TRANSCENDENTAL VALUES OF THE DIGAMMA FUNCTION

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Abstract. Let $\psi(x)$ denote the digamma function, that is, the logarithmic derivative of Euler's Γ-function. Let q be a positive integer greater than 1 and γ denote Euler's constant. We show that all the numbers

$$\psi(a/q)+\gamma, \quad (a,q)=1,\ 1\le a\le q,$$

are transcendental. We also prove that at most one of the numbers

$$\gamma,\ \psi(a/q) \quad (a,q)=1,\ 1\le a\le q,$$

is algebraic.

1. Introduction

The arithmetic nature of the values of the gamma function $\Gamma(z)$ at rational arguments has been the focus of study in number theory for many decades. Apart from the classical result that $\Gamma(1/2)$ is transcendental, being equal to $\sqrt{\pi}$, the earliest result in this study is due to Schneider [19] who proved in 1941 that the number

$$B(a,b):=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \tag{1}$$

is transcendental whenever a, b and $a+b$ are rational numbers which are not integers. In particular, the numbers

$$B(1/4,1/2)=\frac{\Gamma(1/4)^2}{\sqrt{2\pi}}$$

and

$$B(1/3,1/2)=\frac{\sqrt{3}\Gamma(1/3)^3}{2^{4/3}\pi}$$

are transcendental. However, this does not tell us anything about the numbers $\Gamma(1/3)$ and $\Gamma(1/4)$ themselves. The transcendence of $\Gamma(1/3)$ and $\Gamma(1/4)$ was established by Chudnovsky [7] in 1976. More precisely, he proved the stronger result that π and $\Gamma(1/3)$ are algebraically independent and the same is true for π and $\Gamma(1/4)$. In 1996, Nesterenko [16] extended these results by showing that π, $\Gamma(1/3)$ and $e^{\pi\sqrt{3}}$ are algebraically independent and the same is true for $\pi, \Gamma(1/4)$ and e^{π}. Using standard identities satisfied by the Γ-function, one can deduce from this that $\Gamma(1/6)$ is also transcendental. Recently, Grinspan [10] (see also Vasilév [20]) showed that at least two of the three numbers $\pi, \Gamma(1/5)$ and $\Gamma(2/5)$ are algebraically independent. Apart from these results, no further transcendence results are known at present concerning the Γ-function at rational arguments. For a detailed exposition of the "state of the art," we suggest Waldschmidt [22].

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This paper is devoted to the study of the digamma function at rational arguments. The *digamma function* $\psi(z)$ is the logarithmic derivative of the gamma function. It satisfies several functional relations that are easily derived from the properties of the gamma function. Indeed, from the formulas,

$$\Gamma(z+1) = z\Gamma(z), \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z},$$

and

$$\frac{1}{\Gamma(z+1)} = e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n},$$

where γ denotes Euler's constant, we obtain

$$\psi(z+1) = \psi(z) + \frac{1}{z}, \tag{2}$$

$$\psi(1-z) = \psi(z) + \pi \cot \pi z, \tag{3}$$

and

$$\psi(z+1) = -\gamma + \sum_{n=2}^{\infty} (-1)^n \zeta(n) z^{n-1}, \tag{4}$$

where ζ denotes the Riemann zeta function. Let us note that $\psi(1) = -\gamma$ and more generally, for any rational number $z = a/q$ with $(a, q) = 1$ and $1 \le a < q$, we have the celebrated formula of Gauss [9], discovered in 1813:

$$\psi(a/q) = -\gamma - \log 2q - \frac{\pi}{2} \cot \frac{\pi a}{q} + 2 \sum_{0<j\le q/2} \left(\cos \frac{2\pi a j}{q}\right) \log \sin \frac{\pi j}{q}.$$

A simplified treatment of this formula can be found in the papers by Jensen [13] and Lehmer [14]. (We advise the reader that in [14], $\log(k/2)$ in the displayed formula after (20), should be $\log 2k$.) We will also give two proofs of this fact in sections 3 and 6 below. The purpose of this paper is to investigate the transcendental nature of these numbers. More precisely, we prove:

Theorem 1. *Let $q > 1$. Then, $\psi(a/q) + \gamma$ is transcendental for any $1 \le a \le q - 1$.*

Using the series representation (4) of $\psi(z)$ with $z = -a/q$ and a/q respectively, we immediately deduce

Corollary 2. *For any $1 \le a < q$, both the series*

$$\sum_{n=2}^{\infty} \zeta(n)(a/q)^n \quad \text{and} \quad \sum_{n=2}^{\infty} \zeta(n)(-1)^n(a/q)^n$$

are transcendental.

