Lectures on
On the Mean-Value and Omega-Theorems for the Riemann Zeta-Function

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To Professors

Alan Baker, FRS

and

Hugh L. Montgomery

Preface

This monograph is meant to be an appendix to the Chapters VII and VIII of the famous book of E.C. TITCHMARSH (second edition) revised and edited by D.R. HEATH-BROWN. Recently there are a few books on these topics but I am sure that the material presented in this book is new and that this book is not a repetition. There are a few important theorems which are new, especially the asymptotic formula as $H \to \infty$ for

$$\min_{I,|I|=H} \max_{t \in I} |(\zeta(1+it))^z|, z = e^{i\theta},$$

where θ is a real constant and the minimum is taken as I runs over all t-intervals (of fixed length H) contained in $[2, \infty)$. I hope that the style of writing motivates the topic and is also readable. Many of the topics deal with the joint work of mine with Professor R. BALASUBRAMANIAN to whom my indebtedness is due. I owe a lot to the famous book of E.C. TITCHMARCH mentioned already, to the twelve lectures on RAMANUJAN by G.H. HARDY, Distribution of prime numbers by A.E. INGHAM, Riemann zeta-function by K. CHANDRASEKHARAN, and Topics in Multiplicative Number Theory by H.L. MONTGOMERY. Recently I am also indebted to the book Arithmetical functions by K. CHANDRASEKHARAN, Sieve methods by H.-E. RICHERT, A method in the theory of exponential sums by M. JUTILA, and to the two books Riemann zeta-function and Mean values of the Riemann zeta-function by A. IVIĆ. I owe a lot (by way of their encouragement at all stages of my work) to Professors P.X. GALLAGHER, Y. MOTOHASHI, E. BOMBIERI, H.L. MONTGOMERY, D.R. HEATH-BROWN, H.-E. RICHERT, M. JUTILA, K. CHANDRASEKHARAN,

A. BAKER, FRS, A. IVIĆ and M.N. HUXLEY. Particular mention has to be made of Professor M. JUTILA for his many encouraging letters and his help in various ways.

I have to give an excuse for referring to too many papers of myself and that of R. BALASUBRAMANIAN. The excuse is that I deal with an appendix to Chapters VII and VIII of the famous book of E.C. TITCHMARSH as mentioned already and that I can do better justice (by way of a good exposition when writing about our work).

I acknowledge some important help given to me by Dr. A. SANKARANARAYANAN and Sri K. SOUNDARARAJAN.

Next I am indebted to Professor S. RAMANAN for agreeing to publish this material in the Lecture Notes series of TIFR.

Lastly I offer my sincere thanks to Sri D.B. SAWANT for his excellent typing of the manuscript in TEX.

K. RAMACHANDRA

Notation

Except very rarely, the notation is standard. The letters C, c with or without suffixes denote constants. Sometimes we use A, B, D, E, also for constants. T and H will be real variables in the domain $T \geq H \geq C \log\log T$. T will be sufficiently large. Sometimes we use $T \geq H \geq C$. In Chapter II we give explicit constants everywhere with the hope that the results will be useful in many situations. However we have not attempted to get economical constants. The same remarks are true of weak Titchmarsh series with which we deal in Chapter III. The letters w, z and s will be reserved for complex variables. Very often we write $w = u + iv$, $z = x + iy$ and $s = \sigma + it$. But some times there may be exceptions (for example in Chapter VII) in the notation for u, v, x, y, σ and t. The letter k will be often real. Sometimes it is a constant and sometimes it is a variable depending on the context. The letter θ will denote an arbitrary real number. (In the chapter on introductory remarks as well as in Chapter V, θ will denote the least upper bound of the real parts of the zeros of $\zeta(s)$. Of course as is usual $\zeta(s) = \sum_{n=1}^{\infty} n^{-s} (\sigma > 1)$ and its analytic continuations). We write $\mathrm{Exp}(z)$ for e^z. Other standard notation used is as follows:

(1) $f(x) \sim g(x)$ as $x \to x_0$ means $f(x)(g(x))^{-1} \to 1$ as $x \to x_0$, where x_0 is possibly ∞.

(2) $f(x) = O(g(x))$ with $g(x) \geq 0$ means $|f(x)(g(x))^{-1}|$ does not exceed a constant independent of x in the range in context.

(3) $f(x) \ll g(x)$ means $f(x) = O(g(x))$ and $f(x) \gg g(x)$ will mean the

same as $g(x) = O(f(x))$ and $f(x) \asymp g(x)$ will mean both $f(x) \ll g(x)$ and $g(x) \ll f(x)$.

(4) $f(x) = \Omega(g(x))$ will mean $f(x)(g(x))^{-1}$ does not tend to zero. $F(x) = \Omega_+(g(x))$ will mean $\limsup(f(x)(g(x))^{-1}) > 0$. $f(x) = \Omega_-(g(x))$ will mean $\liminf(f(x)(g(x))^{-1}) < 0$. Also $f(x) = \Omega_\pm(g(x))$ will mean both $f(x) = \Omega_+(g(x))$ and $f(x) = \Omega_-(g(x))$. In these Ω results in the text the range in context is as $x \to \infty$.

(5) The letters ϵ, δ, η will denote small positive constants.

Contents

Introductory Remarks

Riemann zeta-function $\zeta(s)(s = \sigma + it)$ is defined in $\sigma > 1$ by **1**

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \left(= \prod_{p} (1 - p^{-s})^{-1} \right) \tag{1}$$

where the product in the parenthesis is over all primes. The identity connecting the series in (1) with the product is the well-known Euler product. Euler knew very much more about the series in (1). He knew things like

$$\begin{aligned}\zeta(s) &= \sum_{n=1}^{\infty} \left(n^{-s} - \int_{n}^{n+1} \frac{du}{u^s} \right) + \sum_{n=1}^{\infty} \int_{n}^{n+1} \frac{du}{u^2} \\ &= \sum_{n=1} \left(n^{-s} - \int_{n}^{n+1} \frac{du}{u^s} \right) + \frac{1}{s-1}, \quad (\sigma > 0).\end{aligned} \tag{2}$$

Also by the repetition of the trick by which we obtained (2) from the series in (1) we can prove that the series in (2) is an entire function (a fact known to Euler). Moreover he knew certainly bounds for the absolute value of the series in (2) and its analytic continuation in the form

$$|\zeta(s) - \frac{1}{s-1}| \ll_A (|t| + 10)^{A+2} \quad (\sigma \geq -A, A \geq 0) \tag{3}$$

and also things like

$$|\zeta(s) - \frac{1}{s-1}| \ll (|t| + 10)^{1-\sigma} \log(|t| + 10), \quad (0 \leq \sigma \leq 1). \tag{4}$$

Euler knew even the functional equation of $\zeta(s)$ (see A. Weil, [104], p. 261-266). However the question of the distribution of the zeros of
2 $\zeta(s)$ was raised by Riemann who initiated some important researches. Riemann conjectured that

$$\zeta(s) \neq 0, (\sigma > \frac{1}{2}), \tag{5}$$

and from the functional equation it was an obvious deduction from this that

$$\zeta(s) = 0 (0 \leq \sigma \leq 1) \text{ implies } \sigma = \frac{1}{2}. \tag{6}$$

This is the famous Riemann conjecture. This being an intractable problem at present (it has withstood the attacks of many important mathematicians like G.H. Hardy for more than a century) we ask: What are some important consequences of (5)? Can we prove any of them without assuming (5)? I mention four outstanding unsolved problems which follow as consequences (5).

First Consequence. For every fixed $\epsilon > 0$, we have

$$\zeta(\frac{1}{2} + it) t^{-\epsilon} \to 0 \text{ as } t \to \infty. \tag{7}$$

Remark 1. In fact J.E. Littlewood proved that (5) implies things like

$$\zeta\left(\frac{1}{2} + it\right) \mathrm{Exp}\left(-\frac{10 \log t}{\log \log t}\right) \to 0 \text{ as } t \to \infty. \tag{8}$$

Remark 2. The latest unconditional result in (7) is with $\epsilon > \dfrac{89}{570}$ due to M.N. Huxley. The truth of (7) for every $\epsilon > 0$ is called Lindelöf hypothesis.

Second Consequence. Consider the rectangle

$$\{\sigma \geq \alpha, 0 \leq t \leq T\} \quad \left(\frac{1}{2} \leq \alpha \leq 1, T \geq 10\right). \tag{9}$$

The number of zeros of $\zeta(s)$ in this rectangle does not exceed

$$C(\epsilon)T^{(2+\epsilon)(1-\sigma)}(\log T)^{100}, \tag{10}$$

for every $\epsilon > 0$, provided we assume (7). (Consequences like this were deduced from (7) for the first time by A.E. Ingham). The unconditional results $\epsilon = \dfrac{2}{3}, \dfrac{1}{2}$ and $\dfrac{2}{5}$ were obtained by A.E. Ingham, H.L. Mont- 3
gomery and M.N. Huxley respectively. The truth of (10) for every $\epsilon > 0$ is called Density hypothesis.

Third Consequence. Let $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$ be the sequence of all primes. Then A.E. Ingham deduced from (10) that

$$p_{n+1} - p_n \leq C(\epsilon)p_n^{\frac{1}{2}+\epsilon} \tag{11}$$

holds for every $\epsilon > 0$ ($C(\epsilon)$ may be different from the one in (10)). His unconditional result $\epsilon = \frac{1}{8}$ in (11) does not need the functional equation or the approximate functional equation. However all the results with $\epsilon < \frac{1}{8}$ which followed later need the functional equation. M.N. Huxley's result $\epsilon = \frac{2}{5}$ in (10) implies an asymptotic formula for the number of primes in $(x, x+h)$ where $h = x^{\lambda}$ with $\lambda > \frac{7}{12}$. D.R. Heath-Brown has an asymptotic formula even when $h = x^{\frac{7}{12}}(\log x)^{-1}$ and slightly better results. All these results depend crucially on the deep result (13) of I.M. Vinogradov. The latest unconditional result in (11) is with $\epsilon > \frac{1}{22}$ due to S.T. Lou and Q. Yao, two Chinese students of H. Halberstam. The unconditional improvements from $\epsilon > \frac{1}{12}$ to $\epsilon > \frac{1}{22}$ are very difficult and involve ideas of H. Iwaniec, M. Jutila and D.R. Heath-Brown.

Fourth Consequence. The consequence (7) of (5) implies

$$\zeta(\sigma + it)t^{-\epsilon} \to 0 \text{ as } t \to \infty \tag{12}$$

for all fixed σ in $\frac{1}{2} \leq \sigma < 1$ and for every fixed $\epsilon > 0$. (For $\sigma = 1$ this is trivially true).

Remark 1. It is a pity that we do not know the truth of (12) for any $\sigma, \left(\frac{1}{2} \leq \sigma < 1\right)$.

Remark 2. The most valuable and the most diffcult result in the whole of the theory of the Riemann zeta-function in the direction of (12) is a
4 result due to I.M. Vinogradov (for reference see A.A. Karatsuba's paper [51]) which states that for $\frac{1}{2} \leq \sigma \leq 1$ we have

$$\left|\zeta(s) - \frac{1}{s-1}\right| \leq ((|t|+10)^{(1-\sigma)^{\frac{3}{2}}} \ \log(|t|+10)) \tag{13}$$

where A is a certain positive constant. Actually Vinogradov proved that in (13) RHS can be replaced by

$$\left((|t|+10)^{(1-\sigma)^{\frac{3}{2}}} + 10\right)^{A} \log(|t|+10).$$

The inequality (13) implies that

$$\pi(x) - li\ x = O(x\,\mathrm{Exp}(-c(\log x)^{\frac{3}{5}}(\log\log x)^{-\frac{1}{5}})), \tag{14}$$

where $\pi(x) = \sum\limits_{p\leq x} 1$, $li\ x = \int\limits_{2}^{x} \frac{du}{\log u}$ and c is a positive numerical constant. This is the best known result to day as regards upper bounds for the LHS of (14). Riemann's hypothesis (5) implies in an easy way that LHS of (14) is $O(x^{\frac{1}{2}} \log x)$. It must be mentioned that $O(x\,\mathrm{Exp}(-c(\log x)^{\frac{1}{2}}))$ is an easy result which follows from (13) with $1-\sigma$ in place of $(1-\sigma)^{\frac{3}{2}}$, which is a very trivial result. The inequality (13) also implies that for $t \geq 200$,

$$\zeta(1+it) = O((\log t)^{\frac{2}{3}}(\log\log t)^{\frac{4}{3}}). \tag{15}$$

Besides proving (13) Vinogradov proved that in (15) we can drop $(\log\log t)^{\frac{4}{3}}$. As a hybrid of this result and (13) H.-E. Richert proved that the R.H.S. in (13) can be replaced by

$$O((|t|+10)^{100(1-\sigma)^{\frac{3}{2}}}(\log(|t|+10))^{\frac{2}{3}}). \tag{16}$$

See also the paper [96] by K. Ramachandra and A. Sankaranarayanan.

Remark 3. Although the best known bound for $|\zeta(1+it)| (t \geq 1000)$ is $O((\log t)^{\frac{2}{3}})$, due to I.M. Vinogradov, we have still a long way to go since

one can deduce in a simple way the bound $O(\log\log t)$ from (5). In fact
5 to deduce this result it is enough to assume that the least upper bound θ for the real parts of zeros of $\zeta(s)$ is < 1. The only information about θ available today is $\frac{1}{2} \leq \theta \leq 1$. An excellent reference article for many of the facts mentioned above is A.A. Karatsuba [51].

It must be mentioned that the results mentioned above serve as a motivation for many result proved in the theory of $\zeta(s)$. However *I* concentrate on what I have called Titchmarsh's phenomenon (i.e. Ω theorems and mean-value theorems). I will also consider a few other problems like the proof (due to J.B. Conrey, A. Ghosh and S.M. Gonek) that $\zeta(s)$ has infinity of simple zeros in $t \geq 1$. In short this monograph is meant to be a short appendix to the famous book of E.C. Titchmarsh on the Riemann zeta-function. I do not deal with complicated results like N. Levinson's result on the critical zeros, and the results of A. Selberg and D.R. Heath-Brown on Levinson's simple zeros, R. Balasubramanian's result on the mean square of $|\zeta(\frac{1}{2} + it)|$ and its latest improvements by D.R. Heath-Brown and M.N. Huxely, D.R. Heath-Brown's result on the mean fourth power and the mean twelfth power, the improvement of the error them (in the fourth power mean result of D.R. Heath-Brown) by N. Zavorotnyi, H.Iwaniec's contribution to the fourth and twelfth power moments and M. Jutila's approach to these and more general problems, the contribution to higher power moments on the lines $\sigma > \frac{1}{2}$ due to S. Graham, the results of N.V. Kuznetsov (to be corrected by Y. Motohashi), and the results of J.L. Hafner, A. Ivić and Y. Motohashi. In short it is meant to be a readable appendix which is not too complicated. Before closing the introduction I would like to mention in connection with the second consequence that after Ingham's contribution $\epsilon = \frac{2}{3}$, a good amount of later researches were inspired by the conditional result (depending on Lindelöf hypothesis) that for every fixed $\epsilon > 0$ and every fixed $\delta > 0$, there holds

$$\lim_{T\to\infty} T^{-\epsilon} N\left(\frac{3}{4} + \delta, T\right) = 0$$

where $N(\alpha, T)$ is the number of zeros of $\zeta(s)$ in the rectangle (9). This conditional result is due to G. Halász and P. Turán. In connection with 6 the second and the third consequences we have to mention the pioneering works of F. Carleson and G. Hoheisel. In connection with the fourth consequence we have to mention the pioneering works of G.H. Hardy, J.E. Littlewood and H. Weyl. For these we refer to the work of A.A. Karatsuba cited above. Regarding the prime number theorems of Hadamard and de la Vallée Poussin we refer the reader once again to the work of Karatsuba.

Notes at the end of Introductory remarks

The equation (8) was proved with some unspecified constant in place of 10 by J.E. Littlewood. By using the method of A. Selberg, K. Ramachandra and A. Sankaranarayanan have shown that it is possible to replace 10 by a constant which is less than $\frac{9}{16}$ (see [94]).

The inequality (10) was first proved with $\epsilon = 2$ by F. Carleson. This result and the earlier results in the direction of (13) due to G.H. Hardy, J.E. Littlewood and (independently by) H. Weyl were used by G. Hoheisel to prove (11) with some $\epsilon < \frac{1}{2}$. The earlier results just referred to, implied that the RHS in (14) could be replaced by

$$O\left(x \operatorname{Exp}(-c\sqrt{\log x \log\log x})\right)$$

due to J.E. Littlewood which is deep but not very different from the results of J. Hadamard and de la Vallée Poussin namely

$$O\left(x \operatorname{Exp}(-c\sqrt{\log x})\right).$$

For these results see E.C. Titchmarsh, [100]. Hereafter we will refer to this as Titchmarsh's book. By looking at the proof of the results of G.H. Hardy, J.E. Littlewood, H. Weyl and also the great improvements by I.M. Vinogradov it will be clear that we do not need machinery like the functional equation or the approximate functional equation in the proof of things like (11) with some $\epsilon < \frac{1}{2}$ (see K. Ramachandra, [64]. The remark there in before §6 is not used in an essential way) and in fact with $\epsilon = \frac{1}{8}$. Among other important reference works are H.L. Montgomery [56], H.E. Richert [99], M. Jutila [47], K. Chandrasekharan [27], [28], Y. Motohashi [60] and A. Ivic [42], [43]. One may also refer to the booklet by K. Ramachandra [65].

Chapter 1

Some Preliminaries

1.1 Some Convexity Principles

Suppose $f(s)$ is an analytic function of $s = \sigma + it$ defined in the rectangle 7

$$R = \{a \leq \sigma \leq b,\ t_0 - H \leq t \leq t_0 + H\}$$

where a and b are constants satisfying $a < b$. We assume that $|f(s)| \leq M$ (with $M \geq 2$; sometimes we assume implicitly that M exceeds a large positive constant) throughout R. A simple method of obtaining better upper bounds for $|f(\sigma_0 + it_0)|$ with $a < \sigma_0 < b$ is to apply maximum modulus principle to

$$f(s_0 + w)e^{w^2} X^w \quad \text{(where } s_0 = \sigma_0 + it_0\text{)}$$

over the rectangle with the sides $\mathrm{Re}\, w = a - \sigma_0$, $\mathrm{Re}\, w = b - \sigma_0$, $\mathrm{Im}\, w = \pm H$ and choose X in an optimal way. We may also consider

$$f(s_0) = \frac{1}{2\pi i} \int f(s_0 + w)e^{w^2} X^w \frac{dw}{w}$$

over the anti-clockwise boundary of the same rectangle. Sometimes after doing this we may consider upper bounds for

$$\int\limits_{|t_0| \leq \frac{1}{2}H} |f(s_0)| dt_0.$$

A better kernel in place of e^{w^2} is $\mathrm{Exp}(w^{2n})$ where n is any positive odd integer or still better $\mathrm{Exp}\left(\left(\sin \frac{w}{1000}\right)^2\right)$, with bigger constants in place of 1000 if necessary. If σ_0 is close to a or b we get a factor from $\frac{1}{w}$ which is very big. However in some cases we may avoid this big factor by appealing to a two variable convexity theorem of R.M. Gabriel. In all these cases, to get worthwhile results it is necessary to have $H \gg \log\log M$ (with a large implied constant). In this section we present a general theorem (without the requirement $H \gg \log\log M$) which enables one to prove, for instance, things like

$$\int_{|v|\leq D} |\zeta\left(\frac{1}{2} + it_0 + iv\right)|^k dv \gg t_0^{-\epsilon}$$

8 where k and ϵ are any two positive constants and D depends only on k and ϵ. The general theorem also gives

$$\int_{|v|\leq D} |\zeta(1 + it_0 + iv)|^k dv \gg (\log t_0)^{-\epsilon}.$$

In a later chapter we will show that here the LHS is actually $\gg D$. We do not know how to prove the same for the previous integral unless $D \gg \log\log t_0$ (when we can prove a better bound as we shall see in a later chapter). First we will be interested in obtaining lower bounds for

$$I(\sigma) = \int\limits_{|v|\leq H} |f(\sigma + it_0 + iv)|^k dv \tag{1.1.1}$$

where $k > 0$ is any real constant.

Theorem 1.1.1. *Suppose there exists a constant d such that $a < d < b$ and that in $d \leq \sigma \leq b$, $|f(s)|$ is bounded both below and above by β and β^{-1} where $\beta \leq 1$ is a positive constant (it is enough to assume this condition for $I(\sigma)$ with H replaced by an arbitrary quantity lying between $\frac{1}{2}H$ and H in place of $|f(s)|$). Let $\epsilon > 0$ be any constant. Then for $H = D$ where D is a certain positive constant depending only on ϵ, k, a, b, d and β we have, for $a \leq \sigma \leq d$,*

$$I(\sigma) \gg M^{-\epsilon}.$$

Next we prove

Theorem 1.1.2. *Let A_0, σ_1 and δ be any three constants satisfying $A_0 > 0$, $a < \sigma_1 < b$, $\delta > 0$ and $H = \delta$. Then for $\sigma = \sigma_1 \pm (\log M)^{-1}$ (whatever be the sign), we have,*

$$|f(\sigma_1 + it_0)|^k \ll M^{-A_0} + I(\sigma) \log M$$

and

$$\int_{|u| \leq \frac{\delta}{2}} |f(\sigma_1 + it_0 + iu)|^k du \ll M^{-A_0} + I(\sigma) \log\log M.$$

We deduce Theorems 1.1.1 and 1.1.2 by two general theorems on convexity which we now proceed to prove. First of all a remark about the real constant $k > 0$. We will (for technical simplicity) assume that k is an integer. To prove the general case we have to proceed as we do here, but we have to use the Riemann mapping theorem (with zero cancelling factors $(\theta(w))^k$ suitably; see Lemmas 2, 3, 4 of § 1.3). If k is an integer we can consider $f(s)$ in place of $(f(s))^k$ without loss of generality. **9**

Let $a \leq \sigma_0 < \sigma_1 < \sigma_2 \leq b$, $0 < D \leq H$, $s_1 = \sigma_1 + it_0$ and let P denote the contour $P_1P_2P_3P_4P_1$ where $P_1 = -(\sigma_1 - \sigma_0) - iD$, $P_2 = \sigma_2 - \sigma_1 - iD$, $P_3 = \sigma_2 - \sigma_1 + iD$ and $P_4 = -(\sigma_1 - \sigma_0) + iD$. Let $w = u + iv$ be a complex variable. We have

$$2\pi i f(s_1) = \int_P f(s_i + w) X^w \frac{dw}{w} \text{ where } X > 0. \tag{1.1.2}$$

We put

$$X = \mathrm{Exp}(Y + u_1 + u_2 + \ldots + u_r) \tag{1.1.3}$$

where $Y \geq 0$ and $(u_1, u_2, \ldots u_r)$ is any point belonging to the r - dimensional cube $[0, C] \times [0, C] \times \ldots \times [0, C]$, C being a positive constant to be chosen later. The contour P consists of the two vertical lines $-V_0$ and V_2 respectively given by P_4P_1 and P_2P_3 and two horizontal lines $Q_1, -Q_2$ respectively given by P_1P_2 and P_3P_4. Averaging the equation (1.1.2) over the cube we get

$$2\pi i f(s_1) = C^{-r} \int_0^C \ldots \int_0^C \int_P f(s_1 + w) \frac{X^w}{w} dw\, du_1 \ldots du_r. \tag{1.1.4}$$

Over V_0 and V_2 we do not do the averaging. But over Q_1 and Q_2 we do average and replace the integrand by its absolute value. We obtain

$$\begin{aligned}
|2\pi f(s_1)| &\leq \mathrm{Exp}(-Y(\sigma_1-\sigma_0)) \int_{V_0} |f(s_1+w)\frac{dw}{w}| \\
&+ \mathrm{Exp}((Y+Cr)(\sigma_2-\sigma_1)) \int_{V_2} |f(s_1+w)\frac{dw}{w}| \\
&+ \frac{2^{r+1}}{C^r D^r} \mathrm{Exp}((Y+Cr)(\sigma_2-\sigma_1))(\max_{w\in Q_1\cup Q_2} |f(s_1+w)|)(\sigma_2-\sigma_0)
\end{aligned}$$

and thus

$$\begin{aligned}
|2\pi f(s_r)| &\leq (\mathrm{Exp}(-Y(\sigma_1-\sigma_0)))I_0 \\
&+ (\mathrm{Exp}(Cr(\sigma_2-\sigma_1)))(\mathrm{Exp}(Y(\sigma_2-\sigma_1)))(I_2+M^{-A}) \\
&+ 2M(\sigma_2-\sigma_0)(\mathrm{Exp}(Y(\sigma_2-\sigma_1)))\left(\frac{2\,\mathrm{Exp}(C(\sigma_2-\sigma_1))}{CD}\right)^r
\end{aligned} \tag{1.1.5}$$

where A is any positive constant and

$$I_0 = \int_{V_0} |f(s_1+w)\frac{dw}{w}| \text{ and } I_2 = \int_{V_2} |f(s_1+w)\frac{dw}{w}|. \tag{1.1.6}$$

Choosing Y to equalise the first two terms on the RHS of (1.1.5), i.e. choose Y by

$$\mathrm{Exp}(Y(\sigma_2-\sigma_0)) = \left(\frac{I_0}{I_2+M^{-A}}\right)\mathrm{Exp}(-Cr(\sigma_2-\sigma_1))$$

i.e.

$$\begin{aligned}
\mathrm{Exp}(Y(\sigma_2-\sigma_1)) = \left(\frac{I_0}{I_2+M^{-A}}\right)^{(\sigma_2-\sigma_1)(\sigma_2-\sigma_0)^{-1}} \\
\mathrm{Exp}(-Cr(\sigma_2-\sigma_1)^2(\sigma_2-\sigma_0)^{-1})
\end{aligned}$$

and noting that

$$(\sigma_2-\sigma_1)-(\sigma_2-\sigma_1)^2(\sigma_2-\sigma_0)^{-1} = (\sigma_2-\sigma_1)(\sigma_1-\sigma_0)(\sigma_2-\sigma_0)^{-1},$$

we obtain

$$
\begin{aligned}
|2\pi f(s_1)| \leq 2 &\left\{ I_0^{\sigma_2-\sigma_1} (I_2 + M^{-A})^{\sigma_1-\sigma_0} \right\}^{(\sigma_2-\sigma_0)^{-1}} \\
&\left\{ \operatorname{Exp}\left(\frac{C(\sigma_2-\sigma_1)(\sigma_1-\sigma_0)}{\sigma_2-\sigma_0} \right) \right\}^r \\
&+ 2M(\sigma_2-\sigma_0) \left(\frac{I_0}{I_2+M^{-A}} \right)^{(\sigma_2-\sigma_1)(\sigma_2-\sigma_0)^{-1}} \\
&\left\{ \frac{2}{CD} \operatorname{Exp}\left(\frac{C(\sigma_2-\sigma_1)(\sigma_1-\sigma_0)}{\sigma_2-\sigma_0} \right) \right\}^r . \qquad (1.1.7)
\end{aligned}
$$

Collecting we state the following convexity theorem.

Theorem 1.1.3. *Suppose $f(s)$ is an analytic function of $s = \sigma+it$ defined in the rectangle $R : \{a \leq \sigma \leq b, t_0 - H \leq t \leq t_0 + H\}$ where a and b are constants with $a < b$. Let the maximum of $|f(s)|$ taken over R be $\leq M$. Let $a \leq \sigma_0 < \sigma_1 < \sigma_2 \leq b$ and let A be any large positive constant. Let r be any positive integer, $0 < D \leq H$ and $s_1 = \sigma_1 + it_0$. Then for any positive constant C, we have,* **11**

$$
\begin{aligned}
|2\pi f(s_1)| \leq 2 &\left\{ I_0^{\sigma_2-\sigma_1} (I_2 + M^{-A})^{\sigma_1-\sigma_0} \right\}^{(\sigma_2-\sigma_0)^{-1}} \\
&\left\{ \operatorname{Exp}\left(\frac{C(\sigma_2-\sigma_1)(\sigma_1-\sigma_0)}{\sigma_2-\sigma_0} \right) \right\}^r \\
&+ 2M^{A+2}(\sigma_2-\sigma_0) \left(2\left(1 + \left(\log\left(\frac{D}{\sigma_1-\sigma_0} \right) \right)^* \right) \right)^{(\sigma_2-\sigma_1)(\sigma_2-\sigma_0)^{-1}} \times \\
&\times \left\{ \frac{2}{CD} \operatorname{Exp}\left(\frac{C(\sigma_2-\sigma_1)(\sigma_1-\sigma_0)}{\sigma_2-\sigma_0} \right) \right\}^r \qquad (1.1.8)
\end{aligned}
$$

where

$$
I_0 = \int_{|v|\leq D} \left| f(\sigma_0 + it_0 + iv) \frac{dv}{\sigma_0 - \sigma_1 + iv} \right| \qquad (1.1.9)
$$

and

$$
I_2 = \int_{|v|\leq D} \left| f(\sigma_2 + it_0 + iv) \frac{dv}{\sigma_2 - \sigma_1 + iv} \right|, \qquad (1.1.10)
$$

and we have written $(x)^* = \max(0, x)$ for any real number x.

Proof. We have used $I_2 + M^{-A} \geq M^{-A}$ and $(\sigma_2 - \sigma_1)(\sigma_2 - \sigma_0)^{-1} \leq 1$ and if $D \geq \sigma_1 - \sigma_0$,

$$I_0 \leq M \int_{|v| \leq D} \left| \frac{dv}{\sigma_0 - \sigma_1 + iv} \right| \leq 2M \left\{ \int_0^{\sigma_1 - \sigma_0} \frac{dv}{\sigma_1 - \sigma_0} + \int_{\sigma_1 - \sigma_0}^{D} \frac{dv}{v} \right\}.$$

This completes the proof of Theorem 1.1.3. □

In (1.1.8) we replace t_0 by $t_0 + \alpha$ and integrate with respect to α in the range $|\alpha| \leq D$, where now $2D \leq H$. LHS in now $J(\sigma_1)$ defined by

$$J(\sigma_1) = 2\pi \int_{|\alpha| \leq D} |f(\sigma_1 + it_0 + i\alpha)| d\alpha. \tag{1.1.11}$$

Next

$$\int_{|\alpha| \leq D} \left(I_0^{\sigma_2 - \sigma_1} (I_2 + M^{-A})^{\sigma_1 - \sigma_0} \right)^{(\sigma_2 - \sigma_0)^{-1}}$$
$$\leq \left(\int_{|\alpha| \leq D} I_0 d\alpha \right)^{(\sigma_2 - \sigma_1)(\sigma_2 - \sigma_0)^{-1}} \left(\int_{|\alpha| \leq D} (I_2 + M^{-A}) d\alpha \right)^{(\sigma_1 - \sigma_0)(\sigma_2 - \sigma_0)^{-1}}.$$

Now

$$\int_{|\alpha| \leq D} I_0 d\alpha = \int_{|v| \leq D} \int_{|\alpha| \leq D} \left| f(\sigma_0 + it_0 + i\alpha + iv) \frac{d\alpha\, dv}{\sigma_0 - \sigma_1 + iv} \right|$$
$$\leq \left(\int_{|v| \leq 2D} |f(\sigma_0 + it_0 + iv)| dv \right) \int_{|v| \leq D} \left| \frac{dv}{\sigma_0 - \sigma_1 + iv} \right|$$
$$\leq 2 \left(1 + \left(\log \left(\frac{D}{\sigma_1 - \sigma_0} \right) \right)^* \right) I(\sigma_0) \tag{1.1.12}$$

12 where

$$I(\sigma_0) = \int_{|v| \leq 2D} |f(\sigma_0 + it_0 + iv)| dv. \tag{1.1.13}$$

Proceeding similarly, with

$$I(\sigma_2) = \int_{|v| \leq 2D} |f(\sigma_0 + it_0 + iv)| dv \tag{1.1.14}$$

we have

$$\int_{|\alpha|\leq D}(I_2 + M^{-A})d\alpha \leq 2DM^{-A} + 2\left(1 + \left(\log\left(\frac{D}{\sigma_2 - \sigma_1}\right)\right)^*\right)I(\sigma_2). \tag{1.1.15}$$

Thus we have the following corollary.

Theorem 1.1.4. *In addition to the conditions of Theorem 1.1.3, let* $2D \leq H$ *and let* $J(\sigma_1)$*,* $I(\sigma_0)$ *and* $I(\sigma_2)$ *be defined by (1.1.11), (1.1.13) and (1.1.14). Then, we have,*

$$\begin{aligned}
2\pi J(\sigma_1) \leq 4&\left\{I(\sigma_0)\left(1 + \left(\log\left(\frac{D}{\sigma_1 - \sigma_0}\right)\right)^*\right)\right\}^{(\sigma_2-\sigma_1)(\sigma_2-\sigma_0)^{-1}} \times \\
&\times \left\{I(\sigma_2)\left(1 + \left(\log\left(\frac{D}{\sigma_2 - \sigma_1}\right)\right)^*\right) + M^{-A}\right\}^{(\sigma_1-\sigma_0)(\sigma_2-\sigma_0)^{-1}} \times \\
&\left\{\mathrm{Exp}\left(\frac{C(\sigma_2 - \sigma_1)(\sigma_1 - \sigma_0)}{\sigma_2 - \sigma_0}\right)\right\}^r + 4M^{A+2}(\sigma_2 - \sigma_0)\times \\
&\left\{1 + \left(\log\left(\frac{D}{\sigma_1 - \sigma_0}\right)\right)^*\right\}^{(\sigma_2-\sigma_1)(\sigma_2-\sigma_0)^{-1}} \times \\
&\times \left\{\frac{2}{CD}\,\mathrm{Exp}\left(\frac{C(\sigma_2 - \sigma_1)(\sigma_1 - \sigma_0)}{\sigma_2 - \sigma_0}\right)\right\}^r.
\end{aligned} \tag{1.1.16}$$

Proof of Theorem (1.1.1). *In Theorem 1.1.4 replace D by* $D/2$ *and assume that* $J(\sigma_1)$ *is bounded below (by* $\frac{1}{2}\beta D$*) and* $I(\sigma_2)$ *is bounded above by* $\beta^{-1}D$ *(these conditions are implied by the conditions of Theorem 1.1.1). Put* $C = 1$*,* $r = [\epsilon \log M] + 1$ *and* $D = \mathrm{Exp}(\epsilon^{-1}E)$ *where E is a large constant. Let* σ_0, σ_1 *and* σ_2 *be constants satisfying* $a \leq \sigma = \sigma_0 < \sigma_1 < \sigma_2 \leq b$*. We see that the second term on the RHS of (1.1.16) is* $\leq M^{-A}$ *so that*

$$\begin{aligned}
\frac{1}{2}\beta D \leq J(\sigma_1) \leq 4&\left\{I(\sigma_0)\left(1 + \left(\log\left(\frac{D}{\sigma_1 - \sigma_0}\right)\right)^*\right)\right\}^{(\sigma_2-\sigma_1)(\sigma_2-\sigma_0)^{-1}} \times \\
\times &\left\{\beta^{-1}D\left(1 + \left(\log\left(\frac{D}{\sigma_2 - \sigma_1}\right)\right)^*\right)\right\}^{(\sigma_1-\sigma_0)(\sigma_2-\sigma_0)^{-1}} \times
\end{aligned}$$

$$\times \left\{ \operatorname{Exp}\left(\frac{(\sigma_2 - \sigma_1)(\sigma_1 - \sigma_0)}{\sigma_2 - \sigma_0} \right) \right\}. \tag{1.1.17}$$

This proves Theorem (1.1.1).

13 **Proof of Theorem (1.1.2).** In theorems 1.1.3 and 1.1.4 choose $D = \delta$ (δ any positive constant), $\sigma_2 - \sigma_1 = (\log M)^{-1}$, $\sigma_2 = \sigma$, $\sigma_0 = a$, $r = [\log M]$, $C =$ a large constant times δ^{-1}. We obtain the first part of Theorem 1.1.2 namely the + sign. To obtain the second part we argue as in the proof of Theorems (1.1.3) and (1.1.4) but now with

$$\frac{1}{2\pi i} \int f(s_1 + w) X^{-w} \frac{dw}{w}$$

along the same contour P with the same X as before (note X^{-w} in the present integrand). The rest of the details are similar.

1.2 A Lemma in Complex Function Theory

In this section we prove

Theorem 1.2.1. *Let n be any positive integer, $B > 0$ arbitrary, $r > 0$ arbitrary. Let $f(z)$ be analytic in $|z| \leq r$ and let the maximum of $|f(z)|$ in this disc be $\leq M$. Let $0 \leq x < r$, $C = Bnx$, $r_0 = \sqrt{r^2 - x^2}$, and $\alpha = 2(C^{-1} \sinh C + \cosh C)$. Then (for any fixed combination of signs $\pm$) we have*

$$\begin{aligned} |f(0)| \leq & \left(\frac{\alpha B n r_0}{\pi} \right) \left(\frac{1}{2r_0} \int_{-r_0}^{r_0} |f(\pm x + iy)| dy \right) \\ & + \left(\frac{2}{Br} \right)^n \left\{ 1 + (e^C + 1)\left(\pi^{-1} \sin^{-1} \left(\frac{x}{r} \right)\right) \right\} M. \end{aligned} \tag{1.2.1}$$

Putting $x = 0$ we obtain the following

Corollary. *We have*

$$|f(0)| \leq \left(\frac{4Bnr}{\pi} \right) \left(\frac{1}{2r} \int_{-r}^{r} |f(iy)| dy \right) + \left(\frac{2}{Br} \right)^n M. \tag{1.2.2}$$

In particular with $B = 6r^{-1}$, $M \geq 3$, $A \geq 1$, *and* $n =$ *the integer part of* $(A + 1)\log M + 1$, *we have*

$$|f(0)| \leq \frac{24}{\pi}((A+1)\log M + 1)\left(\frac{1}{2r}\int_{-r}^{r}|f(iy)|dy\right) + M^{-A}. \tag{1.2.3}$$

Remark 1. The equations (1.2.1), (1.2.2) and (1.2.3) are statements 14
about $|f(z)|$. It is possible to extend (1.2.1) and hence (1.2.2) and (1.2.3) to more general functions than $|f(z)|$ with some other constants in place of α, $\frac{4}{\pi}$, $\frac{24}{\pi}$. (For example to $|f(z)|^k$ where $k > 0$ is any real number).

Remark 2. The Corollary shows that

$$|f(0)| \leq (24A\log M)\left(\frac{1}{2r}\int_{-r}^{r}|f(iy)|dy\right) + M^{-A}. \tag{1.2.4}$$

We ask the question "Can we replace $\log M$ by a term of smaller order say by $\sqrt{\log M}$ (or omit it altogether) at the cost of increasing the constant $24A$?". The answer is no. See the Remark 6 in § 1.7.

Remark 3. The method of proof is nearly explained in § 1.1. As for the applications we can state for example the following result. Let $3 \leq H \leq T$. Divide the interval T, $T + H$ into intervals I of length r each. We can assume $0 < r \leq 1$ and omit a small bit at one of the ends. Then for any integer constant $k \geq 1$, (the result is also true if k is real by Theorem 1.2.2), we have

$$\sum_{I}\max_{i\in I}\left|\zeta\left(\frac{1}{2}+it\right)\right|^k \ll \frac{Ak\log T}{r}\int\limits_{T-r}^{T+H+r}\left|\zeta\left(\frac{1}{2}+it\right)\right|^k dt + r^{-1}HT^{-Ak} \tag{1.2.5}$$

the implied constant being absolute. We may retain only one term on the LHS of (1.2.5) and if we know for example that RHS of (1.2.5) is $\ll HT^{\epsilon}$ then it would follow that

$$\mu\left(\frac{1}{2}\right) \leq \frac{1}{k}\lim_{T\to\infty}\left(\frac{\log H}{\log T}\right). \tag{1.2.6}$$

Since what we want holds for $H = T^{1/3}$ and $k = 2$ and any $r (0 < r \leq 1)$ we obtain the known result $\mu(\frac{1}{2}) \leq \frac{1}{6}$ due to H. Weyl, G.H. Hardy and J.E. Littlewood. Similar remarks apply to L-functions and so on.

Remark 4. The results of this section as well as some of the results
15 section § 1.1 are improvements and generalizations of some lemmas in Ivić's book (see page 172 of this book. Here the results concern Dirichlet series with a functional equation and are of a special nature).

Remark 5. In (1.2.4) we have corresponding results with $|f(iy)|$ on the RHS replaced by $|f(x+iy)|$. These follow from Theorem 1.2.1.

Proof of the Theorem 1.2.1. Let P, Q, R, S denote the points $-ri, ri, r$ and $-r$ respectively. Then we begin with

Lemma 1. *Let $X = \operatorname{Exp}(u_1 + \ldots + u_n)$ where $B > 0$ is arbitrary and $0 \leq u_j \leq B$ for $j = 1, 2, \ldots, n$. Then*

$$f(0) = \frac{1}{2\pi i}\left\{\int_{PQ} f(w)\frac{X^w - X^{-w}}{w}dw + \int_{QSP} f(w)\frac{X^w}{w}dw + \int_{PRQ} f(w)\frac{X^{-w}}{w}dw\right\} \tag{1.2.7}$$

where the integrations are respectively along the straight line PQ, along the semi-circular portion QSP of the circle $|w| = r$, and along the semi-circular portion PRQ of the circle $|w| = r$.

Proof. With an understanding of the paths of integration similar to the ones explained in the statement of the lemma we have by Cauchy's theorem that the integral of $f(w)\frac{X^w}{w}$ over $PQSP$ is $2\pi i f(0)$ provided we deform the contour to $P'Q'SP'$ where $P'Q'$ is parallel to PQ and is close to PQ (and to the right of it) and the points P' and Q' lie on the circle $|w| = r$. Also with the same modification the integral of $f(w)\frac{X^{-w}}{w}$ over $PRQP$ is zero. These remarks complete the proof of the lemma. □

Lemma 2. *Denote by I_1, I_2, I_3 the integrals appearing in Lemma 1. Let $\langle du\rangle$ denote the element of volume $du_1 du_2 \ldots du_n$ of the box $\mathcal{B}$ defined by $0 \leq u_j \leq B(i = 1, 2, \ldots n)$. Then*

$$B^{-n}\left|\int_{\mathcal{B}}\left(\frac{1}{2\pi i}(I_2 + I_3)\right)\langle du\rangle\right| \leq \left(\frac{2}{Br}\right)^n M. \tag{1.2.8}$$

Proof. Trivial since $|X^w| \leq 1$ and $|X^{-w}| \leq 1$ on QSP and PRQ respectively. $\square$

Lemma 3. *Let $w = x + iy$ where x and y are any real numbers, and* **16**
$0 \leq L = \log X \leq Bn$. *Then*

$$\left|\frac{e^{wL} - e^{-wL}}{wL}\right| \leq \frac{1}{C}(e^C - e^{-C}) + e^C + e^{-C} \tag{1.2.9}$$

where $C = Bn|x|$.

Proof. LHS in the lemma is

$$\begin{aligned}
&\left|\left\{e^{xL}(\cos(yL) + i\sin(yL)) - e^{-xL}(\cos(yL) - i\sin(yL))\right\}(wL)^{-1}\right| \\
&\quad \leq \left|\frac{e^{xL} - e^{-xL}}{xL}\right| \left|\cos(yL)\right| + (e^{xL} + e^{-xL})\left|\frac{\sin(yL)}{yL}\right| \\
&\quad \leq \frac{1}{C}(e^C - e^{-C}) + e^C + e^{-C}.
\end{aligned}$$

This completes the proof of the lemma. $\square$

In order to obtain the theorem we note that on the line PQ we have $x = 0$. We now assume that $x > 0$ and more the line of integration to $\operatorname{Re} w = \pm x$, (whatever be the sign) namely the intercept made by this line in the disc $|w| \leq r$. On this line we pass to the absolute value and use Lemma 3. We get the first term on the RHS of (1.2.1). For the two circular portions connecting this path with the straight lien PQ we integrate over the box $\mathcal{B}$ and get

$$\begin{aligned}
&B^{-n}\left|\int_{\mathcal{B}}\left(\frac{1}{2\pi i}\int f(w)\frac{X^w - X^{-w}}{w}dw\right)\langle du\rangle\right| \\
&\qquad \leq \left(\frac{2}{Br}\right)^n (e^C + 1)(\pi^{-1}\sin^{-1}\left(\frac{x}{r}\right))M.
\end{aligned}$$

This with lemma 2 completes the proof of the theorem 1.2.1.

The result referred to in Remark 1 is as follows.

Theorem 1.2.2. *Let k be any positive real number. Let $f(z)$ be analytic in $|z| \leq 2r$ and there $|f(z)|^k \leq M (M \geq 9)$. Let $x = r(\log M)^{-1}$, and let x_1 be any real number with $|x_1| \leq x$. Put $r_0 = \sqrt{4r^2 - x_1^2}$. Then with $A \geq 1$ we have*

$$|f(0)|^k \leq 2e^{84A} M^{-A} + \frac{24}{(2\pi)^2} e^{84A} \log M \left(\frac{1}{2r_0} \int\limits_{-r_0}^{r_0} |f(x_1 + iy)|^k dy \right). \tag{1.2.10}$$

17 **Remark 1.** It is easy to remember a somewhat crude result namely

$$|f(0)|^k \leq e^{90A} \left\{ M^{-A} + (\log M) \left(\frac{1}{2r_0} \int_{-r_0}^{r_0} |f(x_1 + iy)|^k dy \right) \right\}. \tag{1.2.11}$$

Remark 2. In Theorem (1.2.1) the constants are reasonably small where as in Theorem (1.2.2) they are big. We have not attempted to get optimal constants.

For these results see the last section of this chapter.

1.3 Gabriel's Convexity Theorem

In this section I reproduce without any essential charges, the proof of the following important theorem due to R.M. Gabriel [33].

Theorem 1.3.1. *Let D be a simply connected domain symmetrical about a straight line L lying in D. Let the boundary of D be a simple curve $K = K_1 + K_2$ where K_1 and K_2 lie on opposite sides of L. If $f(z)$ is regular in D and continuous on K, then*

$$\left(\int_L |f(z)|^{\frac{2}{a+b}} |dz| \right)^{\frac{a+b}{2}} \leq\leq \left(\int_{K_1} |f(z)^{\frac{1}{a}} |dz| \right)^{\frac{a}{2}} \left(\int_{K_2} |f(z)^{\frac{1}{b}} |dz| \right)^{\frac{b}{2}}$$

where $a > 0$ and $b > 0$ are any two real numbers.

Putting $a = b = \frac{1}{q}$, we get as a special case

Theorem 1.3.2. *Let $q > 0$ be any real number. Then in the notation of Theorem 1.3.1, we have*

$$\int_L |f(z)|^q |dz| \leq \left(\int_{K_1} |f(z)|^q |dz| \right)^{\frac{1}{2}} \left(\int_{K_2} |f(z)|^q |dz| \right)^{\frac{1}{2}}$$

Remark. The assertion of the theorem still holds if $|f(z)|^q$ is replaced by $|\varphi(z)|\ |f(z)|^q$, where $\varphi(z)$ is any function analytic inside D such that $|\varphi(z)|$ is continuous on the boundary of D. To see this replace $f(z)$ by $(f(z))^j (\varphi(z))^r$ and q by qj^{-1} where j and r are positive integers and j and r tend to infinity in such a way that $rj^{-1} \to q^{-1}$. **18**

Lemma 1. *Theorem 1.3.1 is true if $f(z)$ has no zeros in D.*

Proof. Without loss of generality we may take L to be a portion of the real axis cutting K in A, B. Let $\phi(z)$ satisfy the conditions of the Lemma. Now if $\overline{\phi}(z)$ is the conjugate of $\phi(\overline{z})$, where $\overline{z}$ is the conjugate of z, then by a known theorem, $\overline{\phi}(z)$ is regular in D and continuous on K. Further, for a z on L, $|\phi(z)|^2 = \phi(z)\overline{\phi}(z)$. Hence, by Cauchy's theorem,

$$\begin{aligned}
\int_L |\phi(z)|^2 dz &= \left| \int_{AB} \phi(z)\overline{\phi}(z) dz \right| \\
&= \left| \int_{K_1} \phi(z)\overline{\phi}(z) dz \right| \leq \int_{K_1} |\phi(z) \mid \overline{\phi}(z) \mid dz| \\
&\leq \left(\int_{K_1} |\phi(z)|^p |dz| \right)^{1/p} \left(\int_{K_1} |\overline{\phi}(z)|^{p'} |dz| \right)^{1/p'}, \left(\frac{1}{p} + \frac{1}{p'} = 1, \right) \\
&= \left(\int_{K_1} |\phi(z)|^p |dz| \right)^{\frac{1}{p}} \left(\int_{K_2} |\phi(z)|^{p'} |dz| \right)^{1/p'}
\end{aligned}$$

since K_2 is the conjugate of K_1 with respect to the real axis. Next, if the $f(z)$ of the Theorem 1.3.1 has no zero in D, $\phi(z) = f^{1/(a+b)}(z)$ is regular in D and continuous on K. Hence, taking $p = (a+b)/a$, $p' = (a+b)/b$, we have

$$\left(\int_L |f(z)|^{\frac{2}{a+b}} |dz| \right)^{\frac{a+b}{2}} \leq \left(\int_{K_1} |f(z)|^{\frac{1}{a}} |dz| \right)^{\frac{a}{2}} \left(\int_{K_2} |f(z)^{\frac{1}{b}} |dz| \right)^{\frac{b}{2}}$$

This proves Lemma 1 completely. □

Lemma 2. *The domain D of the z-plane can be transformed conformally onto $|w| < 1$ by the transformation $z = A(w)$ which possesses a unique inverse analytic transformation $w = A^{-1}(z)$. Further the boundary is transformed continuously onto the boundary.*

Proof. Follows by a well-known fundamental theorem of Riemann. For the proof of this theorem and references to the work of Riemann see for instance Titchmarsh's book [101] (1952) or L. Ahlfor's book [1] (see Theorems 10 and 11 on pages 172 and 174). □

Lemma 3. *Let $0 < \delta < 1$. Let $F(w) = f(A(w))$. Let $w = 0$ be a*
19 *zero of order $m (m \geq 0)$ of $F(w)$. Denote the other zeros (counted with multiplicity) of $F(w)$ in $|w| \leq 1-\delta$, by $\{\rho\}$. Let the number of zeros (other than $w = 0$) in $|w| \leq 1-\delta$, be n. (We will let $\delta \to 0$ finally). Put*

$$\theta(w) = \frac{F(w)}{\psi(w)}, \text{ and } \psi(w) = \frac{w^m \prod \left(1 - \frac{w}{\rho}\right)}{(1-\delta)^{m-n} w^n \prod \left(1 - \frac{(1-\delta)^2}{w\bar{\rho}}\right)}$$

Then we have

(1) $\theta(w)$ *has no zeros in* $|w| \leq 1-\delta$,

(2) $|\psi(w)| = 1$ *on* $w = 1-\delta$,

(3) $|\theta(w)| = |F(w)|$ *on* $w = 1-\delta$,

and

(4) $|\psi(w)| \leq 1$ *in* $|w| \leq 1-\delta$.

