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Chowla's Problem on the Non-Vanishing of Certain Infinite Series and Related Questions

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In honour of Professor R. P. Bambah on his 80th birthday

1. Chowla's problem

In 1969, Chowla raised the following question:

Suppose $f(n)$ is a rational valued, periodic function mod q where $q(>1)$ is a prime number. Then does the infinite series

$$S = \sum_{n=1}^{\infty} \frac{f(n)}{n} \tag{1.1}$$

never vanish?

Throughout the paper, we assume that f is a non-vanishing number theoretic function. Much earlier, in 1949, Chowla [C] himself showed that

$$S \neq 0$$

if f is an odd function and q prime with $\frac{q-1}{2}$ also prime.

Around 1970, Siegel removed the restriction that $\frac{q-1}{2}$ is prime in the result of Chowla. In 1973, Baker, Birch and Wirsing [BBW] solved Chowla's question completely. In fact, Chowla himself solved the question around the same time. The result of Baker, Birch and Wirsing is more general which we state below.

Theorem A ([BBW]). *Suppose $f(n)$ is an algebraic valued, periodic function mod q. Then the infinite sum S defined in* (1.1) *does not vanish if f satisfies the following conditions.*

(i) $f(r) = 0$ *if* $1 < \gcd(r, q) < q$
(ii) *The cyclotomic polynomial Φ_q is irreducible over* $\mathbb{Q}(f(1), \ldots, f(q))$.

The affirmative answer to Chowla's question is immediate since (i) is vacuous when q is prime and (ii) follows when f is rational valued as it is well known that Φ_q is irreducible over $\mathbb{Q}$. It was shown in [BBW] that conditions (i) and (ii) are necessary. For example, let $q = p^2$ where p is prime and let f be defined by

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = (1 - p^{1-s})^2 \zeta(s)$$

where $\zeta(s)$ is the Riemann Zeta function. For $p = 2$, we get

$$1 - \frac{3}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{3}{6} + \frac{1}{7} + \frac{1}{8} + \cdots = 0$$

with period 4. This shows that (i) is necessary. Let χ, χ' be quadratic characters mod 12 with conductors 3 and 4, respectively, i.e.,

$$\chi(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod 3 \\ -1 & \text{if } n \equiv 2 \pmod 3 \\ 0 & \text{if } n \equiv 0 \pmod 3 \end{cases}$$

and

$$\chi'(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod 4 \\ -1 & \text{if } n \equiv 3 \pmod 4 \\ 0 & \text{if } n \equiv 0, 2 \pmod 4). \end{cases}$$

Let $f = 2\chi - \sqrt{3}\chi'$. Then $S = 0$ since

$$L(1, \chi) = \frac{\pi}{2\sqrt{3}}, \ L(1, \chi') = \frac{\pi}{3}.$$

Here $\mathbb{Q}(f(1), \ldots, f(12)) = \mathbb{Q}(\sqrt{3})$ and $\Phi_{12} = 1 - X^2 + X^4 = (X^2 - \sqrt{3}\,X + 1)(X^2 + \sqrt{3}\,X + 1)$. Thus (ii) is also necessary.

2. Transcendental infinite sums

The sum (whenever the series converges) S can be written as a linear form in logarithms of algebraic numbers with algebraic coefficients. More precisely,

$$S = \sum_{n=1}^{\infty} \frac{f(n)}{n} = -\frac{1}{q} \sum_{s=1}^{q-1} \sum_{r=1}^{q} f(r) \xi^{-rs} \log(1 - \xi^s)$$

where $\xi = e^{2\pi i/q}$. See [Le] or [ASST] for a proof. By the famous result of Baker [Ba] on linear forms in logarithms S is either 0 or transcendental. This was observed in [ASST]. From this it follows that

$$L(1, \chi) \text{ is transcendental}$$

for any non-principal character χ mod q. The non-vanishing of any such $L(1, \chi)$ is a basic result in Dirichlet's famous theorem on primes in arithmetic progression. The transcendence of $L(1, \chi)$ for odd characters χ is also well known for long by the class number formula for quadratic fields. In [ASST] several other infinite series were shown to be transcendental. For instance, series like

$$\sum_{n=1}^{\infty} \frac{1}{(3n+1)(3n+2)(3n+3)}, \quad \sum_{n=1}^{\infty} \frac{F_n}{2^n}$$

represent transcendental numbers. Here F_n denotes the n-th Fibonacci number. In the same year when Theorem A appeared, Lehmer investigated series (1) in connection with extended Euler constants. He showed that

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(2n+1)(4n+1)} = \frac{\pi}{3};$$

$$\sum_{n=1}^{\infty} \frac{1}{(6n+1)\ldots(6n+6)} = \frac{1}{4320}(192\log 2 - 81\log 3 - 7\pi\sqrt{3}).$$

Now we know that the above example represents a transcendental number, thanks to the theorem of Baker. Thus the question of non-vanishing of S becomes important. For a survey of these results and other related results, we refer to [ASST], [A], [AS] and [Tij].