Theorem 3. *Let $q > 1$. At most one of the $\varphi(q) + 1$ numbers in the set*

$$\{\gamma\} \cup \{\psi(a/q) : \quad (a, q) = 1, 1 \le a \le q\},$$

is algebraic.

These theorems do not tell us anything about the algebraicity of γ. However, if γ is algebraic, then all the numbers $\psi(a/q)$ with $(a,q) = 1$ and $q > 1$ are transcendental. Let us observe that from (3), at least one of $\psi(a/q)$ or $\psi(1-a/q)$ is transcendental since π is transcendental and $\cot \pi a/q$ is algebraic. Presumably, all of the numbers listed in the theorem are transcendental and linearly independent over $\mathbb{Q}$. In this direction, we can prove:

Theorem 4. *Let K be an algebraic number field over which the q-th cyclotomic polynomial is irreducible. The numbers*

$$\psi(a/q) + \gamma, \quad (a,q) = 1, 1 \le a \le q$$

are linearly independent over K. Furthermore, the K-vector space spanned by γ and the $\varphi(q)$ numbers

$$\psi(a/q) : \quad (a,q) = 1, 1 \le a \le q$$

has dimension at least $\varphi(q)$.

Let us remark that the q-th cyclotomic polynomial $\Phi_q(x)$ is irreducible over an algebraic number field K if and only if $K \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$, where ζ_q denotes a primitive q-th root of unity. Indeed, the extension $K(\zeta_q)/K$ is Galois with group isomorphic to the subgroup of $\mathrm{Gal}(\mathbb{Q}(\zeta_q)/\mathbb{Q})$ having $K \cap \mathbb{Q}(\zeta_q)$ as its fixed field (see for example, Theorem 29 of [3]). Thus, in order that $\Phi_q(x)$ be irreducible over K, it is necessary and sufficient that $[K(\zeta_q) : K] = \varphi(q)$ and this can only happen if and only if $K \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. In particular, we see that if $[K : \mathbb{Q}]$ is coprime to $\varphi(q)$, then $\Phi_q(x)$ is irreducible over K.

Theorem 4 has important implications for the classical theory of Dirichlet L-functions. More precisely, we deduce that:

Corollary 5. *Suppose $(q, \varphi(q)) = 1$. Let $K = \mathbb{Q}(\xi)$ where ξ is a primitive $\varphi(q)$-th root of unity. Then, any combination*

$$\sum_{\chi \ne \chi_0} a_\chi L(1,\chi),$$

with $a_\chi \in K$, not all zero, is transcendental.

When $(q, \varphi(q)) = 1$, the linear independence over $\mathbb{Q}$ of $L(1,\chi)$ as χ ranges over the non-principal characters mod q, was proved by Baker, Birch and Wirsing in [5].

Theorem 4 suggests that it is reasonable to make the following conjecture:

Conjecture 6. *Let K be an algebraic number field over which the q-th cyclotomic polynomial is irreducible. Then, the numbers $\psi(a/q)$ with $(a,q) = 1$ and $1 \le a < q$ are linearly independent over K.*

Let us note that the conjecture is not true without the coprimality condition. Indeed, we have

$$\psi(1/2) = -\gamma - 2\log 2, \quad \psi(1/4) = -\gamma - \pi/2 - 3\log 2, \quad \psi(3/4) = -\gamma + \pi/2 - 3\log 2,$$

so that

$$\psi(1) + \psi(1/4) - 3\psi(1/2) + \psi(3/4) = 0.$$

Let us define a complex number to be a *Baker period* if it is a $\overline{\mathbb{Q}}$-linear combination of logarithms of algebraic numbers. A non-zero Baker period is necessarily transcendental by Baker's theorem (see Lemma 12 below).

We will prove:

Theorem 7. *Either γ is a Baker period or Conjecture 6 is true.*

A conjecture of Kontsevich and Zagier [12] predicts that γ is not a Baker period. In fact, their conjecture is even stronger (see section 1.1 and the last paragraph of section 4.3 of [12] where they conjecture that γ is not even an exponential period).

Mahler [15] obtained various results about the transcendence of certain numbers involving γ. For example, he proved that

$$\frac{\pi Y_0(2)}{2J_0(2)} - \gamma$$

is transcendental, where J_0 and Y_0 are the Bessel functions of the first and second kind of order zero. More generally, he proved that for any algebraic $\alpha \neq 0$, the number

$$\frac{\pi Y_0(\alpha)}{2J_0(\alpha)} - \log \frac{\alpha}{2} - \gamma$$

is transcendental. Our theorems have the same flavour.

