ON GENERALIZATIONS OF PROBLEMS OF RECAMAN AND POMERANCE

L. HAJDU AND N. SARADHA

Abstract. Answering a question of Balasubramanian, we find all primes $p$ for which there exist $p$ consecutive primes forming a complete residue system (mod $p$). On the other hand, under the prime $\ell$-tuple conjecture we show that for any $k \geq 2$, there exist infinitely many sets of $\varphi(k)$ consecutive primes forming reduced residue classes (mod $k$). The problems considered are generalizations of those of Recaman and Pomerance, respectively.

1. Introduction

Let $2 = p_1 < p_2 < \cdots$ denote the sequence of all primes. Let $k$ and $l$ be positive integers with gcd$(k, l) = 1$. Denote by $p(k, l)$ the least prime $p \equiv l$ (mod $k$). We write $P(k)$ for the maximal value of $p(k, l)$ for all $l$.

A prime $p$ is called a Recaman prime, if the first $p$ primes form a complete residue system (mod $p$). Pomerance [11] showed that there are only finitely many Recaman primes. Recently, Hajdu and Saradha [4] proved that the only Recaman prime is $p = 2$. An integer $k \geq 2$ is called a $P$-integer, if the first $\varphi(k)$ primes coprime to $k$ form a reduced residue system (mod $k$). Pomerance [11] proved that there exist only finitely many $P$-integers. Under certain conditions, Hajdu and Saradha [4] and [13] determined all $P$-integers. Hajdu, Saradha and Tijdeman [5] proved that if $k$ is a $P$ integer, then $k \leq 10^{3500}$, and that if the Riemann Hypothesis is true, then the only $P$-integers are given by $k = 2, 4, 6, 12, 18, 30$. Finally, this was unconditionally verified by Yang and Togbé [14].

After the talk of the first author in the DMANT 2015 meeting, Balasubramanian proposed the variation of the above problems where the first $k$ (resp. $\varphi(k)$) primes are replaced by any block of $k$ (resp. $\varphi(k)$) consecutive primes.

2010 Mathematics Subject Classification. 11N13.

Key words and phrases. Recaman’s problem, Pomerance’s problem, primes in residue classes.

Research supported in part by the OTKA grants K100339 and NK101680.
To be more precise, we introduce some new definitions. An integer $k$ is called a $B$-prime if there exist $k$ consecutive primes forming a complete residue system (mod $k$). Further, an integer $k$ is called a $B$-integer, if there exist $\varphi(k)$ consecutive primes forming a reduced residue system (mod $k$).

Note that the Recaman prime 2 is a $B$-prime also. Further the $P$-integers 2, 4, 6, 12, 18, 30 are also $B$-integers. When a prime $k$ is a $B$-prime, we have

$$P(k) \leq p_{\pi(k)+k-1}.$$  

From well known estimates in Prime Number Theory, it is clear that $p_{\pi(k)+k-1} \ll k \log k$. In fact, the implicit constant lies between 1 and 1.04 for $k \geq 10^{38}$. This leads us to make a more general definition as follows. We say that a prime $k$ is a shifted $P_{\alpha}$-prime if there exist $k$ primes not exceeding $\alpha k \log k$ forming a complete residue system. Finally, an integer $k$ is called a shifted $P_{\alpha}$-integer if there exist $\varphi(k)$ primes not exceeding $\alpha k \log k$ forming a reduced residue system (mod $k$).

In this paper, we show that the only $B$-primes are 2, 3, 7 and there is no shifted $P_{\alpha}$-prime with $\alpha = 1.1954$. Pomerance [11, Theorem 2] showed that if $k$ is any positive integer, then

$$P(k) \geq (e^{\gamma} + o(1))\varphi(k) \log k$$

where $\varphi$ denotes the Euler totient function, and $\gamma = 0.577\ldots$ is Euler’s constant. In particular when $k$ is a prime, this gives

$$P(k) \geq (e^{\gamma} + o(1))k \log k.$$ 

Here the implied constant is not explicit and may be very small. By Theorem 2.2 below, we see that

$$P(k) > 1.1954k \log k$$

for all primes $k$. It appears that one needs to take $k > 10^{10^{10}}$, in order to get

$$P(k) \geq e^{\gamma}k \log k$$

by the method in this paper.

