Lectures on Cyclic Homology

By

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Notes by

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Introduction

This book is based on lectures delivered at the Tata Institute of Fundamental Research, January 1990. Notes of my lectures and a preliminary manuscript were prepared by R. Sujatha. My interest in the subject of cyclic homology started with the lectures of A. Connes in the Algebraic K-Theory seminar in Paris in October 1981 where he introduced the concept explicitly for the first time and showed the relation to Hochschild homology. In the year 1984-1985, I collaborated with Christian Kassel on a seminar on Cyclic homology at the Institute for Advanced Study. Notes were made on the lectures given in this seminar. This project was carried further in 1987-1988 while Kassel was at the Institute for Advanced Study and in 1988-1989 while I was at the Max Planck Institut für Mathematik in Bonn. We have a longer and more complete book coming on the subject. The reader is familiar with functions of several variables or sets of n-tuples which are invariant under the full permutation group, but what is special about cyclic homology is that it is concerned with objects or sets which only have an invariance property under the cyclic group. There are two important examples to keep in mind. Firstly, a trace τ on an associative algebra A is a linear form τ satisfying $\tau(ab) = \tau(ba)$ for all a, $b \in A$. Then the trace of a product of $n + 1$ terms satisfies

$$\tau(a_0 \dots a_n) = \tau(a_{i+1} \dots a_n a_0 \dots a_i).$$

We will use this observation to construct the Chern character of K-theory with values in cyclic homology. Secondly, for a group G, we denote by $N(G)_n$ the subset of G^{n+1} consisting of all $(g_0, \dots, g_n)$ with

$g_0 \ldots g_n = 1$. This subset is invariant under the action of the cyclic group on G^{n+1} since $g_0 \ldots g_n = 1$ implies that $g_{i+1} \ldots g_n g_0 \ldots g_i = 1$. This observation will not be used in these notes but can be used to define the chern character for elements in higher algebraic K-theory. This topic will not be considered here, but it is covered in our book with Kassel. This book has three parts organized into seven chapters. The first part, namely chapters 1 and 2, is preliminary to the subject of cyclic homology which is related to classical Hochschild homology by an exact couple discovered by Connes. In chapter 1, we survey the part of the theory of exact couples and spectral sequences needed for the Connes exact couple, and in chapter 2 we study the question of abelianization of algebraic objects and how it relates to Hochschild homology. In the second part, chapters 3, 4, and 5, we consider three different definitions of cyclic homology. In chapter 3, cyclic homology is defined by the standard double complex made from the standard Hochschild complex. The first result is that an algebra A and any algebra Morita equivalent to A, for example the matrix algebra $M_n(A)$, have isomorphic cyclic homology. In chapter 4, cyclic homology is defined by cyclic covariants of the standard Hochschild complex in the case of a field of characteristic zero. The main result is a theorem discovered independently by Tsygan [1983] and Loday-Quillen [1984] calculating the primitive elements in the Lie algebra homology of the infinite Lie algebra $\underline{g\ell}(A)$ in terms of the cyclic homology of A. In chapter 5, cyclic homology is defined in terms of mixed complexes and the Connes' B operator. This is a way, due to Connes, of simplifying the standard double complex, and it is particularly useful for the incorporation of the normalized standard Hochschild into the calculation of cyclic homology. The third part, chapters 6 and 7, is devoted to relating cyclic and Hochschild homology to differential forms and showing how K-theory classes have a Chern character in cyclic homology over a field of characteristic zero. There are two notions of differential forms depending on the commutativity properties of the algebra. In chapter 6, we study the classical Kähler differential forms for a commutative algebra, outline the proof of the classical Hochschild-Kostant-Rosenberg theorem relating differential forms and Hochschild homology, and relate cyclic homology to deRham cohomology. In chapter 7 we study non-commutative differential forms for algebras and indicate how they can be used to define the Chern character of a K-theory class, that is, a class of an idempotent element in $M_n(A)$, using differential forms in cyclic homology. In this way, cyclic homology becomes the natural home for characteristic classes of elements of K-theory for general algebras over a field of characteristic zero. This book treats only algebraic aspects of the theory of cyclic homology. There are two big areas of application of cyclic homology to index theory, for this, see Connes [1990], and to the algebraic K-theory of spaces $A(X)$ introduced by F. Waldhausen. For references in this direction, see the papers of Goodwillie.

I wish to thank the School of Mathematics of the Tata Institute of Fundamental Research for providing the opportunity to deliver these lectures there, and the Haverford College faculty research fund for support. I thank Mr. Sawant for the efficient job he did in typing the manuscript and David Jabon for his help on international transmission and corrections. The process of going from the lectures to this written account was made possible due to the continuing interest and participation of R. Sujatha in the project. For her help, I express my warm thanks.

Contents

Chapter 1

Exact Couples and the Connes Exact Couple

In this chapter we review background material on graded objects, differ- 1
ential objects or complexes, spectral sequences, and on exact couples. Since the Connes' exact couple relating Hochschild and cyclic homology plays a basic role in the theory of cyclic homology, this material will serve as background material and as a means of introducing other technical topics needed in the subsequent chapters. We discuss the basic structure of the Connes' exact couple and the elementary conclusions that can be drawn from this kind of exact couple.

1 Graded objects over a category

Given a category we formulate the notion of graded objects over the category and define the category of graded objects. There are many examples of gradings indexed by groups $\mathbf{Z}$, $\mathbf{Z}/2\mathbf{Z}$, $\mathbf{Z}/8\mathbf{Z}$, or $\mathbf{Z}^r$ which arise naturally. Then, a bigraded object is a $\mathbf{Z}^2$-graded object, that is, an object graded by the group $\mathbf{Z}^2$.

Definition 1.1. Let $\mathcal{C}$ be a category and Θ an abelian group. The category $\mathcal{G}\tau_{\Theta}(\mathcal{C})$, also denoted $\Theta\mathcal{C}$, of Θ-graded objects over $\mathcal{C}$ has for objects $X = (X_{\theta})_{\theta\in\Theta}$ where X is a family of objects X_{θ} in $\mathcal{C}$ indexed by Θ, for morphisms $f : X \to Y$ families $f = (f_\theta)_{\theta\in\Theta}$ of morphisms $f_\theta : X_\theta \to Y_\theta$ in C, and composition gf of $f : X \to Y$ and $g : Y \to Z$ given by $(gf)_\theta = g_\theta f_\theta$ in C.

The identity on X is the family $(1_\theta)_{\theta\in\Theta}$ of identities 1_θ on X_θ. Thus it is easily checked that we have a category, and the morphism sets define a functor of two variables

$$\mathrm{Hom}_{\Theta C} = \mathrm{Hom} : (\Theta C)^{op} \times \Theta C \to (\text{sets})$$

extending $\mathrm{Hom} : C^{op} \times C \to$ (sets) in the sense that for two Θ-graded objects X and Y we have $\mathrm{Hom}_{\Theta C}(X, Y) = \prod_{\theta'\in\Theta} \mathrm{Hom}_C(X_{\theta'}, Y_{\theta'})$. Note
2 that we do not define graded objects as either products or coproducts, but the morphism set is naturally a product. This product description leads directly to the notion of a morphism of degree $\alpha \in \Theta$ such that a morphism in the category is of degree $0 \in \Theta$.

Definition 1.2. With the previous notations for two objects X and Y in ΘC, the set of morphisms of degree $\alpha \in \Theta$ from X to Y is $\mathrm{Hom}_\alpha(X, Y) = \prod_{\theta'\in\Theta} \mathrm{Hom}(X_\theta, Y_{\theta+\alpha})$. If $f : X \to Y$ has degree α and $g : Y \to Z$ has degree β, then $(gf)_\theta = g_{\theta+\alpha} f_\theta$ is defined $gf : X \to Z$ of degree $\alpha + \beta$, i.e. it is a function $(f, g) \mapsto gf$ defined

$$\mathrm{Hom}_\alpha(X, Y) \times \mathrm{Hom}_\beta(Y, Z) \to \mathrm{Hom}_{\alpha+\beta}(X, Z).$$

Thus this Θ-graded Hom, denoted Hom_*, is defined

$$\mathrm{Hom}_* : (\Theta C)^{op} \times \Theta C \to \Theta\ (\text{Sets})$$

as a functor of two variables with values in the category of Θ-graded sets.

Remark 1.3. Recall that a zero object in a category C is an object denoted 0 or $*$, such that $\mathrm{Hom}(X, 0)$ and $\mathrm{Hom}(0, X)$ are sets with one element. A category with a zero object is called a pointed category. The zero morphism $0 : X \to Y$ is the composite $X \to 0 \to Y$.

Remark 1.4. If $\mathcal{A}$ is an additive (resp. abelian) category, then $\Theta\mathcal{A}$ is an additive (resp. abelian) category where the graded homomorphism functor is defined

$$\mathrm{Hom}_* : (\Theta\mathcal{A})^{op} \times \Theta\mathcal{A} \to \Theta(ab)$$

with values in the category of Θ-graded abelian groups. A sequence $X' \to X \to X''$ is exact in $\Theta\mathcal{A}$ if and only if $X'_\theta \to X_\theta \to X''_\theta$ is exact in $\mathcal{A}$ for each $\theta \in \Theta$.

Remark 1.5. Of special interest is the category (k) of k-modules over a
commutative ring k with unit. This category has an internal Hom functor 3
and tensor functor defined

$$\otimes : (k) \times (k) \to (k) \quad \text{and} \quad \mathrm{Hom} : (k)^{op} \times (k) \to (k)$$

satisfying the adjunction formula with an isomorphism

$$\mathrm{Hom}(L \otimes M, N) \simeq \mathrm{Hom}(L, \mathrm{Hom}(M, N))$$

as functors of L, M, and N. These functors extend to

$$\otimes : \Theta(k) \times \Theta(k) \to \Theta(k) \quad \text{and} \quad \mathrm{Hom} : \Theta(k)^{op} \times \Theta(k) \to \Theta(k)$$

satisfying the same adjunction formula by the definitions

$$(L \otimes M)_\theta = \coprod_{\alpha+\beta=\theta} L_\alpha \otimes M_\beta \quad \text{and} \quad \mathrm{Hom}(M, N)_\theta = \prod_{\alpha\in\Theta} \mathrm{Hom}(M_\alpha, N_{\alpha+\theta}).$$

We leave it to the reader to check the adjunction formula, and we come back to the question of the tensor product of two morphisms of arbitrary degrees in the next section, for it uses an additional structure on the group Θ.

Notation 1.6. For certain questions, for example those related to duality, it can be useful to have the upper index convention for an element X of ΘC. This is $X^\theta = X_{-\theta}$ and $\mathrm{Hom}(X, Y)^\theta = \mathrm{Hom}(X, Y)_{-\theta}$. In the classical case of $\Theta = \mathbf{Z}$ the effect is to turn negative degrees into positive degrees. For example in the category (k) the graded dual in degree n is $\mathrm{Hom}(M, k)^n = \mathrm{Hom}(M_n, k)$. The most clear use of this convention is with cohomology which is defined in terms of the dual of the homology chain complex for spaces.

2 Complexes

To define complexes, we need additional structure on the grading abelian group Θ, and this leads us to the next definition.

4 **Definition 2.1.** An oriented abelian group Θ is an abelian group Θ together with a homomorphism $e : \Theta \to \{\pm 1\}$ and an element $\iota \in \Theta$ such that $e(\iota) = -1$.

Definition 2.2. A complex X in a pointed category χ graded by an oriented abelian group Θ is a pair $(X, d(X))$ where X is in $\Theta\chi$ and $d(X) = d : X \to X$ is a morphism of degree $-\iota$ such that $d(X)d(X) = 0$. A morphism $f : X \to Y$ of complexes is a morphism in $\Theta\chi$ such that $fd(X) = d(Y)f$.

The composition of morphisms of complexes is the composition of the corresponding graded objects. We denote the category of complexes in χ graded by the oriented abelian group by $C_\Theta(\chi)$ or simply $C(\chi)$.

In order to deal with complexes, we first need some additive structure on $\mathrm{Hom}(X, Y)$ for two Θ-graded objects X and Y, which are the underlying graded objects of complexes and second, kernels and cokernels, which are used to define the homology functor. To define the homology, the base category must be an abelian category $\mathcal{A}$, for example, the category (k) of k-modules. Then $\Theta\mathcal{A}$ and $C_\Theta(\mathcal{A})$ are abelian categories, and homology will be defined as a functor $H : C_\Theta(\mathcal{A}) \to \Theta\mathcal{A}$. Tha basic tool is the snake lemma which we state now.

Snake Lemma 2.3. *Let $\mathcal{A}$ be an abelian category, and consider a morphism of exact sequences (u', u, u'') all of degree $\nu \in \Theta$*

$$\begin{array}{ccccccc} & & L' & \xrightarrow{f} & L & \xrightarrow{f'} & L'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u''} & & \\ 0 & \longrightarrow & M' & \xrightarrow{g} & M & \xrightarrow{g'} & M'' & & \end{array}$$

Then f and g induce morphisms $k(f) : \ker(u') \to \ker(u)$ and $c(g) : \operatorname{coker}(u') \to \operatorname{coker}(u)$ and the commutative diagram induces a mor-

phism $\delta : \ker(u'') \to \operatorname{coker}(u')$ *of degree* ν *such that the following sequence, called the sequence of the snake, is exact*

$$\ker(u') \to \ker(u) \to \ker(u'') \xrightarrow{\delta} \operatorname{coker}(u') \to \operatorname{coker}(u) \to \operatorname{coker}(u'').$$

Further, if f *is a monomorphism, then* $\ker(u') \to \ker(u)$ *is a mono-* 5
morphism, and g' *is an epimorphism, then* $\operatorname{coker}(u) \to \operatorname{coker}(u'')$ *is an epimorphism. Finally the snake sequence is natural with respect to morphisms of the above diagrams which give arise to the snake sequence. Here a morphism of the diagram is a family of morphisms of each respective object yielding a commutative three dimensional diagram.*

For a proof, see Bourbaki, *Algébre homologique.*

Notation 2.4. Let X be a complex in $C_\Theta(\mathcal{A})$, and consider the kernel-cokernel sequence in $\Theta\mathcal{A}$ of $d(x) = d$, which has degree $-\iota$,

$$0 \to Z(X) \to X \xrightarrow{d} X \to Z'(X) \to 0.$$

This defines two functors $Z, Z' : C_\Theta(\mathcal{A}) \to \Theta\mathcal{A}$, and this sequence is a sequence of functors $C_\Theta(A) \to \Theta A$. Since $d(X)d(X) = 0$, we derive three factorizations of $d(X)$ namely

$$d' = d'(X) : Z'(X) \to X, d'' = d''(X) : X \to Z(X), \quad \text{and}$$
$$\hat{d} = \hat{d}(X) : Z'(X) \to Z(X)$$

from which we have the following diagram, to which the snake sequence applies,

$$\begin{array}{ccccccccc} & & X & \xrightarrow{d} & X & \longrightarrow & Z'(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d''} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle d'} & & \\ 0 & \longrightarrow & Z(X) & \longrightarrow & X & \longrightarrow & X & & \end{array}$$

and the boundary morphism $\delta : \ker(\delta') = H'(X) \to H(X) = \operatorname{coker}(\delta'')$ has zero kernel and cokernel. Thus it is invertible of degree $-\iota$, and it can be viewed as an isomorphism of the functor H' with H up to the question of degree.

The next application of the snake lemma 2.3 is to a short exact sequence $0 \to X' \to X \to X'' \to 0$ of complexes in $C_\Theta(\mathcal{A})$ and this is
6 possible because the following diagram is commutative with exact rows arising from the snake lemma applied to the morphism $(d(X'), d(X), d(X''))$

$$\begin{array}{ccccccc} & & Z'(X') & \longrightarrow & Z'(X) & \longrightarrow & Z'(X'') & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \hat{d}(X')} & & \downarrow{\scriptstyle \hat{d}(X)} & & \downarrow{\scriptstyle \hat{d}(X'')} & & \\ 0 & \longrightarrow & Z(X') & \longrightarrow & Z(X) & \longrightarrow & Z(X'') & & \end{array}$$

Since H' is the kernel of $\hat{d}$ and H is the cokernel of $\hat{d}$, we obtain the exact sequence

$$H'(X'') \to H'(X) \to H'(X'') \xrightarrow{\delta} H(X') \to H(X) \to H(X''),$$

and using the isomorphism $H' \to H$, we obtain an exact triangle which we formulate in the next basic theorem about homology.

Theorem 2.5. *Let* $0 \to X' \to X \to X'' \to 0$ *be a short exact sequence of complexes in* $C_\Theta(\mathcal{A})$*. Then there is a natural morphism* $\delta : H(X'') \to H(X')$ *such that the following triangle is exact*

$$\begin{array}{ccc} H(X') & \longrightarrow & H(X) \\ & \nwarrow \quad \swarrow & \\ & H(X'') & \end{array}$$

Here the degree of δ *is* $-\iota$*, the degree of* d*.*

3 Formal structure of cyclic and Hochschild homology

Definition 3.1. An algebra A over k is a triple $(A, \phi(A), \eta(A))$ where A is a k-module, $\phi(A) : A \otimes A \to A$ is a morphism called multiplication, and $\eta(A) : k \to A$ is a morphism called the unit which satisfies the following axioms:

(1) (associativity) As morphisms $A \otimes A \otimes A \to A$ we have

$$\phi(A)(\phi(A) \otimes A) = \phi(A)(A \otimes \phi(A))$$

where as usual A denotes both the object and the identity morphism on A. **7**

(2) (unit) As morphisms $A \otimes k \to A$ and $k \otimes A \to A$, the morphisms

$$\phi(A)(A \otimes \eta(A)) \quad \text{and} \quad \phi(A)(\eta(A) \otimes A)$$

are the natural isomorphisms for the unit k of the tensor product. Let Θ be an abelian group. A Θ-graded algebra A over k is a triple $(A, \phi(A), \eta(A))$ where A is a Θ-graded k-module, $\phi(A) : A \otimes A \to A$ is a morphism of Θ-graded k-modules, and $\eta(A) : k \to A$ is a morphism of Θ-graded k-modules satisfying the above axioms (1) and (2).

A morphism $f : A \to A'$ of Θ-graded algebras is a morphism of Θ-graded modules such that $\phi(A')(f \otimes f) = f\phi(A)$ as morphisms $A \otimes A \to A'$ and $f\eta(A) = \eta(A')$ as morphisms $k \to A'$. If $f : A \to A'$ and $f' : A' \to A''$ are two morphisms of Θ-graded algebras, then $f'f : A \to A''$ is a morphism of Θ-graded algebras. Let $\mathrm{Alg}_{\Theta,k}$ denote the category of Θ-graded algebras over k, and when $\Theta = 0$, the zero grading, then we denote $\mathrm{Alg}_{0,k}$ by simply Alg_k.

Notation 3.2. For an abelian group Θ and a pointed category C we denote by $(Z \times \Theta)^+(C)$ the full subcategory of $(\mathbf{Z} \times \Theta)(C)$ determined by all $X_\cdot = (X_{n,\theta})$ with $X_{n,\theta} = *$ for $n < 0$ and $(\mathbf{Z} \times \Theta)^-(C)$ the full subcategory determined by all $X_\cdot = (X_{n,\theta})$ with $X_{n,\theta} = *$ for $n > 0$. The intersection $(\mathbf{Z} \times \Theta)^+(C) \cap (\mathbf{Z} \times \Theta)^-(C)$ can be identified with $\Theta(C)$.

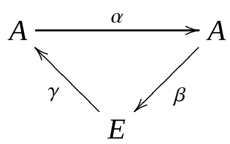
Remark 3.3. As functors, cyclic homology and Hochschild homology, denoted by HC_* and HH_* respectively, are defined

$$HC_* : \mathrm{Alg}_{\Theta,k} \to (\mathbf{Z} \times \Theta)^+(k) \quad \text{and} \quad HH_* : \mathrm{Alg}_{\Theta,k} \to (\mathbf{Z} \times \Theta)^+(k).$$

This is the first indication of what kinds of functors these are.

When he first introduced cyclic homology HC_*, Connes' emphasised that cyclic homology and Hochschild homology were linked with exact sequences which can be assembled into what is called an exact
8 couple. We introduce exact couples with very general gradings to describe this linkage.

Definition 3.4. Let Θ be an abelian group with θ, θ', $\theta'' \in \Theta$ and let $\mathcal{A}$ be a abelian category. An exact couple over $\mathcal{A}$ with degrees θ, θ', θ'' is a pair of objects A and E and three morphisms $\alpha : A \to A$ of degree θ, $\beta : A \to E$ of degree θ', and $\gamma : E \to A$ of degree θ'' such that the following triangle is exact.



In particular, we have $\mathrm{im}(\alpha) = \ker(\beta)$, $\mathrm{im}(\beta) = \ker(\gamma)$, and $\mathrm{im}(\gamma) = \ker(\alpha)$.

Let $(A, E, \alpha, \beta, \gamma)$ and $(A', E', \alpha', \beta', \gamma')$ be two exact couples of degree θ, θ', θ''. A morphism from the first to the second is pair of morphisms (h, f), where $h : A \to A'$ and $f : E \to E'$ are morphisms of degree 0 in $\Theta(\mathcal{A})$ such that $h\alpha = \alpha' h$, $f\beta = \beta' h$, $h\gamma = \gamma' f$. The composition of two morphisms (h, f) and (h', f') is $(h', f')(h, f) = (h'h, f'f)$ when defined. Thus we can speak of the category $ExC(\mathcal{A}; \Theta; \theta, \theta', \theta'')$ of exact couples $(A, E, \alpha, \beta, \gamma)$ in $\Theta(\mathcal{A})$ of degrees θ, θ', θ''.

We can now describe the Cyclic-Hochschild homology linkage in terms of a single functor.

Remark 3.5. The Connes' exact sequence (or exact couple) is a functor

$$(HC_*, HH_*, S, B, I) : \mathrm{Alg}_{\Theta,k} \to ExC((k), \mathbf{Z} \times \Theta, (-2, 0), (1, 0), (0, 0))$$

which, incorporating the remark (3.3) satisfies $HC_n(A) = 0 = HH_n(A)$ for $n < 0$. The special feature of the degrees formally gives two elementary results.

9 **Proposition 3.6.** *The natural morphism $I : HH_0(A) \to HC_0(A)$ is an isomorphism of functors $\mathrm{Alg}_{\Theta,k} \to \Theta(k)$.*

Proof. We have an isomorphism since $\ker(I)$ is zero for reasons of degree and

$$\mathrm{im}(I) = \ker(S : HC_0(A) \to HC_{-2}(A)) = HC_0(A)$$

again, due to degree considerations. This proves the proposition. □

Proposition 3.7. *Let $f : A \to A'$ be a morphism in $\mathrm{Alg}_{\Theta,k}$. Then $HC_*(f)$ is an isomorphism if and only if $HH_*(f)$ is an isomorphism.*

Proof. The direct implication is a generality about morphisms (h, f) of exact couples in any abelian category, namely, if h is an isomorphism, then by the five-lemma f is an isomorphism. Conversely, if we assume that $HC_i(f)$ is an isomorphism for $i < n$ and $HH_*(f)$ is an isomorphism, then $HC_n(f)$ is an isomorphism by the five-lemma applied to the exact sequence

$$HC_{n-1} \xrightarrow{B} HH_n \xrightarrow{I} HC_n \xrightarrow{s} HC_{n-2} \xrightarrow{B} HH_{n-1}.$$

The induction begins with the result in the previous proposition. This proves the proposition. In the next section we study the category of exact couples as a preparation for defining Hochschild and cyclic homology and investigating its properties. We also survey some of the classical examples of exact couples. □

4 Derivation of exact couples and their spectral sequence

The snake lemma (2.3) is a kernel-cokernel exact sequence coming from a morphism of exact sequences. There is another basic kernel-cokernel exact sequence coming from a composition of two morphisms. We announce the result and refer to Bourbaki, *Algébre homologique* for the proof.

Lemma 4.1. *Let $f : X \to Z$ and $g : Z \to Y$ be two morphisms in an abelian category $\mathcal{A}$. Then there is an exact sequence-*

$$0 \to \ker(f) \to \ker(gf) \xrightarrow{f'} \ker(g) \to \operatorname{coker}(f) \xrightarrow{g'} \operatorname{coker}(gf) \to \operatorname{coker}(g) \to 0$$

where $f' : \ker(gf) \to \ker(g)$ is induced by f, $g' : \operatorname{coker}(f) \to \operatorname{coker}(gf)$ is induced by g, and the other three arrows are induced respectively by the identities on X, Z, and Y.

We wish to apply this to an exact couple $(A, E, \alpha, \beta, \gamma)$ in the category $ExC(\mathcal{A}, \Theta; \theta, \theta', \theta'')$ to obtain a new exact couple, called the derived couple. In fact there will be two derived couples one called the left and the other the right derived couple differing by an isomorphism of nonzero degree.