Lehmer [14] has defined the *generalised* Euler constants $\gamma(a, q)$ as

$$\lim_{x\to\infty} \sum_{n\le x, n\equiv a (\bmod q)} \frac{1}{n} - \frac{\log x}{q}.$$

A variation of the argument used to prove Theorem 4 enables us to show:

Theorem 8. *Let $q > 1$. At most one of the $\varphi(q) + 1$ numbers,*

$$\gamma, \gamma(a, q), \quad (a, q) = 1, \quad 1 \le a \le q,$$

is algebraic.

The methods used in this paper and the properties of the digamma function seem to have other interesting applications. In fact, the proof of Theorem 1 relies on the following result, which is of independent interest.

Theorem 9. *If $1 \le a < q$, then*

$$\sum_{n=1}^{\infty} \frac{1}{n(n + a/q)}$$

is transcendental.

We apply this result to the summation of infinite series:

Theorem 10. *Let $A(x)$ and $B(x)$ be polynomials with coefficients in a field $K \subseteq \mathbb{C}$. Suppose that $\deg A \le \deg B - 2 = r - 2$ and that B has simple zeros $\alpha_1, \alpha_2, ..., \alpha_r$. Then, there exist numbers $c_0, ..., c_r \in K(\alpha_1, ..., \alpha_r)$ such that the series*

$$\sum_{n=1}^{\infty}{}' \frac{A(n)}{B(n)} = c_0 + \sum_{i=1}^{r} c_i \psi(\alpha_i),$$

where the summation is over n such that $B(n) \neq 0$. In the special case that $A(x) \in K[x]$ and $B(x) \in \mathbb{Q}[x]$ with simple rational roots, we have that the sum is either in K or is transcendental.

We hasten to point out that theorems similar to Theorems 9 and 10 have been proved in [1].

In a related context, Erdős (see [8]) conjectured that there is no function defined on the integers with period q satisfying $f(q) = 0$, $f(a) = \pm 1$ for $1 \le a \le q-1$ such that

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = 0.$$

As will be shown in section 4, a necessary and sufficient condition for convergence is that

$$\sum_{a=1}^{q} f(a) = 0. \tag{5}$$

This means, we need only consider the case q odd. The case q prime was resolved by a theorem of Baker, Birch and Wirsing [5]. If $2\varphi(q) \ge q$, Okada [17] showed that the conjecture is true. This settles the case when q is a prime power or a product of two primes. In [18], Saradha extended this result and proved that the conjecture holds if $2\varphi(q) \ge q(1 - 1/h)$ where h is the largest prime power dividing q.

We will prove:

Theorem 11. *Let q be a positive odd number. If f is an odd periodic function with period q such that $f(q) = 0$ and*

$$\sum_{a=1}^{q/2} f(a)$$

is odd, then

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \ne 0.$$

In particular, the above assertion is true if $q \equiv 3 \pmod 4$, $f(a)$ is odd valued for $1 \le a \le q-1$, $f(q) = 0$ and f is an odd function.

As a consequence of Theorem 11, we deduce that the Erdős' conjecture holds for q of the form $4k+3$ and odd f. Let us remark that if q is of the form $4k+3$, there is no even function f taking only ± 1 values. Indeed, for convergence, we need

$$\sum_{a=1}^{2k+1} f(a) = 0,$$

and this is impossible.

Our main tools in the paper w ill be the theory of the Hurwitz zeta function and Baker's theory of linear forms in logarithms, both of which we review in the next sections.

2. Preliminaries

A fundamental theorem that we will repeatedly use is the following:

Lemma 12. (A. Baker [4]) *If $\alpha_1, \cdots, \alpha_n \in \overline{\mathbb{Q}} \backslash \{0\}$ and $\beta_1, ..., \beta_n \in \overline{\mathbb{Q}}$, then*

$$\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$$

is either zero or transcendental.

Let us note that here and later, we interpret log as the principal value of the logarithm with argument in $(-\pi, \pi]$.

A creative application of this theorem was made by Baker, Birch and Wirsing [5] to resolve a problem of Chowla [6], which we now describe.

In a lecture at the Stony Brook conference on number theory in the summer of 1969, Sarvadaman Chowla [6] posed the following question. Does there exist a rational-valued function $f(n)$, periodic with prime p such that

$$\sum_{n=1}^{\infty} \frac{f(n)}{n}$$

converges and equals zero? In 1973, Baker, Birch and Wirsing [5] answered this question in the following theorem:

Proposition 13. *If f is a non-vanishing function defined on the integers with algebraic values and period q such that $f(n) = 0$ whenever $1 < (n, q) < q$ and the q-th cyclotomic polynomial is irreducible over $\mathbb{Q}(f(1), ..., f(q))$, then*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0.$$

In particular, if f is rational valued, the second condition holds trivially since the q-th cyclotomic polynomial is irreducible over $\mathbb{Q}$. If q is prime, then the first condition is vacuous. Thus, the theorem resolves Chowla's question. In 2001, Adhikari, Saradha, Shorey and Tijdeman [1] noted that the theory of linear forms in logarithms can be used to show that in fact, the sum in the above theorem is transcendental whenever it converges. In particular, the special value $L(1, \chi)$ of Dirichlet's L-function attached to a non-trivial character mod q is transcendental. We describe the results contained in [1] but in a slightly generalized context. We will streamline these results by using the theory of the Hurwitz zeta function, which we review in the next section.

