</th>
<th>$N_0$</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>17</td>
<td>2</td>
<td>1.9902</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>26</td>
<td>3</td>
<td>1.9056</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>37</td>
<td>38</td>
<td>1.8570</td>
</tr>
<tr>
<td>1</td>
<td>7</td>
<td>2</td>
<td>51</td>
<td>1.8249</td>
</tr>
<tr>
<td>2</td>
<td>140</td>
<td>2</td>
<td>29</td>
<td>1.6982</td>
</tr>
<tr>
<td>2</td>
<td>816</td>
<td>2</td>
<td>985</td>
<td>1.6422</td>
</tr>
<tr>
<td>3</td>
<td>37</td>
<td>7</td>
<td>55</td>
<td>1.8663</td>
</tr>
</tbody>
</table>
Remark 1.5. The main idea in the proofs of Theorems 1.1 and 1.3 is to dilate (with or without translating) the contour considered by Rickert for representing the Padé approximants as contour integrals. See Lemma 4.2. Further, when $\nu = \frac{1}{2}, \frac{1}{4}$, precise estimates for the coefficients of the approximants are used as in [3]. See Lemma 6.1.

To present our next result, we introduce some notation. Let $a_0 < a_1 < a_2$ be integers with one of them equal to zero. Suppose that $N$ is a positive integer exceeding $M = \max(|a_0|, |a_1|, |a_2|)$. For $0 \leq j \leq 2$, put

$$\xi_j = \left(1 + \frac{a_j}{N}\right)^{\nu}.$$  

Denote by $\theta_1$ and $\theta_2$ the two values from $\xi_0, \xi_1$ and $\xi_2$, which are not equal to 1. Define

$$
e_1 = \lcm((a_1 - a_0)(a_2 - a_0), (a_1 - a_0)(a_2 - a_1), (a_2 - a_0)(a_2 - a_1)),
\ne_2 = \lcm(a_1 - a_0, a_2 - a_0, a_2 - a_1),
\ne_3 = \prod_{\text{prime } p \mid n} p^{\max\{\ord_p(n/e_2)+1/(p-1),0\}},
\ne_4 = e_1 e_2 e_3.
$$

In the following theorem we show that, under certain conditions, one can obtain positive numbers $c$ and $\lambda$ such that (2) holds for all integers $p_1, p_2, q$ with $q > 0$. Although the method leads to explicit values of $c$, these values are very small and we do not present them here. We will only give the values of $\lambda$ explicitly.

Theorem 1.6. Put

$$G_0 = \frac{4(2N + 2M + a_1 - a_0)}{N(a_1 - a_0)^2(2a_2 - a_0 - a_1)},

G_1 = \begin{cases} 
\frac{4(2N + 2M + a_1 - a_0)}{N(a_1 - a_0)^2(2a_2 + a_0 - 3a_1)} & \text{if } a_1 - a_0 \leq a_2 - a_1 \\
\frac{4(2N + 2M + a_2 - a_1)}{N(a_2 - a_1)^2(3a_1 - 2a_0 - a_2)} & \text{if } a_1 - a_0 > a_2 - a_1,
\end{cases}

G_2 = \frac{4(2N + 2M + a_2 - a_1)}{N(a_2 - a_1)^2(a_2 - 2a_0 + a_1)},

P = \max(G_0, G_1, G_2),

and

$$D = \begin{cases} 
e_4 N & \text{if } \nu = 1/2 \\
9\ne_4 N/8 & \text{if } \nu = 1/4 \\
3\ne_4 N/2 & \text{otherwise.}
\end{cases}$$


Then (2) holds with

\[ \lambda = 1 + \frac{\log(PD)}{\log(6.68(N-M)^3/D)} \]

provided the denominator is positive.

Examples.

1. By taking \( N = 1713^2, \nu = 1/2 \) and \( (a_0, a_1, a_2) = (-6, 0, 2) \), we obtain

\[ \lambda(\sqrt{3}, \sqrt{24251}) \leq 1.8254. \]

2. By taking \( N = 7940^2, \nu = 1/2 \) and \( (a_0, a_1, a_2) = (-4, 0, 16) \), we obtain

\[ \lambda(\sqrt{11}, \sqrt{13634}) \leq 1.8705. \]

Remark 1.7. The proof of Theorem 1.6 depends on expressing the contour integral appearing in the Padé approximation as an infinite integral. We connect this integral to the values of the classical Beta function, which are then estimated using Stirling’s formula.

In Sections 2-6 we present several lemmas for the proofs of the theorems. These lemmas deal with simultaneous rational approximation to algebraic numbers

\[ \left( 1 + \frac{a_j}{N} \right)^\nu, \quad a_j \in \mathbb{Z}, \quad 0 < \nu < 1 \]

for \( 0 \leq j \leq m \) with \( m \geq 2 \) and one of the \( a_j \)'s equals zero. The theorems are obtained in Section 7 by specializing to \( m = 2 \).

2. Approximating Forms

The lemma presented in this section is used to find effective lower bounds for simultaneous rational approximations to real numbers \( \theta_1, \ldots, \theta_m \) by the construction of suitable independent approximation forms. This is a mild variant of Lemma 2.1 from [11].

