Squares and factorials in products of factorials

F. Luca
Fundación Marcos Moshinsky, UNAM
Circuito Exterior, C.U., Apdo. Postal 70-543
Mexico D.F. 04510, Mexico
fluca@matmor.unam.mx

N. Saradha
Tata Institute of Fundamental Research
School of Mathematics
Homi Bhabha Road, Mumbai-400 005, India
saradha@math.tifr.res.in

T. N. Shorey
Department of Mathematics
Indian Institute of Technology
Powai, Mumbai-400 076, India
shorey@math.iitb.ac.in

Abstract

We study a question of Erdős and Graham on products of factorials being a square or again a factorial.

1 Introduction

For any integer $n > 1$ denote by $P(n)$ and $Q(n)$ the greatest prime factor and the square free part of n and put $P(1) = Q(1) = 1$. Put also $\omega(n)$ for the number of distinct prime factors of n with $\omega(1) = 0$. For any prime p, let $\mathrm{ord}_p(n)$ denote the largest power of p dividing n. For any finite set A of positive integers let

$$m(A) = \prod_{a \in A} a!$$

and $M(A)$ denote the largest integer in A. Further let $|A|$ denote the cardinality of A and $A(X)$, the set of elements of A which do not exceed X. Let $x > 1$ and put $\log_1 x = \max\{\log x, 1\}$ and for $n > 1$, define $\log_n x = \log_1(\log_{n-1} x)$. When $k = 1$ we omit the subscript of $\log_k x$, and thus we understand that all logarithms that appear are defined and larger than or equal to 1.

1.1 A Diophantine Equation

In 1975, Erdős and Selfridge [11] proved a remarkable result that a product of two or more positive integers is never a square or a higher power. Continuing this line of investigation, Erdős and Graham [9] asked if product of two or more disjoint blocks of consecutive integers can be a square or higher power. They studied products of factorials being a square i.e., finding positive integer solutions to the Diophantine equation

$$\prod_{k=1}^{t} a_k! = y^2 \tag{1}$$

in $a_1 \geq a_2 \geq \cdots \geq a_t > 1$ and $y > 1$. By canceling an even number of equal factorials on the left hand side, we may assume that

$$a_1 > a_2 > \cdots > a_t > 1.$$

In [9], the above equation was studied when a_1 is given and t small. If $A = \{a_1, \ldots, a_t\}$, then the left hand side of (1) is $m(A)$ and $M(A) = a_1$. It was shown in [9] that the set of possible values of $m(A)$ is sparse. Hence, it may very well be that (1) has no or very few solutions. Define $F_0 = \emptyset$ and for $k \geq 1$, let

$$F_k = \{n : \text{ there exists } A \text{ with } |A| \leq k,\ M(A) = n \text{ and } m(A) = \square\}$$

and

$$D_k = F_k \setminus F_{k-1}.$$

Suppose the equation (1) has a solution in $a_1, \ldots, a_t$. Then $a_1 \in F_t$. Thus, in order to study the above equation, Erdős and Graham investigated the sets F_k and D_k. They showed that the sets F_k do not contain any prime; $D_1 = F_1 = \{1\}$; F_2 contains all squares while D_2 contains all squares > 1. Further, D_3 contains integers of the form

(i) $a^2 Q(b!),\ a > 1,\ b > 1$, and

(ii) ux^2, where x is a solution of the Pell's equation

$$ux^2 - vy^2 = 1$$

with $uv = Q(a!)$ for some $a > 1$ and $\gcd(u, v) = 1$.