3. The Hurwitz zeta function

We first derive a necessary and sufficient condition for the sum in Proposition 13 to converge. To this end, we use the Hurwitz zeta function. Recall that this function is defined as the series

$$\zeta(s, x) := \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}, \quad 0 < x \leq 1,$$

for $\Re(s) > 1$. The series $\zeta(s, 1)$ is the familiar Riemann zeta function. Hurwitz [11] proved that this function extends to the entire complex plane except at $s = 1$ where it has a simple pole and residue 1. If $x = a/q$, with $1 \leq a < q$, it satisfies the following functional equation (see for example, page 80 of [11] or page 261 of [2]):

$$\zeta(s, a/q) = 2\Gamma(1-s)(2\pi q)^{s-1} \sum_{b=1}^{q} \sin\left(\frac{\pi s}{2} + \frac{2\pi ba}{q}\right) \zeta(1-s, b/q). \tag{6}$$

We also have the following interesting equations:

$$\zeta(0, x) = \frac{1}{2} - x, \quad \zeta'(0, a/q) = \log \Gamma(a/q) - \frac{1}{2} \log(2\pi). \tag{7}$$

Moreover, we have the important fact:

$$\lim_{s\to 1^+} \zeta(s,x) - \frac{1}{s-1} = -\frac{\Gamma'(x)}{\Gamma(x)}.$$

This is easily seen as follows:

$$\lim_{s\to 1^+} \zeta(s,x) - \zeta(s) = \frac{1}{x} + \sum_{n=1}^{\infty}\left(\frac{1}{n+x} - \frac{1}{n}\right).$$

From the Hadamard factorization of $1/\Gamma(z)$,

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n},$$

we have by logarithmic differentiation,

$$-\psi(z) = \gamma + \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{n+z} - \frac{1}{n}\right). \tag{8}$$

Thus,

$$\lim_{s\to 1^+} \zeta(s,x) - \zeta(s) = -\gamma - \psi(x).$$

Observe that in the special case that $x = 1$, we deduce that

$$\Gamma'(1) = -\gamma.$$

Recall that by partial summation, we have

$$\zeta(s) = s\int_1^{\infty} \frac{[x]}{x^{s+1}}dx.$$

The integral can be written as

$$\frac{s}{s-1} - s\int_1^{\infty}\frac{\{x\}}{x^{s+1}}dx,$$

so that

$$\lim_{s\to 1^+} \zeta(s) - \frac{1}{s-1} = 1 + \int_1^{\infty}\frac{\{x\}}{x^2}dx.$$

This last integral is easily evaluated as

$$\lim_{N\to\infty}\int_1^N \frac{x-[x]}{x^2}dx = \lim_{N\to\infty}\log N - \sum_{n=1}^{N-1} n\int_n^{n+1}\frac{dx}{x^2} = \lim_{N\to\infty}\log N - \sum_{n=2}^{N}\frac{1}{n} = 1-\gamma.$$

Thus,

$$\lim_{s\to 1^+} \zeta(s) - \frac{1}{s-1} = \gamma.$$

Putting everything together, we obtain

Proposition 14.

$$\lim_{s\to 1^+} \zeta(s,x) - \frac{1}{s-1} = -\psi(x)$$

We also record here our derivation of the digamma relation of Gauss. (See [14] for a derivation via finite Fourier series.)

Proposition 15.

$$-\psi(a/q)-\gamma=\log 2q-2\sum_{b=1}^{q/2}\left(\cos\frac{2\pi ba}{q}\right)\log\sin\frac{\pi b}{q}+\frac{\pi}{2}\cot\frac{\pi a}{q}.$$

Proof. This is easily derived from the functional equation (6) of the Hurwitz zeta function as follows. We have

$$(2\pi q)^{1-s}\zeta(s,a/q)=2\Gamma(1-s)\sum_{b=1}^{q}\sin\left(\frac{\pi s}{2}+\frac{2\pi ba}{q}\right)\zeta(1-s,b/q).$$

Expanding both sides about $s=1$ and comparing the residue term, we get

$$1=-2\sum_{b=1}^{q}\zeta(0,b/q)\cos\frac{2\pi ba}{q}.$$

Using this, we compare the constant terms in the Laurent expansion at $s=1$ to get

$$-\psi(a/q)-\log 2\pi q=\gamma+2\sum_{b=1}^{q}\zeta'(0,b/q)\cos\frac{2\pi ba}{q}+\frac{\pi}{2}\sin\frac{2\pi ba}{q}\zeta(0,b/q).$$

