HERMITIAN SYMMETRIC SPACE, FLAT BUNDLE AND HOLOMORPHICITY CRITERION

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ABSTRACT. Let $K \backslash G$ be an irreducible Hermitian symmetric space of noncompact type and $\Gamma \subset G$ a closed torsionfree discrete subgroup. Let $X$ be a compact Kähler manifold and $\rho : \pi_1(X, x_0) \to \Gamma$ a homomorphism such that the adjoint action of $\rho(\pi_1(X, x_0))$ on $\text{Lie}(G)$ is completely reducible. A theorem of Corlette associates to $\rho$ a harmonic map $X \to K \backslash G/\Gamma$. We give a criterion for this harmonic map to be holomorphic. We also give a criterion for it to be anti–holomorphic.

1. Introduction

Let $G$ be a noncompact simple Lie group of adjoint type and $K \subset G$ a maximal compact subgroup, such that $K \backslash G$ is an irreducible Hermitian symmetric space of noncompact type. The Lie algebra of $G$ will be denoted by $\mathfrak{g}$. Let $\Gamma$ be a closed torsionfree discrete subgroup of $G$. Take a compact connected Kähler manifold $X$; fix a base point $x_0 \in X$. Let

$$\rho : \pi_1(X, x_0) \to \Gamma$$

be a homomorphism such that the adjoint action of $\rho(\pi_1(X, x_0))$ on $\mathfrak{g}$ is completely reducible. This $\rho$ produces a $C^\infty$ principal $G$–bundle $E_G \to X$ equipped with a flat connection $D$. A reduction of structure group of $E_G$ to $K$ is given by a map $X \to K \backslash G/\Gamma$. Note that $K \backslash G/\Gamma$ is a Kähler manifold; it need not be compact.

A theorem of Corlette says that there is a $C^\infty$ reduction of structure group of $E_G$ to $K$ such that corresponding map $H_D : X \to K \backslash G/\Gamma$ is harmonic [Co]. Our aim here is to address the following:

- When is $H_D$ holomorphic?
- When is $H_D$ anti–holomorphic?

Let $H : X \to K \backslash G/\Gamma$ be a $C^\infty$ map giving a reduction of structure group of $E_G$ to $K$. We give a criterion under which $H$ is holomorphic or anti–holomorphic (see Theorem 4.3). Since a holomorphic or anti–holomorphic map between Kähler manifolds is harmonic, Theorem 4.3 gives criterion for $H_D$ to be holomorphic or anti–holomorphic (see Corollary 4.4).

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2. Harmonic map to $G/K$

A Lie group is called simple if its Lie algebra is so [He, page 131]. Let $G$ be a connected real noncompact simple Lie group whose center is trivial. It is known that if $K_1$ and $K_2$ are two maximal compact subgroups of $G$, then there is an element $g \in G$ such that $K_1 = g^{-1}K_2g$ [He, page 256, Theorem 2.2(ii)]. In particular, any two maximal compact subgroups of $G$ are isomorphic. Assume that $G$ satisfies the condition that the dimension of the center of a maximal compact subgroup $K$ of it is positive. This in fact implies that the center of $K$ is isomorphic to $S^1 = \mathbb{R}/\mathbb{Z}$ [He, page 382, Proposition 6.2].

Fix a maximal compact subgroup $K \subset G$. Consider the left–translation action of $K$ on $G$. The above conditions on $G$ imply that the corresponding quotient $K\backslash G$ is an irreducible Hermitian symmetric space of noncompact type. Conversely, given any irreducible Hermitian symmetric space of noncompact type, there is a group $G$ of the above type such that $K\backslash G$ is isometrically isomorphic to it [He, page 381, Theorem 6.1(i)]. In fact, this gives a bijection between the isomorphism classes of Lie groups of the above type and the holomorphic isometry classes of irreducible Hermitian symmetric spaces [He, page 381, Theorem 6.1(i)]. The Lie algebra of $G$ will be denoted by $\mathfrak{g}$.

Fix a closed torsionfree discrete subgroup $\Gamma \subset G$. Therefore, the two-sided quotient

$$M_\Gamma := K\backslash G/\Gamma$$

(2.1)

is a connected Kähler manifold.