Erdős and Graham predicted that there are perhaps only finitely many elements of D_3 apart from the two classes of integers given in (i) and (ii). This is still *open*. There are examples of the form $a_1!a_2!a_3!$ which are squares with $a_1 = a_2 + 3$. For instance, we have

$$10!7!6!;\ 50!47!3!;\ 50!47!4!$$

which are squares. Erdős and Graham asked if there are other examples. In [6], Dujella, Najman, Saradha and Shorey gave more examples by showing that the equation

$$a_1!a_2!a_3! = y^2 \text{ with } a_1 = a_2 + 3 \text{ and } a_3 \le 100$$

has only the solutions

$$(a_1, a_3) \in \{(10, 6), (50, 3), (50, 4), (324, 26), (352, 13), (442, 18), (2738, 26)\}.$$

Note that the first three solutions in the above list correspond to the three examples given by Erdős and Graham. Thus $10, 50, 324, 352, 442, 2738$ belong to D_3. On the other hand, Erdős and Graham showed that D_3 is very sparse by proving

$$D_3(X) = o(X).$$

In this paper, we improve this result as follows.

Theorem 1. *We have*

$$|D_3(X)| = O\left(\frac{X}{\exp\left(c_0(\log X)^{1/4}(\log_2 X)^{3/4}\right)}\right)$$

for some constant $c_0 > 0$.

1.2 Integers n divisible by $P(n)$ to a large exponent

In order to study the set D_3, Erdős and Graham investigated integers n with $\mathrm{ord}_{P(n)}(n) > 1$. Suppose $p^{(k)}$ denotes the least prime exceeding k. Let $k \geq 3$ and $n + k \geq p^{(k)}$ and

$$\Delta(n,k) = n(n+1)\cdots(n+k-1).$$

In [11], it is proved that for any given $\ell \geq 2$, there exists a prime $p > k$, such that

$$\mathrm{ord}_p(\Delta(n,k)) \not\equiv 0 \pmod{\ell}.$$

In fact, it is *conjectured* therein that if $k \geq 4$ there exists a prime $p > k$, such that

$$\mathrm{ord}_p(\Delta(n,k)) = 1.$$

The assumptions $k \geq 3$ in the above theorem and $k \geq 4$ in the above conjecture are necessary in view of the existence of infinitely many solutions to Pell's equations and of the example

$$\binom{50}{3} = 140^2.$$

Let

$$P(n,k) = P(\Delta(n,k)); O(n,k) = \mathrm{ord}_{P(n,k)}(\Delta(n,k))$$

and

$$S = \{n : \ O(n,k) > 1 \text{ for some } k \geq 1\}.$$

In [9] (see Fact 4, p.343), it is shown that

$$|S(X)| = o(X).$$

We improve this result as

Theorem 2.

$$|S(X)| = O\left(\frac{X}{\exp(c_1(\log X)^{1/4}(\log_2 X)^{3/4})}\right)$$

for some constant $c_1 > 0$.

1.3 The *abc* conjecture

For any $n > 1$, let $N(n)$, the radical of n be defined as $N(n) = \prod_{p|n} p$. Put $N(1) = 1$. The well known *abc* conjecture is the following statement. Suppose a, b and c are pairwise co-prime positive integers satisfying

$$a + b = c.$$

Let $\epsilon > 0$. Then there exists a number κ depending only on ϵ such that

$$c < \kappa N(abc)^{1+\epsilon}. \tag{2}$$

In 1975, Baker [3] proposed an explicit version of the *abc* conjecture. Suppose (2) holds. Then

$$c < \frac{6}{5} N(abc) \frac{(\log N(abc))^\omega}{\omega!}.$$

In [13], Laishram and Shorey showed that Baker's conjecture implies that

$$c < N(abc)^{7/4}. \tag{3}$$

1.4 An application of explicit $abc-$ conjecture

In [14], Luca considered the question of when the left hand side of (1) is another factorial; i.e., the equation

$$a_1!a_2! \cdots a_t! = n! \tag{4}$$

with $n \geq a_1 \geq a_2 \geq \cdots \geq a_t > 1$. He showed under the *abc* conjecture that this equation has only finitely many non-trivial solutions. We may assume that $n - a_1 \geq 2$ otherwise the equation can be solved immediately. In (4), writing $k = n - a_1, m = a_1 + 1$, we get

$$a_2!a_3! \cdots a_t! = m(m + 1) \cdots (m + k - 1) \tag{5}$$

with $m > a_2 \geq a_3 \cdots \geq a_t > 1$. Thus, we have $m \geq 3$ and $k \geq 2$. It is clear that if we bound a_2 then this equation has only finitely many solutions. We shall use (3) to prove the following explicit version of Luca's result. Our proof is different: Mateev's result on linear forms in logarithms in Luca's proof has been replaced by Lemma 7 (ii).