Finding upper bound for $P(k)$ is a well known problem. Linnik [8] showed that

$$P(k) \leq ck^L$$

where $c$ and $L$ are effectively computable constants. There is a huge literature on finding the best constant $L$.

In 1992, Heath-Brown [6] had shown that $L$ can be taken as 5.5. This has been improved to 5 by Xylouris [16] (see Theorem 2.1, p. 12)
in 2011. A conjecture of Chowla [1] says that $L$ is $1 + \epsilon$ for arbitrary $\epsilon > 0$. Observe that as $\alpha$ increases, the set of shifted $P_\alpha$-primes (or integers) becomes larger and larger. Under Chowla’s conjecture, we see that $\alpha$ (as a function of $k$) must be of the order $k^\epsilon$ so that all primes (or integers) $k$ may become shifted $P_\alpha$-primes (or integers). On the other hand, if $k$ is a $B$-integer, then we need to find $\varphi(k)$ consecutive primes coprime to $k$. Assuming the prime $\ell$-tuple conjecture of Hardy and Littlewood, we deduce that every integer $k$ is a $B$-integer, and in fact one can choose appropriate blocks of $\varphi(k)$ consecutive primes in infinitely many ways. We note that for $k = 2, 3, 4, 6$ this assertion easily follows unconditionally.

2. Results

**Theorem 2.1.** The only $B$-primes are given by $2, 3, 7$.

**Theorem 2.2.** There is no shifted $P_\alpha$-prime with $\alpha = 1.1954$.

The above two results are contained in the following theorem.

**Theorem 2.3.** Let $k$ be a prime with the property that there exist $k$ primes not exceeding $\max(p_{\pi(k)} + k, 1.1954k \log k)$ which form a complete residue system. Then $k \in \{2, 3, 7, 11\}$.

To get the assertions of Theorems 2.1 and 2.2 we first deduce that

$$\max(p_{\pi(k)} + k, 1.1954k \log k) = \begin{cases} p_{\pi(k)} + k, & \text{if } k < 6691068 \\ 1.1954k \log k, & \text{otherwise}. \end{cases}$$

Further we find that $2, 3, 7$ are $B$-primes since

$$\{2, 3\}, \{3, 5, 7\}, \{7, 11, 13, 17, 19, 23, 29\}$$

form complete residue systems, respectively. Also $2, 3, 7$ are not shifted $P_\alpha$-primes with $\alpha = 1.1954$ since $\pi(1.1954k \log(k)) < k$ in these cases. Further, $11$ is not a $B$-integer, since no set of $11$ consecutive primes forms a complete residue system (mod 11).

Using the argument in the proof of [4, Theorem 2], one may obtain the following result which we state without proof.

Let $\alpha$ be a fixed positive number. Suppose $k$ is a shifted $P_\alpha$-integer with the least prime factor of $k$ exceeding $\log(k)$. Then there exists an effectively computable number $c(\alpha)$ depending only on $\alpha$ such that $k < c(\alpha)$.

The above result leads us to speculate if there are only finitely many $B$-integers. We show below that the contrary is true under the prime $\ell$-tuple conjecture of Hardy and Littlewood. In fact, assuming the
conjecture we deduce that every integer \( k \) is a \( B \)-integer, and one can choose appropriate blocks of \( \varphi(k) \) consecutive primes in infinitely many ways. We note that for \( k = 2, 3, 4, 6 \) this assertion easily follows unconditionally.

Before formulating our next theorem, we recall the prime \( \ell \)-tuple conjecture. A finite set \( A \) of integers is called admissible, if for any prime \( p \), no subset of \( A \) forms a complete residue system \((\text{mod } p)\).

**Conjecture 2.1** (The prime \( \ell \)-tuple conjecture).

Let \( \{a_1, \ldots, a_\ell\} \) be an admissible set of integers. Then there exist infinitely many positive integers \( n \) such that \( n + a_1, \ldots, n + a_\ell \) are all primes.