First, observe that $\alpha : A \to A$ factorizes naturally as the composite of the natural epimorphism $A \to \operatorname{coker}(\gamma)$, an invertible morphism $\alpha^\# : \operatorname{coker}(\gamma) \to \ker(\beta)$, and the natural monomorphism $\ker(\beta) \to A$. Secondly, since $(\beta\gamma)(\beta\gamma) = 0$, we have an induced morphism $\beta\gamma : \operatorname{coker}(\beta\gamma) \to \ker(\beta\gamma)$ whose kernel and cokernel are naturally isomorphic to $H(E, \beta\gamma)$ by the snake exact sequence as is used in 2.4. Finally, there is a natural factorization of $\beta\gamma : \operatorname{coker}(\beta\gamma) \to \ker(\beta\gamma)$ as a quotient $\gamma^\# : \operatorname{coker}(\beta\gamma) \to A$ of γ composed with a restriction $\beta^\# : A \to \ker(\beta\gamma)$ of β. Then we have $\ker(\beta) = \ker(\beta^\#)$ and $\operatorname{coker}(\gamma) = \operatorname{coker}(\gamma^\#)$. Now we apply (4.1) to the factorization of $\beta\gamma = \beta^\#\gamma^\#$ and consider the middle four terms of the exact sequence

$$H(E, \beta\gamma) \xrightarrow{\gamma^0} \ker(\beta) \xrightarrow{\delta} \operatorname{coker}(\gamma) \xrightarrow{\beta^0} H(E, \beta\gamma).$$

Definition 4.2. We denote $ExC(\mathcal{A}, \Theta; \theta, \theta', \theta'')$ by simply $ExC(\theta, \theta', \theta'')$. The left derived couple functor defined

$$ExC(\theta, \theta', \theta'') \to ExC(\theta, \theta', \theta'' - \theta)$$

assigns to an exact couple $(A, E, \alpha, \beta, \gamma)$, the exact couple

$$(\operatorname{coker}(\gamma), H(E, \beta\gamma), \alpha_\lambda, \beta_\lambda, \gamma_\lambda)$$

where $\alpha_\lambda = \delta\alpha^\#, \beta_\lambda = \beta^0$, and $\gamma_\lambda = (\alpha^\#)^{-1}\gamma^0$, using the above notations. The right derived couple functor defined

$$ExC(\theta, \theta', \theta'') \to ExC(\theta, \theta' - \theta, \theta'')$$

assigns to an exact couple $(A, E, \alpha, \beta, \gamma)$, the exact couple 11

$$(\ker(\beta), H(E, \beta\gamma), \alpha_\rho, \beta_\rho, \gamma_\rho)$$

where $\alpha_\rho = \alpha^\#\delta, \beta_\rho(\alpha^\#)^{-1}$, and $\gamma_\rho = \gamma^0$ using the above notations.

Observe that $(\alpha^\#, H(E, \beta\gamma))$ is an invertible morphism

$$(\text{coker}(\gamma), H(E, \beta\gamma), \alpha_\lambda, \beta_\lambda, \gamma_\lambda) \to (\ker(\beta), H(E, \beta\gamma), \alpha_\rho, \beta_\rho, \gamma_\rho)$$

which shows that the two derived couple functors differ only by the degree of the morphism. The only point that remains, is to check exactness of the derived couple at $H(E, \beta\gamma)$, and for this we use (4.1) as follows. The composite of

$$\gamma^\# : \text{coker}(\beta\gamma) \to A \quad \text{and} \quad \beta^\# : A \to \ker(\beta\gamma)$$

is $\beta\gamma : \text{coker}(\beta\gamma) \to \ker(\beta\gamma)$, and by (4.1) we have a six term exact sequence

$$0 \to \ker(\gamma^\#) \to H(E) \xrightarrow{\gamma^0} \ker(\beta) \xrightarrow{\delta} \text{coker}(\gamma) \xrightarrow{\beta^0} H(E) \to \text{coker}(\beta^\#) \to 0.$$

Hence the following two sequences

$$H(E) \xrightarrow{\gamma_\lambda} \text{coker}(\gamma) \xrightarrow{\alpha_\lambda} \text{coker}(\gamma) \xrightarrow{\beta_\lambda} H(E)$$

and

$$H(E) \xrightarrow{\gamma_\rho} \ker(\beta) \xrightarrow{\alpha_\rho} \ker(\beta) \xrightarrow{\beta_\rho} H(E).$$

are exact. It remains to show that the derived couple is exact at $H(E)$. For this, we start with the exact sequence $A \xrightarrow{\beta} E \xrightarrow{\gamma} A$ of the given exact couple and observe that $\text{im}(\beta\gamma) \subset \text{im}(\beta) = \ker(\gamma) \subset \ker(\beta\gamma)$. Hence the sequence $\text{coker}(\gamma) \to \ker(\beta\gamma)/\,\text{im}(\beta\gamma) = H(E) \to \ker(\beta)$ is exact where the first arrow is β^0 and the second is γ^0. Using the invertible morphism $\alpha^\#$, we deduce that the left and right derived couples are exact couples. This completes the discussion of definition (4.2). 12

Remark 4.3. Let $C_{\Theta,-\iota}(\mathcal{A})$ denote the category of complexes over $\mathcal{A}$, graded by Θ, and with differential of degree $-\iota$. We have used the functor $ExC(\mathcal{A},\Theta : \theta,\theta',\theta'') \to C_{\Theta,\theta'+\theta''}(\mathcal{A})$ which assigns to an exact couple $(A,E,\alpha,\beta,\gamma)$ the complex $(E,\beta\gamma)$. Further, composing with the homology functor, we obtain $H(E)$ which is the second term in the derived couple of $(A,E,\alpha,\beta,\gamma)$.

Remark 4.4. Now we iterate the process of obtaining the derived couple. For an exact couple $(A,E) = (A,E,\alpha,\beta,\gamma)$ in $ExC(\theta,\theta',\theta'')$, we have a sequence of exact couples (A^r,E^r) where $(A,E) = (A^1,E^1)$, (A^r,E^r) is the derived couple of (A^{r-1},E^{r-1}), and $E^{r+1} = H(E^r,d^r)$ with $d^r = \beta\gamma^r$. As for degrees (A^r,E^r) is in $ExC(\theta,\theta',\theta''-(r-1)\theta)$ for a sequence of left derived couples and in $ExC(\theta,\theta'-(r-1)\theta,\theta'')$ for a sequence of right derived couples. In either case the complex (E^r,d^r) is in $C_{\Theta,\theta'+\theta''-(r-1)\theta}(\mathcal{A})$, and the sequence of complexes (E^r,d^r) is an example of a spectral sequence because of the property that $E^{r+1} = H(E^r,d^r)$. We can give a direct formula for the terms E^r as subquotients of $E = E^1$. Firstly, we know that

$$\begin{aligned} E^2 &= H(E^1,\beta\gamma) = \ker(\beta\gamma)/\operatorname{im}(\beta\gamma) = \gamma^{-1}(\ker(\beta)/\beta(\operatorname{im}(\gamma)) \\ &= \gamma^{-1}(\operatorname{im}(\alpha))/\beta(\ker(\alpha)), \end{aligned}$$

and by analogy, the general formula is the following:

$$E^r = \gamma^{-1}(\operatorname{im}(\alpha^{r-1}))/\beta(\ker(\alpha^{r-1})).$$

We leave the proof of this assertion to the reader.

5 The spectral sequence and exact couple of a filtered complex

The most important example of an exact couple and its associated spectral sequence is the one coming from a filtered complex.

13 **Definition 5.1.** A filtered object X in a category C is an object X together with a sequence of subobjects, F_pX or $F_p(X)$, indexed by the integers

$\cdots \to F_{p-1}X \to F_pX \to \cdots \to X$. A morphism $f : X \to Y$ of filtered objects in C is a morphism $f : X \to Y$ in C which factors for each p by $F_p(f) : F_pX \to F_pY$.

The factorization $F_p(f)$ is unique since $F_pY \to Y$ is a monomorphism. The composition gf in C of two morphisms $f : X \to Y$ and $g : Y \to Z$ of filtered objects is again a morphism of filtered objects. Thus we can speak of the category $F \cdot C$ of filtered objects over C.

Remark 5.2. We are interested in the category $\mathcal{F} \cdot C_{\Theta}(A)$ of filtered complexes. In particular we construct a functor

$$E^0 : \mathcal{F} \cdot C_{\Theta,-\iota}(\mathcal{A}) \to C_{\mathbf{Z}\times\Theta,(0,-\iota)}(\mathcal{A})$$

by assigning to the filtered complex X the complex $E^0(X)$, called the associated graded complex, with graded term

$$E^0_{p,\theta} = F_pX_\theta / F_{p-1}X_\theta$$

and quotient differential in the following short exact sequence

$$0 \to F_{p-1}X \to F_pX \to E^0_p \to 0$$

in the category $C_{\Theta}(\mathcal{A})$. The homology exact triangle is a sequence of Θ-graded exact triangles which can be viewed as a single $(\mathbf{Z}\times\Theta)$-graded exact triangle and this exact triangle is an exact couple

$$\begin{array}{ccc} H_*(F_*X) & \xrightarrow{\ \alpha\ } & H_*(F_*X) = A^1_{*,*} \\ & {\scriptstyle\gamma}\nwarrow \quad \swarrow{\scriptstyle\beta} & \\ & H_*(E^0_*) = E^1_{*,*} & \end{array}$$

where the $\mathbf{Z} \times \Theta$-degree of α is $(1, 0)$, of β is $(0, 0)$, and of γ is $(-1, -\iota)$. The theory of the previous section says that we have a spectral sequence (E^r, d^r) and the degree of d^r is $(-r, -\iota)$. Moreover, we have defined a functor (A^r, E^r) on the category $\mathcal{F} \cdot C_{\Theta,-\iota}(\mathcal{A}) \to ExC(\mathcal{A}, \Theta; (1, 0), 0,$ **14**
$(-r, -\iota))$ such that (A^{r+1}, E^{r+1}) is the left derived couple of (A^r, E^r).

In the case where $\Theta = \mathbf{Z}$, the group of integers, and $\iota = +1$, there is a strong motivation to index the spectral sequence with the filtration index p, as above, and the complementary index $q = \theta - p$ where θ denotes the total degree of the object. In particular, we have $H_{p+q}(E^0_{p,*}) = E^1_{p,q}$ in terms of the complementary index. The complementary index notation is motivated by the Leray-Serre spectral sequence of a map $p : E \to B$ where the main theorem asserts that there is a spectral sequence with $E^2 p, q = H_p(B, H_q(F))$ coming from a filtration on the chains of the total space $E.$, F being the fibre of the morphism p.

Remark 5.3. The filtration on a filtered complex X defines a filtration on the homology $H(X)$ of X by the relation that

$$F_p H_*(X) = \mathrm{im}(H_*(F_p X) \to H_*(X)).$$

Now this filtration has something to do with the terms $E^r_{p,*}$ of the spectral sequence. We carry this out for the following special case which is described by the following definition.

Definition 5.4. A filtered object X in a pointed category is positive provided $F_p X = 0$ (cf. (1.3)) for $p < 0$. A filtered Θ-graded object X has a locally finite filtration provided for each $\theta \in \Theta$ there exists integers $m(\theta)$ and $n(\theta)$ such that

$$F_p X_\theta = 0 \quad \text{for} \quad p < m(\theta) \quad \text{and} \quad F_p X_\theta = X_\theta \quad \text{for} \quad n(\theta) < p.$$

Proposition 5.5. *Let X be a locally finite filtered Θ-graded complex X over an abelian category $\mathcal{A}$. Then for a given $\theta \in \Theta$ and filtration index p, if $r > \max(n(\theta) + 1 - p, p - m(\theta - \iota))$, then we have $E^r_{p,\theta} = E^{r+1}_{p,\theta} = \ldots = F_p H_\theta(X)/F_{p-1} H_\theta(X) = E^0 H_\theta(X)$.*

15 *Proof.* We use the characterization of the terms E^r given at the end of (4.4). For $A^1_{p,\theta} \xrightarrow{\beta} E^1_{p,\theta} \xrightarrow{\gamma} A^1_{p-1,\theta-\iota}$ we form a subquotient using $\alpha^{r-1} : A^1_{p,\theta} \to A^1_{p+r-1,\theta}$ and $\alpha^{r-1} : A^1_{p-r,\theta-\iota} \to A^1_{p-1,\theta-\iota}$ where $A^1_{p+r-1,\theta} = H_\theta(X)$ and $A^1_{p-r,\theta-\iota} = 0$ under the above conditions on r. Thus the term $E^r = \gamma^{-1}(\mathrm{im}(\alpha^{r-1}))/\beta(\ker(\alpha^{r-1}))$ has the form

$$E^r_{p,\theta} = \gamma^{-1}(\mathrm{im}(0))/\beta(\ker(\alpha^{r-1})) = \mathrm{im}(\beta)/\beta(\ker(H_\theta(F_p X) \to H_\theta(X)),$$

and this is isomorphic under β to the quotient

$$A^1_{p,\theta}/((\ker(H_\theta(F_pX) \to H_\theta(X)) + \operatorname{im}(H_\theta(F_{p-1}X) \to H_\theta(F_pX))).$$

This quotient is mapped isomorphically by α^{r-1} to the following subquotient of $H_\theta(X)$, which is just the associated graded object for the filtration on $H(X)$ defined in (5.3),

$$\operatorname{im}(H_\theta(F_pX) \to H_\theta(X))/\operatorname{im}(H_\theta(F_{p-1}X) \to H_\theta(X)) = E^0H_\theta(X).$$

This proves the proposition. □

This proposition and the next are preliminaries to the spectral mapping theorem.

Proposition 5.6. *Let $f : L \to M$ be a morphism of locally finite filtered Θ-graded objects over an abelian category $\mathcal{A}$. If the morphism of associated $\mathbf{Z}\times\Theta$-graded objects $E^0(f) : E^0(L) \to E^0(M)$ is an isomorphism, then $f : L \to M$ is an isomorphism.*

Proof. For $F_pL_\theta = 0$, $F_pM_\theta = 0$ if $p < m(\theta)$ and $F_pL_\theta = L_\theta$, $F_pM_\theta = M_\theta$ if $p > n(\theta)$ we show inductively on p from $m(\theta)$ to $n(\theta)$ that $F_pf : F_pL_\theta \to F_pM_\theta$ is an isomorphism. To begin with, we note that by hypothesis $F_{m(\theta)} = E^0_{m(\theta),\theta}$ is an isomorphism. If the inductive statement is true for $p - 1$, then it is true for p by applying the "5-lemma" to the short exact sequence

$$0 \to F_{p-1,\theta} \to F_{p,\theta} \to E^0_{p,\theta} \to 0.$$

Since the induction is finished at $n(\theta)$, this proves the proposition. **16**
□

This proposition is true under more general circumstances which we come back to after the next theorem.

Theorem 5.7. *Let $f : X \to Y$ be a morphism of locally finite filtered Θ-graded complexes over an abelian category $\mathcal{A}$. If for some $r \geq 0$ the term $E^r(f) : E^r(X) \to E^r(Y)$ is an isomorphism, then $H(f) : H(X) \to H(Y)$ is an isomorphism.*

Proof. Since $E^{r+1} = H(E^r)$ as functors, we see that all $E^{r'}(f)$ are isomorphisms for $r' \geq r$. For given $\theta \in \Theta$ and filtration index p we know by (5.5) that $E^r_{p,\theta} = E^0_p H_\theta$ for r large enough. Hence $E^0 H(f)$ is an isomorphism, and by (5.6) we deduce that $H(f)$ is an isomorphism. This proves the theorem. □

This theorem illustrates the use of spectral sequences to prove that a morphism of complexes $f : X \to Y$ induces an isomoprhism $H(f) : H(X) \to H(Y)$. The hypothesis of locally finite filtration is somewhat restrictive for general cyclic homology, but the general theorem, which is contained in Eilenberg and Moore [1962], is clearly given in their article. The modifications involve limits, injective limits as p goes to plus infinity and projective limits as p goes to minus infinity. We explain these things in the next section on the filtered complex related to a double complex.

6 The filtered complex associated to a double complex

For the theory of double complexes we use the simple $\mathbf{Z} \times \mathbf{Z}$ grading which is all we need in cyclic homology. Firstly, we consider an extension of (5.6) for filtered objects which are constructed from a graded object.

Remark 6.1. Let $\mathcal{A}$ denote an abelian category with countable products and countable coproducts. For a $\mathbf{Z}$-graded object SX_p we form the object $X. = \prod_{i \leq a} X_i \times \coprod_{a<i} X_i$ with filtration $F_p X. = \prod_{i \leq p} X_i$. The definition of
17 $X.$ is independent of a. With these definitions the natural morphisms

$$X. \to \lim_{\leftarrow p} X./F_p X. \quad \text{and} \quad \lim_{\to p} F_p X. \to X.$$

are isomorphisms. In general for any filtered object X these natural morphisms are defined. If the first morphism is an isomorphism, then X is called complete, if the second morphism is an isomorphism, then X is called cocomplete, and if the two morphisms are isomorphisms, then

X is called bicomplete. With these definitions we have the following extension of (5.6) not proved here.

Remark 6.2. Let $f : L \to M$ be a morphism of bicomplete filtered objects over an abelian category with countable products and coproducts. If $E^0(f) : E^0(L) \to E^0(M)$ is an isomorphism, then $f : L \to M$ is an isomorphism of filtered objects.

Now we consider double complexes and their associated filtered complexes which will always be constructed so as to be bicomplete.

Definition 6.3. Let $\mathcal{A}$ be an abelian category. A double complex $X_{\cdot\cdot}$ over $\mathcal{A}$ is a $\mathbf{Z} \times \mathbf{Z}$-graded object with two morphisms $d' = d'(X)$, $d'' = d''(X) : X_{\cdot\cdot} \to X_{\cdot\cdot}$ of degree $(-1, 0)$ and $(0, -1)$ respectively satisfying $d'd' = 0$, $d''d'' = 0$, and $d'd'' + d''d' = 0$. A morphism of double complexes $f : X_{\cdot\cdot} \to Y_{\cdot\cdot}$ is a morphism of graded objects such that $d'(Y)f = fd'(X)$ and $d''(Y)f = fd''(X)$. With the composition of graded morphisms we define the composition of morphisms of double complexes. We denote the category of double complexes over $\mathcal{A}$ by $DC(\mathcal{A})$.

There are two functors $DC(\mathcal{A}) \to \mathcal{F} \cdot C(\mathcal{A})$ from double complexes to bicomplete filtered complexes corresponding to a filtration on the first variable or on the second variable.

Definition 6.4. Let $X_{\cdot\cdot}$ be a double complex over the abelian category $\mathcal{A}$. We form:

(1) the filtered graded object ${}^{I}X_{\cdot}$ with **18**

$$ {}^{I}X_n = \prod_{i+j=n, i\le a} X_{i,j} \times \coprod_{i+j=n, i>a} X_{i,j} \quad \text{and} \quad {}^{I}F_pX_n = \prod_{i+j=n, i\le p} X_{i,j}, $$

(2) the filtered graded object ${}^{II}X_{\cdot}$ with

$$ {}^{II}X_n = \prod_{i+j=n, j\le a} \times \coprod_{i+j=n, j>a} X_{i,j} \quad \text{and} \quad {}^{II}F_pX_n = \coprod_{i+j=n, j\le p} X_{i,j} $$

and in both cases the differential is $d = d' + d''$, making the filtered graded objects into bicomplete filtered complexes.

Remark 6.5. Using the complementary degree indexing notation considered in (5.2), we see that

$$E^0_{p,q}({}^I X) = X_{p,q} \text{ with } d^0 = d'' \text{ and } E^0_{p,q}({}^{II} X) = X_{q,p} \text{ with } d^0 = d',$$

and the E^1 terms are the partial homology modules of the double complex with respect to d'' and d respectively. The d^1 differentials are induced by d' and d'' respectively, and the E^2 term is the homology of (E^1, d^1), and

$${}^I E^2_{p,q} = H_p(H_q(X, d''), d') \quad \text{and} \quad {}^{II} E^2_{p,q} = H_q(H_p(X, d'), d'').$$

These considerations in this section are used in the full development of cyclic homology, and they are included here for the sake of completeness.

Chapter 2

Abelianization and Hochschild Homology

IN THIS CHAPTER we first consider abelianization in the contexts of 19
associative algebras and Lie algebras together with the adjunction properties of the related functors. In degree zero, Hochschild and cyclic homology of an algebra A are isomorphic and equal to a certain abelianization of A which involves the related Lie algebra structure on A. We will give the axiomatic definition of Hochschild homology $H_*(A,M)$ of A with values in an A-bimodule M, discuss existence and uniqueness, and relate in degree zero $H_0(A,A)$, the Hochschild homology of A with values in the A-bimodule A, to the abelianization of A. The k-modules $H_*(A,A)$ are the absolute Hochschild homology k-modules $HH_*(A)$ which were considered formally in the previous chapter in conjunction with cyclic homology.

1 Generalities on adjoint functors

Abelianization is defined by a universal property relative to the subcategory of abelian objects. The theory of adjoint functors, which we sketch now, is the formal development of this idea of a universal property, and this theory also gives a means for constructing equivalences between categories. We approach the subject by considering morphisms between the identity functor and a composite of two functors.

For an object X in a category, we frequently use the symbol X also for the identity morphism $X \to X$ along with 1_X, and similarly, for a category $\mathcal{X}$ the identity functor on $\mathcal{X}$ is frequently denoted $\mathcal{X}$. Let (sets) denote the category of sets.

Remark 1.1. Let $\mathcal{X}$ and $\mathcal{Y}$ be two categories and $S : \mathcal{X} \to \mathcal{Y}$ and $T : \mathcal{Y} \to \mathcal{X}$ be two functors. Morphisms of functors $\beta : \mathcal{X} \to TS$ are in bijective correspondence with morphisms

$$b : \mathrm{Hom}_{\mathcal{Y}}(S(X), Y) \to \mathrm{Hom}_{\mathcal{X}}(X, T(Y))$$

20 as functors of X in $\mathcal{X}$ and Y in $\mathcal{Y}$ with values in (sets). A morphism β defines b by the relation $b(g) = T(g)\beta(X)$ and b defines β by the relation $b(1_{S(X)}) = \beta(X) : X \to TS(X)$. In the same way, morphisms of functors $\alpha : ST \to \mathcal{Y}$ are in bijective correspondence with morphisms

$$a : \mathrm{Hom}_{\mathcal{X}}(X, T(Y)) \to \mathrm{Hom}_{\mathcal{Y}}(S(X), Y)$$

as functors of X and Y with values in (sets). A morphism α defines a by the relation $a(f) = \alpha(Y)S(f)$, and a defines α by the relation

$$a(1_{T(Y)}) = \alpha(Y) : ST(Y) \to Y.$$

Definition 1.2. An adjoint pair of functors is a pair of functors $S : \mathcal{X} \to \mathcal{Y}$ and $T : \mathcal{Y} \to \mathcal{X}$ together with an isomorphism of functors of X in $\mathcal{X}$ and Y in $\mathcal{Y}$

$$b : \mathrm{Hom}(S(X), Y) \to \mathrm{Hom}(X, T(Y)),$$

or equivalently, the inverse isomorphism

$$a : \mathrm{Hom}(X, T(Y)) \to \mathrm{Hom}(S(X), Y).$$

The functor S is called the left adjoint of T and T is the right adjoint of S.

This situation is denoted $(a, b) : S \dashv T(\mathcal{X}, \mathcal{Y})$ or just $S \dashv T$.

In terms of the morphisms $\beta(X) = b(1_{S(X)}) : X \to TS(X)$ and $\alpha(Y) = a(1_{T(Y)}) : ST(Y) \to Y$ we calculate, for $f : X \to T(Y)$

$$b(s(f)) = T(a(f))\beta(X) = T(\alpha(Y))TS(f)\beta(X) = T(\alpha(Y))\beta(T(Y))f$$

and for $g : S(X) \to Y$

$$a(b(g)) = \alpha(Y)S(b(g)) = \alpha(Y)ST(g)S(\beta(X)) = g\alpha(S(X))S(\beta(X))$$

Remark 1.3. With the above notations we have **21**

$$b(a(f)) = f \quad \text{if and only if} \quad T(\alpha(Y))\beta(T(Y)) = 1_{T(Y)} \quad \text{and}$$
$$a(b(g)) = g \quad \text{if and only if} \quad \alpha(S(X)S(\beta(X)) = 1_{S(X)}.$$

An adjoint pair of functors can be defined as a pair of functors $S : \mathcal{X} \to \mathcal{Y}$ and $T : \mathcal{Y} \to \mathcal{X}$ together with two morphisms of functors

$$\beta : \mathcal{X} \to TS \quad \text{and} \quad \alpha : ST \to \mathcal{Y}$$

satisfying $\alpha(S(Y))S(\beta(X)) = 1_{S(X)}$ and $T(\alpha(Y))\beta(T(Y)) = 1_{T(Y)}$. This situation is denoted $(\alpha, \beta) : S \dashv T : (\mathcal{X}, \mathcal{Y})$ or just $(\alpha, \beta) : S \dashv T$.

Remark 1.4. If $S : \mathcal{X} \to \mathcal{Y}$ is the left adjoint of $T : \mathcal{Y} \to \mathcal{X}$, then for the dual categories $S : \mathcal{X}^{op} \to \mathcal{Y}^{op}$ is the right adjoint of $T : \mathcal{Y}^{op} \to \mathcal{X}^{op}$.

The relation between adjoint functors and universal constructions is contained in the next proposition.

Proposition 1.5. *Let $S : \mathcal{X} \to \mathcal{Y}$ be a functor, and for each object Y in $\mathcal{Y}$, assume that there exists an object $t(Y)$ in $\mathcal{X}$ and a morphism $\alpha(Y) : S(t(Y)) \to Y$ such that for all $g : S(X) \to Y$, there exists a unique morphism $f : X \to t(Y)$ such that $\alpha(Y)S(f) = g$. Then there exists a right adjoint functor $T : \mathcal{Y} \to \mathcal{X}$ of S such that for each object Y in $\mathcal{Y}$ the object $T(Y)$ is $t(Y)$ and*

$$a : \mathrm{Hom}(X, T(Y)) \to \mathrm{Hom}(S(X), Y)$$

is given by $a(f) = \alpha(Y)S(f)$.

Proof. To define T on moprhisms, we use the universal property. If $v : Y \to Y'$ is a morphism in $\mathcal{Y}$, then there exists a unique morphism $T(v) : T(Y) \to T(Y')$ such that $\alpha F(Y')(S(T(v)) = v\alpha(Y)$ as morphisms $ST(Y) \to Y'$. The reader can check that this defines a functor T, and the rest follows from the fact that the universal property asserts that a is a bijection. This proves the proposition. □ **22**

Proposition 1.5*. *Let $T : \mathcal{Y} \to \mathcal{X}$ be a functor, and for each object X in $\mathcal{X}$ assume there exists an object $s(X)$ in $\mathcal{Y}$ and a morphism $\beta(X) : X \to T(s(X))$ such that for all $f : X \to T(Y)$ there exists a unique $g : s(X) \to Y$ such that $T(g)\beta(X) = f$. Then there exists a left adjoint functor $S : \mathcal{X} \to \mathcal{Y}$ such that for each object X in $\mathcal{X}$ the object $S(X)$ is $s(X)$ and*

$$b : \mathrm{Hom}(S(X), Y) \to \mathrm{Hom}(X, T(Y))$$

is given by $b(g) = T(g)\beta(X)$.