Theorem 3. *Assume that* (5) *holds with some* $m \geq 3, k \geq 2$ *and* $a_2 \geq 2$. *Under the explicit abc conjecture, there exists an absolute constant* k_1 *such that*

$$a_2 \leq e^{24} \ for \ k \geq k_1$$

and

$$a_2 \leq \max\{e^{10}, 10k_1 \log k_1\} \ for \ k \leq k_1.$$

In his result, Luca uses the following fact (see Lemma 1 of [14]) mentioned by Erdős and Graham in [10].

Fact: *Let* $f(n) : \mathbb{N} \to \mathbb{R}_+$ *be any function that tends to infinity when* n *tends to infinity. Then there are only finitely many nontrivial solutions of* (5) *with* $a_2 > f(m) \log m$.

As a proof of this is not readily available, we shall give a proof in this paper. In fact, we show the following.

Theorem 4. *Assume that* (5) *holds with some* $m \geq 3, k \geq 2$ *and* $a_2 \geq 2$. *Then there exists an absolute constant* k_2 *such that*

(i) $a_2 \leq 1.5 \log m$ *if* $k \geq k_2$. *Further*

(ii) $a_2(\log a_2 - 1) \leq k \log(2m)$.

2 Preliminary Lemmas

We begin with a lemma that gives upper and lower bounds for the prime counting function $\pi(X)$.

Lemma 5. *Let* $X > 1$. *Then*

(i) $\pi(X) > \dfrac{X}{\log X}$;

(ii) $\pi(X) \leq \dfrac{X}{\log X} + \dfrac{1.2762X}{(\log X)^2}$;

(iii) $\displaystyle\sum_{p \leq X} \frac{\log p}{p} < \log X$;

(iv) $\displaystyle\sum_{p \leq X} \log p < 1.00003X$.

Inequality (i) is well-known, (ii) and (iii) are due to Dusart [5] and (iv) is from [15]. The next lemma we require is by Canfield, Erdős and Pomerance [4] on smooth numbers. For real numbers $2 \leq y \leq x$ put

$$\Psi(x, y) = |\{n \leq x : P(n) \leq y\}|.$$

Lemma 6. *Let $\varepsilon > 0$ be fixed. Assume that $x \geq y \geq (\log x)^{1+\varepsilon}$. Put $u = \log x / \log y$. Then the estimate*

$$\Psi(x, y) = \frac{x}{\exp((1 + o(1))u \log u)}$$

holds as $u \to \infty$.

The following lemma collects some results on $P(n, k)$.

Lemma 7. *(i) $P(n, k) > k$ if $n > k$.*

(ii) $P(n, k) > 2k \log k/7$ if $n > n_0$ and $n, n + 1, \ldots, n + (k - 1)$ are all composite. with n_0 a large absolute constant.

Further, there exist positive absolute constants c_2, c_3, c_4, c_5 satisfying the following inequalities:

(iii) $P(n, k) \geq c_2 k \log k \log_2 k / \log_3 k$ if $n > k^{3/2}$ and $\log k > e^e$.

(iv) $P(n, k) \geq c_3 k^{1+c_4(\log k/ \log n)^2}$ if $k^{3/2} \leq n \leq k^{c_5(\log k)^{1/2}/ \log_2 k}$.

Proof. The first result (i) is due to Sylvester [17]. Erdős [7], proved (ii) with an unknown constant in place of 2/7. Therefore we give a proof of (ii) below. Shorey [16], using (iv) and linear forms in logarithms proved (iii). The result in (iv) is due to Jutila [12] and is based on exponential sums. We now prove (ii).