**Remark.** By a recent, deep result of Maynard [9] we know that for each \( \ell \), the above conjecture holds for a positive proportion of admissible \( \ell \)-tuples.

**Theorem 2.4.** Suppose that the prime \( \ell \)-tuple conjecture is true. Then for every integer \( k \geq 2 \) one can find infinitely many sets of \( \varphi(k) \) consecutive primes forming a reduced residue system \((\text{mod } k)\).

**Remark.** In fact, in the proof of Theorem 2.4 we need the numbers \( n + a_1, \ldots, n + a_\ell \) occurring in the prime \( \ell \)-tuple conjecture to be consecutive primes. In case of \( \ell = 2 \), by deep and celebrated results of Zhang [17] and Pintz [10] we know this to be true for infinitely many admissible sets \( \{a_1, a_2\} \), even with \( a_1 = 0 \). In case of general \( \ell \), such a variant is known to follow from the following quantitative version of the prime \( \ell \)-tuple conjecture, also made by Hardy and Littlewood. Let \( A_0 = \{a_1, \ldots, a_\ell\} \) be an admissible set with \( a_1 < a_2 < \cdots < a_\ell \). Put

\[
I_0 = \{n \in \mathbb{N} : a_1 \leq n \leq a_\ell\} \quad \text{and} \quad A'_0 = I_0 \setminus A_0.
\]

For every prime \( p \) let \( v_p \) be the number of residue classes \((\text{mod } p)\) met by \( A_0 \). Clearly, for all \( p \) we have \( 1 \leq v_p \leq p - 1 \). Put

\[
\delta_{A_0} := \prod_{p \text{ prime}} \frac{1 - \frac{v_p}{p}}{1 - \frac{1}{p}}.
\]

Note that here the product on the right hand side is convergent for any admissible set. Further if \( A_0 \subseteq B \), then \( \delta_{A_0} \geq \delta_B \). Let

\[
S = \{n \in \mathbb{N} : n + a_1, \cdots, n + a_\ell \text{ are all primes}\}
\]

and

\[
S(X) = \{n \in S : n \leq X\}.
\]
Then the quantitative version of the prime \(\ell\)-tuple conjecture of Hardy and Littlewood asserts that
\[
|S(X)| = (\delta_{A_0} + o(1)) \frac{X}{(\log X)^\ell}.
\]
Now we explain how this implies that there are infinitely many integers \(n\) for which \(n + a_1, \ldots, n + a_\ell\) are all consecutive primes. Let
\[
S_1 = \{n \in S : n + a_1, \ldots, n + a_\ell \text{ are not consecutive primes}\}.
\]
It is enough to show that
\[
|S_1(X)| = o\left(\frac{X}{(\log X)^\ell}\right).
\]
If \(n \in S_1\), then there exists \(a \in A'_0\) such that \(n + a\) is prime. Also \(A'^{(a)}_0 := A_0 \cup \{a\}\) is an admissible set. For \(a \in A'_0\), let
\[
S'^{(a)}_1 = \{n \in S_1 : n + a_1, \ldots, n + a_\ell, n + a \text{ are all primes}\}.
\]
Then
\[
S_1 = \bigcup_{a \in A'_0} S'^{(a)}_1.
\]
Thus
\[
|S_1(X)| \leq \sum_{a \in A'_0} (\delta_{A'^{(a)}_0} + o(1)) \frac{X}{(\log X)^{\ell+1}}
\leq (\delta_A + o(1))(a_\ell - a_1) \frac{X}{(\log X)^{\ell+1}} = o\left(\frac{X}{(\log X)^\ell}\right)
\]
for \(X \to \infty\) as desired. However, in the proof of Theorem 2.4 we avoid the use of the quantitative version of the conjecture. In fact, we apply an elementary argument showing that the prime \(\ell\)-tuple conjecture itself implies the existence of infinitely many \(n\) such that the numbers \(n + a_1, \ldots, n + a_\ell\) are consecutive primes.

As a simple corollary of Theorem 2.4, we obtain

**Corollary 2.1.** Suppose that the prime \(\ell\)-tuple conjecture is true. Then every integer \(k \geq 2\) is a \(B\)-integer.