Lemma 2.1. Let \( \theta_1, \ldots, \theta_m \in \mathbb{R} \). Put \( \theta_0 = 1 \). Suppose that there are positive real numbers \( \ell, p, d, L, P, D, r_0 \) having the following property. For each integer \( r \geq r_0 \geq 0 \), there exist \( p_{ijr} \in \mathbb{Q} \) \((0 \leq i, j \leq m)\), with non-zero determinant and \( C_r \in \mathbb{Z} \) such that for all \( i, j = 0, \ldots, m \), we have

\[ C_r \leq dD^r, \quad C_r p_{ijr} \in \mathbb{Z}, \]

\[ |p_{ijr}| \leq pP^r \]
and

$$\left| \sum_{j=0}^{m} p_{ij} \theta_j \right| \leq \frac{\ell}{L^r}. \quad (7)$$

If

$$\frac{L}{D} > 1,$$

then for all non-zero integer tuples \((p_1, \ldots, p_m, q)\), \(q \geq \left(\frac{L}{D}\right)^{r_0}\), we have

$$\max \left( \left| \frac{\theta_1 - p_1}{q} \right|, \ldots, \left| \frac{\theta_m - p_m}{q} \right| \right) > \frac{c}{q^{1+\chi}}, \quad (8)$$

where

$$\chi = \frac{\log(PD)}{\log(L/D)} \quad (9)$$

and

$$c = 1/(2(m + 1)p_dPD(\max(1, 2d))^{\chi}).$$

Further for all \(q \geq 1\), we can take \(\chi\) as in \((9)\) and

$$c = 1/(2(m + 1)p_dPD(\max(1, 2d))^{\chi W^{1+\chi}}),$$

where

$$W = \left\lfloor \left(\frac{L}{D}\right)^{r_0} \right\rfloor + 1.$$

**Remark 2.1.** Our aim is to find \(\chi\) as small as possible. For this, we see from \((9)\) that \(P\) and \(D\) should be small and \(L\) should be large. The auxiliary parameters \(p, d\) and \(\ell\) are introduced to facilitate this.

**Proof of Lemma 2.1.** Let \(p_1, \ldots, p_m, q\) be integers with \(q \geq \left(\frac{L}{D}\right)^{r_0}\). Put \(p_0 = q\) and

$$\delta = \max \left( \left| \frac{\theta_1 - p_1}{q} \right|, \ldots, \left| \frac{\theta_m - p_m}{q} \right| \right).$$

Take

$$r = 1 + \left\lfloor \frac{\log(Cq)}{\log(L/D)} \right\rfloor,$$

where

$$C = \max(1, 2d).$$

Then

$$\left(\frac{L}{D}\right)^r > Cq \geq q \geq \left(\frac{L}{D}\right)^{r_0}.$$
Since \( L > D \), this implies that \( r \geq r_0 \). Therefore, by hypothesis, there exist numbers \( p_{ijr} \in \mathbb{Q} \), \( 0 \leq i, j \leq m \), with \( \det(p_{ijr}) \neq 0 \), and \( C_r \in \mathbb{Z} \) satisfying the conditions (5)–(7). Then

\[
\sum_{j=0}^{m} p_{ijr}p_j = \left| q \sum_{j=0}^{m} p_{ijr}\theta_j - q \sum_{j=0}^{m} p_{ijr}(\theta_j - p_j/q) \right| \leq q\ell L^{-r} + (m + 1)qpP^r\delta.
\]

(10)

Since \( \det(p_{ijr}) \neq 0 \), there exists \( i \) such that \( \sum_{j=0}^{m} p_{ijr}p_j \) is a nonzero rational number and further, its denominator divides \( C_r \), which is \( \leq dD^r \). Therefore we get

\[
\sum_{j=0}^{m} p_{ijr}p_j \geq (dD^r)^{-1}.
\]

(11)

Note that \( (L/D)^r \geq Cq \geq 2\ell dq \). Comparing (10) and (11) we get

\[
(m + 1)qpP^r\delta \geq \frac{1}{dD^r} - \frac{q\ell}{L^r} \geq \frac{1}{2dD^r}
\]

implying that

\[
\delta \geq \frac{1}{2(m + 1)pdq(PD)^r} \geq \frac{1}{2(m + 1)pdqPD(Cq)^x}
\]

\[
= \frac{c}{q^{1+x}}.
\]

This proves the first part of the lemma. If \( q < \left( \frac{L}{D} \right)^{r_0} \) we apply (8) with \( q \) replaced by \( Wq \) and \( p_i \) replaced by \( Wp_i \), for \( i = 1, \ldots, m \). This gives the second part of the lemma.

\[\Box\]

3. Rickert’s Contour Integral

Let \( a_0 < a_1 < \ldots < a_m \) be integers, with one of them equal to zero. Suppose that \( |x|^{-1} > \max_{0 \leq i \leq m} |a_i| \). Let \( r \) be a non-negative integer. For \( i = 0, \ldots, m \), consider the contour integral

\[
I_i(x, r, \gamma) = \frac{1}{2\pi\sqrt{-1}} \int_{\gamma} (1 + zx)^{r+\nu} (z - a_i)(A(z))^r dz,
\]

where \( A(z) = (z - a_0) \ldots (z - a_m) \) and \( \gamma \) is a closed, counter-clockwise contour containing \( a_0, \ldots, a_m \), but not passing through any of them. Then \( I_i(x) = I_i(x, r, \gamma) \) defines a function of \( x \) which is analytic near the origin. The following lemma summarizes the results in Lemmas 3.1, 3.3 and 3.4 of [11].
Lemma 3.1. Let $0 \leq i \leq m$. Then

(1) $I_i(x)$ has a zero of order at least $(m+1)r$ at the origin.