In addition, using (7), and that

$$\sum_{b=1}^{q}\cos\frac{2\pi ba}{q}=0,\quad \text{for } 1\le a<q, \tag{9}$$

we obtain

$$-\psi(a/q)-\log 2\pi q=\gamma+2\sum_{b=1}^{q-1}\left(\log\Gamma\left(\frac{b}{q}\right)\right)\cos\frac{2\pi ba}{q}+\pi\sum_{b=1}^{q}\left(\frac{1}{2}-\frac{b}{q}\right)\sin\frac{2\pi ba}{q}. \tag{10}$$

Now

$$\sum_{b=1}^{q}\left(\frac{1}{2}-\frac{b}{q}\right)\sin\frac{2\pi ba}{q}=\frac{\cot\pi a/q}{2}, \tag{11}$$

which is easily derived (see for example, Lemma A of [14]). This allows us to simplify the second sum on the right hand side of the formula. Finally, we can use

$$\log\Gamma\left(\frac{b}{q}\right)+\log\Gamma\left(1-\frac{b}{q}\right)=\log\pi-\log\sin\frac{\pi b}{q}.$$

By pairing b with $q-b$ in the summation and noting that when q is even, the corresponding summand for $b=q/2$ is zero, and using

$$\sum_{b=1}^{q/2}\cos\frac{2\pi ba}{q}=-\frac{1}{2},$$

we deduce the result. This completes the proof. □

Thus, the functional equation of the Hurwitz zeta function is one way of deriving the formula of Gauss. In the section 5, we will give another (simpler) derivation.

4. The Chowla Problem

We are now ready to prove:

Theorem 16. *Let f be any function defined on the integers and with period q. Then,*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n}$$

converges if and only if

$$\sum_{a=1}^{q} f(a) = 0, \tag{12}$$

and in the case of convergence, the value of the series is

$$-\frac{1}{q}\sum_{a=1}^{q} f(a)\psi(a/q).$$

Proof. We have

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \sum_{a=1}^{q} f(a) \sum_{n=0}^{\infty} \frac{1}{(qn+a)^s} = q^{-s} \sum_{a=1}^{q} f(a)\zeta(s, a/q).$$

Since the Hurwitz zeta function admits an analytic continuation to the entire complex plane with a simple pole at $s = 1$ and residue 1, we deduce that the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

extends to all values $s \in \mathbb{C}$ except possibly at $s = 1$ where it has a simple pole and residue

$$\frac{1}{q}\sum_{a=1}^{q} f(a).$$

Therefore, (12) is a necessary and sufficient condition for the convergence of the series at $s = 1$. Moreover, the value of the series is by Proposition 14,

$$-\frac{1}{q}\sum_{a=1}^{q} f(a)\psi(a/q).$$

□

This gives us an interesting corollary even in the classical case:

Proposition 17. *If χ is a non-trivial character mod q, then*

$$L(1,\chi) = -\frac{1}{q}\sum_{a=1}^{q} \chi(a)\psi(a/q).$$

The latter result has also been observed by Lehmer[14].

If we apply Theorem 16 to the function $f(n) = e^{2\pi i n b/q}$, with $1 \le b \le q-1$, we obtain:

Proposition 18.

$$\log(1 - e^{2\pi i b/q}) = \frac{1}{q}\sum_{a=1}^{q} e^{2\pi i a b/q}\psi(a/q).$$

Given a function f defined on the integers, with period q, we define

$$\hat{f}(n) = \frac{1}{q}\sum_{a=1}^{q} f(a)e^{2\pi i a n/q}.$$

By orthogonality, we have

$$f(n) = \sum_{a=1}^{q} \hat{f}(a)e^{2\pi i a n/q}.$$

Let us note that the condition (12) is equivalent to $\hat{f}(0) = 0$. We immediately deduce:

Theorem 19. *Suppose that f is a periodic function with period q satisfying (12). Then,*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = -\frac{1}{q}\sum_{a=1}^{q} f(a)\psi(a/q) = -\sum_{a=1}^{q-1} \hat{f}(a)\log(1 - e^{2\pi i a/q}).$$

As in [1], we apply Baker's theorem that an algebraic linear combination of logarithms of algebraic numbers is either zero or transcendental. Thus, we obtain:

Theorem 20. *Let f be any function defined on the integers and with period q satisfying (12). Then,*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n}$$

converges and equals

$$-\sum_{a=1}^{q-1} \hat{f}(a)\log(1 - e^{2\pi i a/q}).$$

In particular, if f takes algebraic values, the series is either zero or transcendental.

