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ON ISOSPECTRAL ARITHMETICAL SPACES

By C. S. RAJAN

Abstract. We study the relationship between the arithmetic and the spectrum of the Laplacian for manifolds arising from congruence arithmetic subgroups of $SL(1,D)$, where $D$ is an indefinite quaternion division algebra defined over a number field $F$.

We give new examples of isospectral but nonisometric compact, arithmetically defined varieties, generalizing the class of examples constructed by Vigneras. These examples are based on an interplay between the simply connected and adjoint group and depend explicitly on the failure of strong approximation for the adjoint group. The examples can be considered as a geometric analogue and also as an application of the concept and results on $L$-indistinguishability for $SL(1,D)$ due to Labesse and Langlands.

We verify that the Hasse-Weil zeta functions are equal for the examples of isospectral pair of arithmetic varieties we construct giving further evidence for an archimedean analogue of Tate’s conjecture, which expects that the spectrum of the Laplacian determines the arithmetic of such spaces.

1. Introduction. Let $M$ be a compact, connected Riemannian manifold. The spectrum of $M$ consists of the collection of nonzero eigenvalues counted with multiplicity of the Laplace-Beltrami operator acting on the space of smooth functions on $M$. The inverse spectral problem is the investigation of the properties of the manifold that can be recovered from a knowledge of the spectrum. Two compact connected Riemannian manifolds $M_1$ and $M_2$ are said to be isospectral if the spectrums of $M_1$ and $M_2$ coincide. The first question that arises in inverse spectral theory is whether the spectrum determines a Riemannian manifold up to isometry. This is known to be false and the first counter-examples were constructed by Milnor in the case of flat tori. When the spaces are compact hyperbolic surfaces, counterexamples were constructed by Vigneras [V] and later by Sunada [S]. In this paper, our aim is to investigate the inverse spectral problem in the arithmetical context of compact, locally symmetric manifolds, especially those arising from co-compact congruence lattices in $G(F) = SL(1,D)(F)$, where $D$ is an indefinite quaternion division algebra over a number field $F$.

The main observation is that of introducing the conjugation of a compact open subgroup $K$ of $G(A_f)$ by an element of the adelic points of the adjoint group, and to observe that this process yields spectrally indistinguishable manifolds. We establish this observation (see Thereom 2) under some hypothesis on the nature of $D$ and the compact open subgroup $K$. This result can be considered as a
geometric analogue of the study initiated by Langlands together with Labesse and Shelstad on the problems arising out of the difference between stable conjugacy and conjugacy: in [LL], the multiplicity of the representations $\pi^g$ in the space of cusp forms on $SL(2, \mathbb{A}_F)$ as a function of $g$ is studied, where $\pi$ is a representation of $SL(2, \mathbb{A}_F)$ and $g \in GL(2, \mathbb{A}_F)$.

As a corollary to the above observation, we give examples of isospectral but nonisometric compact, locally symmetric manifolds as above, generalizing the class of isospectral but nonisometric hyperbolic surfaces and three dimensional manifolds constructed by Vigneras [V]. For $K$ as above, the condition for the existence of nonisometric spaces is that the finite adelic points of the adjoint group is not exhausted by the union of translates of the normalizer of the image of $K$ by rational elements. This always happens if the normalizer of $K$ is sufficiently small by the failure of strong approximation in the adjoint group. In contrast to the method of this paper, Vigneras’ method is a combination of geometry and arithmetic. It depends on the relationship of the spectrum of the Laplacian with the length spectrum for compact hyperbolic surfaces. Vigneras’ method works when the lattices are associated to maximal orders, and is difficult to generalize to more general groups or to more general lattices.

One of the motivating questions for us has been the heuristic that the spectrum of invariant operators should determine the arithmetic of compact locally symmetric manifolds arising from arithmetic lattices. For the family of arithmetic curves associated to congruence arithmetic lattices coming from quaternion division algebras over totally real fields, Shimura constructed a theory of canonical models [Sh]. In particular, these spaces can be defined over number fields. The question was raised in [PR] whether the Hasse-Weil zeta functions of the canonical models of isospectral arithmetically defined curves are equal? The conjecture is based on the analogy that the hyperbolic Laplacian can be considered as the Frobenius at infinity. It can be seen that two compact hyperbolic surfaces are isospectral if and only if the corresponding lattices in $PSL(2, \mathbb{R})$ are representation equivalent. This ties up the hyperbolic geometry with representation theory and arithmetic. Further, the theorem of Faltings proving Tate’s conjecture asserts that if the eigenvalues of the Frobenius elements acting on the $l$-adic cohomology groups of two smooth, projective curves defined over a number field coincide, then the Jacobians of the curves are isogenous. Thus the conjecture can be considered as an archimedean analogue of Tate’s conjecture. In [PR], this conjecture was verified for the examples constructed by Sunada and Vigneras. Using the basic fact in the theory of canonical models that adelic conjugation corresponds to Galois conjugation of the canonical model, we verify the conjecture that the spectrum determines the Hasse-Weil zeta function for the examples constructed in this paper. On the other hand, the results of this paper can be used to reformulate the conjecture for the nonconnected Shimura varieties rather than the connected ones.

To prove Theorem 2, by the generalized Sunada criterion for isospectrality due to DeTurck and Gordon, it is sufficient to show that the lattices corresponding
to $K$ and it’s conjugate are representation equivalent. Using strong approximation for $SL(1, D)$, the proof of the representation equivalence of the two lattices is an application of the formula for the multiplicity of a representation of $G(A)$ in the space of cusp forms on $G(A)$ due to Labesse and Langlands.