Proof. We deduce $(1.5)^*$ immediately by applying (1.5) to the dual category. □

2 Graded commutativity of the tensor product and algebras

Let Θ denote an abelian group with a morphism $\epsilon : \Theta \to \{\pm 1\}$, and define a corresponding bimultiplicative $\epsilon : \Theta \times \Theta \to \{\pm 1\}$, by the requirement that

$$\epsilon(\theta, \theta') = \begin{cases} +1 & \text{if } \epsilon(\theta) = 1 \text{ or } \epsilon(\theta') = 1 \\ -1 & \text{if } \epsilon(\theta) = -1 \text{ and } \epsilon(\theta') = -1. \end{cases}$$

Definition 2.1. The commuting morphism σ or σ_ϵ of the tensor product $\times : \Theta(k) \times \Theta(k) \to \Theta(k)$ relative to ϵ is the morphism

$$\sigma(L, M) = \sigma : L \otimes M \to M \otimes L$$

defined for $x \otimes y \in L_\theta \otimes M_{\theta'}$ by the relation

$$\sigma_\epsilon(x \otimes y) = \epsilon(\theta, \theta')(y \otimes x).$$

Observe that $\sigma(M, L)\sigma(L, M) = L \otimes M$, the identity on the object $L \otimes M$.

Definition 2.2. A Θ-graded k-algebra A is commutative (relative to ϵ) provided $\phi(A)\sigma_\epsilon(A, A) = \phi(A) : A \otimes A \to A$. The full subcategory of
23 $\mathrm{Alg}_{\Theta,k}$ determined by the commutative algebras is denoted by $C\,\mathrm{Alg}_{\Theta,k}$.

Remark 2.3. Let A be a Θ-graded k-algebra. For $a \in A_\theta$ and $b \in A_{\theta'}$ the Lie bracket of a and b is

$$[a, b] = ab - \epsilon(\theta, \theta')ba$$

which is an element of $A_{\theta+\theta'}$. Let $[A, A]$ denote the Θ-graded k - submodule of A generated by all Lie brackets $[a, b]$ for a, $b \in A$. Observe that A is commutative if and only if $[A, A^c] = 0$. Let (A, A) denote the two-sided ideal generated by $[A, A]$.

Definition 2.4. A Θ-graded Lie algebra over k is a pair $\underline{g}$ together with a graded k-linear map $[,] : \underline{g} \otimes \underline{g} \to \underline{g}$, called the Lie bracket, satisfying the following axioms:

(1) For $a \in \underline{g}_\theta$ and $b \in \underline{g}_{\theta'}$ we have

$$[a, b] = -\epsilon(\theta, \theta')[b, a].$$

(2) (Jacobi identity) for $a \in \underline{g}_{\theta'}$, $b \in \underline{g}_{\theta'}$, and $c \in \underline{g}_{\theta''}$ we have

$$\epsilon(\theta, \theta'')[a, [b, c]] + \epsilon(\theta', \theta)[b, [c, a]] + \epsilon(\theta'', \theta')[c, [a, b]] = 0.$$

A morphism $f : \underline{g} \to \underline{g}'$ of Θ-graded Lie algebras over k is a graded k-module morphism such that $f([a, b]) = [f(a), f(b)]$ for all a, $b \in \underline{g}$. Since the composition of morphisms of Lie algebras is again a morphism of Lie algebras, we can speak of the category $\mathrm{Lie}_{\Theta,k}$ of Θ-graded Lie algebras over k and their morphisms. Following the lead from algebras, we define a Lie algebra $\underline{g}$ to be commutative if $[,] = 0$ on $\underline{g} \otimes \underline{g}$, or equivalently, $[\underline{g}, \underline{g}]$ is the zero k-submodule where $[\underline{g}, \underline{g}]$ denotes the k-submodule generated by all Lie brackets $[a, b]$. The full subcategory of commutative Lie algebras is denoted $C\,\mathrm{Lie}_{\Theta,k'}$ and it is essentially the category $\Theta(k)$ of Θ-graded modules.

Example 2.5. If A is a Θ-graded k-algebra, then A with the Lie bracket $[a, b] = ab - \epsilon(\theta, \theta')ba$ for $a \in \underline{g}_{\theta'}$, $b \in \underline{g}_{\theta'}$ is a Θ-graded Lie algebra **24**
which we denote by Lie (A). This defines a functor

$$\mathrm{Lie} : \mathrm{Alg}_{\Theta,k} \to \mathrm{Lie}_{\Theta,k}\,.$$

3 Abelianization of algebras and Lie algebras

In this section we relate several categories by pairs of adjoint functors. For completeness, we include (gr), the category of groups and group morphisms together with the full subcategory (ab) of abelian groups. Also (ab) and $(\mathbf{Z})$ are the same categories. We continue to use the notation of the previous section for the group Θ which indexes the grading.

Definition 3.1. Abelianization is the left adjoint functor to any of the following inclusion functors

$$C\,\mathrm{Alg}_{\Theta,k} \to \mathrm{Alg}_{\Theta,k},\ C\,\mathrm{Lie}_{\Theta,k} \to \mathrm{Lie}_{\Theta,k}, \quad \text{and} \quad (ab) \to (gr).$$

Proposition 3.2. *Each of the inclusion functors*

$$C\,\mathrm{Alg}_{\Theta,k} \to \mathrm{Alg}_{\Theta,k},\ C\,\mathrm{Lie}_{\Theta,k} \to \mathrm{Lie}_{\Theta,k} \quad \textit{and} \quad (ab) \to (gr)$$

have left adjoint functors

$$\mathrm{Alg}_{\Theta,k} \to C\,\mathrm{Alg}_{\Theta,k},\ \mathrm{Lie}_{\Theta,k} \to C\,\mathrm{Lie}_{\Theta,k'} \quad \textit{and} \quad (gr) \to (ab).$$

each of them denoted commonly by $(\ )^{ab}$.

Proof. If the inclusion functor is denoted by J, then we will apply $(1.5)^*$ to $T = J$ and form the commutative $s(A) = A/(A,A)$, $s(L) = L/[L,L]$ and $s(G)/(G,G)$ algebra, Lie algebra, and group respectively by dividing out by commutators. In the case of an algebra A, the commutator ideal (A,A) is the bilateral ideal generated by $[A,A]$ and $\beta(A)$:
25 $A \to J(s(A)) = A/(A,A)$ is the quotient morphism. For each morphism $f : A \to J(B)$ where B is a commutative algebra $f((A,A)) = 0$ and hence it defines a unique $g : s(A) \to B$ in $C\,\mathrm{Alg}_k$ such that $J(g)\beta(A) = f$. Hence there exists a left adjoint functor S of J which we denote by $S(A) = A^{ab}$. The same line of argument applies to Lie algebras where $\underline{g}^{ab} = \underline{g}/[\underline{g},\underline{g}]$ and $[\underline{g},\underline{g}]$ is the Lie ideal of all brackets $[a,b]$ and groups where $G^{ab} = G/(G,G)$ and (G,G) is the normal subgroup of G generated by all commutators $(s,t) = sts^{-1}t^{-1}$ of $s,\ t \in G$. This proves the proposition. □

Now we consider functors from the category of algebras to the category of Lie algebras and the category of groups.

Notation 3.3. We denote the composite of the functor Lie : $\mathrm{Alg}_{\Theta,k} \to \mathrm{Lie}_{\Theta,k}$, which assigns to an algebra A the same underlying k-module together with the Lie bracket $[a,b]$, with the abelianization functor of this Lie algebra $\mathrm{Lie}(A)^{ab}$, and denote it by $A^{\alpha\beta}$. This is just the graded k-module $[A,A]$.

We remark that there does not seem to be standard notation for A divided by the k-module generated by the commutators, and we have hence introduced the notation $A^{\alpha\beta}$. Note that the quotient $A^{\alpha\beta}$ is not an algebra but an abelian Lie algebra, that is, a graded k-module.

Remark 3.4. The importance of $A^{\alpha\beta}$ lies in the fact that it is isomorphic to the zero dimensional Hochschild homology, as we shall see in (6.3)(2), and thus to the zero dimensional cyclic homology, see 1(3.6).

Remark 3.5. The multiplicative group functor $(\ )^* : \mathrm{Alg}_k \to (gr)$ is defined as the subset consisting of $u \in A$ with an inverse $u^{-1} \in A$ and the group law being given by multiplication in A. It is the right adjoint of the group algebra functor $k[\ ] : (gr) \to \mathrm{Alg}_k$ where $k[G]$ is the free module with the set G as basis and multiplication given by the following formula on linear combinations in $k[G]$, **26**

$$\left(\sum_{t\in G} a_t t\right)\left(\sum_{r\in G} b_r r\right) = \sum_{s\in G}\left(\sum_{tr=s} a_t b_r\right) s.$$

The adjunction condition is an isomorphism

$$\mathrm{Hom}(k[G], A) \to \mathrm{Hom}(G, A^*).$$

4 Tensor algebras and universal enveloping algebras

Adjoint functors are also useful in describing free objects or universal objects with respect to a functor which reduces structure. These are called structure reduction functors, stripping functors, or forgetful functors.

Proposition 4.1. *The functor* $J : \mathrm{Alg}_{\Theta,k} \to \Theta(k)$ *which assigns to the graded algebra* (A, ϕ, η) *the graded* k*-module* A *has a left adjoint* $T : \Theta(k) \to \mathrm{Alg}_{\Theta,k}$ *where* $T(M)$ *is the tensor algebra on the graded module* M.

Proof. From the n^{th} tensor power $M^{n\otimes}$ of a graded module M. For each morphism $f : M \to J(A)$ of graded modules where A is an algebra we have defined $f_n : M^{n\otimes} \to J(A)$ as $f_n = \phi_n(A) f^{n\otimes}$, where $\phi_n(A) : A^{n\otimes} \to A$ is the n-fold multiplication.

We give $T(M) = \coprod_n M^{n\otimes}$ the structure of algebra $(T(M), \phi, \eta)$ where $\eta : k = M^{0\otimes} \to T(M)$ is the natural injection into the coproduct and $\phi | M^{p\otimes} \otimes M^{q\otimes}$ is the natural injection of $M^{(p+q)\otimes}$ into $T(M)$ defining $\phi : T(M) \otimes T(M) \to T(M)$. For a morphism $f : M \to J(A)$ the sum of the $f_n : M^{n\otimes} \to J(A)$ define a morphism $g : T(M) \to A$ of algebras. The adjunction morphism is $\beta(M) : M \to J(T(M))$ the natural injection of $M^{1\otimes} = M$ into $J(T(M))$. Clearly $J(g)\beta(M) = f$ and this defines the bijection giving the adjunction from the universal property. This proves the proposition. □

27 Now we consider the question of abelianization of the tensor algebra. Everything begins with the commutativity symmetry $\sigma : L \otimes M \to M \otimes L$ of the tensor product.

Algebra abelianization of $T(M)$ 4.2. The abelianization $T(M)^{ab}$ of the algebra $T(M)$, like $T(M)$, is of the form $\prod_{0 \le n} S_n(M)$ where $S_n(M) = (M^{n\otimes})_{\mathrm{Sym}_n}$ is the quotient of the n^{th} tensor power of M by the action of the symmetric group Sym_n permuting the factors with the sign $\epsilon(\theta, \theta')$ coming from the grading. This follows from the fact that the symmetric group Sym_n is generated by transpositions of adjacent indices, and thus $(T(M), T(M))$ is generated by

$$x \otimes y - \epsilon(\theta, \theta') y \otimes x \quad \text{for} \quad x \in M_\theta, \quad y \in M_{\theta'}$$

as a two sided ideal.

Lie algebra abelianization $T(M)^{\alpha\beta}$ of $T(M)$. 4.3. We form Lie$(T(M))$ and divide by the Θ-graded k-submodule $[T(M), T(M)]$ to obtain $T(M)^{\alpha\beta}$, which like $T(M)$ and $T(M)^{ab} = S(M)$, is of the form $\coprod_{0\le n} L_n(M)$ where $L_n(M) = (M^{n\otimes})_{\mathrm{Cyl}_n}$ is the quotient of the n^{th} tensor power of M by the action of the cyclic group Cyl_n permuting the factors cyclically with the sign $\epsilon(\theta, \theta')$ coming from the grading. In $M^{n\otimes}$, we must divide by elements of the form

$$[x_1\otimes\cdots\otimes x_p, x_{p+1}\otimes\cdots\otimes x_n] = x_1\otimes\cdots\otimes x_n - \epsilon(\theta,\theta')x_{p+1}\otimes\cdots\otimes x_n\otimes x_1\otimes\cdots\otimes x_p$$

where $x_1 \otimes \cdots \otimes x_p \in (M^{p\otimes})_\theta$ and $x_{p+1} \otimes \cdots \otimes x_n \in (M^{(n-p)\otimes})_{\theta'}$. These elements generate $[T(M), T(M)]$ and they are exactly the elements mapping to zero in the quotient, under the action of the cyclic group Cyl_n on $M^{n\otimes}$.

Proposition 4.4. *The functor* Lie : $\mathrm{Alg}_{\Theta,k} \to \mathrm{Lie}_{\Theta,k}$ *has a left adjoint functor* $U : \mathrm{Lie}_{\Theta,k} \to \mathrm{Alg}_{\Theta,k}$.

Proof. The functor Lie starts with the functor J of (4.1) which has $T(\underline{g})$ as its left adjoint functor. This is not enough because $\underline{g} \to T(\underline{g})$ is not a 28
morphism of Lie algebras, so we form the quotient $u(\underline{g})$ of $T(\underline{g})$ by what is needed to make it a Lie algebra morphism, namely the two sided ideal generated by all

$$x \otimes y - \epsilon(\theta, \theta')y \otimes x = [x, y] \quad \text{for} \quad x \in \underline{g}_\theta, y \in \underline{g}_{\theta'}.$$

The resulting algebra $U(\underline{g})$ has the universal property which follows from the universal property for $T(M)$ in (4.1). This proves the proposition. □

Definition 4.5. The algebra $U(\underline{g})$ is called the universal enveloping algebra of the Lie algebra $\underline{g}$.

Example 4.6. The abelianization $U(\underline{g})^{ab} = U(\underline{g}^{ab})$ while $U(\underline{g})^{\alpha\beta}$ is $U(\mathcal{G})_{\}'}$ the universal quotient where the action of $\underline{g}$ on $U(\underline{g})$ is trivial.

Example 4.7. The abelianization $k[G]^{ab} = k[G^{ab}]$ while $k[G]^{\alpha\beta}$ is $k[G]_G$, the universal quotient where the action of G on $k[G]$ is trivial. This is just a free module on the conjugacy classes of G.

5 The category of A-bimodules

Let A be a Θ-graded algebra over k with multiplication $\phi(A) : A \otimes A \to A$ and unit $\eta(A) : k \to A$.

Definition 5.1. A left A-module M is a Θ-graded k-module M, together with a morphism $\phi(M) : A \otimes M \to M$ such that

(1) (associativity) as morphisms $A \otimes A \otimes M \to M$ we have $\phi(M)(A \otimes \phi(M)) = \phi(M)(\phi(A) \otimes M)$, and

(2) (unit property) the composite $(\phi(M)(\eta(A) \otimes M)$ is the natural morphism $k \otimes M \to M$.

A morphism $f : M \to M'$ of left A-modules is a graded k-linear morphism satisfying $f\phi(M) = \phi(M')(A \otimes f)$. The composition of two morphisms of left A-modules as k-modules is a morphism of left A-modules. Thus we can speak of the category ${}_A\mathrm{Mod}$ of left A-modules and their morphisms.

29 **Definition 5.2.** A right A-module L is a Θ-graded k-module L together with a morphism $\phi(L) : L \otimes A \to M$ satisfying an associativity and unit property which can be formulated to say that L together with $\phi(L)\sigma(A, L)$ is a left A^{op}-module where $A^{op} = (A, \phi(A)\sigma(A, A), \eta(A))$. A morphism of right A-modules is just a morphism of the corresponding left A^{op}-modules, and composition of k-linear morphisms induces composition of right A-modules. Thus we can speak of the category Mod_A of right A-modules and their morphisms.

We have the natural identification of categories ${}_A\mathrm{Mod} = \mathrm{Mod}_{(A^{op})}$ and ${}_{(A^{op})}\mathrm{Mod} = \mathrm{Mod}_A$.

Definition 5.3. An A-bimodule M is a Θ-graded k-module together with two morphisms $\phi(M) : A \otimes M \to M$ making M into a left A-module, and $\phi'(M) : M \otimes A \to M$ making M into a right A-module, such that, as morphisms $A \otimes M \otimes A \to M$ we have

$$\phi(M)(A \otimes \phi'(M)) = \phi'(M)(\phi(M) \otimes A).$$

A morphism of A-bimodules $f : M \to M'$ is a k-linear morphism which is both a left A-module morphism and a right A-module morphism. The composition as k-linear morphisms is the composition of A-bimodules. Thus we can speak of the category ${}_A\operatorname{Mod}_A$ of A-bimodules.

We have the natural identification of categories ${}_A\operatorname{Mod}_A = {}_{A\otimes(A^{op})}\operatorname{Mod} = \operatorname{Mod}_{(A^{op})\otimes A}$ in terms of left and right modules over A tensored with its opposite algebra A^{op}.

Definition 5.4. Let M be an A-bimodule. Let $[A, M]$ denote the graded k-submodule generated by all elements of the form

$$[a, x] = ax - \epsilon(\theta, \theta')xa$$

where $a \in A_{\theta,x} \in M_{\theta'}$. As a graded k-module we denote by $M^{\alpha\beta} = M/[A, M]$.

If $f : M \to M'$ is a morphism of A-bimodules, the $f([A, M]) \subset$ **30**
$[A, M']$ and f induces on the quotient $f^{\alpha\beta} : M^{\alpha\beta} \to M'^{\alpha\beta}$, and this defines a functor ${}_A\operatorname{Mod}_A \to \Theta(k)$ which is the largest quotient of an A-bimodule M such that the left and right actions are equal. It is a kind of abelianization, in the sense that for the A-bimodule A the result $A/[A, A]$ is just the abelianization of the Lie algebra $\operatorname{Lie}(A)$.

Remark 5.5. In fact the abelization functor is just a tensor product. Any A-bimodule is a left $A\otimes A^{op}$-module and A^{op} is a right $A\otimes A^{op}$-module. Then $M^{\alpha\beta}$ is just $A^{op}\otimes_{(A\otimes A^{op})} M$, because the tensor product over $A\otimes A^{op}$ is the quotient of $A\otimes M$ divided by the submodule generated by $ab\otimes x - a\otimes bx$ for $a \in A^{op}$, $x \in M$, and $b \in A\otimes A^{op}$, that is, by relations of the form $a\otimes x - 1\otimes ax$ and $a\otimes x - 1\otimes xa$.

In fact $M \to M^{\alpha\beta}$ is a functor $\Theta\,\mathrm{Bimod} \to \Theta(k)$. Here $\Theta\,\mathrm{Bimod}$ is the category of pairs (A, M) where A is Θ-graded algebra over k and M is an A-bimodule, and the morphisms are $(u, f) : (A, M) \to (A', M')$ where $u : A \to A'$ is a morphism of algebras and $f : M \to M'$ is k-linear such that $f\phi(M) = \phi(M')(A\otimes f)$ and $f\phi'(M) = \phi'(M')(f\otimes A)$. Observe that when u is the identity on A, then $f : M \to M'$ is a morphism ${}_A\operatorname{Mod}_A$.

Remark 5.6. The abelianization functor $M^{\alpha\beta}$, being a tensor product, has the following exactness property. If $L \to M \to N \to 0$ is an exact sequence in ${}_A\mathrm{Mod}_A$, then $L^{\alpha\beta} \to M^{\alpha\beta} \to N^{\alpha\beta} \to 0$ is exact in $\Theta(k)$. Even if $L \to M$ is a monomorphism, it is not necessarily the case that $L^{\alpha\beta} \to M^{\alpha\beta}$ is a monomorphism.

Since $M^{\alpha\beta}$ is only right exact, the functor generates a sequence of functors of (A, M) in Θ Bimod, denoted $H_n(A, M)$ and called Hochschild homology of A with values in the module M, such that $H_0(A, M)$ is isomorphic to $M^{\alpha\beta}$. More precisely, in the following section we have a theorem which gives an axiomatic characterisation of Hochschild homology.

6 Hochschild homology

31 **Definition 6.1.** An A-bimodule M is called extended provided it is of the form $A \otimes X \otimes A$ where X is a graded k-module.

Remark 6.2. There is a natural morphism to A-bimodule k-module $\mathrm{Hom}_{(A)}(A \otimes X \otimes A, M')$, denoted

$$a : \mathrm{Hom}_{\Theta(k)}(X, M') \to \mathrm{Hom}_{(A)}(A \otimes X \otimes A, M'),$$

defined by the relation

$$a(f) = \phi'(M')(\phi(M') \otimes A)(A \otimes f \otimes A) = \phi(M')(A \otimes \phi'(M')).$$

Moreover, a is an isomorphism defining $S(X) = A \otimes X \otimes A$ as a left adjoint functor to the stripping functor ${}_A\mathrm{Mod}_A \to \Theta(k)$ which deletes the A-bimodule structure leaving a Θ graded k-module. The extended modules have an additional property, namely that for an exact sequence

$$0 \to M' \to M \to A \otimes X \otimes A \to 0$$

which is k-split exact, we have the short exact sequence

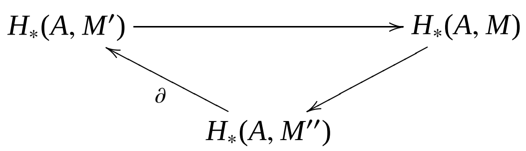
$$0 \to M'^{\alpha\beta} \to M^{\alpha\beta} \to (A \otimes X \otimes A)^{\alpha\beta} \to 0.$$

This follows from the fact that under the hypothesis, we have a splitting of the A-bimodule sequence given by a morphism $A \otimes X \otimes X \otimes A \to M$.

The reader can easily check that the projectives in the category $_A\mathrm{Mod}_A$ are direct summands of extended modules $A \otimes X \otimes A$ where X is a free Θ-graded k-module.

Theorem 6.3. *There exists a functor* $H : \Theta\,\mathrm{Bimod} \to \mathbf{Z}(\Theta(k))$ *together with a sequence of morphisms* $\partial : H_q(A, M'') \to H_{q-1}(A, M')$ *in* $\Theta(k)$ *associated to each exact sequence split in* $\Theta(k)$ *of* A*-modules* $0 \to M' \to M \to M'' \to 0$ *such that*

(1) *the following exact triangle is exact* 32



and ∂ *is natural in* A *and the exact sequence,*

(2) *in degree zero* $H_0(A, M)$ *is naturally isomorphic to* $M^{\alpha\beta} = M/[A, M]$

(3) *if* M *is an extended* A*-bimodule, then* $H_q(A, M) = 0$ *for* $q > 0$.

Finally two such functors are naturally isomorphic in a way that the morphisms ∂ *are preserved.*

Proof. Since $M^{\alpha\beta}$ is isomorphic to the tensor product $A^{op} \otimes_{(A\otimes A^{op})} M$, the functor $H_*(A, M)$ can be defined as $\mathrm{Tor}_*^{A\otimes A^{op}}(A^{op}, M)$, not as the absolute Tor, but as a k-split relative Tor functor. Since this concept is not so widely understood, we give an explicit version by starting with a functorial resolution of M by extended A-bimodules. The first term is the resolution is $A \otimes M \otimes A \to M$ given by scalar multiplication and M in $A \otimes M \otimes A$ viewed as a Θ-graded k-module. The next term is $A \otimes W(M) \otimes A \to A \otimes M \otimes A$, where $W(M) = \{\ker(A \otimes M \otimes A \to M)\}$, and the process continues to yield a complex $Y_*(M) \to M$ depending functorially on M. We can define $H_*(A, M) = H_*(Y_*(M)^{\alpha\beta})$, and to check the properties, we observe that for an exact sequence of A-bimodules which is k-split

$$0 \to M' \to M \to M'' \to 0$$

the corresponding sequence of complexes

$$0 \to Y_*(M')^{\alpha\beta} \to Y_*(M)^{\alpha\beta} \to Y_*(M'')^{\alpha\beta} \to 0$$

is exact, and the homology exact triangle results give property (1) for the homology $H_*(A, M)$. The relation (2) that $H_0(A, M) = M^{\alpha\beta}$ follows
33 from the right exactness of the functor. Finally (3) results from the last statement in (6.2). □

The uniqueness of the functor H_q is proved by induction on q using the technique call dimension shifting. We return to the canonical short exact sequence associated with any A-bimodule M

$$0 \to W(M) \to A \otimes M \otimes A \to M \to 0.$$

This gives an isomorphism $H_q(A, M) \to H_{q-1}(A, W(M))$ for $q > 1$, and an isomorphism $H_1(A, M) \to \ker(H_0(A, W(M)) \to H_0(A, A \otimes M \otimes A))$. In this way the two theories are seen to be isomorphic by induction on the degree. This proves the theorem.

Chapter 3

Cyclic Homology and the Connes Exact Couple

WE START WITH the standard Hochschild complex and study the in- 34
ternal cyclic symmetry in this complex. This leads to the cyclic homology double complex $CC_{\bullet\bullet}(A)$ for an algebra A which is constructed from two aspects of the standard Hochschild complex and the natural homological resolution of finite cyclic groups. In terms of this double complex, we define cyclic homology as the homology of the associated single complex, and since the Hochschild homology complex is on the vertical edge of this double complex, we derive the Connes' exact couple exploiting the horizontal degree 2 periodicity of the double complex.

The standard Hochschild complex comes from a simplicial object which has an additional cyclic group symmetry, formalized by Connes when he introduced the notion of a cyclic object. As introduction to cyclic objects is given.