Proof. We need only observe that if f takes algebraic values, so does $\hat{f}$. Thus, the last conclusion follows from Lemma 12. □

5. Proof of Theorem 11

We first show that for f odd, the series

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \tag{13}$$

is an algebraic multiple of π. By Theorem 16, we may write the sum of the series as

$$-\frac{1}{q}\sum_{a=1}^{q-1} f(a)\psi(a/q) = -\frac{1}{q}\sum_{a=1}^{q/2}\left(f(a)\psi\left(\frac{a}{q}\right) + f(q-a)\psi\left(1-\frac{a}{q}\right)\right).$$

Since f is odd, this becomes

$$-\frac{1}{q}\sum_{a=1}^{q/2} f(a)\left(\psi\left(\frac{a}{q}\right) - \psi\left(1-\frac{a}{q}\right)\right).$$

By (3), this simplifies to

$$\frac{\pi}{q}\sum_{a=1}^{q/2} f(a)\cot\frac{\pi a}{q}.$$

To prove Theorem 11, we note that

$$\cot\frac{\pi a}{q} = i\frac{e^{\pi ia/q}+e^{-\pi ia/q}}{e^{\pi ia/q}-e^{-\pi ia/q}} = i - \frac{2i}{\zeta_q^a - 1},$$

where $\zeta_q = e^{2\pi i/q}$, we may view the right hand side as an element in the field $K = \mathbb{Q}(i,\zeta_q)$. Since q is odd, $\zeta_q^a - 1$ is coprime to 2. In fact, if q is a prime power, and $1 \le a \le q-1$, then $\zeta_q^a - 1$ divides q and if q is not a prime power, then $\zeta_q^a - 1$ is a unit (see for example, [21]). In any case, the elements are coprime to 2. Thus, if $\mathcal{O}_K$ denotes the ring of integers of K, we see that

$$\sum_{a=1}^{q/2} f(a)\left(i - \frac{2i}{\zeta_q^a - 1}\right) \equiv i\sum_{a=1}^{q/2} f(a) \not\equiv 0 (\bmod 2\mathcal{O}_K),$$

since i is a unit and the sum, by hypothesis, is odd. This completes the proof of the first assertion. If q is of the form $4k+3$, then $(q-1)/2$ is odd and consequently, if $f(a)$ is odd-valued for $1 \le a \le q-1$, we have

$$\sum_{a=1}^{q/2} f(a)$$

is odd. Thus, in this case,

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0.$$

This completes the proof of Theorem 11.

6. Proof of Theorem 1

We begin with the following lemma:

Lemma 21.

$$-\psi(a/q) - \gamma = \log q - \sum_{b=1}^{q-1} e^{-2\pi iba/q}\log(1 - e^{2\pi ib/q}).$$

Proof. Using standard orthogonality relations, we "invert" the formula given in Proposition 18. To this end, we use the identity

$$\Gamma(z)\Gamma(z+1/q)\cdots\Gamma(z+(q-1)/q) = q^{1/2-qz}(2\pi)^{(q-1)/2}\Gamma(qz).$$

Logarithmically differentiating this, and setting $z = 1/q$, we get

$$\sum_{a=1}^{q} \psi(a/q) = -q\log q - \gamma q,$$

where we have used the fact $\Gamma'(1) = -\gamma$. Thus,

$$\log q + \gamma = -\frac{1}{q}\sum_{a=1}^{q} \psi(a/q),$$

and from Proposition 18,

$$\log(1 - e^{2\pi ib/q}) = \frac{1}{q}\sum_{a=1}^{q-1} e^{2\pi iab/q}\psi(a/q).$$

Multiplying these equations by $e^{-2\pi ibc/q}$ and summing from $b = 1$ to q, we obtain using the orthogonality relations for the roots of unity,

$$-\psi(c/q) - \gamma = \log q - \sum_{b=1}^{q-1} e^{-2\pi ibc/q} \log(1 - e^{2\pi ib/q}),$$

which completes the proof. □

Let us note that Lemma 21 can be used to give another proof of the digamma formula of Gauss. Indeed, we need only note that

$$1 - e^{2\pi ib/q} = -(e^{\pi ib/q} - e^{-\pi ib/q})e^{\pi ib/q} = -2i(\sin b/q)e^{\pi ib/q}.$$

Thus, the principal value of the logarithm is

$$\log(1 - e^{2\pi ib/q}) = \log\left(2\sin\frac{\pi ba}{q}\right) - \pi i\left(\frac{1}{2} - \frac{b}{q}\right).$$

Inserting this into the formula in Lemma 21, and using (9) and (11), we obtain the formula of Gauss.