**Remark.** It is obvious that 2 is a \(B\)-integer. Since for \(k = 3, 4, 6\) there are only two coprime residue classes, and both classes contain infinitely many primes, there must be infinitely many “switches” between these classes in pairs of consecutive primes. Hence \(k = 3, 4, 6\) are (unconditionally) also \(B\)-integers.

In view of the above remarks and theorems, we propose the following

**Conjecture 2.2.** Every integer \(k \geq 2\) is a \(B\)-integer.
3. Lemmas

The proof of Theorem 2.3 follows similar line of arguments as the proof of [4, Theorem 2]. We record here three lemmas necessary for the proof. The first lemma is from Rosser and Schoenfeld [12].

Lemma 3.1. Let \( p_n \) denote the \( n \)-th prime. Then

1. \( p_n > n(\log(n) + \log_2(n) - \frac{3}{2}) \) for \( n > 1 \);
2. \( p_n < n(\log(n) + \log_2(n)) \) for \( n \geq 6 \).

Here and henceforth, \( \log_2(n) \) denotes \( \log \log(n) \) for any real number \( n > 1 \). For \( n \geq 1 \) the Jacobsthal function \( g(n) \) is defined as the smallest integer such that any sequence of \( g(n) \) consecutive integers contains an element which is coprime to \( n \). This function has been studied by many authors, and good lower as well as upper bounds are known (see e.g. [7], [15], [11], [3] and [2] for some results and history). Further, the exact values of \( g(n) \) when \( n \) is the product of the first \( h < 50 \) primes is given in [3, Table 1].

It was observed by Jacobsthal that for integers \( k \) with \( \ell(k) > \log(k) \) we have \( g(k) = \omega(k) + 1 \) where \( \ell(k) \) is the least prime divisor of \( k \), and \( \omega(k) \) is the number of distinct prime divisors of \( k \). In particular this is true if \( k \) is a prime i.e., \( g(k) = 2 \) in this case. Further, \( g(k) \geq \omega(k) + 1 \) is obviously valid for any \( k \). We shall use these assertions throughout the paper without any further reference. The following lemma is Proposition 1.1 of Hagedorn [3].

Lemma 3.2. We have

\[
g\left( \prod_{i=1}^{h} p_i \right) \geq 2p_{h-1} \quad \text{for } h > 2.
\]

The next result due to Pomerance [11] is an important ingredient in this problem.

Lemma 3.3. Let \( k \) and \( m \) be integers with \( 0 < m \leq \frac{k}{1+g(k)} \) and \( \gcd(m, k) = 1 \). Then \( P(k) > (g(m) - 1)k \).

4. Proofs

Proof of Theorem 2.3. We restrict to \( k \) prime so that \( g(k) = 2 \). First take \( k \geq 10^{39} \). By (2),

\[
\max(p_{\sigma(k)+k-1}, 1.1954k \log k) = 1.1954k \log k.
\]

Put

\[
h = \left\lfloor \frac{0.9688 \log(k)}{\log_2(k)} \right\rfloor + 1.
\]
Then
\[ h < \frac{0.9946 \log(k)}{\log_2(k)} \]
giving
\[ \log(h) < \log_2(k) - \log_3(k) \quad \text{and} \quad \log_2(h) < \log_3(k). \]
This by Lemma 3.1 (ii) implies
\[ p_h < 0.9946 \log(k) < \log(k). \]
Let \( m \) be the product of the first \( h \) primes coprime to \( k \). Since \( p_h < \log(k) < k \), we see that \( m \) is indeed the product of all the first \( h \) primes. Hence
\[ m < p^h < e^{0.9946 \log(k)} < \frac{k}{3}. \]
Thus by Lemmas 3.2 and 3.3, we have
\[ P(k) > (g(m) - 1)k \geq (2p_{h-1} - 1)k. \]
Now
\[ h - 1 \geq 0.9688 \frac{\log(k)}{\log_2(k)} - 1 > 0.943 \frac{\log(k)}{\log_2(k)}. \]
Hence by Lemma 3.1 (i)
\[ p_{h-1} \geq X \left( \log(X) + \log_2(X) - \frac{3}{2} \right) \]
where \( X = 0.943 \frac{\log(k)}{\log_2(k)} \). Let
\[ F(k) = 2X \left( \log(X) + \log_2(X) - \frac{3}{2} - \frac{1}{2X} \right) k - 1.1954k \log(k). \]
Then \( F(k) = k \log(k) f(k) \) with
\[ f(k) := \frac{1.886}{\log_2(k)} \left( \log(X) + \log_2(X) - \frac{3}{2} - \frac{1}{2X} \right) - 1.1954. \]
Observe that \( f(k) \) is an increasing function of \( k \) and hence \( f(k) \geq f(10^{93}) \), since \( k \geq 10^{93} \). As \( f(10^{93}) \geq 0.0005 \), we find that \( F(k) > 0 \) which implies that \( P(k) > 1.1954k \log(k) \). Hence \( k \) is not a \( P_\alpha \)-prime with \( \alpha = 1.1954 \). This proves the theorem for \( k \geq 10^{93} \).