Proof. The statements (1), (2), (3) are obvious and (4) follows from (2) by maximum modulus principle since $\psi(w)$ is analytic in $|w| \leq 1-\delta$. □

Lemma 4. *The inverse image of $|w| = 1-\delta$ together with the inverse image of L' (the image of L-contained in $|w| \leq 1-\delta$) under the transformation $z = A(w)$ approaches K continuously as $\delta \to 0$.*

Remark. For references to earlier versions of Lemmas 3 and 4 see the paper of GABRIEL cited above.

Proof. Follows from Lemma 2.

Lemmas 1 to 4 complete the proof of Theorem 1.3.1.

As before let $z = x + iy$ be a complex variable. We employ a in a meaning different from the one in Theorem 1.3.1. We now slightly extend this as follows. Consider the rectangle $0 \leq x \leq (2^n + 1)a$ (where n is a non-negative integer and a is a positive number), and $0 \leq y \leq R$. Suppose that $f(z)$ is analytic inside the rectangle $\{0 \leq x \leq (2^n + 1)a, 0 \leq y \leq R\}$ and that $|f(z)|$ is continuous on its boundary. Let I_x denote the 20
integral $\int_0^R |f(z)|^q dy$ where as before $z = x + iy$. Let Q_a denote the maximum of $|f(z)|^q$ on $\{0 \leq x \leq \alpha, y = 0, R\}$. Then we have as a first application of the theorem of Gabriel,

$$I_\alpha \leq (I_0 + 4aQ_{2a})^{\frac{1}{2}} (I_{2a} + 4aQ_{2a})^{\frac{1}{2}}.$$

We prove by induction that if $b_m = 2^m + 1$, then

$$I_a \leq \left(I_0 + 2^{2(m+1)}aQ_{ab_m}\right)^{\frac{1}{2}} \left(I_a + 2^{2(m+1)}aQ_{ab_m}\right)^{\frac{1}{2}-\frac{1}{2^{m+1}}}$$
$$\left(I_{ab_m} + 2^{2(m+1)}aQ_{ab_m}\right)^{\frac{1}{2^{m+1}}}.$$

We have as a first application of Gabriel's theorem this result with $m = 0$. Assuming this to be true for m we prove it with m replaced by $m + 1$. We apply Gabriel's Theorem to give the bound for I_{ab_m} in terms of I_a and $I_{ab_{m+1}}$. We have

$$I_{ab_m} \leq (I_a + 2b_{m+a}aQ_{ab_{m+1}})^{\frac{1}{2}} (I_{ab_{m+1}} + 2ab_{m+1}Q_{ab_{m+1}})^{\frac{1}{2}}$$

since as we can easily check $b_{m+1} = b_m + b_m - 1$. We add $2^{2(m+1)}aQ_{ab_m}$ to both sides and use that for $A > 0$, $B > 0$,$Q > 0$ we have

$$\sqrt{AB} + Q \leq \sqrt{(A + Q)(B + Q)}$$

which on squaring both sides reduces to a consequence of $(\sqrt{A} - \sqrt{B})^2 \geq 0$. Thus

$$I_{ab_m} + 2^{2(m+1)}aQ_{ab_m} \leq$$
$$\left(I_a + a(2b_{m+1} + 2^{2(m+1)})Q_{ab_{m+1}}\right)^{\frac{1}{2}} \left(I_{ab_{m+1}} + a\left(2b_{m+1} + 2^{2(m+1)}\right) Q_{ab_{m+1}}\right)^{\frac{1}{2}}$$

Now $2b_{m+1} + 2^{2(m+1)} \leq 2^{2(m+2)}$ i.e. $2(2^{m+1} + 1) \leq 3 \cdot 2^{2(m+1)}$ which is true. Since $\frac{1}{2} - \frac{1}{2^{m+1}} + \frac{1}{2^{m+2}} = \frac{1}{2} - \frac{1}{2^{m+2}}$ the induction is complete and the required result is proved. We state it as a $\square$

21 **Convexity Theorem 1.3.3.** For $m = 0, 1, 2, \ldots, n$ we have

$$I_a \leq (I_0 + 2^{2(m+1)} aQ_{ab_m})^{\frac{1}{2}} (I_a + 2^{2(m+1)} aQ_{ab_m})^{\frac{1}{2} - \frac{1}{2^{m+1}}}$$
$$\times (I_{ab_m} + 2^{2(m+1)} aQ_{ab_m})^{\frac{1}{2^{m+1}}}.$$

Remark. The remark below Theorem 1.3.2 is applicable here also.

1.4 A Theorem of Montgomery and Vaughan

Theorem 1.4.1. *(Montromery and Vaughan) Suppose $R \geq 2$; $\lambda_1, \lambda_2, \ldots, \lambda_R$ are distinct real numbers and that $\delta_n = \min_{m \neq n} |\lambda_n - \lambda_m|$. Then if $a_1, a_2, \ldots, a_R$ are complex numbers, we have*

$$\left| \sum \sum_{m \neq n} \frac{a_m \bar{a}_n}{\lambda_m - \lambda_n} \right| \leq \frac{3\pi}{2} \sum_n |a_n|^2 \delta_n^{-1}. \tag{1.4.1}$$

Remark. We can add any positive constant to each of the λ_n and so we can assume that all the λ_n are positive and distinct. The proof of the theorem is very deep and it is desirable to have a simple proof within the reach of simple calculus. For a reference to the paper of H.L. Montgomery and R.C. Vaughan see E.C. Titchmarsh [100].

In almost all applications it suffices to restrict to the special case $\lambda_n = \log(n + \alpha)$ where $0 \leq \alpha \leq 1$ is fixed and $n = 1, 2, \ldots, R$. Also the constant $3\pi/2$ is not important in many applications. It is the object of this section to supply a very simple proof in this special case with a larger constant in place of $3\pi/2$. Accordingly our main result is

Theorem 1.4.2. *Suppose $R \geq 2$, $\lambda_n = \log(n + \alpha)$ where $0 \leq \alpha \leq 1$ is fixed and $n = 1, 2, \ldots, R$. Let $a_1, \ldots a_R$ be complex numbers. Then, we have,*

$$\left| \sum \sum_{m \neq n} \frac{a_m \bar{a}_n}{\lambda_m - \lambda_m} \right| \leq C \sum n |a_n|^2, \tag{1.4.2}$$

where C is an absolute numerical constant which is effective.

Remark 1. Instead of the condition $\lambda_n = \log(n+\alpha)$ we can also work with the weaker condition $n(\lambda_{n+1} - \lambda_n)$ is both $\gg 1$ and $\ll 1$. Also no attempt is made to obtain an economical value for the constants such as C. 22

Remark 2. Theorem 1.4.2 with $\alpha = 0$ and the functional equation of $\zeta(s)$ are together enough to deduce in a simple way the result that for $T \geq 2$,

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt = \left(\frac{1}{2\pi^2}\right) T(\log T)^4 + O(T(\log T)^3). \tag{1.4.3}$$

The result (1.4.3) was first proved by A.E. Ingham by a very complicated method.

We prove the following result.

Theorem 1.4.3. *If* $\{a_n\}$ *and* $\{b_n\}(n = 1, 2, 3, \ldots R)$ *are complex numbers where* $R \geq 2$ *and* $\lambda_n = \log(n+\alpha)$ *where* $0 \leq \alpha \leq 1$ *is fixed, then*

$$\left|\sum\sum_{m\neq n} \frac{a_m \bar{b}_n}{\lambda_m - \lambda_n}\right| \leq D\left(\sum n|a_n|^2\right)^{1/2} \left(\sum n|b_n|^2\right)^{1/2}$$

where D is an effective positive numerical constant.

We begin with

Lemma 1. *We have*

$$\left|\sum\sum_{m\neq n} \frac{a_m \bar{a}_m}{m-n}\right| \leq \pi \sum |a_n|^2.$$

Proof. We remark that $2\pi\int_0^1 \left(\int_0^y |\sum a_m e^{2\pi imx}|^2 dx\right) dy = \pi \sum |a_n|^2 - E/i$, where E/i is the real number for which $|E| \leq \pi \sum |a_n|^2$ is to be proved. Note that since the integrand is nonnegative, $2\pi\int_0^1 (\int_{-y}^y |\sum a_n e^{2\pi inx}|^2 dx)$ $dy = 2\pi \sum |a_n|^2$ is an upper bound. Thus

$$0 \leq \pi \sum |a_n|^2 - \frac{E}{i} \leq 2\pi \sum |a_n|^2$$

and this proves the lemma. □

Lemma 2. *We have*

$$\left|\sum\sum_{m\neq n}\frac{a_m\bar{b}_n}{m-n}\right| \leq 3\pi\left(\sum|a_n|^2\right)^{1/2}\left(\sum|b_n|^2\right)^{1/2}$$

23 *Proof.* Now $2\pi\int_0^1(\int_0^y(\sum a_m e^{2\pi mx})(\sum \bar{b}_n e^{-2\pi inx})dx)dy = \pi\sum(a_m\bar{b}_m) - E/i$ gives the result since by Holder's inequality

$$\begin{aligned}\left|\pi\sum(a_m\bar{b}_m) - \frac{E}{i}\right| &\leq 2\pi\left(\int_0^1\left(\int_0^y|\sum a_m e^{2\pi imx}|^2 dx\right)dy\right)^{1/2}\\ &\quad\times\left(\int_0^1\left(\int_0^y|\sum \bar{b}_n e^{2\pi inx}|^2 dx\right)dy\right)^{1/2}\\ &\leq 2\pi\left(\sum|a_n|^2\right)^{1/2}\left(\sum|b_n|^2\right)^{1/2}\end{aligned}$$

on using Lemma 1. This completes the proof of Lemma 2. □

We next deduce Theorem 1.4.3 from Lemma 2 as follows. We divide the range $1 \leq n \leq R$ by introducing intervals $I_i = [2^{i-1}, 2^i)$ and the pairs (m, n) with $m \neq n$ into those lying in $I_i \times I_j$. We now start with

$$\begin{aligned}&\int_0^1\left(\int_0^y\left(\sum a_m e^{2\pi i\lambda_m x}\right)\left(\sum \bar{b}_n e^{2\pi i\lambda_n x}\right)dx\right)dy\\ &= \frac{1}{2}\sum a_m\bar{b}_m - \frac{E}{2\pi i} + \sum_{\substack{k,\ell\\ k\geq 1,\ell\geq 1}}\frac{1}{2\pi i}\sum_{(m,n)\in I_k\times I_\ell}\int_0^1\frac{a_m\bar{b}_n e^{2\pi i(\lambda_m-\lambda_n)y}}{\lambda_m-\lambda_n}dy\end{aligned}$$

where E is the quantity for which we seek an upper bound, and hence we have the fundamental inequality

$$\begin{aligned}\left|\frac{E}{2\pi i}\right| &\leq \frac{1}{2}\sum|a_m\bar{b}_m| + \frac{1}{2\pi}\sum_{k,\ell}\left|\sum_{(m,n)\in I_k\times I_\ell}\int_0^1\frac{a_m\bar{b}_n e^{2\pi i(\lambda_m-\lambda_n)y}}{\lambda_m-\lambda_n}dy\right|\\ &\quad+\left(\int_0^1\int_{-y}^y|\sum a_m e^{2\pi i\lambda_m x}|^2 dx\, dy\right)^{1/2}\end{aligned}$$

$$\times\left(\int_0^1\int_{-y}^{y}|\sum \overline{b}_n e^{-2\pi i\lambda_m x}|^2 dx\, dy\right)^{1/2}$$

$$= \sum_1 + \sum_2 + \left(\sum_3\right)^{1/2}\left(\sum_4\right)^{1/2}, \text{ in an obvious notation.}$$

We remark that if $|k-l| \geq 3$ then

$$\left|\sum_{(m,n)\in I_k\times I_\ell}\int_0^1 \frac{a_m\overline{b}_n e^{2\pi i(\lambda_m-\lambda_n)y}}{\lambda_m-\lambda_n}dy\right|$$

$$\ll \left(\sum_{(m,n)\in I_k\times I_\ell}|a_m\overline{b}_n|\right) \text{maximum}_{(m,n)\in I_k\times I_\ell}(\lambda_m-\lambda_n)^{-2}$$

$$(k-l)^{-2}\sum_{(m,n)\in I_k\times I_\ell}|a_m\overline{b}_n| \ll (k-l)^{-2}S_k^{1/2}T_\ell^{1/2}$$

where $S_k = \sum\limits_{n\in I_k} n|a_n|^2$ and $T_\ell = \sum\limits_{n\in I_\ell} n|b_n|^2$. Hence the contribution to **24**
$\sum_2$ from k,ℓ with $|k-\ell| \geq 3$ is $\sum\limits_{|k-\ell|\geq 3}(S_k^{1/2}T_\ell^{1/2}/(k-\ell)^2) \ll (\sum\limits_k S_k)^{1/2}$ $(\sum\limits_k T_k)^{1/2}$. Now we consider those terms of $\sum_2$ with $|k-l| < 3$. A typical term is

$$\int\limits_0^1 \sum_{(m,n)\in I_k\times I_\ell}\frac{a_m\overline{b}_n e^{2\pi i(\lambda_m-\lambda_n)y}}{\lambda_m-\lambda_n}dy$$

Here the inner sum is

$$N\left(\sum_{(m,n)\in I_k\times I_\ell}\frac{a'_m\overline{b'_n}}{(N\lambda_m)-(N\lambda_n)}\right)$$

where $a'_m = a_m e^{2\pi i\lambda_m y}$ and $b'_n = b_n e^{2\pi i\lambda_n y}$, and N is any positive number. Observe that if $N = 2^{k+800}$, then the integral parts of $N\lambda_m(m \in (I_k \cup I_\ell))$ differ each other by at least 3. Also in the denominator we replace $N\lambda_m - N\lambda_n$ by $[N\lambda_m]-[N\lambda_n]$ the consequent error being

$$O\left(N\sum\frac{|a_m b_n|}{([N\lambda_m]-[N\lambda_n])^2}\right)$$

which is easily seen to be $O(S_k^{1/2}T_\ell^{1/2})$. Next by Lemma 2 we see that

$$N\sum_{(m,n)\in I_k\times I_\ell}\frac{a'_m\overline{b'_n}}{[N\lambda_m]-[N\lambda_n]}=O(S_k^{1/2}T_\ell^{1/2}).$$

Thus we see that if $|k-\ell|<3$, the contribution of $\sum\limits_{(m,n)\in I_k\times I_\ell}\ldots$ to $\sum_2$ is $O(S_k^{1/2}T_\ell^{1/2})$. Combining all this one sees easily that

$$\sum_2=O((\sum n|a_n|^2)^{1/2}\times(\sum n|b_n|^2)^{1/2}).$$

The method of estimation of $\sum_2$ shows that

$$\sum_3=\sum|a_n|^2+O(\sum n|a_n|^2)=O(\sum n|a_n|^2)$$

and

$$\sum_4=O(\sum n|b_n|^2).$$

Trivially $\sum|a_nb_n|\leq(\sum|a_n|^2)^{1/2}(\sum|b_n|^2)^{1/2}$ and so

$$E=O\left(\left(\sum n|a_n|^2\right)^{1/2}\left(\sum n|b_n|^2\right)^{1/2}\right).$$

25 This completes the proof of Theorem 1.4.3.

1.5 Hadamard's Three Circles Theorem

We begin by stating the following version of the maximum modulus principle.

Theorem 1.5.1. *Let $f(z)$ be a non-constant analytic function defined on a bounded domain D. Let, for every $\xi\in$ boundary of D, and for every sequence $\{z_n\}$ with $z_n\in D$ which converges to $\xi\in$ boundary of D,*

$$\overline{\lim}|f(z_n)|\leq M.$$

Then

$$|f(z)|<M \text{ for all } z\in D.$$

Remark. We do not prove this theorem (for a reference see the notes at the end of this chapter).

Theorem 1.5.2 (HADAMARD). *Let $f(z)$ be an analytic function regular for $r_1 \le |z| \le r_3$. Let $r_1 < r_2 < r_3$ and let M_1, M_2, M_3 be the maximum of $|f(z)|$ on the circles $|z| = r_1$, $|z| = r_2$ and $|z| = r_3$ respectively. Then*

$$M_2^{\log(r_3/r_1)} \le M_1^{\log(r_3/r_2)} M_3^{\log(r_2/r_1)}$$

Proof. Put $\lambda = \frac{m}{n}$ where m and n are integers with $n \ge 1$. Let $\phi(z) = (f(z)z^{\lambda})^n$. By maximum modulus principle applied to $\phi(z)$, we have

$$M_2^n r_2^m \le \max(M_1^n r_1^m, M_3^n r_3^m).$$

Thus

$$M_2 \le \max(M_1(r_1/r_2)^{m/n}, M_3(r_3/r_2)^{m/n}).$$

Now let λ be any real number. We let m/n approach λ through any sequence of rational numbers. Hence we get

$$M_2 \le \max(M_1(r_1/r_2)^{\lambda}, M_3(r_3/r_2)^{\lambda}).$$

for all real numbers λ. We now chose λ by

$$\lambda \log(r_1/r_2) + \log M_1 = \lambda \log(r_3/r_2) + \log M_3$$

i.e. by $\lambda = (\log M_3 - \log M_1)(\log r_1 - \log r_3)^{-1}$. Thus we have **26**

$$\log M_2 \le \log M_1 + \frac{(\log M_3 - \log M_1)(\log r_2 - \log r_1)}{\log r_3 - \log r_1}$$

This completes the proof of Theorem 1.5.2. □

1.6 Borel-Caratheodory Theorem

Theorem 1.6.1 (BOREL-CARATHÉODORY). *Suppose $f(z)$ is analytic in $|z - z_0| \le R$ and on the circle $z = z_0 + \mathrm{R}e^{i\theta}(0 \le \theta \le 2\pi)$, we have,* $\mathrm{Re}\, f(z) \le U$. *Then in $|z - z_0| \le r < R$ we have*

$$|f(z) - f(z_0)| \le \frac{2r(U - \mathrm{Re}\, f(z_0))}{R - r} \tag{1.6.1}$$

and, for $j \geq 1$

$$|\frac{f^{(j)}(z)}{j!}| \leq \frac{2R}{(R-r)^{j+1}}(U - Re\, f(z_0)). \tag{1.6.2}$$

Proof. Let $f(z) = \sum\limits_{n=0}^{\infty} a_n(z-z_0)^n$ and $\varphi(z) = f(z) - f(z_0)$. Clearly $a_0 = f(z_0)$. Let $a_n = |a_n|e^{i\alpha_n}$, $0 \leq \alpha_n < 2\pi$ for $n \geq 1$. On $|z - z_0| = R$ we have

$$\mathrm{Re}\,\varphi(z) = \frac{1}{2}\sum_{n=1}^{\infty}|a_n|R^n(e^{ni\theta+i\alpha_n} + e^{-ni\theta-i\alpha_n})$$

and so for any fixed $k = 1, 2, \ldots$ we have

$$\begin{aligned}
|a_k|R^k\pi &= \int_0^{2\pi}(\mathrm{Re}\,\varphi(z_0 + \mathrm{Re}^{i\theta}))\left(1 + \frac{1}{2}(e^{ki\theta+i\alpha_k} + e^{-ki\theta-i\alpha_k})\right)d\theta \\
&\leq \int_0^{2\pi}(U - \mathrm{Re}\ f(z_0))\left(1 + \frac{1}{2}(e^{ki\theta+i\alpha_k} + e^{-ki\theta-i\alpha_k})\right)d\theta \\
&= 2\pi(U - \mathrm{Re}\ f(z_0)).
\end{aligned}$$

Thus

$$|a_k| \leq 2R^{-k}(U - \mathrm{Re}\ f(z_0)), k = 1, 2, 3, \ldots. \tag{1.6.3}$$

27 Now for $|z - z_0| \leq r < R$, we have,

$$|f(z) - f(z_0)| \leq \sum_{n=1}^{\infty} 2(U - \mathrm{Re}\ f(z_0))\left(\frac{r}{R}\right)^n = \frac{2r}{R-r}(U - \mathrm{Re}\ f(z_0))$$

and this proves (1.6.1). Also for $j = 1, 2, \ldots$

$$\begin{aligned}
|f^{(j)}(z)| = |\varphi^{(j)}(z)| &\leq \sum_{n=j}^{\infty} n(n-1)\ldots(n-j+1)|a_n(z-z_0)^{n-j}| \\
&\leq 2\sum_{n=j}^{\infty} n(n-1)\ldots(n-j+1)(U - \mathrm{Re}\ f(z_0))R^{-n}r^{n-j} \\
&= 2\left(\frac{d}{dr}\right)^j \sum_{n=0}^{\infty}(U - \mathrm{Re}\ f(z_0))\left(\frac{r}{R}\right)^n
\end{aligned}$$

$$= \frac{2R}{(R-r)^{j+1}}(U - \operatorname{Re} f(z_0))(j!).$$

This proves (1.6.2) and hence Theorem 1.6.1 is completely proved. □

1.7 A Lemma in Complex Function Theorey (Continued)

In this section we prove theorem 1.2.2 and state (with proof) a theorem which is sometimes useful. But before proving Theorem 1.2.2 we make four remarks.

Remark 3. Let $k_1, k_2, \ldots, k_m$ be any set of positive real numbers. Let $f_1(z), f_2(z), \ldots, f_m(z)$ be analytic in $|z| \le 2r$, and there

$$|(f_1(z))^{k_1} \ldots (f_m(z))^{k_m}| \le M \ (M \ge 9).$$

Then Theorem 2 holds good with $|f(z)|^k$ replaced by $|(f_1(z))^{k_1} \cdots (f_m(z))^{k_m}|$

Remark 4. A corollary to our result mentioned in Remark 3 was pointed out to us by Professor J.P. Demailly. It is this: Theorem 1.2.2 holds good with $|f(z)|^k$ replaced by $\operatorname{Exp}(u)$ where u is any subharmonic function. To prove this it suffices to note that the set of functions of the form $\sum\limits_{j=1}^{m} k_j \log |f_j(z)|$ is dense in L^1_{loc} in the set of subharmonic func- **28**
tions. (This follows by using Green-Riesz representation formula for u and approximating the measure Δ_u by finite sums of Dirac measures).

Remark 5. Consider $k = 1$ in Theorem 1.2.2. Put $\varphi(z) = f^{(\ell)}(z)$ the ℓ^{th} derivative of $f(z)$. Then our method of proof gives

$$|\varphi(0)| \le CM^{-A} + C(\log M)^{\ell+1} \left(\frac{1}{4r} \int_{-4r}^{4r} |f(iy)| dy \right),$$

where C depends only on A and ℓ.

Remark 6 (Due to J.-P. Demailly)**.** In view of the example $f(z) = (\frac{e^{nz}-1}{nz})^2$, where n is a large positive integer and $r = 1$, the result of Remark 5 is best possible.

We now prove Theorem 1.2.2. The proof consists of four steps.

Step 1. First we consider the circle $|z| = r$. Let

$$0 < 2x \leq r \tag{1.7.1}$$

and let PQS denote respectively the points $re^{i\theta}$ where $\theta = -\cos^{-1}(\frac{2x}{r})$, $\cos^{-1}(\frac{2x}{r})$ and π. By the consideration of Riemann mapping theorem and the zero can-cellation factors we have for a suitable meromorphic function $\phi(z)$ (in $PQSP$) that (we can assume that $f(z)$ has no zeros on the boundary)

$$F(z) = (\phi(z) f(z))^k \tag{1.7.2}$$

is analytic in the region enclosed by the straight line PQ and the circular arc QSP. Further $\phi(z)$ satisfies

$$|\phi(z)| = 1 \tag{1.7.3}$$

on the boundary of $PQSP$ and also

$$|\phi(0)| \geq 1. \tag{1.7.4}$$

Let

$$X = \text{Exp}(u_1 + u_2 + \ldots + u_n) \tag{1.7.5}$$

29 where $u_1, u_2, \ldots, u_n$ vary over the box $\mathcal{B}$ defined by

$$0 \leq u_j \leq B (j = 1, 2, \ldots, n),$$

and $B > 0$.

We begin with

Lemma 1. *The function $F(z)$ defined above satisfies*

$$F(0) = I_1 + I_2 \tag{1.7.6}$$

where

$$I_1 = \frac{1}{2\pi i}\int_{PQ} F(z)X^z\frac{dz}{z} \tag{1.7.7}$$

and

$$I_2 = \frac{1}{2\pi i}\int_{QSP} F(z)X^z\frac{dz}{z} \tag{1.7.8}$$

where the lines of integration are the straight lien PQ and the circular arc QSP.

Proof. Follows by Cauchy's theorem. ◻

Lemma 2. *We have*

$$|I_1| \leq \frac{e^{2Bnx}}{2\pi}\int_{PQ} |(f(z))^k\frac{dz}{z}|. \tag{1.7.9}$$

Proof. Follows since $|X^z| \leq e^{2Bnx}$ and also $|\phi(z)| = 1$ on PQ. ◻

Lemma 3. *We have,*

$$|B^{-n}|\int_{\mathcal{B}} I_2 du_1 \ldots du_n| \leq e^{2\mathcal{B}nx}\left(\frac{2}{Br}\right)^n M. \tag{1.7.10}$$

Proof. Follows since on QSP we have $|\phi(z)| = 1$ (and so $|F(z)| \leq M$) and also

$$|B^{-n}|\int_{\mathcal{B}}\left(\int_{QSP} X^z\frac{dz}{2\pi iz}\right)du_1 \ldots du_n| \leq \left(\frac{2}{Br}\right)^n.$$

◻

Lemma 4. *We have,* **30**

$$|f(0)|^k \leq e^{2Bnx}\left(\frac{2}{Br}\right)^n M + \frac{e^{2Bnx}}{2\pi}\int_{PQ} |(f(z))^k\frac{dz}{z}|. \tag{1.7.11}$$

Proof. Follows by Lemmas 1, 2 and 3. ◻

Step 2. Next in (1.7.11), we replace $|f(z)|^k$ by an integral over a chord P_1Q_1 (parallel to PQ) of $|w| = 2r$, of slightly bigger length with a similar error. Let x_1 be any real number with

$$|x_1| \le x. \tag{1.7.12}$$

$$\begin{cases} \text{Let } P_1Q_1R_1 \text{ be the points } 2re^{i\theta} \\ \text{where } \theta = -\cos^{-1}\left(\frac{x_1}{2r}\right), 0 \text{ and } \cos^{-1}\left(\frac{x_1}{2r}\right). \\ \text{(If } x_1 \text{ is negative we have to consider the points} \\ \theta = -\frac{\pi}{2} - \sin^{-1}\left(\frac{x_1}{2r}, 0\right) \text{ and } \frac{\pi}{2} + \sin^{-1}\left(\frac{x_1}{2r}\right)\text{).} \end{cases} \tag{1.7.13}$$

Let X be as in (1.7.5). As before let

$$G(w) = (\psi(w)f(w))^k \tag{1.7.14}$$

be analytic in the region enclosed by the circular arc $P_1R_1Q_1$ and the straight line Q_1P_1 (we can assume that $f(z)$ has no zeros on the boundary $P_1R_1Q_1P_1$). By the consideration of Riemann mapping theorem and the zero cancelling factors there exists such a meromorphic function $\psi(w)$ (in $P_1R_1Q_1P_1$) with the extra properties,

$$|\psi(w)| = 1 \text{ on the boundary of } P_1R_1Q_1P_1 \text{ and } |\psi(z)| \ge 1. \tag{1.7.15}$$

Lemma 5. *We have with z on PQ,*

$$G(z) = I_3 + I_4$$

where

$$I_3 = \frac{1}{2\pi i}\int_{Q_1P_1} G(w)X^{-(w-z)}\frac{dw}{w-z} \tag{1.7.16}$$

31 *and*

$$I_4 = \frac{1}{2\pi i}\int_{P_1R_1Q_1} G(w)X^{-w(w-2)}\frac{dw}{w-z}. \tag{1.7.17}$$

Proof. Follows by Cauchy's theorem. □

Lemma 6. *We have with z on PQ*

$$|I_3| \le \frac{e^{3Bnx}}{2\pi}\int_{P_1Q_1}\left|(f(w))^k\frac{dw}{w-z}\right| \tag{1.7.18}$$

Proof. Follows since $|X^{-(w-z)}| \leq e^{3Bnx}$ and $|\psi(w)| = 1$ on P_1Q_1. □

Lemma 7. *We have with z on PQ,*

$$|B^{-n}\int_{\mathcal{B}} I_4 du_1 \ldots du_n| \leq e^{3Bnx}\left(\frac{2}{Br}\right)^n M. \tag{1.7.19}$$

Proof. Follows since on $P_1R_1Q_1$ we have $|\psi(w)| = 1$ (and so $|G(w)| \leq M$) and also

$$|B^{-n}\int_{\mathcal{B}}\int x^{-w(-z)}\frac{dw}{2\pi i(w-z)}du_1 \ldots du_n| \leq \left(\frac{2}{Br}\right)^n.$$

□

Lemma 8. *We have with z on PQ,*

$$|f(z)|^k \leq e^{3Bnx}\left(\frac{2}{Br}\right)^n M + \frac{e^{3Bnx}}{2\pi}\int_{P_1Q_1}|f(w)|^k|\frac{dw}{w-z}|. \tag{1.7.20}$$

Proof. Follows from Lemmas 5, 6 and 7. □

Step 3. We now combine Lemmas 4 and 8.

Lemma 9. *We have*

$$|f(0)|^k \leq e^{2Bnx}\left(\frac{2}{Br}\right)^n M + J_1 + J_2 \tag{1.7.21}$$

where

$$J_1 = \frac{e^{5Bnx}}{2\pi}\left(\frac{2}{Br}\right)^n M\int_{PQ}|\frac{dz}{z}|, \tag{1.7.22}$$

and **32**

$$J_2 = \frac{e^{5Bnx}}{(2\pi)^2}\int_{P_1Q_1}|f(w)|^k\left(\int_{PQ}|\frac{dz}{z(w-z)}|\right)|dw|. \tag{1.7.23}$$

Lemma 10. *We have*

$$\int_{PQ}|\frac{dz}{z}| \leq 2 + 2\log\left(\frac{r}{2x}\right). \tag{1.7.24}$$

Proof. On PQ we have $z = 2x + iy$ with $|y| \leq r$ and $2x \leq r$. We split the integral into $|y| \leq 2x$ and $2x \leq |y| \leq r$. On these we use respectively the lower bounds $|z| \geq 2x$ and $|z| \geq y$. The lemma follows by these observations. □

Lemma 11. *We have for w on P_1Q_1 and z on PQ,*

$$\int_{PQ} \left|\frac{dz}{z(w-z)}\right| \leq \frac{6}{x}. \tag{1.7.25}$$

Proof. On PQ we have $\operatorname{Re} z = 2x$ and on P_1Q_1 we have $|\operatorname{Re} w| \leq x$ and so $|\operatorname{Re}(w-z)| \geq x$. We have

$$\left|\frac{dz}{z(w-z)}\right| \leq \left|\frac{dz}{z^2}\right| + \left|\frac{dz}{(w-z)^2}\right|.$$

Writing $z = 2x + iy$ we have

$$\int_{PQ} \left|\frac{dz}{z^2}\right| \leq \frac{2}{(2x)^2} 2x + 2\int_{2x}^{\infty} \frac{dy}{y^2} = \frac{2}{x}.$$

Similarly

$$\int_{PQ} \left|\frac{dz}{(w-z)^2}\right| \leq 2\left(\frac{1}{x} + \int_{x}^{\infty} \frac{dy}{y^2}\right) = \frac{4}{x}.$$

This completes the proof of the lemma. □

Step 4. We collect together the results in Steps 3 and 4 and choose the
33 parameters B and n and this will give Theorem 1.2.2. Combining Lemmas 9, 10 and 11 we state the following lemma.

Lemma 12. *We have*

$$|f(0)|^k \leq e^{2Bnx}\left(\frac{2}{Br}\right)^n M + \frac{e^{5Bnx}}{\pi}\left(\frac{2}{Br}\right)^n \left(1 + \log\frac{r}{2x}\right) M$$
$$+ \frac{e^{5Bnx}}{(2\pi)^2} \cdot \frac{6}{x} \int_{P_1Q_1} |(f(w))^k dw|, \tag{1.7.26}$$

where $0 < 2x \leq r$, x_1 is any real number with $|x_1| \leq x$, n any natural number and B is any positive real number and P_1Q_1 is the straight line joining $-r_0$ and r_0 where $r_0 = \sqrt{4r^2 - x_1^2}$.

Next we note that $1 + \log \frac{r}{2x} \leq \frac{r}{2x}$ and so by putting $x = r(\log M)^{-1}$ the first two terms on the RHS of (1.7.26) together do not exceed

$$\left(\frac{2}{Br}\right)^n e^{5Bnx}\left(1 + \frac{1}{2\pi}\log M\right)M \leq 2\left(\frac{2}{Br}\right)^n e^{5Bnx} M \log M.$$

Also,

$$\frac{6}{x} = \frac{6\log M}{r} = 6\log M\left(\frac{2r_0}{r}\right)\frac{1}{2r_0} \leq (24\log M)\left(\frac{1}{2r_0}\right).$$

Thus RHS of (1.7.26) does not exceed

$$2\left(\frac{2}{Br}\right)^n e^{5Bnx} M\log M + \left(\frac{24}{(2\pi)^2}e^{5Bnx}\log M\right)\left(\frac{1}{2r_0}\int_{P_1Q_1}|(f(w))^k dw|\right).$$

We have chosen $x = r(\log M)^{-1}$. We now choose B such that $Br = 2e$ and $n = [C\log M] + 1$, where $C \geq 1$ is any real number. We have $5Bnx \leq \frac{5Bnr}{\log M} \leq 10e(C+1) \leq 28(C+1)$ and also

$$\left(\frac{2}{Br}\right)^n \leq e^{-C\log M} = M^{-C}.$$

With these choices of x, B, n we see that RHS of (1.7.26) does not exceed

$$2M^{-C}e^{28(C+1)}M\log M + \left(\frac{24}{(2\pi)^2}e^{28(C+1)}\log M\right)\left(\frac{1}{2r_0}\int_{P_1Q_1}|(f(w))^k dw|\right).$$

Putting $C = A + 2$ we obtain Theorem 1.2.2 since $C + 1 \leq 3A$. This **34**
completes the proof of Theorem 1.2.2.

Lastly we note

Theorem 1.7.1. *Let $f(z)$ be analytic in $|z| \leq R$. Then for any real $k > 0$, we have,*

$$|f(0)|^k \leq \frac{1}{\pi R^2}\int_{|z|\leq R}|f(z)|^k dx\, dy \qquad (1.7.27)$$

Remark. The remark below Theorem 1.3.2 is applicable here also. We have only to replace q by k.

Proof. We begin by remarking that the theorem is true for $k = 1$. Because let $z = re^{i\theta}$ where $0 < r \leq R$. Then by Cauchy's theorem we have

$$f(0) = \frac{1}{2\pi i}\int_i^{2\pi} f(re^{i\theta})id\theta \tag{1.7.28}$$

and so

$$|f(0)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|d\theta. \tag{1.7.29}$$

Multiplying this by $r\,dr$ and integrating from $r = 0$ to $r = R$ we obtain (1.2.7). Now let $f(0) \neq 0$ (otherwise there is nothing to prove) and

$$\phi(z) = \left\{ f(z)\prod_{\rho}\left(\frac{r^2 - \bar{\rho}z}{r(z-\rho)}\right)\right\}^k \tag{1.7.30}$$

where ρ runs over all the zeros of $f(z)$ satisfying $|\rho| \leq r$. The function $\phi(z)$ is analytic (selecting any branch) in $|z| \leq r$ and so (1.7.29) holds with $f(z)$ replaced by $\phi(z)$. Notice that on $|z| = r$, we have $|f(z)|^k = |\phi(z)|$ and also that

$$|\phi(0)| = |f(0)|^k \left(\prod_{\rho}\frac{r}{|\rho|}\right)^k \geq |f(0)|^k.$$

Hence (1.7.29) holds with $|f(z)|$ replaced by $|f(z)|^k$ and hence we are led to (1.7.27) as before. □

Notes at the end of Chapter 1

§ 1.1. The author learnt of convexity principles from A. Selberg who 35
told him about a weaker kernel function. The stronger kernel functions like $\text{Exp}(w^2)$ or $\text{Exp}(w^{4a+2})$ ($a \geq 0$ integer) became known to the author through P.X. Gallagher. The kernel function $\text{Exp}((\sin w)^2)$ was noticed by the author who used it extensively in various situations. It should be mentioned that

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} X^w \,\text{Exp}(w^2) dw \text{ and } \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} X^w \,\text{Exp}(w^2)\frac{dw}{w}$$

are non-negative (a fact which the author learnt from D.R. Heath-Brown). These things coming from the kernel function $\text{Exp}(w^2)$ are sometimes useful. The two inequalities preceeding (1.1.1) (taken with the remark after the second) seem to be new. Also the technique of averaging over "Cubes" seems to be new. This section is based on the paper [6] of R. Balasubramanian and K. Ramachandra.

§ 1.2. The results of this section are improvements and generalisations of some lemmas (in A. Ivić, [42]) due to D.R. Heath-Brown. We will hereafter refer to this book as Ivićs book. This section as well as § 1.7 are based on the papers [7] [8] of R. Balasubramanian and K. Ramachandra.

§ 1.3. The convexity Theorem (1.3.3) is very useful in a later chapter. This section is based on the paper [33] of R.M. Gabriel. See also the appendix to the paper [66] of K. Ramachandra. For the convexity Theorem (1.3.3) see p. 13 of the paper [9] of R. Balasubramanian and K. Ramachandra.

§ 1.4. We do not prove (1.4.1) with the constant $\frac{3\pi}{2}$ although we use it in later chapters. We prove it will some unspecified constant in place of $\frac{3\pi}{2}$. The proof of Theorem 1.4.2 is based on the paper [67] of K. Ramachandra. For Remark 2 below this theorem see K. Ramachandra [68].

§ 1.5. For Theorem 1.5.1 we refer the reader to (K. Chandrasekharan, [27]). The proof of Theorem 1.5.2 given here is due to R. Balasubramanian. For the general principles of complex function theory 36 necessary for Chapters 1 and 7 one may refer to K. Chandrasekharan [27] or E.C. Titchmarsh [101] or L. Ahlfors [1].

Chapter 2

Some Fundamental Theorems on TiTchmarsh Series and Applications

2.1 Introduction

In this chapter we prove three fundamental theorems on "TITCHMARSH **37**
SERIES". These concern lower bounds for

$$\frac{1}{H}\int_0^H |F(it)|dt \text{ and } \frac{1}{H}\int_0^H |F(it)|^2 dt,$$

where $H \geq 10$ and $F(s)(s = \sigma + it)$ is defined by

$$F(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s} \ (\sigma \geq A + 2),$$

where $A > 0$ is an integer constant and the complex numbers a_n are subject to $a_1 = 1$, $|a_n| \leq (nH)^A$ $(n \geq 2)$ and the real numbers λ_n are subject to $\lambda_1 = 1$ and $\frac{1}{C} \leq \lambda_{n+1} - \lambda_n \leq C$ $(n \geq 1)$ where $C \geq 1$ is a constant. $F(it)$ is defined by the condition that $F(s)$ shall be continuable analytically in $(\sigma \geq 0, 0 \leq t \leq H)$. These conditions define a "TITCHMARSH SERIES". Some times as in this chapter we impose a growth condition on certain horizontal lines. But in a later chapter we will manage without the growth condition at the const of imposing a more stringent condition than $|a_n| \leq (nH)^A$. All these results have important applications. Theorem 1 will be proved as a preparation to the proof of a more complicated (but neat) Theorem 3. Both these deal with lower bounds for the mean square of $|F(it)|$ while Theorem 2 deals with the mean of $|F(it)|$. As handy results for application we state a corollary below each of the three main theorems. We begin by stating a main lemma.

2.2 Main Lemma

Let r be a positive integer $H \geq (r+5)U$, $U \geq 2^{70}(16B)^2$ and N and M positive integers subject to $N > M \geq 1$. Let $b_m (m \leq M)$ and $c_n (n \geq N)$ be complex numbers and $A(s) = \sum\limits_{m \leq M} b_m \lambda_m^{-s}$. Let $B(s) = \sum\limits_{n \leq N} c_n \lambda_n^{-s}$ be absolutely convergent in $\sigma \geq A+2$ and continuable analytically in
38 $\sigma \geq 0$. Write $g(s) = A(-s)B(s)$,

$$G(s) = U^{-r} \int_0^U du_r \ldots \int_0^U du_1 (g(s+i\lambda))$$

(here and elsewhere $\lambda = u_1 + u_2 + \ldots + u_r$). Assume that there exist real numbers T_1 and T_2 with $0 \leq T_1 \leq U$, $H - U \leq T_2 \leq H$, such that

$$|g(\sigma + iT_1)| + |g(\sigma + iT_2)| \leq \operatorname{Exp}\operatorname{Exp}\left(\frac{U}{16B}\right)$$

uniformly in $0 \leq \sigma \leq B$. (As stated already $B = A+2$). Let

$$S_1 = \sum_{m \leq M, n \geq N} |b_m c_n| \left(\frac{\lambda_m}{\lambda_n}\right)^B 2^r \left(U \log \frac{\lambda_n}{\lambda_m}\right)^{-r},$$

and

$$S_2 = \sum_{m \leq M, n \geq N} |b_m c_n| \left(\frac{\lambda_m}{\lambda_n}\right)^B.$$

Then

$$|\int_{2U}^{H-(r+3)U} G(it)dt| \leq |U^{-r} \int_0^U du_r \ldots \int_0^U du_1 \int_{2U+\lambda}^{H-(r+3)U+\lambda} g(it)dt|$$

$$\leq 2B^2U^{-10} + 54BU^{-1}\int_0^H |g(it)|dt$$
$$+ (H + 64B^2)S_1 + 16B^2 \operatorname{Exp}\left(-\frac{U}{8B}\right)S_2.$$

To prove this main lemma we need five lemmas. After proving these we complete the proof of the main lemma.

Lemma 2.2.1. *Let $z = x + iy$ be a complex variable with $|x| \leq \frac{1}{4}$. Then, we have,*

(a) $|\operatorname{Exp}((\operatorname{Sin} z)^2)| \leq e^{\frac{1}{2}} < 2$ *for all* y

and

(b) *If* $|y| \geq 2$,

$$|\operatorname{Exp}((\operatorname{Sin} z)^2)| \leq e^{\frac{1}{2}}(\operatorname{Exp}\operatorname{Exp}|y|)^{-1} < 2(\operatorname{Exp}\operatorname{Exp}|y|)^{-1}.$$

Proof. We have **39**

$$\begin{aligned}\operatorname{Re}(\operatorname{Sin} z)^2 &= -\frac{1}{4}\operatorname{Re}\{(e^{i(x+iy)} - e^{-i(x+iy)})^2\}\\ &= -\frac{1}{4}\operatorname{Re}\{e^{2ix-2y} + e^{-2ix+2y} - 2\}\\ &= \frac{1}{2} - \frac{1}{4}\left\{(e^{-2y} + e^{2y})\cos(2x)\right\}.\end{aligned}$$

But in $|x| \leq \frac{1}{4}$, we have $\cos(2x) = \cos(|2x|) \geq \cos\frac{1}{2} \geq \cos\frac{\pi}{6} \geq \frac{\sqrt{3}}{2}$. The rest of the proof is trivial since (i) $\cosh y$ is an increasing function of $|y|$ and (ii) for $|y| \geq 2$

$$\operatorname{Exp}\left(-\frac{\sqrt{3}}{8}e^{2|y|}\right) \leq (\operatorname{Exp}\operatorname{Exp}|y|)^{-1}$$

since $e^2 > (2.7)^2$ and $\frac{8}{\sqrt{3}} < \frac{8\times 1.8}{3} = 4.8$ and so $e^2 > \frac{8}{\sqrt{3}}$. The lemma is completely proved. □

Lemma 2.2.2. *For any two real numbers k and σ with $0 < |\sigma| \leq 2B$, we have,*

$$\int_{-\infty}^{\infty} |\operatorname{Exp}\left(\operatorname{Sin}^2\left(\frac{ik-\sigma-iu_1}{8B}\right)\right)\frac{du_1}{ik-\sigma-iu_1}| \leq 12 + 4\log|\frac{2B}{\sigma}|.$$

Proof. Split the integral into three parts J_1, J_2 and J_3 corresponding to $|u_1 - k| \geq 2B$, $|\sigma| \leq |u_1 - k| \leq 2B$ and $|u_1 - k| \leq \sigma$. The contribution to J_1 from $|u_1 - k| \geq 16B$ is (by (b) of Lemma 2.2.1)

$$\begin{aligned}
&\leq \frac{2e^{\frac{1}{2}}}{16B}\int_{16B}^{\infty}\operatorname{Exp}\left(-\frac{u_1}{8B}\right)du_1\\
&= e^{\frac{1}{2}}\int_{2}^{\infty}\operatorname{Exp}(-u_1)du_1 = \operatorname{Exp}\left(-\frac{3}{2}\right).
\end{aligned}$$

The contribution to J_1 from $2B \leq |u_1 - k| \leq 16B$ is (by (a) of Lemma 2.2.1)

$$\leq e^{\frac{1}{2}}\int_{2B\leq|u_1-k|\leq 16B}|u_1-k|^{-1}du_1 = 2e^{\frac{1}{2}}\log 8 = 6e^{\frac{1}{2}}\log 2.$$

Now

$$\begin{aligned}
6e^{\frac{1}{2}}\log 2 + \operatorname{Exp}\left(-\frac{3}{2}\right) &< 6\left(1 + \frac{1}{2} + \frac{1}{2\cdot 2^2} + \frac{1}{6\cdot 2^2}\right)\\
&\left(\frac{1}{2} + \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^2}\right) + \left(\frac{1}{2\cdot 7}\right)^{3/2} < 8.
\end{aligned}$$

40 Thus $|J_1| \leq 8$. Using (a) of Lemma 2.2.1 we have $|J_2| \leq 4\log|\frac{2B}{\sigma}|$. In J_3 the integrand is at most $e^{\frac{1}{2}}\sigma^{-1}$ in absolute value and so $|J_3| \leq 2e^{\frac{1}{2}} \leq 4$. Hence the lemma is completely proved. □

Lemma 2.2.3. *If $n > m$, we have, for all real k,*

$$|\int_0^U du_r \ldots \int_0^U du_1\left(\frac{\lambda_m}{\lambda_n}\right)^{i(k+\lambda)}| \leq 2^r\left(\log\frac{\lambda_n}{\lambda_m}\right)^{-r}.$$

Proof. Trivial. □

Lemma 2.2.4. *For all real t and all $D \geq B$, we have,*

$$|G(D+it)| \leq S_1 \text{ and } |g(D+it)| \leq S_2.$$

Proof. We have, trivially,

$$|g(D+it)| \leq \sum_{m \leq M, n \geq N} |b_m c_n| \left(\frac{\lambda_m}{\lambda_n}\right)^D$$

and the second result follows on observing that $\frac{\lambda_m}{\lambda_n} < 1$ and so $\left(\frac{\lambda_m}{\lambda_n}\right)^D \leq \left(\frac{\lambda_m}{\lambda_n}\right)^B$. Next

$$G(D+it) = U^{-r} \int_0^U du_r \ldots \int_0^U du_1 (g(D+it+i\lambda))$$

$$= U^{-r} \sum_{m \leq M, n \geq N} b_m c_n \left(\frac{\lambda_m}{\lambda_n}\right)^D \int_0^U du_r \ldots \int_0^U du_1 \left(\frac{\lambda_m}{\lambda_n}\right)^{i(t+\lambda)}.$$

Using Lemma 2.2.3 and observing $\left(\frac{\lambda_m}{\lambda_n}\right)^D \leq \left(\frac{\lambda_m}{\lambda_n}\right)^B$ the first result follows. □

Lemma 2.2.5. *Let $0 < \sigma \leq B$ and $2U \leq t \leq H - (r+3)U$. Then, for $H \geq (r+5)U$ and $U \geq (20)!(16B)^2$, we have,*

$$|G(\sigma+it)| \leq BU^{-10} + U^{-1}\left(2 + 4\log\frac{2B}{\sigma}\right)\int_0^H |g(it)|dt$$

$$+ 16S_1 \log(2B) + 8BS_2 \operatorname{Exp}\left(-\frac{U}{8B}\right).$$

41

Remark. $(20)! < 2^{70}$.