By a result of Baker, Harman and Pintz [2], the interval $[n - n^{.53} + 1, n]$ contains a prime for $n > n_0$. Thus, $k < n^{.53}$ since $n, n+1, \ldots, n+k-1$ are all composite. Hence, $n > k^{3/2}$. Consider the binomial coefficient $\binom{n+k-1}{k}$. We have

$$\binom{n+k-1}{k} = \prod p^a$$

where the product is taken over prime powers p^a such that $p^a \| \binom{n+k-1}{k}$. It is well known that for any prime p if $p^a \| \binom{n+k-1}{k}$, then $p^a \le n+k-1$. Hence,

$$\binom{n+k-1}{k} \le (n+k-1)^{\pi(P(n,k))}.$$

On the other hand,

$$\binom{n+k-1}{k} = \frac{n(n+1)\cdots(n+k-1)}{k!} \ge \left(\frac{n+k-1}{k}\right)^k.$$

Suppose $P(n,k) \le 2k\log k/7$. Then

$$\left(\frac{n+k-1}{k}\right)^k \le (n+k-1)^{\pi(2k\log k/7)}. \tag{6}$$

Let $k \ge 2220$. Using estimates for $\pi(x)$ from (i) and (ii) of Lemma 5 with $x = 2k\log k/7$, we get

$$\frac{n+k-1}{k} \le (n+k-1)^{\frac{2}{7}+\frac{1.2762\times 2}{7\log k}}.$$

Simplifying, we arrive at

$$(n+k-1)^{\frac{5}{7}-\frac{.365}{\log k}} \le k.$$

This is a contradiction for $n > k^{3/2}$. For $k < 2220$, we use (6) with exact values of $\pi(x)$ to get the assertion. $\square$

3 Proof of Theorem 1

Let c_6, c_7 and c_8 denote numbers which can be effectively computable. Let $A = \{a_1, a_2, a_3\}$ with $a_1 > a_2 > a_3 > 1$ and $a_1!a_2!a_3! = \square$. We put $j := a_3,\ n := a_1$ and $k := a_1 - a_2$ to get

$$n(n-1)\cdots(n-k+1)j! = \square, \tag{7}$$

where $j = a_3 < n-k = a_2$. We have $j \ge 2$. Let X be large. The cardinality of $D_3(X)$ is exactly the number of $n \le X$ such that there exist positive integers j, k with $j < n-k$ and relation (7) holds. Let $\mathcal{A}_1$ be the set of all positive integers. Then it is clear that

$$|\mathcal{A}_1(X^{.9})| = O(X^{.9}). \tag{8}$$

So, it remains to estimate the cardinality of $D_3(X)\backslash\mathcal{A}_1(X^{.9}) := D_3'(X)$. By the result of Baker, Harman and Pintz [2], if $X > X_0$, and $k \geq n^{0.53}$, then some $n-j$ is a prime and hence (7) is impossible. Thus, $k < n^{0.53} \leq X^{0.53}$. Since $n > X^{0.9}$, we get that

$$k < X^{0.53} < n^{0.53/0.9} < (n-k+1)^{2/3} \qquad \text{for} \qquad X > X_0. \tag{9}$$

Let

$$P = \prod_{j/2<q\leq j} q.$$

By the Prime Number Theorem, we have that $P > e^{c_6 j}$. Since certainly $n(n-1)\cdots(n-k+1) \geq P > e^{c_6 j}$, it follows that $n-i+1 > e^{c_6 j/k}$ for some $i = 1,\ldots,k$. We thus get that

$$X > e^{c_6 j/k},$$

so

$$k \log X \gg j. \tag{10}$$

Let

$$Z = \exp((\log X)^{3/4}(\log_2 X)^{1/4}).$$

Case 1. Assume that $k \leq Z$. By (10), we have that $j < Z^2$ for all $X > X_0$. Let

$$\mathcal{A}_2(X) = \{n \in D_3'(X), P(n) \geq Z^2\}.$$

Then for any $n \in \mathcal{A}_2(X)$, we have $P^2(n)|n$ by (7). Hence,

$$|\mathcal{A}_2(X)| \leq \sum_{Z^2<p\leq\sqrt{X}} \frac{X}{p^2} \ll X\int_{Z^2}^{\infty}\frac{dt}{t^2} \ll \frac{X}{Z^2}.$$