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2. Sunada’s criterion and representation equivalence. Let $G$ be a Lie group, and let $\Gamma$ be a cocompact lattice in $G$. The existence of a cocompact lattice implies that $G$ is unimodular. Let $R_\Gamma$ denote the right regular representation of $G$ on the space $L^2(\Gamma \backslash G)$ of square integrable functions with respect to the projection of the Haar measure on the space $\Gamma \backslash G$:

$$(R_\Gamma g f)(x) = f(x g), \quad f \in L^2(\Gamma \backslash G), \quad g, x \in G.$$  

Definition 2.1. Let $G$ be a Lie group and $\Gamma_1$ and $\Gamma_2$ be two co-compact lattices in $G$. The lattices $\Gamma_1$ and $\Gamma_2$ are said to be representation equivalent in $G$ if the regular representations $R_{\Gamma_1}$ and $R_{\Gamma_2}$ of $G$ are isomorphic.

We have the following generalization of Sunada’s criterion for isospectrality proved by DeTurck and Gordon in [DG]:

Proposition 1. Let $G$ be a Lie group acting on the left as isometries of a Riemannian manifold $M$. Suppose $\Gamma_1$ and $\Gamma_2$ are discrete, co-compact subgroups of $G$ acting freely and properly discontinuously on $M$, such that the quotients $\Gamma_1 \backslash M$ and $\Gamma_2 \backslash M$ are compact Riemannian manifolds. If the lattices $\Gamma_1$ and $\Gamma_2$ are representation equivalent in $G$, then $\Gamma_1 \backslash M$ and $\Gamma_2 \backslash M$ are isospectral for the Laplacian acting on the space of smooth functions.

Remark. When $G$ is a finite group, this result is due to Sunada [S]. Sunada’s construction is based on an analogous construction in algebraic number theory: let $G$ be the Galois group of a finite Galois extension $L$ over the rationals. Suppose there exists two subgroups $\Gamma_1$ and $\Gamma_2$ of $G$ such that the representations $R_{\Gamma_1}$ and $R_{\Gamma_2}$ are isomorphic. Then the invariant fields $L^{\Gamma_1}$ and $L^{\Gamma_2}$ have the same Dedekind zeta function.

The conclusion in Proposition 1 can be strengthened to imply “strong isospectrality”, i.e., the spectrums coincide for natural self-adjoint differential operators.
besides the Laplacian acting on functions. In particular, this implies the isospectrality of the Laplacian acting on $p$-forms.

One of the heuristics underlying this paper is that suitable strong isospectrality assumptions on a pair of compact locally symmetric spaces should determine the “arithmetic” associated to these spaces (although it does not determine the geometry of such spaces).

3. Adelic conjugation of lattices. Let $F$ be a number field and let $D$ be a quaternion algebra over $F$.

Let $G = SL_1(D), \tilde{G} = GL_1(D), G = PGL_1(D)$.

Let $K_\infty$ be a maximal compact subgroup of $G_\infty = G(F \otimes \mathbb{R})$ and let $M = G_\infty/K_\infty$ be the non-compact symmetric space associated to $G$. For a number field $F$, let $A_F$ denote the adele ring of $F$, and $A_f$ the subring of finite adeles. Let $K$ be a compact open subgroup of $G(A_F)$.

In this section, we assume that $D$ and $K$ satisfy the following hypothesis:

H1. $D$ is a division algebra, and there is at least one finite place $v_0$ of $F$ at which $D$ is ramified, i.e., $D \otimes F_{v_0}$ remains a division algebra.

H2. $D$ is indefinite: there is at least one archimedean place $v$ of $F$, at which $D \otimes F_v \simeq M_2(\mathbb{R})$, where for a place $v$ of $F$, $F_v$ denotes the completion of $F$ at $v$.

H3. There is a factorisation of the form,

$$K = K_{v_0}K_{v_0},$$

where $K_{v_0}$ is a compact, open subgroup of $G(F_{v_0})$, and is a normal subgroup of $D_{v_0}$. The group $K_{v_0}$ has no $v_0$ component, i.e., for any element $x \in K_{v_0}$, the $v_0$ component $x_{v_0} = 1$.

The heart of this paper is the following theorem, whose proof is given in Section 6:

**Theorem 2.** Let $D$ be a quaternion division algebra over a number field $F$ and let $K$ be a compact open subgroup of $G(A_F)$ satisfying the properties H1, H2, H3 given above. Then for any element $x \in GL_1(D)(A_f)$, the lattices $\Gamma_K$ and $\Gamma_{K^x}$ are representation equivalent in $G_\infty$.

**Remark.** The construction can be considered as a geometric analogue of the concept of $L$-indistinguishability arising out of the difference between stable conjugacy and conjugacy due to Labesse, Langlands and Shelstad [LL, Shl]. Let $\pi$
be a representation of \( G(\mathbf{A}_F) \). For an element \( g \in \tilde{G}(\mathbf{A}_F) \), let \( \pi^g(x) := \pi(g x g^{-1}) \), \( x \in G(\mathbf{A}) \) be the conjugate representation of \( \pi \) by \( g \). The representations \( \pi \) and \( \pi^g \) are said to be \( L \)-indistinguishable (see [LL] and Section 5). In [LL], Labesse and Langlands consider the multiplicity with which the conjugate representations \( \pi^g \) occur in the space of automorphic representation of \( G(\mathbf{A}_F) \) as a function of \( g \). We can call the lattices \( \Gamma_K \) and \( \Gamma_{K^x} \) as ‘stably conjugate’ lattices, and the theorem says that stably conjugate lattices give rise to spectrally indistinguishable manifolds. But the terminology can be misleading, as it is not clear that any element of \( \Gamma_K \) is stably conjugate to an element of \( \Gamma_{K^x} \) up to an element of the center of \( G \). This fact is true for cases of surfaces and three dimensional manifolds after Theorem 2, by comparing the spectrum with the length spectrum which is given by the absolute value of the trace (see [V]).