Proof. We note, by Cauchy's theorem, that

$$2\pi i g(\sigma+it+i\lambda) = \int_{iT_1}^{B+1+iT_1} + \int_{B+1+iT_1}^{B+1+iT_2} - \int_{iT_2}^{B+1+iT_2} - \int_{iT_1}^{iT_2}$$

$$\left\{g(w)\,\mathrm{Exp}\left(\mathrm{Sin}^2\left(\frac{w-\sigma-it-i\lambda}{8B}\right)\right)\right\}\frac{dw}{w-\sigma-it-i\lambda}$$
$$J_1 + J_2 - J_3 - J_4 say.$$

We write

$$\begin{aligned}2\pi iG(\sigma+it) &= 2\pi iU^{-r}\int_0^U du_r\ldots\int_0^U du_1(g(\sigma+it+i\lambda))\\ &= U^{-r}\int_0^U du_r\ldots\int_0^U du_1(J_1+J_2-J_3-J_4)\\ &= J_5+J_6-J_7-J_8 \text{ say.}\end{aligned}$$

Let us look at J_5. In J_1 (also in J_3) $|g(w)| \leq \mathrm{Exp}\,\mathrm{Exp}(\frac{U}{16B})$ (by the definition of T_1 and T_2). Also by using Lemma 2.2.1 (b) (since $|\,\mathrm{Re}\,w - \sigma| \leq B+1 \leq 2B$, and $|\,\mathrm{Im}(w-it-i\lambda)| \geq U \geq (20)!(16B)^2$), we have,

$$|\,\mathrm{Exp}\left(\left(\frac{w-\sigma-it-i\lambda}{8B}\right)\right)| \leq 2\,\mathrm{Exp}\left(-\frac{U}{8B}\right).$$

Hence

$$\begin{aligned}|J_1| &\leq \frac{2(B+1)}{U}\,\mathrm{Exp}\left(\mathrm{Exp}\,\frac{U}{16B} - \mathrm{Exp}\,\frac{U}{8B}\right)\\ &\leq \frac{2(B+1)}{U}\,\mathrm{Exp}\left(-\left(\mathrm{Exp}\,\frac{U}{16B}\right)\left(\mathrm{Exp}\,\frac{U}{16B}-1\right)\right)\\ &\leq \frac{B}{2}U^{-10},\end{aligned}$$

since $U \geq (20)!(16B)^2$ and so $\mathrm{Exp}\,\frac{U}{16B} - 1 \geq 1$ and $\mathrm{Exp}\left(-\,\mathrm{Exp}\,\frac{U}{16B}\right) \leq \mathrm{Exp}\left(-\,\mathrm{Exp}\,U^{\frac{1}{2}}\right) \leq \mathrm{Exp}(-U^{\frac{1}{2}}) \leq (20)!U^{-10}$. Thus $|J_5| \leq \frac{1}{2}BU^{-10}$.

Similarly, $|J_7| \leq \frac{1}{2}BU^{-10}$. Next

$$\begin{aligned}J_8 = U^{-r}\int_{iT_1}^{iT_2} g(w)dw\int_0^U du_r\ldots\int_0^U du_2\int_0^U\\ \mathrm{Exp}\left(\sin^2\left(\frac{w-\sigma-it-i\lambda}{8B}\right)\right)\frac{du_1}{w-\sigma-it-i\lambda}.\end{aligned}$$

42 We note that $w-\sigma-it-i\lambda = ik-\sigma-iu_1$ where $k = \mathrm{Im}\,w-t-u_2\ldots-u_r$.

Hence the u_1-integral is in absolute value (by Lemma 2.2.2)

$$\leq 12 + 4\log\frac{2B}{\sigma}.$$

This shows that

$$\begin{aligned}
|J_8| &\leq U^{-r}\int_{iT_1}^{iT_2}|g(w)dw|\left\{U^{r-1}(12+4\log\frac{2B}{\sigma})\right\}\\
&\leq U^{-1}(12+4\log\frac{2B}{\sigma})\int_0^H |g(it)|dt
\end{aligned}$$

Finally we consider J_6.

$$\begin{aligned}
J_6 &= U^{-r}\int_0^U du_r\ldots\int_0^U du_1\int_{B+1+iT_1}^{B+1+iT_2} g(w)\\
&\quad \operatorname{Exp}\left(\operatorname{Sin}^2\left(\frac{w-\sigma-it-i\lambda}{8B}\right)\right)\frac{dw}{w-\sigma-it-i\lambda}\\
&= U^{-r}\int_0^U du_r\ldots\int_0^U du_1\int_{B+1-\sigma+iT_1-it-i\lambda}^{B+1-\sigma+iT_2-it-i\lambda} g(w+\sigma+it+i\lambda)
\end{aligned}$$

Using Lemma 2.2.1 (b) we extend the range of integration of w to $(B+1-\sigma-i\infty, B+1-\sigma+i\infty)$ and this gives an error which is at most

$$\begin{aligned}
&U^{-r}\int_0^U du_r\ldots\int_0^U du_1\int_{|\operatorname{Im} w|\geq U, \operatorname{Re} w=B+1-\sigma}\\
&\quad\left|g(w+\sigma+it+i\lambda)\operatorname{Exp}\left(\operatorname{Sin}^2\left(\frac{w}{8B}\right)\right)\frac{dw}{w}\right|.
\end{aligned}$$

By Lemma 2.2.4 this is

$$\leq S_2 U^{-r}\int_0^U du_r\ldots\int_0^U du_1\int_{|\operatorname{Im} w|\geq U, \operatorname{Re} w=B+1-\sigma}|\operatorname{Exp}\left(\operatorname{Sin}^2\frac{w}{8B}\frac{dw}{w}\right)|.$$

Here the innermost integral is (by Lemma 2.2.1 (b))

$$\leq\frac{4}{U}\int_U^\infty\operatorname{Exp}\left(-\frac{u}{8B}\right)du\leq\int_U^\infty\operatorname{Exp}\left(-\frac{u}{8B}\right)du = 8B\operatorname{Exp}\left(-\frac{U}{8B}\right).$$

Thus the error does not exceed $8BS_2 \operatorname{Exp}\left(-\frac{U}{8B}\right)$ and so

$$
\begin{aligned}
|J_6| \leq U^{-r} \int_0^U du_r \ldots \int_0^U du_1 \int_{B+1-\sigma-i\infty}^{B+1-\sigma+i\infty} g(w+\sigma+it+i\lambda) \\
\operatorname{Exp}\left(\operatorname{Sin}^2\left(\frac{w}{8B}\right)\right)\frac{dw}{w} + 8BS_2 \operatorname{Exp}\left(-\frac{U}{8B}\right) \\
= |U^{-r} \int_{B+1-\sigma-i\infty}^{B+1-\sigma+i\infty} \operatorname{Exp}\left(\operatorname{Sin}^2\left(\frac{w}{8B}\right)\right)\frac{dw}{w} \int_0^U du_r \ldots \\
\int_0^U du_1 g(w+\sigma+it+i\lambda)| + 8BS_2 \operatorname{Exp}\left(-\frac{U}{8B}\right) \\
= \left|\int_{B+1-\sigma-i\infty}^{B+1-\sigma+i\infty} G(w+\sigma+it) \operatorname{Exp}\left(\operatorname{Sin}^2\left(\frac{w}{8B}\right)\right)\frac{dw}{w}\right| \\
+ 8BS_2 \operatorname{Exp}\left(-\frac{U}{8B}\right).
\end{aligned}
$$

43 Using the first part of Lemma 2.2.4 we obtain

$$
\begin{aligned}
|J_6| &\leq S_1 \int_{B+1-\sigma-i\infty}^{B+1-\sigma+i\infty} |\operatorname{Exp}\left(\operatorname{Sin}^2\left(\frac{w}{8B}\right)\right)\frac{dw}{w}| + 8BS_2 \operatorname{Exp}\left(-\frac{U}{8B}\right) \\
&\leq S_1\left(12 + 4\log\frac{2B}{B+1-\sigma}\right) + 8BS_2 \operatorname{Exp}\left(-\frac{U}{8B}\right)
\end{aligned}
$$

by using Lemma 2.2.2. Thus

$$
|J_6| \leq 16S_1 \log(2B) + 8BS_2 \operatorname{Exp} \operatorname{Exp}\left(-\frac{U}{8B}\right).
$$

This completes the proof of the lemma. □

We are now in a position to complete the proof of the main lemma. We first remark that

$$
\begin{aligned}
4\int_0^B \log\frac{2B}{\sigma} d\sigma &= 4B\log 2 + 4\sqrt{2}\int_0^B \left(\frac{B}{\sigma}\right)^{\frac{1}{2}} d\sigma \\
&< 4\left(\frac{1}{4} + \frac{1}{2.2^2} + \frac{1}{3.2^2}\right)B + (8\times 1.415)B < 15B.
\end{aligned}
$$

By Cauchy's theorem, we have,

$$\begin{aligned}\int_{2U}^{H-(r+3)U} G(it)idt &= \int_{i(2U)}^{i(H-(r+3)U)} G(s)ds\\ &= \int_{i(2U)}^{B+i(2U)} G(s)ds + \int_{B+i(2U)}^{B+i(H-(r+3)U)} G(s)ds-\\ &\quad - \int_{i(H-(r+3)U)}^{B+i(H-(r+3)U)} G(s)ds\\ &= J_1 + J_2 - J_3 \text{ say.}\end{aligned}$$

Using the estimate given in Lemma 2.2.5, we see that **44**

$$\begin{aligned}|J_1| \leq \int_0^B \Bigg(BU^{-10} &+ \frac{(12+4\log\frac{2B}{\sigma})}{U}\int_0^H |g(it)|dt\\ &+16(\log(2B))S_1 + 8BS_2 \operatorname{Exp}\left(-\frac{U}{8B}\right)\Bigg) d\sigma\\ \leq B^2U^{-10} &+ \frac{12B+15B}{U}\int_0^H |(g(it))|dt + 16BS_1\log(2B)\\ 8B^2S_2 &\operatorname{Exp}\left(-\frac{U}{8B}\right).\end{aligned}$$

The same estimate holds for $|J_3|$ also. For $|J_2|$ we use the estimate given in Lemma 2.2.4 to get

$$|J_2| \leq HS_1.$$

This completes the proof of the main lemma.

2.3 First Main Theorem

Let A, B, C be as before $0 < \epsilon \leq \frac{1}{2}$, $r \geq [(200A+200)\epsilon^{-1}]$, $|a_n| \leq n^A H^{\frac{r\epsilon}{8}}$. Then $F(s) = \sum_{n=1}^{\infty} a_n\lambda_n^{-s}$ is analytic in $\sigma \geq A+2$. Let $K \geq 30$, $U = H^{1-\frac{\epsilon}{2}} + 50B\log\log K_1$. Assume that

$$H \geq (120B^2C^{2A+4}(4rC^2)^r)^{\frac{400}{\epsilon}} + (100rB)^{20}\log\log K_1,$$

and that there exist T_1, T_2 with $0 \leq T_1 \leq U$, $H - U \leq T_2 \leq H$ such that

$$|F(\sigma + iT_1)| + |F(\sigma + iT_2| \leq K$$

uniformly in $0 \leq \sigma \leq B$ where $B(s)$ is assumed to be analytically continuable in $\sigma \geq 0$. Then

$$\int_0^H |F(it)|^2 dt \geq (H - 10rC^2 H^{1-\frac{\epsilon}{4}} - 100rB \log\log K_1) \sum_{n \leq H^{1-\epsilon}} |a_n|^2,$$

where

$$K_1 = \left(\sum_{n \leq H^{1-\epsilon}} |a_n| \lambda_n^B \right) K + \left(\sum_{n \leq H^{1-\epsilon}} |a_n| \lambda_n^B \right)^2.$$

45 **Corollary.** *Let A and C be as in the introduction § 2.1., $0 < \epsilon \leq \frac{1}{2}$, $r \geq [(200A + 200)\epsilon^{-1}]$, $|a_n| \leq n^A H^{\frac{r\epsilon}{8}}$. Then $F(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s}$ is analytic in $\sigma \geq A + 2$. Let $K \geq 30$, $U_1 = H^{1-\frac{\epsilon}{2}}$. Assume that $K_1 = (HC)^{12A} K$,*

$$H \geq (120(A+2)^2 C^{2A+4} (4rc^2)^r)^{\frac{100}{\epsilon}} + (100r(A+2))^{20} \log\log K_1,$$

and the there exist T_1, T_2 with $0 \leq T_1 \leq U_1$, $H - U_1 \leq T_2 \leq H$, such that uniformly in $\sigma \geq 0$ we have

$$|F(\sigma + iT_1)| + |F(\sigma + iT_2)| \leq K,$$

where $F(s)$ is assumed to be analytically continuable in $(\sigma \geq 0, 0 \leq t \leq H)$. Then

$$\frac{1}{H} \int_0^H |F(it)|^2 dt \geq (1 - 10rC^2 H^{-\frac{\epsilon}{4}} - 100rBH^{-1} \log\log K_1) \sum_{n \leq H^{1-\epsilon}} |a_n|^2.$$

Remark 1. We need the conditions $H \geq (r+5)U$, $U \geq 2^{70}(16B)^2$ in the application of the main lemma. All such conditions are satisfied by our lower bound choice for H. We have not attempted to obtain economical lower bounds.

Remark 2. Taking $F(s) = (\zeta(\frac{1}{2} + it + iT))^k$ in the first main theorem we obtain the following as an immediate corollary. Let $C(\epsilon, k) \log\log T \leq H \leq T$. Then for all integers $k \geq 1$.

$$\frac{1}{H}\int\limits_{T}^{T+H} \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2k} dt \geq (1-\epsilon) \sum_{n \leq H^{1-\epsilon}} (d_k(n))^2 n^{-1} \geq (C'_k - 2\epsilon)(\log H)^{k^2},$$

where

$$C'_k = (\Gamma(k^2+1))^{-1} \prod_p \left\{(1-p^{-1})^{k^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(k+m)}{\Gamma(k) m!}\right)^2 p^{-m}\right\}.$$

(This is because it is well-known that

$$\sum_{n \leq X} (d_n(n))^2 n^{-1} = \left\{C'_k + O\left(\frac{1}{\log X}\right)\right\} (\log X)^{k^2}).$$

Our third main theorem gives a sharpening of this. The third main theorem is sharper than the conjecture (stated by K. Ramachandra [70] in Durham conference 1979). The conjecture (as also the weaker form of the conjecture proved by him in the conference) would only give **46**

$$\frac{1}{H}\int_T^{T+H} \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2k} dt \gg_k (\log H)^{k^2} \text{ in } C(k)\log\log T \leq H \leq T.$$

But the third main Theorem gives

$$\frac{1}{H}\int_T^{T+H} \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2k} dt \geq C'_k (\log H)^{k^2} + O\left(\frac{\log\log T}{H} (\log H)^{k^2}\right) + O(\log H)^{k^2}$$

where the O-constants depend only on k.

Remark 3. The first main theorem gives a lower bound for $\frac{1}{H}\int_T^{T+H} |\zeta(\frac{1}{2}+it)|^{2k} dt$ uniformly in $1 \leq k \leq \log H$, $T \geq H \geq 30$ and $C \log\log T \leq H \leq T$. From this it follows (as was shown in R. Balasubramanian [2]) that for $C \log\log T \leq H \leq T$ we have uniformly

$$\max_{T \leq t \leq T+H} \left|\zeta\left(\frac{1}{2} + it\right)\right| > \text{Exp}\left(\frac{3}{4}\sqrt{\frac{\log H}{\log\log H}}\right)$$

if C is choosen to be a large positive constant. On Riemann hypothesis we can deduce from the first main theorem the following more general result. Let θ be fixed and $0 \leq \theta < 2\pi$. Put $z = e^{i\theta}$. Then (on Riemann hypothesis), we have,

$$\max_{T\leq t\leq T+H} |\left(\zeta\left(\frac{1}{2}+it\right)\right)^{z}| > \operatorname{Exp}\left(\frac{3}{4}\sqrt{\frac{\log H}{\log\log H}}\right)$$

where is LHS is interpreted as $\lim\limits_{\sigma\to\frac{1}{2}+0}$ of the same expression with $\frac{1}{2}+it$ replaced by $\sigma+it$. This result with $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$ gives a quantitative improvement of some results of J.H. Mueller [62].

Proof. Write $M = [H^{1-\epsilon}]$, $N = M+1$, $A(s) = \sum\limits_{m\leq M} \bar{a}_m\lambda_m^{-s}$, $\bar{A}(s) =$
47 $\sum\limits_{m\leq M} a_m\lambda_m^{-s}$, $B(s) = \sum\limits_{n\geq N} a_n\lambda_n^{-s}$. Then we have, in $\sigma \geq A+2$,

$$F(s) = \bar{A}(s) + B(s).$$

Also,

$$\begin{aligned}|F(it)|^2 &= |\bar{A}(it)|^2 + 2\operatorname{Re}(A(-it)B(it)) + |B(it)|^2\\ &\geq |\bar{A}(it)|^2 + 2\operatorname{Re}(g(it))\end{aligned}$$

where $g(s) = A(-s)B(s)$. Hence

$$\begin{aligned}\int_0^H |F(it)|^2 dt &\geq U^{-r}\int_0^U du_r \ldots \int_0^U du_1 \int_{2U+\lambda}^{H-(r+3)U+\lambda} |F(it)|^2 dt\\ &\geq U^{-r}\int_0^U du_r \ldots \int_0^U du_1 \int_{2U+\lambda}^{H-(r+3)U+\lambda} (|\bar{A}(it)|^2\\ &\quad + 2\operatorname{Re} g(it))dt = J_1 + 2J_2 \text{ say.}\end{aligned}$$

Now $\log\left(\frac{\lambda_{n+1}}{\lambda_n}\right) = -\log\left(1-\left(1-\frac{\lambda_n}{\lambda_{n+1}}\right)\right) \geq \frac{\lambda_{n+1}-\lambda_n}{\lambda_{n+1}} \geq (2nC^2)^{-1}$. Hence by Montgomery-Vaughan theorem,

$$J_1 \geq \int_{2U}^{H-(r+3)U} |\bar{A}(it)|^2 dt$$

$$\geq \sum_{n\leq M}(H-(r+5)U-100C^2n)|a_n|^2.$$

We have

$$\begin{aligned}
|g(s)| &= |A(-s)B(s)| = |A(-s)(F(s)-A(s))| \\
&\leq \left(\sum_{n\leq H^{1-\epsilon}} |a_n|\lambda_n^B\right)K + \left(\sum_{n\leq H^{1-\epsilon}} |a_n|\lambda_n^B\right)^2 \\
&= K_1.
\end{aligned}$$

By the main lemma, we have,

$$\begin{aligned}
|J_2| &\leq |U^{-r}\int_0^U du_r \ldots \int_0^U du_1 \int_{2U+\lambda}^{H-(r+3)U+\lambda} g(it)dt| \\
&\leq \frac{2B^2}{U^{10}} + \frac{54B}{U}\int_0^H |g(it)|dt + (H+64B^2)S_1 + 16B^2 S_2 \operatorname{Exp}(-\frac{U}{8B})
\end{aligned} \tag{2.3.1}$$

We simplify the last expression in (2.3.1). We can assume that 48

$$\int_0^H |F(it)|^2 dt \leq H \sum_{n\leq H^{1-\epsilon}} |a_n|^2$$

(otherwise the result is trivially true). Hence

$$\begin{aligned}
\int_0^H |g(it)|dt &= \int_0^H |A(-it)B(it)|dt \\
&\leq \int_0^H |A(-it)|^2 dt + \int_0^H |B(it)|^2 dt \\
&\leq \int_0^H |A(-it)|^2 dt + \int_0^H |F(it)-\bar{A}(it)|^2 dt \\
&\leq 3\int_0^H |A(-it)|^2 dt + 2\int_0^H |F(it)|^2 dt \\
&\leq 3\sum_{n\leq M}(H+100C^2n)|a_n|^2 + 2H\sum_{n\leq M}|a_n|^2
\end{aligned}$$

$$\leq (300C^2 + 5)H \sum_{n\leq M} |a_n|^2.$$

$$
\begin{aligned}
S_2 &\leq \sum_{m\leq M, n\geq N} |b_m c_n| \left(\frac{\lambda_m}{\lambda_n}\right)^{A+2} \\
&\leq \sum_{m\leq M, n\geq N} |a_m a_n| \left(\frac{\lambda_m}{\lambda_n}\right)^{A+2} \\
&\leq \sum_{m\leq M, n\geq N} m^A H^{\frac{r\epsilon}{8}} n^A H^{\frac{r\epsilon}{8}} (C^2 m n^{-1})^{A+2} \\
&\leq H^{\frac{r\epsilon}{4}} C^{2A+4} \sum_{m\leq M} m^{2A+2} \sum_{n\geq N} n^{-2} \\
&\leq H^{\frac{r\epsilon}{4}+2A+3} C^{2A+4} \text{ since } \frac{\pi^2}{6} - 1 < 1.
\end{aligned}
$$

Now

$$S_1 \leq \left(U \log \frac{\lambda_N}{\lambda_M}\right)^{-r} 2^r S_2$$

and

$$\log \frac{\lambda_N}{\lambda_M} \geq \frac{1}{2} \frac{\lambda_N - \lambda_M}{\lambda_M} \geq (2C^2 M)^{-1},$$

$$U \log\left(\frac{\lambda_N}{\lambda_M}\right) \geq (2C^2)^{-1} H^{\frac{\epsilon}{2}}.$$

49 Thus

$$
\begin{aligned}
|J_2| \leq \frac{2B^2}{U^{10}} &+ 54B(300C^2 + 5)HU^{-1} \sum_{n\leq M} |a_n|^2 \\
&(H + 64B^2) H^{-\frac{r\epsilon}{4}+2A+3} 2^r (2C^2)^r C^{2A+4} \\
&+ 16B^2 \operatorname{Exp}\left(-\frac{U}{8B}\right) H^{\frac{r\epsilon}{4}+2A+3} C^{2A+4}. \text{(Note } a_1 = \lambda_1 = 1\text{)}.
\end{aligned}
$$

So

$$(r + 5)U + 100C^2 H^{1-\epsilon} + 2|J_2| \left(\sum_{n\leq M} |a_n|^2\right)^{-1}$$

$$
\begin{aligned}
&\leq (r+5)H^{1-\frac{\epsilon}{2}} + 100C^2H^{1-\epsilon} + 100Br\log\log K_1\\
&\qquad + \frac{4B^2}{H^5} + 108B(300C^2+5)H^{\frac{\epsilon}{2}} + 128(2^r)(2C^2)^r B^2 H^{2A+4-50A}C^{2A+4}\\
&\qquad + 32B^2C^{2A+4}r!(8B)^r H^{2A+3+\frac{r\epsilon}{2}-\frac{r}{2}}\\
&\leq 100Br\log\log K_1 + rC^2H^{1-\frac{\epsilon}{4}}\left\{\frac{r+5}{rC^2H^{\frac{\epsilon}{4}}} + \frac{100C^2}{H^{\frac{3\epsilon}{4}}} + \frac{4B^2}{H^5}\right.\\
&\qquad + \frac{108B(300C^2+5)}{H^{1-\frac{3\epsilon}{4}}} + 128(2^r)(2C^2)^r B^2 H^{-1}C^{2A+4}\\
&\qquad \left. +32B^2C^{2A+4}r!(8B)^r H^{-1}\right\}\\
&\leq 100Br\log\log K_1 + 10C^2rH^{1-\frac{\epsilon}{4}}.
\end{aligned}
$$

This completes the proof of the theorem. □

2.4 Second Main Theorem

We assume the same conditions as in the first main theorem except that we change the definition of U to $U = H^{\frac{7}{8}} + 50B\log\log K_2$. Then there holds

$$\int_0^H |F(it)|dt \geq H - 10rH^{\frac{7}{8}} - 100rB\log\log K_2,$$

where $K_2 = K + 1$.

Corollary. *Let A and C be as in the introduction § 2.1, $|a_n| \leq (nH)^A$. Then $F(s) = \sum\limits_{n=1}^{\infty} a_n\lambda_n^{-s}$ is analytic in $\sigma \geq A+2$. Let $K \geq 30$,*

$$H \geq (3456000A^2C^3)^{640000A} + (240000A)^{20}\log\log(K+1),$$

and that there exist T_1, T_2 with $0 \leq T_1 \leq H^{\frac{7}{8}}$, $H - H^{\frac{7}{8}} \leq T_2 \leq H$, such **50**
that uniformly in $\sigma \geq 0$ we have

$$|F(\sigma+iT_1)| + |F(\sigma+iT_2)| \leq K$$

where $F(s)$ is assumed to be analytically continuable in $(\sigma \geq 0, 0 \leq t \leq H)$. Then

$$\frac{1}{H}\int_0^H |F(it)|dt \geq 1 - 8000AH^{-\frac{1}{8}} - 240000A^2H^{-1}\log\log(K+1).$$

The corollary is obtained by putting $\epsilon = \frac{1}{2}$, $r = 800A$ in the second main theorem.

Remark. Conditions like $H \geq (r+5)U$, $U \geq 2^{70}(16B)^2$ are taken care of by the inequality for H.

Proof. We have,

$$\begin{aligned}
&\int_0^H |F(it)|dt \geq U^{-r}\int_0^U du_r \ldots \int_0^U du_1 \int_{2U+\lambda}^{H-(r+3)U+\lambda} |F(it)|dt \\
&\geq U^{-r}\operatorname{Re}\left(\int_0^U du_r \ldots \int_0^U du_1 \int_{2U+\lambda}^{H-(r+3)U+\lambda} F(it)dt\right) \\
&= U^{-r}\operatorname{Re}\left\{\int_0^U du_r \ldots \int_0^U du_1 \int_{2U+\lambda}^{H-(r+3)U+\lambda} (1 + A(-it)B(it))dt\right\}
\end{aligned}$$

(where $A(s) \equiv 1$ (i.e. $a_1 = 1 = M$) and $B(s) = F(s) - 1) = J_1 + \operatorname{Re} J_2$ say. Clearly $J_1 \geq H - (r+5)U$. For J_2 we use the main lemma.

$$|J_2| \leq \frac{2B^2}{U^{10}} + \frac{54B}{U}\int_0^H |g(it)|dt + (H + 64B^2)S_1 + 16B^2 \operatorname{Exp}\left(-\frac{U}{8B}\right)S_2. \tag{2.4.1}$$

As in the proof of the first main theorem we can assume $\int_0^H |F(it)|dt \leq H$ and so $\int_0^H |g(it)|dt \leq 2H$. We have $|g(s)| \leq K + 1 = K_2$. Now

$$S_2 \leq H^{\frac{r\epsilon}{4}+2A+3}C^{2A+4},$$

and $U\log\left(\frac{\lambda_N}{\lambda_M}\right) = U\log\lambda_2 \geq (2C)^{-1}U$,

$$S_1 \leq 2^r S_2(U\log\lambda_2)^{-r} \leq 2^r S_2((2C)^{-1}U)^{-r}.$$

This shows that

$$\begin{aligned}
&(r+5)U + |J_2| \\
&\leq (r+5)U + \frac{2B^2}{U^{10}} + \frac{54B}{U}2H + \frac{(H+64B^2)}{(2C^{-1}U)^r}2^r C^{2A+4}H^{\frac{r\epsilon}{4}+2A+3} \\
&16B^2 \operatorname{Exp}\left(-\frac{U}{8B}\right)H^{\frac{r\epsilon}{4}+2A+3}C^{2A+4}
\end{aligned}$$

$$\le 100rB\log\log K_2 + rH^{\frac{7}{8}}\left\{\frac{r+5}{r} + \frac{2B^2}{rH^{\frac{69}{8}}} + \frac{108B}{H^{\frac{3}{4}}}\right\}$$
$$+ (H+64B^2)2^r C^{2A+4} H^{\frac{r\epsilon}{4}+2A+3+\frac{r}{16}-(r+1)\frac{7}{8}}$$
$$+ 16B^2 C^{2A+4}(8B)^r r! H^{\frac{r\epsilon}{4}+2A+3-\frac{7r}{8}}$$
$$\le 10rH^{\frac{7}{8}} + 100rB\log\log K_2,$$

when H satisfies the inequality of the theorem. □ 51

2.5 Third Main Theorem

Let $\{a_n\}$ and $\{\lambda_n\}$ be as in the introduction and $|a_n| \le (nH)^A$ where $A \ge 1$ is an integer constant. Then $F(s) = \sum\limits_{n=1}^{\infty} a_n\lambda_n^{-s}$ is analytic in $\sigma \ge A+2$. Suppose $F(s)$ is analytically continuable in $\sigma \ge 0$. Assume that (for some $K \ge 30$) there exist T_1 and T_2 with $0 \le T_1 \le H^{\frac{7}{8}}$, $H - H^{\frac{7}{8}} \le T_2 \le H$ such that $|F(\sigma+iT_1)| + |F(\sigma+iT_2)| \le K$ uniformly in $0 \le \sigma \le A+2$. Let

$$H \ge (4C)^{9000A^2} + 520000A^2\log\log K_3.$$

Then

$$\int_0^H |F(it)|^2 dt \ge \sum_{n\le\alpha H}(H - (3C)^{1000A}H^{\frac{7}{8}} - 130000A^2\log\log K_3 - 100C^2 n)|a_n|^2,$$

where $\alpha = (200C^2)^{-1}2^{-8A-20}$ and

$$K_3 = \left(\sum_{n\le H}|a_n|\lambda_n^B\right)K + \left(\sum_{n\le H}|a_n|\lambda_n^B\right)^2.$$

Corollary. *Let A and C be as in the introduction § 2.1, $|a_n| \le (nH)^A$. Then $F(s) = \sum\limits_{n=1}^{\infty} a_n\lambda_n^{-s}$ is analytic in $\sigma \ge A+2$. Let $K \ge 30$, $K_2 = (HC)^{12A}K$,* 52

$$H \ge (4C)^{9000A^2} + 520000A^2\log\log K_2,$$

and that there exist T_1, T_2 *with* $0 \leq T_1 \leq H^{\frac{7}{8}}$, $H - H^{\frac{7}{8}} \leq T_2 \leq H$, *such that uniformly in* $\sigma \geq 0$ *we have*

$$|F(\sigma + iT_1)| + |F(\sigma + iT_2)| \leq K,$$

where $F(s)$ *is assumed to be analytically continuable in* $(\sigma \geq 0, 0 \leq t \leq H)$. *Then*

$$\frac{1}{H}\int_0^H |F(it)|^2 dt \geq \sum_{n \leq \alpha H} (1 - (3C)^{1000A} H^{-\frac{1}{8}} - 130000A^2 H^{-1} \log\log K_2 \\ -100C^2 H^{-1} n)|a_n|^2,$$

where $\alpha = (200C^2)^{-1} 2^{-8A-20}$.

To prove this theorem we need the following two lemmas.

Lemma 2.5.1. *In the interval* $[\alpha H, (1600C^2)^{-1} H]$ *there exists an* X *such that*

$$\sum_{X \leq n \leq X + H^{\frac{1}{4}}} |a_n|^2 \leq H^{-\frac{1}{4}} \sum_{n \leq X} |a_n|^2,$$

provided $H \geq 2^{1000A^2} C^{50A}$.

Proof. Assume that such an X does not exist. Then for all X in $[\alpha H$, $(1600C^2)^{-1} H]$,

$$\sum_{X \leq n \leq X + H^{\frac{1}{4}}} |a_n|^2 > H^{-\frac{1}{4}} \sum_{n \leq \sum} |a_n|^2. \tag{2.5.1}$$

Let $L = \alpha H$, $I_j = [2^{j-1}L, 2^j L]$ for $j = 1, 2, \ldots 8A + 17$. Also let $I_0 = [1, L]$. Put $S_j = \sum\limits_{n \in I_j} |a_n|^2 (j = 0, 1, 2, \ldots, 8A + 17)$. For $j \geq 1$ divide the
53 interval I_j into maximum number of disjoint sub-intervals each of length $H^{\frac{1}{4}}$ (discarding the bit at one end). Since the lemma is assumed to be false the sum over each sub-interval is $\geq H^{-\frac{1}{4}} S_{j-1}$. The number of sub-intervals is $\geq [2^{j-1} L H^{-\frac{1}{4}}] - 1 \geq 2^{j-2} L H^{-\frac{1}{4}}$ (provided $2^{j-1} L H^{-\frac{1}{4}} - 2 \geq 2^{j-2} L H^{-\frac{1}{4}}$, i.e. $2^{j-2} L H^{-\frac{1}{4}} \geq 2$ i.e. $\alpha H^{\frac{3}{4}} \geq 4$ i.e. $H \geq (4\alpha^{-1})^{\frac{4}{3}}$). It

follows that $S_j \geq 2^{j-2} L H^{-\frac{1}{2}} S_{j-1}$. By induction $S_j \geq (\frac{1}{2} L H^{-\frac{1}{2}})^j S_0$. Since $S_0 \geq 1$ we have in particular

$$S_{8A+17} \geq \left(\frac{1}{2}\alpha H^{\frac{1}{2}}\right)^{8A+17} \geq \left(\frac{1}{2}\alpha\right)^{8A+17} H^{4A+\frac{1}{2}\cdot 17}.$$

On the other hand

$$S_{8A+17} = \sum_{\alpha_1 H \leq n \leq \alpha_2 H} |a_n|^2 \leq \sum_{n \leq \alpha_2 H} (nH)^{2A},$$

where $\alpha_1 = 16^{-1}(200C^2)^{-1}$ and $\alpha_2 = 8^{-1}(200C^2)^{-1}$. Thus $S_{8A+17} \leq H^{4A+1}$. Combining the upper and lower bounds we are led to

$$H^{\frac{1}{2}\cdot 15} \leq (2\alpha^{-1})^{8A+17} \tag{2.5.2}$$

provided $H \geq (4\alpha^{-1})^{\frac{4}{3}}$ (the latter condition is satisfied by the inequality for H prescribed by the Lemma). But (2.5.2) contradicts the inequality prescribed for H by the lemma. This contradiction proves the Lemma. □

From now on we assume that X is as given by Lemma 2.5.1.

Lemma 2.5.2. *Let* $\bar{A}(s) = \sum_{n \leq X} a_n \lambda_n^{-s}$, $E(s) = \sum_{X \leq n \leq X + H^{\frac{1}{4}}} a_n \lambda_n^{-s}$ *and* $B(s) = F(s) - \bar{A}(s) - E(s)$. *Clearly in* $\sigma \geq A+2$ *we have* $B(s) = \sum_{n \geq X + H^{\frac{1}{4}}} a_n \lambda_n^{-s}$. *Let* $H \geq 2^{1000A^2} C^{50A}$, $U = H^{\frac{7}{8}} + 100B \log\log K_3$, $K_3 \geq 30$ *and* $H \geq (2r+5)U$. *Then we have the following five inequalities.*

(a) $\int_0^H |\bar{A}(it)|^2 dt \leq 100C^2 H \sum_{n \leq X} |a_n|^2$,

(b) $\int_{2U+rU}^{H-(r+3)U} |\bar{A}(it)|^2 dt \geq \sum_{n \leq X} (H - (2r+5)U - 100C^2 n)|a_n|^2$,

(c) $\int_0^H |E(it)|^2 dt \leq 100C^2 H^{\frac{3}{4}} \sum_{n \leq X} |a_n|^2$, **54**

(d) $\int_0^H |B(it)|^2 dt \leq 1000C^2 H \sum_{n \leq X} |a_n|^2$, *and finally*

(e) $\int_0^H |A(-it)B(it)|dt \leq 400C^2 H \sum_{n\leq X} |a_n|^2$,

where (d) and (e) are true provided

$$\int_0^H |F(it)|^2 dt \leq H \sum_{n\leq X} |a_n|^2.$$

Proof. The inequalities (a) and (b) follow from the Montgomery - Vaughan theorem. From the same theorem

$$\begin{aligned}\int_0^H |E(it)|^2 dt &\leq \sum_{X\leq n\leq X+H^{\frac{1}{4}}} (H + 100C^2 n)|a_n|^2 \\ &\leq 100C^2 H \sum_{X\leq n\leq X+H^{\frac{1}{4}}} |a_n|^2\end{aligned}$$

and hence (c) follows from Lemma 2.5.1. Since

$$|B(it)|^2 \leq 9(|F(it)|^2 + |\bar{A}(it)|^2 + |E(it)|^2)$$

the inequality (d) follows from (a) and (c). Lastly (e) follows from (a) and (d). Thus the lemma is completely proved. □

We are now in a position to prove the theorem. We write (with $\lambda = u_1 + u_2 + \ldots + u_r$ as usual)

$$\int_0^H |F(it)|^2 dt \geq U^{-r} \int_0^U du_r \ldots \int_0^U du_1 \int_{2U+\lambda}^{H-(r+3)U+\lambda} |F(it)|^2 dt$$

(where $(r+5)U \leq H$ and $0 \leq u_i \leq U$. In fact we assume $(2r+5)U \leq H$).
55 Now

$$|F(it)|^2 \geq |\bar{A}(it)|^2 + 2\,\mathrm{Re}(A(-it)B(it)) + 2\,\mathrm{Re}(A(-it)E(it)) + 2\,\mathrm{Re}(\bar{B}(-it)E(it)),$$

where $\bar{B}(s)$ is the analytic continuation of $\sum_{n\geq X+H^{\frac{1}{4}}} a_n \lambda_n^{-s}$. Accordingly

$$\int_0^H |F(it)|^2 dt \geq J_1 + J_2 + J_3 + J_4 \tag{2.5.3}$$

where

$$J_1 = \int_0^H |\bar{A}(it)|^2 dt,\ J_2 = 2\,\mathrm{Re}\int_0^H (A(-it)B(it))dt,$$
$$J_3 = 2\,\mathrm{Re}\int_0^H (A(-it)E(it))dt \text{ and } J_4 = 2\,\mathrm{Re}\int_0^H (\bar{B}(-it)E(it))dt.$$

By Lemma 2.5.2 (b), we have,

$$J_1 \geq \sum_{n\leq X}(H-(2r+5)U-100C^2n)|a_n|^2.$$

Also by Lemma 2.5.2 ((a) and (c)), we have,

$$|J_3| \leq 2\int_0^H |A(-it)E(it)|dt \leq 200C^2H^{\frac{7}{8}}\sum_{n\leq X}|a_n|^2.$$

Similarly by Lemma 2.5.2 ((c) and (d)),

$$|J_4| \leq 800C^2H^{\frac{7}{8}}\sum_{n\leq X}|a_n|^2.$$

For J_2 we use the main lemma. We choose $U = H^{\frac{7}{8}} + 100B\log\log K_3$. We have $g(s) = A(-s)B(s)$. We have

$$|g(s)| \leq \left(\sum_{n\leq H}|a_n|\lambda_n^B\right)K + \left(\sum_{n\leq H}|a_n|\lambda_n^B\right)^2 = K_3.$$

By Lemma 2.5.2 ((e)) we have **56**

$$\int_0^H |g(it)|dt \leq 400C^2H\sum_{n\leq X}|a_n|^2.$$

Again

$$S_2 \leq \sum_{m\leq X, n\geq X+H^{\frac{1}{4}}}|a_m||a_n|\left(\frac{\lambda_m}{\lambda_n}\right)^{A+2}$$

$$\sum_{m\le X, n\ge X+H^{\frac{1}{4}}} (mH)^A (nH)^A (C^2 mn^{-1})^{A+2} \le C^{2A+4} H^{4A+3}.$$

Put $x = \frac{\lambda_N}{\lambda_M} - 1$ where $N = \left[X + H^{\frac{1}{4}}\right]$, $M = [X]$. Then $0 < x < \frac{2c(n-M)}{c^{-1}M} < \frac{3C^2 H^{\frac{1}{4}}}{\alpha H} < \frac{1}{2}$ under the conditions on H imposed in the theorem. Hence

$$\begin{aligned} U \log\left(\frac{\lambda_N}{\lambda_M}\right) &\ge \frac{U}{2}\left(\frac{\lambda_N - \lambda_M}{\lambda_M}\right) \\ &\ge \frac{U}{2}\left(\frac{N - M - 3}{C^2 M}\right) \\ &\ge \frac{1}{2} H^{\frac{7}{8}}\left(\frac{H^{\frac{1}{4}} - 3}{C^2 H}\right) \\ &\ge \frac{H^{\frac{1}{8}}}{3C^2}, \end{aligned}$$

(under the conditions on H imposed in the theorem). Thus

$$S_1 \le 2^r S_2 H^{-\frac{r}{8}} (3C^2)^r.$$

We choose $r = 100A + 100$ and check that $U \ge 2^{70}(16B)^2$, and that $H \ge (2r+5)U$. Thus by applying the main Lemma we obtain

$$\begin{aligned} |\frac{1}{2} J_2| \le &\left\{\frac{2B^2}{U^{10}} + \frac{54B}{U}(400C^2 H) + \frac{(H + 64B^2)2^r C^{2A+4} H^{4A+3}}{((3C^2)^{-1} H^{\frac{1}{8}})^r}\right. \\ &+16B^2 \operatorname{Exp}\left(-\frac{U}{8B}\right) C^{2A+4} H^{4A+3} \sum_{n\le X} |a_n|^2. \end{aligned}$$

Hence

$$\int_0^H |F(it)|^2 \ge \sum_{n\le \alpha H} (H - D - 100C^2 n)|a_n|^2,$$

where

$$D = (2r+5)U + 1000C^2 H^{\frac{7}{8}} + \frac{4B^2}{U^{10}} + \frac{43200C^2 BH}{U}$$

$$
\begin{aligned}
&+ (H + 64B^2)2^{r+1}C^{2A+4}(3C^2)^r H^{4A+3-\frac{r}{8}}\\
&+ 32B^2 \operatorname{Exp}\left(-\frac{U}{8B}\right) C^{2A+4}H^{4A+3}\\
&< 130000A^2 \log\log K_3 + 405AH^{\frac{7}{8}} + 1000C^2 H^{\frac{7}{8}} + 36A^2 H^{\frac{7}{8}}\\
&+ 43200C^2(3A)H^{\frac{7}{8}}\\
&+ 600A^2 H(2^{100A+101})C^{2A+4}3^{100A+100}C^{200A+200}H^{4A+3-12A-12}\\
&+ 300A^2 C^{2A+4}(720)(56)(24A)^8 H^{\frac{7}{8}}\\
&\le 130000A^2 \log\log K_3 + H^{\frac{7}{8}}\left\{405A + 1000C^2 + 36A^2 + 129600AC^2\right.\\
&\left.+600A^2C^{406A}3^{401A} + 3^{58}A^{10}C^{6A}\right\}\\
&\le 130000A^2 \log\log K_3 + (3C)^{1000A}.
\end{aligned}
$$

This proves the theorem completely. 57

Notes at the end of Chapter II

The previous history of the fundamental theorems proved in this chapter is as follows. In 1928 E.C. Titchmarsh proved (see Titchmarsh's book [100] p. 174 and also E.C. Titchmarsh [103]) that for every fixed integer $k \geq 1$ and any $\delta > 0$, we have

$$\delta \int_0^\infty |\zeta\left(\frac{1}{2} + it\right)|^{2k} e^{-\delta t} dt \gg \left(\log \frac{1}{\delta}\right)^{k^2},$$

where the constant implied by $\gg$ depends only on k. Later these ideas were developed by the author to prove Ω-theorems for short intervals on the line $\sigma = \frac{1}{2}, \frac{1}{2} < \sigma < 1$, and on $\sigma = 1$ (see K. Ramachandra [69]). These were taken up again by the author who introduced "TITCHMARSH SERIES" and proved very general theorems which gave

$$\frac{1}{H} \int_T^{T+H} |\zeta\left(\frac{1}{2} + it\right)|^{2k} dt \gg (\log H)^{k^2}, \ (k \geq 1 \text{ fixed integer}),$$

where $H \gg \log\log T$ and at the same time gave Ω-theorems for short 58 intervals on the said lines. These results were presented by the author in Durham Conference (1979) (See K. Ramachandra, [70]). The three fundamental theorems proved in this chapter are due to R. Balasubramanian and K. Ramachandra. (See R. Balasubramanian and K. Ramachandra, [10], [9] and also R. Balasubramanian, [2]). The word Titchmarsh's phenomenon is used to mean the swayings of $|\zeta(s)|$ as t varies and σ is fixed.

Chapter 3

Titchmarsh's Phenomenon

3.1 Introduction

We have used the term "TITCHMARSH'S PHENOMENON" for the swayings of $|\zeta(\sigma + it)|$ as σ is fixed and t varies. More generally we consider the swayings of $|(\zeta(\sigma + it))^z|(= f_\sigma(t)$ say) where $z = e^{i\theta}$, θ and σ being constants (with $\frac{1}{2} \leq \sigma \leq 1$ and $0 \leq \theta < 2\pi$; we will always use z for $e^{i\theta}$). First of all a remark about extending σ to be a constant with $\sigma < \frac{1}{2}$. By the functional equation (see equation (4.12.3) on page 78 of E.C. Titchmarsh [100]), we obtain for $t \geq 20$ and $\sigma < \frac{1}{2}$, 59

$$f_\sigma(t) = |(\zeta(\sigma + it))^z| \asymp \left(\left(\frac{t}{2\pi}\right)^{(\frac{1}{2}-\sigma)\operatorname{Cos}\theta + t\operatorname{Sin}\theta} e^{-t\operatorname{Sin}\theta}\right)|(\zeta(1 - \sigma - it))^z|,$$

and so RH implies $f_\sigma(t) \to \infty$ as $t \to \infty$ provided that $0 \leq \theta < \pi$, (because $\log|(\zeta(1-\sigma-it))^z| = \operatorname{Re}\log(\zeta(1-\sigma-it))^z = o(\log t)$ if $\sigma < \frac{1}{2}$; see equation (14.2.1) on page 336 of E.C. Titchmarsh [100]). Also when $\pi \leq \theta < 2\pi$ we have $z = -e^{i\phi}$ where $0 \leq \phi < \pi$. Hence (on RH) when $\pi \leq \theta < 2\pi$ and $\sigma < \frac{1}{2}$, we have $f_\sigma(t) \to 0$ as $t \to \infty$. The main question which we ask in this chapter is this: Let us fix σ in $\frac{1}{2} \leq \sigma \leq 1$, and put

$$f(H) = \min_{|I|=H} \max_{t \in I} f_\sigma(t).$$

Then does $f(H) \to \infty$ as $H \to \infty$? If so can we obtain an asymptotic formula for $f(H)$? We obtain a completely satisfactory answer if $\sigma = 1$.