(2) There are polynomials $q_{ijr}(x)$ ($0 \leq j \leq m$) of degree at most $r$ such that

$$I_i(x) = \sum_{j=0}^{m} q_{ijr}(x)(1 + a_jx)^{\nu}.$$ 

Here, $q_{ijr}(x)(1 + a_jx)^{\nu}$ is the residue of the integrand in (12) at $z = a_j$ for $j = 0, \ldots, m$. Also,

$$\det_{0 \leq i,j \leq m} (q_{ijr}(x)) \neq 0.$$ 

Thus for every $i$ with $0 \leq i \leq m$ and $r \geq 0$, the polynomials $q_{ijr}(x)$ are the Padé approximants to the function $(1 + a_jx)^{\nu}$, $0 \leq j \leq m$. The polynomials $q_{ijr}(x)$ were given explicitly in [11] as

$$q_{ijr}(x) = \sum_{h_0 + \ldots + h_m = \nu, \ r_j - 1, \ h_i \geq 0} \left(\frac{r}{h_j}\right)x^{h_j}(1 + a_jx)^{r-h_j} \prod_{0 \leq l \leq m, \ l \neq j} \frac{1}{(a_j - a_l)^{r_l + h_l}},$$

where $r_i = r + \delta_i$ where $\delta_i$ is the Kronecker delta function. We will now take $x = 1/N$ where $N$ is a natural number exceeding $\max_{0 \leq i \leq m} |a_i|$. For $0 \leq i, j \leq m$ put

$$\xi_j = \left(1 + \frac{a_j}{N}\right)^{\nu}$$

and

$$q_{ijr} = q_{ijr}(1/N).$$

Note that one of the $\xi_i$'s, say $\xi_u$ is equal to 1. For $0 \leq i, j \leq m$, put

$$\theta_j = \begin{cases} 1 & \text{if } j = 0 \\ \xi_{j-1} & \text{if } 1 \leq j \leq u \\ \xi_j & \text{if } u + 1 \leq j \leq m \end{cases}$$

and

$$p_{ijr} = \begin{cases} q_{iur} & \text{if } j = 0 \\ q_{i(j-1)r} & \text{if } 1 \leq j \leq u \\ q_{ijr} & \text{if } u + 1 \leq j \leq m \end{cases}$$

We approximate $\theta_i$'s simultaneously by using Lemma 2.1. For this purpose, we need to determine the values of $d, p, l, D, P, L$ occurring in Lemma 2.1. Thus we need to obtain upper bounds of the shape given in Lemma 2.1 for $C_r, |p_{ijr}|$ and $\left|\sum_{j=0}^{m} p_{ijr}\theta_j\right|$. This is equivalent to
finding such bounds for $C_r$, $|q_{ijr}|$ and $\left| \sum_{j=0}^{m} q_{ijr} \xi_j \right|$ as $(p_{\omega r}, \ldots, p_{imr})$ is a permutation of $(q_{i0r}, \ldots, q_{imr})$ for all $i, r$.

4. Determination of $p$ and $P$

Let $M = \max_{0 \leq t \leq m} |a_t|$. Put $R_0 = \frac{a_1 - a_0}{2}$, $R_m = \frac{a_m - a_{m-1}}{2}$ and for $j = 1, \ldots, m-1$, define

$$R_j = \min \left( \frac{a_j - a_{j-1}}{2}, \frac{a_{j+1} - a_j}{2} \right).$$

Let $0 \leq i, j \leq m$. If $j = m$ or $R_j = \frac{a_j - a_{j-1}}{2}$, take

$$T_{ji} = \begin{cases} a_j - a_{j-1} & \text{if } i = j \\ a_j - 2a_i + a_{j-1} & \text{if } i < j \\ 2a_i - 3a_j + a_{j-1} & \text{if } i > j. \end{cases}$$

If $j = 0$ or $R_j = \frac{a_{j+1} - a_j}{2}$, take

$$T_{ji} = \begin{cases} a_{j+1} - a_j & \text{if } i = j \\ 3a_j - 2a_i - a_{j+1} & \text{if } i < j \\ 2a_i - a_j - a_{j+1} & \text{if } i > j. \end{cases}$$

Further we define

$$g_{ji} = \frac{2R_j(N + M + R_j)^{\nu}}{|\xi_j| T_{ji} N^{\nu}}$$

and

$$G_j = \frac{2^{m+1}(N + M + R_j)}{N \prod_{i=0}^{m} T_{ji}}.$$

Lemma 4.1. Condition (6) in Lemma 2.1 is satisfied by taking

$p = \max_{i,j} (g_{ji})$ and $P = \max_j G_j$.