More generally, let \( \Gamma_1, \Gamma_2 \) be two lattices in \( H(\mathbb{R}) \), where \( H \) is a reductive algebraic group over \( \mathbb{R} \). Call \( \Gamma_1 \) and \( \Gamma_2 \) to be weakly stably conjugate if every element \( \gamma_1 \in \Gamma_1 \) is stably conjugate (i.e., conjugate inside the group \( H(\mathbb{C}) \)) to an element of \( \Gamma_2 \) up to a central element of \( H(\mathbb{R}) \). The following question can be raised: if \( \Gamma_1 \) and \( \Gamma_2 \) are weakly stably conjugate cocompact lattices in \( G(\mathbb{R}) \), are they representation equivalent? It is possible that the question may have an affirmative answer with either a weaker conclusion: given an irreducible, unitary representation \( \pi \) of \( G(\mathbb{R}) \), the sum of the multiplicities of the representations in the (Arthur) \( L \)-packet of \( \pi \) occurring in \( R_{\Gamma_1} \) and \( R_{\Gamma_2} \) are equal; or with a stronger hypothesis: the stable conjugacy classes are counted with an appropriate notion of multiplicity.

But as far as producing examples of isospectral spaces are concerned, the problem remains of producing examples of pairs of non-conjugate but weakly stably conjugate lattices.

Remark. The proof of the theorem given here uses adelic methods and depends quite crucially on hypothesis \( H3 \) that \( K \) is required to satisfy. For example, an analogous question can be raised for the cuspidal spectrum of \( SL(2) \) for which we do not know an answer: let \( K \) be a compact open subgroup of \( SL_2(\mathbf{A}_{F,f}) \). For any \( x \in GL_2(\mathbf{A}_{F,f}) \) are the cuspidal spectrums of \( L^2(\Gamma_K \setminus SL_2(F \otimes \mathbb{R})) \) and \( L^2(\Gamma_{K^x} \setminus SL_2(F \otimes \mathbb{R})) \) equal? In this situation, hypothesis \( H3 \) cannot be assumed as the groups \( SL_2(k) \) are simple modulo center for any infinite field \( k \).

3.1. Isospectral spaces. The following corollary gives examples of isospectral but nonisometric compact Riemannian manifold:

**Corollary 1.** With the notation as in Theorem 2, assume further that \( K \) is small enough so that \( \Gamma_K \) and \( \Gamma_{K^x} \) are torsion-free. Let \( N(\bar{K}) \) denote the normalizer of \( \bar{K} \) in \( \bar{G}(\mathbf{A}_{F,f}) \). Suppose \( x \) is an element in \( \bar{G}(\mathbf{A}_{F,f}) \) such that \( \bar{x} \) does not belong to the set \( N(\bar{K})\bar{G}(F) \). Then \( X_K \) and \( X_{K^x} \) are (strongly) isospectral, but are not isometric.
Proof. Suppose on the contrary, that $X_K$ and $X_{K'}$ are isometric. Then there exists $\bar{g} \in \bar{G}(\mathbb{R})$ such that

$$\bar{g}^{-1}\Gamma_{K'}\bar{g} = \Gamma_K,$$

where $\Gamma_K$, $\Gamma_{K'}$ is the projection of $\Gamma_K$, $\Gamma_{K'}$ to $\mathcal{O}_\infty$. Since the lattices $\Gamma_K$ and $\Gamma_{K'}$ are arithmetic and commensurable, it follows by a theorem of Margulis that $\bar{g} \in \bar{G}(F)$. Since the kernel of the projection map $\bar{G} \to G$ is a split torus, by Hilbert Theorem 90, there is an element $\tilde{g} \in \tilde{G}(F)$ satisfying,

$$\tilde{g}^{-1}\Gamma_{K'}\tilde{g} = \Gamma_K.$$

Since $D$ is indefinite, by the strong approximation theorem for $SL(1, D)$, the lattice $\Gamma_K$ sitting diagonally inside $G(\mathbb{A}_F)$ is dense in $K$. Hence

$$\tilde{g}^{-1}K\tilde{g} = K.$$

Projecting to $\bar{G}$, we obtain

$$\bar{g}^{-1}\bar{x}^{-1}\bar{K}\bar{g} = \bar{K},$$

where $\bar{K}$ denotes the image of $K$ in $\bar{G}(\mathbb{A}_F)$. This implies that $\bar{x} \in N(\bar{K})\bar{G}(F)$, contradicting our choice of $\bar{x}$. \hfill \Box

Remark. The normalizer $N(\bar{K})$ is a compact open subgroup of $\bar{G}(\mathbb{A}_F)$. By the failure of strong approximation for the adjoint group $PGL_1(D)$, the hypothesis that $\bar{x}$ does not belong to the double coset $N(\bar{K})\bar{G}(F)$ is satisfied provided $N(K)$ is small enough. For example $K$ can be taken to be a sufficiently deep Hecke congruence subgroup corresponding to a finite collection of split primes for $D$.