Thus, for large X,

$$|\mathcal{A}_2(X)| \ll \frac{X}{Z^2} = \frac{X}{\exp\left(2(\log X)^{3/4}(\log_2 X)^{1/4}\right)}. \tag{11}$$

Let $\mathcal{A}_3(X)$ be the complement of $\mathcal{A}_2(X)$ in $D_3'(X)$. Put

$$u = \log X/\log(Z^2) = 0.5(\log X)^{1/4}(\log_2 X)^{-1/4}.$$

Then

$$\log u = (0.25 + o(1))\log_2 X$$

as $X \to \infty$. Since the parameters $x = X$ and $y = Z^2$ fulfill the conditions from Lemma 6 with $\varepsilon = 1$ for all sufficiently large X, we get that the number of such $n \le X$ is at most

$$\psi(X, Z^2) \le \frac{X}{\exp((1+o(1))u\log u)} < \frac{X}{\exp(0.1(\log X)^{1/4}(\log_2 X)^{3/4})}$$

for $X > X_0$. This together with (11) implies that the number of integers in $D_3'(X)$ with $k \le Z$ is bounded by

$$O\left(\frac{x}{\exp\left(0.1(\log X)^{1/4}(\log_2 X)^{3/4}\right)}\right). \tag{12}$$

This deals with the case $k \le Z$.

Case 2. Let $k > Z$. With c_5 being the constant from Lemma 7, we have

$$\begin{aligned} k^{c_5(\log k)^{1/2}/\log_2 k} &= \exp(c_5(\log k)^{3/2}/\log_2 k) \\ &> \exp(c_7(\log X)^{9/8}(\log_2 X)^{-5/8}) > X \end{aligned}$$

for $X > X_0$. Thus, using also (9), we have

$$k^{3/2} \le n - k + 1 < X < k^{c_5(\log k)^{1/2}/\log_2 k},$$

so, according to Lemma 7 (iv), we have

$$P(n,k) > c_3 k^{1+c_4(\log k/\log n)^2} > c_3 k^{1+c_4(\log k/\log X)^2} > k$$

since $k > Z$ and X is large. Suppose

$$P(n,k) \le j.$$

Then, by (10), we have

$$k\log X \gg j \ge P(n,k) \gg k^{1+c_4(\log k/\log X)^2},$$

giving

$$\log k \ll (\log X)^{2/3}(\log_2 X)^{1/3} < \log Z$$

implying $k < Z$, a contradiction. Thus, $P(n,k) > j$. Further $P(n,k)$ divides only one term in $\{n, n-1, \ldots, n-k+1\}$ since $P(n,k) > k$. Then, by (7), there exists a term say, $n - i_0 + 1$ for some i_0 with $i_0 \in \{1, \ldots, k\}$, such that $(P(n,k))^2 \mid (n - i_0 + 1)$. Let now $\mathcal{A}_4(X)$ be the set of such n. Thus, the subset of integers in $D_3'(X)$ with $k > Z$ is a subset of $\mathcal{A}_4(X)$ which in turn is a subset of integers $n \le x$ such that there exist $k \in [Z, X^{0.53}]$ and a prime $p \in I = [Z^{1+c_4(\log Z/\log X)^2}, X^{1/2}]$ with $p > k^{1+c_7(\log k/\log X)^2}$ and $p^2 \mid (n - i_0 + 1)$ for some $i_0 \in \{0, 1, \ldots, k-1\}$. Let $p \in I$ be fixed. Then n is such that $n - i_0 + 1 \equiv 0 \pmod{p^2}$ for some $i_0 \le k$, where

$$\begin{aligned} p &> k^{1+c_7(\log k/\log X)^2} \\ &> k\exp(c_7(\log Z)^3/(\log X)^2) \\ &= k\exp(c_8(\log X)^{1/4}(\log_2 X)^{3/4}). \end{aligned}$$