Put $\lambda(\theta) = \prod\limits_{p} \lambda_p(\theta)$ where

$$\lambda_p(\theta) = \left(1 - \frac{1}{p}\right)\left(\sqrt{1 - \frac{\mathrm{Sin}^2\theta}{p^2}} - \frac{\mathrm{Cos}\,\theta}{p}\right)^{-\mathrm{Cos}\,\theta} \mathrm{Exp}\left(\mathrm{Sin}\,\theta\,\mathrm{Sin}^{-1}\left(\frac{\mathrm{Sin}\,\theta}{p}\right)\right).$$

Then we prove that

$$|f(H)e^{-\gamma}(\lambda(\theta))^{-1} - \log\log H| \leq \log\log\log H + O(1).$$

60 (Putting $\theta = 0$ and π we obtain stronger forms of two results due to earlier authors. For the earlier history see Theorems 8.9(A) and 8.9(B) on page 197 and also pages 208 and 209 of E.C. Titchmarsh [100]). However we are completely in the dark if $\frac{1}{2} \leq \sigma < 1$. We can only prove that if $T \geq H \geq C \log\log T$ where C is a certain positive constant, then

$$\max_{T \leq t \leq T+H} f_\sigma(t) > \begin{cases} \mathrm{Exp}\left(\frac{3}{4}\sqrt{\frac{\log H}{\log\log H}}\right) & \text{if } \sigma = \frac{1}{2}; \\ \mathrm{Exp}\left(\frac{\alpha_\sigma(\log H)^{1-\sigma}}{\log\log H}\right) & \text{if } \frac{1}{2} < \sigma < 1, \end{cases}$$

where α_σ is a certain positive constant. The result just mentioned does not need RH if $\theta = 0$. But we need RH if $0 < \theta < 2\pi$. When $\sigma = 1$ our results are completely satisfying since the results are all free from RH. It may also be mentioned (see the two equations proceeding (1.1.1)) that we do not need RH in proving (that if $\sigma < \frac{1}{2}$)

$$\max_{|t-t_0| \leq C(\epsilon)} |\zeta(\sigma + it)| \gg t_0^{\frac{1}{2}-\sigma-\epsilon}$$

where $C(\epsilon)$ depends only on $\epsilon (0 < \epsilon < 1)$, and is positive. But RH gives $|\zeta(\sigma+it)| \gg t^{\frac{1}{2}-\sigma-\epsilon}$. The results (with the condition $T \geq H \geq C \log\log T$ mentioned above will be proved by an application of the corollary to the third main theorem (in § 2.5) and will form the substance of § 3.2 and § 3.3. However we include in § 3.3 some other "Upper bound results" regarding the maximum of $f_\sigma(t)$ over certain intervals still with $\frac{1}{2} < \sigma < 1$). These results will also be used in the proof of the result on $\sigma = 1$, which forms the subject matter of § 3.4.

A new approach to the simpler question (than whether $f(H) \to \infty$ and so on) is due to H.L. Montgomery. He developed a method of proving an Ω result for $f_\sigma(t)$ namely

$$f_{\frac{1}{2}}(t) = \Omega\left(\mathrm{Exp}\left(\frac{1}{20}\sqrt{\frac{\log t}{\log\log t}}\right)\right) \text{ on } RH,$$

and **61**

$$f_\sigma(t) = \Omega\left(\mathrm{Exp}\left(\frac{1}{20}\left(\sigma - \frac{1}{2}\right)^{\frac{1}{2}} \frac{(\log t)^{1-\sigma}}{(\log\log t)^\sigma}\right)\right) \text{ if } \frac{1}{2} < \sigma < 1$$

the latter being independent of R.H. In the Ω result in $\frac{1}{2} < \sigma < 1$ it is possible (using Montgomery's method) to replace $c(\sigma - \frac{1}{2})^{\frac{1}{2}}$ by $c(\sigma - \frac{1}{2})^{\frac{1}{2}}(1-\sigma)^{-1}$. The method also succeeds in getting Ω results for $\mathrm{Re}(e^{i\theta}\zeta(1+it))$. For references to these results see the notes at the end of this chapter.

3.2 On The line $\sigma = \frac{1}{2}$

As a first theorem we prove

Theorem 3.2.1 (on RH)**.** *Let $z = e^{i\theta}$ where θ is a constant satisfying $0 \leq \theta < 2\pi$. If $H \leq T$ and $H(\log\log T)^{-1}$ exceeds a certain positive constant, then*

$$\max_{T \leq t \leq T+H} |(\zeta(\frac{1}{2} + it))^z| \geq \mathrm{Exp}\left(\frac{3}{4}\sqrt{\frac{\log H}{\log\log H}}\right).$$

Remark 1. We do not read RH if $z = 1$.

Remark 2. We prove the theorem with a certain positive constant $< \frac{3}{4}$ in place of $\frac{3}{4}$. Replacing Lemma 3.2.2 of the present section by a more powerful lemma due to R. Balasubramanian we get $\frac{3}{4}$. See the notes at the end of this chapter.

Lemma 3.2.1. *Given any $t \geq 10$ there exists a real number τ with $|t-\tau| \leq 1$, such that*

$$\frac{\zeta'(\sigma + i\tau)}{\zeta(\sigma + i\tau)} = ((\log t)^2) \tag{3.2.1}$$

uniformly in $-1 \leq \sigma \leq 2$. Hence

$$\log \zeta(\sigma + i\tau) = O((\log t)^2) \tag{3.2.2}$$

uniformly in $-1 \leq \sigma \leq 2$.

Proof. See Theorem 9.6(A), page 217 pf E.C. Titchmarsh [100]. □

Lemma 3.2.2. *Let*

$$F(s) = \left(\zeta\left(\frac{1}{2} + s\right)\right)^{kz} = \sum_{n=1}^{\infty} \frac{a_n}{n^s} (\sigma \geq 2), \tag{3.2.3}$$

62 *where $k \geq 10000$ is any integer. Let $x \geq 1000$,*

$$\prod_{k^3 \leq p \leq k^4} \left(1 + \frac{k^2}{p^s}\right) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}. \tag{3.2.4}$$

Then

$$\sum_{n \leq x} \frac{b_n}{n} \leq \sum_{n \leq x} |a_n|^2, \tag{3.2.5}$$

$$\prod_{k^3 \leq p \leq k^4} \left(1 + \frac{k^2}{p}\right) > \mathrm{Exp}(k^2 \log \frac{5}{4}), \tag{3.2.6}$$

and

$$\prod_{k^3 \leq p \leq k^4} \left(1 + \frac{k^2}{p^{1+\delta}}\right) < \mathrm{Exp}(k^2 e^{-C_1}), \tag{3.2.7}$$

where $\delta = \frac{C_1}{\log k}$ and C_1 is any positive constant. Also

$$\sum_{n \leq x} \frac{b_n}{n} > \mathrm{Exp}\left(k^2 \log \frac{5}{4}\right) - x^{\delta} \sum_{n > x} \frac{b_n}{n^{1+\delta}} \tag{3.2.8}$$

$$> \frac{1}{2}\operatorname{Exp}\left(k^2 \log\frac{5}{4}\right) \tag{3.2.9}$$

provided

$$k \geq C_2\left(\frac{\log x}{\log\log x}\right)^{\frac{1}{2}} \tag{3.2.10}$$

where C_2 is a positive constant.

Proof. The equation (3.2.5) is trivial. The equations (3.2.6) and (3.2.7) follow from $\log(1+y) < y$ and $\log(1+y) > y - \frac{1}{2}y^2$ for $0 < y < 1$. The equation (3.2.8) is trivial whereas (3.2.9) follows if $x^\delta \leq \operatorname{Exp}(k^2 e^{-C_1})$ and C_1 is large. This leads to the condition (3.2.10) for the validity of (3.2.9). □

Lemma 3.2.3. *We have, with $k = [C_3(\log H)^{\frac{1}{2}}(\log\log H)^{-\frac{1}{2}}]$,*

$$\left(\frac{1}{2}\sum_{n\leq\alpha H}|a_n|^2\right)^{\frac{1}{2k}} > \operatorname{Exp}(k\log(100/99)) \tag{3.2.11}$$

where α is an in corollary to the third main theorem (see § 2.5) and C_3 is a certain positive constant. **63**

Proof. Follows from (3.2.5), (3.2.9) and (3.2.10). □

Lemma 3.2.4. *The condition $|a_n| < (nH)^A$ is satisfied for some integer constant $A > 0$.*

Proof. Trivial since, for large H, we have

$$\frac{|a_n|}{n^2} \leq \sum_{n=1}^{\infty}\frac{|a_n|}{n^2} \leq \sum_{n=1}^{\infty}\frac{d_k(n)}{n^2} = (\zeta(2))^k \leq H^2.$$

Lemmas 3.2.1, 3.2.2, 3.2.3 and 3.2.4 complete the proof of Theorem 3.2.1, since by Lemma 3.2.1 we have

$$(\zeta(\sigma+it))^z = \operatorname{Exp}(O((\log t)^2)) \quad \left(\sigma \geq \frac{1}{2}\right) \tag{3.2.12}$$

on two suitable lines (necessary for the application of the third main theorem) $t = t_1$ and $t = t_2$. (It is here that we need the condition that $H(\log\log T)^{-1}$ should exceed a large positive constant). □

Theorem 3.2.2 (On RH)**.** *For all H exceeding a suitable positive constant, we have,*

$$\max_{\sigma\geq\frac{1}{2}}\max_{T\leq t\leq T+H}|(\zeta(\sigma+it))^z| > \mathrm{Exp}\left(\frac{3}{4}\sqrt{\frac{\log H}{\log\log H}}\right),$$

z being as in Theorem 3.2.1.

Remark. As in Theorem 3.2.1 we do not need RH if $z = 1$. Also the previous remark regarding the constant $\frac{3}{4}$ stands.

Proof. Assume that Theorem 3.2.2 is false. Then

$$\max_{\sigma\geq\frac{1}{2}}\max_{T\leq t\leq T+H}|(\zeta(\sigma+it))^z| < H.$$

64 This is enough to prove (by the method of proof of Theorem 3.2.1) that on $\sigma = \frac{1}{2} + (\log H)^{-1}$ we have

$$\max_{T\leq t\leq T+H}|(\zeta(\sigma+it))^z| > \mathrm{Exp}\left(\frac{3}{4}\sqrt{\frac{\log H}{\log\log H}}\right).$$

This completes the proof of Theorem 3.2.2. □

If we are particular about the line $\sigma = \frac{1}{2}$ (and smaller H than in Theorem 3.2.1) we have

Theorem 3.2.3. *From the interval $[T, 2T]$ we can exclude $T(\log T)^{-20}$ intervals of length $(\log T)^2$ with the property that in the remaining intervals I, we have*

$$\max_{s\in(\frac{1}{2},\infty)\times I}|\zeta(s)|^2 \leq (\log T)^{30}. \tag{3.2.13}$$

Let $H \leq (\log T)^2$ and I_0 be any interval of length H contained in I. Then for $z = e^{i\theta}$, we have

$$\max_{t\in I_0}|(\zeta(\frac{1}{2}+it))^z| > \mathrm{Exp}\left(\frac{3}{4}\sqrt{\frac{\log H}{\log\log H}}\right)$$

provided only that $H(\log\log\log T)^{-1}$ exceeds a suitable positive constant and $z = 1$. If $z \neq 1$, then we have to assume RH and to replace $\frac{1}{2}$ by $\frac{1}{2} + (\log\log T)^{-1}$.

Remark 1. The previous remark about the constant $\frac{3}{4}$ stands. The question of proving Theorem 3.2.3 (when $z \neq 1$) without replacing $\frac{1}{2}$ by $\frac{1}{2} + (\log\log T)^{-1}$ is an open question.

Remark 2. We can for $z \neq 1$ replace $\frac{1}{2}$ by $\frac{1}{2} + (\log\log T)^{-1000}$.

Remark 3. Theorems such as 3.2.1, 3.2.2 and 3.2.3 have applications to the variation of $\arg\zeta(s)$ over short t-intervals.

3.3 On the Line σ with $\frac{1}{2} < \sigma < 1$

As a simple application of the corollary to the third main theorem (see § 2.5) we first prove Theorems 3.3.1, 3.3.2, and 3.3.3 to follows. **65**

Theorem 3.3.1. *Let $f_\sigma(t) = |(\zeta(\sigma+it))^z|$ where $z = e^{i\theta}$ and $T^{\frac{1}{3}} \leq H \leq T$. Then for $\frac{1}{2} < \alpha < 1$, there holds*

$$\max_{T \leq t \leq T+H} f_\alpha(t) > \operatorname{Exp}\left(C_1 \frac{(\log H)^{1-\alpha}}{\log\log H}\right)$$

where C_1 is a certain positive constant.

Remark. If $z = 1$ then the restriction on H can be relaxed to $H \leq T$ and that $H(\log\log T)^{-1}$ shall be bounded below by a certain positive constant. However when $z \neq 1$, we need to assume RH to uphold the corresponding result.

Theorem 3.3.2 (on RH)**.** *For all H exceeding a suitable positive constant, we have, for $\frac{1}{2} < \alpha < 1$,*

$$\max_{\sigma \geq \alpha} \max_{T \leq t \leq T+H} |(\zeta(\sigma + it))^z| > \operatorname{Exp}\left(C_1 \frac{(\log H)^{1-\alpha}}{\log\log H}\right)$$

z and C_1 being as before.

Remark. If $z = 1$ RH is not necessary.

Theorem 3.3.3. *Divide $[T, 2T]$ into abutting intervals I of fixed length H (ignoring a bit at one end) where H exceeds a sufficiently large constant and $H(\log\log T)^{-1}$ is bounded above by any fixed constant. Then there exists a positive constant β' such that with the exception of $T(\mathrm{Exp}\,\mathrm{Exp}(\beta' H))^{-1}$ intervals I, we have,*

$$\max_{t\in I} |(\zeta(\alpha + it))^z| > \mathrm{Exp}\left(C_1 \frac{(\log H)^{1-\alpha}}{\log\log H}\right)$$

where as usual $z = e^{i\theta}$.

Remark 1. Note that this theorem does not depend on RH.

Remark 2. We can replace $[T, 2T]$ by $[T, T + T^{\frac{1}{3}}]$. The number of exceptions will then be $T^{\frac{1}{3}}(\mathrm{Exp}\,\mathrm{Exp}(\beta' H))^{-1}$.

66 First we give details of proof for Theorem 3.3.1 and then briefly sketch the proof of Theorem 3.3.3. The proof of Theorem 3.3.2 is similar to that of Theorem 3.2.2. We begin with

Lemma 3.3.1. *Let $\frac{1}{2} \geq \beta \leq 1$ and $H = T^{\frac{1}{3}}$. Then the number of zeros of $\zeta(s)$ in $(\sigma \geq \beta,\ T \leq t \leq T + H)$ is*

$$\ll H^{4(1-\beta)/(3-2\beta)}(\log T)^{100} \tag{3.3.1}$$

where the constant implied by the Vinogradov symbol $\ll$ is absolute.

Proof. This is a consequence of a deep result of R. Balasubramanian [3] on the mean square of $|\zeta(\frac{1}{2} + it)|$. (See Theorem 6 on page 576 of his paper; see also K. Ramachandra [65]). □

Lemma 3.3.2. *Let α and β be constants satisfying $\frac{1}{2} < \beta < \alpha < 1$. Then there exists a t-interval I contained in $T \leq t \leq T + H$ of length T^{δ} (where $\delta > 0$ depends only on α and β) such that the region $(\sigma \geq \beta, t \in I)$ is free from zeros of $\zeta(s)$.*

Proof. Follows from lemma 3.3.1. □

Lemma 3.3.3. *Let I_0 denote the t-interval obtained from I by removing, on both sides, intervals of length $\frac{1}{100}T^{\delta}$. Then in $(\sigma \geq \alpha, t \in I_0)$, we have,*

$$\log \zeta(s) = O(\log T)$$

Proof. Follows by Borel-Caratheodory Theorem (see Theorem 1.6.1). □

Lemma 3.3.4. *We apply the corollary to the third main theorem (see § 2.5) to the interval I_0 in place of $T \leq t \leq T+H$. Then (with $z = e^{i\theta}$), we have,*

$$\max_{(\sigma=0, t\in I_0)} |(\zeta(\alpha+s))^z| \geq \left(\frac{1}{2}|a_n|^2\right)^{\frac{1}{2k}} \tag{3.3.2}$$

where n is any integer not exceeding $\alpha' H$ where α' is the α of the corollary to the third main theorem and $k \geq 1000$ is $O(\log H)$ and the numbers a_n are defined by 67

$$F(s) = (\zeta(\alpha+s))^{kz} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}. \tag{3.3.3}$$

Proof. It is easily seen (as before) that the conditions for the application of the corollary to the third main theorem are satisfied. □

Lemma 3.3.5. *Let*

$$n = \prod p \tag{3.3.4}$$

where p runs over all the primes in the interval $[(\frac{k}{4})^{\frac{1}{\alpha}}, (\frac{k}{2})^{\frac{1}{\alpha}}]$. Then

$$|a_n|^2 \geq \operatorname{Exp}\left(\frac{C_1 k^{\frac{1}{\alpha}}}{\log k}\right)$$

where C_1 is a positive constant.

Proof. Note that $k^2 p^{-2\alpha}$ is bounded below by a constant > 1 and that the number of primes in (3.3.4) is $\gg k^{\frac{1}{\alpha}}(\log k)^{-1}$. This proves Lemma 3.3.4. □

Lemma 3.3.6. *Let* $k = [C_2(\log H)^{\alpha}]$ *where* $C_2 > 0$ *is a small constant. Then, we have,*

$$n \leq \alpha' H, \tag{3.3.5}$$

and so RHS of 3.3.2 exceeds

$$\operatorname{Exp}\left(\frac{C_3(\log H)^{1-\alpha}}{\log\log H}\right) \tag{3.3.6}$$

where $C_3 > 0$ *is a constant.*

Theorem 3.3.1 follows from (3.3.2) and (3.3.6). We now briefly sketch the proof of Theorem 3.3.3. Let β be a constant satisfying $\frac{1}{2} < \beta < \alpha < 1$. The number of zeros of $\zeta(s)$ in $(\sigma \geq \beta, T \leq t \leq 2T)$ is $< T^{1-\delta}$ for some positive constant δ. Omit those intervals I for which $(\sigma \geq \beta, t \in I)$ contains a zero of $\zeta(s)$. (The number of intervals omitted
68 is $\leq T^{1-\delta}$). Denote the remaining intervals by I'. For these, we have, by standard methods (see Theorem 1.7.1) the inequality

$$\sum_{I'} \max_{\sigma \geq \beta, t \in I'} |\zeta(\sigma + it)|^2 \ll T$$

and so the number of intervals I' for which the maximum exceeds Exp Exp$(2\beta' H)$ does not exceed $T(\operatorname{Exp}\operatorname{Exp}(2\beta' H))^{-1}$. Omit these also. Call the remaining intervals I''. Applying Borel-Caratheodory theorem (see Theorem 1.6.1), we find that for suitable t-intervals at both ends of I'' and $\sigma \geq \alpha$ we have $\log\zeta(s) = O(\operatorname{Exp}(2\beta' H))$ and so $(\zeta(s))^z = O(\operatorname{Exp}\operatorname{Exp}(2\beta' H))$. We can now apply the corollary to the third main theorem to obtain Theorem 3.3.3 since plainly the number of omitted intervals is

$$\leq T^{1-\delta} + T(\operatorname{Exp}\operatorname{Exp}(2\beta' H))^{-1} \leq T(\operatorname{Exp}\operatorname{Exp}(\beta' H))^{-1}$$

if β' is small enough.

In the remainder of this section namely § 3.3, we concentrate on proving the following theorem. (For the history of this theorem and other interesting results see R. Balasubramanian and K. Ramachandra [12]). This theorem will be used in § 3.4. For the sake of convenience we adopt the notation of the paper of R. Balasubramanian and myself cited just now and whenever we apply this theorem we take care to see that there is no confusion of notation.

Theorem 3.3.4. *Let α be a fixed constant satisfying $\frac{1}{2} < \alpha < 1$ and $E > 1$ an arbitrary constant. Let $C \leq H \leq T/100$ and $K = \mathrm{Exp}\left(\frac{D \log H}{\log\log H}\right)$ where C is a large positive constant and D an arbitrary positive constant. Then there are $\geq TK^{-E}$ disjoint intervals I of length K each, contained in $[T, 2T]$ such that*

$$\frac{(\log K)^{1-\alpha}}{(\log\log K)^{\alpha}} \ll \max_{t \in I} |\log \zeta(\alpha + it)| \ll \frac{(\log K)^{1-\alpha}}{(\log\log K)^{\alpha}}.$$

Furthermore

$$\max_{\sigma \geq \alpha, t \in I} |\log \zeta(\sigma + it)| \ll \frac{(\log K)^{1-\alpha}}{(\log\log K)^{\alpha}}.$$

Remark. Here as elsewhere $\log \zeta(s)$ is the analytic continuation along lines parallel to the σ-axis (we chose those and only those lines which do not contain a zero or a pole of $\zeta(s)$) of $\log \zeta(s)$ in $\sigma \geq 2$. **69**

We first outline the proof of this theorem and reduce it to the proof of Theorem 3.3.5 and 3.3.6 below. Let β_0, β_1 and β be constants satisfying $\frac{1}{2} < \beta_0 < \beta_1 < \beta < \alpha < 1$. It is well-known that

$$\frac{1}{T} \int_T^{2T} \left|\zeta\left(\frac{1}{2} + it\right)\right|^2 dt = O(\log T).$$

From this it follows that there are $\gg TH^{-1}$ disjoint intervals I_0 for t (ignoring a bit at one end) each of length $H + 20(\log H)^2$ contained in $[T, 2T]$ for which

$$\int_{2 \geq \sigma \geq \beta_0} \int_{t \in I_0} |\zeta(s)|^2 dt d\sigma \ll H \tag{3.3.7}$$

and

$$\int_{t \in I_0} |\zeta(\beta_1 + it)|^2 dt \ll H. \tag{3.3.8}$$

From (3.3.7) and (3.3.8) it follows by standard methods (explained in my booklet on Riemann zeta-function published by Ramanujan Institute, now using Jensen's theorem see E.C. Titchmarsh [101] on page 125; See also Chapter 7 of the present book) that $N(\beta, I_O)$ is the number of zeros ρ of $\zeta(s)$ with $\operatorname{Re}\rho \geq \beta$ and $\operatorname{Im}\rho$ lying in I_0. Hence if we divide I_0 into abutting intervals (ignoring a bit at one end) I_1 each of length $H^\theta + 20(\log H)^2$ whee $\theta = \frac{\delta}{2}$, the number of intervals I_1 is $\sim H^{1-\theta}$. Out of these we omit those I_1 for which $(\sigma \geq \beta, t \text{ in } I_1)$ contains a zero of $\zeta(s)$. (They are not more than a constant times $H^{1-\delta}$ in number). We now consider a typical interval I_1 which is such that $(\sigma \geq \beta, t \text{ in } I_1)$ is zero-free. Let us designate this t-interval by $[T_0 - 10(\log H)^2, T_0 + H^\theta + 10(\log H)^2]$. Put

$$H_1 = H^\theta \text{ and } k = \left[\frac{C_1 \log H}{\log\log H}\right] \tag{3.3.9}$$

70 where C_1 is a small positive constant. Then we prove the following Theorem.

Theorem 3.3.5. *We have,*

$$\int_{T_0}^{T_0+H_1} |\log \zeta(\alpha + it)|^{2k} dt > C_2^k A_k^{2k} H_1 \tag{3.3.10}$$

and

$$\int_{T_0}^{T_0+H_1} |\log \zeta(\alpha + it)|^{4k} dt < C_3^{2k} A_{2k}^{4k} H_1 < 2^{4k} C_3^{2k} A_k^{4k} H_1 \tag{3.3.11}$$

where $A_k = k^{1-\alpha}(\log k)^{-\alpha}$, and C_2, C_3 are positived constants independent of C_1.

Corollary. *Divide $[T_0, T_0 + H_1]$ into equal (abutting) intervals I each of length K (neglecting a bit at one end). Then the number N of intervals I for which*

$$\int_I |\log \zeta(\alpha + it)|^{2k} dt > \frac{1}{4} C_2^k A_k^{2k} K \tag{3.3.12}$$

satisfies $N \geq -1 + \frac{1}{16}(\frac{C_2}{4C_3})^{2k} H_1 K^{-1}$ and so, in these intervals

$$\max_{t \in I} |\log \zeta(\alpha + it)| \gg A_k.$$

Proof of the Corollary. Put $J = \int_I |\log\zeta(\alpha + it)|^{2k} dt$. Then

$$\sum_1 J > \frac{1}{2} C_2^k A_k^{2k} H_1$$

since the contribution from the neglected bit is not more than $K^{\frac{1}{2}}(2^{4k} C_3^{2k} A_k^{4k} H_1)^{\frac{1}{2}}$ on using (3.3.11). Let

$$\sum_1 J = \sum_{I, J \leq \frac{1}{4} C_2^k A_k^{2k} K} J$$

and $\sum_2$ the sum over the remaining intervals I. Then $\sum_2 J > \frac{1}{4} C_2^k A_k^{2k} H_1$. **71**
Put $\sum_2 1 = N$. Then by Hölder's inequality we have

$$\begin{aligned} \frac{1}{4} C_2^k A_2^{2k} H_1 &< N^{\frac{1}{2}} \left(\sum_2 J^2 \right)^{\frac{1}{2}} \\ &\leq N^{\frac{1}{2}} \left(\sum_2 \int_I |\log\zeta(\alpha + it)|^{4k} dt K \right)^{\frac{1}{2}} \\ &\leq N^{\frac{1}{2}} K^{\frac{1}{2}} (2^{4k} C_3^{2k} A_k^{4k} H_1)^{\frac{1}{2}}. \end{aligned}$$

Hence $N \geq \frac{1}{16} \left(\frac{C_2}{4C_3} \right)^{2k} H_1 K^{-1}$. This proves the corollary.

Theorem 3.3.6. *Let J_1 be the maximum over* $(\operatorname{Re} s \geq \alpha, \operatorname{Im} s$ *in* $I)$ *of* $|\log\zeta(s)|^{2k}$. *Then with the notation introduced above and $I = [a, b]$, we have,*

$$\begin{aligned} E_2 J_1 &\leq (\log H)^2 \sum_2 \int_{\alpha - 1 \leq t \leq b + 1} \int_{\sigma \geq \alpha - (\log H)^{-1}} |\log\zeta(s)|^{2k} d\sigma\, dt \\ &\leq 2(\log H)^2 \int \int_{\substack{T_0 - 1 \leq t \leq T_0 + H_1 + 1 \\ \sigma \geq \alpha - (\log H)^{-1}}} |\log\zeta(s)|^{2k} d\sigma\, dt \\ &\leq 2(\log H)^2 C_4^k A_k^{2k} H_1, \end{aligned} \tag{3.3.13}$$

where $C_4 > 0$ is independent of C_1.

Corollary. *Of any* $\frac{N}{2}$ *of the summands* J_1 *appearing in* $\sum_2$ *the minimum* J_1 *does not exceed*

$$\frac{2(\log H)^2 C_4^k A_k^{2k} H_1}{-1+\frac{1}{16}(\frac{C_2}{4C_3})^{2k} H_1 K^{-1}}. \tag{3.3.14}$$

Hence the $\max\limits_{t\in I} |\log\zeta(\alpha+it)|$ *over those intervals* I *is* $\ll A_k$.

Combining corollaries to Theorems 3.3.5 and 3.3.6, we have $\geq -1+\frac{1}{32}(\frac{C_2}{4C_3})^{2k} H_1 K^{-1} (= M$ say) intervals I contained in I_1 for which there holds

$$A_k \ll \max_{t\in I} |\log\zeta(\alpha+it)| \ll A_k. \tag{3.3.15}$$

Now by choosing C_1 small we have $M \gg H_1 K^{-E}$ where $H_1 = H^\theta$ and the number of intervals I_1 is $\sim H^{1-\theta}$. Since I_1 is contained in I_0 and the number of intervals I_0 is $\gg TH^{-1}$ we have in all

$$H^\theta K^{-E} H^{1-\theta} T H^{-1} = TK^{-E} \tag{3.3.16}$$

72 disjoint intervals I of length K each, where (3.3.15) holds. This completes the proof of Theorem 3.3.4 provided we prove Theorems 3.3.5 and 3.3.6

We now develop some preliminaries to the proofs of Theorems 3.3.5 and 3.3.6. From (3.3.7) using the fact that the absolute value of an analytic function at a point does not exceed its mean-value over a disc (say of radius $(\log H)^{-1}$) round that point as centre, we obtain

$$|\zeta(s)| \leq H^2 \text{ in } (\sigma \geq \beta + (\log H)^{-1}, T_0 - 9(\log H)^2 \leq t \leq T_0 + H_1 + 9(\log H)^2).$$

Hence in this region $\operatorname{Re}\log\zeta(s) \leq 2\log H$. Now $\log\zeta(2+it) = O(1)$ and hence by Borel-Caratheodory theorem, we have,

$$\log\zeta(s) = O(\log H) \text{ in } (\sigma \geq \frac{1}{2}(\alpha+\beta), T_0 - 8(\log H)^2 \leq t \leq T_0 + H_1 + 8(\log H)^2).$$

We now put

$$X = (\log H)^B \tag{3.3.17}$$

where B is a large positive constant. We have

$$\sum_p p^{-s}\operatorname{Exp}\left(-\frac{p}{X}\right) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \log\zeta(s+w) X^w \Gamma(w)dw + O(1)$$

where $\sigma = \alpha$ and $T_0 - 7(\log H)^2 \leq t \leq T_0 + H_1 + 7(\log H)^2$. Here first break off the portion $|\operatorname{Im} w| \geq (\log H)^2$ and move the rest of the line of integration to $\operatorname{Re} w$ given by $\operatorname{Re}(s + w) = \frac{1}{2}(\alpha + \beta)$. Also observe that

$$\sum_{p \geq X^2} p^{-s} \operatorname{Exp}\left(-\frac{p}{X}\right) = O(1).$$

Collecting our results we have (since $|\Gamma(w + 1)| \ll \operatorname{Exp}(-|\operatorname{Im} w|)$),

$$\log \zeta(s) = \sum_{p \leq X^2} p^{-s} \operatorname{Exp}\left(-\frac{p}{X}\right) + O(1) \tag{3.3.18}$$

where $\sigma = \alpha$ and $T_0 - 7(\log H)^2 \leq t \leq T_0 + H_1 + 7(\log H)^2$. Let **73**

$$X^{2k} \leq H_1^{\frac{1}{2}} \text{ and } \left(\sum_{p \leq X^2} p^{-s} \operatorname{Exp}\left(-\frac{p}{X}\right)\right)^k = \sum_{p \leq X^{2k}} a_k(n) n^{-s} = F(s), \tag{3.3.19}$$

say.

Then we have

$$|F(s)|^2 \leq (|\log \zeta(s)| + C_5)^{2k} \leq 2^{2k} |\log \zeta(s)|^{2k} + (2C_5)^{2k}, \tag{3.3.20}$$

and also

$$|\log \zeta(s)|^{2k} \leq 2^{2k} |F(s)|^2 + (2C_5)^{2k}. \tag{3.3.21}$$

We now integrate these equations from $t = T_0$ to $t = T_0 + H_1$. Also we note that these inequalities are valid even when $\frac{11}{10} \geq \operatorname{Re} s \geq \alpha - (\log H)^{-1}$, $T_0 - 6(\log H)^2 \leq t \leq T_0 + H_1 + 6(\log H)^2$. Now in $\sigma \geq \frac{11}{10}$, we have $|\log \zeta(s)| \ll 2^{-\sigma}$ and so

$$\int\int_{\sigma \geq \frac{11}{10}, T_0 - 1 \leq t \leq T_0 + H_1 + 1} |\log \zeta(s)|^{2k} d\sigma dt \leq C_6^k \int\int 2^{-2k\sigma} d\sigma dt \leq H_1 C_6^k. \tag{3.3.22}$$

Therefore in order to prove Theorem 3.3.6 it suffices to consider

$$2^{2k} \int\int |F(s)|^2 d\sigma\, dt + H_1 C_7^k \tag{3.3.23}$$

where the area integral extends over

$$\left(\frac{11}{10} \geq \operatorname{Re} s \geq \alpha - (\log H)^{-1}, T_0 - 1 \leq t \leq T_0 + H_1 + 1\right).$$

By a simple computation, we have since $X^{2k} \leq H_1^{\frac{1}{2}}$,

$$G(\sigma) \ll \frac{1}{H_1} \int_{T_0-1}^{T_0+H_1+1} |F(s)|^2 dt \ll G(\sigma) \tag{3.3.24}$$

where

$$G(\sigma) = \sum_{n \leq X^{2k}} (a_k(n))^2 n^{-2\sigma}. \tag{3.3.25}$$

74 Thus in order to prove Theorems 3.3.5 and 3.3.6, we see (by (3.3.20), (3.3.21), (3.3.24) and (3.3.25)) that we have to obtain upper and lower bounds for $(G(\sigma))^{\frac{1}{(2k)}}$. (Things similar to $G(\sigma)$ were first studied by H.L. Montgomery. See the notes at the end of this chapter). Let $p_1 = 2$, $p_2 = 3, \ldots, p_k$ be the first k primes. By prime number theorem

$$p_1 p_2 \ldots p_k = \operatorname{Exp}(p_k + O(k)) = \operatorname{Exp}(k \log k + k \log\log k + O(k)). \tag{3.3.26}$$

Taking only the contribution to $G(\sigma)$ from $n = p_1 \ldots p_k$, we have, since $\operatorname{Exp}(-\frac{p_i}{X}) \geq \frac{1}{2}(i = 1, 2, \ldots, k)$,

$$(G(\sigma))^{\frac{1}{(2k)}} \geq \left(\frac{(k!)^2 2^{-2k}}{(p_1 \ldots p_k)^{2\sigma}}\right)^{\frac{1}{(2k)}} \gg \frac{k^{1-\sigma}}{(\log k)^{\sigma}} = A_k(\sigma) \text{ say.} \tag{3.3.27}$$

This proves the lower bound

$$G(\sigma) \geq (A_k(\sigma))^{2k} C_8^{2k}. \tag{3.3.28}$$

As regards the upper bound we write

$$\sum_{p \leq X^2} p^{-s} \operatorname{Exp}\left(-\frac{p}{X}\right) = \sum_1 + \sum_2 \tag{3.3.29}$$

where $\sum_1$ extends over $p \leq k \log k$ and $\sum_2$ the rest.

Note that

$$|F(s)|^2 \leq 2^{2k}|\sum_1|^{2k} + 2^{2k}|\sum_2|^{2k}. \tag{3.3.30}$$

Put

$$\left(\sum_1\right)^k = \sum_{n=1}^{\infty}\frac{b_k(n)}{n^s} = F_1(s) \text{ say}, \tag{3.3.31}$$

and

$$\left(\sum_2\right)^k = \sum_{n=1}^{\infty}\frac{c_k(n)}{n^s} = F_2(s) \text{ say}. \tag{3.3.32}$$

By a simple computation we have

$$\frac{1}{H_1}\int_{T_0-1}^{T_0+H_1+1}|F_1(s)|^2 dt \ll G_1(\sigma) \text{ and } \frac{1}{H_1}\int_{T_0-1}^{T_0+H_1-1}|F_2(s)|^2 dt \ll G_2(\sigma) \tag{3.3.33}$$

where **75**

$$G_1(\sigma) = \sum\frac{(b_k(n))^2}{n^{2\sigma}} \leq \left(\sum\frac{b_k(n)}{n^\sigma}\right)^2 = \left(\sum_{p\leq k\log k} p^{-\sigma}\operatorname{Exp}\left(-\frac{p}{X}\right)\right)^{2k} \tag{3.3.34}$$

and

$$G_2(\sigma) = \sum\frac{(c_k(n))^2}{n^{2\sigma}} \leq k!\sum\frac{c_k(n)}{n^{2\sigma}} \leq k!\left(\sum_{p\leq k\log k} p^{-2\sigma}\operatorname{Exp}\left(-\frac{p}{X}\right)\right)^{2k}. \tag{3.3.35}$$

If $\sigma < 1$, we have easily,

$$(G_1(\sigma))^{\frac{1}{(2k)}} \ll \frac{(k\log k)^{1-\sigma}}{\log k} = \frac{k^{1-\sigma}}{(\log k)^\sigma} \tag{3.3.36}$$

and by Stirling's approximation to $k!$ we have also

$$(G_2(\sigma))^{\frac{1}{(2k)}} \ll k^{\frac{1}{2}}\left(\frac{k(\log k)^{1-2\sigma}}{\log k}\right)^{\frac{1}{2}} = \frac{k^{1-\sigma}}{(\log k)^\sigma}. \tag{3.3.37}$$

This proves the upper bound

$$(G(\sigma))^{\frac{1}{(2k)}} \ll \left(\frac{1}{H_1}\int_{T_0-1}^{T_0+H_1+1}|F(s)|^2 dt\right)^{\frac{1}{(2k)}} \ll A_k(\sigma) \qquad (3.3.38)$$

which in turn gives an upper bound for $(G(\sigma))^{\frac{1}{(2k)}}$ if $\beta_0 \leq \sigma \leq 1-\delta_1$ uniformly in σ for every small constant $\delta_1 > 0$. If $\frac{11}{10} \geq \sigma \geq 1-\delta_1$ the bounds for the area integral are negligible if δ_1 is small since it is

$$\leq 2\left(\sum \frac{a_k(n)}{n^{\sigma_1}}\right)^2 \leq 2\left(\sum_{p\leq X^2} p^{-\sigma_1}\operatorname{Exp}\left(-\frac{p}{X}\right)\right)^{2k}$$

where $\sigma_1 = 1-\delta_1$.

This completes the proof of Theorems 3.3.5 and 3.3.6. Thus the proof of Theorem 3.3.4 is complete.

3.4 Weak Titchmarsh series and Titchmarsh's Phenomenon on the line $\sigma = 1$

76 The main object of this section is to prove the asymptotic formula for $f(H)$ (of course with $\sigma = 1$). This is a long story and we will state it as a theorem at the end of this section. We find it convenient to split up this section into part A (weak Titchmarsh series), part B (application to lower bound), part C (upper bound) and part D (the main theorem).

PART A

Weak Titchmarsh Series. Let $0 \leq \epsilon < 1$, $D \geq 1$, $C \geq 1$ and $H \geq 10$. Put $R = H^{\epsilon}$. Let $a_1 = \lambda_1 = 1$ and $\{\lambda_n\}(n = 1, 2, 3, \ldots)$ be any sequence of real numbers with $\frac{1}{C} \leq \lambda_{n+1} - \lambda_n \leq C(n = 1, 2, 3, \ldots)$ and $\{a_n\}(n = 1, 2, 3, \ldots)$ any sequence of complelx numbers satisfying

$$\sum_{\lambda_n \leq X} |a_n| \leq D(\log X)^R$$

for all $X \geq 3C$. Then for complex $s = \sigma + it(\sigma > 0)$ we define the analytic function $F(s) = \sum\limits_{n=1}^{\infty} a_n \lambda_n^{-s}$ as a weak Titchmarsh series associated with the parameters occuring in the definition.

Theorem 3.4A.1 (FOURTH MAIN THEOREM)**.** *For a weak Titchmarsh series $F(s)$ with $H \geq 36C^2H^{\epsilon}$, we have*

$$\lim \inf_{\sigma \to +0} \int_0^H |F(\sigma + it)| dt \geq H - 36C^2H^{\epsilon} - 12CD.$$

Theorem 3.4A.2 (FIFTH MAIN THEOREM)**.** *For a weak Titchmarsh series $F(s)$ with $\log H \geq 4320C^2(1-\epsilon)^{-5}$, we have,*

$$\lim \inf_{\sigma \to +0} \int_0^H |F(\sigma + it)|^2 dt \geq \sum_{n \leq M} \left(H - \frac{H}{\log H} - 100C^2 n \right) |a_n|^2 - 2D^2$$

where $M = (36C^2)^{-1} H^{1-\epsilon} (\log H)^{-4}$. **77**

Remarks. The two theorems just mentioned have been referred to in the published papers as the fourth and the fifth main theorems. (See the notes at the end of this chapter). Theorem 3.4A.1 will be used later.

Proof of Theorem 3.4A.1 We can argue with $\sigma > 0$ and then pass to the limit as $\sigma \to +0$. But formally the notation is simplified if we treat as though $F(s)$ is convergent absolutely if $\sigma = 0$ and there is no loss of generality. Let r be a positive integer and $0 < U \leq r^{-1}H$. Then since $|F(s)| \geq 1 + \mathrm{Re}(F(s))$, we have (with $\lambda = u_1 + \ldots + u_r$),

$$\begin{aligned}
\int_0^H |F(it)| dt &\geq U^{-r} \int_0^U du_r \ldots \int_0^U du_1 \int_{\lambda}^{H-rU+\lambda} |F(it)| dt \\
&\geq U^{-r} \int_0^U du_r \ldots \int_0^U du_1 \int_{\lambda}^{H-rU+\lambda} \{1 + \mathrm{Re}(F(it))\} dt \\
&\geq H - rU - 2^{r+1} U^{-r} J
\end{aligned}$$

where $J = \sum\limits_{n=2}^{\infty} |a_n| (\log \lambda_n)^{-r-1}$. Now $J = S_0 + \sum\limits_{j=1}^{\infty} S_j$ where $S_0 = \sum\limits_{\lambda_n \leq 3C} |a_n| (\log \lambda_n)^{-r-1}$ and $S_j = \sum\limits_{3^j C \leq \lambda_n \leq 3^{j+1} C} |a_n| (\log \lambda_n)^{-r-1}$. In S_0 we use

$\lambda_n \geq \lambda_2 \geq 1 + C^{-1}$ and so $(\log \lambda_n)^{-r-1} \leq (2C)^{r+1}$ and we obtain $S_0 \leq D(2C)^{r+1}(3C)^r$. Also, we have,

$$\begin{aligned} S_j &\leq D(\log(3^{j+1}C))^R(\log(3^jC))^{-r-1} \\ &\leq D2^R(\log(3^jC))^{R-r-1}, (\text{since } 3^{j+1}C \leq (3^jC)^2). \\ &\leq D2^R j^{-2} \text{ by fixing } r = [3R]. \end{aligned}$$

Thus for $r = [3R]$ we have

$$\begin{aligned} J &\leq D(2C)^{r+1}(3C)^R + 2D2^R, (\text{since } \sum_{j=1}^{\infty} j^{-2} < 2), \\ &\leq 3D(2C)^{r+1}(3C)^R. \end{aligned}$$

78 Collecting we have,

$$\begin{aligned} 2^{r+1}U^{-r}J &\leq 12CD(3C)^R\left(\frac{4C}{U}\right)^r \\ &\leq 12CD\left(\frac{12C^2}{U}\right)^R \text{ if } U \geq 4C \\ &\leq 12CD \text{ by fixing } U = 12C^2. \end{aligned}$$

The only condition which we have to satisfy is $rU \leq H$ which is secured by $H \geq 36C^2H^\epsilon$. This completes the proof of Theorem 1.

Proof of Theorem 3.4A.2

We write $\lambda = u_1 + \ldots + u_r$, where $0 \leq u_i \leq U$ and $0 < U \leq r^{-1}H$. We put $M_1 = [M]$, $A(s) = \sum\limits_{m \leq M_1} a_m\lambda_m^{-s}$ and $B(s) = \sum\limits_{n \leq M_1+1} a_n\lambda_n^{-s}$ so that $F(s) = A(s)+B(s)$. For the moment we suppose M to be a free parameter with the restriction $3 \leq M \leq H$. We use

$$|F(it)|^2 \geq |A(it)|^2 + 2\,\mathrm{Re}(A(it)\overline{B(it)}).$$

Now by a well-known theorem of H.L. Montgomery and R.C. Vaughan (see Theorems 1.4.1 and 1.4.2) we have

$$\int_{\lambda}^{H-rU+\lambda} |A(it)|^2 dt \geq \sum_{n \leq M}(H - rU - 100C^2n)|a_n|^2.$$

Next the absolute value of

$$2U^{-r}\int_0^U du_r \ldots \int_0^U du_1 \int_\lambda^{H-rU+\lambda} (A(it)\overline{B(it)})dt \tag{3.4A.1}$$

does not exceed

$$\begin{aligned} &2^{r+2}U^{-r}\sum_{m\le M_1, n\ge M_1+1} |a_m\bar{a}_n|\left(\log\frac{\lambda_n}{\lambda_m}\right)^{-r-1}\\ &\le 2^{r+2}U^{-r}\left(\sum_{m\le M_1}|a_n|\right)\left(\sum_{n\ge M_1+1}|a_n|\left(\log\frac{\lambda_n}{\lambda M_1}\right)^{r+1}\right). \end{aligned}$$

Here the m-sum is $\le D(\log\lambda_{M_1})^R \le D(\log(3MC))^R$, since $\lambda_{M_1}\le M_1C\le MC$. It is enough to choose $M\ge 1$ for the bound for the m-sum. The n-sum can be broken up into $\lambda_n\le 3\lambda_{M_1}$ and $3^j\lambda_{M_1}<\lambda_n\le 3^{j+1}\lambda_{M_1}(j=1,2,3,\ldots)$. Let us denote these sums by S_0 and S_j. Now since **79**

$$\left(\log\frac{\lambda_n}{\lambda_{M_1}}\right)\ge\left(\log\frac{\lambda_{M_1+1}}{\lambda_{M_1}}\right)\ge\log\left(1+\frac{1}{C\lambda_{M_1}}\right)\ge(2C\lambda_{M_1})^{-1}\ge(2C^2M)^{-1},$$

we obtain

$$S_0\le D(\log(3\lambda_{M_1}))^R(2C^2M)^{r+1}\le D(\log(3MC))^R(2C^2M)^{r+1}.$$

Also

$$\begin{aligned} S_j &\le D(\log(3^{j+1}\lambda_{M_1}))^R(j\log 3)^{-r-1}\\ &\le D(j\log 3+\log(3MC))^R j^{-r-1}\\ &\le 2^RD(j\log 3)^R(\log(3MC))^R j^{-r-1}\\ &\le 4^RD(\log(3MC))^R j^{-2},\ \text{if } r\ge R+1, \end{aligned}$$

and so (since $\sum\limits_{j=1}^{\infty} j^{-2}<2$),

$$\left(\sum_m\ldots\right)\left(\sum_n\ldots\right)\le D^2(\log(3MC))^R(\log(3MC))^RY$$

(where $Y = (2C^2M)^{r+1} + 2(4^R)$)

$$\leq D^2(\log(3MC))^{2R}((2C^2M)^{r+1} + 2(4^R)).$$

Hence the absolute value of the expression (3.4A.1) does not exceed

$$D^2(\log(3MC))^{2R}\left((8C^2M)\left(\frac{4C^2M}{U}\right)^r + 2\left(\frac{4^R}{U^r}\right)\right)^r \tag{3.4A.2}$$

$$\leq D^2\left\{8C^2M\left(\frac{4C^2M(\log(3MC))^2}{U}\right)^{R+\log(8C^2M)} + 2\left(\frac{4}{U}\right)^R\right\}$$

if $U \geq 4C^2M$ and $r \geq R + \log(8C^2M)$. We put $U = 12C^2M(\log(3MC))^2$ and obtain for (3.4A.2) the bound $D^2\{1 + 1\} \leq 2D^2$. The conditions to be satisfied are $M \geq 1$ and

$$12C^2M(\log(3MC))^2(R + \log(8C^2M) + 1) \leq H.$$

80 In fact we can satisfy $Ur \leq \frac{H}{\log H}$ by requiring

$$12C^2M(\log(3MC))^2(R + \log(8C^2M) + 1) \leq \frac{H}{\log H}.$$

This is satisfied if

$$36C^2M(\log(8C^2M))^3R \leq H(\log H)^{-1}.$$

Let $8C^2M \leq H$. Then $36C^2MR \leq H(\log H)^{-4}$ gives what we want. We choose $M = (36C^2)^{-1}H^{1-\epsilon}(\log H)^{-4}$. Clearly this satisfies $8C^2M \leq H$. In order to satisfy $M \geq 1$ we have to secure that

$$(36C^2)^{-1}\frac{((1-\epsilon)(\log H))^5}{120}(\log H)^{-4} \geq 1$$

i.e. $\log H \geq 4320C^2(1-\epsilon)^{-5}$.

This completes the proof of Theorem 3.4A.2.

PART B

The main result of part B is

Theorem 3.4B.1. *We have (with $z = e^{i\theta}$)*

$$\min_{T\geq 1}\max_{T\leq t\leq T+H}|(\zeta(1+it))^{z}| \geq e^{\gamma}\lambda(\theta)(\log\log H - \log\log\log H) + O(1), \tag{3.4B.1}$$

where $H \geq 10000$ and

$$\lambda(\theta) = \prod_{p}\lambda_p(\theta) \tag{3.4B.2}$$

and

$$\lambda_p(\theta) = \left(1-\frac{1}{p}\right)\left(\frac{\sqrt{p^2-\operatorname{Sin}^2\theta}+\operatorname{Cos}\theta}{p-p^{-1}}\right)^{\operatorname{Cos}\theta}\operatorname{Exp}\left(\operatorname{Sin}\theta\operatorname{Sin}^{-1}\left(\frac{\operatorname{Sin}\theta}{p}\right)\right). \tag{3.4B.3}$$

Remark. Note that 81

$$\left(\frac{\sqrt{p^2-\operatorname{Sin}^2\theta}+\operatorname{Cos}\theta}{p-p^{-1}}\right)^{\operatorname{Cos}\theta} = \left(\sqrt{1-\frac{\operatorname{Sin}^2\theta}{p^2}}-\frac{\operatorname{Cos}\theta}{p}\right)^{-\operatorname{Cos}\theta} \tag{3.4B.4}$$

The outline of the proof of this theorem is as follows. By Theorem 3.4A.2 with $\epsilon = \frac{1}{3}$, $k_0 = kz$ and $F(s) = (\zeta(1+s))^{k_0}$ we have

$$\frac{1}{H}\int_{T}^{T+H}|(\zeta(1+it))^{k_0}|^2 dt \geq \frac{1}{2}\sum_{n\leq H^{\frac{1}{4}}}\frac{|d_{k_0}(n)|^2}{n^2} \tag{3.4B.5}$$

uniformly in $T \geq 1$, and k any positive integer satisfying $1 \leq k \leq \log H$, provided H exceeds an absolute positive constant. Denote by S the RHS of (3.4B.5). We prove (by considering "the maximum term" in S) the following Theorem.