Remark. These examples generalize the class of examples constructed by Vigneras of isospectral but nonisometric compact Riemann surfaces. For compact hyperbolic surfaces (resp. hyperbolic three manifolds), the spectrum of the Laplacian can be related to the (resp. complex) length spectrum. For the lattices arising from maximal orders in $D$, it is possible to obtain explicit formula for the multiplicities of lengths of periodic geodesics. Using these formulae, the above corollary follows in dimensions two and three and for lattices arising from maximal orders. The failure of strong approximation is reflected in the fact that the lattices correspond to non-conjugate maximal orders inside the quaternion algebra. In particular, it is required that the underlying field has a nontrivial class number. Vigneras’ method seems difficult to generalize to higher dimensions and to lattices not arising from maximal orders. In contrast, the above corollary is applicable even for indefinite quaternion division algebras over rationals.
4. Spectral and arithmetical equivalence. In this section, we take $F$ to be a totally real number field and $D$ a quaternion division algebra over $F$. Let $\tau_1, \ldots, \tau_r$ be the real embeddings corresponding to archimedean places of $F$ at which $D$ splits, and let $F'$ be the reflex field of $(F, \tau_1, \ldots, \tau_r)$. For any compact open subgroup $\tilde{K}$ of $\tilde{G}(A_{F,f})$, let $F_{\tilde{K}}$ be the abelian extension of $F'$ defined as in \cite[page 157]{Sh}. Let $K = \tilde{K} \cap G(A_{F,f})$. In \cite{Sh}, Shimura defines a canonical model for the spaces $M_K$ over the field $F_{\tilde{K}}$. As a consequence of the main theorem of canonical models in \cite[Theorem 2.5, page 159, Section 2.6]{Sh} and Theorem 2, we obtain the following corollary providing more evidence in support of the conjectures made in \cite{PR} that the spectral zeta function determines the arithmetical zeta function:

**Corollary 2.** Let $F$ be a totally real number field and $\tilde{K}$ be a compact open subgroup of $\tilde{G}(A_{F,f})$. Assume that $\tilde{K}$ satisfies the hypothesis of Theorem 2 and is such that the lattices $\Gamma_{K'}$ are torsion-free. Then the spaces $M_{K'}$ for $x \in \tilde{G}(A_{F,f})$ are isospectral and have the same Hasse-Weil zeta function for the canonical model defined by Shimura.

If $D$ is ramified at all real places except one, then the Jacobians of $M_K$ and $M_{K'}$ are conjugate by an automorphism of $\bar{\mathbb{Q}}$.

There are a couple of remarks to be made about the conjecture made in \cite{PR}. If we fix our attention on $K$, there can be more than one choice of $\tilde{K}$ with $K = \tilde{K} \cap G(A_f)$. Thus the choice of a canonical model on the space $M_K$ is not uniquely determined by $K$, and we can expect the equality of the zeta functions only after a finite extension of the base field.

A second observation is that the zeta function of Shimura varieties and its relationship to that of $L$-functions of automorphic representations are better behaved for the non-connected Shimura varieties associated to inner forms of $GL(2)$, than for the connected Shimura varieties. Consider the space

$$M_{\tilde{K}} = \tilde{G}(F)_+ \backslash M \times \tilde{G}(A_{F,f}) / \tilde{K}$$

where $\tilde{G}(F)_+$ consists of those elements with totally positive determinant. The non-connected space $M_{\tilde{K}}$ has a canonical model in the sense of Shimura and Deligne \cite{D} over the reflex field $F'$. The connected components of $M_{\tilde{K}}$ are of the form $M_{K'}$ as $x$ ranges over $\tilde{G}(A_{F,f})$. We make the following conjecture:

**Conjecture 1.** Let $\tilde{K}_1, \tilde{K}_2$ be compact open subgroups of $\tilde{G}(A_f)$, such that the lattices $\Gamma_{K_1}$ and $\Gamma_{K_2}$ are torsion-free modulo their centers. If the spaces $M_{K_1}$ and $M_{K_2}$ are isospectral, then the Hasse-Weil zeta functions of the canonical models of $M_{K_1}$ and $M_{K_2}$ are equal.

The conjecture is well posed thanks to Theorem 2. This clarifies and makes precise the arithmetical conjecture made in \cite{PR}.
 Remark. The initial hope behind the conjectures made in [PR], was that isospectral compact Riemann surfaces (with respect to the hyperbolic metric) have isogenous Jacobians. This naive conjecture proves to be false for the example of Shimura curves constructed above, as the respective Jacobians become isogenous after twisting by an automorphism of $\bar{Q}$.

It is tempting to conjecture that this will be the exception: if two compact Riemann surfaces are isospectral, then the Jacobian of one is isogenous to a conjugate of the Jacobian of the other by an automorphism $\sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q})$, where $\sigma$ preserves the spectrum of the Riemann surface. If moreover $\sigma$ is not identity or of order two, then the pair of Riemann surfaces arise from an arithmetical context, i.e., are Shimura curves as considered in this paper.

In particular, a more general formulation of Conjecture 1 can be formulated in the case of Riemann surfaces, by assuming that the compact open subgroups $\tilde{K}_1$ and $\tilde{K}_2$ are subgroups of $\tilde{G}_1(\mathbb{A}_F)$, where $\tilde{G}_i = \text{GL}_1(D_i)$, $i = 1, 2$ and $D_i$ are quaternion division algebras defined respectively over number fields $F_i$. By a theorem of A. Reid [Re], if two such compact Riemann surfaces are isospectral, then $F_1 = F_2$ and $D_1 = D_2$. Thus by Reid’s theorem, the more general conjecture reduces to Conjecture 1 for the ‘arithmetical’ Riemann surfaces. In this context, we refer also to the recent examples produced by Lubotzky, Samuels and Vishne of isospectral spaces arising from congruence lattices in higher rank groups but which are not commensurable [LSV].

5. Multiplicity formula of Labesse and Langlands. Our aim in this section is to recall the multiplicity formula of Labesse and Langlands [LL], giving the multiplicity $m(\pi)$ with which a representation $\pi$ of $\text{SL}_1(D)(\mathbb{A}_F)$ occurs in the space of cusp forms on $\text{SL}_1(D)(\mathbb{A}_F)$. This formula will be required for the proof of Theorem 2. The references for the material contained in this section are [LL] and [Shl].