Proof. Fix $j$ with $0 \leq j \leq m$. Let $\gamma_j$ denote the circle of radius $R_j$ centered at $a_j$. Then

$$q_{ijr}(x)(1 + a_j x)^{\nu} = \frac{1}{2\pi \sqrt{-1}} \oint_{\gamma_j} \frac{(1 + z x)^{r+\nu}}{(z - a_i)(A(z))^r} \, dz.$$  

Notice that, for $z \in \gamma_j$,

$$|1 + z N^{-1}| \leq |1 + a_j N^{-1}| + R_j N^{-1} \leq 1 + MN^{-1} + R_j N^{-1}.$$
and

\[ |z - a_i| |A(z)|^r = |z - a_j|^{r+\delta_j} \prod_{h=0}^{j-1} |z - a_h|^{r+\delta_h} \prod_{h=j+1}^{m} |z - a_h|^{r+\delta_h} \]

\[ \geq R_j^{r+\delta_j} \prod_{h=0}^{j-1} (a_j - a_h - R_j)^{r+\delta_h} \prod_{h=j+1}^{m} (a_h - a_j - R_j)^{r+\delta_h}. \]

Case 1: Let \( R_j = \frac{a_j - a_{j-1}}{2} \). In this case,

\[ |z - a_i| |A(z)|^r \geq \frac{(a_j - a_{j-1})^{r+\delta_j}}{2^{(m+1)r+1}} \prod_{h=0}^{j-1} (a_j - 2a_h + a_{j-1})^{r+\delta_h} \]

\[ \times \prod_{h=j+1}^{m} (2a_h - 3a_j + a_{j-1})^{r+\delta_h}. \]

By (14) and (15), we get

\[ |q_{ijr}| \leq g_{ji} G_j^r. \]

Case 2: Let \( R_j = \frac{a_{j+1} - a_j}{2} \). In this case

\[ |z - a_i| |A(z)|^r \geq \frac{(a_{j+1} - a_j)^{r+\delta_j}}{2^{(m+1)r+1}} \prod_{h=0}^{j-1} (3a_j - 2a_h - a_{j+1})^{r+\delta_h} \]

\[ \times \prod_{h=j+1}^{m} (2a_h - a_j - a_{j+1})^{r+\delta_h}. \]

Again, using (14) and (15), we get

\[ |q_{ijr}| \leq g_{ji} G_j^r. \]

This proves the lemma. \( \square \)

If the integers \( a_j \)'s are equally spaced, we deduce the following result.

**Corollary 4.1.** Let \( a_0, \ldots, a_m \) satisfy \( a_{j+1} - a_j = a_j - a_{j-1} \) for all \( j = 1, \ldots, m - 1 \). Then condition (6) in Lemma 2.1 is satisfied by taking

\[ p = \frac{1}{\min_{0 \leq j \leq m} |\theta_j|} \left( 1 + \frac{M + R}{N} \right)^{R} \quad \text{and} \quad P = \frac{(m + 2)^2 (N + M + R)}{2^{m+2} N R^{m+1} [m/2]! (m - [m/2])!}. \]

Here \( R = \frac{a_1 - a_0}{2} \).
Proof. Since $T_{ji} \geq 2R$ for all $i$, we get that

$$g_{ji} \leq \frac{1}{\min_{0 \leq j \leq m} |\xi_j|} \left(1 + \frac{M + R}{N} \right)^\nu. \quad (16)$$

Further

$$G_j = \frac{N + M + R}{NR^{m+1} \left(\prod_{h=0}^{j-1} (2j - 1 - 2h)\right) \left(\prod_{h=j+1}^m (2h - 1 - 2j)\right)}.$$

The denominator in the above expression equals

$$\frac{R^{m+1}(2j)!(2m - 2j)!}{2^m j!(m - j)!}.$$

From the inequality

$$\left(\begin{array}{c} 2t \\ t \end{array}\right) \geq \frac{4^t}{t + 1}$$

for the central binomial coefficient, it follows that

$$\left(\begin{array}{c} 2j \\ j \end{array}\right) \left(\begin{array}{c} 2m - 2j \\ m - j \end{array}\right) > \frac{4^m}{(j + 1)(m - j + 1)} \geq \frac{4^{m+1}}{(m + 2)^2}.$$

Also $\min_{0 \leq j \leq m} j!(m - j)! = [m/2]!([m - m/2])!$. Therefore the denominator of $G_j$ is at least

$$\frac{2^{m+2}R^{m+1}[m/2]!(m - [m/2])!}{(m + 2)^2}.$$

This implies that

$$G_j \leq \frac{(m + 2)^2 (N + M + R)}{2^{m+2}NR^{m+1}[m/2]!(m - [m/2])!}. \quad (17)$$

The corollary follows from (16), (17) and Lemma 4.1. \hfill \square

Now we consider the case $m = 2$ in Corollary 4.1. In this case, we take the lemniscate used by Rickert as the contour and get better values for $p$ and $P$.