3. Values connected with the Gamma function

Let us consider the series

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

where f satisfies the conditions of Theorem A. This can be written as

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \sum_{a=1}^{q} f(a) \left\{ \sum_{n=0}^{\infty} \frac{1}{(qn+a)^s} \right\} = \sum_{a=1}^{q} f(a) \frac{1}{q^s} \left\{ \sum_{n=0}^{\infty} \frac{1}{(n+a/q)^s} \right\}.$$

The last series within { } is the well known Hurwitz-Zeta function and we have

$$\sum_{n=1}^{\infty} \frac{1}{(n+a/q)^s} = \frac{1}{s-1} - \frac{\Gamma'(a/q)}{\Gamma(a/q)} + \text{terms involving } (s-1).$$

Thus

$$S = \sum_{n=1}^{\infty} \frac{f(n)}{n} = lt_{s\to 1} \sum_{a=1}^{q} \frac{f(a)}{q} \left\{ \frac{1}{s-1} - \frac{\Gamma'(a/q)}{\Gamma(a/q)} + \text{terms involving } (s-1) \right\}$$

$$= -\frac{1}{q} \sum_{a=1}^{q} f(a) \frac{\Gamma'(a/q)}{\Gamma(a/q)},$$

since $\sum_{a=1}^{q} f(a) = 0$ is a necessary and sufficient condition for the convergence of S. By the considerations in Section 2, we get that

$$\sum_{a=1}^{q} f(a) \frac{\Gamma'(a/q)}{\Gamma(a/q)}$$

is transcendental.

The above connection of S with values of Γ'/Γ was pointed out by Professor M. Ram Murty during the author's lecture in the conference at Chandigarh. This connection has been noticed by Lehmer [Le] in 1973. He has also connected S with some extended Euler constants. This view point was developed in great detail in [RS].

4. Necessary and Sufficient condition for the sum S to vanish

In 1982, Okada [O] provided a criterion of all functions f for which (ii) of Theorem A holds and $S = 0$. This criterion is a system of $\varphi(q) + \omega(q)$ homogeneous linear equations in $f(1), \dots, f(q)$ with rational coefficients where $\varphi(q)$ is the Euler totient function and $\omega(q)$ is the number of distinct prime divisors of q. This criterion of Okada was used by Tijdeman [Tij] to show that

(a) $S \neq 0$ if f is completely multiplicative
(b) $S \neq 0$ if f is multiplicative and $| f(p^k) | < p - 1$ for every prime p.

In 2003, Saradha and Tijdeman [ST] re-phrased the criterion of Okada so that it is more convenient for application. We give this criterion below. For any integer $n \geq 1$ and any prime p, let $v_p(n)$ denote the exact power of p dividing n.

Theorem B ([ST]). *Let Φ_q be irreducible over $\mathbb{Q}(f(1), \dots, f(q))$. Let M be the set of positive integers which are composed of prime factors of q and let*

$$\varepsilon(r, p) = \begin{cases} v_p(q) + \frac{1}{p-1} & \text{if } v_p(r) \geq v_p(q) \\ v_p(r) & \text{otherwise.} \end{cases}$$

Then $S = 0$ if and only if

$$\sum_{m \in M} \frac{f(am)}{m} = 0 \text{ for every } a \text{ with } 0 < a < q, \ \gcd(a, q) = 1 \tag{4.1}$$

and

$$\sum_{r=1}^{q} f(r)\varepsilon(r, p) = 0 \text{ for every prime divisor } p \text{ of } q \tag{4.2}$$

Applying Theorem B, explicit conditions were given in [ST] under which series like

$$\sum_{n=0}^{\infty} \frac{(-1)^n(an + b)}{(qn + b_1)(qn + b_2)} \text{ or } \sum_{n=0}^{\infty} \frac{an + b}{(qn + s_1)(qn + s_2)(qn + s_3)}$$

do not vanish. Here a, b are algebraic numbers, b_1, b_2, s_1, s_2 and s_3 are integers.