Let $X$ be a compact connected orientable real manifold. Fix a base point $x_0 \in X$. Let

$$\beta : \tilde{X} \longrightarrow X$$

(2.2)

be the universal cover of $X$ associated to $x_0$. The right–action of the fundamental group $\pi_1(X, x_0)$ on $\tilde{X}$ will be denoted by "·". Let

$$\rho : \pi_1(X, x_0) \longrightarrow \Gamma$$

(2.3)

be a homomorphism such that the adjoint action of $\rho(\pi_1(X, x_0)) \subset G$ on the Lie algebra $\mathfrak{g}$ is completely reducible. Associated to $\rho$ we have a principal $G$–bundle $E_G \longrightarrow X$ equipped with a flat connection $D$. We note that $E_G$ is the quotient of $\tilde{X} \times G$ (see (2.2)) where two points $(x_1, g_1), (x_2, g_2) \in \tilde{X} \times G$ are identified if there is an element $z \in \pi_1(X, x_0)$ such that $x_2 = x_1 \cdot z$ and $g_2 = \rho(z)^{-1}g_1$. The natural projection $\tilde{X} \times G \longrightarrow \tilde{X}$ produces the projection $E_G \longrightarrow X$ from the quotient space, and the right–translation action of $G$ on $\tilde{X} \times G$ produces the action of $G$ on $E_G$. The trivial connection on the trivial principal $G$–bundle $\tilde{X} \times G \longrightarrow \tilde{X}$ descends to the connection $D$ on the principal $G$–bundle $E_G$.

The above construction gives the following:

**Corollary 2.1.** The pulled back principal $G$–bundle $\beta^*E_G$ is identified with the trivial principal $G$–bundle $\tilde{X} \times G$ by sending any $(x, g) \in \tilde{X} \times G$ to $(x, z)$, where $z$ is the equivalence class of $(x, g)$. This identification takes the connection $\beta^*D$ on $\beta^*E_G$ to the trivial connection on the trivial principal $G$–bundle $\tilde{X} \times G$. 
A Hermitian structure on $E_G$ is a $C^\infty$ reduction of structure group $E_K \subset E_G$ to the maximal compact subgroup $K \subset G$. For such a reduction of structure group $E_K$, we have the reduction of structure group
\[ \beta^* E_K \subset \beta^* E_G = \tilde{X} \times G \rightarrow \tilde{X}, \]
where $\beta$ is the projection in (2.2). Therefore, a Hermitian structure on $E_G$ is given by a $\pi_1(X, x_0)$–equivariant $C^\infty$ map
\[ h : \tilde{X} \rightarrow G/K \] (2.4)
with $\pi_1(X, x_0)$ acting on the left of $G/K$ via the homomorphism $\rho$.

Let $\iota : G/K \rightarrow K \backslash G$ be the isomorphism that sends a coset $gK$ to the coset $Kg^{-1}$. Let
\[ E_K \subset E_G \]
be the reduction of structure group to $K$ corresponding to a map $h$ as in (2.4). Since $h$ is $\pi_1(X, x_0)$–equivariant, the composition $\iota \circ h$ descends to a map
\[ H_D : X \rightarrow M_\Gamma = K \backslash G/\Gamma \] (2.5)
(see (2.1)).

Fix a Riemannian metric $g_X$ on $X$. A theorem of Corlette says that there is a Hermitian structure $E'_K \subset E_G$ such that the above map $h$ is harmonic with respect to $\beta^* g_X$ and the natural invariant metric on $G/K$ [Co, page 368, Theorem 3.4] (in the special case where $\dim X = 2$ and $G = \text{PSL}(2, \mathbb{R})$, this was proved in [Do]). Note that $h$ is harmonic if and only if $H_D$ in (2.5) is harmonic with respect to $g_X$ and the natural metric on $M_\Gamma$. If $E'_K$ is another Hermitian structure such that corresponding map $\tilde{X} \rightarrow G/K$ is also harmonic, then there is an automorphism $\delta$ of the principal $G$–bundle $E_G$ such that
\begin{itemize}
  \item $\delta$ preserves the flat connection $D$ on $E_G$, and
  \item $\delta(E_K) = E'_K$.
\end{itemize}
In other words, if $h' : \tilde{X} \rightarrow G/K$ is the $\pi_1(X, x_0)$–equivariant map corresponding to this $E'_K$, then then there is an element $g \in G$ such that
\begin{itemize}
  \item $g$ commutes with $\rho(\pi_1(X, x_0))$, and
  \item $h'(y) = gh(y)$ for all $y \in \tilde{X}$.
\end{itemize}
Note that the first condition that $g$ commutes with $\rho(\pi_1(X, x_0))$ implies that the map $y \mapsto gh(y)$ intertwines the actions of $\pi_1(X, x_0)$ on $\tilde{X}$ and $G/K$.