Thus,

$$i_0 \le k < \frac{p}{\exp\left(c_8(\log X)^{1/4}(\log_2 X)^{3/4}\right)}. \tag{13}$$

For a p fixed, the number of $m = n - i_0 + 1 \le X$ which are multiples of p^2 is $\le X/p^2$, and multiplying this bound with the number of possibilities for i_0 when p is fixed given by (13), we get that the number of possibilities for n is

$$\ll \frac{X}{p\exp\left(c_8(\log X)^{1/4}(\log_2 X)^{3/4}\right)}.$$

Summing the above bound over p, we get that

$$\begin{aligned} |\mathcal{A}_4(X)| &\ll \frac{X}{\exp\left(c_8(\log X)^{1/4}(\log_2 X)^{3/4}\right)} \sum_{p\in I} \frac{1}{p} \\ &\ll \frac{X\log_2 X}{\exp\left(c_9(\log X)^{1/4}(\log_2 X)^{3/4}\right)}. \end{aligned} \tag{14}$$

This completes the case $k > Z$.

The desired estimate for $D_3(X)$ follows from estimates (8), (12) and (14) with $c_0 = \min\{0.1, c_9\}$. □

4 Proof of Theorem 2

The proof of Theorem 2 is similar to the proof of Theorem 1 with some modification. We give only some essential steps. Let $c_{10}, \dots, c_{13}$ be effectively computable positive constants. Let

$$B = \{n : n \in [a, a+k-1] \text{ for some } a \text{ and } k \text{ such that } O(a,k) > 1.\}$$

In other words, B counts those integers which lie in some interval $[a, a+k-1]$ with the greatest prime factor $P(a,k)$ dividing the product $\Delta(a,k)$ to an exponent exceeding 1. Note that $S \subseteq B$. We shall denote by $I_n(a,k)$ the interval $[a, a+k-1]$ containing n with $1 < O(a,k) := O_n(a,k)$. When $n \in B$, note that all the integers in $I_n(a,k)$ are also in B and $a+k-1 \geq 4$. Further, we may assume that $a > k$. Otherwise, $a \leq k$ and there exists a prime between $(a+k-1)/2$ and $a+k-1$ by Bertrand's postulate, and any such prime divides $\Delta(a,k)$ only to the first power. In particular, this is true for $P(a,k)$ which contradicts the fact that $O(a,k) > 1$. Thus, by Lemma 7(i), we find that

$$P_n(a,k) > k \text{ whenever } n \in B. \tag{15}$$

As in the proof of Theorem 1, it is enough to estimate the number of integers n in

$$B(X) \backslash \mathcal{A}_1(X^{.9}) := B'(X).$$

Let p be a prime. Suppose that

$$C_p(X) = \{n \in B'(X) : P_n(a,k) = p\}.$$

We shall give an upper bound for $|C_p(X)|$. Let $n \in C_p(X)$ and $I_n(a,k)$ be the corresponding interval. Thus, $a \in C_p(X)$. By (15), $p > k$ divides only one number in $I_n(a,k)$, and it divides it to an exponent ≥ 2. Suppose some $a+j$ is a prime. Then

$$a \leq a+j \leq p \leq \sqrt{a+k-1},$$

implying $a^2 \leq a+k-1 < 2a-1$, a contradiction. Thus, no term in $I_n(a,k)$ is a prime. As in the proof of Theorem 1, by the result of [2], we get

$$k < (a+k-1)^{.53} \leq (X+k-1)^{.53}$$

for X sufficiently large. Since $n > X^{.9}$, n and hence a are also sufficiently large. Thus,