In his thesis Tengely [Te] has shown that condition (4.1) in Theorem B can be written as a finite sum

$$\sum_{r|q} \varphi\left(\frac{q}{r}\right) f(r) = 0$$

under the additional hypothesis

$$f(m) = f(n) \text{ whenever } v_p(m) = v_p(n) \text{ for every prime } p|q. \tag{4.3}$$

5. Erdős problem

The following question is one of the myriad problems posed by Erdős (see [A]):

Suppose f is a periodic number theoretic function with period q such that

$$f(n) = \begin{cases} \pm 1 & \text{if } n \not\equiv 0 (\text{mod } q) \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

Then does the series S vanish?

The condition $f(q) = 0$ in Erdős problem is necessary as shown by Tengely recently. After a computer search he has shown that $q = 36$ is the least integer for which $S = 0$ when $f(n)$ for $1 \le n \le 36$ takes the following sequence of values

$$\{1, -1, -1, -1, -1, 1, 1, 1, -1, 1, -1, -1, 1, -1, 1, -1, -1, 1, 1, 1, -1, 1, -1,$$
$$-1, 1, -1, -1, -1, -1, 1, 1, 1, 1, 1, -1, 1 \dots\}$$

Okada [O] used his criterion mentioned in Section 4 to show that if f satisfies (5.1), then $S \neq 0$ whenever

$$2\varphi(q) \ge q. \tag{5.2}$$

Thus $S \neq 0$ whenever $\omega(q) \le 2$. In [S], condition (5.2) was relaxed as

$$2\varphi(q) \ge q(1 - \frac{1}{h}) \tag{5.3}$$

where $h = \max_{1 \le i \le r}(p_i^{\alpha_i})$ with $q = p_1^{\alpha_1} \dots p_r^{\alpha_r}$. This relaxed condition enabled us to cover more values of q for which $S \neq 0$. For example, it was shown that

$$S \neq 0 \text{ whenever } q \le 1154 \text{ and } q \neq 525,\ 735,\ 945.$$

The condition (5.3) was used to show that all $f(a)$'s with $\gcd(a, q) = 1$ take the same value. In fact, this holds if (4.3) is assumed.

Thus we show

Theorem 5.1. *Suppose f is a periodic number theoretic function with period q such that* (5.1) *holds. Let $q = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$. Assume further that*

$$f(a) = f(b) \textit{ whenever } \gcd(a, q) = \gcd(b, q) = 1. \tag{5.4}$$

Then $S \neq 0$ *provided*

$$2\varphi(q) > q\left(1 - \frac{1}{h_0}\right)$$

where $h_0 = \min_{1 \le i \le r} p_i^{\alpha_i}$.

Proof. Suppose $S = 0$. By Theorem B, (4.2) is valid. By the convergence of S, we have

$$\sum_{h=1}^{q} f(h) = 0.$$

Thus

$$\sum_{\substack{h=1\\ \gcd(h,q)=1}}^{q} f(h) = -\sum_{\substack{h=1\\ \gcd(h,q)>1}}^{q} f(h) \tag{5.5}$$

By the assumption (5.4)

$$\mid L.H.S \mid = \varphi(q). \tag{5.6}$$

Let i be such that $p_i^{\alpha_i} = h_0$. By (4.2), we get

$$\begin{aligned} &\sum_{p_i \| h} f(h) + 2\sum_{p_i^2 \| h} f(h) + \cdots + (\alpha_i - 1)\sum_{p_i^{\alpha_i - 1} \| h} f(h) \\ &+ \left(\alpha_i + \frac{1}{p_1 - 1}\right)\sum_{p_i^{\alpha_i} | h} f(h) = 0. \end{aligned} \tag{5.7}$$

Further if $p_i \parallel q$, then $\sum_{p_i|h} f(h) = 0$. From (5.5), (5.6) and (5.7), we get

$$\begin{aligned} \varphi(q) = \Bigg| &- \sum_{\substack{h=1\\ gcd(h,q)>1\\ p_i \nmid h}}^{q} f(h) - \delta_i \Bigg(- \sum_{p_i^2 \| h} f(h) - \cdots - (\alpha_i - 2) \\ &\times \sum_{p_i^{\alpha_i - 1} \| h} f(h) - \left(\alpha_i - 1 + \frac{1}{p_i - 1}\right)\sum_{p_i^{\alpha_i} | h} f(h)\Bigg)\Bigg| \end{aligned}$$

where

$$\delta_i = \begin{cases} 0 & \text{if } p_i \parallel q \\ 1 & \text{otherwise.} \end{cases}$$