1 The standard complex

In Chapter 2 § 6, we considered an axiomatic characterization of Hochschild homology and then remarked that it is a split Tor functor over $A \otimes A^{op}$. The Tor functors are defined, and in some cases also calculated, using a projective resolution which in this case is a split projective resolution made out of extended modules. We consider a particular resolution using the most natural extended A-bimodules, $A \otimes A^{q\otimes} \otimes A = C'_q(A)$ made out of tensor powers of A. The morphisms in the resolution are defined using the extended multiplications $\phi_i : C'_q(A) \to C'_{q-1}(A)$ defined by

$$\phi_i(a_0 \otimes \cdots \otimes a_{q+1}) = a_0 \otimes \cdots \otimes a_i \cdot a_{i+1} \otimes \cdots \otimes a_{q+1} \quad \text{for} \quad i = 0, \ldots, q.$$

The A-bimodule structure on $C'_q(A)$ is given by the extended A-bimodule structure where for $a \otimes a' \in A \otimes A^{op}$ we have

$$(a \otimes a')(a_0 \otimes \cdots \otimes a_{q+1}) = (aa_0) \otimes \cdots \otimes (a_{q+1}a'),$$

35 and from this it is clear that ϕ_i is an A-bimodule morphism. Finally, note that the morphism $\phi_0 : C'_0(A) \to A$ is the usual multiplication morphism on A.

Definition 1.1. The standard split resolution of A as an A-bimodule is the complex $(C'_*(A), b') \to A$ of A-bimodules over A where with the above notations $b' : C'_q(A) \to C'_{q-1}(A)$ is given by $b' = \sum_{0 \le i \le q} (-1)^i \phi_i$.

Proposition 1.2. *The standard split resolution of A is a split projective resolution of A by A-bimodules.*

Proof. By construction each $C'_q(A)$ is an extended A-bimodule. Next, we have $b'b' = 0$ because an easy check shows that

$$\phi_i \phi_j = \phi_{j-1} \phi_i \quad \text{for} \quad i < j,$$

and this gives $b'b' = 0$ by an argument where $(q + 1)q$ terms cancel in pairs. Finally the complex is split acyclic with the following homotopy $s : C'_q(A) \to C'_{q+1}(A)$ given by $s(a_0 \otimes \cdots \otimes a_{q+1}) = 1 \otimes a_0 \otimes \cdots \otimes a_{q+1}$. Since $\phi_0 s = 1$ and $\phi_{i+1} s = s\phi_i$ for $i \ge 0$, we obtain $b's + sb' = 1$, the identity. This proves the proposition. □

To calculate the Hochschild homology with the resolution, we must apply the functor R, where $R(M) = A \otimes_{(A \otimes A^{op})} M$, to the complex of the resolution. Now, for an extended A-bimodule $A \otimes X \otimes A$ the functor has the value $R(A \otimes X \otimes A) = A \otimes X$ as a k-module.

Definition 1.3. The standard complex $C_*(A)$ for an algebra A over k is, with the above notation $C_*(A) = R(C'_*(A), b')$.

In particular, we have $C_q(A) = A^{(q+1)\otimes}$ and for $d_i = R(\phi_i)$ the differential of the complex is $b = \sum\limits_{0\leq i\leq q} (-1)^i d_i$ where $d_i : C_q(A) \to C_{q-1}(A)$ is given by the following formulas

$$d_i(a_0 \otimes \cdots \otimes a_q) = a_0 \otimes \cdots \otimes (a_i a_{i+1}) \otimes \cdots \otimes a_q \quad \text{for} \quad 0 \leq i < q$$
$$d_q(a_0 \otimes \cdots \otimes a_q) = (a_q a_0) \otimes a_q \otimes \cdots \otimes a_{q-1}.$$

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The last formula, the one for d_q, reflects how the identification of $A\otimes X$ with $R(A\otimes X\otimes A)$ is made from the right action of A on A becoming the left action on A^{op}. Again we have $d_i d_j = d_{j-1} d_i$ for $i < j$.

Further, as a complex over k, we see clearly that $C_q(A) = C'_{q-1}(A)$ with $d_i = \phi_i$ for $i < q$. If $b' = \sum\limits_{0\leq i<q} (-1)^i d_i : C_q(A) \to C_{q-1}(A)$, then from (1.2) we deduce immediately that $(C_*(A), b')$ is acyclic. In terms of b', it is clear that $b = b' + (-1)^q d_q$.

Remark 1.4. The Hochschild homology $HH_*(A) = H_*(A, A)$ of A can be calculated as $H_*(C_*(A))$, the homology of the standard complex of A.

2 The standard complex as a simplicial object

Remark 2.1. Besides the operators d_i on the standard complex $C_*(A)$, there are operators s_j where $s_j : C_q(A) \to C_{q+1}(A)$ for $0 \leq j \leq q$ defined by the following formula

$$s_j(a_0 \otimes \cdots a_q) = a_0 \otimes \cdots \otimes a_j \otimes 1 \otimes a_{j+1} \otimes \cdots \otimes a_q \quad \text{for} \quad 0 \leq j \leq q.$$

With both the operators d_i and s_j, the standard complex becomes what is called a simplicial k-module. We define now the general concept of a simplicial object over a category.

Definition 2.2. Let C be a category. A simplicial object X_* in the category C is a sequence of objects X_q in C together with morphisms

$d_i : X_q \to X_{q-1}$ for $q > 0$ and $s_j : X_q \to X_{q+1}$ for $0 \leq i, j \leq q$ satisfying the following relations

(1) $d_i d_j = d_{j-1} d_i$ for $0 < j - 1$,

(2) $s_j s_i = s_i s_{j-1}$ for $0 < j - i$,

(3) $$d_i s_j = \begin{cases} s_{j-1} d_i & \text{for } 0 < j - i \leq q \\ \text{identity} & \text{for } -1 \leq j - i \leq 0 \\ s_j d_{i-1} & \text{for } j - i < -1. \end{cases}$$

37 A morphism $f : X_* \to Y_*$ of simplicial objects over the category $\mathcal{C}$ is a sequence $f_q : X_q \to Y_q$ of morphisms in $\mathcal{C}$ such that $d_i f = f d_i$ and $s_j f = f s_j$, i.e. a sequence of morphisms commutating with the simplicial operations. Composition of $f : X_* \to Y_*$ and $g : Y_* \to Z_*$ is the sequence $g_q f_q$ defined $gf : X_* \to Z_*$.

Simplicial objects in a category $\mathcal{C}$, morphisms of simplicial objects, and composition of morphisms define the category $\Delta(\mathcal{C})$ of simplicial objects in $\mathcal{C}$.

Originally, simplicial objects arose in the context of the singular complex of a space which is an example of a simplicial set, and by considering the k-module in each degree with the singular simplexes as basis, we come to a simplicial k-module $C_*(A)$ associated with an algebra A over k.

Already, for the standard simplicial k-module $C_*(A)$ we have associated a positive complex with boundary operator defined in terms of the operators d_i. This can be done for any simplicial object over an abelian category. Let $C^+(\mathcal{A})$ denote the category of positive complexes over an abelian category $\mathcal{A}$.

Notation 2.3. For a simplicial object X_* in an abelian category $\mathcal{A}$ we use the following notations

$$b = d = \sum_{0 \leq i \leq q} (-1)^i d_i$$

$$b' = d' = \sum_{0 \leq i < q} (-1)^i d_i,$$

$$s = (-1)^q s_q : \quad X_q \to X_{q+1}.$$

Remark 2.4. The functors which assign to a simplicial object X_* in $\Delta(\mathcal{A})$ either the complex (X_*, d) or the complex (X_*, d'), and to morphisms in $\Delta(\mathcal{A})$ the corresponding morphisms of complexes, are each functors defined $\Delta(\mathcal{A}) \to C^+(\mathcal{A})$. By a direct calculation, s is a homotopy operator for d' of the identity to zero, that is, **38**

$$d's + sd' = 1.$$

This means that (X_*, d') is an acyclic complex or equivalently $H_*(X_*, d') = 0$.

Notation 2.5. We define a filtration F^*X and two subcomplexes $D(X)$ and $N(X)$ of the complex (X, d) associated with the simplicial object X in $\mathcal{A}$. For the filtration in degree n, we define

$$F^p X_n = \bigcap_{n-p<i\leq n, 0<i} \ker(d_i).$$

The subcomplex of degeneracies $D_n(X)$ in degree n is the subobject of X_n generated by $\operatorname{im}(s_i)$ for $i = 0, \ldots, n-1$, and the Moore subcomplex $N_n(X)$ in degree n is $F^n X_n$. In other words, the Moore subcomplex is the intersection of the filtration $N(X) = \bigcap_q F^q(X)$, and the boundary d is just $d_0 : N_q(X) \to N_{q-1}(X)$.

The next theorem is proved by retracting F^pX into $F^{p+1}X$ with a morphism of complexes homotopic to the inclusion morphism of $F^{p+1}X$ into F^pX. For the proof of the theorem, we refer to MacLane 1963, VIII. 6.

Theorem 2.6. *Let X be a simplicial object in an abelian category $\mathcal{A}$. The following composite is an isomorphism*

$$N_*(X) \to X_* \to X_*/D_*(X),$$

and the induced homology morphisms

$$H_*(N(X)) \to H_*(X) \quad \textit{and} \quad H_*(X) \to H_*(X/D(X))$$

are each isomorphisms.

Normalized standard complex 2.7. Let A be an algebra over k. The subcomplex of degeneracies in degree q is $DC_q(A)$ and is generated by
39 all elements $a_0 \otimes \cdots \otimes a_q$ such that $a_i = 1$ for some i, $1 \leq i \leq q$. Thus there is a natural isomorphism of $\overline{C}_q(A) = C_q(A)/DC_q(A)$ with $A \otimes \overline{A}^{q\otimes}$. The graded k-module $\overline{C}_*(A)$ has a quotient complex structure, and by (2.6) the quotient morphism $C_*(A) \to \overline{C}_*(A)$ induces an isomorphism in homology, i.e. $HH_*(A) \to H_*(\overline{C}_*(A))$ is an isomorphism. The complex $\overline{C}_*(A)$ is called the normalized standard complex. In the case of the standard complex, the fact that $C_*(A) \to \overline{C}_*(A)$ induces an isomorphism in homology can be seen directly, by noting that $\overline{C}_*(A)$ is obtained as $A^{op} \otimes_{(A \otimes A^{op})} \overline{C}'_*(A)$ in the quotient resolution of $(C_*(A), b')$ where $\overline{C}'_*(A)$ is defined by

$$\overline{C}'_q(A) = A \otimes \overline{A}^{q\otimes} \otimes A.$$

The normalized complex is useful for comparing Hochschild homology with differential forms. We treat this in greater detail later.

3 The standard complex as a cyclic object

Remark 3.1. Besides the operators making $C_*(A)$ into a simplicial k-module, there is a cyclic permutation operator $t : C_q(A) \to C_q(A)$ defined by the following formula

$$t(a_0 \otimes \cdots \otimes a_q) = a_q \otimes a_0 \otimes \cdots \otimes a_{q-1}.$$

With the simplicial operators and this cyclic permutation in each degree, the standard complex becomes what is called a cyclic k-module. We now define the general concept of a cyclic object in a category.

Definition 3.2. Let $\mathcal{C}$ be a category. A cyclic object $X_\bullet$ in the category $\mathcal{C}$ is a simplicial object together with a morphism $t_q : X_q \to X_q$ for each $q \geq 0$ satisfying:

(1) The $(q+1)^{th}$-power $(t_q)^{q+1} = X_q$, the identity on X_q,

(2) As morphisms $X_q \to X_{q-1}$ we have $d_i t_q = t_{q-1} d_{i-1}$ for $i > 0$ and $d_0 t_q = d_q$, and

40 (3) As morphisms $X_q \to X_{q+1}$ we have $s_j t_q = t_{q+1} s_{j-1}$ for $j > 0$ and $s_0 t_q = (t_{q+1})^2 s_q$.

A morphism $f : X. \to Y.$ of cyclic objects in C is a morphism of the simplicial objects $f : X \to Y$ associated with the cyclic objects such that $t_q f_q = f_q t_q$ as morphisms of $X_q \to Y_q$. The composition of cyclic morphisms as simplicial morphisms is again a cyclic morphism. We denote the category of all cyclic objects in C and their morphisms by $\Lambda(C)$.

For each algebra A, we denote the cyclic object determined by the standard complex by $C.(A)$. We leave it to the reader to check that the above axioms (1), (2) and (3) are satisfied. The following discussion is carried out for $C.(A)$, but in fact, it holds for any cyclic object over an abelian category.

Notation 3.3. Let $T = (-1)^q t : C_q(A) \to C_q(A)$, and observe that both T^{q+1} and t^{q+1} are equal to the identity map on $C_q(A)$. Let $N : C_q(A) \to C_q(A)$ be defined by $N = 1+T+T^2+\cdots+T^q$, and observe that $N(1-T) = 0 = (1-T)N$. In order to prove the next commutativity proposition, it is handy to have the following operator $J = d_0 T : C_q(A) \to C_{q-1}(A)$, because it satisfies the relations

$$\begin{cases} T^i J T^{-i-1} = (-1)^i d_i \quad \text{for} \quad 0 \le i < q \\ T^q J T^{-q-1} = J \end{cases}$$

Proposition 3.4. *For an algebra A the following diagrams are commutative,*

$$\begin{array}{ccc} C_q(A) & \xrightarrow{N} & C_q(A) \\ \downarrow b & & \downarrow b' \\ C_{q-1}(A) & \xrightarrow{N} & C_{q-1}(A) \end{array} \qquad \begin{array}{ccc} C_q(A) & \xrightarrow{1-T} & C_q(A) \\ \downarrow b' & & \downarrow b \\ C_{q-1}(A) & \xrightarrow{1-T} & C_{q-1}(A). \end{array}$$

Proof. We first note that

$$b(1-T) = \left(\sum_{i=0}^{q}(-1)^i d_i\right)\left(1-(-1)^q t_{q+1}\right) = \sum_{i=0}^{q-1}(-1)^i d_i - (-1)^{q-1}\sum_{i=0}^{q-1}(-1)^i t_q d_i,$$

since $d_i t_{q+1} = t_q d_{i-1}$ for $0 < i \le n$ and $d_0 t_{q+1} = d_q$. But the last expres- **41**
sion is just $(1 - T)b'$ proving that the second diagram is commutative.

For the commutativity of the first diagram, we use $NT^i = T^i N = N$ for all i. Using the operator J introduced above in (3.3), we have

$$\begin{aligned} b'N &= JT^{-1}N + TJT^{-2}N + \cdots + T^{q-1}JT^{-q}N \\ &= JN + TJN + \cdots + T^{n-1}JN = (1 + T + \cdots + T^{q-1})JN = NJN, \end{aligned}$$

and similarly

$$\begin{aligned} Nb &= NJT^{-1} + NTJT^{-2} + \cdots + NT^qJT^{-q-1} \\ &= NJT^{-1} + NJT^{-2} + \cdots + NJT^{-q-1} \\ &= NJ(T^{-1} + T^{-2} + \cdots + T^{-q-1}) \\ &= NJN. \end{aligned}$$

This proves the proposition. □

Remark 3.5. This proposition is the basis for forming a double complex in the next section. Since $(C_*(A), b')$ is an acyclic complex, we consider two complexes coming from the standard complex and each giving Hochschild homology. From (3.4) the double complexes with two vertical columns $(C_*(A), b) \xleftarrow{1-T} (C_*(A), -b')$ and $(C_*(A), -b') \xleftarrow{N} (C_*(A), b)$ where $(C_*(A), b)$ is in horizontal degree 0 have associated total single complexes with homology equal to Hochschild homology. Using the spectral sequence of a filtered complex, we see by filtering on the horizontal degree that we get Hochschild homology for the homology of the associated total complex because $E^1_{0,q} = HH_q(A)$ and $E^q_{p,q} = 0$ otherwise.

4 Cyclic homology defined by the standard double complex

Definition 4.1. Let $C.(A)$ denote the cyclic object associated with the standard complex of an algebra A over k. The standard double complex $CC..(A)$ associated with this cyclic object and hence also with A is the

first quadrant double complex which is the sequence of vertical columns made up of even degrees by $(C_*(A), b)$ and odd degrees by $(C_*(A), b')$, 42
with horizontal structure morphisms given by $1 - T$ and N as indicated in the following display

$$C_*(A), b \xleftarrow{1-T} C_*(A), -b' \xleftarrow{N} C_*(A), b \xleftarrow{1-T} C_*(A), -b' \xleftarrow{N} C_*(A), b \leftarrow ..$$

which is periodic of period 2 horizontally to the right, starting with $p = 0$ in the double complex. The corresponding cyclic complex $CC_\bullet(A)$ is the associated total complex of $CC_{\bullet\bullet}(A)$.

Observe that by (3.4), $CC_{\bullet\bullet}(A)$ is a double complex, since we have already remarked that $(1 - T)N = 0 = N(1 - T)$. This construction is made with just the cyclic object structure, and thus can be made for any cyclic object in an abelian category.

Definition 4.2. Let A be an algebra over k. The cyclic homology $HC_*(A)$ of A is the homology $H_*(CC_\bullet(A))$ of the standard total complex of the standard double complex of A.

Remark 4.3. The standard double complex $CC_{\bullet\bullet}(A)$, its associated total complex $CC_\bullet(A)$, and the cyclic homology $HC_*(A)$, are all functors of A on the category of algebras over k, since the standard cyclic object $C_\bullet(A)$ is functorial in A from the category of algebras over k to the category of cyclic k-modules $\Lambda(k)$.

Connes' exact couple 4.4. From the 2-fold periodicity of the double complex $CC_{\bullet\bullet}(A)$, we have a morphism $\sigma : CC_{\bullet\bullet}(A) \to CC_\bullet(A)$ of bidegree $(-2, 0)$, giving a morphism $\sigma : CC_\bullet(A) \to CC_\bullet(A)$ of degree -2 and a short exact sequence of complexes

$$0 \to \ker(\sigma) \to CC_\bullet(A) \xrightarrow{\sigma} CC_\bullet(A) \to 0.$$

The homology of $\ker(\sigma)$ was considered in (3.5) and we have

$$H_*(\ker(\sigma)) = HH_*(A).$$

The homology exact triangle of this short exact sequence of com- 43 plexes is the Connes' exact triangle

$$\begin{array}{ccc} HC_*(A) & \xrightarrow{\ S\ } & HC_*(A) \\ & {\scriptstyle I}\nwarrow \quad \swarrow{\scriptstyle B} & \\ & HH_*(A) & \end{array}$$

where $S = H_*(\sigma)$ so $\deg(S) = -2$, $\deg(B) = +1$, and $\deg(I) = 0$. Moreover, this defines an functor from the category of algebras over k to the category of positively $\mathbf{Z}$-graded exact couples $ExC(-2, +1, 0)$ over the category of k-modules (k).

Remark 4.4. The entire discussion in this chapter could have been carried out with Θ-graded k-algebras A. The Θ-grading plays no role in any of the definitions. In particular, we have completed the definition of cyclic homology and the Connes' exact couple introduced in 1(3.5) namely

$$(HC_*, HH_*, S, B, I) : \mathrm{Alg}_{\Theta,k} \to ExC((k), \mathbf{Z} \times \Theta, (-2, 0), (1, 0), (0, 0)).$$

Also, the fact that $I : HH_0(A) \to HC_0(A)$ is an isomorphism holds, (see 1(3.6)), and if $f : A \to A'$ is a morphism of algebras, then $HC_*(f)$ is an isomorphism if and only if $HH_*(f)$ is an isomorphism, see 1(3.7).

5 Morita invariance of cyclic homology

Let A and B be two algebras, and let ${}_A\mathrm{Mod}_B$ denote the category of bimodules with A acting on the left and with B acting on the right. In other words ${}_A\mathrm{Mod}_B$ is the category of left $A \otimes B^{op}$-modules or the category of right $A^{op} \otimes B$-modules.

Definition 5.1. A Morita equivalence between two algebras A and B is given by two bimodules, P in ${}_A\mathrm{Mod}_B$ and Q in ${}_B\mathrm{Mod}_A$ together with isomorphisms

$$w_A : P \otimes_B Q \to A \quad \text{and} \quad w_B : Q \otimes_A P_A \to B$$

44 in the categories ${}_A\mathrm{Mod}_A$ and ${}_B\mathrm{Mod}_B$ respectively. Two algebras A and B are said to be Morita equivalent provided there exists a Morita equivalence between A and B.

The bimodules P and Q define six different functors:

(a) for left modules, $\phi_{P:B}\mathrm{Mod} \to_A \mathrm{Mod}$ and $\phi_{Q:A}\mathrm{Mod} \to_B \mathrm{Mod}$ defined by $\phi_P(M) = P \otimes_B M$ and $\phi_Q(M') = Q \otimes_A M'$,

(b) for right modules $\psi_P : \mathrm{Mod}_A \to \mathrm{Mod}_B$ and $\psi_Q : \mathrm{Mod}_B \to \mathrm{Mod}_A$ defined by $\psi_Q(L) = L \otimes_A P$ and $\psi_Q(L') = L' \otimes_B Q$, and

(c) for bimodules $\phi_{P,Q} :_A \mathrm{Mod}_A \to_B \mathrm{Mod}_B$ and $\phi_{Q,P} :_B \mathrm{Mod}_B \to_A \mathrm{Mod}_A$ defined by $\phi_{P,Q}(M) = Q \otimes_A M \otimes_A P$ and $\phi_{Q,P}(N) = P \otimes_B N \otimes_B Q$.

Proposition 5.2. *Let A and B be two algebras, and let (P, Q, w_A, w_B) be a Morita equivalence. Then the following hold:*

(1) *The functors $\phi_P :_B \mathrm{Mod} \to_A \mathrm{Mod}$ and $\phi_Q :_A \mathrm{Mod} \to_B \mathrm{Mod}$ are inverse to each other up to equivalence.*

(2) *The functors $\psi_P : \mathrm{Mod}_A \to \mathrm{Mod}_B$ and $\psi_Q : \mathrm{Mod}_B \to \mathrm{Mod}_A$ are inverse to each other up to equivalence.*

(3) *The functors $\phi_{P,Q} :_A \mathrm{Mod}_A \to_B \mathrm{Mod}_B$ and $\phi_{Q,P} :_B \mathrm{Mod}_B \to_A \mathrm{Mod}_A$ are inverse to each other up to equivalence.*

Also, there are natural isomorphisms induced by w_A and w_B between the functors defined on ${}_A\mathrm{Mod}_A \times_A \mathrm{Mod}_A$, namely

$$\phi_{P,Q}(M) \otimes_{B\otimes B^{op}} \phi_{P,q}(N) \to M \otimes_{A\otimes A^{op}} N,$$

and the corresponding derived functors

$$\mathrm{Tor}_*^{B\otimes B^{op}}(\phi_{P,Q}(M), \phi_{P,Q}(N)) \to \mathrm{Tor}_*^{A\otimes A^{op}}(M, N).$$

Proof. As an indication of the proof, we consider an A-bimodule M. There is a natural isomorphism

$$\phi_{Q,P}(\phi_{P,Q}(M)) = (P \otimes_B Q) \otimes_A M \otimes_A (P \otimes_B Q) \to A \otimes_A M \otimes_A A = M,$$

and similarly there is a natural isomorphism $\phi_{P,Q}\phi_{Q,P} \simeq \mathrm{id}$. 45

The isomorphism between two bimodule tensor products is just an associativity law for tensor products. This canonical isomorphism extends to the derived functors from uniqueness properties of the derived functors. This proves the proposition. □

Corollary 5.3. *Morita equivalent algebras A and B have isomorphic Hochschild homology.*

Example 5.4. The algebras A and the matrix algebra $M_n(A)$ are Morita equivalent. To see this, we observe that the module of n by q matrices $M_{n,q}(A)$ is a left $M_n(A) \otimes M_q(A)^{op}$-module and matrix multiplication factors by a tensor product over $M_q(A)$ as follows

$$\begin{array}{ccc} M_{n,q}(A) \otimes M_{q,s}(A) & \xrightarrow{\text{matrix multiplication}} & M_{n,s}(A) \\ & \searrow \qquad \nearrow_{f} & \\ & M_{n,q} \otimes_{M_q(A)} M_{q,s}(A) & \end{array}$$

Assertion. The morphism f in the previous diagram is an isomorphism of $M_n(A) \otimes M_s(A)^{op}$-modules. Clearly f is a bimodule morphism. To see the isomorphism assertion, we can reduce to the case $n = s = 1$ and consider $f : M_{1,q}(A) \otimes_{M_q(A)} M_{q,1}(A) \to M_{1,1}(A) = A$ and calculate

$$f\left((a_1, \ldots, a_q) \otimes \begin{pmatrix} b_1 \\ \vdots \\ b_q \end{pmatrix}\right) = f\left((a_1, \ldots, a_q) \otimes \begin{pmatrix} b_1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ b_q & 0 & \ldots & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) =$$

$$= f((c, 0, \ldots, 0) \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = c = a_1 b_1 + \cdots + a_q b_q.$$

It is clear from this computation that f is a bijection.

The Morita equivalence between A and $M_q(A)$ is given by $(M_{1,q}(A)$, $M_{q,1}(A)$, f, $f)$. There is a morphism of cyclic sets from the standard
46 complex for $M_n(A)$ to the standard complex for A.

Definition 5.5. The Dennis trace map

$$\mathrm{Tr} : M_n(A)^{(q+1)\otimes} \to A^{(q+1)\otimes}$$

is given by

$$\mathrm{Tr}(a(0) \otimes \cdots \otimes a(q)) = \sum_{1 \le i_0, \ldots, i_q \le n} a_{i_0 i_1}(0) \otimes \cdots \otimes a_{i_q i_0}(q).$$

Theorem 5.6. *The Dennis trace map induces isomorphisms $HH_*(M_n(A)) \to HH_*(A)$ and $HC_*(M_n(A)) \to HC_*(A)$.*

Proof. It is an isomorphism on Hochschild homology by (5.3), and since this isomorphism is given by a morphism of cyclic objects, the induced map is an isomorphism on cyclic homology by the criterion 1(3.7). This proves the theorem. □

Remark 5.7. In McCarthy [1988], there is a proof that in general Morita equivalent algebras have isomorphic cyclic homology.

Reference: Compte Rend Acad Sci, 307 (1988), pp. 211-215.