Next consider \( 6691068 \leq k < 10^{93} \). By (2),
\[ \max(P_{\pi(k)+k-1}, 1.1954k \log k) = 1.1954k \log(k). \]
Suppose \( k \in [10^{43}, 10^{93}) \). The largest integer \( h \) such that \( p_h < \log(10^{43}) \) is 25. Taking
\[ m = \prod_{j=1}^{25} p_j, \]
we find that \( \gcd(m, k) = 1 \) and 
\[
m < \frac{10^{43}}{3} \leq \frac{k}{g(k)+1}.
\]
From [3, Table 1], \( g(m) = 258 \). Hence by Lemma 3.3, 
\[
P(k) > 257k > 1.1954 \times 93 \log(10)k > 1.1954 \times k \log(k).
\]
This proves the proposition for \( k \in [10^{43}, 10^{93}] \). Let \( k \in [10^a, 10^b] \). In Table 4, we give the values of \((a, b), h, g(m)\) from [3, Table 1] where \( m = \prod_{i=1}^h p_i \) so that 
\[
p_h < \log(10^a), P(k) > 1.1954k \log(k).
\]
Then the assertion of the theorem follows for \( k \) in this interval. Thus

<table>
<thead>
<tr>
<th>( h )</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>11</th>
<th>14</th>
<th>18</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g(m) )</td>
<td>26</td>
<td>34</td>
<td>40</td>
<td>58</td>
<td>90</td>
<td>132</td>
</tr>
<tr>
<td>((a, b))</td>
<td>(8,9)</td>
<td>(9,10)</td>
<td>(10,14)</td>
<td>(14,19)</td>
<td>(19,27)</td>
<td>(27,43)</td>
</tr>
</tbody>
</table>

Table 1

we conclude that \( k < 10^8 \). Further, we take \( k \in [6691068, 10^8] \) with \( h = 7, g(m) = 26 \) to get the assertion of the theorem.

Next, we take \( 90107 \leq k < 6691068 \). In this case, we find that 
\[
p_{\pi(k)+k-1} < 1.25k \log(k).
\]
Then we take \( h = 6, g(m) = 22 \) to exclude these values of \( k \) by Lemma 3.3.

Thus \( k < 90107 \). For these values of \( k \) we give a computational argument. Let \( k \) be fixed. Suppose \( S_k \) denotes the set of residues mod \( k \) of all the primes upto \( p_{\pi(k)+k-1} \). If 
\[
|S_k| = k
\]
then, \( k \) may be a \( B \)-prime. We check that (3) is valid only for \( k = 2, 3, 7, 11 \). Further 11 is not a \( B \)-prime as there is no set of 11 consecutive primes among the first 15 primes which yields a complete residue system. On the other hand, 2,3,7 give consecutive primes forming a complete residue system as mentioned in Section 2. This proves the theorem.