5.1. Langlands parameters. Let $F$ be a global or a local field of characteristic zero, $W_F$ be the Weil group of $F$, and $W'_F$ be the Weil-Deligne group of $F$:

$$W'_F = \begin{cases} W_F \times SU(2) & \text{if } F \text{ is non-archimedean,} \\ W_F & \text{if } F \text{ is archimedean.} \end{cases}$$

By abelian reciprocity, we will identify $W'_F$ with $C_F$, where

$$C_F = \begin{cases} F^* & \text{if } F \text{ is local,} \\ J_F/F^* & \text{if } F \text{ is global,} \end{cases}$$

where for a global field $F$, $J_F$ is the idele group associated to $F$. 

For a connected reductive group $H$ defined over $F$, let $^L H^0$ be the connected reductive group over $\mathbb{C}$ whose root datum is the dual root datum of the root datum of $H$ over $\bar{F}$. Let $^L H$ denote the Langlands dual group defined over the complex numbers associated to $H$ over $F$. The Langlands dual group $^L H$ is defined as a semidirect product,

$$^L H = ^L H^0 \times W'_F.$$ 

Let $\Phi(H)$ be the set of equivalence classes of admissible homomorphisms (see [B]) of $W'_F$ into $^L H$.

In our case, where $H$ is an inner form of $SL_2$, the connected component $^L H^0$ is isomorphic to $PGL_2(\mathbb{C})$. Further, the Langlands dual group for $GL(2)$ defined over $F$ is just the direct product $GL(2, \mathbb{C}) \times W'_F$. Let $\sigma: W'_F \to \tilde{G}$ denote the natural projection map. By a lemma of Langlands (or more generally by the results of [Lab], [Ra]), any admissible homomorphism $\phi \in \Phi(G)$ admits a lifting $\tilde{\phi}: W'_F \to \tilde{G}$, such that $\sigma \circ \tilde{\phi} = \phi$.

The particular class of parameters that are of interest to us are those for which $\phi$ is an induced representation. Let $E$ be a quadratic extension of $F$, and $\theta$ a character of $W_E$. Consider the induced representation

$$I(\theta) = \text{Ind}_{W'_F}^{W'_E}(\theta): W_F \to \tilde{G},$$

where we consider $\theta$ as a character of $W'_E$ trivial on $SU(2)$.

Denote by $\sigma$ the Galois conjugation on $E$ over $F$, and by $\theta^\sigma$ the character $\theta^\sigma(\gamma) = \theta(\sigma(\gamma))$ for $\gamma \in C_E$, where we consider $\theta$ as a character of $C_E \cong W_E^b$. The representation $I(\theta)$ is irreducible precisely when $\theta^\sigma \neq \theta$. If $F$ is a local field and $G$ is anisotropic over $F$, then the parameter $I(\theta)$ is admissible for $G$ if and only if it is irreducible.

When $E/F$ is a quadratic extension of global fields, $D$ a quaternion division algebra over $F$, and $\theta$ an idele class character of $J_E/E^*$, the notion of admissibility is defined as follows:

**Definition 5.1.** An idele class character $\theta$ of $J_E/E^*$ is said to be **admissible** with respect to the quaternion algebra $D$, if $\theta^\sigma \neq \theta$ and the following is satisfied: let $v$ be a place of $F$ at which $D$ does not split. Then the place $v$ should either be inert or ramified in $E$. Let $w$ be the unique place of $E$ dividing $v$. We further require that the local component $\theta_w$ satisfies $\theta^\sigma_w \neq \theta_w$.

Let $S_\phi$ be the centralizer in $^L G^0$ of the image of $\phi$, $S_\phi^0$ be the connected component of the identity in $S_\phi$, and let $C_\phi = S_\phi/S_\phi^0$ be the group of connected components of $S_\phi$. The group $C_\phi$ is a finite abelian group and is isomorphic to either $1$, $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. 

5.2. Local theory. We recall the local theory. Let $F$ be a local field of characteristic zero and $D$ be a quaternion algebra over $F$. Let $\pi$ be a representation of $G(F)$. For $g \in \hat{G}(F)$, let $\pi^g$ denote the conjugate representation defined by $\pi^g(h) = \pi(ghg^{-1})$ for $h \in G(F)$.

Definition 5.2. Two irreducible, admissible representations $\pi_1$ and $\pi_2$ of $G(F)$ are said to be $L$-indistinguishable if $\pi_2 \cong \pi_1^g$ for some $g \in \hat{G}(F)$. Equivalently if both $\pi_1$ and $\pi_2$ occur in the restriction of an irreducible, admissible representation $\tilde{\pi}$ of $\hat{G}(F)$ to $G(F)$.

Given an irreducible, admissible representation $\pi$ of $G(F)$ (or $\tilde{\pi}$ of $\hat{G}(F)$), the $L$-packet $L(\pi)$ of $\pi$ is defined to be the collection of all irreducible, admissible representations of $G(F)$ which are $L$-indistinguishable from $\pi$.

Since the index of $F^* G(F)$ inside $\hat{G}(F)$ is finite, any irreducible, admissible representation $\tilde{\pi}$ of $\hat{G}(F)$ decomposes as a finite direct sum of irreducible, admissible representations of $G(F)$. Hence $L$-packets are of finite cardinality. It can be seen that the cardinality of an $L$-packet is either 1, 2 or 4.

If $\phi$ is an admissible parameter for $G$, and $\tilde{\phi}$ is a lifting of $\phi$ to $\hat{G}$, let $\tilde{\pi}(\tilde{\phi})$ be the irreducible, admissible representation of $\hat{G}(F)$ associated to $\tilde{\phi}$ by the local Langlands correspondence. Let $L(\phi)$ be the $L$-packet given by the collection of irreducible, admissible representations occurring in the decomposition of $\tilde{\pi}(\tilde{\phi})$ to $G(F)$. This depends only on the parameter $\phi$.