Chapter 4

Cyclic Homology and Lie Algebra Homology

CYCLIC HOMOLOGY WAS introduced in the previous chapter using 47 a double complex $C_{*,*}(A)$ with columns made up of standard Hochschild complexes $(C_*(A), b)$ and $(C_*(A), b')$. The cyclic structure gave a morphism of complexes $1 - T : (C_*(A), b) \to (C_*(A), b')$ which was also used to define the double complex $C_{*,*}(A)$. In the case of characteristic zero we will show that cyclic homology $HC_*(A)$ can be calculated in terms of the homology of $\mathrm{coker}(1 - T)$ and the homology of $\ker(1 - T)$. In this way we recover the original definition of Connes for cyclic cohomology as the cohomology of the dual of one of these complexes.

Then we sketch the Loday-Quillen and Tsygan theorem which says that the primitive elements in the homology of the Lie algebra $H_*(\underline{gl}(A))$ is isomorphic to the cyclic homology of A shifted down one degree. This is one of the main theorems in the subject of cyclic homology.

1 Covariants of the standard Hochschild complex under cyclic action

We start with a remark about endomorphisms of finite order.

Proposition 1.1. *Let $T : L \to L$ be an endomorphism of an object in an additive category such that $T^n = 1$, the identity on L. For $N = 1 + T + T^2 + \cdots + T^{n-1}$ we have the following representation of n times the identity on L*

$$n = N + (-(T + 2T^2 + \cdots + T^{n-1}))(1 - T).$$

Proof. We apply the differential operator $t\frac{d}{dt}$ to the relation

$$(1 - t^n) = (1 - t)(1 + t + \cdots + t^{n-1})$$

to obtain the relation

$$-nt^n = -t(1 + t + \cdots + t^{n-1}) + (1 - t)(t + 2t^2 + \cdots + (n - 1)t^{n-1}).$$

48 Substituting T for t and using $T^n = 1$ and $TN = N = NT$ we obtain the stated result. This proves the proposition. □

Recall that in the cyclic homology double complex for an algebra A the horizontal rows going in the negative direction in degree $q = n - 1$ are of the form

$$\ldots \xrightarrow{N} A^{n\otimes} \xrightarrow{1-T} A^{n\otimes} \xrightarrow{N} A^{n\otimes} \xrightarrow{1-T} A^{n\otimes} \to 0$$

where $T(a_1 \otimes \cdots \otimes a_n) = (-1)^{n-1} a_n \otimes a_1 \otimes \cdots a_{n-1}$. Now when the ground ring k is a $\mathbf{Q}$-algebra, so that the n in the previous proposition can be inverted, we have the identity

$$1 = \frac{1}{n}N + \theta(1 - T) \quad \text{where} \quad \theta = -\frac{1}{n}(T + 2T^2 + \cdots + (n - 1)T^{n-1}).$$

This leads to the following proposition.

Proposition 1.2. *Let A be an algebra over a ring k which is a $\mathbf{Q}$-algebra. Let $(A^{n\otimes})_{1-T} = \operatorname{coker}(1 - T)$, in other words, the coinvariants of the action of the cyclic group $\mathbf{Z}/n\mathbf{Z}$ acting through T on $A^{n\otimes}$. Then the following sequence of k-modules is exact*

$$\ldots \xrightarrow{N} A^{n\otimes} \xrightarrow{1-T} A^{n\otimes} \xrightarrow{N} A^{n\otimes} \xrightarrow{1-T} A \to (A^{n\otimes})_{1-T} \to 0,$$

and the following sequence of complexes over k is exact

$$\ldots \xrightarrow{1-T} C_*(A), b \xrightarrow{N} C_*(A), b' \xrightarrow{1-T} C_*(A), b \to C_*(A)_{1-T'} b \to 0.$$

Proof. Every thing follows from the homotopy formula for N and $1-T$, $1 = \frac{1}{n}N + \theta(1-T)$, except for the observation that $1-T$ and N are morphisms of complexes and this is contained in 3(3.4). This proves the proposition. □

Remark 1.3. The sequence of complexes in (1.2) being exact leads to the following isomorphism involving $(C_*(A)_{1-T}, b)$ namely

$$(C_*(A)_{1-T}, b) \to \operatorname{im}(N) \subset (C_*(A), b').$$

Further, we have a morphism of the assembled double complex into 49
the complex of covariants

$$CC_*(A) \to C_*(A)_{1-T}, b$$

which also maps the double complex filtration arising from the vertical grading into the degree filtration. In other words for

$$F_pCC_n(A) = \coprod_{i \le p, i+j=n} C_i(A) \to F_pC_n(A)_{1-T}$$

where

$$F_pC_n(A)_{1-T} = \begin{cases} C_n(A)_{1-T} & \text{for } p \le n \\ 0 & \text{for } p > n. \end{cases}$$

For these filtrations, looking at the associated graded E^0, we arrive at the quotient morphism $E^0_{p,0}CC_p(A) \to E^0_{p,0}C_p(A)_{1-T}$. The differential d^0 is zero in both complexes while E^1 of the mapping of the complexes is just the horizontal exact sequence in $CC_{**}(A)$. Thus by (1.2) we have an isomophism of the E^2-terms which is the homology of the E^1-terms. By the basic mapping theorem on spectal sequences, see 1(5.7), we have te following theorem.

Theorem 1.4. *Let A be an algebra over a ring k which is a* **Q***-algebra. The quotient morphism of complexes*

$$CC_*(A) \to C_*(A)_{1-T}, b$$

induces an isomorphism

$$HC_*(A) = H_*(CC_*(A)) \to H_*(C_*(A)_{1-T}, b)$$

of cyclic homology onto the homology of the standard complex with the cyclic action divided out.

2 Generalities on Lie algebra homology

50 From an algebraic point of view, cyclic homology is important for its relation to Hochschild homology and also Lie algebra homology. Indeed in Chapter 2, § 3 we showed how both concepts were related to abelianization.

Definition 2.1. Let $\underline{g}$ be a Lie algebra over k with universal enveloping algebra $U(\underline{g})$. The homology $H_*(\underline{g}, M)$ of $\underline{g}$ with values in the $\underline{g}$-module M is the Tor functor

$$H_*(\underline{g}, M) = \mathrm{Tor}_*^{U(\underline{g})}(k, M).$$

The absolute Lie algebra homology is $H_*(\underline{g}) = H_*(\underline{g}, M)$.

Recall that a $\underline{g}$-module or representation of $\underline{g}$ is just a $U(\underline{g})$-module by the universal property of the universal enveloping algebra $U(\underline{g})$.

Remark 2.2. In degree zero, Lie algebra homology is just

$$H_0(\underline{g}, M) = k \otimes_{U(\underline{g})} M = M/[\underline{g}, M]$$

where $[\underline{g}, M]$ is the k-submodule of M generated by all $[u, x]$ where $u \in \underline{g}, x \in M$. In particular $H_0(\underline{g}) = k$. Moreover it is the case that $H_1(\mathcal{G}) = \underline{g}^{ab} = \underline{g}/[\underline{g}, \underline{g}]$ which is easily seen from the following resolution which can be used to calculate Lie algebra homology.

Standard complex 2.3. Let $\underline{g}$ be a Lie algebra and M a $\underline{g}$-module. The standard complex $C_*(\underline{g}, M)$ for $\underline{g}$ with values in M as a graded k-module is $\Lambda^*(\underline{g}) \otimes M$ where $\Lambda^*(\underline{g})$ is the graded exterior algebra on the k-module $\underline{g}$ together with the differential given by the formula

$$d((u_1 \wedge \ldots \wedge u_n) \otimes x) = \sum_{1 \le i \le n} (-1)^i (u_1 \wedge \ldots \wedge \hat{u}_i \wedge \ldots \wedge u_n) \otimes u_i x +$$

$$+ \sum_{1\le i<j\le n} (-1)^{i+j+1}([u_i, u_j] \wedge u_1 \wedge \ldots \hat{u}_i \wedge \ldots \wedge \hat{u}_j \wedge \ldots \wedge u_n) \otimes x.$$

We leave it to the reader to check that $d^2 = 0$ by direct computation using the Jacobi law and $[u, v]x = u(vx) - v(ux)$. In Cartan and Eilenberg, Chapter XIII, (7.1) it is proved that $H_*(C_*(\underline{g}, M)) = H_*(\underline{g}, M)$ which is 51
defined by the Tor functor.

We will be primarily interested in the case where $M = k$. Then the standard complex is denoted by just $C_*(\underline{g})$, and as a graded k-module it is the exterior module $\Lambda^*(\underline{g})$ with differential given by

$$d(u_1 \wedge \ldots \wedge u_n) = \sum_{1\le i<j\le n} (-1)^{i+j+1}[u_i, u_j] \wedge u_1 \wedge \ldots u_i \wedge \ldots \wedge u_j \wedge \ldots \wedge u_n$$

since $u1 = 0$ in the $\underline{g}$-module k.

Remark 2.3. The exterior k-module $\Lambda^*(V)$ has both an algebra structure given by exterior multiplication and a coalgebra structure given by

$$\begin{aligned}\Delta(u_1 \wedge \ldots \wedge u_n) = {} & (u_1 \wedge \ldots \wedge u_n) \otimes 1 \\ & + \sum_{1\le i\le n-1} (u_1 \wedge \ldots \wedge u_i) \otimes (u_{i+1} \wedge \ldots \wedge u_n) \\ & + 1 \otimes (u_1 \wedge \ldots \wedge u_n).\end{aligned}$$

The algebra structure is not compatible with the differential on $\Lambda^*(\underline{g})$ since, for example, $[u, v] = d(u \wedge v)$, and it would have to equal

$$d(u \wedge v) = du \wedge v - u \wedge dv = 0$$

in order to have a differential algebra structure. On the other hand $C_*(\underline{g})$ with the exterior coalgebra structure is compatible with d making $C_*(\underline{g})$ into a differential coalgebra. In the case where k is a field or more generally $H_*(\underline{g})$ is k-flat so that the Künneth morphism is an isomorphism, the Lie algebra homology $H_*(\underline{g})$ is a commutative coalgebra over k.

Concerining the calculations given in (2.2), we observe that $d = 0$ on $C_0(\underline{g})$ and $C_1(\underline{g})$ while $d(u \wedge v) = [u, v]$. Thus $H_0(\underline{g}) = 0$ and

$$H_1(\underline{g}) = \operatorname{coker}(d : C_2(\underline{g}) \to C_1(\underline{g})) = \mathcal{G}/[\underline{g}, \underline{g}] = \underline{g}^{ab}.$$

3 The adjoint action on homology and reductive algebras

52 **Notation 3.1.** Let $\mathrm{Rep}(\underline{g})$ denote the category of $\underline{g}$-modules. On the tensor product $L \otimes M$ over k of two $\underline{g}$-modules L and M we have a natural $\underline{g}$-module structure given by the relation

$$u(x \otimes y) = (ux) \otimes y + x \otimes (uy) \quad \text{for} \quad u \in \underline{g}, x \in L, \quad \text{and} \quad y \in M.$$

Hence tensor powers, symmetric powers, and exterior powers of $\underline{g}$-modules have natural $\underline{g}$-module structures. For example on $\Lambda^q M$ the $\underline{g}$-module structure is given by the relation

$$u(x_1 \wedge \ldots \wedge x_q) = \sum_{1 \le i \le q} x_1 \wedge \ldots \wedge (ux_j) \wedge \ldots \wedge x_q.$$

Example 3.2. The k-module $\underline{g}$ is a $\underline{g}$-module with the action called the adjoint action, denoted $ad(u) : \underline{g} \to \underline{g}$ for $u \in \underline{g}$, where

$$ad(u)(x) = [u, x] \quad \text{for} \quad u, x \in \underline{g}.$$

Observe that the Jacobi identity gives the $\underline{g}$-module condition

$$ad([u, v])(x) = ad(u)(ad(v)(x)) - ad(v)(ad(u)(x))$$

or $[[u, v], x] = [u, [v, x]] - [v, [u, x]]$ for $u, v, x \in \underline{g}$.

Combining the previous two considerations, we see that $\underline{g}$ acts on the graded module $C_*(\underline{g}) = \Lambda^*(\underline{g})$ of the standard Lie algebra complex. Each element $u \in \underline{g}$ defines a grading preserving map

$$ad(u) : C_*(\underline{g}) \to C_*(\underline{g}),$$

and by exterior multiplication, a morphism of degree +1 denoted $e(u) : C_*(\underline{g}) \to C_*(\underline{g})$ defined by

$$e(u)(x_1 \wedge \ldots \wedge x_q) = u \wedge x_1 \wedge \ldots \wedge x_1.$$

53 The relation of the differential d on $C_*(\underline{g})$ to the adjoint action $ad(u)$ and the exterior multiplication $e(u)$ are contained in the next proposition. The details of this proposition are left to the reader.

Proposition 3.3. *For $u \in \underline{g}$ the adjoint action $ad(u)$ commutes with d, that is, $(ad(u))d = d(ad(u))$ so that $C_*(\underline{g})$ is a complex of $\underline{g}$-modules and for exterior product $e(u)$ we have*

$$ad(u) = de(u) + e(u)d. \qquad (*)$$

In low degrees $d : C_2(\underline{g}) \to C_1(\underline{g})$ commutes with $ad(u)$ by the Jacobi identity, and the homotopy formula (*) holds on $C_1(\underline{g})$ by the relation $ad(u)(x) = [u, x] = de(u)(x)$ and on $C_2(\underline{g})$ by the Jacobi formula.

The action of $\underline{g}$ on the standard complex $C_*(\underline{g})$ induces an action on $H_*(\underline{g})$. In view of the homotopy formula (*) this action $ad(u)$ is homotopic to zero, and this gives the following corollary.

Corollary 3.4. *The action of $\underline{g}$ on $H_*(\underline{g})$ is zero, that is, the homology $\underline{g}$-module is the trivial module.*

Definition 3.5. A $\underline{g}$-module M is simple or irreducible provided the only submodules are the trivial ones 0 and M. A $\underline{g}$-module M is semisimple or completely reducible if it satisfies the following equivalent conditions:

(a) M is a direct sum of simple modules,

(b) M is a sum of simple submodules, and

(c) every submodule L of M has a direct summand, i.e. there is another submodule L, with $L \oplus L'$ isomorphic to M.

The above definition applies to any abelian category, for example, all representations of a group. For a proof of the equivalence of (a), (b), and (c) see Cartan and Eilenberg.

We will not make a definition in a nonstandard form, but it is exactly **54**
what is needed for applications.

Definition 3.6. A Lie subalgebra $\underline{g}$ of a Lie algebra $\underline{s}$ is reductive in $\underline{s}$ provided all exterior powers $\Lambda^q \underline{s}$ are semisimple as $\underline{g}$-modules with the exterior power of the adjoint action of $\underline{g}$ on $\underline{s}$. A Lie algebra $\underline{g}$ is reductive provided $\underline{g}$ is reductive in itself.

Proposition 3.7. *Let $\underline{g}$ be a reductive Lie subalgebra of a Lie algebra $\underline{s}$. Then the quotient morphism*

$$C_*(\underline{s}) \to C_*(\underline{s}) \otimes_{U(\underline{g})} k = C_*(\underline{s})_{\underline{g}}$$

is a homology isomorphism.

Proof. The kernel of the quotient $C_*(\underline{s}) \to C_*(\underline{s})_{\underline{g}}$ onto the $\underline{g}$ - coinvariants is the direct sum of an acyclic complex and one with zero differential. The factor with the zero differential must be zero by (3.4). Hence the kernel is acyclic, and the morphism is a homology isomorphism. This proves the proposition. □

Example 3.8. Let A be a k-module, and let $\underline{g\ell_n}(A)$ denote the Lie algebra over k of n matrices with entries in A with the usual Lie bracket $[u, v] = uv - vu$ for $u, v \in \underline{g\ell_n}(A)$. Then the Lie subalgebra $\underline{g\ell_n}(k)$ is reductive in $\underline{g\ell_n}(A)$, and in particular, $\underline{g\ell_n}(k)$ is a reductive Lie algebra. This is the basic example for the relation between the cyclic homology of A and the Lie algebra homology of $\underline{g\ell}(A) = \varinjlim \underline{g\ell_n}(A)$. We have to be a little careful with the limits because $\underline{g\ell}(k)$ is not reductive in $\underline{g\ell}(A)$. On the other hand we have the following result by passing to limits.

Proposition 3.9. *Let A be an algebra over k, a field of characteristic zero. Then the quotient morphism of complexes*

$$\theta_A : C_*(\underline{g\ell}(A)) \to C_*\underline{g\ell}(A))_{\underline{g\ell}(k)}$$

induces an isomorphism in homology.

4 The Hopf algebra $H_*(\underline{g\ell}(A), k)$ and additive algebraic K-theory

55 The algebra structure on $H_*(\underline{g\ell}(A))$ comes from the direct sum of matrices namely the morphisms of Lie algebras

$$\underline{g\ell}_n(A) \oplus \underline{g\ell}_n(A) \to \underline{g\ell}_{2n}(A) \to \underline{g\ell}(A).$$

The natural isomorphism $C_*(\underline{g}_1) \otimes C_*(\underline{g}_2) \to C_*(\underline{g}_1 \otimes \underline{g}_2)$ composes with the induced morphism of the inclusion to give a morphism of differential coalgebras

$$C_*(\underline{g\ell}_n(A)) \otimes C_*(\underline{g\ell}_n(A)) \to C_*(\underline{g\ell}_{2n}(A))$$

which in the limit over n gives a multiplication

$$C_*(\underline{g\ell}(A)) \otimes C_*(\underline{g\ell}(A)) \to C_*(\underline{g\ell}(A)).$$

Remark 4.1. This multiplication induces a morphism of homology which when composed with the Künneth morphism yields a multiplication $H_*(\underline{g\ell}(A))$ namely

$$H_*(\underline{g\ell}(A)) \otimes H_*(\underline{g\ell}(A)) \to H_*(\underline{g\ell}(A)).$$

Now we put together this multiplication and the isomorphism of (3.9) to obtain the following theorem.

Theorem 4.2. *With the coalgebra structure and multiplication on C_* $(\underline{g\ell}(A))$, the quotient morphism induces on $C_*(\underline{g\ell}(A))_{\underline{g\ell}(k)}$ a differential Hop algebra structure and the isomorphism $H_*(\underline{g\ell}(A)) \to H_*(C_*$ $(A)_{\underline{g\ell}(k)})$ shows that the multiplication on $C_*(\underline{g\ell}(A))$ induces a Hopf algebra structure on $H_*(\underline{g\ell}(A))$.*

Proof. The differential coalgebra structure and the multiplication given by direct sum of matrices is defined on the quotient by θ_A and can be seen directly from the definition. The multiplication defined by special choices of direct sum on $C_*(\underline{g\ell}(A))$ is not associative, but in the quotient these choices all reduce to the same morphism which gives associativity. This proves the theorem. □

Before going on to the calculation of $H_*(\underline{g\ell}(A))$ using cyclic ho- **56**
mology, we indicate how this is an additive K-theory by analogy with algebraic K-theory as defined by Quillen. The K-groups $K_*(A)$ of a ring A are the homotopy groups of a certain space

$$K_*(A) = \pi_*(BGL(A)^+)$$

where the space $BGL(A)^+$ comes from A a series of three steps

$$A \mapsto GL(A) \mapsto BGL(A) \mapsto BGL(A)^+$$

where $GL(A) = \varinjlim GL_n(A)$ is the infinite linear group, B is the classifying space of the group $GL(A)$, and $BGL(A)^+$ is the result of applying the Quillen plus construction. The map $BGL(A) \to BGL(A)^+$ is a homology isomorphism and $\pi_1(BGL(A)^+)$ is the abelianization of $\pi_1(BGL(A)) = GL(A)$. From the relations of algebraic K-theory with extensions of groups, the work of Kassel and Loday 1982 suggested that there should be an additive analogue of K-theory using the homology of Lie algebras.

The analogue for Lie algebras of the three steps in algebraic K-theory over k is to begin with an algebra A over k and perform the following three steps

$$A \mapsto \underline{g\ell}(A) \mapsto C_*(\underline{g\ell}(A)) \mapsto C_*(\underline{g\ell}(A))_{\underline{g\ell}(k)}.$$

The quotient coalgebra construction $C_*(\underline{g\ell}(A)) \to C_*(\underline{g\ell}(A))_{\underline{g\ell}(k)}$ is like the plus construction $BGL(A) \to BGL(A)^+$ in the sense that the map is an isomorphism of the homology coalgebras and $C_*(\underline{g\ell}(A))_{\underline{g\ell}(k)}$ has a Hopf algebra structure where by analogy the plus construction $BGL(A)^+$ is an H-space.

There is no Lie algebra homotopy groups, but the rational homotopy can be calculated from the homology in the case of an H-space. This is the basic theorem of Milnor-Moore in rational homotopy.

Theorem 4.3. *Let X be a path connected H-space. The rational Hure-*
57 *wicz morphisms $\phi : \pi_*(X) \otimes \mathbf{Q} \to PH_*(X, \mathbf{Q})$ is an isomorphism of graded Lie algebras onto the primitive elements PH_* in homology.*

Remark 4.4. The above considerations together with the Milnor-Moore theorem led Feigin and Tsygan [1985] to introduce the following definition of the additive K-groups of algebras A over a filed k of characteristic zero

$$K_*^{\mathrm{add}}(A) = PH_*(C_*(\underline{g\ell}(A))_{\underline{g\ell}(k)}).$$

5 Primitive elements $PH_*(\underline{g\ell}(A))$ and cyclic homology of A

In this section k will always denote a field of characteristic zero. We begin with two preliminaries. The first is based on Appendix 2 of the rational homotopy theory paper of Quillen [1969].

Proposition 5.1. *On the category of cocommutative differential Hopf algebras A over k, the natural morphism $H(P(A)) \to P(H(A))$ is an isomorphism where $x \in P(A)$ means $\Delta(x) = x \otimes 1 + 1 \otimes x$.*

Proof. Quillen proves rather directly that for a differential Lie algebra $\underline{L}$ with universal enveloping $U(\underline{L})$ differential Hopf algebra that $U(H(\underline{L})) \to H(U(\underline{L}))$ is an isomorphism. Now U and P are inverse functors between differential Lie algebras and cocommutative differential Hopf algebras by a basic structure theorem of Milnor and moore 1965. This proves the proposition. □

The second preliminary is basic invariant theory over a field of characteristic zero.

Basic invariant theory 5.2. Let V be an n-dimensional vector space over k, denote $\underline{g\ell}(V) = \mathrm{End}(V)$ as a Lie algebra over k, and denote the symmetric group on q letters by Sym_q. There is a map $\phi : k[\mathrm{Sym}_q] \to \mathrm{End}(V^{q\otimes}) = \underline{g\ell}(V)^{q\otimes}$ where

$$\phi(\sigma)(x_1 \otimes \cdots \otimes x_q) = x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(q)} \quad \text{for} \quad \sigma \in \mathrm{Sym}_q\,.$$

The basic assertion of invariant theory is the following morphisms **58**
are isomorphisms for $n = \dim(V) \geq q$

$$k[\mathrm{Sym}_q] \to (\underline{g\ell}(V)^{q\otimes})^{\underline{g\ell}(V)} \to (\underline{g\ell}(V)^{q\otimes})_{\underline{g\ell}(V)}.$$

The symmetric group Sym_q acts on $\underline{g\ell}(V)^{q\otimes}$ by conjugation through ϕ and this ϕ is Sym_q equivariant as is seen from a direct calculation.

A basis of V is equivalent to an isomorphism $\underline{g\ell}(V) \to \underline{g\ell}_n(k)$ and $\underline{g\ell}(V{\otimes}A) \to \underline{g\ell}(V){\otimes}A \to \underline{g\ell}_n(A)$ for any k-algebra. The next proposition is the first link between Lie algebra chains and certain tensor powers of A.

Proposition 5.2. *If* $n = \dim(V) \geq q$*, then we have an isomorphism of* k*-modules,*

$$\Lambda^q(\underline{g\ell}(V) \otimes A)_{\underline{g\ell}(V)} \simeq (k[\mathrm{Sym}_q] \otimes A^{q\otimes}) \otimes_{\mathrm{Sym}_q} (sgn)$$

where Sym_q *acts by conjugation on* $k[\mathrm{Sym}_q]$ *and* (sgn) *is the one dimensional sign representation.*

Proof. We can write the exterior power

$$\begin{aligned}\Lambda^q(\underline{g\ell}(V) \otimes A)_{\underline{g\ell}(V)} &= [(\underline{g\ell}(V) \otimes A)^{q\otimes} \otimes_{\mathrm{Sym}_q} (sgn)]_{\underline{g\ell(V)}}\\ &= [(\underline{g\ell}(V)^{q\otimes})_{g\ell}(V) \otimes A^{q\otimes}] \otimes_{\mathrm{Sym}_q} (sgn).\end{aligned}$$

Using (5.2), we tensor ϕ with $A^{q\otimes}$ and (sgn) to obtain an isomorphism

$$\{k[\mathrm{Sym}_q] \otimes A^{q\otimes}\} \otimes_{\mathrm{Sym}_q} (sgn) \to \{(\underline{g\ell}(V)^{q\otimes}\} \otimes_{\mathrm{Sym}_q} (sgn).$$

This proves the proposition. □

In terms of this isomorphism we decompose $\Lambda^q(\overline{g\ell}(V) \otimes A)_{\underline{g\ell}(V)}$ using the decomposition of $k[\mathrm{Sym}_q]$ under conjugation. There will be one factor for each conjugacy class of Sym_q. The elements of the form $x = [\sigma] \otimes a$ where $[\sigma]$ is the conjugacy class of the element σ and $a =$
59 $a_1 \otimes \cdots \otimes a_q$ with $a_i \in A$ generate $(k[\mathrm{Sym}_q] \otimes A^{q\otimes}) \otimes_{\mathrm{Sym}_q} (sgn)$, and the diagonal morphism on this element is given by shuffles as

$$\Delta(x) = \sum_{\{1,\ldots,n\}=I \coprod J, \sigma(I)=I, \sigma(J)=J} ([\sigma|_I] \otimes a_I) \otimes ([\sigma|_J] \otimes a_J)$$

where $x = [\sigma] \otimes a$, $a_I = \otimes_{i \in I} a_i$, and $a_J = \otimes_{j \in J a_j}$.

