Lectures on
A Method in the Theory of Exponential Sums

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Preface

These lectures were given at the Tata Institute of Fundamental Research in October - November 1985. It was my first object to present a self-contained introduction to summation and transformation formulae for exponential sums involving either the divisor function $d(n)$ or the Fourier coefficients of a cusp form; these two cases are in fact closely analogous. Secondly, I wished to show how these formulae - in combination with some standard methods of analytic number theory - can be applied to the estimation of the exponential sums in question.

I would like to thank Professor K. Ramachandra, Professor R. Balasubramanian, Professor S. Raghavan, Professor T.N. Shorey, and Dr. S. Srinivasan for their kind hospitality, and my whole audience for interest and stimulating discussions. In addition, I am grateful to my colleagues D.R. Heath-Brown, M.N. Huxley, A. Ivic, T. Meurman, Y. Motohashi, and many others for valuable remarks concerning the present notes and my earlier work on these topics.

Notation

The following notation, mostly standard, will occur repeatedly in these notes.

γ	Euler's constant.
s	$= \sigma + it$, a complex variable.
$\zeta(s)$	Riemann's zeta-function.
$\Gamma(s)$	The gamma-function.
$\chi(s)$	$= 2^s \pi^{s-1} \Gamma(1-s) \sin(\pi s/2)$.
$J_n(z), Y_n(z), K_n(z)$	Bessel functions.
$e(\alpha)$	$= e^{2\pi i\alpha}$.
$e_k(\alpha)$	$= e^{2\pi i\alpha/k}$.
$\mathrm{Res}(f, a)$	The residue of the function f at the point a.
$\int\limits_{(c)} f(s)\,ds$	The integral of the function f over the line $\mathrm{Re}\ s = c$.
$d(n)$	The number of positive divisors of the integer n.
$a(n)$	Fourier coefficient of a cusp form.
$\varphi(s)$	$= \sum\limits_{n=1}^{\infty} a(n)n^{-s}$.
κ	The weight of a cusp form.
$\tilde{a}(n)$	$= a(n)n^{-(\kappa-1)/2}$.
r	$= h/k$, a rational number with $(h, k) = 1$ and $k \geq 1$.
$\bar{h}$	The residue $\pmod{k}$ defined by $h\bar{h} \equiv 1 \pmod{k}$.
$E(s, r)$	$= \sum\limits_{n=1}^{\infty} d(n)e(nr)n^{-s}$.
$\varphi(s, r)$	$= \sum\limits_{n=1}^{\infty} a(n)e(nr)n^{-s}$.
$\Vert \alpha \Vert$	The distance of α from the nearest integer.
$\sideset{}{'}\sum\limits_{n\leqq x} f(n)$	$= \sum\limits_{1\leqq n\leqq x} f(n)$, except that if x is an integer, then the term $f(x)$ is to be replaced by $\frac{1}{2}f(x)$.
$\sideset{}{'}\sum\limits_{a\leqq n\leqq b} f(n)$	A sum with similar conventions as above if a or b is an integer.

$D(x)$	$= \sum_{n\leqq x}{}' d(n).$
$A(x)$	$= \sum_{n\leqq x}{}' a(n).$
$D(x,\alpha)$	$= \sum_{n\leqq x}{}' d(n)e(n\alpha).$
$A(,\alpha)$	$= \sum_{n\leqq x}{}' a(n)e(n\alpha).$
$D_a(x)$	$= \frac{1}{a!}\sum_{n\leqq x}{}' d(n)\ (x-n)^a.$
$A_a(x), D_a(x,\alpha), A_a(x,\alpha)$	are analogously defined.
ϵ	An arbitrarily small positive constant.
A	A constant, not necessarily the same at each occurrence.
$C^n[a,b]$	The class of functions having a continuous nth derivative in the interval $[a,b]$.

The symbols $0(), \ll,$ and $\gg$ are used in their standard meaning. Also, $f \asymp g$ means that f and g are of equal order of magnitude, i.e. that $1 \ll f/g \ll 1$. The constants implied by these notations depend at most on $\in$.

Contents

Introduction

ONE OF THE basic devices (usually called "process B"; see [13], § 2.3) in van der Corput's method is to transform an exponential sum into a new shape by an application of van der Corput's lemma and the saddle-point method. An exponential sum 1

$$\sum_{a<n\le b} e(f(n)), \tag{0.1}$$

where $f \epsilon C^2[a,b]$, $f''(x) < 0$ in $[a,b]$, $f'(b) = \alpha$, and $f'(a) = \beta$, is first written, by use of van der Corput's lemma, as

$$\sum_{\alpha-\eta<n<\beta+\eta} \int\limits_a^b e(f(x) - nx)\,dx + 0(\log(\beta - \alpha + 2)), \tag{0.2}$$

where $\eta \epsilon$ (0.1) is a fixed number. The exponential integrals here are then evaluated approximately by the saddle-point method in terms of the saddle points $x_n \epsilon (a,b)$ satisfying $f'(x_n) = n$.