To prove Theorem 1, we need to establish the following inequality:

Lemma 22. *For $1 \leq a < q$, we have*

$$\sum_{n=1}^{\infty} \frac{1}{n(qn+a)} < \frac{q}{a^2}.$$

Proof. Indeed, we have $qn(qn + a) > an(an + a)$ so that

$$\sum_{n=1}^{\infty} \frac{1}{qn(qn+a)} < \frac{1}{a^2}\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

and the latter series telescopes to 1. □

Now we can prove Theorem 1. By Lemma 21,

$$-\psi(a/q) - \gamma = \log q - \sum_{b=1}^{q} e^{-2\pi iba/q} \log(1 - e^{2\pi ib/q}).$$

By Lemma 12, this number is either zero or transcendental. From (8), we have

$$-\psi(a/q) - \gamma = \frac{q}{a} + \sum_{n=1}^{\infty}\left(\frac{1}{n+a/q} - \frac{1}{n}\right) = \frac{q}{a} - \frac{a}{q}\sum_{n=1}^{\infty}\frac{1}{n(n+a/q)}.$$

The quantity on the right is non-zero, for otherwise, we would have by Lemma 22,

$$\frac{1}{a^2} = \sum_{n=1}^{\infty} \frac{1}{qn(qn+a)} < \frac{1}{a^2},$$

a contradiction. Thus, the number is transcendental. This completes the proof of Theorem 1.

7. Proofs of Theorem 3 and Theorem 8

First we prove Theorem 3.

If we choose two distinct residue classes $a, b \bmod q$, among the coprime residue classes and the zero residue class, and set $f(a) = 1$, $f(b) = -1$ with f zero otherwise, then, f satisfies the conditions of Theorem 20. Thus, the sum is either zero or transcendental. However, the former case is ruled out by Proposition 13. Thus, it is transcendental. By Theorem 19, the sum is equal to

$$\frac{1}{q}\left(\psi(b/q) - \psi(a/q)\right).$$

In this way, we see that the difference of any two values in the set

$$\gamma, \psi(a/q), \quad 1 \leq a \leq q, \quad (a,q) = 1,$$

is transcendental. Thus, if there were at least two algebraic numbers in this set, we derive a contradiction. This completes the proof.

To prove Theorem 8, we will use the formula

$$\gamma(a,q) = -\frac{1}{q}(\psi(a/q) + \log q),$$

which is Theorem 7 in [14]. Consequently, for any function with period q, we have by Theorem 16

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = \sum_{a=1}^{q} f(a)\gamma(a,q).$$

As in the previous proof, we choose two distinct residue classes $a, b \bmod q$, among the coprime residue classes and the zero residue class and set $f(a) = 1, f(b) = -1$ with f zero otherwise. The argument is now as before.

8. Proofs of Theorem 4 and Corollary 5

We will now prove Theorem 4. Suppose there are numbers $c_a \in K$ such that

$$\sum_{(a,q)=1} c_a(\psi(a/q) + \gamma) = 0.$$

Define a q-periodic function f as follows. Let $f(a) = c_a$ for $(a,q) = 1$,

$$f(q) = -\sum_{(a,q)=1} c_a,$$

and zero otherwise. Then,

$$\sum_{a=1}^{q} f(a)\psi(a/q) = 0,$$

since $\psi(1) = -\gamma$. But then, by Theorem 16,

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = 0,$$

which contradicts Proposition 13.

To prove the second part of the theorem, suppose that the conclusion of the theorem is false. Let $a_0 = 0$ and let $a_1, ..., a_{\varphi(q)}$ be the coprime residue classes mod q.

Let v be the column vector with components γ and $\psi(a_i/q)$, with $1 \le i \le \varphi(q)$. Consider the map

$$T : K^{\varphi(q)+1} \to \mathbb{C}$$

given by the dot product:

$$T(w) = w \cdot v.$$

The dimension of the image of this map is the number of K-linearly independent elements amongst the numbers

$$\gamma, \psi(a/q), \quad (a,q) = 1, \quad 1 \le a \le q.$$

Suppose there are two linearly independent elements in the kernel, say,

$$w_0 = (c_0, c_1, ..., c_{\varphi(q)})$$

and

$$w_1 = (d_0, d_1, ..., d_{\varphi(q)}).$$

Define $f_0(0) = c_0$ and $f_0(a_i) = c_i$ and zero for the other residue classes, and extend the definition by periodicity. Similarly, define $f_1(0) = d_0$ and $f_1(a_i) = d_i$ and zero for the other residue classes, and extend the definition by periodicity. Then the sum