Theorem 3.4B.2. *We have*

$$\max_{1\le k\le \log H}\left(S^{\frac{1}{(2k)}}\right) \geq e^{\gamma}\lambda(\theta)(\log\log H - \log\log\log H) + O(1). \qquad (3.4B.6)$$

Remark. This would complete the proof of Theorem 3.4B.1.

We select a single term of S as follows. To start with we recall that we have to impose $1 \leq k \leq \log H$, $k_0 = kz$. We select n as follows. Let $n \geq 2$ and let $n = \prod_p p^m$ be the prime factor decomposition of n. Then (in the notation of Theorem 3.4A.2) we have

$$a_1 = 1, \text{ and } a_n = \prod_p a_{p^m} = \prod_p \frac{k_0(k_0+1)\ldots(k_0+m-1)}{m!\,p^m}, \qquad (3.4B.7)$$

by using the Euler product for $\zeta(s)$. For each $p(\leq k)$ we select an $m = m(p)$ for which $|a_{p^m}|$ is nearly maximum. Then we have to satisfy $n = \prod_{p\le k} p^m \leq H^{\frac{1}{4}}$. In fact we choose k as large as possible with these properties. We now proceed to the details.

82 **Lemma 4B.1.** *Let, for each $p \leq k$,*

$$\ell = k\left(\frac{\operatorname{Cos}\theta + \sqrt{p^2 - \operatorname{Sin}^2\theta}}{p^2 - 1}\right) = \frac{k}{q} \text{ say, and } m = [\ell]. \qquad (3.4B.8)$$

Then, putting $n = \prod_{p\le k} p^m$, we have

$$\frac{1}{2k}\log|a_n|^2 = \frac{1}{2k}\sum_{p\le k}\{-2m\log m + 2m + O(\log m) - 2m\log p + E(k,m)\} \qquad (3.4B.9)$$

where

$$E(k,m) = \sum_{\nu=0}^{m-1}\log(k^2 + \nu^2 + 2k\nu\operatorname{Cos}\theta). \qquad (3.4B.10)$$

Proof. Follows from the formula

$$\log m! = m\log m - m + O(\log m).$$

□

Lemma 4B.2. *We have,*

$$E(k,m) = 2m\log k + k\int_0^{\frac{1}{q}} \log(1+u^2+2u\operatorname{Cos}\theta)du + O\left(\frac{1}{p}\right). \quad (3.4B.11)$$

Proof. We have

$$\begin{aligned} E(k,m) = \sum_{v=0}^{m-1} \Big\{ & \log(k^2+v^2+2kv\operatorname{Cos}\theta) \\ & - \int_v^{v+1} \log(k^2+u^2+2ku\operatorname{Cos}\theta)du \Big\} \\ & + \int_0^m \log(k^2+u^2+2ku\operatorname{Cos}\theta)du. \end{aligned}$$

Here the sum on the right is easily seen to be $O\left(\frac{1}{p}\right)$. The integral on the right is

$$2m\log k + \int_0^m \log\left(1+\frac{u^2}{k^2}+2\frac{u}{k}\operatorname{Cos}\theta\right)du.$$

Here we can replace the upper limit m of the integral by ℓ with an error $O(\frac{m}{k}) = O(\frac{1}{p})$. The lemma now follows by a change of variable. □

Lemma 4B.3. *We have,* **83**

$$\frac{1}{k}\sum_{p\le k} \log m = O\left(\frac{1}{\log k}\right) \quad (3.4B.12)$$

and

$$\frac{1}{k}\sum_{p\le k}\frac{1}{p} = O\left(\frac{1}{\log k}\right)$$

Proof. Follows by prime number theorem. □

Lemma 4B.4. *We have,*

$$\begin{aligned} & \frac{1}{2k}\sum_{p\le k}\{-2m\log m + 2m - 2m\log p + 2m\log k\} \\ & \quad = \sum_{p\le k}\left\{-\frac{1}{q}\log\frac{p}{q} + \frac{1}{q}\right\} + O\left(\frac{1}{\log k}\right). \quad (3.4B.13) \end{aligned}$$

Proof. On the LHS we can replace m by ℓ with a total error

$$\leq \frac{1}{2k} \sum_{p \leq k} O(\log m) = O(\frac{1}{\log k}).$$

The rest is

$$\sum_{p \leq k} \left\{ -\frac{1}{q} \log \frac{k}{q} + \frac{1}{q} - \frac{1}{q} \log p + \frac{1}{q} \log k \right\}$$

which gives the lemma. □

Lemma 4B.5. *We have,*

$$\frac{1}{2k} \sum_{p \leq k} k \int_0^{\frac{1}{q}} \log(1 + u^2 + 2u \operatorname{Cos} \theta) du$$
$$= \operatorname{Re} \sum_{p \leq k} \left(\frac{1 + \frac{1}{q} e^{i\theta}}{e^{i\theta}} \log \left(1 + \frac{1}{q} e^{i\theta} \right) - \frac{1}{q} \right) \tag{3.4B.14}$$

Proof. Trivial. □

Lemma 4B.6. *We have,*

$$\frac{1}{2k} \log |a_n|^2 = \log \log k + \gamma + \log \lambda(\theta) + O\left(\frac{1}{\log k} \right), \tag{3.4B.15}$$

where $\lambda(\theta)$ is as in Theorem 4B.1.

84 *Proof.* By Lemmas 4B.1, 4B.2, 4B.3 and 4B.4 we see that LHS of (3.4B.15) is, (with an error $O(\frac{1}{\log k})$),

$$\operatorname{Re} \sum_{p \leq k} \left\{ -\frac{1}{q} \log \frac{p}{q} + \frac{1}{q} \log \left(1 + \frac{1}{q} e^{i\theta} \right) + e^{-i\theta} \log \left(1 + \frac{1}{q} e^{i\theta} \right) \right\}.$$

Now the contribution from the first two terms (in the curly bracket) to the sum is

$$\operatorname{Re} \sum_{p \leq k} \frac{1}{q} \log \left| \frac{q + e^{i\theta}}{p} \right| = 0,$$

since

$$|q + e^{i\theta}|^2 = \left(\frac{p^2 - 1}{\sqrt{p^2 - \operatorname{Sin}^2\theta + \operatorname{Cos}\theta}} + \operatorname{Cos}\theta \right)^2 + \operatorname{Sin}^2\theta$$
$$= \left(\frac{p^2 - 1 + \operatorname{Cos}^2\theta + \operatorname{Cos}\theta\sqrt{p^2 - \operatorname{Sin}^2\theta}}{\sqrt{p^2 - \operatorname{Sin}^2}\theta + \operatorname{Cos}\theta} \right)^2 + \operatorname{Sin}^2\theta$$
$$= p^2.$$

The third term contributes

$$\sum_{p\le k} \left(\operatorname{Cos}\theta \log\frac{p}{q} + \operatorname{Sin}\theta \tan^{-1}\left(\frac{\operatorname{Sin}\theta}{q + \operatorname{Cos}\theta}\right)\right)$$
$$= \sum_{p\le k} \left\{ \log\left(1 - \frac{1}{p}\right) + \operatorname{Cos}\theta \log\frac{p}{q} + \operatorname{Sin}\theta \tan^{-1}\left(\frac{\operatorname{Sin}\theta}{q + \operatorname{Cos}\theta}\right)\right\}$$
$$+ \sum_{p\le k} \log\left(1 - \frac{1}{p}\right)^{-1}.$$

This together with the well-known formula $\prod_{p\le k}\left(1 - \frac{1}{p}\right)^{-1} = e^{\gamma}\log k + O(1)$ proves the lemma. □

Lemma 4B.7. *For the n defined in Lemma 3.3.5, we have,*

$$\log n = \sum_{p\le k} m \log p = k \log k + O(k). \tag{3.4B.16}$$

Proof. Replacement of m by ℓ involves an error $O(k)$ by the prime number theorem. Now $\ell = \frac{k}{q}$ and **85**

$$q = p\left(p - \frac{1}{p}\right)\left(p\sqrt{1 - \frac{\operatorname{Sin}^2\theta}{p^2}} + \operatorname{Cos}\theta\right)^{-1}$$

$$= p\left(p - \frac{1}{p^2}\right)\left(p\sqrt{1 - \frac{\operatorname{Sin}^2\theta}{p^2}} + \frac{\operatorname{Cos}\theta}{p}\right)^{-1}$$
$$= p + O(1).$$

This proves the lemma. □

Lemma 4B.8. *Set* $k = \left[\frac{\log H}{2\log\log H}\right]$. *Then for all H exceeding a large positive constant, we have,*

$$m \leq \frac{H}{200}.$$

Proof. Follows from Lemma 4B.7.

Lemmas 4B.6 and 4B.8 complete the proof of Theorem 3.4B.2 and as remarked already this proves Theorem 3.4B.1 completely. □

PART C

The main result of part C is

Theorem 3.4C.1. *We have (with $z = e^{i\theta}$),*

$$\begin{aligned}&\min_{T\geq 1}\max_{T\leq t\leq T+H}|(\zeta(1+it))^z|\\ &\leq e^{\gamma}\lambda(\theta)(\log\log H + \log\log\log H) + O(1),\end{aligned} \tag{3.4C.1}$$

where $H \geq 10000$ and $\lambda(\theta)$ is as in Theorem 3.4B.1.

We begin by

Lemma 4C.1. *Let* $T = \operatorname{Exp}((\log H)^2)$ *where H exceeds an absolute constant. Then there exists a sub-interval I of $[T, 2T]$ of length $H +$
86 $2(\log H)^{10}$, such that the rectangle ($\sigma \geq \frac{3}{4}, t \in I$) does not contain any zero of $\zeta(s)$ and moreover*

$$\max|\log\zeta(\sigma + it)| = O((\log H)^{\frac{1}{4}}(\log\log H)^{-\frac{3}{4}}) \tag{3.4C.2}$$

the maximum being taken over the rectangle referred to.

Proof. Follows from Theorem 3.3.4 and the result (due to A.E. Ingham [41], see also E.C. Titchmarsh [100], page 236, and p. 293-295 of A. Ivić [42]) that the number of zeros of $\zeta(s)$ in $(\sigma \geq \frac{3}{4}, T \leq t \leq 2T)$ is $O(T^{\frac{3}{4}})$. □

Lemma 4C.2. *Let J be the interval obtained by removing from I intervals of length* $(\log H)^{10}$ *from both ends. Then for $t \in J$, we have,*

$$\log \zeta(1+it) = \sum_{m \geq 1, p} \sum (mp^{ms})^{-1} \operatorname{Exp}\left(-\frac{p^m}{X}\right) + O((\log\log H)^{-1}) \tag{3.4C.3}$$

where $X = \log H \log\log H$ and $s = 1 + it$.

Proof. The lemma follows from the fact that the double sum on the right is

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \log \zeta(s+w) X^w \Gamma(w) dw \tag{3.4C.4}$$

where $w = u + iv$ is a complex variable. Here we break off the portion $|v| \geq (\log H)^9$ with an error $O((\log\log H)^{-1})$ and move the line of integration to $u = -\frac{1}{4}$. Using Lemma 4C.1 it is easily seen that the horizontal portions and the main integral contribute together $O((\log\log H)^{-1})$. □

Lemma 4C.3. *Denote the double sum in (3.4C.3) by S. Then*

$$S = \log \prod_{p \leq X} (1 - p^{-s})^{-1} + O((\log\log H)^{-1}). \tag{3.4C.5}$$

Proof. We use the fact $\operatorname{Exp}(-p^m X^{-1}) = 1 + O(p^m X^{-1})$ if $p^m \leq X$ and $= O(Xp^{-m})$ if $p^m \geq X$. Using this it is easy to see that

$$\begin{aligned} S &= \sum_{p^m \leq X} \sum (mp^{ms})^{-1} + O\left(\sum_{p^m \leq X} \sum X^{-1}\right) + O\left(\sum_{p^m \geq X} \sum X(mp^{2m})^{-1}\right) \\ &= \sum_{p^m \leq X} \sum (mp^{ms})^{-1} + O((\log\log H)^{-1}). \end{aligned}$$

Denoting the last double sum by S_0, we have, 87

$$S_0 - \sum_{p \le X} \log(1-p^{-s})^{-1} = O\left(\sum \sum_{p^m \ge X, m \ge 2} (mp^m)^{-1}\right) = O((\log\log H)^{-1}).$$

□

Lemma 4C.4. *We have, for $t \in J$,*

$$\log \zeta(1+it) = \sum_{p \le X} \log(1-p^{-s})^{-1} + O((\log\log H)^{-1}), \qquad (3.4C.6)$$

where $s = 1 + it$.

Proof. Follows from Lemmas 4C.1, 4C.2 and 4C.3. □

Lemma 4C.5. *Let $0 \le r < 1$, $0 \le \phi < 2\pi$. Then, we have,*

$$\log|(1-re^{i\phi})^{-z}| \le -\operatorname{Cos}\theta \log\left(\sqrt{1-r^2 \operatorname{Sin}^2\theta} - r\operatorname{Cos}\theta\right) + \\ + \operatorname{Sin}\theta \operatorname{Sin}^{-1}(r\operatorname{Sin}\theta). \qquad (3.4C.7)$$

Remark. Put

$$\lambda_p(\theta) = (1-p^{-1})\left(\sqrt{1-p^{-2}\operatorname{Sin}^2\theta} - p^{-1}\operatorname{Cos}\theta\right)^{-\operatorname{Cos}\theta} \\ \operatorname{Exp}\left(\operatorname{Sin}\theta \operatorname{Sin}^{-1}\left(\frac{\operatorname{Sin}\theta}{p}\right)\right). \qquad (3.4C.8)$$

In the lemma replace $re^{i\phi}$ by p^{-s}. Lemmas 4C.4 and 4C.5 complete the proof of Theorem 3.4C.1 since $\sum_{p \ge X} \log \lambda_p(\theta) = O(X^{-1})$ and $\prod_{p \le X}(1 - p^{-1})^{-1} = e^{\gamma}\log X + O(1)$. (See page 81 of K. Prachar [63]).

Proof of Lemma 4C.5. Denote the LHS of (11) by $g(\phi)$. Then

$$g(\phi) = \sum_{n=1}^{\infty} n^{-1} r^n \operatorname{Cos}(n\phi + \theta)$$

$$g'(\phi) = -\sum_{n=1}^{\infty} r^n \operatorname{Sin}(n\phi + \theta)$$

$$= \operatorname{Im}\left\{\frac{-re^{i(\phi+\theta)}(1-re^{-i\phi})}{(1-re^{i\phi})(1-re^{-i\phi})}\right\}.$$

88 Hence $g'(\phi) = 0$ if $\operatorname{Sin}(\phi+\theta) = r\operatorname{Sin}\theta$, i.e. if

$$\phi = -\theta + \operatorname{Sin}^{-1}(r\operatorname{Sin}\theta). \tag{3.4C.9}$$

At this point $g(\phi)$ attains the maximum as we shall show in the end. Now

$$\begin{aligned} g(\phi) &= \operatorname{Re}\left\{-e^{i\theta}\left(\log\sqrt{1-2r\operatorname{Cos}\phi+r^2} - i\operatorname{Sin}^{-1}\frac{r\operatorname{Sin}\phi}{\sqrt{1-2r\operatorname{Cos}\phi+r^2}}\right)\right\} \\ &= -\operatorname{Cos}\theta\log\sqrt{1-2r\operatorname{Cos}\phi+r^2} - \operatorname{Sin}\theta\operatorname{Sin}^{-1}\left(\frac{r\operatorname{Sin}\phi}{\sqrt{1-2r\operatorname{Cos}\phi+r^2}}\right). \end{aligned} \tag{3.4C.10}$$

From (3.4C.9) we have

$$\begin{aligned} \operatorname{Sin}\phi &= r\operatorname{Sin}\theta\operatorname{Cos}\theta - \sqrt{1-r^2\operatorname{Sin}^2\theta}\operatorname{Sin}\theta \\ &= -\operatorname{Sin}\theta(\sqrt{1-r^2\operatorname{Sin}^2\theta} - r\operatorname{Cos}\theta), \\ \operatorname{Cos}\phi &= \sqrt{1-r^2\operatorname{Sin}^2\theta}\operatorname{Cos}\theta + r\operatorname{Sin}^2\theta, \\ 1-2r\operatorname{Cos}\phi+r^2 &= 1-2r\operatorname{Cos}\theta\sqrt{1-r^2\operatorname{Sin}^2\theta} - 2r^2\operatorname{Sin}^2\theta + r^2 \\ &= (\sqrt{1-r^2\operatorname{Sin}^2\theta} - r\operatorname{Cos}\theta)^2, \end{aligned}$$

since $-r^2\operatorname{Sin}^2\theta + r^2\operatorname{Cos}^2\theta = -2r^2\operatorname{Sin}^2\theta + r^2$. Hence

$$g(\phi) \le h(\theta) \tag{3.4C.11}$$

where $h(\theta)$ is the RHS of (3.4C.7), provided $g(\phi)$ attains its maximum for the value ϕ gives by (3.4C.9). We now show that

(a) If $\operatorname{Cos}\theta \ge 0$ then $g(\pi) < h(\theta)$

and

(b) If $\operatorname{Cos}\theta < 0$ then $g(0) < h(\theta)$.

Note that $\operatorname{Sin}\theta \operatorname{Sin}^{-1}(r \operatorname{Sin}\theta) \geq 0$. Hence it suffices to prove (in case (a))

$$\begin{aligned} g(\pi) &= \operatorname{Re}\log\{(1 - re^{i\phi})^{-z}\}_{\phi=\pi} \\ &= -\operatorname{Cos}\theta \log(1 + r) < -\operatorname{Cos}\theta \log(\sqrt{1 - r^2 \operatorname{Sin}^2\theta} - r \operatorname{Cos}\theta) \end{aligned}$$

89 i.e. $\log(1 + r) > \log(\sqrt{1 - r^2 \operatorname{Sin}^2\theta} - r \operatorname{Cos}\theta)$
i.e. $(1 + r + r \operatorname{Cos}\theta)^2 > 1 - r^2 \operatorname{Sin}^2\theta$
i.e. $(1 + r)^2 + 2r(1 + r) \operatorname{Cos}\theta > 1 - r^2$
i.e. $1 + r + 2r \operatorname{Cos}\theta > 1 - r$ (true since $\operatorname{Cos}\theta \geq 0$)
In case (b) it suffices to prove

$$\begin{aligned} g(0) &= \operatorname{Re}\log\{(1 - re^{i\phi})^{-z}\}_{\phi=0} \\ &= -\operatorname{Cos}\theta \log(1 - r) < -\operatorname{Cos}\theta \log(\sqrt{1 - r^2 \operatorname{Sin}^2\theta} - r \operatorname{Cos}\theta) \end{aligned}$$

i.e. $\log(1 - r) < \log(\sqrt{1 - r^2 \operatorname{Sin}^2\theta} - r \operatorname{Cos}\theta)$
i.e. $1 - r < \sqrt{1 - r^2 \operatorname{Sin}^2\theta} - r \operatorname{Cos}\theta$
i.e. $(1 - r + r \cos\theta)^2 < 1 - r^2 \operatorname{Sin}^2\theta$
i.e. $(1 - r)^2 + 2r(1 - r) \operatorname{Cos}\theta < 1 - r^2$
i.e. $1 - r + 2r \operatorname{Cos}\theta < 1 + r$ (which is true).

Thus Lemma 4C.5 is completely proved and hence Theorem 3.4C.1 is completely proved.

PART D

Collecting together the main results of parts B and C we conclude § 3.4 by stating the following theorem.

Theorem 3.4D.1. *The function $f(H)$ defined by*

$$f(H) = \min_{T \geq 1} \max_{T \leq t \leq T+H} |(\zeta(1 + it))^z| \tag{3.4D.1}$$

where $z = e^{i\theta}$ (θ being a constant satisfying $0 \leq \theta < 2\pi$) satisfies the asymptotic estimate

$$|f(H)e^{-\gamma}(\lambda(\theta))^{-1} - \log\log H| \leq \log\log\log H + O(1). \tag{3.4D.2}$$

90 *We recall that* $\lambda(\theta) = \prod_p \lambda_p(\theta)$, *where*

$$\lambda_p(\theta) = \left(1 - \frac{1}{p}\right)\left(\sqrt{1 - \frac{(\mathrm{Sin}\,\theta)^2}{p^2}} - \frac{\mathrm{Cos}\,\theta}{p}\right)^{-\mathrm{Cos}\,\theta} \mathrm{Exp}\left(\mathrm{Sin}\,\theta\,\mathrm{Sin}^{-1}\left(\frac{\mathrm{Sin}\,\theta}{p}\right)\right) \tag{3.4D.3}$$

$\mathrm{Sin}^{-1}\, x$ *being as usual the expansion valid in* $|x| < 1$, *vanishing at* $x = 0$.

Remark. It is an open problem to improve the RHS of (3.4D.2).

Notes at the end of Chapter III

In the year 1928 E.C. Titchmarsh [103] proved (the earlier discoveries in this direction depended on RH, for references see E.C. Titchmarsh [100]) that

$$|\zeta(\sigma + it)| = \Omega(\mathrm{Exp}((\log t)^{1-\sigma-\epsilon})), (\epsilon, \sigma \text{ fixed } \epsilon > 0, \frac{1}{2} \leq \sigma < 1).$$

Extending this method of Titchmarsh, K. Ramachandra [69] proved the lower bound $\gg \mathrm{Exp}((\log H)^{1-\sigma-\epsilon})$ for the maximum of $|\zeta(\sigma + it)|$ taken over $T \leq t \leq T + H$ with $T \geq H \geq (\log T)^{\frac{1}{100}}$. (It was not difficult to relax the lower bound for H to $\gg \log\log T$ with a suitable implied constant). Around the same time (see § 5 of [69] for an explanation of this remark, and further results over short intervals) N. Levinson [53] independently proved that

$$\max_{1 \leq t \leq T} \log|\zeta(\sigma + it)| \gg (\log T)^{1-\sigma}(\log\log T)^{-1}, (\sigma \text{ fixed}, \frac{1}{2} \leq \sigma < 1)$$

and that

$$\max_{1 \leq t \leq T} |\zeta(1 + it)| \geq e^{\gamma} \log\log T + O(1)$$

and also

$$\max_{1 \leq t \leq T} |\zeta(1 + it)|^{-1} \geq \frac{6}{\pi^2} e^{\gamma}(\log\log T - \log\log\log T) + O(1).$$

A few years later H.L. Montgomery [57] developed a new method of 91 proving things like (note that we write $z = e^{i\theta}$)

$$|\left(\zeta\left(\frac{1}{2}+it\right)\right)^z| = \Omega\left(\mathrm{Exp}\left(\frac{1}{20}\sqrt{\frac{\log t}{\log\log t}}\right)\right) \quad \text{(on RH)}$$

and

$$|(\zeta(\sigma+it))^z| = \Omega\left(\mathrm{Exp}\left(\frac{1}{20}\left(\sigma-\frac{1}{2}\right)^{\frac{1}{2}}\frac{(\log t)^{1-\sigma}}{(\log\log t)^{\sigma}}\right)\right) \quad (\sigma \text{ fixed}, \frac{1}{2} < \sigma < 1).$$

It should be mentioned that Montgomery's method needs RH even for $\theta = 0$. Developing the method of K. Ramachandra ([69]), R. Balasubramanian [2] and R. Balasubramanian and K. Ramachandra ([11]) proved Theorem 3.2.1 and in particular the result

$$|\zeta\left(\frac{1}{2}+it\right)| = \Omega\left(\mathrm{Exp}\left(\frac{3}{4}\sqrt{\frac{\log t}{\log\log t}}\right)\right)$$

without any hypothesis. For the result which asserts the replacement of $(\sigma-\frac{1}{2})^{\frac{1}{2}}$ by $c(\sigma-\frac{1}{2})^{\frac{1}{2}}(1-\sigma)^{-1}$, $(\frac{1}{2} < \sigma < 1)$, and some other results see the two papers [90] [91] by K. Ramachandra and A. Sankaranarayanan. We have not included the proof of these results in this monograph. A part from a paper [26] by R. Balasubramanian, K. Ramachandra and A. Sankaranarayanan all the results of this chapter are completely due to various results of R. Balasubramanian and K. Ramachandra developed in various stages most of the time jointly and very rarely individually. Hence it is very well justified to refer to all the theorems of this chapter as joint work of R. Balasubramanian and K. Ramachandra (see the following 15 papers by us: (a) Papers I to IX with the title "On the frequency of Titchmarsh's phenomenon for $\zeta(s)$", (b) Papers I to III with the title "Progress towards a conjecture on the mean-value of Titchmarsh series", (c) one paper with the title "Proof of some conjectures on the mean-value of Titchmarsh series - I", (d) one paper with the title "Proof
92 of some conjectures on the mean-value of Titchmarsh series with applications to Titchmarsh's phenomenon" and (e) one paper with the title "On the zeros of a class of generalised Dirichlet series-III. The paper V of the series (a) uses some ideas of the paper (e)).

Regarding limitation theorems for our method R. Balasubramanian showed in [2]) that it is not possible to get even 0.76 in pace of $\frac{3}{4}$ in Theorem 3.2.1. Also, R. Balasubramanian and K. Ramachandra showed in ([11]) that for $\frac{1}{2} < \sigma < 1$, we cannot get better results than

$$\max_{T \le t \le T+H} \log |\zeta(\sigma + it)| \gg (\log H)^{1-\sigma} (\log\log H)^{-1}$$

where $H(\le T)$ and exceeds a certain constant multiple of $\log\log T$. In (H.L. Montgomery [58]) H.L. Montgomery showed that (by Balasubramanian Ramachandra method) it is not possible to get better results than even

$$\max_{T \le t \le 2T} |\log \zeta(\sigma + it)| \gg (\log T)^{1-\sigma} (\log\log T)^{-1} \quad \left(\sigma \text{ fixed}, \frac{1}{2} < \sigma < 1\right).$$

This shows the supremacy of some aspects of Montgomery's method although it fails for short intervals. It will be of some interest to examine hte limitation of our method for $\sigma = 1$. In view of Levinson's results

$$\max_{1 \le t \le T} |\zeta(1 + it)| > e^{\gamma} \log\log T + O(1)$$

one may conjecture that we may drop the term $\log\log\log H$ in (3.4D.2). But this may be very very difficult to achieve.

Chapter 4

Mean-Value Theorems for the Fractional Powers of $|\zeta(\frac{1}{2}+it)|$

4.1 Introduction

In § 4.2 of this chapter we consider lower bounds for **93**

$$\max_{\sigma\geq\alpha}\left(\frac{1}{H}\int_T^{T+H}|\frac{d^m}{ds^m}(\zeta(s))^{2k}|dt\right)$$

where $\frac{1}{2}\leq\alpha\leq 2$, $m\geq 0$ is an integer constant, k is any complex constant and $T\geq H\gg\log\log T$. Let now $\alpha=\frac{1}{2}$. If Riemann hypothesis (RH) is true then we establish (we mean than our method gives) the lower bound

$$\gg(\log H)^{|k^2|+m}.$$

If we do not assume RH then we can deal only with $k=\frac{p}{q}$ (where $q\geq 1$ and $p>0$ are integers) and $0\leq m\leq k$. In that case we obtain the lower bound

$$\gg(q^{-1}\log H)^{k^2+m}$$

provided $\alpha=\frac{1}{2}+q(\log H)^{-1}$. We can also allow $k(>0)$ to be real with $|k-\frac{p}{q}|\leq(\log\log H)^{-1}$ and $1\leq q\leq 10\log\log H$. Here uniformly we have the lower bound

$$\gg (q^{-1}\log H)^{k^2+m} \geq \left(\frac{\log H}{10\log\log H}\right)^{k^2+m}.$$

So far (in this book) we have not used the functional equation for $\zeta(s)$. In the upper bounds problem for the same integral very little is known. The best known result is that (see § 4.3 of this chapter)

$$\max_{\sigma\geq\alpha}\left(\frac{1}{H}\int_T^{T+H}|\frac{d^m}{ds^m}(\zeta(s))^{2k}|dt\right) \ll (\log T)^{k^2+m}$$

provided $\alpha = \frac{1}{2}$, $k = \frac{1}{2}$ and $H = T^{\lambda_0}$ with any constant λ_0 satisfying $\frac{1}{2} < \lambda_0 \leq 1$. Of course it is enough to prove the upper bound for $m = 0$. The result for general m is deducible from the case $m = 0$ from easy
94 principles (unlike the lower bound). However the result with $k = \frac{1}{n}$, $m = 0$ is true for all $n \geq 1$ (the case $n = 1$ is trivial). In the case $n \geq 3$ we can not talk of upper bounds unless $m = 0$. (We can however manage for all m if $k = \frac{1}{2}$). Except the case $k = 1$ all other cases depend on the functional equation. It will be a great achievement to prove (even assuming RH) results like

$$\frac{1}{T}\int_T^{2T}|\zeta(\frac{1}{2}+it)|^{2k}dt \ll (\log T)^A$$

for some $A > 0$ and some $k > 2$. The biggest integer k for which this is known is $k = 2$ and in this case we have an asymptotic formula for the mean-value. (For this we do not need RH). Trivially given this for any $k > 0$, its truth for all smaller positive k follow by Holder's inequality. Of course RH implies that the upper bound is $\ll \operatorname{Exp}\left(\frac{10k\log T}{\log\log T}\right)$ and it would be of great interest to know whether for any constant $k > 2$ (of course bigger the k the better) the inequality

$$\frac{1}{T}\int_T^{2T}|\zeta(\frac{1}{2}+it)|^{2k}dt \ll_\epsilon T^\epsilon$$

holds for every $\epsilon > 0$ (without assuming RH). This knowledge improves the range for h in the asymptotic formula

$$\pi(x+h)-\pi(x) \sim \frac{h}{\log x}.$$

The truth of

$$\frac{1}{T}\int_T^{2T}|\zeta(\frac{1}{2}+it)|^{2k}dt \ll_{\epsilon,k} T^{\epsilon}$$

for every integer $k > 0$ and every $\epsilon > 0$ is equivalent to Lindelöf hypothesis as can be easily seen from $\zeta'(\frac{1}{2}+it) = O(t)$.

4.2 Lower Bounds

We divide this section conveniently into three parts. Part A deals with statement of the result and remarks, and statement of some preliminary results. Part B deals with a reduction of the problem. Part C deals with completion of the proof.

PART A

Theorem 4.2.1. *Let α and k be real numbers subject to $\frac{1}{2}+q(\log H)^{-1} \leq$* **95**
$\alpha \leq 2$ (where q is a positive integer to be defined presently) and $\delta \leq k \leq \delta^{-1}$ where δ is any positive constant. Let m be any non-negative integer subject to $0 \leq m \leq 2k$ (no restriction on m if $2k$ is an integer). Then

$$\max_{\sigma\geq\alpha}\left(\frac{1}{H}\int_T^{T+H}|\frac{d^m}{ds^m}(\zeta(s))^{2k}|dt\right) > C(\delta,m)\left(\alpha-\frac{1}{2}\right)^{-k^2-m},$$

where $s = \sigma + it$, T and H are subject to $T \geq H \geq H_0 = H_0(\delta, m)$, $C(\delta, m)$ and $H_0(\delta, m)$ being positive constants depending only on δ and m. The integer q is defined as follows. It is any integer subject to $1 \leq q \leq 10\log\log H$, $|k-\frac{p}{q}| \leq (\log\log H)^{-1}$ if $2k-m \geq \delta$, i.e. if $m \leq [2k]-1$, provided $2k \leq [2k]+\delta$ (p being a positive integer). If $2k \geq [2k]+\delta$ then $\frac{p}{q}-k \geq 0$ is another extra condition in addition to $m \leq 2k$ (in place of $m \leq 2k-\delta$). Here after we write $\rho = \frac{p}{q}$.

Corollary. *We have, for a$C'(\delta, m) > 0$ depending only on δ and m,*

$$\frac{1}{H}\int_T^{T+H}|\frac{d^m}{ds^m}(\zeta(s))^{2k}|_{\sigma=\frac{1}{2}}dt > C'(\delta,m)(q^{-1}\log H)^{k^2+m},$$

provided only that $T \geq H \gg \log\log T$ with a certain positive constant implied by the Vinogradov symbol $\gg$.

Remark 1. Our proof of Theorem 4.2.1 depends only on the Euler product and the analytic continuation of $\zeta(s)$ in $(\sigma \geq \alpha, T \leq t \leq T+H)$. Thus it goes through for more general Dirichlet series where Euler product and analytic continuation in the region (just mentioned) are available. In particular it goes through for zeta and L-functions of algebraic number fields.

Remark 2. If we assume Riemann hypothesis ($\zeta(s) \neq 0$ in $\sigma \geq \alpha$, $T \leq t \leq T+H$ will do) then we can prove much more namely this: Let α be real subject to $\frac{1}{2}+(\log H)^{-1} \leq \alpha \leq 2$ and k be any complex number
96 suject to $\delta \leq |k| \leq \delta^{-1}$. Then for all integers $m \geq 0$, we have

$$\max_{\sigma \geq \alpha}\left(\frac{1}{H}\int_T^{T+H}|\frac{d^m}{ds^m}(\zeta(s))^{2k}|dt\right) > C(\delta, m)\left(\alpha - \frac{1}{2}\right)^{-|k^2|-m},$$

where $C(\delta, m) > 0$ depends only on δ and m. As a corollary we can obtain

$$\left(\frac{1}{H}\int_T^{T+H}|\frac{d^m}{ds^m}(\zeta(s))^{2k}|_{\sigma=\frac{1}{2}}dt\right) > C'(\delta, m)(\log H)^{|k^2|+m},$$

where $C'(\delta, m) > 0$ depends only on δ and m and $T \geq H \gg \log\log T$, with a suitable constant implied by $\gg$. (Remark 1 is also applicable).

We prove Theorem 4.2.1 with $\alpha = \frac{1}{2}+q(\log H)^{-1}$ and leave the general α as an exercise. Also we leave the deduction of the corollary to Theorem 4.2.1 as an exercise (we have to use the fact that the integrand in Theorem is bounded above on $\sigma = 2$ and use the convexity result stated in Theorem 4.2.3 below, with the kernel related to $\mathrm{Exp}((\sin s)^2)$).

§ 2. Some Preliminaries.

Before commencing the proof we recall four theorems with suitable notation and remarks. The first is the convexity theorem of R.M. Gabriel. In this section we use q for a positive real number which may or may not be the integer introduced already.

Theorem 4.2.2. *Let $z = x+iy$ be a complex variable. Let D_0 be a closed rectangle with sides parallel to the axes and let L be the closed* *line segment parallel to the y-axis which divides D_0 into two equal parts. Let D_1 and D_2 be the two congruent rectangles into which D_0 is divided by L. Let K_1 and K_2 be the boundaries of D_1 and D_2 (with the line L excluded). Let $F(z)$ be analytic in the interior of D_0 and $|f(z)|$ be continuous on the boundary of D_0. Then, we have,*

$$\int_L |f(z)|^q |dz| \leq \left(\int_{K_1} |f(z)|^q |dz|\right)^{\frac{1}{2}} \left(\int_{K_2} |f(z)|^q |dz|\right)^{\frac{1}{2}},$$

where $q > 0$ is any real number.

Remark. The assertion of the theorem still holds if $|f(z)|^q$ is replaced 97
by $|\varphi(z)||f(z)|^q$, where $\varphi(z)$ is any function analytic inside D_0 and such that $|\varphi(z)|$ is continuous on the boundary of D_0. To see this replace $f(z)$ by $(f(z))^j(\varphi(z))^r$ and q by qj^{-1} where j and r are positive integers and j and r tend to infinity in such a way that $rj^{-1} \to q^{-1}$.

Proof. See Theorem 1.3.2.

We now slightly extend this as follows. Consider the rectangle defined by $0 \leq x \leq (2^n + 1)a$ (where n is a non-negative integer and a any positive real number), and $0 \leq y \leq R$. Let I_x denote the integral $\int_0^R |f(z)|^q dy$, where as before $z = x + iy$. Let Q_α denote the maximum of $|f(z)|^q$ on $(0 \leq x \leq \alpha, y = 0$ and $y = R)$. Then we have (assuming $f(z)$ to be analytic in the interior of $(0 \leq x \leq (2^n + 1)a,\ 0 \leq y \leq R)$ and $|f(z)|$ continuous on its boundary) the following theorem. □

Theorem 4.2.3. *Put $b_\nu = 2^\nu + 1$. Then for $\nu = 0, 1, 2, \ldots, n$ we have,*

$$I_\alpha \leq (I_0 + U)^{\frac{1}{2}} (I_\alpha + U)^{\frac{1}{2} - 2^{-\nu-1}} (I_{ab_\nu} + U)^{2^{-\nu-1}}$$

where $U = 2^{2(\nu+1)} a Q_{ab_\nu}$.

Remark. The remark below Theorem 4.2.2 is applicable here also.

Proof. See Theorem 1.3.3. □

Theorem 4.2.4. *Let $f(z)$ be analytic in $|z| \leq R$, q be any positive real constant (not necessarily the same q and R as before). Then, we have,*

$$|f(0)|^q \leq \frac{1}{\pi R^2} \int_{|z| \leq R} |f(z)|^q dxdy.$$

Remark. The remark below Theorem 4.2.2 is applicable here also.

Proof. This result follows from Cauchy's theorem with proper zero cancellation factors.

The next theorem is a well-known theorem due to H.L. Montgomery and R.C. Vaughan, (see Theorem 1.4.3). □

98 **Theorem 4.2.5.** *Let* $\{a_n\}(n = 1, 2, 3, \ldots)$ *be any sequence of complex numbers which may or may not depend on* $H(\geq 2)$. *Then subject to the convergence of* $\sum\limits_{n=1}^{\infty} n|a_n|^2$, *we have,*

$$\frac{1}{H}\int_0^H |\sum_{n=1}^{\infty} a_n n^{it}|^2 = \sum_{n=1}^{\infty} |a_n|^2 \left(1 + O\left(\frac{n}{H}\right)\right),$$

where the O-constant is absolute.

Remark. This theorem is not very easy to prove but very convenient to use in several important situations. But it should be mentioned that for the proof of Theorem 4.2.1 it suffices to use a rough result for the mean-value of $|\sum\limits_{n\leq N} a_n n^{it}|^2$ where N is a small positive constant power of H. The result which we require in this connection is very easy to prove.

PART B

For any Dirichlet series $F(s) = \sum\limits_{n=1}^{\infty} a_n n^{-s}$ and $Y \geq 1$ we write $F(s, Y) = \sum\limits_{n\leq Y} a_n n^{-s}$. Also we write

$$Z_1 = \left(\frac{d^m}{ds^m}(\zeta^k(s, Y))^2\right)(\zeta^{\rho-k}(s, Y))^2.$$

Note that $\zeta^k(s, Y) \neq (\zeta(s, Y))^k$. We will show in a few lemmas that the proof of Theorem 1 reduces to proving that

$$\frac{1}{H}\int_T^{T+H} |Z_1| dt \gg D^{-\mu''}(q^{-1}\log H)^{k^2+m} \tag{4.2.1}$$

where μ'' is a certain positive constant, $\sigma_0 = \frac{1}{2} + 10q(\log H)^{-1}$, $a = Dq(\log H)^{-1}$ $s = \sigma_0 + a + it$ and D is a large positive constant. The rest of the proof consists in proving (4.2.1).

Lemma 1. *Let* 99

$$\max_{\sigma \geq \frac{1}{2}+q(\log H)^{-1}} \left(\frac{1}{H} \int_T^{T+H} |\frac{d^m}{ds^m}(\zeta(s))^{2k}| dt \right) < (\log H)^{k^2+m}. \tag{4.2.2}$$

Then for $\nu = 0, 1, 2, \ldots, m$*, we have,*

$$\max_{\sigma \geq \frac{1}{2}+q(\log H)^{-1}} \left(\frac{1}{H} \int_T^{T+H} |\frac{d^\nu}{ds^\nu}(\zeta(s))^{2k}| dt \right) < (\log H)^{k^2+m+1}. \tag{4.2.3}$$

Remark. Note that we are entitled to assume that the LHS of (4.2.2) does not exceed $(q^{-1} \log H)^{k^2+m}$ since otherwise Theorem 4.2.1 is proved.

Proof. We have, for $\frac{1}{2} + q(\log H)^{-1} \leq \sigma \leq 2$,

$$\int_2^\sigma \frac{d^m}{ds^m}(\zeta(s))^{2k} d\sigma = \frac{d^{m-1}}{ds^{m-1}}(\zeta(s))^{2k} + O(1).$$

So

$$|\frac{d^{m-1}}{ds^{m-1}}(\zeta(s))^{2k}| \leq \int_2^\sigma |\frac{d^m}{ds^m}(\zeta(s))^{2k}| d\sigma + O(1).$$

Integrating this with respect to t we obtain the result for $\nu = m-1$. Continuing this process we can establish this lemma for $\nu = 0, 1, 2, \ldots, m$. □

Lemma 2. *Divide* $(T+1, T+H-1)$ *into abutting intervals* I *of length* $(\log H)^A$ *(where* $A > 0$ *is a large constant) ignoring a bit at one end. Let* $m(I)$ *be the maximum of* $|\zeta(s)|^{2k}$ *in* $(\sigma \geq \frac{1}{2} + (q+1)(\log H)^{-1}, t \in I)$*. Then we have,*

$$\sum_I m(I) \leq H(\log H)^{k^2+m+4}. \tag{4.2.4}$$

Proof. We observe that the value of $|\zeta(s)|^{2k}$ at any point (where $m(I)$ is attained) is majorised by its mean-value over a disc of radius $(\log H)^{-1}$ with that point as centre. This follows by the application of Theorem 4.2.4. Now by Lemma 1 the proof is complete. □

Lemma 3. *In* $(\sigma \geq \frac{1}{2} + (q+2)(\log H)^{-1}, T+1 \leq t \leq T+H-1)$, *we have,*

$$|\frac{d^m}{ds^m}(\zeta(s))^{2k}| \leq H^{1.5} \text{ and so } |\zeta(s)|^{2k} \leq H^2.$$

Proof. By Lemma 1 the proof follows by arguments similar to the one 100
by which we obtained Lemma 2. □

Lemma 4. *Let* $B > 0$ *be any (large) constant. Then the number of intervals* I *for which* $m(I) \geq (\log H)^B$ *is*

$$\leq H(\log H)^{k^2+m+4-B}.$$

Proof. By Lemma 2 the proof follows. □

Lemma 5. *Let accent denote the sum over those* I *for which* $m(I) < (\log H)^B$. *Also let* $\sigma \geq \frac{1}{2} + (q+3)(\log H)^{-1}$. *Then for* $\delta \leq k \leq \delta^{-1}$, *we have,*

$$\begin{aligned} &\frac{1}{H}\int_T^{T+H} |\frac{d^m}{ds^m}(\zeta(s))^{2k}|dt \\ \gg &\frac{1}{H}\sum_1{}' \left(\int_I |\frac{d^m}{ds^m}(\zeta(s))^{2k} \mid \zeta(s)|^{2\rho-2k} dt - (\log H)^A\right), \end{aligned} \tag{4.2.5}$$

where $\rho = \frac{p}{q}$ *is a rational approximation to* k *such that either (i)* $2(\log\log H)^{-1} \geq 2\rho - 2k \geq 0$ *and* $0 \leq m \leq 2k$ *(no restriction on* m *if* $2k$ *is an integer) or (ii)* $|2\rho - 2k| \leq 2(\log\log H)^{-1}$ *and* $2k - m \geq \eta > 0$ *for some constant* $\eta > 0$. *In both the cases (i) and (ii) it is assumed that* $1 \leq q \leq 10\log\log H$. *The implied constant in the inequality asserted by the lemma is independent of* H, q *and* k.

Remark. There is always a solution of $|\rho - k| \leq (\log\log H)^{-1}$, $1 \leq q \leq 10\log\log H$. This can be easily seen by box principle.

Proof. We introduce the factor $|\zeta(s)|^{2(\rho-k)} \times |\zeta(s)|^{2(k-\rho)}$. Now in case (i)

$$|\zeta(s)|^{2(k-\rho)2k(2k)^{-1}} \geq ((\log H)^{-B})^{(\rho-k)k^{-1}} \gg 1.$$

In case (ii) we have only to consider $\rho - k < 0$ and $2k - m \geq \eta > 0$. We divide I into two parts (iii) that for which $\max|\zeta(s)| \leq (\log H)^{-B'}$ and (iv) the rest, ($B' > 0$ being a suitable large constant).

In (iv) we have

$$|\zeta(s)|^{2(k-\rho)} \geq ((\log H)^{-B'})^{2(k-\rho)} \geq ((\log H)^{-B'})^{2(\log\log H)^{-1}} \gg 1.$$

In case (iii) we plainly omit it and consider **101**

$$\int_{I^*} \left|\frac{d^m}{ds^m}(\zeta(s))^{2k}\right| |\zeta(s)|^{2(\rho-k)} dt,$$

(where $I^* = I \cap (\max|\zeta(s)| > (\log H)^{-B'})$)

$$= \int_I \left|\frac{d^m}{ds^m}(\zeta(s))^{2k}\right| |\zeta(s)|^{2(\rho-k)} dt - \int_{I^{**}} \left|\frac{d^m}{ds^m}(\zeta(s))^{2k}\right| |\zeta(s)|^{2(\rho-k)} dt$$

(where $I^{**} = I \cap (\max|\zeta(s)| \leq (\log H)^{-B'})$). For large B' it is easily seen, since $2k - m \geq \eta > 0$, that the integral over I^{**} is $\leq (\log H)^A$. (Note that in $\sigma \geq \frac{1}{2} + (q+3)(\log H)^{-1}$ the derivatives of order $\leq m$ of $\zeta(s)$ are in absolute value not more than a bounded power of $\log H$, the bound depending only on δ and m). □

Lemma 6. *For any two complex numbers A_0, B_0 and any real number $q > 0$, we have,*

$$|A_0|^{\frac{1}{q}} \leq 2^{\frac{1}{q}} \left(|A_0 - B_0|^{\frac{1}{q}} + |B_0|^{\frac{1}{q}}\right). \tag{4.2.6}$$

Proof. We have $A_0 = A_0 - B_0 + B_0$ and so

$$|A_0| \leq 2\max(|A_0 - B_0|, |B_0|).$$

This gives the lemma. □

Lemma 7. *Let J denote the interval I with intervals of length* $(\log H)^2$ *being removed from both ends. Let Z_1 be as already introduced in the beginning of this section namely,*

$$Z_1 = \left(\frac{d^m}{ds^m}(\zeta^k(s, Y))^2\right)(\zeta^{\rho-k}(s, Y))^2, \tag{4.2.7}$$

and let

$$A_0 = Z_1^q \text{ and } B_0 = \left(\left(\frac{d^m}{ds^m}(\zeta(s))^{2k}\right)(\zeta^{2\rho-2k}(s))\right)^q. \tag{4.2.8}$$

102 *Then for any* $\sigma \geq \frac{1}{2}$, *we have, by Lemma 6.*

$$\begin{aligned} &2^{\frac{1}{q}} \int_J |\left(\frac{d^m}{ds^m}(\zeta(s))^{2k}\right)\zeta^{2\rho-2k}(s)|dt \\ &\geq \int_J |Z_1|dt - 2^{\frac{1}{q}} \int_J |B_0 - A_0|^{\frac{1}{q}} dt. \end{aligned} \tag{4.2.9}$$

where $q > 0$ *is any real number.*

Proof. Follows from

$$2^{\frac{1}{q}}|B_0|^{\frac{1}{q}} \geq |A_0|^{\frac{1}{q}} - 2^{\frac{1}{q}}|B_0 - A_0|^{\frac{1}{q}}.$$

□

Lemma 8. *We have, for* $\sigma \geq \frac{1}{2}$,

$$\sum_I{}' \int_J |Z_1|dt = \int_T^{T+H} |Z_1|dt + O(H(\log H)^{-1}), \tag{4.2.10}$$

provided Y *is a small positive constant power of* H *and also that the constant* $A > 0$ *is large enough.*

Proof. Follows by Holder's inequality applied to the integral over the complementary interval and Montgomery-Vaughan theorem.