**Lemma 4.2.** Suppose

(i) $(a_0, a_1, a_2) = (-a, 0, a)$, then condition (6) in Lemma 2.1 is satisfied by taking

$$p = 1.54608 \left(1 + \frac{2a}{N\sqrt{3}}\right)^\nu \left(1 - \frac{a}{N}\right)^{-\nu} \quad \text{and} \quad P = \frac{3\sqrt{3}}{2a^3} \left(1 + \frac{2a}{N\sqrt{3}}\right).$$
Simultaneous Rational Approximation via Rickert’s Integrals

(ii) \((a_0, a_1, a_2) = (0, a, 2a)\), then condition (6) in Lemma 2.1 is satisfied by taking

\[ p = 1.54608 \left( 1 + \frac{(2 + \sqrt{3})a}{N\sqrt{3}} \right)^r \quad \text{and} \quad P = \frac{3\sqrt{3}}{2a^3} \left( 1 + \frac{(2 + \sqrt{3})a}{N\sqrt{3}} \right). \]

Proof. In the first case, \(A(z) = z(z^2 - a^2)\). Let \(\gamma\) be the lemniscate defined by \(|A(z)| = \frac{2a^3}{3\sqrt{3}}\). By taking \(z = at\), this can be transformed to \(\gamma' : |t(t^2 - 1)| = \frac{2}{3\sqrt{3}}\) around \(-1, 0, 1\). This splits into three contours \(\gamma_0', \gamma_1'\) and \(\gamma_2'\) such that \(\gamma_i'\) is a contour around \(i - 1\) and not including the other two points. Let \(\gamma_i(z) = \gamma_i(z/a)\) for \(0 \leq i \leq 2\). From (14) with \(x = \frac{1}{N}\), we get

\[
|q_{i,jr}(1 + a_jN^{-1})^{r+\nu}| = \left| \frac{1}{2\pi \sqrt{-1}} \int_{\gamma_j} \frac{(1 + zN^{-1})^{r+\nu}}{(z - a_i)(z(z^2 - a^2))^{r+\nu}} \, dz \right| \\
\leq \frac{1}{2\pi} \int_{\gamma_j} \frac{|1 + zN^{-1}|^{r+\nu}}{|z - a_i||z^2 - a^2|^{r+\nu}} \, |dz|.
\]

Thus

\[
|q_{i,jr}(1 + a_jN^{-1})^{r+\nu}| \leq \frac{1}{2\pi a^{3r}} \int_{\gamma_j} \frac{|1 + atN^{-1}|^{r+\nu}}{|t - a_i a^{-1}|(|t^2 - 1|)^r} |dt|.
\]

Now we use some numerical calculations from the proof of \([11, \text{Lemma 4.1}]. On \(\gamma_0'\), we have

\[
|t - a_i a^{-1}| > 0.344
\]

and

\[
|t| \leq 1/\sqrt{3}.
\]

Also, \(\gamma_0'\) has length \(< 2.775\). Therefore

(18)

\[
|q_{i,0r}(1 + a_0N^{-1})^{r+\nu}| \leq 1.2839 \left( 1 + \frac{a}{N\sqrt{3}} \right)^r \frac{(3\sqrt{3})^r}{(2a^3)^r} \left( 1 + \frac{a}{N\sqrt{3}} \right)^r.
\]

Similarly, on \(\gamma_1'\) and \(\gamma_2'\) we have

\[
|t - a_i a^{-1}| > 0.154
\]

and

\[
|t| \leq 2/\sqrt{3}.
\]
Further, the lengths of both $\gamma_1'$ and $\gamma_2'$ can be bounded by 1.496. Therefore, for $j = 1, 2$ we obtain

$$\left| q_{ijr}(1 + a_j N^{-1})^\nu \right| \leq 1.54608 \left( 1 + \frac{2a}{N\sqrt{3}} \right)^\nu \left( \frac{3\sqrt{3}}{2a^3} \right)^r \left( 1 + \frac{2a}{N\sqrt{3}} \right)^r$$

which is also valid for $j = 0$ by (18). Thus we may take

$$p = 1.54608 \left( 1 + \frac{2a}{N\sqrt{3}} \right)^\nu \left( 1 - \frac{a}{N} \right)^{-\nu} \text{ and } P = \frac{3\sqrt{3}}{2a^3} \left( 1 + \frac{2a}{N\sqrt{3}} \right).$$

Similarly, when $(a_0, a_1, a_2) = (0, a, 2a)$, we have $A(z) = z(z-a)(z-2a)$. Taking $\gamma_j = \gamma_j' \left( \frac{z-a}{a} \right)$ and following the above argument, we get

$$p = 1.54608 \left( 1 + \frac{(2 + \sqrt{3})a}{N\sqrt{3}} \right)^\nu \text{ and } P = \frac{3\sqrt{3}}{2a^3} \left( 1 + \frac{(2 + \sqrt{3})a}{N\sqrt{3}} \right).$$

\[ \square \]

5. Determination of $\ell$ and $L$

**Lemma 5.1.** Condition (7) in Lemma 2.1 is satisfied by taking

$$\ell = \frac{e^{1.1} N}{(N - M) \sin(\pi \nu)} \left( 1 + \frac{1}{m} \right)^{-1/2} \text{ and } L = 0.99m(N-M)^{m+1} \left( 1 + \frac{1}{m} \right)^{m+1}$$

if $r \geq 1100$.