$$\sum_{n=1}^{\infty} \frac{f_0(n)}{n}$$

is zero by Theorem 16, provided that

$$c_0 + c_1 + \cdots + c_{\varphi(q)} = 0.$$

This contradicts Proposition 13. So we must have

$$c_0 + c_1 + \cdots + c_{\varphi(q)} \neq 0.$$

Similarly,

$$d_0 + d_1 + \cdots + d_{\varphi(q)} \neq 0.$$

Set

$$\lambda = \frac{c_0 + c_1 + \cdots + c_{\varphi(q)}}{d_0 + d_1 + \cdots + d_{\varphi(q)}}$$

and define

$$f(0) = c_0 - \lambda d_0, \quad f(a_i) = c_i - \lambda d_i,$$

and zero for the other residue classes. Extending f by periodicity, we see that

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = \sum_{n=1}^{\infty} \frac{f_0(n)}{n} - \lambda \frac{f_1(n)}{n} = 0.$$

Moreover,

$$\sum_{a=1}^{q} f(a) = 0,$$

and by Proposition 13, this is a contradiction. This completes the proof of Theorem 4.

To prove Corollary 5, let $K = \mathbb{Q}(\xi)$ where ξ is a primitive $\varphi(q)$-th root of unity. The Dirichlet characters (mod q) take values in K. Now, if

$$\sum_{\chi \neq \chi_0} a_\chi L(1, \chi) \tag{14}$$

vanishes, with $a_\chi \in K$, then for

$$f = \sum_{\chi \neq \chi_0} a_\chi \chi,$$

we have by Theorem 16, that

$$\sum_{(a,q)=1} f(a)(\psi(a/q) + \gamma) \tag{15}$$

vanishes. As noted in [5], the condition $(q, \varphi(q)) = 1$ implies that the q-th cyclotomic polynomial is irreducible over K so we have a contradiction to the first part of Theorem 4. Thus, the sum in (14) does not vanish. Consequently, the sum (15) does not vanish. As this sum is a Baker period, by Lemma 12, it is transcendental. This completes the proof of Corollary 5.

9. Proof of Theorem 7

To prove Theorem 7, let us suppose that the second assertion is false and that the numbers $\psi(a/q)$ with $1 \leq a < q$ and $(a, q) = 1$ are linearly dependent over K. Thus, for some a_0 coprime to q, we have

$$\psi(a_0/q) = \sum_{a \neq a_0} c_a \psi(a/q),$$

for some $c_a \in K$. If all the c_a are zero, then by Theorem 3, $\psi(a_0/q) + \gamma = \gamma$ is transcendental and by Lemma 21, it is a Baker period. So, in this case, we are done. So let us assume that not all c_a are zero. We may also assume that

$$\lambda := \sum_{a \neq a_0} c_a \neq 1$$

for otherwise, we have by Theorem 16,

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = 0,$$

with $f(n) = c_a$ whenever $n \equiv a \pmod{q}$, $a \neq a_0$, $f(n) = -1$ if $n \equiv a_0 \pmod{q}$ and zero otherwise. By Proposition 13, this is a contradiction. Hence,

$$\psi(a_0/q) + \lambda\gamma = \sum_{a \neq a_0} c_a \left(\psi(a/q) + \gamma\right).$$

Consequently,

$$(\lambda - 1)\gamma = -(\psi(a_0/q) + \gamma) + \sum_{a \neq a_0} c_a \left(\psi(a/q) + \gamma\right),$$

and each of the terms on the right hand side is a linear form in logarithms of algebraic numbers, by Lemma 21. Since the left hand side is non-zero, the right-hand side is non-vanishing and so by Lemma 12, we deduce that γ is transcendental. This completes the proof.

10. Proof of Theorem 10

This proof is similar to the methods in [1] with minor changes. By using partial fractions, we may write

$$\frac{A(n)}{B(n)} = \sum_i \frac{c_i}{n+\alpha_i}$$

where $c_i = A(\alpha_i)/B'(\alpha_i) \in K(\alpha_1, ..., \alpha_r)$. The restriction on the degree of A shows that

$$\sum_i c_i = 0.$$

Thus, we have

$$\sum_{n=1}^{\infty}{}' \frac{A(n)}{B(n)} = \sum_i c_i \sum_{n=1}^{\infty} \left(\frac{1}{n+\alpha_i} - \frac{1}{n} \right).$$

The inner sum on the right hand side is by (8)

$$-\psi(\alpha_i) - \gamma - \frac{1}{\alpha_i}$$

from which the result is now immediate. For the second assertion, we can express the sum as

$$\beta - \sum_i c_i \left(\psi(\alpha_i) + \gamma\right)$$

where $\beta \in K$ and the α_i are rational numbers. By (2), we may assume that the α_i are between 0 and 1 without any loss of generality. Now we apply Lemma 21 and Lemma 12 to obtain the result.

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