Fix an additive character $\psi$ of $F$. Depending on the choice of $\psi$, Labesse and Langlands define a pairing,

\[
\langle \cdot, \cdot \rangle : S_{\phi}/S_{\phi}^0 \times L(\phi) \to \mathbb{Z}.
\]

We make a provisional definition:

Definition 5.3. The pairing (5.1) is said to be nondegenerate if the pairing identifies $L(\phi)$ (non-canonically) as the dual group of $S_{\phi}/S_{\phi}^0$, i.e., $L(\phi)$ and $C_{\phi}$ are of equal cardinality and the map sending an element $\pi \in L(\phi)$ to the character $s \mapsto \langle s, \pi \rangle$, $s \in C_{\phi}$ is a bijection from $L(\phi)$ to the dual group of characters $\hat{C}_{\phi}$ of $C_{\phi}$.

We have:

Proposition 3. Let $F$ be a nonarchimedean local field of characteristic zero. The pairing 5.1 is nondegenerate, except when $D$ is a quaternion division algebra over $F$ and the parameter $\tilde{\phi}$ is induced of the form $I(\theta)$, where $\theta$ is a character of a quadratic extension $E$ of $F$ such that $\theta^q/\theta$ is a non-trivial quadratic character.

In the exceptional case, the $L$-packet consists of one representation $\pi$ occurring with multiplicity two in the restriction of $\tilde{\pi}(I(\theta))$ to $G(F)$, and

\[
C_{\phi} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.
\]
The pairing in this exceptional case is given by,

\[ \langle 1, \pi \rangle = 2, \quad \langle s, \pi \rangle = 0, \quad s \neq 1, \quad s \in C_\phi. \]

For the proof of Theorem 2, the following case of the pairing (5.1) is crucial: let \( D \) be a quaternion division algebra over \( F \), \( E \) a quadratic extension of \( F \), and \( \theta \) a character of \( E^* \) such that \( \theta/\theta^\sigma \) is non-trivial and not quadratic. Then the \( L \)-packet \( L(\phi) = \{ \pi(I(\theta))^+, \pi(I(\theta))^- \} \) consists of two representations and \( C_\phi \simeq \{1, \epsilon\} \) is of cardinality two, such that the pairing is given by

\[ \langle 1, \pi(I(\theta))^+ \rangle = 1, \quad \langle 1, \pi(I(\theta))^- \rangle = 1 \]
\[ \langle \epsilon, \pi(I(\theta))^+ \rangle = 1, \quad \langle \epsilon, \pi(I(\theta))^- \rangle = -1. \]

5.3. Global theory. Let \( F \) be a number field, and \( D \) be a quaternion division algebra over \( F \). If \( \Pi \) is any irreducible, admissible representation of \( G(A_F) \), then \( \Pi \) can be decomposed

\[ \Pi = \otimes_{v \in \Sigma_F} \Pi_v, \]

as a restricted tensor product of the local components \( \Pi_v \) of \( \Pi \) at the places \( v \) of \( F \). The local component \( \Pi_v \) is an irreducible, admissible representation of \( G(F_v) \), such that at almost all finite places \( v \) of \( F \), the representation \( \Pi_v \) is an unramified representation of \( G(F_v) \).

**Definition 5.4.** Two irreducible, admissible representations \( \Pi_1 \) and \( \Pi_2 \) of \( G(A_F) \) are said to be \( L \)-indistinguishable if for all places \( v \) of \( F \), the local components \( \Pi_{1,v} \) and \( \Pi_{2,v} \) are \( L \)-indistinguishable, and \( \Pi_{1,v} \simeq \Pi_{2,v} \) at almost all places of \( F \).

The \( L \)-packet \( L(\Pi) \) of an irreducible, admissible representation \( \Pi \) of \( G(A_F) \) is the collection of all irreducible, admissible representations of \( G(A_F) \) which are \( L \)-indistinguishable from \( \Pi \). Equivalently it is the orbit of \( \Pi \) under the action of \( \tilde{G}(A_F) \).

5.3.1. Dihedral representations. Let \( E/F \) be a quadratic extension of \( F \), and \( \theta \) a character of \( C_E \) satisfying \( \theta^\sigma \neq \theta \). Generalizing earlier construction of Hecke and Mass, Jacquet and Langlands associate a cuspidal automorphic representation \( \tilde{\Pi}(I(\theta)) \) of \( GL_2(A_F) \) such that at any place \( v \) of \( F \), the Langlands parameter of the local component \( \tilde{\Pi}(I(\theta))_v \) is the local component \( I(\theta)_v \). By the Jacquet-Langlands correspondence [JL], this can be lifted to an automorphic representation, denoted again by \( \tilde{\Pi}(I(\theta)) \) of \( GL_2(D(A_F)) \) provided \( \theta \) is admissible with respect to \( D \).

**Definition 5.5.** An irreducible admissible representation \( \Pi \) of \( G(A_F) \) is said to be dihedral, if there exists a quadratic extension \( E \) of \( F \) and an admissible idele class character \( \theta \) of \( E \) with respect to \( D \) such that at any place \( v \) of \( F \), the
local component $\Pi_v$ occurs in the restriction of the local component $\tilde{\Pi}(I(\theta))_v$ to $G(F_v)$. Further, at almost all finite places $v$ of $F$ where $D$ is unramified, the subgroup $SL(2, O_{F_v})$ has a nonzero fixed vector in the space of $\Pi_v$. We will say that $\Pi$ occurs in the restriction of $\tilde{\Pi}(\theta)$ to $G(A_F)$.

Given a representation $\Pi$ of $G(A_F)$, let $m(\Pi)$ be the multiplicity of $\Pi$ occurring in the space of cusp forms on $G(A_F)$.