Remark 5.3. An element $x = [\sigma] \otimes a$ is primitive, i.e. $\Delta(x) = x \otimes 1 + 1 \otimes x$ if and only if $\sigma \in U_q$, the conjugacy class of the cyclic permutation $(1, \ldots, q)$. As a Sym_q-set, the conjugacy class U_q is isomorphic to $\mathrm{Sym}_q / \mathrm{Cyl}_q$ where Cyl_q is the cyclic subgroup generated by $(1, \ldots, q)$. Thus we have an isomorphism between the following k-modules $(k[U_q] \otimes A^{q\otimes} \otimes_{\mathrm{Sym}_q} (sgn)$ and $(k[\mathrm{Sym}_q / \mathrm{cyl}_q] \otimes A^{q\otimes}) \otimes_{\mathrm{Sym}_q} (sgn)$. We can summarize the above discussion with the following calculation of the primitive elements of $\Lambda^*(\underline{g\ell}(V) \otimes A)_{\underline{g\ell}(V)}$ in a given degree.

Proposition 5.4. *The submodule* $P\Lambda^q(\underline{g\ell}(V)\otimes A)_{\underline{g\ell}(V)}$ *of primitive elements for* $q \leq n = \dim(V)$ *is isomorphic to*

$$A^{q\otimes} \otimes_{\mathrm{Cyl}_q} (sgn) = C_{q-1}(A)_{1-T}, \text{ the cyclic homology chains.}$$

A further analysis of the isomorphisms involved shows that the differential in the Lie algebra homology induces the quotient of the Hochschild differential, or the cyclic homology differential. Thus we are led to the basic result of Tsygan [1983] and Loday-Quillen [1984] in characteristic zero.

Theorem 5.5. *The vector space of primitive elements in Lie algebra homology* $PH_q(C_*(\underline{g\ell}(A))_{\underline{g\ell}(k)}) = PH_q(\underline{g\ell}(A))$ *is isomorphic to the cyclic homology vector space* $HC_{q-1}(A)$*.*

Chapter 5

Mixed Complexes, the Connes Operator B, and Cyclic Homology

THE DOUBLE COMPLEX $CC_{*,*}(A)$ has acyclic columns in odd de- **60**
grees, and this property leads to the concept of a mixed complex. Thus we effectively suppress part of the cyclic homology complex $CC_*(A)$. In the second section this new definition is shown to be equivalent to the old one. Yet another way of simplifying the Connes-Tsygan double complex is to normalize the Hochschild complexes, and this is considered in § 3.

1 The operator B and the notion of a mixed complex

Let A be an algebra over k. The last simplicial operator defines a homotopy operator $s : C_q(A) \to C_{q+1}(A)$ by the relation $s = (-1)^q s_q$. It has the basic property that $sb' + b's = 1$, and this is a general property of simplicial objects over an abelian category.

Definition 1.1. Let A be an algebra over k. The Connes operator is $B = (1 - T)sN : C_q(A) \to C_{q+1}(A)$ on the standard complex.

For any cyclic object $X_\bullet$ the Connes operator is

$$B = (1 - T)sN : X_q \to X_{q+1}$$

a morphism of degree $+1$. The corresponding diagram is

$$\begin{array}{ccc} X_{q+1} & \xleftarrow{1-T} & X_{q+1} & & \\ & & b' \uparrow \downarrow s & & \\ & & X_q & \xleftarrow{N} & X_q \end{array}$$

Proposition 1.2. *Let $X_\bullet$ be a cyclic object over an abelian category $\mathcal{A}$. The Connes operator B of degree $+1$ and the usual boundary operator b satisfy the following relations*

$$b^2 = 0, B^2 = 0, \quad \textit{and} \quad Bb + bB = 0.$$

61 *Proof.* The first relation was already contained in 3(2.4), and the second $BB = (1-T)sN(1-T)sB = 0$ since $N(1-T) = 0$ by 3(3.3). For the last relation we calculate using 3(3.4)

$$\begin{aligned} Bb + bB &= (1 - T)(sNb + b(1 - T)sN = (1 - T)sb'N + (1 - T)b's N \\ &= (1 - T)(sb' + b's)N = (1 - T)N = 0. \end{aligned}$$

This proves the proposition. □

Remark 1.3. For the standard cyclic object $C_\bullet(A)$ associated with an algebra A, the following formula defines B on an element,

$$\begin{aligned} B(a_0 \otimes \cdots \otimes a_q) = &\sum (-1)^{iq}(a_{q-i} \otimes \cdots \otimes a_q \otimes a_0 \otimes \cdots \otimes a_{q-1-i} \otimes 1) - \\ &- \sum (-1)^{(i-1)q}(1 \otimes a_{q-i} \otimes \cdots \otimes a_q \otimes a_0 \otimes \cdots \otimes a_{q-1-i}). \end{aligned}$$

This leads to a new structure called a mixed complex which is a complex with two operators one of degree -1 and one of degree $+1$ which commute in the graded sense, that is, anticommute in the ungraded sense. This is the relation $Bb + bB = 0$. Each mixed complex has homology in the usual sense with its operator of -1. Using the two operators, we can associate a second complex, which can be thought of as the total complex of a double complex associated with the mixed complex. The homology of this complex is called the cyclic homology of the mixed complex. This terminology is justified because the cyclic homology of a mixed complex associated with a cyclic object will shown to be isomorphic to the cyclic homology of the cyclic object as defined in the previous chapter. A second point justifying the terminology is that there is a Connes exact couple relating the ordinary and cyclic homology of a mixed complex.

There are two advantages in considering mixed complexes. The complex defining cyclic homology of the mixed complex is smaller than $CC_{\bullet\bullet}(X)$ for a cyclic object. Then there are mixed complexes which do not come from cyclic objects which are useful, namely the one corresponding to the normalized standard complex $\overline{C}_*(A)$ for Hochschild **62**
homology.

2 Generalities on mixed complexes

Definition 2.1. Let $\mathcal{A}$ be an abelian category. A mixed complex X is a triple (X_*, b, B) where X_* is a **Z**-graded object in $\mathcal{A}$, $b : X_* \to X_*$ is a morphism of degree -1, and $B : X_* \to X_*$ is a morphism of degree $+1$ satisfying the relations

$$b^2 = 0, B^2 = 0, Bb + bB = 0.$$

A morphism $f : X \to Y$ of mixed complxes is a morphism of graded objects such that $bf = fb$ and $Bf = fB$. A mixed complex is positive if $X_q = 0$ if $q < 0$.

Let $\text{Mix}(\mathcal{A})$ denote the category of mixed complexes and $\text{Mix}^+(\mathcal{A})$ the full subcategory of positive mixed complexes.

Remark 2.2. We have the following functors associated with mixed complexes. Let $\mathcal{A}$ denote an abelian category.

(1) The functor which assigns to a mixed complex (X, b, B) the complex (X, b) is defined $\text{Mix}(\mathcal{A}) \to C(\mathcal{A})$ and it restricts to $\text{Mix}^+(\mathcal{A}) \to C^+(\mathcal{A})$ to the full subcategories of positive objects. When

it is composed with the homology functor $H : C(\mathcal{A}) \to Gr_Z(\mathcal{A})$, it defines the homology $H(X)$ of the mixed object X.

(2) The functor which assigns to a cyclic object $X_\bullet$ the mixed object (X, b, B) as in (1.2) is defined $\Lambda(\mathcal{A}) \to \mathrm{Mix}^+(\mathcal{A})$, and when composed with $\mathrm{Mix}^+(\mathcal{A}) \to C^+(\mathcal{A})$ gives the usual simplicial differential object whose homology is the ordinary homology of the cyclic object.

(3) Finally the standard cyclic object $C_\bullet(A)$ associated with an algebra A over a ring k is a functor defined $\mathrm{Alg}_k \to \Lambda(k)$ which can be composed with the above functors to give a positive mixed complex of k-modules whose homology is in turn its Hochschild homology.

63 Now we wish to define a functor $\mathrm{Mix}^+(\mathcal{A}) \to C^+(\mathcal{A})$ whose homology is to be the cyclic homology. There is a similar construction for $\mathrm{Mix}(\mathcal{A}) \to C(\mathcal{A})$ which is not given since it is not needed for our purposes.

Definition 2.3. Let (X, b, B) be a positive mixed complex over an abelian category $\mathcal{A}$. The cyclic complex $(X[B], b_B)$ associated with the mixed complex (X, b, B) is defined as a graded object by $X[B]_n = X_n \oplus X_{n-2} \oplus \dots$ which is a finite sum since X is positive and $b_B : X[B]_n \to X[B]_{n-1}$ is defined using the projections $p_i : X[B]_n \to X_i$ by the relation $p_i b_B = bp_{i+1} + Bp_{i-1}$. The cyclic homology $HC_*(X)$ of the mixed complex X_* is $HC_*(X) = H_*(X[B])$, the homology of cyclic complex associated with X_*.

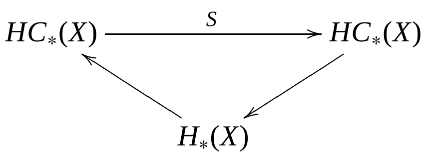
If the abelian category $\mathcal{A} = (k)$, the category of k-modules, then the boundary in the cyclic complex can be described by its image on elements $(x_n, x_{n-2}, x_{n-4}, \dots) \in X[B]_n$, and the above definition gives

$$b_B(x_n, x_{n-2}, x_{n-4}, \dots) = (b(x_n) + B(x_{n-2}), b(x_{n-2}) + B(x_{n-4}), \dots).$$

Remark 2.4. To (X, b, B), a positive mixed complex over an abelian complex $\mathcal{A}$, we associate an exact sequence

$$0 \to (X, b) \xrightarrow{i} (X[B], b_B) \to (s^{-2}X[B], b_B) \to 0$$

where $i : X_n \to X[B]_n$ is defined by $p_n i = X_n$ and $p_i i = 0$ for $i < n$. Observe that i is a monomorphism of complexes with quotient of $X[B]$ equal to $s^{-2}X[B]$ which is $X[B]$ shifted down by 2 degrees. The exact triangle of this short exact sequence is the Connes exact couple for mixed complexes



and as usual $\deg(S) = -2$, $\deg(B) = +1$, and $\deg(I) = 0$.

If we can show that the Connes exact sequence of the previous Re- **64**
mark (2.4) is the same as the Connes exact sequence for a cyclic object in terms of $CC_{\cdot}(X)$, then we have a new way of calculating cyclic homology for a cyclic object and hence also for an algebra. This we do in the next section.

First we remark that the above construction of the complex $(X[B], b_B)$ from a mixed complex (X, b, B) can be viewed as the total complex of a double complex $\mathcal{B}(X)$.

Definition 2.5. Let (X, b, B) be a positive mixed complex over an abelian category $\mathcal{A}$. The Connes double complex $\mathcal{B}(X)$ associated with X is defined by the requirement that $\mathcal{B}(X)_{p,q} = X_{q-p}$ for $p, q \geq 0$ and 0 otherwise, the differential $d' = B$ and $d'' = b$.

The double complex $\mathcal{B}(X)$ is concentrated in the 2^{nd} octant of the first quadrant, that is, above the line $p = q$ in the first quadrant.

$$
\begin{array}{ccccccccccc}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
X_q & \overset{B}{\longleftarrow} & X_{q-1} & \overset{B}{\longleftarrow} & X_{q-2} & \overset{B}{\longleftarrow} & \cdots & \overset{B}{\longleftarrow} & X_1 & \overset{B}{\longleftarrow} & X_0 \\
\downarrow b & & \downarrow b & & \downarrow b & & & & \downarrow b & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & & \\
\downarrow & & \downarrow & & \downarrow & & & & & & \\
X_2 & \overset{B}{\longleftarrow} & X_1 & \overset{B}{\longleftarrow} & X_0 & & & & & & \\
\downarrow b & & \downarrow b & & & & & & & & \\
X_1 & \overset{B}{\longleftarrow} & X_0 & & & & & & & & \\
\downarrow b & & & & & & & & & & \\
X_0 & & & & & & & & & &
\end{array}
$$

The associated single complex of the double complex $\mathscr{B}(X)$ is just $X[B]$, b_B. Once again one can see the double periodicity which arises by deleting the first column.

3 Comparison of two definition of cyclic homology for a cyclic object

65 We have two functors defined on category $\Lambda(\mathscr{A})$ of cyclic objects over an abelian category $\mathscr{A}$ with values in the category of positive complexes $C^+(\mathscr{A})$ over $\mathscr{A}$. The first is $CC_*(X)$, the associated complex of the cyclic homology double complex $CC_{\bullet\bullet}(X)$, and the second is $X_\bullet[B]$ where $X_\bullet$, b, B is the mixed complex associated with X, see (1.1) and (1.2)

Notation 3.1. For a cyclic object $X_\bullet$ over an abelian category $\mathscr{A}$ we define a comparison morphism $f : X_\bullet[B] \to CC_*(X)$ by the following relations in degree n. For $f_n : X_\bullet[B]_n \to CC_n(X)$ we require that

$$
pr_i f = \begin{cases} pr_i & \text{for } i \text{ even} \\ s'N\, pr_{i-1} & \text{for } i \text{ odd} \end{cases}
$$

where is degree n the diagram takes the form

$$\begin{array}{ccc} X.[B]_n = \coprod_i X_{n+2i} & \longrightarrow & \coprod_i X_{n+i} \\ & & \Big\downarrow {\scriptstyle pr_i} \\ & & X_{n+i}. \end{array}$$

If $X.$ is a cyclic k-module, then this definition can be given in terms of elements,

$$f(x_n, x_{n-2}, x_{n-4}, \ldots) = (x_n, s'Nx_{n-2}, x_{n-2}, s'Nx_{n-4}, x_{n-r}, \ldots).$$

Lemma 3.2. *The graded morphism $f : X.[B] \to CC_*(X)$ is a morphism of differential objects.*

Proof. There is a general argument that says that abelian categories can be embedded in a category of modules. The result is that we can check the commutativity of f with boundary morphisms using elements. Now the differential of

$$f(x_n, x_{n-2}, x_{n-4}, \ldots) = (x_n, s'Nx_{n-2}, x_{n-2}, s'Nx_{n-4}, x_{n-4}x \ldots)$$

is the element **66**

$$(bx_n + (1-t)s'Nx_{n-2'} - b's'Nx_{n-2} + Nx_{n-2}, \ldots).$$

If we apply f to the element

$$b_B(x_n, x_{n-2}, x_{n-4}, \ldots) = (bx_n + Bx_{n-2}, bx_{n-2} + Bx_{n-4}, \ldots),$$

then we obtain

$$(bx_n + Bx_{n-2}, s'Nbx_{n-2} + s'NBx_{n-4}, \ldots).$$

A direct inspection shows that the differential of $f(x_n, x_{n-2}, \ldots)$ and $f(b_B(x_n, x_{n-2}, \ldots))$ have the same even coordinates. For the odd indexed coordinates, we calculate

$$s'Nbx_{n-2} + s'NBx_{n-4} = s'Nbx_{n-2} + s'N(1-t)s'Nx_{n-4}$$

$$= s'b'Nx_{n-2}$$
$$= Nx_{n-2} - b's'Nx_{n-2}.$$

This shows that f is a morphism of complexes and proves the lemma. □

The following result shows that the two definitions of cyclic homology are the same.

Theorem 3.3. *Let $X_\bullet$ be a cyclic object in an abelian category $\mathcal{A}$. The above comparison morphism $f : X_\bullet[B] \to CC_*(X)$ induces an isomorphism $H(f) : H_*(X_\bullet[B]) \to HC_*(X)$.*

Proof. The first index of the double complex $X_{\bullet\bullet}$ determines a filtration $F_pCC_*(X)$ on $CC_*(X)$ where

$$F_pCC_n(X) = \coprod_{i+j=n, i\le p} X_{i,j}$$

and there is a related filtration $F_pX_\bullet[B]$ on $X_\bullet[B]$

$$F_pX_\bullet[B]_n = \coprod_{2i\le p} X_{n-2i}.$$

67 From the definition of f, we check that f is filtration preserving. The morphism $E^0(f)$ is a monomorphism and $d^0 = b$ with $E^0_{2k,*}(f)$ and isomorphism, $E^0_{2k+1,*}X_\bullet[B] = 0$, and $E^0_{2k+1,*}CC(X)$ acyclic. Thus $E^1(f)$ is an isomorphism. By 1(5.6) the induced morphism $H_*(f)$ is an isomorphism. This proves the theorem. □

Remark 3.4. The morphism f considered above can be viewed as $f : \mathcal{B}(X)_* = X_\bullet[B] \to CC_*(X)$. These complexes come from double complexes with a periodic structure. The first vertical column of $\mathcal{B}(X)$ maps to the total subcomplex of $CC_*(X)$ determined by the first two vertical columns of $CC_{\bullet\bullet}(X)$. The resulting subcomplexes have homology equal to Hochschild homology while the quotient complexes have the form of $\mathcal{B}(X)_*$ and $CC_*(X)$ respectively. We arrive at a sharper form of the isomorphism in (3.3), namely that f induces an isomorphism of the Connes' exact couple defined by mixed complexes onto the Connes' exact couple defined by the cyclic homology double complex.

4 Cyclic structure on reduced Hochschild complex

In 3(2.6), we remarked that for a simplicial k-module X, the subcomplex $D(X)$ generated by degenerate elements was contractible, and thus the quotient morphism induces an isomorphism on homology

$$H_*(X) \to H_*(X/D(X)).$$

For the standard complex $C_*(A)$ of an algebra A the quotient complex $C_*(A)/DC_*(A)$ is the reduced standard complex $\overline{C}_*(A)$ where

$$\overline{C}_q(A) = A \otimes \overline{A}^{q\otimes}$$

as noted in 3(2.7). To study the cyclic homology $HC_*(A)$ with the reduced standard complex, we use the mixed complex construction and the following formula for the Connes' operator B.

Proposition 4.1. *The operators b and B on the standard complex $C_*(A)$ define operators b and B on the quotient reduced standard complex $\overline{C}_*(A)$ given by the formulas* 68

$$\begin{aligned} b(a_0 \otimes \cdots \otimes a_q) = {} & a_0a_1 \otimes a_2 \otimes \cdots \otimes a_q \\ & + \sum_{0<i<q} (-1)^i a_0 \otimes \cdots \otimes a_ia_{i+1} \otimes \cdots \otimes a_q \\ & (-1)^q a_qa_0 \otimes a_1 \otimes \cdots \otimes a_{q-1} \end{aligned}$$

where the ambiguity in a_0a_1 and in a_qa_0 is cancelled with the terms $i = 1$ and $i = q - 1$ respectively in the sum and

$$B(a_0 \otimes \cdots \otimes a_q) = \sum_{1\le i\le q} (-1)^{iq} 1 \otimes a_i \otimes \cdots \otimes a_q \otimes a_0 \otimes \cdots \otimes a_{i-1}.$$

Proof. The first formula is just a quotient of the usual formula, and for the second we calculate immediately that

$$sN(a_0 \otimes \cdots \otimes a_q) = \sum_{1\le i\le q} (-1)^{iq} 1 \otimes a_i \otimes \cdots \otimes a_q \otimes a_0 \otimes \cdots \otimes a_{i-1}.$$

The statement follows from the fact that $tsN(a_0 \otimes \cdots a_q) = 0$ in the reduced complex with 1 in the nonzero place giving a degenaracy and the formula $B = (1 - t(sN$. This proves the proposition. □

Now we rewrite the b, B double complex for the reduced standard complex $\overline{C}_*(A)$ where $\overline{C}_q(A) = A \otimes \overline{A}^{q\otimes}$. It is in this form that we will compare it with complexes of differential forms in the next two chapters.

$$
\begin{array}{ccccccccccc}
\cdots & & \cdots & & \cdots & & \cdots\ \cdots\ \cdots & & \cdots & & \cdots\ \cdots \\
A \otimes \overline{A}^{(q+1)\otimes} & \overset{B}{\longleftarrow} & A \otimes \overline{A}^{q\otimes} & \overset{B}{\longleftarrow} & A \otimes \overline{A}^{(q-1)\otimes} & \overset{B}{\longleftarrow} & & \overset{B}{\longleftarrow} & A \otimes \overline{A} & \overset{B}{\longleftarrow} & A \\
\downarrow b & & \downarrow b & & \downarrow b & & & & \downarrow b & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & & \cdots\ \cdots\ \cdots & & A & & \\
\downarrow b & & \downarrow b & & \downarrow b & & & & & & \\
A \otimes \overline{A}^{2\otimes} & \overset{B}{\longleftarrow} & A \otimes \overline{A} & \overset{B}{\longleftarrow} & A & & & & & & \\
\downarrow b & & \downarrow b & & & & & & & & \\
A & & A & & & & & & & &
\end{array}
$$

Chapter 6

Cyclic Homology and de Rham Cohomology for Commutative Algebras

THIS CHAPTER DEALS with the relations between Hochschild homology and de Rham cohomology for commutative algebras. In the case of algebras over a field of characteristic zero, we can go further to prove that the de Rham cohomology groups occur as components in a direct sum expression for cyclic homology. We begin with a discussion of differential forms and show how closely related they are to Hochschild homology. Then we introduce a product structure on $HH_*(A)$ in the special case where A is commutative. This gives us a comparison morphism between graded algebras, and then we sketch the Hochschild-Kostant-Rosenberg theorem which says that this morphism is an isomorphism for smooth algebras. We then calculate the cyclic homology of smooth algebras over a filed of characteristic zero. This is a case where the first derived couple of the Connes' exact couple splits and the first differential is the exterior differential of forms. **69**

Finally, we continue with a discussion of the algebra $A = C^\infty(M)$ of smooth functions on a manifold and prove Connes' theorem, which says roughly that this smooth case is parallel to the algebra case.

1 Derivations and differentials over a commutative algebra

In this section, let A denote a commutative algebra over k.

Definition 1.1. Let M be an A-module. A derivation D of A with values in M is a k-linear map $D : A \to M$ such that

$$D(ab) = aD(b) + bD(a) \quad \text{for} \quad a, b \in A.$$

Let $\mathrm{Der}_k(A, M)$ or just $\mathrm{Der}(A, M)$ denote the k-module of all derivations of A with values in M.

70 The module $\mathrm{Der}(A, M)$ has a left A-module structure where cD is defined by $(cD)(a) = cD(a)$ for c, $a \in A$. For $M = A$ the k-module $\mathrm{Der}(A, A)$ has the structure of a Lie algebra over k, where the Lie bracket is given by $[D, D'] = DD' - D'D$ for $D, D' \in \mathrm{Der}(A, A)$. A simple check shows that $[D, D']$ satisfies the derivation rule on products.

Definition 1.2. The A-module of Kähler differentials is a pair, $(\Omega^1_{A/k}, d)$ where $\Omega^1_{A/k}$, or Ω^1_A or simply Ω^1, is an A-module and $d : A \to \Omega^1_{A/k}$ is a derivation such that for any derivation $D : A \to M$, there exists a unique A-linear morphism $f : \Omega^1_{A/k} \to M$ with $D = fd$.

The derivation d defines an A-linear morphism

$$\mathrm{Hom}_A(\Omega^1_{A/k}, M) \to \mathrm{Der}_k(A, M)$$

by assigning to $f \in \mathrm{Hom}_A(\Omega^1_{A/k}, M)$ the derivation $fd \in \mathrm{Der}_k(A, M)$. The universal property is just the assertion that this morphism is an isomorphism of A-modules. The universal property shows that two possible A-modules of differentials are isomorphic with a unique isomorphism preserving the derivation d.

There are two constructions of the module of derivations $\Omega^1_{A/k}$. The first one as the first Hochschild homology k-module of A and the second by a direct use of the derivation property.

Construction of $\Omega^1_{A/k}$1. 1.3. Let I denote the kernel of the multiplication morphism $\phi(A) : A \otimes A \to A$. To show that

$$\Omega^1_{A/k} = I/I^2 = HH_1(A),$$

we give I/I^2 an A-module structure by $ax = (1 \otimes a)x = (a \otimes 1)x$, observing that $1 \otimes a - a \otimes 1 \in I$ and $(1 \otimes a - a \otimes 1)x \in I^2$ for $a \in A$, $x \in I$. We define

$$d : A \to I/I^2 \quad \text{by} \quad d(a) = (1 \otimes a - a \otimes 1) \bmod I^2 \quad \text{for} \quad a \in A,$$

and check that it is a derivation by **71**

$$\begin{aligned} d(ab) &= 1 \otimes ab - ab \otimes 1 \\ &= (1 \otimes a)(1 \otimes b - b \otimes 1) + (b \otimes 1)(1 \otimes a - a \otimes 1) \\ &= ad(b) + bd(a). \end{aligned}$$

To verify the universal property, we consider a derivation $D : A \to M$, and note that $f(a \otimes b) = aD(b)$ defined on $A \otimes A$ restricts to I. Since $D(1) = 0$, we see that $f(d(a)) = f(1 \otimes a - a \otimes 1) = D(a)$ or $fd = D$. The uniqueness of f follows from the fact that I, and hence also I/I^2, is generated by the image of d. This is seen from the following relation,

$$\sum_i a_i \otimes b_i = \sum_i (a_i \otimes 1)(1 \otimes b_i - b_i \otimes 1) = \sum_i a_i db_i$$

which holds for $\sum_i a_i \otimes b_i \in I$ or equivalently if $\sum_i a_i b_i = 0$ in A. Finally, we note that $f(I^2) = 0$ by applying f to $(\sum_i a_i \otimes b_i)(1 \otimes c - c \otimes 1)$ to obtain

$$\begin{aligned} f\left(\sum_i a_i \otimes b_i c - \sum_i a_i c \otimes b_i\right) &= \sum_i a_i D(b_i c) - \sum_i a_i c D(b_i) \\ &= \left(\sum_i a_i b_i\right) D(c) = 0. \end{aligned}$$

Thus $d : A \to I/I^2$ is a module of differentials.

Construction of $\boldsymbol{\Omega^1_{A/k}}$ II. 1.4. Let L be the A-submodule of $A \otimes A$ generated by all $1 \otimes ab - a \otimes b - b \otimes a$ for $a, b \in A$ where the A-module structure on $A \otimes A$ is given by $c(a \otimes b) = (ca) \otimes b$ for $c \in A$, $a \otimes b \in A \otimes A$. Next, we define $d : A \to (A \otimes A)/L$ by $d(b) = (1 \otimes b) \bmod L$ and from the nature of the generators of L, it is clearly a derivation. Further, if $D \in \mathrm{Der}_k(A, M)$, then $f : (A \otimes A)/L \to M$ defined by $f(a \otimes b \bmod L) = aD(b)$ is a well-defined morphism of A-modules, and it is the unique one with the property that $fd = D$.