From now on q will be the denominator of ρ. We now apply Theorem 4.2.3 with $\frac{1}{q}$ in place of q and state a lemma. □

Lemma 9. *Write*

$$f_0(s) = \left(\left(\frac{d^m}{ds^m}(\zeta(s))^{2k}\right)\zeta^{2\rho-2k}(s)\right)^q - Z_1^q \tag{4.2.11}$$

and $w = u + iv$ for a complex variable and put $K(w) = \mathrm{Exp}(w^2)$. Also write $\tau = (\log H)^2$ and $f(s,w) = f_0(s+w)K(w)$. Then, we have for $t \in J$ and $a > 0$,

$$\begin{aligned}\int_{|v|\le r} |f(s,w)|^{\frac{1}{q}}_{u=0}|dw| \le \left(\int_{|v|\le r} |f(s,w)|^{\frac{1}{q}}_{u=-a}|dw| + H^{-10}\right)^{\frac{1}{2}} \times \\ \times \left(\int_{|v|\le r} |f(s,w)|^{\frac{1}{q}}_{u=0}|dw| + H^{-10}\right)^{\frac{1}{2}-2^{-n-1}} \\ \times \left(\int_{|v|\le r} |f(s,w)|^{\frac{1}{q}}_{u=ab_n-\alpha}|dw| + H^{-10}\right)^{2^{-n-1}}, \qquad (4.2.12)\end{aligned}$$

provided $\sigma_0 > \frac{1}{2}$ and $a(2^n + 1)$ is bounded above. Here s, σ_0 and a are **103**
as in (4.2.1).

Proof. Follows by Theorem 4.2.3. □

Lemma 10. *If $a > 0$ and ab_n is bounded above, we have,*

$$\begin{aligned}\int_t\int_v |f(s,w)|^{\frac{1}{q}}_{u=0}|dw|dt \le \left(\int_t\int_v |f(s,w)|^{\frac{1}{q}}_{u=-a}|dw|dt + H^{-5}\right)^{\frac{1}{2}} \times \\ \times \left(\int_t\int_v |f(s,w)|^{\frac{1}{q}}_{u=0}|dw|dt + H^{-5}\right)^{\frac{1}{2}-2^{-n-1}} \times \\ \times \left(\int_t\int_v |f(s,w)|^{\frac{1}{q}}_{u=ab_n-a}|dw|dt + H^{-5}\right)^{2^{-n-1}}. \qquad (4.2.13)\end{aligned}$$

The limits of integration are determined by $t \in J$ and $|v| \le \tau$ with $\tau = (\log H)^2$.

Proof. Follows by Holder's inequality.

Summing over J (counted by the accent) and applying Holder's inequality we state a lemma. □

Lemma 11. *We have, with $\sigma \ge \frac{1}{2} + 10q(\log H)^{-1}$,*

$$\sideset{}{'}\sum_J \int_t\int_v |f(s,w)|^{\frac{1}{q}}_{u=0}|dw|dt \le \left(\sideset{}{'}\sum_J \int\int \ldots_{u=-a}|dw|dt + H^{-3}\right)^{\frac{1}{2}} \times$$

$$\times\left(\sum_J{}' \int\int \ldots_{u=0} |dw|dt + H^{-3}\right)^{\frac{1}{2}-2^{-n-1}} \times$$

$$\times\left(\sum_J{}' \int\int \ldots_{u=ab_n-a} |dw|dt + H^{-3}\right)^{2^{-n-1}} \tag{4.2.14}$$

104 *provided* $a > 0$ *and* ab_n *is bounded above.*

We now complete the reduction step as follows. In Lemma 11 either LHS $\le H^{-3}$ in which case the quantity

$$2^{\frac{1}{q}} \sum_J{}' \int_J |B_0 - A_0|^{\frac{1}{q}} dt$$

is small enough to assert that the sum over J (accented ones) of the quantity on the LHS of (4.2.9) exceeds

$$\frac{1}{2}\int_T^{T+H} |Z_1| dt$$

where $\sigma = \sigma_0 + a$. On the other hand if in Lemma 11, LHS $\ge H^{-3}$,

$$\left(\sum_J{}' \int_t\int_v |f(s,w)|^{\frac{1}{q}}_{u=0} |dw|dt\right)^{\frac{1}{2}+2^{-n-1}}$$
$$\le 2\left(\int_{T+r}^{T+H-r}\int_v |f(s,w)|^{\frac{1}{q}}_{u=-a}|dw|dt + H^{-3}\right)^{\frac{1}{2}} \times$$
$$\times\left(\int_{T+r}^{T+H-r}\int_v |f(s,w)|^{\frac{1}{q}}_{u=ab_n-a}|dw|dt + H\right)^{2^{-n-1}}$$

Now by using the fact that $|K(w)| \ll \mathrm{Exp}(-v)^2$ for all v, and also that $|K(w)| \gg 1$ for $|v| \le 1$, we obtain

$$\left(\sum_I{}' \int_{t\in J} |f_0(s)|^{\frac{1}{q}}_{\sigma=\sigma_0+a} dt\right)^{\frac{1}{2}+2^{-n-1}} \ll \left(\sum_I{}' \int_{t\in I} |f_0(s)|^{\frac{1}{q}}_{\sigma=\sigma_0} dt + H^{-3}\right)^{\frac{1}{2}} \times$$

$$\times\left(\sum_{I}{}' \int_{t\in I} |f_0(s)|^{\frac{1}{q}}_{\sigma=\sigma_0+ab_n} dt + H^{-3}\right)^{2^{-n-1}}. \qquad (4.2.15)$$

105 (From now on we stress that the constant implied by the Vinogradov symbols $\ll$ and $\gg$ depend only on δ and m). From now on we assume that

$$\max_{\sigma\geq\frac{1}{2}+q(\log H)^{-1}}\left(\frac{1}{H}\int_{T}^{T+H} |\frac{d^m}{ds^m}(\zeta(s))^{2k}|dt\right) \leq (q^{-1}\log H)^{k^2+m}. \quad (4.2.16)$$

From this it follows (since Y is a small positive constant power of H) as we shall see (by Lemma 5 and remark following it) that (see Part C for explanations)

$$\frac{1}{H}\sum_{1}{}' \int_{t\in I} |f_0(s)|^{\frac{1}{q}}_{\sigma=\sigma_0} dt \ll (q^{-1}\log H)^{k^2+m}$$

Also we shall see that (by choosing n such that $\sigma_0 + ab_n$ lies between 2 large positive constants)

$$\frac{1}{H}\sum_{I}{}' \int_{t\in I} |f_0(s)|^{\frac{1}{q}}_{\sigma=\sigma_0+ab_n} dt \ll H^{-\mu' q^{-1}}$$

where μ' is a certain positive constant. If $\sigma_0 = \frac{1}{2} + 10q(\log H)^{-1}$ and $a = Dq(\log H)^{-1}$, D being a large positive constant (our estimations will be uniform in D) we have $2^n Dq(\log H)^{-1}$ lies between two positive constants and so $2^n \asymp (Dq)^{-1}\log H$ and so $2^{-n-1} \asymp Dq(\log H)^{-1}$. Hence it would follow that

$$\frac{1}{H}\sum_{I}{}' \int_{t\in J} |f_0(s)|^{\frac{1}{q}}_{\sigma=\sigma_0+a} dt \ll (q^{-1}\log H)^{k^2+m} e^{-\mu D}$$

where μ is a certain positive constant.

The rest of the work consists in proving that

$$\frac{1}{H}\int_{T}^{T+H} |Z_1| dt \gg D^{-\mu''}(q^{-1}\log H)^{k^2+m}$$

where μ'' is a certain positive constant.

PART C

106 We recall that

$$Z_1 = \left(\frac{d^m}{ds^m}(\zeta^k(s,Y))^2\right)(\zeta^{\rho-k}(s,Y))^2.$$

We now write

$$Z_2 = \frac{d^m}{ds^m}(\zeta^\rho(s,Y))^2 \tag{4.2.17}$$

and go on to prove that (for $\frac{1}{2} < \sigma \leq 2$)

$$\frac{1}{H}\int_T^{T+H}|Z_1 - Z_2|dt \ll \left((\log\log H)^{-1} + H^{-\mu q^{-1}(\sigma-\frac{1}{2})}\right)\left(\sigma - \frac{1}{2}\right)^{-k^2-m}, \tag{4.2.18}$$

where μ is a positive constant which we may take to be the same as before. This will be done in three stages to be stated in Lemma 12. We introduce

$$Z_3 = \left(\frac{d^m}{ds^m}\left(\left(\zeta^{\frac{1}{q}}(s,Y_1)\right)^q\right)^{2k}\right)\left(\left(\zeta^{\frac{1}{q}}(s,Y_1)\right)^q\right)^{2\rho-2k} \tag{4.2.19}$$

where Y_1^q is a small positive constant power of H, and

$$Z_4 = \frac{d^m}{ds^m}\left(\left(\zeta^{\frac{1}{q}}(s,Y_1)\right)^q\right)^{2\rho}, \tag{4.2.20}$$

We remark first of all that Z_1, Z_2, Z_3 and Z_4 are Dirichlet polynomials with Y and Y_1^q being small positive constant powers of H and so the contribution of the integrals of $|Z_j|^2$ from any interval of length $(\log H)^2$ contained in $(T, T+H)$ is $O(H(\log H)^{-1})$. With this remark and some standard application of Cauchy's theorem we can prove

Lemma 12. *We have, for* $\frac{1}{2} < \sigma \leq 2$,

$$\frac{1}{H}\int_T^{T+H}|Z_1 - z_3|dt \ll H^{-\mu q^{-1}(\sigma-\frac{1}{2})}\left(\sigma - \frac{1}{2}\right)^{-k^2-m}, \tag{4.2.21}$$

$$\frac{1}{H}\int_T^{T+H}|Z_3 - Z_4|dt \ll (\log\log H)^{-1}\left(\sigma - \frac{1}{2}\right)^{-k^2-m}, \tag{4.2.22}$$

107 *and*

$$\frac{1}{H}\int_{T}^{T+H}|Z_4-Z_2|dt \ll H^{-\mu q^{-1}(\sigma-\frac{1}{2})}\left(\sigma-\frac{1}{2}\right)^{-k^2-m}. \tag{4.2.23}$$

Corollary. *We have, for* $\frac{1}{2}<\sigma\leq 2$,

$$\frac{1}{H}\int_{T}^{T+H}|Z_1-Z_2|dt \ll \left((\log\log H)^{-1}+H^{-\mu q^{-1}(\sigma-\frac{1}{2})}\right)\left(\sigma-\frac{1}{2}\right)^{-k^2-m}. \tag{4.2.24}$$

Proof of Lemma 12. We first consider the first and third assertions. The Dirichlet polynomials in the integrands have the property that sufficiently many terms (i.e. $[H^{\mu' q^{-1}}]$ terms for a certain positive constant μ') in the beginning vanish. Hence it suffices (by the method by which we proved that

$$\frac{1}{H}\sideset{}{'}\sum_{I}\int_{t\in J}|f_0(s)|^{\frac{1}{q}}_{\sigma=\sigma_0+a}dt \ll (q^{-1}\log H)^{k^2+m}e^{-\mu D}$$

holds) to check that for $\sigma=\sigma_0$ the estimates

$$\frac{1}{H}\int_{T}^{T+H}|Z_j|dt \ll \left(\sigma-\frac{1}{2}\right)^{-k^2-m}, (j=1,2,3,4),$$

hold. Now Z_3 and Z_4 have the following property: Z_3 is the same as Z_4 except that "the coefficients" differ by $O(k-\rho)$. Hence the second assertion follows if we prove that the mean-value of the absolute value of "the terms" are $\ll (\sigma-\frac{1}{2})^{-k^2-m}$ (the explanation of the terms in the inverted commas will be given presently). We begin by checking the mean-value estimates for $|Z_j|(j=1$ to $4)$. For any function $f_1(s)$ analytic in $(\sigma>\frac{1}{2}, T\leq t\leq T+H)$ we have, by Cauchy's theorem,

$$\left|\frac{d^m}{ds^m}f_1(s)\right| \leq \frac{1}{2\pi}\int_{|w|=r}\left|\frac{f_1(s+w)dw}{w^{m+1}}\right|,$$

where $r = \frac{1}{2}(\sigma - \frac{1}{2})$ and $T + 2r \leq t \leq T + H - 2r$. To prove the mean-value assertion about $|Z_1|$ we put $f_1(s) = \frac{d^m}{ds^m}(\zeta^k(s, Y))^2$ and observe that the mean-value of

$$|(\zeta^k(s + w, Y))^2(\zeta^{\rho-k}(s, Y))^2|$$

108 with respect to t in $T + (\log H)^2 \leq t \leq T + H - (\log H)^2$ is $O((\sigma - \frac{1}{2})^{-k^2})$ (uniformly with respect to w) and so the mean-value of $|Z_1|$ is $O((\sigma - \frac{1}{2})^{-k^2-m})$. Similar result about $|Z_2|$ follows since $\rho - k = O((\log\log H)^{-1})$.

Now let us look at $f_3(s) = \frac{d^m}{ds^m}(f_2(s))^k$ where $f_2(s) = (\zeta^{\frac{1}{q}}(s, Y_1))^{2q}$. When $m = 1$, $f_3(s) = k(f_2(s))^{k-1} f_2'(s)$. When $m = 2$ it is $k(k-1)(f_2(s))^{k-2}(f_2'(s))^2 + k(f_2(s))^{k-1} f_2''(s)$ and so on. By induction we see that for general m, we have,

$$f_3(s) = \sum_{j_1+j_2+\ldots+j_\nu=m} g_{j_1,\ldots,j_\nu}(k)(f_2(s))^{k-\nu}(f_2^{(j_1)}(s))(f_2^{(j_2)}(s))\ldots(f_2^{(j_\nu)}(s))$$

where the $g's$ depend only on $j_1, \ldots, j_\nu$ and k. To obtain Z_3 we have to multiply $f_3(s)$ by $(f_2(s))^{\rho-k}$. Hence Z_4 is the same as Z_3 with $g's$ replaced by $g_{j_1,\ldots,j_\nu}(\rho)$. Thus

$$g_{j_1,\ldots,j_\nu}(k) - g_{j_1,\ldots,j_\nu}(\rho) = O(k - \rho) = O((\log\log H)^{-1}).$$

The terms like $(f_2(s))^{k-\nu}(f_2^{(j_1)}(s))\ldots(f_2^{(j_\nu)}(s))$ contribute $O((\sigma-\frac{1}{2})^{-k^2-m})$ by using Cauchy's theorem as before. Thus Lemma 12 is completely proved.

Lemma 13. *For* $\frac{1}{2} + C(\log H)^{-1} \leq \sigma \leq 2$ *(C being a large positive constant), we have,*

$$\frac{1}{H}\int_T^{T+H} |Z_2| dt \gg (\sigma - \frac{1}{2})^{-k^2-m}. \tag{4.2.25}$$

Proof. We recall that $Z - 2 = \frac{d^m}{ds^m} f_4(s)$ where $f_4(s) = (\zeta^\rho(s, Y))^2$ and Y is a small positive constant power of H. By Montgomery-Vaughan theorem

$$C_1\left(\sigma - \frac{1}{2}\right)^{-k^2} \leq \frac{1}{H}\int_T^{T+H} |f_4(s)| dt \leq C_2\left(\sigma - \frac{1}{2}\right)^{-k^2}$$

where $C_2 > C_1 > 0$ are constants, provided $\sigma \geq \frac{1}{2} + C(\log H)^{-1}$. If $m = 0$ we are through, (otherwise $2k \geq 1$ and so $k \geq \frac{1}{2}$). Let $\beta' = (2C_2C_1^{-1})^{-4}(\alpha' - \frac{1}{2}) + \frac{1}{2}$ where $\alpha' > \frac{1}{2}$. Then

$$\begin{aligned}
&\frac{1}{H}\int_T^{T+H} |f_4(s)|_{\sigma=\beta'}\, dt - \frac{1}{H}\int_T^{T+H} |f_4(s)|_{\sigma=\alpha'}\, dt \\
&\geq C_1\left(\left(2C_2C_1^{-1}\right)^{-4}\left(\alpha' - \frac{1}{2}\right)\right)^{-k^2} - C_2\left(\alpha' - \frac{1}{2}\right)^{-k^2} \\
&\geq C_1(2C_2C_1^{-1})\left(\alpha' - \frac{1}{2}\right)^{-k^2} - C_2\left(\alpha' - \frac{1}{2}\right)^{-k^2} \\
&= C_2\left(\alpha' - \frac{1}{2}\right)^{-k^2}
\end{aligned}$$

Also 109

$$\begin{aligned}
&\frac{1}{H}\int_T^{T+H} (|f_4(s)|_{\sigma=\beta'} - |f_4(s)|_{\sigma=\alpha'})dt \\
&\leq \frac{1}{H}\int_T^{T+H} |f_4(\beta' + it) - f_4(\alpha' + it)|dt \\
&\leq \frac{1}{H}\int_{\beta'}^{\alpha'}\left(\int_T^{T+H} |f_4'(u + it)|dt\right) du.
\end{aligned}$$

Thus there exists a number γ' with $\beta' < \gamma' < \alpha'$ such that if $m = 1$ and $\sigma = \gamma'$ the lower bounds is

$$C_2\left(\alpha' - \frac{1}{2}\right)^{-k^2}(\alpha' - \beta')^{-1} \gg \left(\gamma' - \frac{1}{2}\right)^{-k^2-1}$$

and by induction there is a number $\sigma = \alpha'_m$ where the lower bound is $\gg \left(\alpha'_m - \frac{1}{2}\right)^{-k^2-m}$ for general m (since the upper bound required at each stage of induction is available by a simple application of Cauchy's theorem). Now from a given α'_m we can pass onto general σ by an application of Theorem 4.2.3. (Note that $|Z_2|$ is bounded both above and below when σ is large enough. We can select two suitable value of σ). Hence Lemma 13 is completely proved. □

Lemma 14. *We have, for* $\frac{1}{2}+Dq(\log H)^{-1} \leq \sigma \leq 2$, *($D>0$ being a large constant),*

$$\frac{1}{H}\int_T^{T+H} |Z_1| dt$$
$$\geq C'_m\left(\sigma-\frac{1}{2}\right)^{-k^2-m} - C''_m\left(\sigma-\frac{1}{2}\right)^{-k^2-m} H^{-\mu q^{-1}(\sigma-\frac{1}{2})}$$
$$- C'''_m\left(\sigma-\frac{1}{2}\right)^{-k^2-m}(\log\log H)^{-1},$$

110 *where* C'_m, C''_m *and* C'''_m *are positive constants independent of* D. *Also*

$$\frac{1}{H}\int_T^{T+H} (|Z_1|_{\sigma=\frac{1}{2}+Dq(\log H)^{-1}}) dt \gg (q^{-1}\log H)^{k^2+m}.$$

Proof. Follows from Lemma 13 and the corollary to Lemma 12. This proves our main theorem (namely Theorem 4.2.1) completely. □

4.3 Upper Bounds

The object of this section is to prove the following theorem.

Theorem 4.3.1. *Let k be a constant of the type $\frac{1}{j}$ where $j(\geq 2)$ is an integer. Let $H = T^{\frac{1}{2}+\epsilon}$ where $\epsilon(0<\epsilon<\frac{1}{2})$ is any constant. Then, we have,*

$$\frac{1}{H}\int_T^{T+H} |\zeta(s)|^{2k}_{\sigma=\frac{1}{2}} dt \ll (\log T)^{k^2}, \tag{4.3.1}$$

$$\frac{1}{H}\int_T^{T+H} |\zeta^{(m)}(s)|_{\sigma=\frac{1}{2}} dt \ll (\log T)^{\frac{1}{4}+m} \tag{4.3.2}$$

and

$$\frac{1}{H}\int_T^{T+H} |\frac{d^m}{ds^m}(\zeta(s))^{2k}|_{\sigma=\frac{1}{2}+(\log T)^{-1}} dt \ll (\log T)^{k^2+m}. \tag{4.3.3}$$

The last inequality however assumes RH. In the last two assertions of the theorem $m(\geq 0)$ is an integer constant.

Remark. It is not hard to prove the modified results of the type

$$\max_{\sigma \geq \frac{1}{2}} \left(\frac{1}{H} \int_T^{T+H} |\zeta(s)|^{2k} dt \right) \ll (\log T)^{k^2}, \tag{4.3.4}$$

and similar results. These can be proved by convexity principles.

The letter r denotes a large positive constant which will be chosen in the end. We need the properties of $\zeta(s)$ only in the region ($\sigma \geq \frac{1}{2} - \delta, T \leq t \leq T + H$) for some arbitarily small constant $\delta > 0$. We begin by introducing the following notation.

$$M(\sigma) = \frac{1}{H} \int_T^{T+H} |\zeta(\sigma + it)|^{2k} dt \tag{4.3.5}$$

$$A(\sigma) = \frac{1}{H} \int_T^{T+H} |\sum_{n \leq H} \frac{d_k(n)}{n^{\sigma + it}}|^2 dt \tag{4.3.6}$$

and 111

$$M_1(\sigma) = \frac{1}{H} \int_T^{T+H} |\zeta(\sigma + it)| - \left(\sum_{n \leq H} \frac{d_k(n)}{n^{\sigma + it}} \right)^j |^{2k} dt. \tag{4.3.7}$$

Remark. Since $\zeta(\frac{1}{2} + it) = 0(t^{\frac{1}{4}} (\log t)^2)$ (this follows for example by the functional equation or otherwise) it follows that the quantities $M(\sigma)$, $A(\sigma)$ and $M_1(\sigma)$ get multiplied by $1 + o(1)$ when we change the limits of integration by an amount $O((\log T)^2)$.

We begin by proving three lemmas.

Lemma 1. *We have*

$$M(\sigma) \leq 2^{2k} (M_1(\sigma) + A(\sigma)).$$

Proof. The lemma follows from

$$\zeta(s) = \zeta(s) - \left(\sum_{n \leq H} \frac{d_k(n)}{n^s} \right)^j + \left(\sum_{n \leq H} \frac{d_k(n)}{n^s} \right)^j.$$

$\square$

Lemma 2. *Let* $\sigma_0 = \frac{1}{2} + r(\log T)^{-1}$ *and* $M = M(1-\sigma_0)$. *Then, we have,*

$$M(\sigma_0) \sim M \operatorname{Exp}(-2kr). \tag{4.3.8}$$

Proof. The lemma follows by the functional equation. (Functional equation and this consequence will be proved in the appendix at the end of this book). □

Lemma 3. *We have at least one of the following two possibilities:*

$$M(1-\sigma_0) \ll A(\sigma_0) + A(1-\sigma_0) \tag{4.3.9}$$

or

$$M_1(\sigma_0) \ll M_1(1-\sigma_0) \operatorname{Exp}(-\lambda) \tag{4.3.10}$$

112 *where* $\lambda = 4kr(\log H)(\log T)^{-1}$ *and in the second of these possibilities the constant implied by* $\ll$ *is independent of* r.

Proof. We apply the convexity Theorem 4.2.3 in a manner similar to what we did in Lemmas 9, 10 and 11 of part B of § 4.2. Let $w = u + iv$ be a complex variable,

$$f(s,w) = \left(\zeta(s+w) - \left(\sum_{n\le H} \frac{d_k(n)}{n^{s+w}}\right)^j\right) \operatorname{Exp}(w^2)$$

and

$$I(\sigma,u) = \frac{1}{H}\int_{(t)}\int_{|v|\le r} |f(s,w)|^{2k} dv\, dt$$

where $\tau = (\log T)^2$ and the t-range of integration is $T+\tau \le t \le T+H-\tau$.

Exactly as before it follows that

$$M_1(\sigma_0) \le C_1(M_1(1-\sigma_0)+E)^{\frac{1}{2}}(M_1(\sigma_0)+E)^{\frac{1}{2}-2^{-n-1}}$$
$$(M_1(1-\sigma_0+(2\sigma_0-1)(2^n+1))+E)^{2^{-n-1}}$$

where $E = (\operatorname{Exp}((\log T)^3))^{-1}$ and $C_1(>1)$ is independent of n and r provided that $1-\sigma_0+(2\sigma_0-1)(2^n+1)$ is bounded above. If either $M_1(1-\sigma_0) \le 1$ or $M_1(\sigma_0) \le 1$ we end up (by using Lemma 1) with

$$M(1-\sigma_0) \le 2^{2k}(A(1-\sigma_0)+1) \text{ or } M(\sigma_0) \le 2^{2k}(A(\sigma_0)+1)$$

respectively and so by Lemma 2 we end up, in any case, with

$$M(1-\sigma_0) \ll A(\sigma_0) + A(1-\sigma_0).$$

In the remaining case we are led to

$$M_1(\sigma_0) \leq 2C_1(M_1(1-\sigma_0))^{\frac{1}{2}}(M_1(\sigma_0))^{\frac{1}{2}-2^{-n-1}}(H^{-L_0+C_2}+E)^{2^{-n-1}}$$

where $L_0 = 2k(2\sigma_0 - 1)(2^n + 1)$ and $C_2(> 1)$ is independent of n and r provided, of course, that L_0 is bounded above. Hence we are led to

$$M_1(\sigma_0) \leq 4C_1^2(M_1(1-\sigma_0))^{L^*}(H^{-L_0+C_2}+E)^{2^{-n}L^*}$$

where $L^* = (1 + 2^{-n})^{-1}$. We choose n in such a way that $2^n r(\log T)^{-1}$ is a large constant C_3 which is $\asymp r^2$. We have $M_1(1-\sigma_0) = O(\log T)$ and $(1+2^{-n})^{-1} = 1+O(2^{-n}) = 1+O(rC_3^{-1}(\log T)^{-1})$. Now $2^n = C_3 r^{-1}\log T$ and $E = (\mathrm{Exp}((\log T)^3))^{-1} \leq H^{-L_0+C_2}$. Thus since 113

$$(L_0 - C_2)2^{-n}L^* = 4kr(\log T)^{-1} + O(r^{-1}(\log T)^{-1}),$$

we have,

$$M_1(\sigma_0) \ll M_1(1-\sigma_0)H^{-4kr(\log T)^{-1}}.$$

This proves the lemma. □

We can now complete the proof of Theorem 4.3.1 as follows. By Lemma 2, we have

$$M_1(\sigma_0) \sim M\,\mathrm{Exp}(-2kr).$$

By the second possibility of Lemma 3, we have

$$M_1(\sigma_0) \ll (M(1-\sigma_0) + A(1-\sigma_0))\,\mathrm{Exp}(-\lambda)$$

i.e.

$$M_1(\sigma_0) \ll (M + A(1-\sigma_0))\,\mathrm{Exp}(-\lambda).$$

Now by Lemma 1, we have

$$M_1(\sigma_0) \geq 2^{-2k}M(\sigma_0) - A(\sigma_0) \geq 2^{-4k}M\,\mathrm{Exp}(-2kr) - A(\sigma_0)$$

for large r. Thus

$$2^{-4k} M \operatorname{Exp}(-2kr) - A(\sigma_0) \leq C_4(M + A(1-\sigma_0)) \operatorname{Exp}(-\lambda),$$

where $C_4(\geq 1)$ is a constant independent of r. Hence

$$M\{2^{-4k} \operatorname{Exp}(-2kr) - C_4 \operatorname{Exp}(-\lambda)\} \leq A(\sigma_0) + C_4 A(1-\sigma_0) \operatorname{Exp}(-\lambda).$$

In this equation we note that for $r \geq r_0(\epsilon)$ the coefficient of M on the LHS is bounded below by a positive constant. Now by fixing r to be a large constant we obtain

$$M \ll A(\sigma_0) + A(1-\sigma_0).$$

114 Using Theorem 4.2.5 we see that $M \ll (\log T)^{k^2}$. Now by convexity Theorem 4.2.3 and the fact that $M(\sigma)$ is bounded for $\sigma \geq 2$, we see that the first assertion of Theorem 4.3.1 proved. The remaining two assertions of Theorem 4.3.1 follow from things like

$$\zeta^{(m)}(s) = \frac{m!}{2\pi i} \int \frac{\zeta(w)}{(w-s)^{m+1}} dw$$

(where the integration is over the circle $|w-s| = \frac{1}{3}(\log T)^{-1}$) and by convexity Theorem 4.2.3.

Notes at the end of Chapter V

§ 4.1 and § 4.2. From $\zeta(s) = \sum\limits_{n \leq 10T} n^{-s} + O(T^{-\sigma})$, valid uniformly, for example, in $(\frac{1}{4} \leq \sigma \leq 2, T \leq t \leq 2T)$ and from Montgomery-Vaughan Theorem it follows that

$$\frac{1}{T} \int\limits_{T}^{2T} \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt = \log T + O(1).$$

However to prove

$$\frac{1}{T} \int_{T}^{2T} \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt = (2\pi^2)^{-1} (\log T)^4 + O((\log T)^3),$$

it seems that the functional equation is unavoidable. This latter result (originally a difficult result due to A.E. Ingham) was proved in a fairly simple way (but still using the functional equation) by K. Ramachandra (See A. Ivic [42]).

In the direction of lower bounds the earliest general result (see page 174 of Titchmarsh [100]) is due to E.C. Titchmarsh who proved that for $0 < \delta < 1$, we have

$$\int_0^\infty \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} e^{-\delta t} dt \gg_k \frac{1}{\delta}\left(\log\frac{1}{\delta}\right)^{k^2}$$

for all integers $k \geq 1$. As a corollary this gives 115

$$\lim_{T\to\infty} \sup\left(\left(\frac{1}{T}\int_T^{2T}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt\right)(\log T)^{-k^2}\right) > 0,$$

for all integers $k \geq 1$.

The history of Theorem 4.2.1 is as follows. The general problem of obtaining lower bounds for

$$\max_{\sigma\geq\alpha}\left(\frac{1}{H}\int_T^{T+H}\left|\frac{d^m}{ds^m}(\zeta(s))^{2k}\right| dt\right) \quad \left(k > 0, \frac{1}{2} \leq \alpha \leq 2\right)$$

where $T \geq H \gg \log\log T$ and $m(\geq 0)$ is an integer constant, was solved by K. Ramachandra with an imperfection factor $(\log\log H)^{-C}$ (see [76]). This imperfection was removed by him in a later paper (see [77]) for all positive integers $2k$. The Next step was teken by D.R. Heath-Brown (see [37]) who proved that for all rational constants $2k > 0$, we have

$$\frac{1}{T}\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt \gg_k (\log T)^{k^2}.$$

Ramachandra's proof of his theorem mentioned above (valid for short intervals $T, T+H$ and integer constants $2k > 0$) did not use Gabriel's two variable convexity theorem. (This depends upon Riemann mapping theorem). The present proof of Theorem 4.2.1 (due to K. Ramachandra, see [78]) uses all these ideas in addition to those of Ramachandra's paper

([79]) and Gabriel's theorem in the form established by him and R. Balasubramanian namely Theorem 4.2.3 (see their paper [67]). However we do not use the method of obtaining auxiliary zero-density estimates for $\zeta(s)$ adopted in K. Ramachandra [76].

§ 4.3. The main difference between § 4.2 and § 4.3 is that § 4.3 depends crucially on the functional equation $\zeta(s)=\chi(s)\zeta(1-s)$ where $|\chi(s)| \asymp t^{\frac{1}{2}-\sigma}$ uniformly in $a_1 \leq \sigma \leq b_1$, $t \geq 2$ where a_1 and b_1 are any two constants. (Owing to the presence of $n(\frac{1}{2}-\sigma)$ in $|\chi(s)| \asymp t^{n(\frac{1}{2}-\sigma)}$
116 upper bound problems are hopeless if $n > 2$ i.e. if $k > 2$ in the mean-value problem). These results will be proved in the appendix at the end. Another result which we have used frequently is

$$\frac{1}{x}\sum_{n\leq x}|d_k(n)|^2 = C(\log x)^{|k^2|-1}(1+O((\log x)^{-1}))$$

valid for complex constants k and $C = C(k) > 0$. These will also be proved in the appendix. Regarding the history of Theorem 4.3.1, the result (4.3.2) was first proved by K. Ramachandra. (See [80]). Later D.R. Heath-Brown proved that

$$\frac{1}{T}\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt \ll (\log T)^{k^2}$$

where $k=\frac{1}{j}$ ($j \geq 2$ an integer constant). (See [37]). Another point of interest is the proof of (see K. Ramachandra [80])

$$\frac{1}{H}\int_T^{T+H} \left|\zeta^{(m)}\left(\frac{1}{2}+it\right)\right| dt \ll (\log T)^{\frac{1}{4}+m}$$

valid for $H = T^{\frac{1}{4}+\epsilon}$ and any integer constant $m \geq 0$ and arbitrary real constant $\epsilon (0 < \epsilon \leq \frac{1}{4})$. This result depends on RH. We do not have the least idea for proving the same result with $H = T^{\frac{1}{6}+\epsilon}$. It should be mentioned that the proof of Ramachandra's result with $H = T^{\frac{1}{4}+\epsilon}$ above (assuming RH) is incomplete. To complete the proof we have to use Gabriel's convexity theorem in the form Theorem 4.2.3. Theorem 4.2.3 was suggested by the version of Gabriel's thoerem used by D.R. Heath-Brown in his paper cited above and Ramachandra regards the upper bound theorem with $H = T^{\frac{1}{4}+\epsilon}$ (on RH) as joint work with him.

It is of some interest to determine the constants in

$$(\log T)^{k^2} \ll \frac{1}{T}\int_0^T |\zeta\left(\frac{1}{2}+it\right)|^{2k} dt \ll (\log T)^{k^2}$$

where $k = \frac{1}{j}$ ($j \geq 2$ an integer). An important result due to M. Jutila (see [48]) says that the constants implied by $\gg$ and $\ll$ are independent of k. Using this A. Ivic and A. Perelli proved that the mean-value in question **117**
(for real positive k) $\to 1$ if $k \leq (\psi(T)\log\log T)^{-\frac{1}{2}}$ where $\psi(T)$ is any function which $\to \infty$ as $T \to \infty$. (See [45]).

The best result on lower bounds (for integral $k > 0$) is due to K. Soundararajan and a particular case of it reads

$$\frac{1}{T}\int_0^T |\zeta\left(\frac{1}{2}+it\right)|^6 dt \geq (24.59) + o(1)\sum_{n\leq T}(d_3(n))^2 n^{-1}$$

(information by private communication).

Chapter 5

Zeros of $\zeta(s)$

5.1 Introduction

118

In this chapter we deal with three results. In § 5.2 we deal with a simple proof (due to K. Ramachandra) of the inequality $\theta \geq \frac{1}{2}$, where θ is the least upper bound of the real parts of the zeros of $\zeta(s)$. (Trivially from the Euler product we have $\theta \leq 1$). This proof (which does not use Borel-Caratheodory theorem and Hadamard's three circle theorem) has some advantages. We will make some remarks about the proof which uses the two theorems in the brackets. It has the advantage that it generalises very much. In § 5.3 we mention some localisation of theorems of Littlewood and Selberg. These localisations are due to K. Ramachandra and A. Sankaranarayanan whose proof has the advantage that it generalises very much. Lastly § 5.4 deals with a proof due to J.B. Conrey, A. Ghosh and S.M. Gonek, that $\zeta(s)$ has infinity of simple zeros in $\sigma \geq 0$. Only the last section uses the functional equation and some difficult machinery viz. asymptotics of $\Gamma(s)$ and so on. These will be proved in the appendix in the last chapter.

5.2 Infinitude of Zeros in $t \geq 1$

First we give a simple proof of the inequality $\theta \geq \frac{1}{2}$ and then remark about another proof. We will prove the following theorem.

Theorem 5.2.1. *We have*

$$\theta \geq \frac{1}{4} \tag{5.2.1}$$

Remark 1. The method of proof actually gives

$$\theta \geq \frac{1}{2}. \tag{5.2.2}$$

To see this we have only to replace $\frac{1}{4}$ in our proof by $\frac{1}{2} - \delta$ where $\delta (0 < \delta < \frac{1}{4})$ is any small constant. Also we can prove the existence of at least one zero in $(\sigma \geq \frac{1}{2} - \delta, T \leq t \leq T + T^{\epsilon})$ for $T \geq T_0(\epsilon, \delta)$ and constants ϵ, δ with $0 < \epsilon < 1$, $0 < \delta < \frac{1}{4}$.

Remark 2. All that we use in our proof is the Euler product and analytic
119 continuation in $\sigma \geq \frac{1}{10}$ and the bound $|\zeta(s)| \leq t^A (t \geq 2)$ where A is any constant. Hence (5.2.2) holds good for the zeta and L-funcitons of any algebraic number field. We do not need the functional equation. In fact we may dispense with the Euler product and prove some worthwhile results (see the notes at the end of this chapter).

Remark 3. Our method shows that $\zeta(s)$ has $\gg T(\log\log T)^{-1}$ zeros in $(\sigma \geq \frac{1}{2} - 20(\log\log\log T)(\log T)^{-1}, T \leq t \leq 2T)$. (See the notes at the end of this chapter).

Lemma 1. *Let $w = u + iv$ be a complex variable. Then for $y > 0$, we have,*

$$\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \frac{y^w dw}{w(w+1)} = 1 - \frac{1}{y} \quad \textit{or} \quad 0 \tag{5.2.3}$$

according as $y \geq 1$ or $y \leq 1$.

Proof. If $y \geq 1$ we apply Cauchy's theorem and obtain that the LHS of (5.2.3) is

$$1 - \frac{1}{y} + \frac{1}{2\pi i}\int_{-R-i\infty}^{-R+i\infty} \frac{y^w dw}{w(w+1)} \tag{5.2.4}$$

and it is easily seen that as $R \to \infty$, the last integral tends to zero. If $y < 1$, we apply Cauchy's theorem and obtain that the LHS of (5.2.3) is

$$\frac{1}{2\pi i}\int_{R-i\infty}^{R+i\infty} \frac{y^w dw}{w(w+1)} \tag{5.2.5}$$

which tends to zero as $R \to \infty$. $\square$

Lemma 2. *Let $s = \sigma + it$ and let $d_k(n)$ be defined for any complex constant k by $(\zeta(s))^k = \sum_{n=1}^{\infty} d_k(n) n^{-s}$ where $\sigma \geq 2$. If $0 < k \leq 1$ then* (i) $0 < d_k(n) \leq 1$ *for all n and* (ii) $d_k(p) = k$ *for all primes p.*

Proof. The lemma follows from

$$(1 - p^{-s})^{-k} = 1 + kp^{-s} + \frac{k(k+1)}{2!} p^{-2s} + \cdots$$

and the fact that **120**

$$(\zeta(s))^k = \prod_p (1 - p^{-s})^{-k}.$$

$\square$

Lemma 3. *Let $T \geq 10$ and $\zeta(s) \neq 0$ in $(\sigma \geq \frac{1}{4}, \frac{1}{2}T \leq t \leq \frac{5}{2}T)$. Put $G(s) = (\zeta(s))^k$ where $k = q^{-1}$ and $q \geq 1$ is an integer constant. For $X \geq 1$ define $A(s) = A(s, X)$ and $b_n = b_n(X)$ by*

$$\frac{1}{2\pi i} \int_{1-i\infty}^{1+\infty} \frac{G(s+w) X^w (2^w - 1) dw}{w(w+1)} = \sum_{n=1}^{\infty} b_n n^{-s} = A(s), \qquad (5.2.6)$$

where $s = \frac{1}{4} + it$ and $T \leq t \leq 2T$. Then

(i) $b_n = d_k(n)(\frac{n}{2X})$ *for* $1 \leq n \leq X$,

(ii) $b_n = d_k(n)(1 - \frac{n}{2X})$ *for* $X \leq n \leq 2X$, *and*

(iii) $b_n = 0$ *for* $n \geq 2X$.

In particular $|b_n| \leq 1$ for all n and $b_p = \frac{p}{2qX}$ for primes $p \leq X$.

Proof. Lemma 3 follows from Lemma 1 since on $u = 1$ the series for $G(s + w)$ is absolutely convergent. $\square$

Lemma 4. *We have, for* $T \geq 10$,

$$\frac{1}{T} \int_T^{2T} \left|A\left(\frac{1}{4} + it\right)\right|^2 dt \geq \sum_{p \leq X} \left(\frac{p}{2qX}\right)^2 p^{-\frac{1}{2}} - 16X^3T^{-1}. \qquad (5.2.7)$$

Proof. We multiply $A(s)$ by its complex conjugate and integrate term by term. The LHS of (5.2.7) is thus seen to be

$$\sum_{n\leq 2X} b_n^2 n^{-\frac{1}{2}} + \frac{2\psi}{T}\sum_{m\neq n} b_m b_n (mn)^{-\frac{1}{4}} |\log\frac{m}{n}|^{-1},$$

where $|\psi| \leq 1$. If $m > n$, we have, $\log\frac{m}{n} = -\log(1-(1-\frac{n}{m})) \geq I - \frac{n}{m} = \frac{m-n}{m} \geq (2X)^{-1}$. Hence we have the lower bound

$$\sum_{p\leq X} b_p^2 p^{-\frac{1}{2}} - \frac{4X}{T}\sum\sum_{m\neq n} 1$$

121 and the required lower bound follows. □

Lemma 5. *For $X \geq 350$, we have, with usual notation,*

$$\pi(X) - \pi\left(\frac{X}{2}\right) > (6\log X)^{-1}(X - 18X^{\frac{1}{2}}). \tag{5.2.8}$$

Remark. S. Ramanujan proved this result in an easy and elementary way. It is possible to resort to using simpler and easier results in place of this lemma, but we do not do it here.

Lemma 6. *Suppose that $X = T^{\frac{1}{3}}$ and that T exceeds a large absolute constant. Then for every fixed integer $q \geq 1$,*

$$\max_{T\leq t\leq 2T} |A\left(\frac{1}{4}+it\right)| \gg X^{\frac{1}{4}}(\log X)^{-\frac{1}{2}}. \tag{5.2.9}$$

Proof. We have

$$\sum_{p\leq X}\left(\frac{p}{2qX}\right)^2 p^{-\frac{1}{2}} \geq \sum_{\frac{1}{2}X\leq p\leq X}\cdots$$

and so Lemma 6 follows from Lemmas 4 and 5. □

Lemma 7. *We have, for $s = \frac{1}{4} + it$, $T \leq t \leq 2T$ the inequality*

$$|A(s)| \leq \frac{1}{2\pi}\int_{(w)} |\frac{G(s+w)X^w(2^w-1)}{w(w+1)} dw| \tag{5.2.10}$$

where the contour of integration is the union of straight line segments obtained by joining the points $1-i\infty$, $1-\frac{1}{2}iT$, $-\frac{1}{2}iT$, $\frac{1}{2}iT$, $1+\frac{1}{2}iT$, $1+i\infty$ in this order.

Proof. The lemma follows by Cauchy's theorem. □

We now fix t to be the point in $T \leq t \leq 2T$ at which $|A(\frac{1}{4}+it)|$ attains its maximum. We estimate the integral in Lemma 7 from above. This will lead to a contradiction as we will see. We begin with

Lemma 8. (i) *On $u = 1$ we have $|\zeta(s+w)| \leq 5$.*

(ii) *In $\sigma \geq \frac{1}{4}$, $u \geq 0$, $|v| \leq \frac{1}{2}T$ we have, $|\zeta(s+w)| \leq 100T$, for $T \geq 1000$.* **122**

Proof. We have $|\zeta(s+w)| \leq \zeta(\frac{5}{4}) \leq 1 + \int_1^\infty u^{-\frac{5}{4}} du = 5$. This proves (i). Next

$$\zeta(s+w) = \sum_{n=1}^{\infty} (n^{-s-w} - \int_n^{n+1} u^{-s-w} du) + (s+w-1)^{-1}$$

and the fact that the infinite series here is

$$(s+w) \sum_{n=1}^{\infty} \int_n^{n+1} \left(\int_n^u v^{-s-w-1} dv \right) du$$

complete the proof of (ii). □

Lemma 9. *Let T exceed a large constant, $X = T^{\frac{1}{3}}$ and $q = 100$. Then the inequality asserted by Lemma 7 is false if $t(T \leq t \leq 2T)$ is fixed such that $|A(\frac{1}{4}+it)|$ is maximum.*

Proof. The contribution from the segments on $u = 1$ is

$$\leq \int_{|v| \geq \frac{1}{2}T} 15X \frac{dv}{v^2} = \frac{60X}{T} < 1.$$

The contribution from the two horizontal segments on $v = \pm\frac{1}{2}T$ is

$$\leq 6(100T)^{q^{-1}} \left(\frac{1}{2} T \right)^{-2} \leq 1.$$

The contribution from the remaining part on $u = 0$ is

$$\leq (100T)^{q^{-1}} \left\{ \int_{|v| \leq 1} |\frac{2^{iv} - 1}{v}| dv + \int_{|v| \geq 1} \frac{|2^{iv} - 1|}{v^2} dv \right\}$$

$$\leq (100T)^{q^{-1}}\{4+4\} \leq 8(100T)^{q^{-1}}.$$

The contradiction is now immediate.

Lemma 9 completes the proof of Theorem 5.2.1. □

By using Borel-Caratheodory's Theorem 1.6.1, Hadamard's three circle Theorem 1.5.2 and things like the third main theorem of § 2.5 (or
123 easier theorems) we can prove the following theorem.

Theorem 5.2.2. *Let $\{\lambda_n\}$ be a sequence satisfying $1 = \lambda_1 < \lambda_2 < \ldots$ where $C_1^{-1} \leq \lambda_{n+1} - \lambda_n \leq C_1$ (for some constant $C_1 \geq 1$), and let $\{a_n\}$ be any sequence of complex numbers such that the series $\sum\limits_{n=1}^{\infty} (a_n \lambda_n^{-s})^2$ has a finite abscissa of absolute convergence say C_2. By replacing a_n by $a'_n = a_n \lambda_n^{C_2 - \frac{1}{2}}$ if necessary we can assume, as we do, that $C_2 = \frac{1}{2}$. Suppose that $F(s) = \sum\limits_{n=1}^{\infty} a_n \lambda_n^{-s}$ (which is certainly absolutely convergent in $\sigma > 1$) be continuable analytically in $(\sigma \geq \frac{1}{2} - \delta, t \geq t_0)$ where $\delta (> 0)$ and $t_0 (\geq 10)$ are some constants and there $|F(s)| < t^A$, for some constant $A \geq 10$. Let $\epsilon (> 0)$ be any constant and $H = T^{\epsilon}$. Then there exists an infinite sequence $\{T_\nu\} (\nu = 1, 2, 3, \ldots)$ such that $T_\nu \to \infty$ and if $T = T_\nu$, then the rectangle $(\sigma \geq \frac{1}{2} - \delta, t \in I)$ contains at least one zero of $F(s)$ provided I is any sub-internal of $(T, 2T)$ of length H. In particular if θ is the least upper bound of the real parts of the zeros of $F(s)$ then $\theta \geq \frac{1}{2}$.*

Remark . The proof of this theorem essentially due to Littlewood. Roughly speaking it is enough to disprove the analogue of Lindelöf hypothesis on $\sigma = \frac{1}{2} - \delta$, for $F(s)$. This is done by the third main theorem of § 2.5 (or easier theorems). For details of proof see the proof of Theorem 14.2 on pages 336 and 337 of E.C. Titchmarsh [100]. By imposing some very mild extra conditions we can even take $\delta = C_3 (\log\log t)^{-1}$, where $C_3 (> 0)$ is a certain constant. But this needs the results of § 5.3. (See the notes at the end of this chapter).

5.3 On Some Theorems of Littlewood and Selberg

Now we ask the following question: What are upper and lower bounds for *Re* $\log F(s)$ and *Im* $\log F(s)$ if there are no zeros in certain rectangles? We mean the "localised analogue" of the results of Littlewood and Selberg which they prove assuming RH. (See for example Theorem 14.14 (B) and equation (14.14.5) on pages 354 and 355 of E.C. Titchmarsh [100]).

We state two theorems in this direction. **124**

Theorem 5.3.1. *Let* $s = \sigma + it$ *and*

$$F(s) = \sum_{n=1}^{\infty} a_n n^{-s} = \prod_p (1 - w(p)p^{-s})^{-1}, \tag{5.3.1}$$

where p *runs over all primes and* $w(p)$ *are arbitrary complex numbers (independent of* s*) with absolute value not exceeding* 1*. Suppose* α *and* δ *are positive constans satisfying* $\frac{1}{2} \leq \alpha \leq 1 - \delta$ *and that in* $(\sigma \geq \alpha - \delta, T - H \leq t \leq T + H)$*,* $F(s)$ *can be continued analytically and there* $|F(s)| < T^A$*. Here* $T \geq T_0$*,* $H = C \log\log\log T$ *and* A*,* T_0 *and* C *are large positive constants of which* C *depends on* T_0 *and* A*. Let* $F(s) \neq 0$ *in* $(\sigma > \alpha, T - H \leq t \leq T + H)$*. Then for* $\alpha \leq \sigma \leq \alpha + C_1(\log\log T)^{-1}$ *and* $T - \frac{1}{2}H \leq t \leq T + \frac{1}{2}H$*, we have, uniformly in* σ*,*

(a) $\log|F(\sigma + it)|$ *lies between* $C_2(\log T)(\log\log T)^{-1}$ *and* $-C_3(\log T)(\log\log T)^{-1}\log\{C_4((\sigma - \alpha)\log\log T)^{-1}\}$ *and*

(b) $|\arg F(\sigma+it)| \leq C_5(\log T)(\log\log T)^{-1}$*, where* C_1, C_2, C_3, C_4 *and* C_5 *are certain positive constants.*

Corollary 1. *For* $\alpha + C_1(\log\log T)^{-1} \leq \sigma \leq 1 - \delta$*,* $t = T$*, we have*

$$|\log F(\sigma + it)| \leq C_6(\log T)^{\theta}(\log\log T)^{-1},$$

where $\theta = (1 - \sigma)(1 - \alpha)^{-1}$ *and* C_6 *is a certain constant. The inequality holds uniformly in* σ*.*

Corollary 2. *For $\alpha \leq \sigma \leq 1-\delta$, $t = T$, we have,*

$$|F(\sigma+it)| \leq \mathrm{Exp}(C_6(\log T)^{\theta}(\log\log T)^{-1})$$

where θ is the same as in Corollary 1. (C_6 may not be the same as before).