**Proof.** Following [4, Lemma 3.1, p. 1723], it is possible to write

$$I_i \left( \frac{1}{N} \right) = \frac{(-1)^{mr} e^{\pi \nu - \frac{1}{4}} (1 - e^{2\pi \nu - 1})}{2\pi \sqrt{-1} N^{(m+1)r}} \int_0^\infty \frac{x^{r+\nu}}{(x+1+\frac{a_i}{N})(\prod_{l=0}^m (x+1+\frac{a_l}{N}))^r} dx.$$ 

Observe that $|x+1+\frac{a_i}{N}| = (x+1)|1+\frac{a_i}{N(x+1)}| \geq (x+1)(1-M/N)$. Therefore

$$I_i \left( \frac{1}{N} \right) \leq \frac{1}{\pi N^{(m+1)r}} \int_0^\infty \frac{x^{r+\nu}}{(x+1)(m+1)^r+1(1-M/N)(m+1)^{r+1}} dx$$

(19)  \[ \frac{N}{\pi(N-M)^{m+1}} B(r+\nu+1, mr - \nu) \]
where $B(m, n)$ is the classical Beta function, see [6, Exercise 4, p. 228] for the above assertion. It is well known that

$$B(r + \nu + 1, mr - \nu) = \frac{\pi(r + \nu)\ldots\nu(mr - \nu)\ldots(1 - \nu)}{\sin(\pi\nu)((m + 1)r)!} \leq \frac{\pi(r + 1)!(mr)!}{\sin(\pi\nu)((m + 1)r)!}$$

since $0 < \nu < 1$. If $x \geq 2$ is an integer, then by Stirling’s formula [1, Formula 6.1.38], we have

$$(x + 0.5) \log x - x + 0.9 < \log x! < (x + 0.5) \log x - x + 1.$$ Using this, we obtain

$$\log \left( \frac{(r + 1)!(mr)!}{((m + 1)r)!} \right) \leq (r + 1.5) \log(r + 1) + (mr + 0.5) \log(mr + r) - (mr + r + 0.5) \log(mr + r) + 0.1 = r \log(1 + 1/r) + 1.5 \log(r + 1) - r \log m - (mr + r + 0.5) \log(1 + 1/m) + 0.1.$$ Hence,

$$\frac{(r + 1)!(mr)!}{((m + 1)r)!} \leq e^{0.1}(1 + 1/r)^r(r + 1)^{1.5/2} \leq e^{1.1} \left(1 + \frac{1}{m}\right)^{-1/2} \left(\frac{(r + 1)^{1.5/r}}{m(1 + 1/m)^{m+1}}\right)^r.$$ Combining (19)–(21) and using $(r + 1)^{-1.5/r} \geq 0.99$ for $r \geq 1100$, we get the assertion of the lemma.

For some special numbers $a_i$ the values of $\ell$ and $L$ can be improved as shown in the lemma below.

**Lemma 5.2.** Suppose that the set $S = \{a_0, \ldots, a_m\}$ satisfies one of the following two properties:

(i) All the elements of $S$ are non-negative.
(ii) $a_i \in S$ if and only if $-a_i \in S$.

Then condition (7) in Lemma 2.1 is satisfied by taking

$$\ell = \left|\frac{1 + \nu}{m + 1}\right| \frac{(m + 1)^{m+1}N}{m^m(N - a_m)} \quad \text{and} \quad L = \frac{(m + 1)^{m+1}(N - a_0)\ldots(N - a_m)}{m^m}.$$ 

**Proof.** We follow [11, Lemma 3.1]. Let $\gamma$ be a closed contour enclosing all the $a_i$'s, traced counter clockwise. Then

$$I_i(x, r, \gamma) = \sum_{h=(m+1)r}^{\infty} \left(\frac{r + \nu}{h}\right)x^h J_{ih},$$
where
\[ J_{ih} = \frac{1}{2\pi \sqrt{-1}} \oint_{\gamma} \frac{z^{h}}{(z - a_i)(A(z))^{r}} \, dz. \]

Therefore
\[ |I_i\left(\frac{1}{N}, r, \gamma\right)| \leq \sum_{h=(m+1)r}^{\infty} b_h |J_{ih}| N^{-h}, \]

where \( b_h = \left| \left( \frac{r + \nu}{h} \right) \right| \). For \( h \geq r \), \( b_h \) decreases as \( h \) increases. Hence we have
\[ (22) \quad |I_i\left(\frac{1}{N}, r, \gamma\right)| \leq b_{(m+1)r} \sum_{h=(m+1)r}^{\infty} |J_{ih}| N^{-h}. \]

Further, by induction on \( r \) it follows that
\[ (23) \quad b_{(m+1)r} \leq \left| \left( \frac{1 + \nu}{m+1} \right) \right| \left( \frac{m^m}{(m+1)^{m+1}} \right)^{r-1}. \]