\( \square \)

**Proof of Theorem 2.4.** Let \( k \geq 2 \) be an arbitrary integer. We shall show that under the prime \( \ell \)-tuple conjecture, \( k \) is a \( B \)-integer i.e., there exists \( \varphi(k) \) consecutive primes forming a reduced residue system mod \( k \). Let \( A = \{a_1, \ldots, a_{\varphi(k)}\} \) be the set of all positive integers coprime to \( k \) with 
\[
1 = a_1 < \cdots < a_{\varphi(k)} < k.
\]
The set \( A \) may not be an admissible set. We construct an admissible set out of \( A \) as follows. Put
\[
P = \prod_{p \text{ prime } p \mid k, p \leq \varphi(k)} p.
\]

Let \( B = \{b_1, \ldots, b_{\varphi(k)}\} \) be a set of positive integers such that
\[
(4) \quad b_1 = a_1 = 1; \quad b_i \equiv a_i \pmod{k} \quad \text{and} \quad b_i \equiv 1 \pmod{P} \quad (\text{for } i \geq 2).
\]

Firstly, note that such \( b_i \)'s exist by the Chinese Remainder Theorem. Next we show that \( B \) is an admissible set. Since \(|B| = \varphi(k)\) and \( B \) contains integers coprime to \( k \), it is enough to restrict to primes \( p \leq \varphi(k) \) and \( p \nmid k \). Then by (4), every \( b_i \equiv 1 \pmod{p} \), hence \( B \) cannot have a complete residue system \( \pmod{p} \). By applying the prime \( \ell \)-tuple conjecture to \( B \), we find infinitely many \( n > k \) for which
\[
n + b_1, \ldots, n + b_{\varphi(k)}
\]
are all primes and hence coprime to \( k \). But these primes \emph{may not be consecutive primes}. To ensure this, we proceed as follows. Let
\[
M = \max b \quad \text{and} \quad I \text{ the set of positive integers } n \text{ with } n \leq M.
\]

Further let
\[
C = \{c \in I \setminus B : B \cup \{c\} \text{ is admissible}\}.
\]

Let \( t = |C| \) and write \( C' = I \setminus (B \cup C) \). Thus for \( c' \in C' \), \( B \cup \{c'\} \) is not an admissible set. Hence there exists a prime \( p \leq M \) such that \( B \cup \{c'\} \) has a complete residue system \( \pmod{p} \).

Note that \( M > k \) by (4). We construct an admissible set \( S \supseteq B \), such that \( S \cup \{c\} \) is not admissible for any \( c \in C \). If \( t = 0 \) then take \( S = B \). If \( t \geq 1 \), take primes \( q_1 < \cdots < q_t \) exceeding \( M \) and put
\[
Q = \prod_{p < q_1 + \cdots + q_t} p.
\]

Let us enumerate the elements of \( C \) as \( c_1, \ldots, c_t \). Corresponding to each \( c_i \), we construct a set \( D^{(i)} \) as follows. Let \( d^{(i)}_1 \) satisfy
\[
d^{(i)}_1 > M, d^{(i)}_1 \equiv 1 \pmod{Q} \quad \text{and} \quad d^{(i)}_1 \pmod{q_i} \notin B \cup \{c_i\}.
\]

Since \( B \cup \{c_i\} \) is an admissible set, it is possible to find \( d^{(i)}_1 \) as above. Now consider
\[
B \cup \{c_i\} \cup \{d^{(i)}_1\}.
\]
If this has a complete residue system \((\text{mod } q_i)\), then put \(D^{(i)} = \{d^{(i)}_1\}\).

If not, we choose
\[
d^{(i)}_2 > M, \quad d^{(i)}_1 \equiv 1 \left( \mod \frac{Q}{q_i} \right) \quad \text{and} \quad d^{(i)}_2 \equiv 1 \left( \mod q_i \right) \notin B \cup \{c_i\} \cup \{d^{(i)}_1 \left( \mod q_i \right)\}.
\]

If \(B \cup \{c_i\} \cup \{d^{(i)}_1, d^{(i)}_2\}\) has a complete residue system \((\text{mod } q_i)\), take \(D^{(i)} = \{d^{(i)}_1, d^{(i)}_2\}\). Otherwise, we proceed to find \(d^{(i)}_3\) and so on. This process has at most \(q_i - 1 - \varphi(k)\) steps. Thus \(D^{(i)}\) has at most \(q_i - 1 - \varphi(k)\) elements with the property that
\[
B \cup \{c_i\} \cup \{d^{(i)}_1, d^{(i)}_2\} \quad \text{has a complete residue system \((\text{mod } q_i)\) and every element of } D^{(i)} \quad \text{exceeds } M.
\]

Take \(S = B \cup D^{(1)} \cup \cdots \cup D^{(t)}\).