Theorem 5.3.1 is nearly true of functions, very much more general than the ones given by (5.3.1). In this direction we state the following theorem.

125 **Theorem 5.3.2.** *Let $\{\lambda_n\}$ be a sequence satisfying $1 = \lambda_1 < \lambda_2 < \ldots$ where $C_1^{-1} \leq \lambda_{n+1} - \lambda_n \leq C_1$ (for some constant $C_1 \geq 1$) and let $\{a_n\}$ be a sequence of complex numbers such that the series $F(s) = \sum\limits_{n=1}^{\infty} a_n \lambda_n^{-s}$ converges for some complex s and continuable analytically in $(\sigma \geq \alpha - \delta, T-H \leq t \leq T+H)$ and there $|F(s)| < T^A$, $T \geq T_0$, $H = C\log\log\log T$. Here α, δ, A are positive constants with $\alpha > \delta$, T_0 and C are large positive constants. Let $F(s) \neq 0$ in $(\sigma > \alpha, T-H \leq t \leq T+H)$. Then the conclusions* (a) *and* (b) *of Theorem 5.3.1 hold good without any change.*

Next assume that $F(1+it) = O((\log t)^A)$ for all $t \geq t_0$ (t_0 being a constant) and that $H = C(\log\log T)(\log\log\log T)$ in place of the earlier condition on H. Then we have the following corollaries (the inequalities asserted here hold uniformly in σ).

Corollary 1. *For $\alpha + C_1(\log\log T)^{-1} \leq \sigma \leq 1-\delta$, $t = T$, we have,*

$$|\log F(\sigma+it)| \leq C_6(\log T)^{\theta}(\log\log T)^{C_7},$$

where $\theta = (1-\sigma)(1-\alpha)^{-1}$ and C_6 and C_7 are certain positive constants.

Corollary 2. *For $\alpha \leq \sigma \leq 1-\delta$, $t = T$, we have,*

$$|F(\sigma+it)| \leq \mathrm{Exp}(C_6(\log T)^{\theta}(\log\log T)^{C_7})$$

where C_6 and C_7 may not be the same as before.

5.4 Infinitude of Simple Zeros in $t \geq 1$

In this section we prove (following J.B. Conrey, A. Ghosh and S.M. Gonek) that for all T exceeding a large positive constant T_0,

$$Re \sum_{1 \leq Im\, \rho \leq T} \zeta'(\rho) = \frac{T}{4\pi}(\log T)^2 + O(T \log T), \tag{5.4.1}$$

where the sum on the left is over all zeros ρ with $1 \leq Im\, \rho \leq T$ (of course for all such zeros ρ of $\zeta(s)$ we must have (see the last chapter), necessarily $0 \leq Re\, \rho \leq 1$). As a corollary we have the following theorem.

Theorem 5.4.1. *There are infinity of simple zeros of $\zeta(s)$ in $0 \leq \sigma \leq 1$, $t \geq 1$.* 126

Remark. In fact the three authors (mentioned above) prove by their simple method that the sum on the LHS of (5.4.1) is $(4\pi)^{-1}T(\log T)^2 + O(T \log T)$. Their "simple method" uses the functional equation of $\zeta(s)$, the asymptotics of gamma function and etc. It is relatively very simple compared with other methods namely that of N. Levinson, D.R. Heath-Brown and A. Selberg.

Throughout our proof of (5.4.1) we write $L = \log(2T)$. We begin with the remarks that $\sum \zeta'(\rho)$ summed up over zeros ρ with $T - 1 \leq Im\rho \leq T$ is $O_\epsilon(T^{\frac{1}{2}+\epsilon})$ for every $\epsilon > 0$ and that there is a T' satisfying $T - 1 \leq T' \leq T$ for which $|\zeta'(\sigma + iT')(\zeta(\sigma + iT'))^{-1}| \ll L^2$ uniformly in $-1 \leq \sigma \leq 2$. Another result which we will be using is $\zeta(s) = \chi(s)\zeta(1 - s)$ where for $t \geq 1$, $|\chi(s)| \asymp t^{\frac{1}{2}-\sigma}$ uniformly for σ in any closed bounded interval. Yet another result which we will be using is $\chi'(s)(\chi(s))^{-1} = -\log(\frac{t}{2\pi}) + O(t^{-1})$ uniformly for σ in any closed bounded interval. Finally we need the lemma on page 143 of E.C. Titchmarsh [100]. So we find that all the results that we need are in this book with proper references. Except this lemma on page 143 we will prove all the results that we need in the appendix, which forms our last chapter. The method consists in considering the integral

$$\frac{1}{2\pi i}\int_{(s)} F(s)ds, F(s) = (\zeta'(s))^2(\zeta(s))^{-1}, \tag{5.4.2}$$

taken over the anticlockwise boundary of the rectangle obtained by joining (by straight line segments) the points $c+i$, $c+iT'$, $1-c+iT'$, $1-c+i$, $c+i$ in this order. We will fix $c = 1+L^{-1}$. Clearly the integral is the sum $\sum\limits_{\cdots} \zeta'(\rho)$ which figures on the LHS of (5.4.1) plus a quantity $O_\epsilon(T^{\frac{1}{2}+\epsilon})$. We will show that the right vertical line and the two horizontal lines contribute $O_\epsilon(T^{\frac{1}{2}+\epsilon})$ and that the real part of the contribution from the
127 left vertical line is $(4\pi)^{-1}T(\log T)^2 + O(T\log T)$. This proves all that we want. Now

$$F(s) = \frac{\zeta'(s)}{\zeta(s)}\zeta'(s) = \sum\sum_{m\geq 1, n\geq 1} (\Lambda(m)\log n)(mn)^{-s} \tag{5.4.3}$$

and so the right vertical line contributes

$$O(\sum\sum_{\cdots}(\Lambda(m)(\log n)(mn)^{-c}) = O((\zeta'(c))^2(\zeta(c))^{-1}) = O(L^3).$$

The two horizontal lines contribute $O_\epsilon(T^{\frac{1}{2}+\epsilon})$ by the choice of T'. Thus we are left with

$$I_0 = \frac{1}{2\pi i}\int_{1-c+iT'}^{1-c+i} F(s)ds, \tag{5.4.4}$$

$$I_0 = \frac{1}{2\pi}\int_{T'}^{1} F(1-c+it)dt = -\frac{1}{2\pi}\int_{1}^{T'} F(1-c+it)dt. \tag{5.4.5}$$

Here after we may suppose, as we will, that T' is replaced by T since the error is $O_\epsilon(T^{\frac{1}{2}+\epsilon})$. With this we have

$$\overline{I}_0 + O_\epsilon(T^{\frac{1}{2}+\epsilon}) = -\frac{1}{2\pi}\int_{1}^{T} F(1-c-it)dt = -\frac{1}{2\pi i}\int_{c+i}^{c+iT} F(1-s)ds. \tag{5.4.6}$$

We will prove that the last expression involving the integral has the real part $(4\pi)^{-1}T(\log T)^2 + O(T\log T)$. Let us write χ, ζ, ζ' for $\chi(s)$, $\zeta(s)$ and $\zeta'(s)$. We have

$$\zeta(s) = \chi\zeta(1-s), (\zeta(1-s))^{-1} = \chi\zeta^{-1},$$
$$\zeta' = \chi'\zeta(1-s) - \chi\zeta'(1-s) = \chi'\chi^{-1}\zeta - \chi\zeta'(1-s).$$

(Note that $\chi\chi(1-s) = 1$). Hence $\zeta'(1-s) = \chi^{-1}(\chi'\chi^{-1}\zeta - \zeta')$ and so

$$\begin{aligned} F(1-s) &= \frac{\chi}{\zeta}\left(\left(\frac{\chi'}{\chi}\zeta - \zeta'\right)\chi^{-1}\right)^2 = \frac{\chi(1-s)}{\zeta}\left(\frac{\chi'}{\chi}\zeta - \zeta'\right)^2 \\ &= \chi(1-s)\left\{\left(\frac{\chi'}{\chi}\right)^2\zeta - 2\frac{\chi'}{\chi}\zeta' + F(s)\right\}. \end{aligned} \tag{5.4.7}$$

Hence (with an error $O_\epsilon(T^{\frac{1}{2}+\epsilon})$),

$$-\bar{I}_0 = I_1 - 2I_2 + I_3, \tag{5.4.8}$$

where 128

$$I_1 = \frac{1}{2\pi i}\int \chi(1-s)\left(\frac{\chi'}{\chi}\right)^2 \zeta dx, I_2 = \frac{1}{2\pi i}\int \chi(1-s)\frac{\chi'}{\chi}\zeta' ds \quad \text{and}$$

$$I_3 = \frac{1}{2\pi i}\int \chi(1-s)F(s)ds\} \tag{5.4.9}$$

and the integrals being taken from $c + i$ to $c + iT$.

Lemma 1. *If $n < \frac{T}{2\pi}$,*

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \chi(1-s)n^{-s}ds = 2 + O\left(n^{-\frac{1}{2}}\left(\log\frac{T}{2n\pi}\right)^{-1}\right) + O(n^{-\frac{1}{2}}\log T). \tag{5.4.10}$$

If $n > \frac{T}{2\pi}$ and $c > \frac{1}{2}$,

$$\frac{1}{2\pi i}\int_{e-iT}^{c+iT} \chi(1-s)n^{-s}ds = O\left(T^{c-\frac{1}{2}}n^{-c}\left(\log\frac{2n\pi}{T}\right)^{-1}\right) + O(T^{c-\frac{1}{2}}n^{-c}). \tag{5.4.11}$$

Remark 1. This is the lemma on page 143 of E.C. Titchmarsh [100]. Out choice will be, as stated already, $c = 1 + L^{-1}$.

Remark 2. The following result which says more is due to S.M. Gonek. (We state Lemma 1 of his paper the reference to which will be given in the notes at the end of this chapter). There is a constant $c > 0$ (not $1 + L^{-1}$) such that

$$\int_{r(1-c)}^{r(1+c)} \text{Exp[it} \quad \log\left(\frac{t}{er}\right)]\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}} dt$$
$$= (2\pi)^{1-a} r^a \,\text{Exp}\left(-ir + \frac{i\pi}{4}\right) + O(r^{a-\frac{1}{2}})$$

for all real constants a and all real $r \geq r_0(a)$. See also Lemma 3.3 of N. Levinson [54].

Now let us look at the first part of the lemma. Here LHS is by Cauchy's theorem

$$\frac{1}{2\pi i}\left(\int_{c-iT}^{c+iT} + \int_{\frac{1}{2}-iT}^{c-iT} + \int_{c+iT}^{\frac{1}{2}+iT}\right)\chi(1-s)n^{-s}ds$$
$$= \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \chi(1-s)n^{-s}ds + O\left(\int_{\frac{1}{2}}^{c} T^{\frac{1}{2}-(1-\sigma)}n^{-\sigma}d\sigma\right).$$

129

Hence if $\{a_n\}$ is any sequence of complex numbers with $|a_n| \leq (\log(n+2))^A$ (for some constant $A > 0$ which is arbitrary) and $c = 1+L^{-1}$, then we have (with $\tau = (2\pi)^{-1}T$),

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \chi(1-s)\left(\sum_{n<\tau-2} + \sum_{|n-\tau|\leq 2} + \sum_{n>\tau+2}\right)a_n n^{-s}ds$$
$$= 2\sum_{n\leq\tau-2} a_n\left(1 + O\left(n^{-\frac{1}{2}}\left(\log\frac{\tau}{n}\right)^{-1}\right) + O\left(n^{-\frac{1}{2}}\log T\right) + O\left(n^{-c}T^{\frac{1}{2}}\right)\right)$$
$$+ O\left(\int_{c-iT}^{c+iT} T^{\epsilon}T^{-c}(|t|+1)^{\frac{1}{2}}|ds|\right)$$
$$+ O\left(\sum_{n>\tau+2} a_n T^{c-\frac{1}{2}}n^{-c}\left(\log\frac{n}{\tau}\right)^{-1}\right) + O\left(\sum_{n>\tau+2} a_n T^{c-\frac{1}{2}}n^{-c}\right)$$
$$= 2\sum_{n\leq\tau} a_n + O\left(T^{\epsilon}\sum_{n\leq\tau-2}\left(n^{-\frac{1}{2}}\left(\log\frac{\tau}{n}\right)^{-1} + n^{-\frac{1}{2}} + n^{-1}T^{\frac{1}{2}}\right)\right)$$

$$+ O\left(\sum_{n>\tau+2}\left(T^{\frac{1}{2}}|a_n|n^{-c}\left(\log\frac{n}{\tau}\right)^{-1} + |a_n|n^{-c}T^{\frac{1}{2}}\right)\right)$$
$$= 2\sum_{2\leq\tau} a_n + O(T^{\frac{1}{2}+\epsilon}),$$

since for example

$$\sum_{\frac{1}{2}\tau\leq n\leq\tau-2} n^{-\frac{1}{2}}\left(\log\frac{\tau}{n}\right)^{-1} = O\left(T^{-\frac{1}{2}}\sum_{\cdots}\frac{\tau}{\tau-n}\right) = O(T^{\frac{1}{2}+\epsilon}).$$

Thus we have proved

Lemma 2. *Let $c = 1 + L^{-1}$ and $F_0(s) = (\zeta'(s))^2(\zeta(s))^{-1}$ or $\zeta'(s)$ of $\zeta(s)$ and let a_n dbe defined accordingly. Then, we have,* 130

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\chi(1-s)F_0(s)ds = 2\sum_{n\leq\frac{T}{2\pi}} a_n + O(T^{\frac{1}{2}+\epsilon}). \tag{5.4.12}$$

We now prove

Lemma 3. *Under the conditions of Lemma 2, we have,*

$$Re\ I_3 = Re\left(\frac{1}{2\pi i}\int_{c+i}^{c+iT}\chi(1-s)F(s)ds\right) = \frac{T}{4\pi}(\log T)^2 + O(T\log T), \tag{5.4.13}$$

$$Re\left(\frac{1}{2\pi i}\int_{c+i}^{c+iu}\chi(1-s)\zeta'(s)ds\right) = -\frac{u}{2\pi}\log u + O(u), \tag{5.4.14}$$

and

$$Re\left(\frac{1}{2\pi i}\int_{c+i}^{c+iu}\chi(1-s)\zeta(s)ds\right) = \frac{u}{2\pi} + O(u^{\frac{1}{2}+\epsilon}), \tag{5.4.15}$$

where in the last two assertions $1 \leq u \leq T$.

Proof. Let us denote the integrand in (5.4.13) by $G(s)$. In ($|t| \leq 1, \sigma = c$) we have $G(s) = O(L^3)$ and so we can include this in the error term. Next

$$\frac{1}{2\pi i}\int_{c-iT}^{c-i}G(c+it)ds = \frac{1}{2\pi}\int_{-T}^{-1}G(c+it)dt = \frac{1}{2\pi}\int_{1}^{T}G(c-it)dt$$

which is the complex conjugate of

$$\frac{1}{2\pi i}\int_{c+i}^{c+iT} G(s)ds.$$

Hence

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} G(s)ds = 2\, Re\left(\frac{1}{2\pi i}\int_{c+i}^{c+iT} G(s)ds\right) + O((\log T)^3).$$

This proves the first part of the lemma since by prime number theorem

$$\sum_{n\le\tau} a_n = \frac{T}{4\pi}(\log T)^2 + O(T\log T).$$

131 The other two parts follow since while moving the line of integration from $\sigma = c$ to $\sigma = 1 + (\log(2u))^{-1}$, $u \geq 1$ we have the contribution $O(u^{\frac{1}{2}+\epsilon})$ from the horizontal sides.

We have to treat $Re\, I_1$ and $Re\, I_2$. We use $\chi'\chi^{-1} = -\log\tau + O(t^{-1})$ for $t \geq 1$, $\sigma = c$. Since the O-term contributes a small quantity we may replace $\chi'\chi^{-1}$ by $-\log\tau$ and $(\chi'\chi^{-1})^2$ by $(\log\tau)^2$. Now

$$\begin{aligned}
Re\, I_2 &= \frac{1}{2\pi} Re\int_{1,(\sigma=c)}^{T}\chi(1-s)\left(-\log\frac{t}{2\pi}\right)\zeta' dt\\
&= \int_1^T\left(-\log\frac{u}{2\pi}\right) d\, Re\, K_2(u), (\text{where } K_2(u) = \frac{1}{2\pi}\int_{1,(\sigma=c)}^{u}\chi(1-s)\zeta' dt)\\
&= -\log\frac{u}{2\pi} Re\, K_2(u)]_1^T + O\left(\int_1^T\frac{1}{u}|Re\, K_2(u)|du\right)\\
&= -\frac{T}{2\pi}(\log T)^2 + O(T\log T).
\end{aligned}$$

Similarly

$$\begin{aligned}
Re\, I_1 &= \frac{1}{2\pi}\int_{1,(\sigma=c)}^{T}\chi(1-s)\left(\log\frac{t}{2\pi}\right)^2\zeta dt\\
&= \int_1^T\left(\log\frac{u}{2\pi}\right)^2 d\, Re\, K_1(u),\ (\text{where } K_1(u) = \frac{1}{2\pi}\int_{1,(\sigma=c)}^{u}\chi(1-s)\zeta dt)
\end{aligned}$$

$$= \left(\log \frac{u}{2\pi}\right)^2 Re\ K_1(u)]_1^T + O\left(\int_1^T \frac{\log}{u} u\, du\right)$$
$$= \frac{T}{2\pi}(\log T)^2 + O(T \log T).$$

Since $-\overline{I}_0 = I_1 - 2I_2 + I_3$ (with an error $O_\epsilon(T^{\frac{1}{2}+\epsilon})$), we have $Re(-I_0) = (\frac{1}{4\pi} - \frac{2}{2\pi} + \frac{1}{2\pi})T(\log T)^2 + O(T \log T)$ and so

$$Re\ I_0 = \frac{T}{4\pi}(\log T)^2 + O(T \log T).$$

This proves all that we wanted to prove. □

Notes at the End of Chapter c5

132 All references except to the book of E.C. Titchmarsh [100] (revised by D.R. Heath-Brown) are postponed to the notes at the end of the chapter.

§ 5.2. The proof of $\frac{1}{2} \leq \theta \leq 1$ given here is a slightly simplified version of the (Hansraj Gupta memorial) lecture given by me at Aligarh during the 57th Annual Conference of the Indian Mathematical Society held during December 1991. The details of this lecture will appear with the title "A new approach to the zeros of $\zeta(s)$" in Mathematics Student (India) [24]. This method itself has been published by myself and R. Balasubramanian in a very much more general (but complicated) form in two papers $X^{[11]}$ and $XI^{[12]}$ with the same title "on the zeros of a class of generalised Dirichlet series". For the simplest proof see [89].

The difficulty of the generalisation mentioned in Remark 2 is the analogue of the upper bound of $|\zeta(s) - \frac{1}{s-1}|$. For the results on general number fields the only method, known is by using the functional equation.

For the result mentioned in Remark 3 due to K. Ramachandra see [83].

The Lemma 5 is due to S. Ramanujan [97] (see also paper number 24 pages 208-209 of his collected papers [98]). Actually it is enough to prove something like $\pi(X) - \pi(X/2) > X(\log X)^{-2}$ for $X = X_\nu(\nu = 1, 2, 3, \ldots)$ such that $X_\nu \to \infty$. This follows from $\prod\limits_{p\leq x}(1 - p^{-1})^{-1} \geq \sum\limits_{n\leq x} n^{-1}$, on taking logarithms on both sides.

Theorem 5.2.2 is nearly proved in the papers $I^{[21]}$ and $II^{[22]}$ of the series "On the zeros of a class of generalised Dirichlet series". The papers $III^{[10]}$, $IV^{[13]}$, $V^{[23]}$, $VI^{[14]}$, $XIV^{[19]}$ and $XV^{[20]}$ of the same series are more involved and deal with refined developments. All these deal with the zeros in $(\sigma \geq \frac{1}{2} - \delta, T \leq t \leq 2T)$ where $\delta(> 0)$ is any constant. The papers $VII^{[24]}$, $VIII^{[15]}$, $IX^{[16]}$, $X^{[11]}$ and $XI^{[12]}$ concentrate on the same problem with $\delta = \delta(T) \to 0$. In fact XI (as also the papers "On the zeros of $\zeta'(s) - a$"$^{[17]}$ and "On the zeros of $\zeta(s) - a$"$^{[18]}$ to appear) deal with
133 the zeros in $(\sigma \geq \frac{1}{2} + \delta, T \leq t \leq 2T)\delta(> 0)$ being a constant, and further

refinements. Amongst the papers just mentioned in this paragraph the papers I, II, V and VII are due to K. Ramachandra. The rest are all due to R. Balasubramanian and K. Ramachandra. It must be mentioned that the paper VII[24] is very general and deals with the zeros of $\sum_{n=1}^{\infty} a_n \lambda_n^{-s}$ with $\frac{1}{x}\sum_{n\leq x} |a_n|^2 \gg \mathrm{Exp}\left(-\frac{c\log x}{\log\log x}\right)$ for some constant $c > 0$ and all $x \geq 100$, in the rectangle $(\sigma \geq \frac{1}{2} - \delta, T \leq t \leq 2T)$ with $\delta = c'(\log\log T)^{-1}$ for some constant $c'(> 0)$ (and further refinements). This paper depends on the localisation of some theorems of J.E. Littlewood and A. Selberg (who dealt with $\zeta(s)$) to very general Dirichlet series due to K. Ramachandra and A. Sankaranarayanan [92] [94]. These results are stated in § 5.3. Another recent paper XVI[6] by K. Ramachandra and A. Sankaranarayanan adds to our knowledge of the zeros of a class of generalised Dirichlet series in $(\sigma \geq \frac{1}{2}+\delta, T \leq t \leq 2T)$, where $\delta(> 0)$ is any constant.

§ 5.4. The reference to the papers of three authors is J.B. Conrey, A. Ghosh and S.M. Gonek, [31]. The result mentioned in Remark 2 below Lemma 1 is proved in S.M. Gonek [34]. The latest improvement of (5.4.1) is due to A. Fujii (see A. Fujii [32]). It runs as follows: There exist real constants $A_1 > 0$, A_2, A_3 such that the difference

$$\sum_{1\leq Im\ \rho\leq T} \zeta'(\rho) - A_1 T(\log T)^2 - A_2 T(\log T) - A_3 T$$

is $O(Te^{-c\sqrt{\log T}})$ where $c > 0$ is an absolute constant. He also proves that if we assume Riemann's hypothesis then the difference is $O(T^{\frac{1}{2}}(\log T)^{\frac{1}{2}})$.

Chapter 6

Some Recent Progress

6.1 Introduction

In this chapter we shall state without proofs some difficult results (and related results) mentioned in the introductory remarks. Most of the references not mentioned are to be found in E.C. Titchmarsh [100]. 134

6.2 Hardy's Theorem and Further Developments

G.H. Hardy was the first to attack the problem of zeros of $\zeta(s)$ on the critical line. Of course the number N of zeros of $\zeta(s)$ in $(0 \leq \sigma \leq 1, 0 \leq t \leq T)$ is given by the Riemann-von Mongoldt formula

$$N = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).$$

Denote by N_0 the number of zeros of $\zeta(s)$ in $(\sigma = \frac{1}{2}, 0 \leq t \leq T)$. (Both N and N_0 are counted with multiplicity). Hardy proved that $N_0 \to \infty$ as $T \to \infty$. He and J.E. Littlewood proved later that $N_0 \gg T$. A. Selberg developed their method further and by using certain mollifiers proved that $N_0 \gg N$. On the other hand C.L. Siegel developed another method to prove $N_0 \gg T$. By a deep variant of this method N. Levinson [54] proved that limit of $N^{-1}N_0 \geq \frac{1}{3}$ as $T \to \infty$. The references to the works of the other authors mentioned above are to be found in Levinson's paper. A. Selberg and D.R. Heath-Brown (independent of eacg other) pursued the method of Levinson and proved that if N_0^* denotes the number of simple zeros of $\zeta(s)$ in $(\sigma = \frac{1}{2}, 0 \leq t \leq T)$ then limit of $N^{-1}N_0^* \geq \frac{1}{3}$ as $T \to \infty$. Next some work in these directions was done by R. Balasubramanian, J.B. Conrey, D.R. Heath-Brown, A. Ghosh and S.M. Gonek and subsequently by J.B. Conrey who refined the method of Levinson and proved that, as $T \to \infty$, limit $N^{-1}N_0^* \geq \frac{2}{5}$. (Ref. J.B. Conrey [29]). In another direction J.B. Conrey improved the previous results of A. Selberg and M. Jutila and proved (with usual notation) that

$$N(\sigma, T) \ll_\epsilon T^{1-(\frac{8}{7}-\epsilon)(\sigma-\frac{1}{2})} \log T.$$

135 In yet another direction J.B. Conrey improved on the previous results of A. Selberg; R. Balasubramanian, J.B. Conrey, D.R. Heath-Brown. He proved that

$$\int_{\frac{1}{2}}^{1} N(\sigma, T) d\sigma \leq (0.0806 + o(1))T.$$

The last three sults of J.B. Conrey mentioned above were announced in (J.B. Conrey [30]). In a completely different direction (namely pair corelation of the zeros of $\zeta(s)$) H.L. Montgomery proved (on RH) that limit of $N^{-1}N_0^* \geq \frac{2}{3}$ as $T \to \infty$, (see H.L. Montgomery [59]).

6.3 Deeper Problems of Mean-Value Theorems on $\sigma = \frac{1}{2}$

Define $E(T)$ by

$$\frac{1}{2\pi} \int_0^T |\zeta\left(\frac{1}{2} + it\right)|^2 dt = \frac{T}{2\pi} \log \frac{T}{2\pi} + (2\gamma - 1)\frac{T}{2\pi} + E(T)$$

where γ is as usual the Euler's constant. Then A.E. Ingham was the first to show that $E(T) = O(T^{\frac{1}{2}+\epsilon})$. This result was improved in a complicated way to E.C. Titchmarsh who proved that $E(T) = O(T^{\frac{5}{12}+\epsilon})$. After a lapse of nearly 45 years R. Balasubramanian [3] took up the problem and building upon the ideas of Titchmarsh and adding his own ideas proved that $E(T) = O(T^{\frac{1}{3}+\epsilon})$. The latest improvement is due to D.R. Heath-Brown and M.N. Huxley namely $E(T) = O(T^{\frac{7}{22}+\epsilon})$. The reference to their paper is (D.R. Heath-Brown and M.N. Huxley [40]). J.L. Hafner and A. Ivic have proved (ref. J.L. Hafner and A. Ivic [36]) some nice $\Omega_{\pm}$ theorems for $E(T)$. Their results read

$$E(T) = \Omega_{+}\left\{(T\log T)^{\frac{1}{4}}(\log\log T)^{\frac{1}{4}(3+\log 4)}\operatorname{Exp}(-c\sqrt{\log\log\log T})\right\}$$

and

$$E(T) = \Omega_{-}\left\{T^{\frac{1}{4}}\operatorname{Exp}\left(\frac{D(\log\log T)^{\frac{1}{4}}}{(\log\log\log T)^{\frac{3}{4}}}\right)\right\},$$

where $c > 0$ and $D > 0$ are constants. Let

$$E_2(T) = \int_0^T |\zeta\left(\frac{1}{2}+it\right)|^4 dt - TP_4(\log T),$$

where $P_4(\log T)$ is a certain polynomial in $\log T$ of degree 4, Then D.R. **136**
Heath-Brown was the first to prove (for a certain explicit $P_4(\log T)$) that $E_2(T) = O(T^{\frac{7}{8}+\epsilon})$. (Ref. [39]). His method also gave the result of R. Balasubramanian mentioned earlier. The final result $E_2(T) = O(T^{\frac{2}{3}+\epsilon})$ which we can expect in the present state of knowledge was proved by N. Zavorotnyi (Ref. [105]) It should be mentioned that T^{ϵ} has been replaced by a constant power of $\log T$ by A. Ivić and Y. Motohashi (Ref. A. Ivić [43]). The result $E_2(T) = \Omega(T^{\frac{1}{2}})$ (recently Motohashi has proved that $E_2(T) = \Omega_{\pm}(T^{\frac{1}{2}})$) of considerable depth is due to A. Ivic and Y. Motohashi (Ref. [44]). The deep result

$$\int_0^T |\zeta\left(\frac{1}{2}+it\right)|^{12} dt \ll T^2(\log T)^{17}$$

was first proved by D.R. Heath-Brown (Ref. [38]). Later on, H. Iwaniec developed another method and proved a result on the mean value of $|\zeta\left(\frac{1}{2}+it\right)|^4$ over short intervals, which gave as a corollary the result of Heath-Brown just mentioned and also

$$\int_T^{T+H} |\zeta\left(\frac{1}{2}+it\right)|^4 dt \ll H^{1+\epsilon}, H = T^{\frac{2}{3}}.$$

Afterwards M. Jutila and Y. Motohashi (independently) gave different methods of approach to this problem of H. Iwaniec. Thus there are at present three different methods of approach to this problem. (Ref. H. Iwaniec [46]; M. Jutila [47]; M. Jutila [49]; M. Jutila, [50]; Y. Motohashi (several papers of which the following is one [61]). Of these methods Jutila's method works very well for hybrid versions to L-functions and so on. However we do not say more on such questions in this monograph. In 1989, N.V. Kuznetsov published (N.V. Kuznetsov [52]) a proof of

$$\int_0^T |\zeta\left(\frac{1}{2}+it\right)|^8 dt \ll T(\log T)^{16+B},$$

where $B > 4$ is a certain constant. However his proof appears to contain many serious errors. (Professor Y. Motohashi of Japan is trying to
137 correct the mistakes and the result that

$$\int_0^T |\zeta\left(\frac{1}{2}+it\right)|^8 dt \ll T^{\frac{4}{3}+\epsilon}$$

valid for every fixed $\epsilon > 0$ which Motohashi hopes to obtain should be called Kuznetsov-Motohashi theorem if at all Motohashi succeeds in proving it. If however Motohashi succeeds in proving

$$\int_0^T |\zeta\left(\frac{1}{2}+it\right)|^6 dt \ll T^{1+\epsilon}$$

the full credit of such a discovery should go to Motohashi). Before leaving this section it is appropriate that the following two results should be mentioned (and as is common with all the results of this chapter we do not prove them). Of course they have a place in Chapter 4 and we have mentioned it there and we do not prove them. The first is the result

$$(\log T)^{k^2} \ll \frac{1}{T}\int_0^T |\zeta\left(\frac{1}{2}+it\right)|^{2k} dt \ll (\log T)^{k^2}$$

uniformly for all $k = \frac{1}{n}$($n \geq 1$ integer) due to M. Jutila. (Ref. M. Jutila, [48]). This has application to large values of $|\zeta(\frac{1}{2}+it)|$. Another result is due to A. Ivic and A. Perelli. They have proved that if k is any real number with $0 \leq k \leq (\psi(T) \log\log T)^{-\frac{1}{2}}$ where $\psi(T) \to \infty$ as $T \to \infty$, we have,

$$\frac{1}{T}\int_0^T |\zeta\left(\frac{1}{2}+it\right)|^{2k} dt \to 1.$$

(Ref. A. Ivic and A. Perelli, [45]).

6.4 Deeper Problems on Mean-Value Theorems in $\frac{1}{2} < \sigma < 1$

For fixed $\sigma \left(\frac{1}{2} < \sigma < 1\right)$ define

$$E(\sigma, T) = \int_0^T |\zeta(\sigma+it)|^2 dt - \zeta(2\sigma)T - \frac{\zeta(2\sigma-1)\Gamma(2\sigma-1)}{1-\sigma} \operatorname{Sin}(\pi\sigma)T^{2-2\sigma}.$$

This definition is due to A. Ivic (Note that $\lim E(\sigma, T) = E(T)$ as $\sigma \to \frac{1}{2}+0$). K. Matsumoto defines in a slightly different way. K. Matsumoto was the first to use the method of R. Balasubramanian and he proved

$$E(\sigma, T) \ll T^{1/(4\sigma+1)+\epsilon}, \left(\frac{1}{2} < \sigma < \frac{3}{4}\right).$$

(Ref. [55] K. Matsumoto. However in this paper he uses a slightly different method namely the one which uses Atkinson's formula). A. Ivic ([43] p. 90), has shown that $E(\sigma, T) \ll T^{1-\sigma}$. K. Matsumoto has proved (in the paper cited above, see also the A. Ivić [43]) that 138

$$\int_0^T (E(\sigma, T))^2 dt = C(\sigma)T^{\frac{5}{2}-2\sigma} + O(T^{\frac{7}{4}-\sigma}), \left(\frac{1}{2} < \sigma < \frac{3}{4}\right),$$

(where $C(\sigma) > 0$), which implies $E(\sigma, T) = \Omega(T^{\frac{3}{4}-\sigma})$. There are many other interesting results given in A. Ivić [43] mentioned above and the interested reader is referred to this LN. However we have to mention a result of S.W. Graham which seems to have missed the attention of many mathematicians. Let $q \geq 1$ be an integer, $R_q = 2^{q+2} - 2$, and

$\sigma_q = 1-(q+2)R_q^{-1}$. Then his result reads

$$\int_0^T |\zeta(\sigma_q+it)|^{14R_q}dt \ll T^{14}(\log T)^{A(q)},$$

where $A(q)>0$ is a certain constant depending only on q. In particular when $q=2$, we have,

$$\int_0^T |\zeta\left(\frac{5}{7}+it\right)|^{196}dt \ll T^{14}(\log T)^{425}.$$

(Note that this implies that $\mu(\frac{5}{7}) \leq \frac{1}{14}$ and also $\mu(\sigma_q) \leq R_q^{-1}$. These results are close to Theorems 5.12 and 5.13 of E.C. Titchmarsh [100]). Reference to these results is (S.W. Graham, [35]).

6.5 On the Line $\sigma = 1$

Let k be any complex constant, $(\zeta(s))^k = \sum\limits_{n=1}^{\infty} d_k(n)n^{-s}$ when $Re\ s > 2$. Put

$$E(k,1,T) = \int_1^T |(\zeta(1+it))^{2k}|dt - T\sum_{n=1}^{\infty}|d_k(n)|^2n^{-2}.$$

The function $E(1,1,T)$ was studied in great detail in (R. Balasubramanian, A. Ivic and K. Ramachandra [4]). One of the results proved in this paper is

$$E(1,1,T) = -\pi\log T + O\left((\log T)^{\frac{2}{3}}(\log\log T)^{\frac{1}{3}}\right).$$

139 It follows that $E(1,1,T) = \Omega_-(\log T)$. It is also $O(\log T)$. In another paper (R. Balasubramanian, A. Ivic and K. Ramachandra [5]) they have proved many results. A sample result is

$$E(k,1,T) = O\left((\log T)^{|k^2|}\right).$$

Finally we mention a result on the large value of $|\log\zeta(1+it)|$ (K. Ramachandra [88]). The result is this. Let $\epsilon(0<\epsilon<1)$ be any constant, $T \geq 10000$, $X = \mathrm{Exp}\left(\frac{\log\log T}{\log\log\log T}\right)$. Consider the set of points t for which $T \leq t \leq T+e^X$ and $|\log\zeta(1+it)| \geq \epsilon\log\log T$. Then this set is contained in $O_\epsilon(1)$ intervals of length $\frac{1}{X}$.

Chapter 7

Appendix

7.1 Introduction

140

In this chapter we prove some well-known results and usually references will not be given. We prove the functional equation of $\zeta(s)$, the asymptotics of $\Gamma(s)$ and that of $\sum\limits_{n\le x}|d_k(n)|^2$ (k-complex constant) and make some remarks about some useful kernel functions.

7.2 A Fourier Expansion

Let $y > 0$, v a real variable in $(-\infty, \infty)$ and $f(v) = \sum\limits_{n=-\infty}^{\infty} \operatorname{Exp}(-\pi(v + n)^2 y)$. Clearly $f(v)$ is a periodic function whose Fourier series represents the function since $f(v)$ is continuously differentiable. Let $f(v) = \sum\limits_{n=-\infty}^{\infty} a_n \operatorname{Exp}(2\pi i n v)$. Then

$$\begin{aligned} a_n &= \int_0^1 f(v) \operatorname{Exp}(-2\pi i n v) dv \\ &= \sum_{m=-\infty}^{\infty} \int_0^1 \operatorname{Exp}(-\pi(v+m)^2 y - 2\pi i n v) dv \end{aligned}$$

$$= \sum_{m=-\infty}^{\infty} \int_m^{m+1} \text{Exp}(-\pi v^2 y - 2\pi i n v) dv$$
$$= \int_{-\infty}^{\infty} \text{Exp}(-\pi v^2 y - 2\pi i n v) dv$$
$$= \int_{-\infty}^{\infty} \text{Exp}\left(-\pi \left(v + \frac{in}{y}\right)^2 y - \frac{\pi n^2}{y}\right) dv$$
$$= \text{Exp}\left(-\frac{\pi n^2}{y}\right) \int_{-\infty}^{\infty} \text{Exp}\left(-\pi y \left(v + \frac{in}{y}\right)^2\right) dv$$

Lemma. *We have,*

$$\int_{-\infty}^{\infty} \text{Exp}\left(-\pi y \left(v + \frac{in}{y}\right)^2\right) dv = \int_{-\infty}^{\infty} \text{Exp}(-\pi v^2 y) dv.$$

Proof. Integrate $\text{Exp}(-\pi z^2 y)$ over the rectangle obtained by joining $-R$, R, $R + \frac{in}{y}$, $-R + \frac{in}{y}$, $-R$ by straight line segments in this order. We have trivially

$$\int_0^{ny^{-1}} \text{Exp}(-\pi(\pm R + iu)^2 y) du \to 0$$

141 as $R \to \infty$, since the absolute value of the integrand is $\leq \text{Exp}((-R^2 y + \frac{n^2}{y})\pi)$. This proves the lemma. $\square$

Thus we can state

Theorem 7.2.1. *We have, for $y > 0$ and real v,*

$$\sum_{n=-\infty}^{\infty} \text{Exp}(-\pi y(n+v)^2) = \left(\int_{-\infty}^{\infty} \text{Exp}(-\pi v^2 y) dv\right) \sum_{n=-\infty}^{\infty} \text{Exp}\left(-\frac{\pi n^2}{y} + 2\pi i n v\right).$$

As a corollary we state

Theorem 7.2.2. *We have, for $y > 0$,*

$$1 + 2\sum_{n=1}^{\infty} \text{Exp}(-\pi n^2 y) = y^{-\frac{1}{2}} \left(1 + 2\sum_{n=1}^{\infty} \text{Exp}\left(-\frac{\pi n^2}{y}\right)\right), \tag{7.2.1}$$

and

$$\int_{-\infty}^{\infty} \text{Exp}(-\pi v^2) dv = 1. \tag{7.2.2}$$

Proof. Putting $v = 0$ and $y = 1$ in Theorem 7.2.1 we obtain

$$\int_{-\infty}^{\infty} \text{Exp}(-\pi v^2) dv = 1$$

and so

$$\int_{-\infty}^{\infty} \text{Exp}(-\pi v^2 y) dv = y^{-\frac{1}{2}} \int_{-\infty}^{\infty} \text{Exp}(-\pi v^2) dv = y^{-\frac{1}{2}}.$$

Putting $v = 0$ in Theorem 7.2.1 we obtain (7.2.1). □

7.3 Functional Equation

We first introduce as usual

$$\Gamma(s) = \int_0^{\infty} \text{Exp}(-v) v^{s-1} dv, (s = \sigma + it, \sigma > 0). \tag{7.3.1}$$

The analytic continuation is provided by the functional equation

$$\Gamma(s+1) = s\Gamma(s) \quad \text{and so} \quad \Gamma(n+1) = n! \tag{7.3.2}$$

which is obtained on integration of (7.3.1) by parts. We now write 142
(somewhat artificially) for $\sigma > 1$

$$\begin{aligned}
\pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s) &= \sum_{n=1}^{\infty} \int_0^{\infty} (n^2 \pi)^{-\frac{s}{2}} \text{Exp}(-v) v^{\frac{s}{2}} \frac{dv}{v} \\
&= \sum_{n=1}^{\infty} \int_0^{\infty} \text{Exp}(-n^2 \pi v) v^{\frac{s}{2}} \frac{dv}{v} \\
&= \int_0^{\infty} \phi(v) v^{\frac{s}{2}} \frac{dv}{v},
\end{aligned}$$

where $\phi(v) = \sum\limits_{n=1}^{\infty} \text{Exp}(-n^2 \pi v)$. Now by Theorem 7.2.2, we have,

$$1 + 2\phi(v) = v^{-\frac{1}{2}} \left(1 + 2\phi\left(\frac{1}{v}\right)\right).$$

Hence for $s = \sigma + it$, $\sigma > 1$, we have

$$\begin{aligned}
\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) &= \int_1^\infty \phi(v)v^{\frac{s}{2}}\frac{dv}{v} + \frac{1}{2}\int_0^1 (v^{-\frac{1}{2}} - 1)v^{\frac{s}{2}}\frac{dv}{v} \\
&\quad + \int_0^1 v^{-\frac{1}{2}}\phi\left(\frac{1}{v}\right)v^{\frac{s}{2}}\frac{dv}{v} \\
&= \int_1^\infty \phi(v)v^{\frac{s}{2}}\frac{dv}{v} + \frac{1}{2}\left(\frac{2}{s-1} - \frac{2}{s}\right) \\
&\quad + \int_1^\infty v^{\frac{1}{2}}\phi(v)v^{-\frac{s}{2}}\frac{dv}{v} \\
&= -\frac{1}{s(1-s)} + \int_1^\infty \phi(v)\left(v^{\frac{s}{2}} + v^{\frac{1-s}{2}}\right)\frac{dv}{v}.
\end{aligned}$$

This proves the following theorem.

Theorem 7.3.1. *For $v \geq 1$, let $\phi(v) = \sum\limits_{n=1}^{\infty} \operatorname{Exp}(-n^2\pi v)$. Then for $\sigma > 1$, we have,*

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = -\frac{1}{s(1-s)} + \int_1^\infty \phi(v)\left(v^{\frac{s}{2}} + v^{\frac{1-s}{2}}\right)\frac{dv}{v}. \tag{7.3.3}$$

Plainly the last equation is true for all complex s by analytic continuation. Since the RHS is symmetric in s and $1-s$, we have the functional equation namely that LHS is unchanged under the transformation $s \to 1 - s$.

7.4 Asymptotics of $\Gamma(s)$

143 For many important purposes it is necessary to know the behaviour of $\Gamma(s)$ as $|s| \to \infty$. It is also important to know its poles (and zeros if any). For real $s > 0$ we define

$$\Gamma(s) = \lim_{n\to\infty} \Gamma_n(s) \text{ where } \Gamma_n(s) = \int_0^n \operatorname{Exp}(-v)v^{s-1}dv.$$

Now $\operatorname{Exp}(-v) = (\operatorname{Exp}(-\frac{v}{n}))^n \geq (1 - \frac{v}{n})^n$ since for $0 \leq v \leq n$ we have $\log(1 - \frac{v}{n}) \leq -\frac{v}{n}$. Hence

$$\Gamma_n(s) \geq \int_0^n \left(1 - \frac{v}{n}\right)^n v^{s-1}dv = I_n(s) \text{ say.}$$

Again

$$0 \leq \Gamma_n(s) - I_n(s) \leq \int\limits_0^n \left(\mathrm{Exp}(-v) - \left(1 - \frac{v}{n}\right)^n\right) v^{s-1} dv$$

$$\leq \frac{e}{n}\int_0^n v^{s+2}\,\mathrm{Exp}(-v)\frac{dv}{v} \leq \frac{e}{n}\int_0^\infty v^{s+2}\,\mathrm{Exp}(-v)\frac{dv}{v} = \frac{e}{n}\Gamma(s+2),$$

on using

$$\begin{aligned}
&\mathrm{Exp}(-v) - \left(1 - \frac{v}{n}\right)^n = \left(\mathrm{Exp}\left(-\frac{v}{n}\right)\right)^n - \left(1 - \frac{v}{n}\right)^n \\
&= \left(\mathrm{Exp}\left(-\frac{v}{n}\right) - \left(1 - \frac{v}{n}\right)\right)\sum_{\nu=0}^{n-1}\left(\mathrm{Exp}\left(\frac{v}{n}\right)\right)^\nu \left(1 - \frac{v}{n}\right)^{n-\nu-1} \\
&\leq \frac{v^2}{n^2}\cdot n \cdot \mathrm{Exp}\left(-\frac{v}{n}(n-1)\right) \\
&= \frac{ev^2}{n}\,\mathrm{Exp}(-v).
\end{aligned}$$

Hence, as $n \to \infty$,

$$\Gamma(s) = \lim \Gamma_n(s) = \lim I_n(s) = \lim \int_0^n \left(1 - \frac{v}{n}\right)^n v^s \frac{dv}{v}$$

provided $s > 0$. We now determine the last limit. Plainly

$$I_n(s) = n^s \int_0^1 (1-v)^n v^{s-1} dv = n^s J_n(s) \text{ say.}$$

Integrating by parts, we have (for $s > 0$ and $n \geq 1$), **144**

$$\begin{aligned}
J_n(s) &= \frac{v^s}{s}(1-v)^n]_0^1 + n\int_0^1 \frac{v^s}{s}(1-v)^{n-1} dv \\
&= \frac{n}{s} J_{n-1}(s+1)
\end{aligned}$$

$$= \frac{n}{s}\cdot\frac{n-1}{s+1} J_{n-2}(s+2) = \frac{n}{2}\cdot\frac{n-1}{s+1}\dots\frac{n-r}{s+r} J_{n-r-1}(s+r+1)$$

for $n - r - 1 \geq 0$. Putting $r = n - 1$ we obtain

$$J_n(s) = \frac{n!}{s(s+1)\dots(s+n-1)} \cdot \frac{1}{s+n}$$

Hence

$$\begin{aligned}
\frac{1}{s\Gamma(s)} &= \lim_{n\to\infty}\left(n^{-s}\prod_{\nu=1}^{n}\left(1+\frac{s}{\nu}\right)\right\} \\
&= \lim_{n\to\infty}\left(\prod_{\nu=1}^{n}((\mathrm{Exp}(-\frac{s}{\nu}))(1+\frac{s}{\nu}))\right\} \times \lim_{n\to\infty}\mathrm{Exp}\left(s\sum_{\nu=1}^{n}\frac{1}{\nu} - s\log n\right) \\
&= e^{\gamma^s}\prod_{\nu=1}^{\infty}\left\{(1+\frac{s}{\nu})\,\mathrm{Exp}(-\frac{s}{\nu})\right\}
\end{aligned}$$

since $\lim_{n\to\infty}\left(\sum_{\nu=1}^{n}\frac{1}{\nu} - \log n\right) = \gamma$ the Euler's constant. This holds for real $s > 0$ and by analytic continuation for all complex s. Hence we state

Theorem 7.4.1. *We have, for all complex s,*

$$\frac{1}{\Gamma(s)} = se^{\gamma^s}\prod_{\nu=1}^{\infty}\left(\left(1+\frac{s}{\nu}\right)\mathrm{Exp}\left(-\frac{s}{\nu}\right)\right\} \tag{7.4.1}$$

where γ is the well-known Euler's constant.

Since $\frac{\sin\theta}{\theta} = \prod_{n=1}^{\infty}\left(1 - \frac{\theta^2}{n^2\pi^2}\right)$ for all θ we have the following

Corollary 3. *We have*

$$\frac{1}{\Gamma(1+s)\Gamma(1-s)} = \frac{\sin(s\pi)}{s\pi} \quad \textit{and so} \quad \Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(s\pi)}.$$

145 **Corollary 4.** *The function $(\Gamma(s))^{-1}$ is entire. It has simple zeros at $s = 0$, $-1, -2, \dots$ and no other zeros. The residue of $\Gamma(s)$ at $s = n - n$ is $(-1)^n(n!)^{-1}$ as is easily seen by*

$$\Gamma(s) = \int_0^1 \mathrm{Exp}(-v)v^{s-1}dv + \int_1^\infty \mathrm{Exp}(-v)v^{s-1}dv$$

and $\operatorname{Exp}(-v) = \sum\limits_{n=0}^{\infty} \dfrac{(-v)^n}{n!}$.