**Theorem 4. (Labesse-Langlands)** If $\Pi$ is a nondihedral cuspidal automorphic representation $\Pi$ of $G(A_F)$, then for any $g \in \tilde{G}(A_F)$, the representations $\Pi$ and $\Pi^g$ occur with the same multiplicity in the space of cusp forms on $G(A_F)$.

**5.3.2. Multiplicity formula.** We first define an auxiliary integer that occurs in the multiplicity formula of Labesse and Langlands for the multiplicity $m(\Pi)$ of a representation of dihedral type in the space of cusp forms of $G(A_F)$.

Given two parameters, $\tilde{\phi}, \tilde{\phi}' : WF \to GL_2(\mathbb{C})$ call them to be weakly globally equivalent if for any place $v$ of $F$, there exists a character $\chi_v$ of $F_v^*$ such that $\tilde{\phi}_v = \chi_v \tilde{\phi}'_v$; and globally equivalent if there exists an idele class character $\chi$ of $CF$ such that $\tilde{\phi} = \chi \tilde{\phi}'$. Define the integer $d(\tilde{\phi})$ to be the number of weak global equivalence classes modulo global equivalence.

Let $\Pi$ be a representation of $G(A_F)$ occurring in the restriction of $\tilde{\Pi}(I(\theta))$ to $G(A_F)$. Let $\tilde{\phi} = I(\theta)$ be the Langlands parameter associated to an admissible character $\theta$ as defined above. Let $\phi$ denote the projection of $\tilde{\phi}$ to $PGL_2(\mathbb{C})$. Define $d(\Pi) = d(I(\theta))$. It follows from the definition of $L$-indistinguishability, that for $g \in \tilde{G}(A_F)$,

\begin{equation}
(5.4) \quad d(\Pi) = d(\Pi^g).
\end{equation}

For each place $v$ of $F$, let $\phi_v : W_{F_v} \to PGL_2(\mathbb{C})$ denote the local component of the parameter $\phi$. We have natural maps $S_{\phi} \to S_{\phi_v}$, sending $S_{\phi}^0$ to $S_{\phi_v}^0$. Given an element $s \in S_{\phi}$, we denote by $s_v$ both its image in $S_{\phi_v}$ and in the group of connected components $C_{\phi_v}$ of $S_{\phi_v}$. We have,

**Theorem 5. (Multiplicity formula)** Fix an additive character $\psi$ of $A_F$. Let $\Pi$ be a representation of $G(A_F)$ occurring in the restriction of $\tilde{\Pi}(I(\theta))$ to $G(A_F)$. With notation as above, the multiplicity of $\Pi$ in the space of automorphic forms of $G(A_F)$ is given by the formula:

\[ m(\Pi) = \frac{d(\Pi)}{[S_{\phi}/S_{\phi}^0]} \sum_{s \in S_{\phi}/S_{\phi}^0} \prod_{v \in \Sigma_F} \langle s_v, \Pi_v \rangle, \]

where the local pairings are normalized with respect to the additive character $\psi_v$ of $F_v$, the local component of $\psi$ at $v$.

We now make this formula more explicit.
Definition 5.6. [LL, pages 42–44] Let $\theta$ be an idele class character of $J_E/E^*$ admissible with respect to $D$. The character $\theta$ is said to be of type (b) if $\theta/\theta^\sigma$ is a (nontrivial) quadratic character. In this case $C_\phi \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

If $\theta$ is not of type (b), then it is said to be of type (a). Here $C_\phi \simeq \mathbb{Z}/2\mathbb{Z}$, which we identify with $\{1, \epsilon\}$.

Call $\theta$ to be of type $(a')$ if it is of type (a), and there exists a place $v$ of $F$ where $D$ does not split, such that $\theta_w/\theta^\sigma_w$ is a quadratic character of $E_w^*$. Here $w$ is the unique place of $E$ lying over the place $v$ of $F$.

The character $\theta$ is said to be of type $(a'')$ if it is of type $(a)$ and not of type $(a')$.

Combining Theorem 5 together with the local properties of the pairing given by Proposition 3 and Equation 5.2, we obtain the following corollary (Propositions 7.3 and 7.4 of [LL]):

Corollary 3. Let $\Pi = \Pi(\theta)$ be a dihedral representation of $G(\mathbb{A}_F)$, where $\theta$ is an idele class character of a quadratic extension $E$ of $F$, admissible with respect to $D$.

1. If $\theta$ is of type $(a')$, then
   \[
   m(\Pi) = \frac{d(\Pi)}{2} \prod_v \langle 1, \Pi_v \rangle.
   \]

2. If $\theta$ is of type $(a'')$, then
   \[
   m(\Pi) = \frac{d(\Pi)}{2} \left( \prod_v \langle 1, \Pi_v \rangle + \prod_v \langle \epsilon, \Pi_v \rangle \right).
   \]

3. If $\theta$ is of type (b), then
   \[
   m(\Pi) = \frac{d(\Pi)}{4} \prod_v \langle 1, \Pi_v \rangle.
   \]

6. Proof of Theorem 2. In order to prove the theorem, we have to show that
   \[
   L^2(\Gamma_K \backslash G_\infty) \simeq L^2(\Gamma_K^* \backslash G_\infty),
   \]
as $G_\infty$-modules. Fix an unitary representation $\pi$ of $G_\infty$. We need to show the equality of multiplicities:

\[
(6.1) \quad m(\pi, \Gamma_K) = m(\pi, \Gamma_K^*).
\]