$$X+k-1 \geq a+k-1 \geq k^{1.8867} \tag{16}$$

for $X \geq X_0$. Further the number of integers $m \leq X+k-1$ divisible by p^2 is at $\left\lfloor \dfrac{X+k-1}{p^2} \right\rfloor$, and hence

$$|C_p(X)| \leq k\left(\frac{X+k-1}{p^2}\right). \tag{17}$$

As in Theorem 1, we take $Z = \exp((\log X)^{3/4}(\log_2 X)^{1/4})$ and let

$$B'(X) = B'_1(X) \cup B'_2(X),$$

with

$$B'_1(X) = \{n \in B'(X) : k \leq Z\} \text{ and } B'_2(X) = \{n \in B'(X) : k > Z\}.$$

Further, we split $B'_1(X)$ as $B'_{11}(X) \cup B'_{12}(X)$, where

$$\begin{aligned} B'_{11}(X) &= \{n \in B'_1(X) : P_n(a,k) > Z^2\}; \\ B'_{12}(X) &= \{n \in B'_1(X) : P_n(a,k) \leq Z^2\}. \end{aligned}$$

Then,

$$\begin{aligned} |B'_{11}(X)| = \sum_{Z^2 < p \leq \sqrt{2X-1}} |C_p(X)| &\leq \sum_{Z^2 < p \leq \sqrt{2X-1}} k\left(\frac{X+k-1}{p^2}\right) \\ &\ll \frac{kX}{Z\log Z} \ll \frac{X}{Z} \leq \frac{X}{\exp(c_{10}(\log X)^{3/4}(\log_2 X)^{1/4})}, \end{aligned}$$

while by Lemma 6,

$$|B'_{12}(X) \leq \frac{X}{\exp(c_{11}(\log X)^{1/4}(\log_2 X)^{3/4})}.$$

In order to evaluate $|B'_2(X)|$, we proceed as in the proof of Theorem 1 to get

$$k^{c_5(\log k)^{1/2}/\log_2 k} > 2X > X+k-1$$

for $X > X_0$. Now we use (16) and Lemma 7(iv), to find for any $n \in B'_2(X)$, a prime $p = P_n(a,k) \gg k^{1+c_4(\log k/\log X)^2}$ implying that

$$k \ll \frac{p}{\exp(c_{12}(\log X)^{1/4}(\log_2 X)^{3/4})}.$$

Hence,

$$|B_2'(X)| \ll \sum_{p \le \sqrt{2X-1}} k \left(\frac{X+k-1}{p^2} \right)$$

$$\ll \frac{X}{\exp(c_{12}(\log X)^{1/4}(\log_2 X)^{3/4})} \sum_{p<\sqrt{2X-1}} \frac{1}{p} \ll \frac{X \log_2 X}{\exp(c_{13}(\log X)^{1/4}(\log_2 X)^{3/4})}.$$

By combining the estimates for $|B_1'(X)|$ and $|B_2'(X)|$, we obtain the required upper bound. □

5 Proof of Theorem 4

Consider equation (5). We observe that none of the terms in $\Delta(m,k)$ is a prime, since if $m+i=p$ prime for some $0 \le i < k$, then

$$a_1 + 1 \le m + i = p \le a_2,$$

a contradiction. Suppose $m \le k$. By Bertrand's postulate, there is a prime in $((m+k-1)/2, m+k-1]$ and this interval is contained in $[m, m+k-1]$, which is a contradiction. Thus $m > k$. By (5), we have

$$a_2 \log a_2 - a_2 \le \log a_2! \le \log(m+k)^k \le k \log 2m.$$

This gives the second assertion of the theorem.