We show that \(S\) is an admissible set. Firstly,
\[
|S| \leq \varphi(k) + q_1 + \cdots + q_t - t(\varphi(k) + 1) < q_1 + \cdots + q_t
\]
since \(t \geq 1\). Hence we need to consider only primes \(p < q_1 + \cdots + q_t\).

Let \(p\) be such a prime with \(p \neq q_i\) \((1 \leq i \leq t)\). Then by the definition of \(Q\) and the construction of the sets \(D^{(i)}\), all the elements of \(D^{(i)}\) are \(\equiv 1 \left( \mod p \right)\) and as \(1 \in B\) we get
\[
S \equiv B \left( \mod p \right).
\]

(By the above notation we mean \(\{s \left( \mod p \right) : s \in S\} = \{b \left( \mod p \right) : b \in B\}\)). Since \(B\) is an admissible set, we see that \(S\) cannot have a complete residue system \((\text{mod } p)\). Let now \(p = q_i\) for some \(i\) with \(1 \leq i \leq t\). Then
\[
S \equiv B \cup D^{(i)} \left( \mod p \right).
\]

Since \(c_i \notin B \cup D^{(i)}\), by \(q_i > M\) and the construction of \(D^{(i)}\), the set \(S\) does not contain a complete residue system \((\text{mod } p)\). Thus \(S\) is an admissible set.

Note that for \(1 \leq i \leq t\), \(S \cup \{c_i\}\) is not an admissible set since it contains a complete residue system \((\text{mod } q_i)\). Also for any \(c' \in C'\), \(S \cup \{c'\}\) is not an admissible set since \(B \cup \{c'\}\) is not admissible, by definition and in this case there exists a complete residue system \((\text{mod } p)\) for some \(p \leq M\). Summarizing, \(S\) is an admissible set, but \(S \cup \{c\}\) for \(c \in I \setminus B\) is not an admissible set. Thus for any \(c \in I \setminus B\) there exists a prime \(p_c\) such that \(S \cup \{c\}\) contains a complete residue system \((\text{mod } p_c)\). As seen earlier, \(p_c\) can be taken as not exceeding \(M\) or equal to


For any \( c \in I \setminus B \), there exists a complete residue system (mod \( p_c \)) in \( S \cup \{c\} \) and hence in \( \{n+s, s \in S \cup \{c\}\} \) for any \( n \), and in particular for those \( n \) satisfying (5). Thus \( p_c | (n+s) \) for some \( s \in S \cup \{c\} \). Since \( n+s \) for \( s \in S \) are all primes \( > q_t \), this implies that \( s = c \). That is, \( p_c | n+c \), whence \( n+c \) is not a prime for any \( c \in I \setminus B \). This means that \( n+b \) with \( b \in B \) are \( \varphi(k) \) consecutive primes, all coprime to \( k \). Since by the construction of \( B \) these numbers belong to different residue classes (mod \( k \)), we get that \( k \) is a \( B \)-integer. In fact there are infinitely many sets of \( \varphi(k) \) consecutive primes, coprime to \( k \), belonging to different residue classes (mod \( k \)).

5. Acknowledgements

We are grateful to János Pintz for his guiding remarks concerning the prime \( \ell \)-tuple conjecture and related results. We thank the referee for her/his helpful comments and remarks.

REFERENCES


L. Hajdu

University of Debrecen, Institute of Mathematics
H-4010 Debrecen, P.O. Box 12.
Hungary
E-mail address: hajdul@science.unideb.hu

N. Saradha

School of Mathematics, Tata Institute of Fundamental Research,
Dr. Homibhabha Road, Colaba, Mumbai
India
E-mail address: saradha@math.tifr.res.in