## 3. Constructions using flat connection and Hermitian structure

Let $G$ and $K$ be as before. As before, the Lie algebra of $G$ will be denoted by $\mathfrak{g}$. The Lie algebra of $K$ will be denoted by $\mathfrak{k}$. We have the isotypical decomposition
\[ \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \] (3.1)
for the adjoint action of the center of $K$ [He, p. 208, Theorem 3.3(iii)]. Note that the adjoint action of $K$ on $\mathfrak{g}$ preserves this decomposition. Let

$$\phi : \mathfrak{g} \rightarrow \mathfrak{k} \quad \text{and} \quad \psi : \mathfrak{g} \rightarrow \mathfrak{p}$$

be the projections associated to the decomposition in (3.1).

Let $Y$ be a connected complex manifold, $F_G$ a $C^\infty$ principal $G$–bundle on $Y$ and $D^Y$ a flat connection on the principal $G$–bundle $F_G$. Fix a $C^\infty$ reduction of structure group of $F_G$

$$F_K \subset F_G$$
to the subgroup $K$. Let

$$\text{ad}(F_K) = F_K(\mathfrak{k}) := F_K \times^K \mathfrak{k} \rightarrow Y \quad \text{and} \quad F_K(\mathfrak{p}) := F_K \times^K \mathfrak{p} \rightarrow Y$$

be the vector bundles associated to this principal $K$–bundle $F_K$ for the $K$–modules $\mathfrak{k}$ and $\mathfrak{p}$ respectively. Let $\text{ad}(F_G) = F_G \times^G \mathfrak{g} \rightarrow Y$ be the adjoint vector bundle for $F_G$. From (3.1) it follows that

$$\text{ad}(F_G) = \text{ad}(F_K) \oplus F_K(\mathfrak{p}).$$

The connection $D^Y$ on $F_G$ is defined by a $\mathfrak{g}$–valued 1–form on the total space of $F_G$; this form on $F_G$ will also be denoted by $D^Y$. The restriction of this form $D^Y$ to $F_K$ will be denoted by $D'$. Consider the $\mathfrak{k}$–valued 1–form $\phi \circ D'$ on $F_K$, where $\phi$ is the projection in (3.2). It is $K$–equivariant and restricts to the Maurer–Cartan form on the fibers of the principal $K$–bundle $F_K$. Hence $\phi \circ D'$ is a connection on $F_K$; we will denote by $D_K$ this connection on $F_K$. Consider the $\mathfrak{p}$–valued 1–form $\psi \circ D'$ on $F_K$, where $\psi$ is the projection in (3.2). It is $K$–equivariant and its restriction to any fiber of the principal $K$–bundle $F_K$ vanishes identically. Therefore, $\psi \circ D'$ is a $C^\infty$ section $F_K(\mathfrak{p}) \otimes T^*Y$, where $F_K(\mathfrak{p})$ is the vector bundle defined in (3.3), and $T^*Y$ is the real cotangent bundle of $Y$. Let

$$D^{Y,p} \in C^\infty(Y, F_K(\mathfrak{p}) \otimes T^*Y)$$

be this $F_K(\mathfrak{p})$–valued 1–form on $Y$.

Using the complex structure on $Y$ we may decompose any $F_K(\mathfrak{p})$–valued 1–form on $Y$ into a sum of $F_K(\mathfrak{p})$–valued forms of types $(1,0)$ and $(0,1)$. Let

$$D^{Y,p} = D^{1,0} + D^{0,1}$$

be the decomposition of the above $F_K(\mathfrak{p})$–valued 1–form $D^{Y,p}$ into $(1,0)$ and $(0,1)$ parts.