Let $\mathcal{A}$ denote the equivalence classes of automorphic representations of $G(\mathbb{A}_F)$. Given a representation $\Pi$ of $G(\mathbb{A}_F)$, write $\Pi = \Pi_\infty \otimes \Pi_f$, where $\Pi_\infty$ (resp. $\Pi_f$), the archimedean (resp. finite) component of $\Pi$, is a representation of
$G_{\infty}$ (resp. $G(\mathbb{A}_{F_f})$). Since $G$ is simply connected and semisimple, and $G_{\infty}$ is non-compact, by strong approximation we obtain,

$$L^2(\Gamma_K \backslash G_{\infty}) \simeq \oplus_{\Pi \in \mathcal{A}} m(\Pi)\dim(\Pi^K_{\infty})\Pi_{\infty}.$$ 

Here $\Pi^K_{\infty}$ denotes the space of $K$-invariants in the representation space for $\Pi_f$, and $m(\Pi)$ is the multiplicity with which $\Pi$ occurs in the space of automorphic forms of $G(\mathbb{A}_F)$. Consequently,

$$m(\pi, \Gamma_K) = \sum_{\{\Pi \in \mathcal{A} | \Pi_{\infty} = \pi\}} m(\Pi)\dim(\Pi^K_{\infty}).$$

In order to establish Equation 6.1, it is enough to show the following:

$$\sum_{\{\Pi \in \mathcal{A} | \Pi_{\infty} = \pi\}} m(\Pi)\dim(\Pi^K_{\infty}) = \sum_{\{\Pi \in \mathcal{A} | \Pi_{\infty} = \pi\}} m(\Pi)\dim(\Pi^{K_{\pi}}_{\infty}).$$

(6.3)

Fix a representation $\pi_{\infty}$ of $G_{\infty}$. In order to prove Equation 6.3, given a $\Pi$ with archimedean component $\Pi_{\infty} = \pi$, we will produce another representation $\Pi'$ in the same $L$-packet satisfying,

- The archimedean component $\Pi'_{\infty} = \pi$.
- The map $\Pi \mapsto \Pi'$ is injective, and

$$m(\Pi)\dim(\Pi^K_{\infty}) = m(\Pi')\dim(\Pi'^{K_{\pi}}_{\infty}).$$

(6.4)

By Equation 6.2 it follows that $m(\pi, \Gamma_K) \leq m(\pi, \Gamma^{K_{\pi}}_K)$. Reversing the argument we obtain Equation 6.3, and the theorem will be proved.

If $\Pi$ is not dihedral, let $\Pi' = \Pi$. By Theorem 4, $m(\Pi) = m(\Pi')$. Clearly $\dim(\Pi^K_{\infty}) = \dim((\Pi')^{K_{\pi}}_{\infty})$ and Equation 6.4 follows.

Now suppose $\Pi$ is a dihedral representation of $G(\mathbb{A}_F)$ associated to an idèle class character $\theta$ of a quadratic extension $E$ over $F$ of type $(a')$ or type $(b)$. Let $\Pi' = \Pi^\varepsilon$. By parts (1) and (3) of Corollary 3 and Equation 5.4,

$$m(\Pi) = m(\Pi^\varepsilon),$$

and Equation 6.4 follows.

Hence we have to only consider dihedral representations $\Pi$ of the form $\pi(\theta)$ (with archimedean component $\pi$), where $\theta$ is an idèle class character of a quadratic extension $E/F$ of type $(a'')$. In particular, this means that the distinguished place $v_0$ of $F$ and $w_0$ the unique place of $E$ lying above $v_0$, the character $\theta_{w_0}/\theta^\varepsilon_{w_0}$ is not quadratic. By 5.3, at the place $v_0$, there are two elements in the $L$-packet

$$L(I(\theta_{w_0})) = \{ \pi(I(\theta_{w_0}))^+, \pi(I(\theta_{w_0}))^- \}.$$
which pairs perfectly with the group of connected components $C_{I(\theta_{w_0})} \simeq \mathbb{Z}/2\mathbb{Z} = \{1, \epsilon\} \simeq \{1, -1\}$, as given by Equation 5.3. Denote by $\epsilon(\Pi_{v_0})$ the element of the group $\{1, \epsilon\}$ associated to the component $\Pi_{v_0}$ via this perfect pairing.

Let $T$ be the finite subset of finite places $v \neq v_0$ of $F$, such that

$$\langle \epsilon, \pi_v \rangle \neq \langle \epsilon, \pi_v^x \rangle,$$

where $x_v$ is the component at $v$ of the adele $x \in \tilde{G}(\mathbb{A}_f)$.

Define the local components of $\Pi'$ at any place $v \neq v_0$ of $F$ as,

$$\Pi'_v = \Pi'_v.$$

At the place $v_0$ of $F$, we define the local component of $\Pi'$ as,

$$\Pi'_{v_0} = \begin{cases} \pi(I(\theta_{w_0}))^+ & \text{if } \epsilon(\Pi_{v_0})(-1)^{|T|} = 1, \\ \pi(I(\theta_{w_0}))^- & \text{if } \epsilon(\Pi_{v_0})(-1)^{|T|} = -1. \end{cases}$$

It follows from the definition of the local pairings, that

$$\prod_v \langle \epsilon, \Pi_v \rangle = \prod_v \langle \epsilon, \Pi'_v \rangle,$$

and hence by part (2) of Corollary 3 we obtain that

$$m(\Pi) = m(\Pi').$$

Since the representations $\pi(\theta)_{v_0}^{\pm}$ are conjugate under $D_{v_0}^*$, both $\Pi$ and $\Pi'$ are in the same $L$-packet, and by Equation 5.4,

$$d(\Pi) = d(\Pi').$$

By Hypothesis $H3$, $K_{v_0}$ is a normal subgroup of $D_{v_0}^*$. Hence,

$$\dim(\Pi^K_f) = \dim(\Pi'^{K_f}).$$

From these equations, we conclude that Equation 6.4 is satisfied.

This concludes the proof of Theorem 2.
REFERENCES