Corollary 5. *The function $\zeta(s)$ has simple zeros at $s = -2, -4, -6, \ldots$. If has no other zeros in $\sigma > 1$ and also in $\sigma < 0$. The function $\zeta(s) - (s-1)^{-1}$ is entire. At $s = 0, -1 - 3, \ldots, \zeta(s)$ can be expressed in terms of Bernoulli numbers. Hence, for $n = 1, 2, 3, \ldots, \zeta(2n) = \pi^{2n}$ times a rational number.*

Remark. It is easy to prove (though it took a long time in the history of mathematics to prove this) that $\zeta(it) \neq 0$ and $\zeta(1+it) \neq 0$ for all t. But it is not known whether there exists a sequence of zeros with real parts tending to 1. This is likely to remain unsolved for a long time to come.

Proof of Corollary 5. The proof follows by the functional equation and the Euler product. We may use the obvious formula

$$\Gamma(s)\zeta(s) = \int_0^{\infty} \left(\frac{v}{e^v - 1}\right) v^{s-2} dv$$

and integrate it by parts (several times) to prove the statement of the corollary regarding the assertion about $s = 0, -1, -2, \ldots$. In passing we remark that we can also consider $\zeta(s,a) = \sum\limits_{n=0}^{\infty} (n+a)^{-s}$ (where a is a constant with $0 < a \leq 1$) at $s = 0, -1, -2, \ldots$. It may also be remarked that (using the functinal equation for $\zeta(s,a)$) $\dfrac{d}{ds}\zeta(s,a)]_{s=0} = \log\Gamma(a) - \frac{1}{2}\log(2\pi)$.

We now resume the asymptotics of $\Gamma(s)$. We begin with the remark that $\log\Gamma(s)$ is analytic in the complex plane with the straight line $(-\infty, 0]$ removed. So it suffices to study an asymptotic expansion for real $s > 0$, provided we arrive at an expansion which is analytic in the complex plane with the straight line $(-\infty, 0]$ removed. By Theorem 146
7.4.1, we have, for $s > 0$,

$$\log\Gamma(s) = -\log s - \gamma s - \sum_{\nu=1}^{\infty} \left(\log\left(1 + \frac{s}{\nu}\right) - \frac{s}{\nu}\right).$$

Hence

$$\frac{\Gamma'(s)}{\Gamma(s)} = -\frac{1}{s} - \gamma - \sum_{\nu=1}^{\infty} \left(\frac{1}{s+\nu} - \frac{1}{\nu}\right)$$

and

$$\frac{d}{ds}\frac{\Gamma'(s)}{\Gamma(s)} = \sum_{\nu=0}^{\infty}\frac{1}{(s+\nu)^2}. \tag{7.4.2.}$$

Notice that

$$\sum_{\nu=0}^{\infty}\frac{1}{(s+\nu)^2} = \int_0^{\infty}\frac{du}{(s+u)^2} + \sum_{\nu=0}^{\infty}\left(\frac{1}{(s+\nu)^2} - \int_{\nu}^{\nu+1}\frac{du}{(s+u)^2}\right)$$
$$= \frac{1}{s} + \int_0^1 \sum_{\nu=0}^{\infty}\left(\frac{1}{(s+\nu)^2} - \frac{1}{(s+\nu+u)^2}\right)du$$

and that the integrand (in the last integral) is

$$O\left(\sum_{\nu=0}^{\infty}\frac{1}{(s+\nu)^3}\right) = O\left(\sum_{\nu\le s}\frac{1}{(s+\nu)^3} + \sum_{\nu>s}\frac{1}{(s+\nu)^3}\right)$$
$$= O(s^{-2}).$$

Continuing this process we are led to

$$\sum_{\nu=0}^{\infty}\frac{1}{(s+\nu)^2} = \frac{c_1}{s} + \frac{c_2}{s^2} + \frac{c_3}{s^3} + \cdots + \frac{c_N}{s^N} + A(N,s), \tag{7.4.3}$$

where $c_1, c_2, \ldots, c_N$ are certain constants, $N \geq 1$ arbitrary and $A(N,s)$ is analytic in the complex plane with the straight line $(-\infty, 0]$ removed. Also it is easy to prove that for complex s in $|\arg s| \leq \pi - \delta(\delta > 0$ being a fixed constant) we have

$$A(N,s) = O(|s|^{-N-1}), \tag{7.4.4}$$

147 where the O-constant depends only on N and δ. Integrating (7.4.3) twice we obtain

$$\frac{\Gamma'(s)}{\Gamma(s)} = c_0 + c_1\log s + \frac{c_2}{s} + \frac{c_3}{s^2} + \cdots + \frac{c_N}{s^N} + A^*(N,s), \tag{7.4.5}$$

where $A^*(N,s)$ has the same property as (7.4.4) with N replaced by $N-1$. Integrating (7.4.5) again, we obtain

$$\log\Gamma(s) = d_1 s\log s + d_2 s + d_3\log s + d_4 + \sum_{\nu=1}^{N}\frac{d_{-\nu}}{s^{\nu}} + B(N,s), \tag{7.4.6}$$

where d_1, d_2, d_3, d_4 and $d_{-\nu}(\nu = 1$ to $N)$ are constants, and $B(N, s)$ satisfies the condition similar to (7.4.4). Now we use $\log\Gamma(n+1) - \log\Gamma(n) = \log n$ to determine d_1, d_2 and d_3 as follows. We have

$$\begin{aligned}\log n &= d_1((n+1)\log(n+1) - n\log n) + d_2 + d_3 \\ &\qquad (\log(n+1) - \log n) + O\left(\frac{1}{n^2}\right) \\ &= d_1((n+1)\left(\log n + \frac{1}{n} - \frac{1}{2n^2}\right) - n\log n) + d_2 + \frac{d_3}{n} + O\left(\frac{1}{n^2}\right) \\ &= d_1 \log n + d_1(n+1)\left(\frac{1}{n} - \frac{1}{2n^2}\right) + d_2 + \frac{d_3}{n} + O\left(\frac{1}{n^2}\right)\end{aligned}$$

This gives on dividing by $\log n$ and letting $n \to \infty$, that $d_1 = 1$ and so

$$(n+1)\left(\frac{1}{n} - \frac{1}{2n^2}\right) + d_2 + \frac{d_3}{n} = O\left(\frac{1}{n^2}\right).$$

Here $LHS = 1 + d_2 + \frac{1}{n} - \frac{1}{2n} + \frac{d_3}{n} = O(\frac{1}{n^2})$ and so $d_2 = -1$, and $d_3 = -\frac{1}{2}$. Thus

$$\log\Gamma(s) = \left(s - \frac{1}{2}\right)\log s - s + d_4 + \frac{d_{-1}}{s} + \frac{d_{-2}}{s^2} + \cdots + \frac{d_{-N}}{s^N} + B(N, s). \quad (7.4.7)$$

To determine d_4 we use

$$\Gamma\left(n + \frac{1}{2}\right) = \left(n - \frac{1}{2}\right)\ldots\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\Gamma(2n)\Gamma(\frac{1}{2})}{2^n(2n-2)(2n-4)\ldots 2}.$$

i.e.

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{\Gamma(2n)\Gamma(\frac{1}{2})}{2^{2n-1}\Gamma(n)}.$$

Hence d_4 is determined by

$$\begin{aligned}&n\log\left(n + \frac{1}{2}\right) - \left(n + \frac{1}{2}\right) + d_4 + O\left(\frac{1}{n}\right) \\ &= \left(2n - \frac{1}{2}\right)\log(2n) - 2n + \log\Gamma\left(\frac{1}{2}\right) - (2n-1)\end{aligned}$$

$$\log 2 - \left(n - \frac{1}{2}\right)\log n + n + O\left(\frac{1}{n}\right)$$

148 i.e. by

$$n\log n + n\log\left(1 + \frac{1}{2n}\right) - \left(n + \frac{1}{2}\right) + d_4$$
$$= \left(2n - \frac{1}{2}\right)(\log 2 + \log n) - 2n + \log\Gamma\left(\frac{1}{2}\right) - (2n - 1)$$
$$\log 2 - \left(n - \frac{1}{2}\right)\log n + n + O\left(\frac{1}{n}\right)$$

i.e. by

$$n\log\left(1 + \frac{1}{2n}\right) - n - \frac{1}{2} + d_4$$
$$= 2n\log 2 - \frac{1}{2}\log 2 - 2n + \log\Gamma\left(\frac{1}{2}\right) - (2n - 1)\log 2 + n + O\left(\frac{1}{n}\right)$$

i.e. by

$$n\left(\frac{1}{2n}\right) + O\left(\frac{1}{n}\right) - \frac{1}{2} + d_4 = \frac{1}{2}\log 2 + \log\Gamma\left(\frac{1}{2}\right) + O\left(\frac{1}{n}\right),$$

i.e. by

$$d_4 = \frac{1}{2}\log 2 + \frac{1}{2}\log\pi = \log(\sqrt{2\pi}).$$

Thus we have,

Theorem 7.4.2. *We have, in* $|s| \geq 1$, $|\arg s| \leq \pi - \delta$, *where* $\delta(> 0)$ *is a constant, the expansion*

$$\log\Gamma(s) = \frac{1}{2}\log(2\pi) + \left(s - \frac{1}{2}\right)\log s - s + \frac{d_{-1}}{s} + \frac{d_{-2}}{s^2} + \cdots + \frac{d_{-N}}{s^N} + B(N, s) \tag{7.4.8}$$

where $N \geq 1$, *and* $d_{-\nu}(\nu = 1$ *to* $N)$ *are constants,* $B(N, s)$ *is analytic in the said region and further as* $|s| \to \infty$ *we have*

$$B(N, s) = O(|s|^{-N-1}), \tag{7.4.9}$$

where the O-constant depends only on δ *and* N.

Remark. The constants d_{-1}, $d_{-2}, \ldots, d_{-N}$ are rational and can be expressed in terms of Bernoulli numbers. We do not work out these relations. **149**

Corollary 1. *In the same region as mentioned in the theorem, we have,*

$$\Gamma(s) = \sqrt{2\pi} s^{s-\frac{1}{2}} e^{-s} \left(1 + O\left(\frac{1}{|s|}\right)\right).$$

as $|s| \to \infty$.

Corollary 2. *If* $a \leq \sigma \leq b$ *and* $t \geq 1$, *we have,*

$$\Gamma(\sigma + it) = \sqrt{2\pi} t^{\sigma+it-\frac{1}{2}} e^{-\frac{1}{2}\pi t - it + \frac{1}{2} i\pi(\sigma - \frac{1}{2})} \left(1 + O\left(\frac{1}{t}\right)\right)$$

and hence if $\zeta(s) = \chi(s)\zeta(1-s)$, *then we have,*

$$\chi(s) = \left(\frac{2\pi}{t}\right)^{\sigma+it-\frac{1}{2}} e^{i(t+\frac{\pi}{4})} \left(1 + O\left(\frac{1}{t}\right)\right)$$

and

$$\frac{\chi'(s)}{\chi(s)} = -\log\left(\frac{t}{2\pi}\right) + O\left(\frac{1}{t}\right).$$

Proof. From the functional equation for $\zeta(s)$ we obtain with slight work the formula

$$\chi(s) = \frac{1}{2}(2\pi)^s \sec\left(\frac{1}{2} s\pi\right) (\Gamma(s))^{-1}$$

using this and Theorem 7.4.2 the corollary follows. ◻

7.5 Estimate for $\dfrac{\zeta'(s)}{\zeta(s)}$ on Certain Lines in the Critical Strip

Lemma 4. *Let $t_0 \geq 1000$, $s_0 = 2 + it_0$ and let ρ run over the zeros of $\zeta(s)$ satisfying $|\rho - s_0| \leq 3$. Then in the disc $|s - s_0| \leq 3 - \frac{1}{50}$, we have,*

$$\left|\frac{\zeta'(s)}{\zeta(s)} - \sum_{\rho} \frac{1}{s-\rho}\right| \leq 10^{10} \log t_0.$$

Remark. We have prefered here to write a big constant 10^{10} in place of
150 $O(\cdot)$. These constants are unimportant for our purposes.

Proof. Consider the function

$$F(s) = \zeta(s) \prod_{|\rho - s_0| \leq 3} \left(1 - \frac{s - s_0}{\rho - s_0}\right)^{-1}.$$

It is analytic in $|s - s_0| \leq 9$ and on its boundary $|F(s)|$ is clearly $\leq t_0^{10}$. By maximum modulus principle $|F(s)| \leq t_0^{10}$ in $|s - s_0| \leq 3$. Plainly it is analytic in this disc and is free from zeros. Hence in this disc $Re\ \log F(s) \leq 10 \log t_0$. Hence by Borel-Caratheodory theorem (see Theorem 1.6.1) we see that in $|s - s_0| \leq 3 - \frac{1}{100}$, we have

$$|\log F(s)| \leq 10^6 \log t_0,$$

and so by Cauchy's theorem we have in $|s - s_0| \leq 3 - \frac{1}{50}$,

$$\left|\frac{F'(s)}{F(s)}\right| \leq 10^{10} \log t_0.$$

This is precisely the statement of the lemma. □

Lemma 5. *The number of zeros of $\zeta(s)$ in $\sigma \geq -\frac{1}{25}$, $|t - t_0| \leq \frac{1}{1000}$, is $O(\log t_0)$.*

Proof. This lemma can be proved by maximum modulus principle. But this lemma is also a consequence of the following theorem which is useful in many investigations. □

Jensen's Theorem 7.5.1. *Let $f(z)$ be analytic in $|z| \leq R$ and $f(0) \neq 0$. Let $n(r)$ denote the number of zeros of $f(z)$ in $|z| \leq r(\leq R)$. Then*

$$\int_0^R \frac{n(r)}{r}dr = \frac{1}{2\pi i}\int_{|z|=R} \log|f(z)|\frac{dz}{z} - \log|f(0)|.$$

Remark. Our proof shows that if $f(z)$ is meromorphic in $|z| \leq R$ and $f(0) \neq 0$ and $|f(0)| \neq \infty$, and if $n(r)$ is the number of zeros minus the number of poles in $|z| \leq r(\leq R)$ then the same result holds. **151**

Proof. (i) If $f(z)$ has no zeros in $|z| \leq R$ then $\log f(z)$ is analytic and so

$$\log f(0) = \frac{1}{2\pi i}\int_{|z|=R} \log f(z)\frac{dz}{z}.$$

Taking real parts both sides the theorem follows.

(ii) Now if there is a zero z_0 on $|z| = R$ then at a distance δ from this zero $\log|f(z)| = O(\log\frac{1}{\delta})$ and since $\delta\log\frac{1}{\delta} \to 0$ as $\delta \to 0$ we are through in this case.

(iii) Now suppose that $f(z)$ has zeros in $|z| \leq R$, but $f(0) \neq 0$. Put

$$F(z) = f(z)\prod_a \frac{R^2 - \overline{a}z}{R(z-a)}$$

where the product is over all zeros a of $f(z)$ in $|z| < R$. Then since $F(z)$ has no zeros in $0 < |z| < R$ we, have,

$$\log f(0) + \sum_a \log\frac{R}{|a|} = \frac{1}{2\pi i}\int_{|z|=R} \log|f(z)|\frac{dz}{z}$$

since on $|z| = R$ we have

$$|\frac{R^2 - \overline{a}z}{R(z-a)}| = |\frac{z\overline{z} - \overline{a}z}{z(z-a)}| = 1$$

(iv) To prove the theorem it suffices to prove that

$$\sum_a \log \frac{R}{|a|} = \int_0^R \frac{n(r)}{r} dr.$$

Here

$$LHS = \int_0^R \log \frac{R}{r} dn(r) = n(r) \log \frac{R}{r}]_0^R + \int_0^R \frac{n(r)}{r} dr$$

and so the theorem is proved.

□

Remark 1. The last principle used is this. If $\phi(u)$ is continuously differentiable, then

$$\sum_{A \leq n \leq B} \phi(n) a_n \left(= \int_{A-0}^{B+0} \phi(u) d \sum_{n \leq u} a_n \right)$$
$$= \phi(u) \sum_{n \leq u} a_n]_{A-0}^{B+0} - \int_{A-0}^{B+0} \left(\sum_{n \leq u} a_n \right) \phi'(u) du.$$

152

This useful result can be easily verified.

Remark 2. By taking the disc with centre $s_0 = 2 + it_0$ and radius 3, we obtain

$$\int_0^3 \frac{n(r)}{r} dr = \frac{1}{2\pi i} \int_{|s-s_0|=3} \log |\zeta(s)| \frac{ds}{s - s_0} + O(1)$$
$$< 10^{10} \log t_0,$$

where $n(r)$ is the number of zeros of $\zeta(s)$ in $|s - s_0| \leq r (\leq 3)$. Noting that LHS is $\geq N(\alpha) \int_\alpha^3 \frac{dr}{r}$ we obtain Lemma 5 by choosing $\alpha = 3 - \frac{1}{100}$.

Theorem 7.5.1. *Given any* $t_0 \geq 1000$ *there is a* t *satisfying* $|t - t_0| \leq \frac{1}{2}$ *such that for* $s = \sigma + it$ *with* $-2 \leq \sigma \leq 2$, *we have, uniformly*

$$\frac{\zeta'(s)}{\zeta(s)} = O((\log t_0)^2).$$

Proof. The number of zeros of $\zeta(s)$ in $(\sigma \geq -\frac{1}{25}, |t - t_0| \leq \frac{1}{1000})$ is $O(\log t_0)$. Divide this t-interval into abutting intervals all of equal length equal to a small constant times $(\log t_0)^{-1}$ ignoring a bit at one end. It followis that at least one of these is free from zeros of $\zeta(s)$ and so by Lemma 4 the theorem follows for $-\frac{1}{25} \leq \sigma \leq 1 + \frac{1}{25}$. The rest follows by the functional equation. □

7.6 Asymptotics of $|d_k(n)|^2$

We begin with the remark that $d_k(n)$ is defined by $\sum\limits_{n=1}^{\infty} d_k(n)n^{-s} = (\zeta(s))^k$. From the Euler product it is easy to verify that $|d_k(n)|^2 \leq d_{k'}(n)$ where $k' = (|k| + 1)^2$. It is also easy to verify that for each $n \geq 1$, $d_\ell(n)$ is an increasing function of $\ell \geq 0$. Thus if we are interested in an upper bound for $|d_k(n)|^2$ we see that it is majorised by $d_\ell(n)$ where $\ell \geq 0$ is a certain integer. Now the result $d_{\ell_1}(n)d_{\ell_2}(n) \leq d_{\ell_1\ell_2}(n)$ for any two integers $\ell_1 \geq 0$, $\ell_2 \geq 0$ can be verified when n is a prime power and 153
the result for general n follows since for all n_1, n_2 with $(n_1, n_2) = 1$ and $\ell \geq 0$ we have $d_\ell(n_1)d_\ell(n_2) = d_\ell(n_1n_2)$. Thus for any fixed ℓ and all integers $\nu \geq 1$ we have $(d_\ell(n))^\nu \leq d_m(n)$ where $m = \ell^\nu$. Hence

$$(d_\ell(n))^\nu n^{-2} \leq \sum_{n=1}^{\infty} (d_\ell(n))^\nu n^{-2} \leq (\zeta(2))^{\ell^\nu}.$$

Therefore for $n \geq 2$, we have

$$d_\ell(n) \leq (n^2 2^{\ell^\nu})^{\nu^{-1}} \leq n^\epsilon$$

by choosing ν large enough.

From now on we give a brief sketch of the fact that

$$\sum_{n \leq x} |d_k(n)|^2 = C_k^{(0)} x(\log x)^{|k^2|-1}(1 + O((\log x)^{-1})),$$

where

$$C_k^{(0)} = (\Gamma(|k^2|))^{-1} \prod_p \left(\left(1 - \frac{1}{p}\right)^{|k^2|} \sum_{m=0}^{\infty} |d_k(p^m)|^2 p^{-m} \right\}.$$

We begin with a well-known lemma.

Lemma 1. *We have, for $c > 0$, $y > 0$, and $T \geq 10$,*

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} y^s \frac{ds}{s} = (0 \ \ or \ \ 1) + O\left(\frac{y^c}{T|\log y|}\right)$$

according as $0 < y < 1$ or $y > 1$.

Proof. Move the line of integration to $\sigma = R$ or $\sigma = -R$. This leads to the lemma. □

Lemma 2. *Let $1 < c < 2$ and let $x(> 10)$ be half an odd integer. Then for $T \geq 10$, we have,*

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} f(s) x^s \frac{ds}{s} = \sum_{n \leq x} |d_k(n)|^2 + O\left(\frac{x^{c+\epsilon}}{T}\right),$$

154 *where $f(s) = \sum\limits_{n=1}^{\infty} |d_k(n)|^2 n^{-s}$ and the O-constant depends only on ϵ.*

Proof. We have to use $|d_k(n)| \leq n^{\epsilon}$ for $n \geq n_0(\epsilon)$, and Lemma 1 and the inequality $|\log \frac{x}{n}| \gg \frac{|n-x|}{|n+x|}$. From these the lemma follows in a fairly straight forward way. □

Lemma 3. *We have,*

$$f(s) = (\zeta(s))^{|k^2|} \phi(s)$$

where

$$\phi(s) = \prod_p \left\{ \left(1 - \frac{1}{p^s}\right)^{|k^2|} \sum_{m=0}^{\infty} |d_k(p^m)|^2 p^{-ms} \right\} \tag{7.6.1}$$

is analytic in $\sigma \geq 1 - \frac{1}{100}$.

Proof. Trivial. □

We have now to use one deep result due to I.M. Vinogradov namely equation number (13) of introductory remarks. From this it follows (K. Ramachandra [65]) that $\zeta(s) \neq 0$ in $\sigma \geq 1 - \alpha(\log t)^{-\frac{2}{3}}(\log\log t)^{-\frac{1}{3}}$, $|t| \leq T$, where $\alpha > 0$ is a certain constant. By using the result of I.M. Vinogradov it follows that in $(\sigma \geq 1 - (\log T)^{-\frac{2}{3}-\epsilon}, |t| \leq T, |s-1| \geq (\log T)^{-\frac{2}{3}-\epsilon})$ we have $\zeta(s) \neq 0$ and $\zeta(s) = O((\log T)^A)$ where $A = A_k$ is a constant provided $T \geq T_0(\epsilon)$. We will assume hereafter that $T \geq T_0(\epsilon)$. Moving the line of integration from $c = 1 + \epsilon$ to $\sigma = 1 - (\log T)^{-\frac{2}{3}-\epsilon}$, we have by Lemma 2

$$\sum_{n\leq x} |d_k(n)|^2 = \frac{1}{2\pi i}\int_{\substack{\sigma=1-(\log T)^{-\frac{2}{3}-\epsilon}\\ 0<|t|\leq T}} f(s)\frac{x^s}{s}ds + I_0 + O\left(\frac{x^{1+2\epsilon}}{T}\right), \quad (7.6.2)$$

where

$$I_0 = \frac{1}{2\pi i}\int_{\substack{|s-1|=(\log T)^{-\frac{2}{3}-\epsilon}\\ s\neq 1-(\log T)^{-\frac{2}{3}-\epsilon}}} f(s)\frac{x^s}{s}ds, \quad (7.6.3)$$

and the integration in I_0 is anti-clockwise. Here we choose $T = x^{\frac{1}{2}}$ and obtain

Lemma 4. *We have, for any complex constant k,* **155**

$$\sum_{n\leq x} |d_k(n)|^2 = I_0 + O(x(\log x)^{-B}), \quad (7.6.4)$$

where $B(>0)$ *is any arbitrary constant and* I_0 *is as in* (7.6.3) *with* $T = x^{\frac{1}{2}}$.

Proof. Trivial. □

Lemma 5. *We have, for* $|s-1| \leq r_1$ *where* r_1 *is a small constant, with the straight line segment* $[1-r_1, 1]$ *removed,*

$$\frac{f(s)}{s} = C_k\left(\frac{1}{s-1}\right)^{|k^2|}(1 + \lambda_1(s-1) + \cdots + \lambda_r(s-1)^r + O((s-1)^{r+1}))$$

where

$$C_k = \prod_p \left\{\left(1-\frac{1}{p}\right)^{|k^2|}\sum_{m=0}^{\infty}|d_k(p^m)|^2 p^{-m}\right\},$$

and $\lambda_1, \lambda_2, \ldots, \lambda_r$ *are constants depending on k.*

Proof. Trivial. $\square$

Lemma 6. *By deforming the contour properly the contribution to* I_0 *from* $O((s-1)^r)$ *is* $O(x(\log x)^{|k^2|-3})$ *for a sufficiently large constant* r.

Proof. The lemma follows from the observation that for $r \geq |k|^2 + 40$, we have,

$$\int_0^\infty v^{-|k^2|+r} x^{-v} dv = O\left((\log x)^{|k^2|-3}\right).$$

$\square$

Lemma 7. *We have, for* $0 \leq j \leq r$ *and* $T = x^{\frac{1}{2}}$,

$$\int_{(\log T)^{-\frac{2}{3}-\epsilon}}^\infty v^{-k^2+j} x^{-v} dv = O\left((\log x)^{k^2-j-1}\right)$$

Proof. Putting $v \log x = u$ we see that the LHS is equal to

$$(\log x)^{k^2-j-1} \int_{(\log x)(\log T)^{-\frac{2}{3}-\epsilon}} v^{-k^2+j} \operatorname{Exp}(-v) dv$$

156 and the required result follows since $(\log x)(\log T)^{-\frac{2}{3}-\epsilon} \gg (\log x)^{\frac{1}{3}-\epsilon}$ and $\operatorname{Exp}(-v) \ll v^{-k^2-j-30}$.

With the substitution $v \log x = u$ we see now (in view of Lemma 7) that we are led (by I_0) to the integral in lemma below. $\square$

Lemma 8. *With usual notation* (see the remark below) *we have, for any complex* z,

$$\frac{1}{2\pi i} \int_{-\infty}^{0+} v^{-z} \operatorname{Exp}(-v) dv = \frac{\sin(\pi z)}{\pi} \Gamma(1-z) = \frac{1}{\Gamma(z)}.$$

Remark. We recall that the path is the limit as $\delta \to 0$ of the contour obtained by joining by straight line the points $\infty e^{-i\pi}$ to $\delta e^{-i\pi}$ and then continuing by the circular arc $\delta e^{i\theta} (\theta = -\pi$ to $\pi)$ and then by the straight line $\delta e^{i\pi}$ to $\infty e^{i\pi}$.

Proof. Trivial. $\square$

From Lemmas 4 to 8 we get

Theorem 7.6.1. *We have,*

$$\sum_{n\leq x} |d_k(n)|^2 = C_k(\Gamma(|k^2|))^{-1} x(\log x)^{|k^2|-1}(1 + O((\log x)^{-1})).$$

As a corollary we obtain

Theorem 7.6.2. *We have,*

$$\sum_{n\leq x} |d_k(n)|^2 n^{-1} = C_k^{(1)}(\log x)^{|k^2|}(1 + O((\log x)^{-1})),$$

where

$$C_k^{(1)} = (\Gamma(|k^2| + 1))^{-1} \prod_p \left\{\left(1 - \frac{1}{p}\right)^{|k^2|} \sum_{m=0}^{\infty} |d_k(p^m)|^2 p^{-m}\right\}.$$

Proof. The proof follows by

$$\sum_{n\leq x} |d_k(n)|^2 n^{-1} = \int_{1-0}^{x+0} u^{-1} d\left(\sum_{n\leq u} |d_k(n)|^2\right)$$

and integration by parts. □ 157

7.7 Some Useful Reciprocal Relations Involving Certain Kernels

By the term kernel function we mean a function $\varphi(w)$ which tapers off. Examples of $\varphi(w)$ are 1, $\Gamma(w+1)$, $\mathrm{Exp}(w^{4a+2})$, and $\mathrm{Exp}((\sin \frac{w}{8A})^2)$. Here w is a complex variable, $a > 0$ an integer constant, and $A(\geq 10)$ any real constant. The last three kernels decay like $\mathrm{Exp}(-|Im\ w|)$, $\mathrm{Exp}(-|Im\ w|^{4a})$ and $\left(\mathrm{Exp}\,\mathrm{Exp}\,\frac{|Im\ w|}{80A}\right)^{-1}$ (the last mentioned decay is valid in $|Re\ w| \leq A$). Let $\varphi(w)$ be any of these kernels. While applying the maximum modulus principle to an analytic function $f(z)$ we may apply maximum modulus principle to $f(w)\varphi(w-z)$ as a function of w, in a rectangle with z as an interior point. We may also apply the same to $f(z)\varphi(w-z)x^{w-z}$ where $x > 0$ is a free parameter. This leads to convexity. It is well-known that

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} x^w \frac{dw}{w} = 0, \frac{1}{2} \text{ or } 1 \tag{7.7.1}$$

according as $0 < x < 1$, $x = 1$ or $x > 1$. The other helpful evaluations are

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} x^w\Gamma(w+1)\frac{dw}{w} = \mathrm{Exp}\left(-\frac{1}{x}\right), \tag{7.7.2}$$

and

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} x^w \,\mathrm{Exp}(w^2)\frac{dw}{w} = 1 - \pi^{-\frac{1}{2}}\int_{\frac{1}{2}\log x}^{\infty} \mathrm{Exp}(-v^2)dv, \tag{7.7.3}$$

both valid for $x > 0$. Note that

$$\int_{-\infty}^{\infty} \mathrm{Exp}(-v^2)dv = \pi^{\frac{1}{2}} \tag{7.7.4}$$

Remarks. I learnt of (7.7.3) from Professor D.R. Heath-Brown and of the kernel $\mathrm{Exp}(w^{4a+2})$ from Professor P.X. Gallagher. I thought of the kernel $\mathrm{Exp}((\sin\frac{w}{8A})^2)$ myself. I learnt of some convexity principles from Professor A. Selbert. I learnt the proof of the functional equation of
158 $\zeta(s)$ as presented in this chapter from Professor K. Chandrasekharan and K.G. Ramanathan. The treatment of the asymptotics of $\Gamma(s)$ is my own while that of $\sum\limits_{n\le x}|d_k(n)|^2$ is well-known. To prove (7.7.3) denote the LHS by $\Delta(x)$ and consider $x\Delta'(x)$. Then since $\Delta(x) \to 0$ as $x \to 0$ we can come back to $\Delta(x)$ by using $\Delta(x) = \int_0^x (u\Delta'(u))\frac{du}{u}$.

More generally we can write for $x > 0$,

$$\Delta(x) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} x^w\varphi(w)\frac{dw}{w} \tag{7.7.5}$$

where $\varphi(w)$ is any of the kernels mentioned above in the beginning of this section. In case $\varphi(w) = 1$ the function $\Delta(x)$ is non-negative but discontinuous. In the cases (7.7.2) and (7.7.3), $\Delta(x)$ is monotonic and continuous and $0 < \Delta(x) < 1$. In the case $\varphi(w) = \mathrm{Exp}(w^{4a+2})$ we can move the line of integration any where and so we get

$$\Delta(x) = O_B(x^B) \quad \text{and} \quad \Delta(x) = 1 + O_B(x^{-B}) \tag{7.7.6}$$

for any constant $B > 0$. In the case $\varphi(w) = \mathrm{Exp}((\sin\frac{w}{8A})^2)$ we can move the line of integration (but not too far). Thus

$$\Delta(x) = O_A(x^A) \quad \text{and} \quad \Delta(x) = 1 + O_A(x^{-A}), \tag{7.7.7}$$

where A is the constant occuring in the definition of $\varphi(w)$.

An interesting formula is

$$\frac{\varphi(w)}{w} = \int_0^\infty \Delta\left(\frac{1}{x}\right) x^{w-1} dx, \quad (Re\, w > 0). \tag{7.7.8}$$

These are special cases of more general reciprocal transforms (see E.C. Titchmarsh, [102]). See also § 2 of (K. Ramachandra [81]).

Bibliography

[1] L. Ahlfors, Complex Analysis, McGraw-Hill Book Company Inc., (1953). 159

[2] R. Balasubramanian, On the frequency of Titchmarsh's phenomenon for $\zeta(s)$-IV, Hardy-Ramanujan J., 9 (1986), 1-10.

[3] R. Balasubramanian, An improvement of a theorem of Titchmarsh on the mean square of $|\zeta(\frac{1}{2}+it)|$, Proc. London Math. Soc., (3) 36 (1978), 540-576.

[4] R. Balasubramanian, A. Ivić and K. Ramachandra, The mean square of the Riemann zeta-function on the line $\sigma = 1$, L'Ensign. Math., tome 38 (1992), 13-25.

[5] R. Balasubramanian, A. Ivić and K. Ramachandra, An application of the Hooley-Huxley contour, Acta Arith. LXV, 1(1993), 45-51.

[6] R. Balasubramanian and K. Ramachandra, Some local convexity theorems for the zeta-function-like analytic functions, Hardy-Ramanujan J., 11 (1988), 1-12.

[7] R. Balasubramanian and K. Ramachandra, A Lemma in complex function theory-I, Hardy-Ramanujan J., 12 (1989), 1-5.

[8] R. Balasubramanian and K. Ramachandra, A Lemma in complex function theory-II, Hardy-Ramanujan J., 12 (1989), 6-13.

[9] R. Balasubramanian and K. Ramachandra, Proof of some conjectures on the mean-value of Titchmarsh series-II, Hardy-Ramanujan J., 14 (1991), 1-20.

160 [10] R. Balasubramanian and K. Ramachandra, Proof of some conjectures on the mean-value of Titchmarsh series-I, Hardy-Ramanujan J., 13 (1990), 1-20.

[11] R. Balasubramanian and K. Ramachandra, On the frequency of Titchmarsh's phenomenon for $\zeta(s)$-III, Proc. Indian Acad. Sci., Section A, 86 (1977), 341-351.

[12] R. Balasubramanian and K. Ramachandra, On the frequency of Titchmarsh's phenomenon for $\zeta(s)$-V, Arkiv för Matematick, 26(1) (1988), 13-20.

[13] R. Balasubramanian and K. Ramachandra, Progress towards a conjecture on the mean-value of Titchmarsh series-III, Acta Arith., 45 (1986), 309-318.

[14] R. Balasubramanian and K. Ramachandra, On the frequency of Titchmarsh's phenomenon for $\zeta(s)$-VI, Acta Arith., 53 (1989), 325-331.

[15] R. Balasubramanian and K. Ramachandra, On the zeros of a class of generalised Dirichlet series-III, J. Indian Math. Soc., 41 (1977), 301-315.

[16] R. Balasubramanian and K. Ramachandra, On the zeros of a class of generalised Dirichlet series-X, Indag. Math., N.S., 3(4), (1992), 377-384.

[17] R. Balasubramanian and K. Ramachandra, On the zeros of a class of generalised Dirichlet series-XI, Proc. Indian Acad. Sci., (Math. Sci.) 102 (1992), 225-233.

[18] R. Balasubramanian and K. Ramachandra, On the zeros of a class of generalised Dirichlet series-IV, J. Indian Math. Soc., 42 (1978), 135-142.

[19] R. Balasubramanian and K. Ramachandra, On the zeros of a class of generalised Dirichlet series-VI, Arkiv för Matematik, 19 (1981), 239-250. **161**

[20] R. Balasubramanian and K. Ramachandra, On the zeros of a class of generalised Dirichlet series-VIII, Hardy-Ramanujan J., 14 (1991), 21-33.

[21] R. Balasubramanian and K. Ramachandra, On the zeros of a class of generalised Dirichlet series-IX, Hardy-Ramanujan J., 14 (1991), 34-43.

[22] R. Balasubramanian and K. Ramachandra, On the zeros of $\zeta'(s)-a$, (zeros XII), Acta Arith., LXIII. 2, (1993), 183-191.

[23] R. Balasubramanian and K. Ramachandra, On the zeros of $\zeta(s)-a$, (zeros XIII), Acta Arith., LXIII. 4, (1993), 359-366.

[24] R. Balasubramanian and K. Ramachandra, On the zeros of a class of generalised Dirichlet series-XIV, Proc. Indian Acad. Sci., (Math. Sci.) vol. 104 (1994), 167-176.

[25] R. Balasubramanian and K. Ramachandra, On the zeros of a class of generalised Dirichlet series-XV, Indag. Math., NS 5(2) (1994), 129-144.

[26] R. Balasubramanian, K. Ramachandra and A. Sankaranarayanan, On the frequency of Titchmarsh's phenomenon for $\zeta(s) - VIII$, Proc. Indian Acad. Sci., (Math. Sci.), 102 (1992), 1-12.

[27] K. Chandrasekharan, The Riemann zeta-function, LN1, Tata Institute of Fundamental Research, Bombay (1953), Reissued (1962).

[28] K. Chandrasekharan, Arithmetical functions, Springer-Verlag, Berlin-New York (1970).

[29] J.B. Conrey, More than two fifths of the zeros of the Riemann zeta function are on the critical line, J. Reine angew. Math., 399 (1989), 1-26. **162**

[30] J.B. Conrey, At least two fifths of the zeros of the Riemann zeta function are on the critical line, Bull. Amer. Math. Soc., 20 (1989), 79-81.

[31] J.B. Conrey, A. Ghosh and S.M. Gonek, Simple zeros of zeta functions, Colloq. de Théorie analytique des nombres, University of Paris Sud (1985), 77-83.

[32] A. Fujii, *On a conjecture of Shanks*, paper No. 26, Proc. Japan Acad. Sci., (Math. Sci.) Vol. LXX (1994), 109-114.

[33] R.M. Gabriel, Some results concerning the integrals of the modulii of regular functions along certain curves, J. London Math. Soc., 11 (1927), 112-117.

[34] S.M. Gonek, Mean value of the Riemann zeta-function and its derivatives, Invent. Math., 75 (1984), 123-141.

[35] S.W. Graham, Large values of the Riemann zeta-function, Topics in analytic number theory, Proc. Conf. Austin/Texas (1982), (1985), 98-126.

[36] J.L. Hafner and A. Ivić, On the mean square of the Riemann zeta function on the critical line, J. Number Theory, 32 (1989), 151-191.

[37] D.R. Heath-Brown, Fractional moments of the Riemann zeta-function, J. London Math. Soc. (2) 24 (1981), 65-78.

[38] D.R. Heath-Brown, The twelfth power moment of the Riemann zeta-function, Quart. J. Math., Oxford (2) 29 (1978), 443-462.

163 [39] D.R. Heath-Brown, The fourth power moment of the Riemann zeta-function, Proc. London Math. Soc., (3) 38 (1979), 385-422.

[40] D.R. Heath-Brown and M.N. Huxley, Exponential sums with a difference, Proc. London Math. Soc., (3) 61 (1990), 227-250.

[41] A.E. Ingham, On the estimation of $N(\sigma, T)$, Quart. J. Oxford, 11 (1940), 291-292.

[42] A. Ivić, The Riemann zeta-function, John Wiley and Sons, New York (1985).

[43] A. Ivić, Mean values of the Riemann zeta-function, LN82, Tata Institute of Fundamental Research, Bombay (1991), (Published by Springer-Verlag, Berlin-New York).

[44] A. Ivić and Y. Motohashi, On the fourth power moment of the Riemann zeta-function, (to appear).

[45] A. Ivić and A. Perelli, Mean values of certain zeta-functions on the critical line, Litovskij Mat. Sbornik, 29 (1989), 701-714.

[46] H. Iwaniec, Fourier coefficients of cusp forms and the Riemann zeta-function, Expose No. 18, Semin. Théor. Nombres, Universite Bordeaux (1979/80).

[47] M. Jutila, A method in the theory of exponential sums, LN80, Tata Institute of Fundamental Research, Bombay (1987), (Published by Springer-Verlag, Berlin-New York).

[48] M. Jutila, On the value distribution of the zeta-function on the critical line, Bull. London Math. Soc., 15 (1983), 513-518.

[49] M. Jutila, The fourth power moment of the Riemann zeta-function over a short interval in "Coll. Math. Soc. J. Bolyai 54, Number Theory, Budapest 1987", North-Holland, Amsterdam 1989, 221-244.

[50] M. Jutila, Mean value estimates for exponential sums with applications to L-functions, Acta Arith., 57 (1990), 93-114. **164**

[51] A.A. Karatsuba, The distribution of prime numbers, Uspekhi Mat. Nauk., 45: 5 (1990) 81-140.

[52] N.V. Kuznetsov, Sums of Kloosterman sums and the eighth power moment of the Riemann zeta-function, Number Theory and related topics, Ramanujan Cent. Colloq. TIFR, Bombay (1988), Oxford University Press (1989), 57-117.

[53] N. Levinson, Ω-Theorems for the Riemann zeta-function, Acta Arith., 20 (1972), 319-332.

[54] N. Levinson, More than one third of the zeros of the Riemann's zeta-function are on $\sigma = \frac{1}{2}$, Advances in Math., 13 (1974), 383-436.

[55] K. Matsumoto, The mean square of the Riemann zeta-function in the critical strip, Japanese J. Math., New series, 15 (1989), 1-13.

[56] H.L. Montgomery, Topics in multiplicative number theory, Lecture notes in Mathematics, 227, Springer-Verlag, Berlin-New York (1971), IX + 178 pp.

[57] H.L. Montgomery, Extreme values of the Riemann zeta-function, Comment. Math. Helv., 52 (1977), 511-518.

[58] H.L. Montgomery, On a question of Ramachandra, Hardy-Ramanujan J., 5 (1982), 31-36.

[59] H.L. Montgomery, The pair correlation of the zeros of zeta function, Analytic number theory, Proc. Sympos. Pure Math., Vol. XXIV, St. Louis, M.O., (1972), p. 181-193, Amer. Math. Soc., Providence R.I., (1973).

165 [60] Y. Motohashi, On Sieve methods and prime number theory, LN72, Tata Institute of Fundamental Research, Bombay (1983) (Published by Springer-Verlag, Berlin-New York).

[61] Y. Motohashi, An explicit formula for the fourth power mean of the Riemann zeta-function, Acta Math., 170 (1993), 180-220.

[62] J.H. Mueller, On the Riemann zeta-function $\zeta(s)$-gaps between sign changes of $S(t)$, Mathematika, 29 (1983), 264-269.

[63] K. Prachar, Primzahlverteilung, Springer-Verlag, Berlin-Göttingen-Heidelberg, (1957).

[64] K. Ramachandra, Some problems of analytic number theory, Acta Arith., 31 (1976), 313-324.

[65] K. Ramachandra, Riemann zeta-function, Ramanujan Institute, Madras University, Madras (1979), 16 pp.

[66] K. Ramachandra, A brief summary of some results in the analytic theory of numbers-II, Lecture note in Mathematics No. 938, ed. K. Alladi, Springer-Verlag (1981), 106-122.

[67] K. Ramachandra, Some remarks on a theorem of Montgomery and Vaughan, J. Number Theory, 11 (1979), 465-471.

[68] K. Ramachandra, Application of a theorem of Montgomery and Vaughan to the zeta-function, J. London Math. Soc., 10 (1975), 482-486.

[69] K. Ramachandra, On the frequency of Titchmarsh's phenomenon for $\zeta(s)$-I, J. London Math. Soc., 8 (1974), 683-690.

[70] K. Ramachandra, Progress towards a conjecture on the mean-value of Titchmarsh series-I, Durham Sympos. (1979), Recent progress
in analytic number theory, vol. 1, ed. H. Halberstam and C. Hooley, **166**
FRS, Academic Press, London, New York, Toronto, Sydney, San Franscisco, (1981), 303-318.

[71] K. Ramachandra, Progress towards a conjecture on the mean-value of Titchmarsh series-II, Hardy-Ramanujan J., 4 (1981), 1-12.

[72] K. Ramachandra, On the frequency of Titchmarsh's phenomenon for $\zeta(s)$-II, Acta Math. Acad. Sci. Hungarica, vol. 30/1-2 (1977), 7-13.

[73] K. Ramachandra, On the frequency of Titchmarsh's phenomenon for $\zeta(s)$-VII, Annales Acad. Sci. Fenn., Ser. AI Math., 14 (1989), 27-40.

[74] K. Ramachandra, On the frequency of Titchmarsh's phenomenon for $\zeta(s)$-IX, Hardy-Ramanujan J., 13 (1990), 28-33.

[75] K. Ramachandra, Proof of some conjectures on the mean-value of Titchmarsh series with applications to Titchmarsh's phenomenon, Hardy-Ramanujan J., 13 (1990), 21-27.

[76] K. Ramachandra, Some remarks on the mean-value of the Riemann zeta-function and other Dirichlet series-I, Hardy-Ramanujan J., 1 (1978), 1-15.

[77] K. Ramachandra, Some remarks on the mean-value of the Riemann zeta-function and other Dirichlet series-II, Hardy-Ramanujan J., 3 (1980), 1-24.

[78] K. Ramachandra, Some remarks on the mena-value of the Riemann zeta-function and other Dirichlet series-IV, J. Indian Math. Soc., vol. 60 (1994), 107-122.

[79] K. Ramachandra, Mean-value of the Riemann zeta-function and other remarks-III, Hardy-Ramanujan J., 6 (1983) , 1-21.

167 [80] K. Ramachandra, Some remarks on the mean-value of the Riemann zeta-function and other Dirichlet series-III, Annales Acad. Sci. Fenn., Ser AI, 5 (1980), 145-158.

[81] K. Ramachandra, Mean-value of the Riemann zeta-function and other remarks-I, Colloq. Math. Soc. Janos Bolyai, 34 Topics in classical number theory, Budapest (Hungary), (1981), 1317-1347.

[82] K. Ramachandra, A new approach to the zeros of $\zeta(s)$, Math. Student, India, (to appear),

[83] K. Ramachandra, Application of a theorem of Montgomery and Vaughan-II, J. Indian Math. Soc., 59 (1993), 1-11.

[84] K. Ramachandra, On the zeros of a class of generalised Dirichlet series-I, J. Reine u. Angew. Math., 273 (1975), 31-40; Addendum 60.

[85] K. Ramachandra, On the zeros of a class of generalised Dirichlet series-II, J. Reine u. Angew. Math., 289 (1977), 174-180.

[86] K. Ramachandra, On the zeros of a class of generalised Dirichlet series-V, J. Reine u. Angew. Math., 303/304, (1978), 295-313.

[87] K. Ramachandra, On the zeros of a class of generalised Dirichlet series-VII, Annales Acad. Sci. Fenn., Ser AI. Math., 16 (1991), 391-397.

[88] K. Ramachandra, A remark on $\zeta(1 + it)$, Hardy-Ramanujan J., 10 (1987), 2-8.

[89] K. Ramachandra, The simplest quickest and self-contained proof that $\frac{1}{2} \leq \theta \leq 1$, J. Indian Math. Soc., vol. 61 Nos. 1-2 (1995), 7-12.

[90] K. Ramachandra and A. Sankaranarayanan, Note on a paper by H.L. Montgomery-I, Publ. de L'Institut Math., Novelle Ser. tome 50 (64), Beograd (1991), 51-59.

[91] K. Ramachandra and A. Sankaranarayanan, Note on a paper by **168**
H.L. Montgomery-II, Acta Arith., 58 (1991), 299-308.

[92] K. Ramachandra and A. Sankaranarayanan, On some theorems of Littlewood and Selberg-II, Annales Acad. Sci. Fenn., Ser. AI Math., 16 (1991), 131-137.

[93] K. Ramachandra and A. Sankaranarayanan, On some theorems of Littlewood and Selberg-III, Annales Acad. Sci. Fenn., Ser. AI Math., 16 (1991), 139-149.

[94] K. Ramachandra and A. Sankaranarayanan, On some theorems of Littlewood and Selberg-I, J. Number Theory, 44 (1993), 281-291.

[95] K. Ramachandra and A. Sankaranarayanan, On the zeros of a class of generalised Dirichlet series-XVI, Math. Scand., (to appear).

[96] K. Ramachandra and A. Sankaranarayanan, A remark on Vinogradov's mean-value theorem, The J. of Analysis, 3 (1995) (to appear).

[97] S. Ramanujan, A proof of Bertrand's Postulate, J. Indian Math. Soc., 11 (1919), 181-182.

[98] S. Ramanujan, Collected papers, ed. G.H. Hardy, P.V. Seshu Iyer and B.M. Wilson, Chelsea (1962).

[99] H.-E. Richert, Sieve methods LN55, Tata Institute of Fundamental Research, Bombay (1976).

[100] E.C. Titchmarsh, The theory of the Riemann zeta-function, Second edition, Revised and edited by D.R. Heath-Brown, Clarendon Press, Oxford (1986).

169 [101] E.C. Titchmarsh, Theory of functions, Oxford University Press (1952).

[102] E.C. Titchmarsh, Introduction to the theory of Fourier integrals, Oxford (1937).

[103] E.C. Titchmarsh, On an inequality satisfied by the zeta-function of Riemann, Proc. London Math. Soc., (2) 28 (1928), 70-80.

[104] A. Weil, Number Theory, Birkhauser, Boston-Basel-Stuttgart, (1983).

[105] N. Zavorotnyi, On the fourth moment of the Riemann zeta-function (Russian), Preprint, Vladivostok, Far Eastern Research Centre of the Academy of Sciences U.S.S.R., (1986), 36 pp.