Define

$$\omega_{D^Y} := \frac{1}{\sqrt{-1}} (D^{1,0} - D^{0,1}) \in C^\infty(Y, (\text{ad}(F_G) \otimes T^*Y) \otimes_{\mathbb{R}} \mathbb{C}),$$

where $D^{1,0}$ and $D^{0,1}$ are constructed in (3.6). From (3.6) it follows that

$$\omega_{D^Y} \in C^\infty(Y, F_K(\mathfrak{p}) \otimes T^*Y) \subset C^\infty(Y, (\text{ad}(F_G) \otimes T^*Y) \otimes_{\mathbb{R}} \mathbb{C})$$

(see (3.4)). In other words, $\omega_{D^Y}$ is a real 1–form on $Y$ with values in the real vector bundle $F_K(\mathfrak{p}) \subset \text{ad}(F_G) \otimes_{\mathbb{R}} \mathbb{C}$. 
We now work in the set-up of Section 2. Assume that the manifold $X$ is equipped with a complex structure. Identify two elements $(g_1, g_1')$ and $(g_2, g_2')$ of $(K \backslash G) \times G$ if there is an element $\gamma \in \Gamma$ such that $g_2 = g_1 \gamma^{-1}$ and $g_2' = \gamma g_1'$. The corresponding quotient will be denoted by $\mathcal{E}_G$. The group $G$ acts on $\mathcal{E}_G$; the action of any $g \in G$ sends the equivalence class of $(g_1, g_1') \in (K \backslash G) \times G$ to the equivalence class of $(g_1, g_1' g)$. Consider the map

$$\mathcal{E}_G \longrightarrow M_\Gamma = K \backslash G / \Gamma$$

that sends the equivalence class of any $(g, g') \in (K \backslash G) \times G$ to the equivalence class of $g$. It, and the above action of $G$ on $\mathcal{E}_G$, together make $\mathcal{E}_G$ a principal $G$–bundle on $M_\Gamma$. Pull back the left invariant Maurer–Cartan form on $G$ using the projection to the second factor $(K \backslash G) \times G \longrightarrow G$. This pulled back form descends to the quotient space $\mathcal{E}_G$. It is straightforward to check that this descended form defines a connection on the principal $G$–bundle $\mathcal{E}_G$. This connection on $\mathcal{E}_G$ will be denoted by $D^0$. The connection $D^0$ is flat.

Consider the submanifold $\{ (g^{-1}, g) \mid g \in G \} \subset G \times G$. Let

$$\mathcal{N} \subset (K \backslash G) \times G$$

be the image of it under the obvious quotient map. Let

$$\mathcal{E}_K \subset \mathcal{E}_G$$

be the image of $\mathcal{N}$ in the quotient space $\mathcal{E}_G$ (recall that $\mathcal{E}_G$ is a quotient of $(K \backslash G) \times G$). It is straightforward to check that the action of the subgroup $K \subset G$ on $\mathcal{E}_G$ preserves $\mathcal{E}_K$. More precisely, $\mathcal{E}_K$ is a reduction of structure group of the principal $G$–bundle $\mathcal{E}_G \longrightarrow M_\Gamma$ to the subgroup $K \subset G$.

Take $\rho$ as in (2.3). As in Section 2, let $(E_G, D)$ denote the associated flat principal $G$–bundle over the compact complex manifold $X$. Take any $C^\infty$ reduction of structure group

$$E_K \subset E_G$$

to $K \subset G$.

**Proposition 4.1.** The pulled back principal $G$–bundle $H^*_D \mathcal{E}_G$, where $H_D$ is constructed in (2.5), is canonically identified with the principal $G$–bundle $E_G$. This identification between $E_G$ and $H^*_D \mathcal{E}_G$ takes

1. the pulled back connection $H^*_D D^0$ to the connection $D$ on $E_G$, and
2. the reduction $H^*_D \mathcal{E}_K \subset H^*_D \mathcal{E}_G$ (see (4.2)) to the reduction $E_K \subset E_G$.