Thus

$$\varphi(q) \le q - \varphi(q) - \frac{q}{p_i} + \delta_i \left(\frac{q}{p_i^2} - \frac{q}{p_i^3} + 2\left(\frac{q}{p_i^3} - \frac{q}{p_i^4} \right) + \cdots \right.$$
$$\left. + (\alpha_i - 2)\left(\frac{q}{p_i^{\alpha_i - 1}} - \frac{q}{p_i^{\alpha_i}} \right) + \left(\alpha_i - 1 + \frac{1}{p_i - 1} \right) \frac{q}{p_i^{\alpha_i}} \right)$$
$$= q - \varphi(q) - \frac{q}{p_i} + \delta_i \left(\frac{q}{p_i^2} + \frac{q}{p_i^3} + \cdots + \frac{q}{p_i^{\alpha_i}} + \frac{1}{p_i - 1} \frac{q}{p_i^{\alpha_i}} \right)$$
$$= q - \varphi(q) - \frac{q}{p_i} + \delta_i \left(\frac{q(p_i^{\alpha_i - 1} - 1)}{p_i^{\alpha_i}(p_i - 1)} + \frac{q}{p_i^{\alpha_i}(p_i - 1)} \right)$$
$$= q - \varphi(q) - \frac{q}{p_i} + \delta_i \left(\frac{q}{p_i(p_i - 1)} \right).$$

Hence

$$2\varphi(q) \le q \left(1 - \frac{1}{p_i} + \frac{\delta_i}{p_i(p_i - 1)} \right).$$

If $\delta_i = 0$, we have

$$2 \le \frac{q}{\varphi(q)} \left(1 - \frac{1}{p_i} \right) < 2,$$

a contradiction. Let $\delta_i = 1$. Then $\alpha_i > 1$ and

$$2 \le \frac{2p_i^{\alpha_i}}{p_i - 1} \left(1 - \frac{1}{p_i} + \frac{1}{p_i(p_i - 1)} \right)$$

implying

$$p_i^{\alpha_i - 1} - 1 \le \frac{p_i^{\alpha_1 - 1}}{p_i - 1}$$

which is impossible. □

As a consequence of Theorem 5.1, we get the following corollary.

Corollary 5.1. Suppose f satisfies the conditions of Theorem 5.1. Then $S \neq 0$ for all $q < 17325$. Further suppose $\omega(q) = 3$. Then $S \neq 0$ except possibly when q is of the form

$$3^{\alpha_1} 5^{\alpha_2} 7^{\alpha_3} \text{ with } \alpha_1 \ge 3, \alpha_2 \ge 2, \alpha_3 \ge 2 \text{ or}$$
$$3^{\alpha_1} 5^{\alpha_2} 11^{\alpha_3} \text{ with } \alpha_1 \ge 4, \alpha_2 \ge 3, \alpha_3 \ge 2 \text{ or}$$
$$3^{\alpha_1} 5^{\alpha_2} 13^{\alpha_3} \text{ with } \alpha_1 \ge 4, \alpha_2 \ge 3, \alpha_3 \ge 2.$$

Proof. We have

$$\frac{\varphi(q)}{q} = \left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_r}\right).$$

We check that for $q < 17325$,

$$\frac{\varphi(q)}{q} \geq \frac{1}{2}\left(1 - \frac{1}{h_0}\right). \tag{5.8}$$

Thus by Theorem 5.1, $S \neq 0$ for $q < 17325$. Suppose $\omega(q) = 3$. We may assume by Okada's result, that $\varphi(q)/q \leq 1/2$. Hence we find that

$$(p_1, p_2, p_3) \in \{(3, 5, 7), (3, 5, 11), (3, 5, 13)\}.$$

Let $(p_1, p_2, p_3) = (3, 5, 7)$. Then from (5.8), we get

$$\frac{48}{105} \geq \frac{1}{2}\left(1 - \frac{1}{h_0}\right)$$

which gives $h_0 \leq 11$. Thus $S \neq 0$, whenever

$$q \in \{3 \cdot 5^{\alpha_2} 7^{\alpha_3}, 3^{\alpha_1} 5 \cdot 7^{\alpha_3}, 3^{\alpha_1} 5^{\alpha_2} 7, 3^2 5^{\alpha_2} 7^{\alpha_3}\}.$$

When $(p_1, p_2, p_3) = (3, 5, 11)$ or $(3, 5, 13)$, we get $h_0 \leq 33$ and 65, respectively. This yields the possibilities for q as mentioned in the corollary for which S may vanish.

□

It will be desirable to obtain the improvement of Theorem 5.1 without the assumption (5.4).

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