By Lemma 7 (i) and (5), we see that

$$a_2 > k.$$

Let $k \ge k_2$ with k_2 sufficiently large. By Lemma 7 (ii) there exists a prime exceeding $2k \log k/7$ dividing the product. By the left–hand side of (5), this prime must be $\le a_2$. Thus, we get

$$a_2 > 2k \log k/7, \tag{18}$$

giving

$$k \le 3.5 a_2/\log k.$$

We count the power of 2 on either side of (5). The power of 2 on the left hand side is $\ge$ the power of 2 in $a_2!$ which is

$$\left\lfloor \frac{a_2}{2} \right\rfloor + \left\lfloor \frac{a_2}{2^2} \right\rfloor + \left\lfloor \frac{a_2}{2^3} \right\rfloor + \cdots$$

In the product on the right hand side, delete the term in which 2 appears to the highest power. Then the power of 2 in the rest of the terms does not exceed the power of 2 in $k!$, which is at most k. Further, the power of 2 in any term in the product does not exceed

$$\frac{\log(m+k)}{\log 2}.$$

Thus, the power of 2 on the right hand side is at most

$$\frac{\log(m+k)}{\log 2} + k.$$

Combining the above estimates we get

$$.99a_2 - 7 \leq \frac{\log(2m)}{\log 2} + \frac{3.5a_2}{\log k}.$$

Since k_2 is sufficiently large, we get

$$.97a_2 \leq \frac{\log m}{\log 2}$$

implying $a_2 \leq 1.5 \log m$. □

6 Proof of Theorem 3

We will follow the line of argument as in [14] with some change. We will not improve the bound for k as Luca does using Matveev's lower bound for linear forms in logarithms. As in the proof of Theorem 4, no term in $\Delta(m,k)$ is a prime. Observe that primes $> k$ divide at most one term of the product on the right hand side of (5). For every prime $p \leq k$, we delete the term in which it appears to the maximum power. The power with which p divides the rest of the product is at most the power of p in $(k-1)!$. Further, all the primes dividing the product are $\leq a_2$. Thus,

$$\prod_{i=0}^{k-1} N(m+i) \leq (\prod_{k \leq p \leq a_2} p)(\prod_{p<k} p^{\lfloor k/p \rfloor})$$

$$\leq \exp\left(\sum_{k \leq p \leq a_2} \log p + k \sum_{p<k} \frac{\log p}{p}\right).$$

By Lemma 7 (iii) and (iv), we get

$$\prod_{i=0}^{k-1} N(m+i) \le \exp\left(1.00003a_2 + k\log k\right).$$

Choose $m+j_1$ and $m+j_2$ such that $N(m+j_1) \le N(m+j_2)$ are the smallest among $N(m+i)$ for $0 \le i < k$. Then

$$N(m+j_2) \le \left(\prod_{i=0}^{k-1} N(m+i)\right)^{1/(k-1)} \le \exp\left(\frac{1.00003a_2}{k-1} + \frac{k\log k}{k-1}\right). \quad (19)$$

Consider the equation

$$\frac{m+j_1}{d} - \frac{m+j_2}{d} = \frac{j_1-j_2}{d},$$

where $d = \gcd(m+j_1, j_1-j_2)$. Applying inequality (3), we get,

$$\frac{m}{d} \le (N(m+j_1)N(m+j_2)|j_1-j_2|/d)^{7/4}$$

implying, by (19), that

$$\log m \le \frac{7}{4}\left(\frac{2.00006a_2}{k-1} + \frac{2k\log k}{k-1} + \log k\right). \quad (20)$$

Suppose that $k \ge k_1$ where k_1 is sufficiently large. Then, by Lemma 7(ii) and (20), we get

$$k\log m \le 22a_2.$$

We use this bound in Theorem 4(ii) to obtain

$$\log a_2 \le 24.$$

Now let $k < k_1$. We may assume that $a_2 > 10k_1\log k_1$, since otherwise the result follows. Then (20) with $k \ge 2$ implies that

$$k\log m \le 7.00021a_2 + 8.75k\log k.$$

Again using Theorem 4(ii), we find that

$$a_2\log a_2 \le k\log m + a_2 \le 8.00021a_2 + 9.75k\log k$$

giving

$$\log a_2 \le 8.00021 + \frac{9.75k\log k}{a_2} < 10,$$

since $a_2 > 10k_1\log k_1$ and $k < k_1$. This proves the second part of the theorem. □

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