Let
\[ J_i(x) = \sum_{h=0}^{\infty} x^h J_{ih}. \]

Suppose that \( |x| < 1/M \) and that \( \gamma \) is a circle centered at the origin, with radius \( r' \) satisfying \( M < r' < |x|^{-1} \). Arguing as in the proof of [11, Lemma 3.2], we get that
\[ J_i(x) = \frac{1}{2\pi \sqrt{-1}} \oint_{\gamma} \frac{1}{(1 - xz)(z - a_i)(A(z))^{r}} \, dz \]
\[ = -\text{Res}_{z=1/x} \frac{1}{(1 - xz)(z - a_i)(A(z))^{r}} = \frac{1}{(1 - a_i x)A(1/x)^r}. \]

Thus when property (i) holds for \( S \), the coefficients \( J_{ih} \) of \( J_i(x) \) are all positive. Taking \( x = 1/N \) in the above expression, we obtain that
\[ (24) \quad J_i(1/N) = \frac{1}{(1 - a_i N^{-1})A(N)^r} = \sum_{h=0}^{\infty} |J_{ih}| N^{-h}. \]

This is also true when property (ii) holds for \( S \) and \( a_i \) is non-negative. Further, when property (ii) holds, we have \( A(N) = A(-N) \). Now suppose that \( a_i \) is negative. Then we get
\[ (25) \quad \sum_{h=0}^{\infty} |J_{ih}| N^{-h} = |J_i(-1/N)| = \frac{1}{(1 - |a_i| N^{-1})A(N)^r}. \]
Hence from (22)–(25), we find in both cases
\[
\left| I_i \left( \frac{1}{N}, r, \gamma \right) \right| \leq \left| \left( 1 + \nu \right) \left( \frac{m^m}{(m + 1)^m} \right)^{r-1} \right| \frac{1}{|1 - |a_i|N^{-1}|A(N)|^r}
\]
which gives the values of \( \ell \) and \( L \) as claimed. \( \square \)

6. Determination of \( d \) and \( D \)

Define
\[
c_1 = \text{lcm} \{ \prod_{0 \leq h \leq m} |a_h - a_i| : 0 \leq i \leq m \},
\]
\[
c_2 = \text{lcm} \{ |a_\ell - a_t| : 0 \leq \ell < t \leq m \},
\]
\[
c_3 = \prod_{p | n} p^{\max \{ \text{ord}_p(n/c_2) + 1/(p-1), 0 \}},
\]
\[
c_4 = c_1 c_2 c_3.
\]

Further, for \( k \) with \( 1 \leq k < n \), let \( S(k) \) be the set of all primes \( p > \sqrt{nr + s} \), \( \gcd(p, nr) = 1 \), \( \{ \frac{r-1}{p} \} > \max(\frac{nm-k}{nm}, \frac{k}{n}) \) and \( pk \equiv s \) (mod \( n \)). Then it follows from [4, Lemma 4.2, p. 1728] that a denominator for \( p_{ijr} \) can be taken as
\[
(c_4N)^r c_2 r H(n, r)
\]
where
\[
H(n, r) = \left( \prod_{1 \leq k \leq n, p \in S(k)} \prod_{(k,n)=1} p \right)^{-1}.
\]

Further, in [3, Lemma 3.3], Bennett used estimates from Prime Number Theory to show that \( H(2, r) < 168(2/3)^r \) and \( H(4, r) < 679(3/4)^r \). Using these results, we prove the following lemma.

**Lemma 6.1.** Let \( e' \) and \( e'' \) be given by (4).

(i) Condition (5) in Lemma 2.1 is satisfied by taking \( d = c_2 e'' \) and
\[
D = \begin{cases} 
    c_4 N & \text{if } \nu = 1/2 \\
    9c_4 N / 8 & \text{if } \nu = 1/4 \\
    3c_4 N / 2 & \text{otherwise.}
\end{cases}
\]

(ii) If \( m = 2 \) and \( (a_0, a_1, a_2) = (-a, 0, a) \) or \( (0, a, 2a) \), then condition (5) in Lemma 2.1 is satisfied by taking
\[
d = 2ae'' \quad \text{and} \quad D = 4a^3 e' N.\]
Proof. (i) Note that

\[ H(n, r) \leq \begin{cases} 
    e''(2/3)^r & \text{if } \nu = 1/2 \\
    e''(3/4)^r & \text{if } \nu = 1/4 \\
    e'' & \text{otherwise.}
\end{cases} \]

Since a denominator for \( q_{ijr} \) can be taken as \((c_4 N)^r c_2 r H(n, r)\) which is at most

\((c_4 N)^r c_2 1.5^r H(n, r),\)

we can take \( d \) and \( D \) as in the statement of the lemma.