**Proof.** Consider the pulled back reduction of structure group $\beta^* E_K \subset \beta^* E_G$, where $\beta$ is the universal cover in (2.2). From Corollary 2.1 we know that $\beta^* E_G = \widetilde{X} \times G$. Let

$$f' : \beta^* E_K \longrightarrow G$$

be the composition

$$\beta^* E_K \hookrightarrow \beta^* E_G \xrightarrow{\sim} \widetilde{X} \times G \xrightarrow{\text{pr}_2} G,$$
where \( \text{pr}_2 \) is the projection to the second factor. Now define the map

\[
f'' : \beta^*E_K \rightarrow (K \backslash G) \times G, \quad z \mapsto (\widehat{f'}(z)^{-1}, f'(z)),
\]
where \( \widehat{f'}(z)^{-1} \in K \backslash G \) is the image of \( f'(z)^{-1} \) under the quotient map \( G \rightarrow K \backslash G \). The Galois group \( \pi_1(X, x_0) \) for \( \beta \) has a natural right–action of the pullback \( \beta^*E_K \) that lifts the right–action of \( \pi_1(X, x_0) \) on \( \hat{X} \). For any \( z \in \beta^*E_K \) and \( \gamma \in \pi_1(X, x_0) \), we have

\[
f''(z \cdot \gamma) = (\widehat{f'}(z)^{-1} \rho(\gamma), \rho(\gamma)^{-1} f'(z)),
\]
where \( z \cdot \gamma \in \beta^*E_K \) is the image of \( z \) under the action of \( \gamma \). This, and the fact that \( \text{image}(f'') \subset \mathcal{N} \) (defined in (4.1)), together imply that \( f'' \) descends to a map

\[
\tilde{d} : E_K \rightarrow \mathcal{E}_K
\]
(see (4.2)). This map \( \tilde{d} \) is clearly \( K \)–equivariant, and the following diagram is commutative

\[
\begin{array}{ccc}
E_K & \xrightarrow{\tilde{d}} & \mathcal{E}_K \\
\downarrow & & \downarrow \\
X & \xrightarrow{H_D^*} & M
\end{array}
\]

Therefore, we get an isomorphism of principal \( K \)–bundles

\[
f : E_K \rightarrow H_D^* \mathcal{E}_K.
\]  
(4.3)

Since \( E_G \) (respectively, \( \mathcal{E}_G \)) is the extension of structure group of \( E_K \) (respectively, \( \mathcal{E}_K \)) using the inclusion of \( K \) in \( G \), the isomorphism \( f \) in (4.3) produces an isomorphism of principal \( G \)–bundles

\[
\hat{f} : E_G \rightarrow H_D^* \mathcal{E}_G.
\]

It is straightforward to check that \( \hat{f} \) takes the connection \( D \) to the connection \( H_D^*D^0 \). \( \square \)

Define \( \mathcal{E}_K(p) := \mathcal{E}_K \times^K p \rightarrow M_\Gamma \) as in (3.3). Also, construct the associated vector bundle \( E_K(p) \) on \( X \) as done in (3.3).

**Corollary 4.2.** The pulled back vector bundle \( H_D^* \mathcal{E}_K(p) \), where \( H_D \) is defined in (2.5), is canonically isomorphic to \( E_K(p) \).

**Proof.** The isomorphism \( f \) in (4.3) between principal \( K \)–bundles produces an isomorphism between the associated vector bundles. \( \square \)

Let

\[
\omega_{D^0} \in C^\infty(M_\Gamma, \mathcal{E}_K(p) \otimes_\mathbb{R} T^*M_\Gamma)
\]
be the real 1–form constructed as in (3.7) for the pair \( (D^0, \mathcal{E}_K) \). Using the isomorphisms in Proposition 4.1 and Corollary 4.2, together with the homomorphism

\[
(dH_D)^* : H_D^*T^*M_\Gamma \rightarrow T^*X,
\]
the pulled back section \( H_D^* \omega_{D^0} \) produces a section

\[
\widehat{H_D^* \omega_{D^0}} \in C^\infty(X, E_K(p) \otimes_\mathbb{R} T^*X).
\]  
(4.5)
Let
\[ \omega_D \in C^\infty(X, E_K(p) \otimes T^*X) \]
be the real 1–form constructed as in (3.7) for the pair \((D, E_K)\).