Remark 1.5. In the first construction, we saw that $\Omega^1_{A/k} = HH_1(A)$ and in the second construction we see that

$$\Omega^1_{A/k} = \mathrm{coker}(b : C_2(A) = A^{\otimes 3} \to A^{\otimes 2} = C_1(A))$$

72 in the standard complex for calculating Hochschild homology. Now we introduce the algebra of all differential forms in order to study the higher Hochschild homology modules in terms of differential forms.

Definition 1.6. The algebra of differential forms over an algebra A is the graded exterior algebra $\Lambda^*_A \Omega^1_A$ over A, denoted Ω^*_A or $\Omega^*_{A/k}$. The elements of $\Omega^q_A = \Lambda^q_A \Omega^1_A$ are called differential forms of degree q, or simply q-forms over A.

A q-form is a sum of expressions of the form $a_0 da_1 \ldots da_q$ where $a_0, \ldots, a_q \in A$. If Ω^1_A is a free A-module with basis $da_1, \ldots, da_n$, then $\Omega^q_{A/k}$ has a basis consisting of

$$da_{i(1)} \ldots da_{i(q)} \quad \text{for all} \quad i(1) < \ldots < i(q)$$

as an A-module.

Remark 1.7. The algebra $\Omega^*_{A/k}$ is strictly commutative in the graded sense. This means that

$$(1)\ \omega_1 \omega_2 = (-1)^{pq} \omega_2 \omega_1 \quad \text{for} \quad \omega_1 \in \Omega^p_{A/k}, \omega_2 \in \Omega^q_{A/k}$$

(this is commutativity in the graded sense), and

$$(2)\ \omega^2 = 0 \quad \text{for} \quad \omega \quad \text{of odd degree}$$

(this is strict commutativity).

Moreover, the exterior algebra is universal for strictly commutative algebras, in the sense that if $f : M \to H_1$ is a k-linear morphism of a k-module into the elements of degree 1 in a strictly commutative algebra H, then there exists a morphism of graded algebras $h : \Lambda^* M \to H$ with the property that $f = h_{|M} = \Lambda^1 M \to H^1$.

Since $\Omega^1_{A/k} \to HH_1(A)$ is a natural isomorphism by (1.2), we wish to define a strictly commutative algebra structure on $HH_*(A)$ for any commutative algebra A. We do this in the next section, and before that, we describe the exterior derivative which also arises from the universal 73
property of the exterior algebra.

Proposition 1.8. *There exists a unique morphism d of degree $+1$ defined $\Omega^*_{A/k} \to \Omega^*_{A/k}$ satisfying*

(a) $d^2 = 0$

(b) *d is a derivation of degree $+1$, that is,*

$$d(\omega_1\omega_2) = (d\omega_1)\omega_2 + (-1)^p\omega_1(d\omega_2) \quad \textit{for} \quad \omega_1 \in \Omega^p_{A/k}, \omega_2 \in \Omega^q_{A/k}.$$

(c) *d restricted to $A = \Omega^0$ is the canonical derivation $d : A \to \Omega^1$.*

Proof. The uniqueness follows from the relation

$$d(a_0 d_{a_1} \dots da_q) = da_0 da_1 \dots da_q$$

since the elements $a_0 d_{a_1} \dots da_q$ generate $\Omega^q_A = \Lambda^q \Omega^1_A$, and the existence is established with this formula. □

Definition 1.9. For an algebra A over k, the complex $(\Omega^*_{A/k}, d)$ is called the de-Rham complex of A, and the cohomology algebra $H^*(\Omega^*_{A/k}, d)$, denoted $H^*_{DR}(A)$, is called the de Rham cohomology of A over k.

2 Product structure on $HH_*(A)$

The basis for a product structure is usually a Künneth morphism and a Künneth theorem which says when the morphism is an isomorphism. The Künneth morphism usually comes from the morphism α for the homology of a tensor product $X_\bullet \otimes Y_\bullet$ of two complexes.

Definition 2.1. Let $X_\bullet$ and $Y_\bullet$ be two complexes of k-modules. The tensor Künneth morphism is

$$\alpha : H_\bullet(X_\bullet) \otimes H_\bullet(Y_\bullet) \to H_\bullet(X_\bullet \otimes Y_\bullet)$$

defined by the relation $\alpha(u \otimes v) = w$ where $u \in H_p(X)$ is represented by $x \in X_p$, $v \in H_q$ is represented by $y \in Y_q$ and w is represented by $x \otimes y \in (X \otimes Y)_{p+q}$.

74 If k is a field, then α is always an isomorphism. Under the assumption that $X_\bullet$ and $Y_\bullet$ are flat over k, it follows that α is an isomorphism if either $H_\bullet(X_\bullet)$ or $H_\bullet(Y_\bullet)$ is flat over k.

Remark 2.2. Let B and B' be two algebras over k. If L is a right B-module and L' a right B'-module, then $L \otimes L'$ is a right $B \otimes B'$ module, and if M is a left B-module and M' a left B'-module, then $M \otimes M'$ is a left $B \otimes B'$-module. Using the natural associativity and commutativity isomorphisms for the tensor product over k, we have a natural isomorphism

$$\theta : (L \otimes_B M) \otimes (L' \otimes_{B'} M') \to (L \otimes L')_{B \otimes B'}(M \otimes M').$$

If $P_\bullet \to L$ is a projective resolution of L over B, and if $P'_\bullet \to L'$ is a projective resolution of L' over B', then $P_\bullet \otimes P'_\bullet \to L \otimes L'$ is a projective resolution of $L \otimes L'$ over $B \otimes B'$. This assertion holds in either the absolute projective or k-split projective cases. Combining the isomorphism of complexes

$$(P_\bullet \otimes_B M) \otimes (P'_\bullet \otimes_{B'} M') \to (P_\bullet \otimes P') \otimes_{B \otimes B'} (M \otimes M')$$

with the Künneth morphism of (2.1), we obtain the following:

Künneth morphism for Tor 2.3. Let B and B' be two algebras with modules L and M over B and L' and M' over B'. The isomorphism θ extends to a morphism of functors

$$\alpha : \mathrm{Tor}_*^B(L, M) \otimes \mathrm{Tor}_*^{B'}(L', M') \to \mathrm{Tor}_*^{B\otimes B'}(L \otimes L', M \otimes M')$$

which we call the Künneth morphism for the Tor functor. This morphism is defined for both the absolute and k-split Tor functors.

Let A and A' be two algebras, and form the algebras $A^e = A{\otimes}A^{op}$ and $A'^e = A'{\otimes}A'^{op}$. There is a natural commuting isomorphism $(A{\otimes}A')^e \to A^e \otimes A'^e$ which we combine with the Künneth morphism for the Tor to obtain:

Künneth morphism for Hochschild homology 2.4. Let M be an A- 75
bimodule, and let M' be an A'-bimodule. A special case of the Künneth morphism for Tor is

$$\alpha : H_*(A, M) \otimes H_*(A', M') \to H_*(A \otimes A', M \otimes M')$$

called the Künneth morphism for Hochschild homology. In particular, we have $\alpha : HH_*(A) \otimes HH_*(A') \to HH_*(A \otimes A')$.

Definition 2.5. The Künneth_morphisms for Tor and for Hochschild homology satisfy associativity, commutativity, and unit properties which we leave to the reader to formulate. If k is a field, then the Künneth morphism is an isomorphism.

We are now ready to define the product structure $\phi(HH_*(A))$ on $HH_*(A)$ when A is commutative. Recall that an algebra A is commutative if and only if the structure morphism is a morphism of algebras $A \otimes A \to A$.

Definition 2.6. For a commutative k-algebra A the multiplication $\phi(HH_*(A))$ on $HH_*(A)$ is the composite $HH_*(\phi(A))\alpha$ defined by

$$HH_*(A) \otimes HH_*(A) \to HH_*(A \otimes A) \to HH_*(A).$$

From the above considerations $HH_*(A)$ is an algebra which is commutative over $A = HH_0(A)$ in the graded sense.

Remark 2.7. Let $B \to A$ be an augmentation of the commutative algebra B. If $K_* \to A$ is a B-projective resolution of A such that K_* is a differential algebra and $K_* \to A$ is a morphism of differential algebras, then we have the following morphisms

$$(A \otimes_B K_*) \otimes (A \otimes_B K_*) \to (A \otimes A) \otimes_{B \otimes B} (K_* \otimes K_*) \to A \otimes_B K_*$$

where the first is a general commutativity isomorphism for the tensor product and the second is induced by the algebra structures on A, B
76 and K_*. If the composite is denoted by ψ, then the algebra structure on $\mathrm{Tor}^B(A, A)$ is the Künneth morphism composed with $H(\psi)$ in

$$H(A \otimes_B K_*) \otimes H(A \otimes_B K_*) \to H((A \otimes_B K_*) \otimes (A \otimes_B K_*)) \to H(A \otimes_B K_*).$$

Remark 2.8. There is a natural A-morphism of the abelianization of the tensor algebra $T(HH_1(A))$ on $HH_1(A)$, viewed as a graded algebra over $A = HH_0(A)$ with $HH_1(A)$ in degree 1 defined $T_A(HH_1(A))^{ab} \to HH_*(A)$. This is a morphism of commutative algebras. Since the square of every element in $HH_1(A)$ is zero, we have in fact a morphism of the exterior algebra on $HH_1(A)$ into $HH_*(A)$,

$$\psi(A) : \Lambda_A(HH_1(A)) \to HH_*(A).$$

Note that if k is a field of characterisitic different from 2, then the natural algebra morphism $T_A(X)^{ab} \to \Lambda_A(X)$ is an isomorphism when X is graded, with nonzero terms in odd degrees.

In this chapter we will show that $\psi(A)$ is an isomorphism, for a large class of algebras A which arise in smooth geometry.

We conclude by mentioning another way of defining the product on $HH_*(A)$ by starting with a product, called the shuffle product, on the simplicial object $C_*(A)$. In the commutative case $C_*(A)$ is a simplicial k-algebra, i.e. each $C_q(A)$ is a k-algebra and the morphisms d_i and s_j are morphisms of algebras.

Definition 2.9. Let $R.$ be a simplicial k-algebra. The shuffle product $R_p \otimes R_q \to R_{p+q}$ is defined by the following sum for $\alpha \in R_p$, and $\beta \in R_q$,

$$\alpha.\beta = \sum_{\mu,\nu} \epsilon(\mu,\nu)(s_\mu(\alpha)(s_\nu(\beta) \quad \text{in} \quad R_{p+q}$$

where μ, ν is summed over all (q, p) shuffle permutations of $[0, \ldots, p + q - 1]$ of the form $(\mu_1, \ldots, \mu_q, \nu_1, \ldots, \nu_p)$ where $\mu_1 < \ldots < \mu_q$ and $\nu_1 < \ldots < \nu_p$. Also $\epsilon(\mu, \nu)$ denotes the sign of the permutation μ, ν, and **77**
the iterated operators are

$$s_\mu(\alpha) = s_{\mu_q}(\ldots(s_{\mu_1}(\alpha))\ldots) \quad \text{and} \quad s_\nu(\beta) = s_{\nu_p}(\ldots(s_{\nu_1}(\beta))\ldots).$$

Remark 2.10. With the shuffle product on a simplicial k-algebra $R_\bullet$, the differential module $(R_\bullet, d)$ becomes a differential algebra over k. If $R_\bullet$ is a commutative simplicial algebra, then $(R_\bullet, d)$ is a commutative differential algebra. This applies to $HH_*(A)$ for a commutative algebra A, and again we obtain a natural morphism

$$\Lambda^* HH_1(A) \to HH_*(A).$$

Example 2.11. For $\alpha = (a, x), \beta = (a', y)$ the shuffle product is

$$\begin{aligned} \alpha \centerdot \beta = (s_0\alpha) \cdot (s_1\beta) - (s_1\alpha)(s_0\beta) &= (a, x, 1)(a', 1, y) - (a, 1, x)(a', y, 1) \\ &= (aa', x, y) - (aa', y, x). \end{aligned}$$

For $\alpha_j = (a_j, x_j)$ where $j = 1, \ldots, p$ this formula generalizes to

$$\alpha_1 \ldots \alpha_p = \sum_{\alpha \in \mathrm{Sym}_p} sgn(\alpha)(a_1 \ldots a_p, x_{\alpha(1)}, \ldots, x_{\alpha(p)}).$$

3 Hochschild homology of regular algebras

In this section we outline the proof that Hochschild homology is just the Kähler differential forms for a regular k-algebra A, i.e. that $HH_q(A)$ is isomorphic to $\Omega^q_{A/k}$. We start with some background from commutative algebra.

Definition 3.1. A sequence of elements $y_1, \ldots, y_d$ in a commutative k-algebra B is called regular provided the image of y_i in the quotient algebra $B/B(y_1, \ldots, y_{i-1})$ is not a zero divisor.

Let $K(b, B)$ denote the exterior differential algebra on one generator 78
x in degree 1 with boundary $dx = b \in B = K(b, B)_0$. If $y_1, \ldots, y_d$ is a regular sequence of elements, then

$$K(y_i, B/B(y_1, \ldots, y_{i-1}) \to B/B(y_1, \ldots, y_i)$$

is a free resolution of $B/B(y_1, \ldots, y_i)$ by $B(y_1, \ldots, y_{i-1})$-modules.

Notation 3.2. Let B be a commutative algebra, and let $b_1, \ldots, b_m$ be elements of B. We denote by $K(b_1, \ldots, b_m)$ the differential algebra which is the tensor product

$$K(b_1, \ldots, b_m; B) = K(b_1, B) \otimes_B \ldots \otimes_B K(b_m, B).$$

This algebra is zero in degrees $q > m$ and $q < 0$ and free of rank $\binom{n}{q}$ in degree q, further the differential on a basis element is given by

$$d(x_{k(1)} \wedge \ldots \wedge x_{k(q)}) = \sum_{1 \le i \le 1} (-1)^{i-1} b_i (x_{k(1)} \wedge \ldots \wedge x_{k(i)} \wedge \ldots \wedge x_{k(q)}),$$

and the augmentation is defined by $K(b_1, \ldots, b_m; B) \to B/B(b_1, \ldots, b_m)$. Filtering $K(B_1, \ldots, b_m; B)$ in two steps with respect to degrees of $K(b_m, B)$, and looking at the associated spectral sequence, we obtain immediately the following proposition.

Proposition 3.3. *For $b_1, \ldots, b_m$ a sequence of elements in a commutative algebra B the augmentation morphism induces an isomorphism*

$$H_0(K(b_1, \ldots, b_m; B) \to B/B(b_1, \ldots, b_m).$$

If $b_1, \ldots, b_m$ is a regular sequence, then the augmentation morphism induces isomorphisms $H_0(K(b_1, \ldots, b_m; B)) \to B/B(b_1, \ldots, b_m)$ and

$$K(b_1, \ldots, b_m; B) \to B/B(b_1, \ldots, b_m).$$

This resolution is called the **Koszul resolution** of the quotient of B by free B-modules.

79 **Definition 3.4.** An ideal J in a commutative k-algebra B is said to be regular if it is generated by a regular sequence. An algebra A is ϕ-regular provided the kernel I of $\phi(A) : A \otimes A \to A$ is regular in the algebra $B = A \otimes A$.

The next theorem is the first case where we identify the Hochschild homology of a commutative algebra as the exterior algebra on the first Hochschild homology module.

Theorem 3.5. *If A is a commutative ϕ-regular algebra, then the natural morphisms of algebras*

$$\Lambda_A(HH_1(A)) \to HH_*(A)$$

or equivalently

$$\Lambda_A^*(I/I^2) = \Omega_{A/k}^* \to HH_*(A)$$

is an isomorphism of graded commutative algebras.

Proof. By (1.5) we have the natural isomorphisms between $HH_1(A)$, I/I^2 and $\Omega^1_{A/k}$. By hypothesis for $B = A \otimes A$ the previous proposition (3.4) applies and we have a resolution of $A = B/I$ by a differential algebra of free B-modules $K_* = K(b_1, \ldots, b_m; B) \to A$, such that the augmentation morphism is a morphism of algebras. Hence $HH_*(A) = H_*(K(b_1, \ldots, b_m; B) \otimes_B A$ since the coefficients in the formula of (3.2) are in I and the resulting algebra over A is the exterior algebra on I/I^2. This proves the theorem. □

Remark 3.6. The hypothesis of being a ϕ-regular algebra is rather restricted, except in the local case where it is equivalent to the maximal ideal being generated by a regular sequence. This means that the above construction applies to a regular local algebra, i.e. a local algebra whose maximal ideal is generated by a regular sequence.

Definition 3.7. An algebra A over a field k is regular provided each localisation $A_{\mathfrak{P}}$ at a prime ideal $\mathfrak{P}$ is regular.

These are the algebras with the property that their Hochschild ho- 80 mology is the algebra of differential forms. This leads to the theorem of Hochschild, Kostant and Rosenberg.

Theorem 3.8. *The natural morphism of graded commutative algebras* $\Omega^*_{A/k} \to HH_*(A)$ *is an isomorphism for a regular algebra* A *over a field* k.

Proof. For each prime ideal $\mathfrak{P}$ in A, the localisation of this morphism in the statement of the theorem

$$\Omega^*_{A_{\mathfrak{P}}/k} = (\Omega^*_{A/k})_{\mathfrak{P}} \to HH_*(A)_{\mathfrak{P}} = HH_*(A_{\mathfrak{P}})$$

is an isomorphism by (3.5). Hence the morphism is an isomorphism by a generality about localisation at each prime ideal. This proves the main theorem of this section. □

4 Hochschild homology of algebras of smooth functions

In this section we outline the proof that Hochschild homology is just the algebra of differential forms for an algebra A of smooth complex valued functions on a smooth manifold X.

Remark 4.1. Let X be a smooth n-dimensional manifold, and $A = C^\infty(X)$ denote the algebra of smooth complex valued functions on X. Then the Lie algebra of derivations $\mathrm{Der}_C(C^\infty(X))$ is just the space of smooth vector fields on X with complex coefficients, and $\Omega^1_{A/C} = A^1(X)$ is the A-module of 1-forms and $A^q(X)$ is the A-module of q-forms on X. This means that $HH_1(A) = A^1(X)$, by the characterization of $HH_1(A)$ in terms of Kähler 1-forms of a commutative algebra. We will outline the proof that $HH_q(A) = A^q(X)$, the module of q-forms over $A = A^0(X)$, the algebra of smooth functions on X. Thus we have the same calculation in degree 1, and following the lead from the previous section, we see that there must be a resolution of the ideal $\ker(A^0(X)\otimes A^0(X) \to A^0(X))$. This we do by relating this multiplication with $A^0(X \times X) \to A^0(X)$ coming
81 from restriction to the diagonal. Observe that there is an embedding

$$A^0(X) \otimes A^0(X) \to A^0(X \times X)$$

given by assigning to a tensor product of functions, a function of two variables and then using the normal bundle to the diagonal in $X \times X$. The result corresponding to the ϕ-regular algebra construction is the following proposition.

Remark 4.2. Let $E \to Y$ be a complex vector bundle with dual bundle $E^{\frown}$. If $s \in \Gamma(Y, E)$ is a cross section of E, then its inner product with an element of a fibre of $E^{\frown}$ defines a scalar varying from fibre to fibre. We define a morphism $s^{\vdash} : E^{\frown} \to \Lambda^0 E^{\frown}$, the trivial bundle. This $s^{\vdash}$ extends to a complex

$$\ldots \xrightarrow{s^{\vdash}} \Lambda^2 E^{\frown} \xrightarrow{s^{\vdash}} \Lambda^1 E^{\frown} \xrightarrow{s^{\vdash}} \Lambda^0 E^{\frown} \to 0$$

which is exact at all points where $s \neq 0$.

Now assume that Y is a smooth manifold, E is a smooth vector bundle, and X, the set of zeros of s is transverse to the zero section, and that the tangent morphism $ds_y : T_y Y \to E_y$ is surjective. Then X is a submanifold of Y of codimension q where $q = \dim E$ and the normal bundle to the zero set X in Y is isomorphic to $E|_X$.

Proposition 4.3. *With the above notations the complex of Fréchet spaces*

$$R(Y, E) : \ldots \to \Gamma(Y, \Lambda^q E^{\frown}) \xrightarrow{s^{\vdash}} \Gamma(Y, \Gamma^{q-1} E^{\frown}) \to \ldots$$
$$\ldots \xrightarrow{s^{\vdash}} \Gamma(Y, \Lambda^1 E^{\frown}) \xrightarrow{s^{\vdash}} \Gamma(Y) \xrightarrow{res} \Gamma(X) \to 0$$

is contractible.

Proof. The first step is to show that if the result holds locally, then it holds globally. Let $Y = \bigcup_{i \in I} U_i$ be an open covering with a smooth partition of unity $\sum_{i \in I} \eta_i = 1$ where $U_i \supset$ closure of $\eta_1^{-1}((0, 1])$ and $R(U_i, E_{|U_i})$ is contractible with contracting homotopy h_i for each $i \in I$. For $\pi : E \to$ **82**

Y the complex $R(Y, E)$ has a retracting homotopy

$$h(x) = \sum_{i \in I} \eta_i(\pi(x)) h_i(x|_{U_i}).$$

If N is the normal bundle of X in Y, then the induced tangent mapping $ds_x : N_x \to E_x$ is an isomorphism by the transversality hypothesis. Thus locally the bundle is of the form

$$\mathbf{R^q} \times \mathbf{R^q} \times \mathbf{R^p} = T(\mathbf{R^q}) \times \mathbf{R^p} \to \mathbf{R^q} \times \mathbf{R^p}$$

with projection from the middle $\mathbf{R^q}$ coordinate or $T(\mathbf{R^q}) \to \mathbf{R^q}$ with parameters from $\mathbf{R^p}$. □

Remark 4.4. For a submanifold X of Y and a smooth bundle E over Y, the restriction from the space of cross sections induces an isomorphism $\Gamma(X)\Omega_{\Gamma(Y)}\Gamma(Y, E) \to \Gamma(X, E_{|X})$.

Theorem 4.5. *For a smooth manifold we have a natural isomorphism* $HH_q(A^0(X)) \to \Gamma(X, \Lambda^q T^*(X)) = A^q(X)$.

We do not give a proof of this theorem here see Connes [1985].

5 Cyclic homology of regular algebras and smooth manifolds

We calculate the cyclic homology by comparing the basic standard complex with the complex of differential forms. For this, we consider a basic morphism from the standard complex to the complex of differential forms and study to what extent it is a morphism of mixed complexes.

Notation 5.1. The morphism μ is defined in two situations:

(1) Let A be a commutative algebra over a field k of characteristic zero. Denote by $\mu : A^{(q+1)\otimes} \to \Omega^q_{A/k}$ defined by

$$\mu(a_0 \otimes \ldots \otimes a_q) = (1/q!) a_0 da_1 \ldots da_q.$$

83 (2) Let X be a smooth manifold. Denote by $\mu : A^0(X^{q+1}) \to A^q(X)$ defined by

$$\mu(f(x_0, \ldots, x_q)) = (1/q!)\Delta^*(f d_1 f \ldots d_q f)$$

where $d_i f(x_0, \ldots, x_q)$ is the differential of f along the x_i variable in X^{q+1} and $\Delta : X \to X^{q+1}$ is the diagonal map.

Remark 5.2. Both $A^{q+1}\otimes$ and $A^0(X^{q+1})$ are the terms of degree q of cyclic vector spaces and hence the operators b and B are defined. Under the morphism μ we have the following result.

Proposition 5.3. *We have, with the above notations*

$$\mu b = 0 \quad \textit{and} \quad \mu B = d\mu$$

where d is the exterior differential on differential forms.

Proof. Given $a_0 \otimes \cdots \otimes a_q \in A^{(q+1)\otimes}$, we must show that the following sum of differentials is zero,

$$a_0 a_1 da_2 \ldots da_q + \sum_{0<i<q} (-1)^i a_0 da_1 \ldots d(a_i a_{i+1}) \ldots da_q + (-1)^q a_q a_0 da_1 \ldots da_{q+1}.$$

A direct check shows that terms with coefficients $a_0 a_i$ come in pairs with opposite signs. Hence $\mu b = 0$. Since μ factors through $A \otimes \overline{A}^{q\otimes}$, we can calculate by 5(4.1),

$$\begin{aligned}(\mu B)(a_0 \otimes \ldots \otimes a_q) &= \mu(\sum_{0\le i\le q} (-1)^{iq}(1 \otimes a_i \otimes \ldots \otimes a_q \otimes a_0 \otimes \ldots \otimes a_{i+1}) \\ &= (1/(q+1)!)(q+1) da_0 \ldots da_q \\ &= (1/q!) d(a_0 da_1 \ldots da_q) \\ &= d\mu(a_0 da_1 \ldots da_q).\end{aligned}$$

This shows that $\mu B = d\mu$. The above calculation works also for μ in the smooth manifold case. This proves the proposition. □

Remark 5.4. The above morphism μ induces a morphism

$$\mu : HH_q(A) \to \Omega^q_{A/k}$$

which when composed with the natural $\Omega^q_{A/k} \to HH_q(A)$ on the right **84**
is multiplication by $q+1$ on $\Omega^q_{A/k}$. Thus μ is a morphism of mixed complexes

$$\mu : (C_*(A), b, B) \to \Omega^*(A/k, 0, d)$$

which induces an isomorphism $HH_*(A) \to \Omega^*_{A/k}$. Thus the mixed complex of differential forms $(\omega^*_{A/k}, 0, d)$ can be used to calculate the cyclic homology of A or $A^0(X)$.