(ii) Observe that \( c_1 = 2a^2, c_2 = 2a \) and \( c_3 = e_0 \). When \( \nu \neq \frac{1}{2}, \frac{1}{4} \), we bound the denominator of \( q_{ijr} \) by \((c_4 N)^r c_2 1.5^r e''\). Let \( \nu = \frac{1}{2}, \frac{1}{4} \). Hence \( n = 2 \) or \( 4 \). Any summand in (13) with \( m = 2 \) and \( x = 1/N \) is of the form

\[ \Delta_j (N + a_j)^{r-h_j} - \frac{-r_{i\alpha}}{h_\alpha} \left( -r_{i\beta} \right) \frac{h_\beta}{h_\alpha} \]

where \( \Delta_j = (nr + 1) \cdots (nr + 1 - n(h_j - 1)) \), \( 0 \leq \alpha, \beta \leq 2 \), \( \alpha, \beta \) not equal to \( j \) and \( h_\alpha + h_\beta + h_j = r_{ij} - 1 \). Note that 2 does not divide \( \Delta_j \). Now, as in [3], we get that

\[ \frac{h_j!}{2^{\text{ord}_2(h_j)} H(n, r)} \]

divides \( \Delta_j (-r_{i\alpha}) (-r_{i\beta}) \). Hence a denominator for the summand does not exceed

\[ H(n, r) 2^{2r + \delta_{i\alpha} + \delta_{i\beta} - 1} n^h_j N r a^{\delta_{i\alpha} + \delta_{i\beta} + \delta_{ij} - 1 - h_j} \]

\[ \leq 2 H(n, r) (4Na^3) r^{r_\alpha} \max(\text{ord}_2(n/a), 0). \]

Thus a denominator for \( q_{ijr} \) can be taken as \((c_4 N)^r c_2 H(n, r)\). Further, since \( n = 2 \) or \( 4 \), we have \( e_0 \leq n \). Using the above bounds for \( H(n, r) \) we conclude that we may take \( d = 2ae'' \) and \( D = 4a^3 e'/N \).

\[ \square \]

7. PROOF OF THE THEOREMS

For the proofs of the theorems we take \( m = 2 \). Then \( c_i = e_i \) for \( 1 \leq i \leq 4 \).

Proof of Theorem 1.1 Let \((a_0, a_1, a_2) = (-a, 0, a), a \geq 1 \) and \( N \geq 6.794e^2a^3 \). Then by Lemmas 4.2(i), 5.2 and 6.1(ii), we can take

\[ p = 1.54608 \left( 1 + \frac{2a}{N\sqrt{3}} \right) \nu \left( 1 - \frac{a}{N} \right)^{-\nu}, P = \frac{3\sqrt{3}}{2a^3} \left( 1 + \frac{2a}{N\sqrt{3}} \right), \]
\[
\ell = \frac{9N\nu(1-\nu^2)}{8(N-a)}, \quad L = \frac{27}{4}N(N^2-a^2),
\]
(26) \[d = 2ae'' \quad \text{and} \quad D = 4a^3e'N.\]

Thus \(\frac{L}{D} = \frac{27(N^2-a^2)}{16a^3e'}\) and \(L/D > 1\) by the assumption on \(N\). The values of \(c\) and \(\chi\) are calculated from the expressions given by Lemma 2.1. In order that \(\chi < 1\), we require \(PD^2 < L\).

Thus \(\chi < 1\) whenever

\[9(N^2-a^2) > 32\sqrt{3}a^3e'^2N \left(1 + \frac{2a}{N\sqrt{3}}\right).\]

Since \(e' \geq 4/3\) and \(a \geq 1\), the above quadratic inequality holds if \(N \geq 6.794e'^2a^3\). This proves the theorem. \(\Box\)

**Proof of Theorem 1.3** Let \((a_0, a_1, a_2) = (0, a, 2a), a \geq 1\). Then by Lemmas 4.2(ii), 5.2 and 6.1(ii), we can take

\[p = 1.54608 \left(1 + \frac{(2 + \sqrt{3})a}{N\sqrt{3}}\right)^\nu, \quad P = \frac{3\sqrt{3}}{2a^3} \left(1 + \frac{(2 + \sqrt{3})a}{N\sqrt{3}}\right),\]

\[\ell = \frac{9N\nu(1-\nu^2)}{8(N-2a)}, \quad L = \frac{27}{4}N(N^2-3aN+2a^2)\]
and \(d\) and \(D\) as in (26). The values of \(c\) and \(\chi\) are calculated from the expressions given by Lemma 2.1. In this case, \(\chi < 1\) whenever

\[9(N^2-3aN+2a^2) > 32\sqrt{3}a^3e'^2N \left(1 + \frac{(2 + \sqrt{3})a}{N\sqrt{3}}\right)\]

which holds for \(N \geq 8.637e'^2a^3\). This completes the proof of the theorem. \(\Box\)

**Proof of Theorem 1.6** We apply Lemmas 4.1 and 6.1(i) with \(m = 2\) to get the values of \(P\) and \(D\) as in the statement of the theorem. From Lemma 5.1, for \(r \geq 1100\) we get

\[L = 0.99 \times 2(N-M)^3(3/2)^3\]
\[\geq 6.68(N-M)^3.\]

Thus by Lemma 2.1, the value of \(\chi\) can be taken as

\[\log(PD) \quad \text{and} \quad \log(6.68(N-M)^3/D).\]

\(\Box\)
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