**Theorem 4.3.** The map \(H_D\) is holomorphic if and only if \(\tilde{H}^*_D \omega_{D0} = \omega_D\) (see (4.5) and (4.6)).

The map \(H_D\) is anti–holomorphic if and only if \(\tilde{H}^*_D \omega_{D0} = -\omega_D\).

**Proof.** The real tangent bundle \(TM_\Gamma\) of \(M_\Gamma = K\backslash G/\Gamma\) is identified with the associated vector bundle \(E_K(p)\), where \(E_K\) is the principal \(K\)–bundle in (4.2). Also, we have \(H^*_D E_K = E_K\) by Proposition 4.1. Combining these we conclude that the vector bundle \(E_K(p) \to X\) in (3.3) is identified with the pullback \(H^*_D TM_\Gamma\). The \(E_K(p)\)–valued 1–form \(D^p \in C^\infty(X, E_K(p) \otimes T^*X)\), obtained by substituting \((X, D, E_K)\) in place of \((Y, D^Y, F_K)\) in (3.5), coincides with the section given by the differential
\[ dH_D : TX \to H^*_D TM_\Gamma. \]
From this it follows that \(\omega_D\) in (4.6) is the section given by the homomorphism
\[ TX \to H^*_D TM_\Gamma, \quad v \mapsto dH_D(J_X(v)), \]
where \(J_X : TX \to TX\) is the almost complex structure on \(X\).

Let
\[ D^{0,p} \in C^\infty(M_\Gamma, E_K(p) \otimes T^*M_\Gamma) \]
be the section constructed just as \(D^{0,p}\) is constructed in (3.5) after substituting \((E_G, D^{0}, E_K)\) in place of \((Y, D^Y, F_K)\). This \(D^{0,p}\) coincides with the section given by the identity map of \(TM_\Gamma\) (recall that \(TM_\Gamma\) is identified with \(E_K(p)\)). Therefore, the section \(\omega_{D0}\) in (4.4) coincides with the section given by the almost complex structure
\[ J_{M_\Gamma} : TM_\Gamma \to TM_\Gamma \]
on \(M_\Gamma\). Consequently, \(\tilde{H}^*_D \omega_{D0} = \omega_D\) (respectively, \(\tilde{H}^*_D \omega_{D0} = -\omega_D\)) if and only if the differential \(dH_D\) takes the almost complex structure \(J_X\) to \(J_{M_\Gamma}\) (respectively, \(-J_{M_\Gamma}\)). This completes the proof. \(\square\)

Now assume that the complex manifold \(X\) is Kähler. This means that \(X\) is equipped with a Hermitian structure \(g_X\) such that

- the almost complex structure \(J_X\) on \(X\) is orthogonal with respect to \(g_X\), and
- the \((1,1)\) on \(X\) associate to the pair \((g_X, J_X)\) is closed.

If \(A\) and \(B\) are Kähler manifolds, then any holomorphic map \(A \to B\) is harmonic [EL, page 1, § (1.2)(e)], [Li]. This implies that any anti–holomorphic map \(A \to B\) is also harmonic; to see this simply replace the almost complex structure \(J_A\) of \(A\) by the almost complex structure \(-J_A\). Therefore, Theorem 4.3 has the following corollary:
Corollary 4.4. Take the Hermitian structure on $E_G$ given by a map $H_D : X \to K\backslash G/\Gamma$. If $\widetilde{H}_D^*\omega_D = \omega_D$, then the map

$$X \to K\backslash G/\Gamma$$

given by a harmonic metric in [Co] is holomorphic. In that case $H_D$ gives a harmonic metric. Conversely, if $H_D$ gives a harmonic metric and $\widetilde{H}_D^*\omega_D = \omega_D$, then $H_D$ is holomorphic.

Take the Hermitian structure on $E_G$ given by a map $H_D : X \to K\backslash G/\Gamma$. If $\widetilde{H}_D^*\omega_D = -\omega_D$, then the map $X \to K\backslash G/\Gamma$ given by a harmonic metric in [Co] is anti–holomorphic. In that case $H_D$ gives a harmonic metric. Conversely, if $H_D$ gives a harmonic metric and $\widetilde{H}_D^*\omega_D = -\omega_D$, then $H_D$ is anti–holomorphic.

References


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