Theorem 5.5. *Let A be a regular k-algebra over a field k of characteristic zero. Then the cyclic homology is given by*

$$HC_p(A) = \Omega^p_{A/k}/d\Omega^{p-1}_{A/k} \oplus H^{p-2}_{DR}(A) \oplus H^{p-4}_{DR}(A) \oplus \dots$$

Let A be the $\mathbf{C}$-algebra of smooth functions on a smooth manifold. Then the cyclic homology is given by

$$HC_p(A) = A^p(X)/dA^{p-1}(X) \oplus H^{p-2}_{DR}(X) \oplus H^{p-4}_{DR}(X) \oplus \dots$$

In both cases, the projection of $HC_p(A)$ onto the first term is induced by μ and in the Connes' exact sequence, we have:

1. *$I : HH_p(A) \to HC_p(A)$ is the projection of $HH_p(A)$, the p-forms, onto the first factor of $HC_p(A)$,*
2. *$S : HC_p(A) \to HC_{p-2}(A)$ is injection of the first factor of $HC_p(A)$ into the second factor H^{p-2}_{DR} and the other factors map isomorphically on the corresponding factor of HC_{p-2}.*
3. *$B : HC_{p-2}(A) \to HH_{p-1}(A)$ is zero on all factors except the first one where it is $d : \Omega^{p-2}/d\Omega^{p-3} \to \Omega^{p-1}$.*

Finally in the first derived couple of the Connes' exact couple we have $B = 0$ and the exact couple is the split exact sequence

$$0 \to HH_p \to HH_p \oplus HH_{p-2} \oplus HH_{p-4} \oplus \dots \to HH_{p-2} \oplus HH_{p-4} \oplus \dots \to 0.$$

85 *Proof.* Everything in this theorem follows from the fact that we can calculate cyclic homology, Hochschild homology, and the Connes' exact couple with the mixed complex $(\Omega, 0, d)$ and it is an easy generality on mixed complexes with the first differential zero. □

6 The Chern character in cyclic homology

Recall that for topological K-theory, we have a ring homomorphism

$$ch : K(X) \to H^{ev}(X, \mathbf{Q})$$

such that $ch \otimes \mathbf{Q}$ is an isomorphism. Here the superscript ev denotes the homology groups of even degree. We wish to define a sequence of morphisms $ch_m : K_0(A) \to HC_{2m}(A)$ for all m such that $S(ch_m) = ch_{m-1}$, in terms of $S : HC_{2m}(A) \to HC_{2m-2}(A)$. In this section k is always a field of characteristic zero.

Remark 6.1. K-theory is constructed from either vector bundles over a space or from finitely generated projective modules over a ring. The vector bundles under consideration are always direct summands of a trivial bundle. In either case, it is a direct summand which is represented by an element $e = e^2$ in a matrix ring $M_r(A)$ over A. Here A is an arbitrary ring or the algebra of either the continuous functions on the space or of smooth functions on a smooth base manifold. Our approach to the Chern character is motivated by differential geometry where a differential form construction of the Chern character is made from e. The choice of $e = e^2$ is not uniquely defined by the element of K-theory but it amounts to the choice of a connection on a vector bundle.

Proposition 6.2. *If $e = e^2 \in M_r(A)$ for a commutative ring, then in $M_r\Omega^1_{A/k}$ we have the relations*

$$e(de) = de(1-e) \quad \text{and} \quad (de)e = (1-e)de.$$

In particular, $e(de)e = 0$ and $e(de)^2 = (de)^2e$ where $M_r(A)$ acts on 86
$M_r\Omega^1_{A/k}$ by matrix multiplication of a matrix valued form with a matrix valued function on either side.

Proof. We calculate $de = d(e^2) = e(de) + (de)e$ and use this to derive the relations immediately. □

Remark 6.3. For $e = e^2 \in M_r(A)$ we denote by $\Gamma(E) = \mathrm{im}(e) \subset A^r$ where we think of $\Gamma(E)$ as the cross sections of the vector bundle E corresponding to e. The related connection is $D(s) = eds$ for $s \in \Gamma(E)$ where $eds \in \Gamma(E \otimes \Omega^1_{A/k})$, and the curvature is

$$D^2 s = ed(eds) = e(de)^2_s.$$

In order to see how the second formula follows from the first, we calculate

$$ed(eds) = ededs = eded(es) = ede(de)s + e(de)eds = e(de)^2 s.$$

Thus the curvature is given by $D^2 = e(de)^2$ and this means that

$$(D^2)^q = e(de)^2 \dots e(de)^2 = e(de)^{2q}$$

which leads to the following definition by analogy with classical differential geometry.

Definition 6.4. The Chern character form of $e = e^2 \in M_r(A)$ with curvature $D^2 = e(de)^2$ is given by the sum

$$ch(e) = tr(e^{D^2}) = \sum_{q \geq 0} (1/q!)tr(e(de)^{2q}).$$

Now we will see how this Chern character form defines a class in cyclic homology. The guiding observation is the fact that up to a scalar, $tr(e(de)^{2q})$ is $\mu(tr(e^{(2q+1)\otimes}))$ where μ was introduced in (5.1) of the previous section. We have two preliminary results in the cyclic homology complex.

87 **Proposition 6.5.** *Let A be an algebra over a field k. For an element $a \in A$ and $a^{(q+1)\otimes} \in C_q(A)$ in the standard complex, we have*

$$(t-1)(a^{(q+1)\otimes}) = -2a^{(q+1)\otimes} \textit{ for } q \textit{ odd}$$

$$= 0 \text{ for } q \text{ even.}$$

For $e = e^2 \in A$ *and* $e^{(q+1)\otimes} \in C_q(A)$ *we have the relation*

$$b(e^{(q+1)\otimes}) = e^{q\otimes} \text{ for } q \text{ even}$$
$$= 0 \text{ for } q \text{ odd.}$$

Proof. The first formula follows from the relation $t(a^{(q+1)\otimes}) = (-1)^q a^{(q+1)\otimes}$. Since $e = ee$ the sum $b(e^{(q+1)\otimes})$ is an alternating sum of $q + 1$ terms $e^{q\otimes}$, and they either cancel to yield zero or reduce to $e^{q\otimes}$. This proves the proposition. □

Corollary 6.6. *If* $e = e^2 \in M_r(A)$, *then the boundary*

$$b(tr(e^{(2q+1)\otimes})) = 0 \text{ in } C_{2q-1}(A)/\operatorname{im}(1 - t).$$

Thus $tr(e^{(2q+1)\otimes})$ *defines a class* $ch_q(e) \in HC_{2q}(A)$, *for* $e = e^2 \in M_r(A)$ *and this is the Chern character form upto a scalar factor. This was the aim of this section, and we finish with the following summary assertion.*

Theorem 6.7. *Let* $e = e^2 \in M_r(A)$ *with Chern character form* $ch_q(e) = (1/q!)tr(e(de)^{2q})$ *in degree* $2q$. *Then in degree* $2q$ *we have*

$$\mu(ch_q(e)) = ch_q(e) \text{ in } HC_{2q}(A).$$

Moreover, under $S : HC_{2q}(A) \to HC_{2q-2}(A)$, *we have for this Chern character class,* $S(ch_q(E)) = ch_{q-1}(E)$.

Chapter 7

Noncommutative Differential Geometry

IN THE PREVIOUS chapter, we developed the close relationship between differential forms and de Rham cohomology on one hand and Hochschild and cyclic homology on the other hand, for commutative algebras. In this chapter, we explore the relationship in the general case, using the concept of the bimodule of differential forms, which we denote by $\Omega^1(A/k)$. As before, these forms are related to I, the kernel of the multiplication map $\phi(A) : A \otimes A \to A$, and in fact in this case, we have $\Omega^1(A/k) = I$. **88**

1 Bimodule derivations and differential forms

In this section let A denote an algebra over k.

Definition 1.1. Let M be an A-bimodule. A derivation D of A with values in M is a k-linear map $D : A \to M$ such that

$$D(ab) = aD(b) + D(a)b \text{ for } a, b \in A.$$

We denote by $\mathrm{Der}_k(A, M)$ or just $\mathrm{Der}(A, M)$ the k-module of all bimodule derivations of A with values in M.

Unlike in the commutative case, $\mathrm{Der}(A, M)$ has no A-linear structure, but $\mathrm{Der}(A, A)$ is a Lie algebra over k with Lie bracket given by $[D, D'] = DD' - D'D$ for $D, D' \in \mathrm{Der}(A, A)$.

Definition 1.2. The A-bimodule of bimodule differentials is a pair $(\Omega^1(A/k), d)$ where $\Omega^1(A/k)$, or simply $\Omega^1(A)$ or Ω, is an A-bimodule and the morphism $d : A \to \Omega^1(A/k)$ is a bimodule derivation such that, for any derivation $D : A \to M$ there exists a unique A-linear morphism $f : \Omega^1(A/k) \to M$ such that $D = fd$. The bimodule derivation d defines a k-linear morphism

$$\mathrm{Hom}_A(\Omega^1(A/k), M) \to \mathrm{Der}_k(A, M)$$

89 by assigning to each morphism $f \in \mathrm{Hom}_A(\Omega^1(A/k), M)$ of A-bimodules the bimodule derivation $fd \in \mathrm{Der}_k(A, M)$. The universal property is just the assertion that this morphism is an isomorphism of A-modules. As usual, the universal property shows that two possible k-modules of differentials are isomorphic with a unique isomorphism preserving the derivation d. As in the previous chapter, there are two constructions of the module of derivations $\Omega^1(A/k)$. The first uses $I = \ker(\phi(A))$ and the second uses the relations coming directly from the derivation property. They are tied together with an acyclic standard resolution.

Construction of $\boldsymbol{\Omega^1(A/k)}$ I. 1.3. Let I denote the kernel of the multiplication morphism $\phi(A) : A \otimes A \to A$. We define $\Omega^1(A/k) = I$ and $d : A \to I$ by $d(a) = 1 \otimes a - a \otimes 1$ for $a \in A$ and check that it is a derivation by

$$\begin{aligned} d(ab) &= 1 \otimes ab - ab \otimes 1 \\ &= (1 \otimes a)(1 \otimes b - b \otimes 1) + (1 \otimes a - a \otimes 1)(b \otimes 1) \\ &= ad(b) + d(a)b \end{aligned}$$

where the left action of A on $I \subset A \otimes A$ is given by $ax = (1 \otimes a)x$ and the right action by $xb = x(b \otimes 1)$ in I for $x \in I$. To verify the universal property, we consider a derivation $D : A \to M$. If $\sum_i a_i \otimes b_i \in I$ or in other words $\sum_i a_i b_i = 0$, then we have

$$\sum_i a_i(Db_i) + \sum_i (Da_i)b_i = 0$$

from the derivation rule, and we define $f : I \to M$ by

$$f\left(\sum_i a_i \otimes b_i\right) = \sum_i a_i D(b_i) = -\sum_i D(a_i) b_i.$$

Now $f(d(a)) = f(1 \otimes a - a \otimes 1) = 1D(a) - aD(1) = D(a)$, and hence $fd = D$. Thus $(\Omega^1(A/k), d)$ is a module of bimodule differentials.

Construction of $\Omega^1(A/k)$ II. 1.4. Following the idea of 6(1.4), we should consider the k-submodule L of $A \otimes A \otimes A$ generated by all elements of the form $a_0 a_1 \otimes a_2 \otimes a_3 - a_0 \otimes a_1 a_2 \otimes a_3 + a_0 \otimes a_1 \otimes a_2 a_3$ which is just $b'(a_0 \otimes a_1 \otimes a_2 \otimes a_3)$ for the differential $b' : C_3(A) \to C_2(A)$ in the **90**
standard acyclic complex for the algebra A. Since $(C_*(A), b')$ is acyclic, we have a natural isomorphism

$$A^{3\otimes}/L = \operatorname{coker}(b' : A^{4\otimes} \to A^{3\otimes}) \to \ker(b' = \phi(A) : A \otimes A \to A) = I.$$

To see the universal property for $d(a) = 1 \otimes a \otimes 1 \bmod L$, we note first that d is a derivation by the properties of the generators of L and for a derivation $D : A \to M$ we define a morphism $f : A^{3\otimes}/L \to M$ by the relation $f(a \otimes b \otimes c \bmod L) = aD(b)c$.

Remark 1.5. The module $\Omega^1(A/k)$ is generated by elements adb for a, $b \in A$ with the left A-module structure given by

$$a'(adb) = (a'a)db$$

and the right A-module structure given by

$$(adb)a' = ad(ba') - (ab)da'$$

for $a, a', b \in A$.

Now we proceed to define the bimodule of q-forms by embedding $\Omega^1(A/k)$ in a kind of tensor algebra derived from the A-bimodule structure. In this case, we factor tensor products over k as tensor products over A, but we do not introduce any commutativity properties in the algebra since A is not commutative.

Definition 1.6. Let M be an A-bimodule. The bimodule tensor algebra $T_A(M)$ is the graded algebra where in degree n

$$T_A(M)_n = M \otimes_A \dots (n) \dots \otimes_A M$$

with algebra structure over k given by a direct sum of the natural quotients $T_A(M)_p \otimes T_A(M)_q \to T_A(M)_{p+q}$. In particular $T_A(M)_n$ is generated by elements

$$x_1 \otimes_A \cdots \otimes_A x_n = x_1 \dots x_n \quad \text{for} \quad x_1, \dots, x_n \in M,$$

91 and in degree zero $T_A(M)_0 = A$.

2 Noncommutative de Rham cohomology

Now we apply the above constructions, not directly to the algebra A, but to $k \oplus A$ viewed as a supplemented algebra with augmentation ideal A itself.

Notation 2.1. Let $A^\sharp$ denote the algebra $k \oplus A$ given by inclusion $k \to A^\sharp = k \oplus A$ on the first factor. Since $A^\sharp$ is supplemented, we have a splitting $s : A^\sharp \to A^\sharp \otimes A^\sharp$, of the exact sequence

$$0 \to \Omega^1(A^\sharp) \to A^\sharp \otimes A^\sharp \to A^\sharp \to 0$$

defined by $s(a) = a \otimes 1$. Thus there is a natural morphism $\Omega^1(A^\sharp) \to \mathrm{coker}(s)$ and we have the following result.

Proposition 2.2. *We have a natural isomorphism*

$$\delta : A \oplus (A \otimes A) \to \Omega^1(A^\sharp)$$

where $\delta(a, 0) = da$ *and* $\delta(0, a \otimes b) = adb = a(1 \otimes b - b \otimes 1)$. *The right A-module structure is given by* $(a_0 da_1)a = a_0 d(a_1 a) - a_0 a_1 da$. *Now we define the algebra of all noncommutative forms.*

Definition 2.3. The algebra of noncommutative differential forms is the following tensor algebra $T(\Omega^1(A^\sharp))$ over $A^\sharp$. This is a graded algebra and d extends uniquely to d on this tensor algebra satisfying $d^2 = 0$. More explicitly, we have the following description.

Proposition 2.4. *We have a natural isomorphism*

$$\delta : A^\sharp \otimes A^{p\otimes} = A^{p\otimes} \oplus A^{(p+1)\otimes} \to \Omega^p(A^\sharp)$$

where $\delta(a_1 \otimes \cdots \otimes a_p) = da_1 \dots da_p$ *and* $\delta(a_0 \otimes \cdots \otimes a_p) = a_0 da_1 \dots da_p$. *The right* $A^\sharp$*-module structure on* $\Omega^p(A^\sharp)$ *is given by the formula*

$$\begin{aligned}(da_1 \dots da_p)b &= da_1 \dots d(a_p b) - da_1 \dots d(a_{p-1}a_p)db \\ &\quad + da_1 \dots d(a_{p-2}a_{p-1})da_p + \cdots + (-1)^p a_1 da_2 \dots da_p db.\end{aligned}$$

Moreover, $H^*(\Omega^*(A^\sharp)) = k$ *which is illustrated with the following* 92
diagram

$$\begin{array}{ccccccccccc} k & & A & & A^{2\otimes} & & A^{(p-1)\otimes} & & A^{\otimes p} & & \\ \oplus & \nearrow d & \oplus & \nearrow d & \oplus & \cdots\cdots & \oplus & \nearrow d & \oplus & & \cdots \\ A & & A^{2\otimes} & & A^{3\otimes} & & A^{p\otimes} & & A^{(p+1)\otimes} & & \end{array}$$

Definition 2.5. The noncommutative de Rham cohomology of an algebra A over a field is $H^*_{NDR}(A) = H^*(\Omega^*(A^\sharp)^{\alpha\beta})$, the cohomology of the Lie algebra abelianization of the differential algebra of noncommutative differential forms over $A^\sharp$. More precisely, for $\omega \in \Omega^p(A^\sharp)$ and $\omega' \in \Omega^q(A^\sharp)$ we form the (graded) commutator $[\omega, \omega'] = \omega - (-1)^{pq}\omega'\omega$ and denote by $[\Omega^*(A^\sharp), \Omega^*(A^\sharp)]$ the Lie subalgebra generated by all commutators. The Lie algebra abelianization of the algebra of differential forms is

$$\Omega^*(A^\sharp)^{\alpha\beta} = \Omega^*(A^\sharp)/\{k \oplus [\Omega^*(A^\sharp), \Omega^*(A^\sharp)]\}.$$

To obtain an other version of $\Omega^*(A^\sharp)^{\alpha\beta}$, we use the following result.

Proposition 2.6. *Let* S *be a set of generators of an algebra* B*. For a* B*-module* M *we have* $[B, M] = \sum_{b \in S} [b, M]$.

Proof. First, we calculate

$$\begin{aligned}[bb', x] &= (bb')x - x(bb') \\ &= b(b'x) - (b'x)b + b'(xb) - (xb)b'\end{aligned}$$

$$= [b, b'x] + [b', xb].$$

Thus it follows that $[bb', x] \in [b, M] + [b', M]$. Hence the set of all $b \in B$ with $[b, M] \subset \sum_{b \in S} [b, M]$ is a subalgebra of B containing S, and therefore it is B. This proves the proposition. □

Corollary 2.7. *The abelianization of the algebra of differential forms is*

$$\Omega^*(A^\sharp)^{\alpha\beta} = \Omega^*(A^\sharp)/\{k + [A, \Omega^*(A^\sharp)] + [dA, \Omega^*(A^\sharp)].$$

93 **Definition 2.8.** Let A be an algebra over k. The noncommutative de Rham cohomology of A is

$$H^*_{NDR}(A) = H^*(\Omega^*(A^\sharp)^{\alpha\beta}).$$

Since $\Omega^*(A^\sharp)^{\alpha\beta}$ is a functor from the category of algebras over k to the category of cochain complexes over k, the noncommutative de Rham cohomology is a graded k-module, but is does not have any natural algebra structure.

3 Noncommutative de Rham cohomology and cyclic homology

Now we relate the noncommutative de Rham cohomology with cyclic homology over a field k of characteristic zero following ideas from the theory of commutative algebras where the morphism μ is used.

Notation 3.1. Again we denote by

$$\mu : C_q(A) \to \Omega^*(A^\sharp)^{\alpha\beta}$$

the morphism $\mu(a_0 \otimes \cdots \otimes a_q) = (1/q!) a_0 da_1 \ldots da_q$.

Proposition 3.2. *The morphism μ satisfies the following identities*

1. $\mu b(a_0 \otimes \cdots \otimes a_{q+1}) = ((-1)^{q+1}/q!)[a_{q+1}, a_0 da_1 \ldots da_q]$
2. $\mu(1-t)(a_0 \otimes \cdots \otimes a_q) \equiv (1/q!)[a_0 da_1 \ldots da_{q-1}, da_q] \bmod d\Omega^{q-1}(A^\sharp)$.

Proof. The composite μb is zero for a commutative algebra, see 6(5.3), but this time the sum will not have the same cancellations in the last two terms. We have

$$\begin{aligned} q!\mu b(a_0 \otimes \cdots \otimes a_{q+1}) = a_0 a_1 da_2 \dots da_{q+1} + \\ \sum_{0<i<q+1} (-1)^i a_0 da_1 \dots d(a_i a_{i+1}) \dots da_{q+1}) \\ + (-1)^{q+1} a_{q+1} a_0 da_1 \dots da_q \\ = (-1)^q a_0 da_1 \dots da_q a_{q+1} + (-1)^{q+1} a_{q+1} a_0 da_1 \dots da_q \\ = (-1)^{q+1} [a_{q+1}, a_0 da_1 \dots da_q]. \end{aligned}$$

For the second formula we have the calculation **94**

$$\begin{aligned} \mu(1-t)(a_0 \otimes \cdots \otimes a_q) &= \mu(a_0 \otimes \cdots \otimes a_q) - (-1)^q \mu(a_q \otimes a_0 \otimes \cdots \otimes a_{q-1}) \\ &= (1/q!)(a_0 da_1 \dots da_q - (-1)^q a_q da_0 \dots da_{q-1}) \\ &\equiv (1/q!)(a_0 da_1 \dots da_q + (-1)^q da_q a_0 da_1 \dots da_{q-1}) \bmod d\Omega^{q-1} \\ &\equiv (1/q!)[a_0 da_1 \dots da_{q-1}, da_q] \bmod d\Omega^{q-1}. \end{aligned}$$

From this proposition we state the following theorem of Connes' where only the question of injectivity in the first assertion is not covered by the above proposition. As for the second assertion, this is a deeper result of Connes which we do not go into, see Connes [1985]. □

Theorem 3.3. *The morphism μ induces an isomorphism*

$$\mu : A^{(q+1)\otimes}/((1-t)A^{(q+1)\otimes} + bA^{(q+2)\otimes}) \to \Omega^q/(d\Omega^{q-1} + [dA, \Omega^{q-1}] + [A, \Omega^q])$$

where, as usual, $\Omega^q = \Omega^q(A^\natural)$. The left hand side has $HC_q(A)$ as a submodule and μ restricted to the submodule

$$\mu : \ker(B) = \operatorname{im}(S) \to H^q_{NDR}(A)$$

is an isomorphism on the noncommutative de Rham cohomology of A viewed as a submodule of $\Omega^q/(d\Omega^{q-1} + [dA, \Omega^{q-1}] + [A, \Omega^q])$.

4 The Chern character and the suspension in noncommutative de Rham cohomology

Example 4.1. Let $A = ke$ where $e = e^2$ is the identity in the algebra A and an idempotent in $A^\sharp = k \oplus ke$. Then $\Omega^1(A^\sharp/k)$ is free on two generators de and ede, and

$$\begin{aligned}\Omega^q(A^\sharp/k)^{\alpha\beta} &= k.e(de)^q \text{ for } q = 2i\\ &= 0 \text{ for } q \text{ odd.}\end{aligned}$$

95 **Remark 4.2.** With this calculation we can carry out the construction of $ch_q(e)$ for $e^2 = e \in A$ for an arbitrary algebra A over k. Namely, we map the universal e to the special $e \in A$, and this lifts to $\Omega^*(ke^\sharp) \to \Omega^*(A^\sharp)$ as differential algebras by the universal property of the tensor product and hence to

$$\Omega^*(ke^\sharp)^{\alpha\beta} \to \Omega^*(A^\sharp)^{\alpha\beta}$$

as complexes and to $H^*_{NDR}(ke) \to H^*_{NDR}(A)$. The image of $d(de)^{2q}/q!$ is $ch_q(E)$. Now we consider the S operator in noncommuative de Rham theory which has the property that

$$\frac{S\,(e(de)^{2q})}{q!} = \frac{e(de)^{2q-2}}{(q-1)!}$$

Remark 4.3. The natural isomorphism $A \to A \otimes ke$ extends to a morphism of differential algebras

$$\Omega^*(A^\sharp) \to \Omega^*(A^\sharp) \otimes \Omega^*(ke^\sharp)$$

with quotient morphism

$$\Omega^*(A^\sharp)^{\alpha\beta} \to \Omega^*(A^\sharp)^{\alpha\beta} \otimes \Omega^*(ke^\sharp)^{\alpha\beta}$$

which on degree q is given by

$$\Omega^q(A^\sharp)^{\alpha\beta} \to \oplus_i \Omega^{q-2i}(A^\sharp)^{\alpha\beta} \otimes \Omega^{2i}(ke^\sharp)^{\alpha\beta}.$$

Now we consider the map picking out the coefficient of $e(de)^2$ which we call $S : \Omega^q(A^\sharp)^{\alpha\beta} \to \Omega^{q-2}(A^\sharp)^{\alpha\beta}$. Observe that S is compatible with d and we have the following formula.

Proposition 4.4. *For* $a_0 da_1 \dots da_q \in \Omega^q(A^\sharp)^{\alpha\beta}$ *we have*

$$S(a_0 da_1 \dots da_q) = \sum_{1 \le i \le q-1} a_0 da_1 \dots da_{i-1}(a_i a_{i+1}) da_{i+2} \dots da_q.$$

Proof. Let $\tau : \Omega^2(ke^\sharp)^{\alpha\beta} \to k$ be the linear functional such that **96**

$$\tau((de)^2) = 0 \text{ and } \tau(e(de)^2) = 1.$$

Then

$$\begin{aligned} S(a_0 da_1 \dots da_q) = {} & (1 \otimes \tau)[(a_0 \otimes e)(da_1 \otimes e + a_1 \otimes de) \cdots \\ & (da_q \otimes e + a_q \otimes de)] + (1 \otimes \tau) \\ & \left[\left(\sum_{1 \le i \le q-1} a_0 da_1 \dots da_{i-1}(a_i a_{i+1}) da_{i+2} \dots da_q\right) \otimes e(de)^2\right] \\ = {} & \sum_{1 \le i \le q-1} a_0 da_1 \dots da_{i-1}(a_i a_{i+1}) da_{i+2} \dots da_q. \end{aligned}$$

This proves the proposition. □

Corollary 4.5. *We have* $S(ch_q) = ch_{q-1}$.

Proof. Using (4.4) we calculate

$$\begin{aligned} S(e(de)^{2q}) &= e^3(de)^{2q-2} + e(de)ee(de)^{2q-2} + \cdots \\ &= qe(de)^{2q-2} \end{aligned}$$

and hence we have the result indicated above, that

$$S\left(\frac{e(de)^{2q}}{q!}\right) = \left(\frac{e(de)^{2q-2}}{(q-1)!}\right).$$

This is the statement of the corollary. □

Bibliography

THIS IS A short list of some of the basic references in the subject. For further references, see P. Cartier [1985], Séminaire Bourbaki 621 (fév. 1984): Asterisque 121-122 (1985), 123-146, or the book of Connes [1990]. Most of the references are related to the algebraic aspect of the theory. 97

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