If the sum (0.1) is represented as a series by Poisson's summation formula, then the sum in (0.2) can be interpreted as the "interesting" part of this series, consisting of those integrals which have a saddle point in (a,b), or at least in a slightly wider interval.

The same argument applies to exponential sums of the type

$$\sum_{a\le n\le b} d(n)g(n)e(f(n)) \tag{0.3}$$

as well. The role of van der Corput's lemma or Poisson's summation formula is now played by Voronoi's summation formula

$$\sum_{a\le n\le b}{}' d(n)f(n) = \int\limits_a^b (\log x + 2\gamma) f(x)\,dx + \sum_{n=1}^{\infty} d(n) \int\limits_a^b f(x)\alpha(nx)\,dx,$$

$$\alpha(x) = 4K_\circ(4\pi x^{1/2}) - 2\pi Y_\circ(4\pi x^{1/2}). \tag{0.4}$$

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The well-known asymptotic formulae for the Bessel functions $K_\circ$ and $\gamma_\circ$ imply an approximation for $\alpha(nx)$ in terms of trigonometric functions, and, when the corresponding exponential integrals in (0.4) with $g(x)e(f(x))$ in place of $f(x)$-are treated by the saddle-point method, a certain exponential sum involving $d(n)$ can be singled out, the contribution of the other terms of the series (0.4) being estimated as an error term. The leading integral normally represents the expected value of the sum in question.

As a technical device, it may be helpful to provide the sum (0.3) with suitable smooth weights $\eta(n)$ which do not affect the sum too much but which make the series in Voronoi formula for the sum

$$\sum_{a\le n\le b}{}' \eta(n)d(n)g(n)e(f(n))$$

absolutely convergent.

Another device, at first sight nothing but a triviality, consists of replacing $f(n)$ in (0.3) by $f(n) + rn$, where r is an integer to be chosen suitably, namely so as to make the function $f'(x) + r$ small in $[a, b]$. This formal modification does not, of course, affect the sum itself in any way, but the outcome of applying Vornoi's summation formula and the saddle-point method takes quite a new shape.

The last-mentioned argument appeared for the first time [16], where a transformation formula for the Dirichlet polynomial

$$S(M_1, M_2) = \sum_{M_1\le m\le M_2} d(m)m^{-1/2-it} \tag{0.5}$$

3 was derived. An interesting resemblance between the resulting expres-

sion for $S(M_1, M_2)$ and the well-known formula of F.V. Atkinson [2] for the error term $E(T)$ in the asymptotic formula

$$\int_0^T |\zeta\left(\frac{1}{2}+it\right)|^2\,dt = (\log(T/2\pi) + 2\gamma - 1)T + E(T) \tag{0.6}$$

was clearly visible, especially in the case $r = 1$. This phenomenon has, in fact, a natural explanation. For differentiation of (0.6) with respect to T, ignoring the error term $o(\log^2 T)$ in Atkinson's formula for $E(T)$, yields heuristically an expression for $|\zeta(\frac{1}{2}+it)|^2$, which can be indeed verified, up to a certain error, if

$$|\zeta\left(\frac{1}{2}+it\right)|^2 = \zeta^2\left(\frac{1}{2}+it\right)\chi^{-1}(\frac{1}{2}+it)$$

is suitably rewritten invoking the approximate functional equation for $\zeta^2(s)$ and the transformation formula for $S(M_1, M_2)$ (for details, see Theorem 2 in [16]).

The method of [16] also works, with minor modifications, if the coefficients $d(m)$ in (0.5) are replaced by the Fourier coefficients $a(m)$ of a cusp form of weight κ for the full modular group; the Dirichlet polynomial is now considered on the critical line $\sigma = \kappa/2$ of the Dirichlet series

$$\varphi(s) = \sum_{n=1}^{\infty} a(n)n^{-s}.$$

This analogy between $d(m)$ and $a(m)$ will prevail throughout these notes, and in order to avoid repetitions, we are not going to give details of the proofs in both cases. As we shall see, the method could be generalized to other cases, related to Dirichlet series satisfying a functional **4**
equation of a suitable type. But we are leaving these topics aside here, for those two cases mentioned above seem to be already representative enough.

The transformation formula of [16] has found an application in the proof of the mean twelfth power estimate

$$\int_0^T |\zeta\left(\frac{1}{2}+it\right)|^{12}\,dt \ll T^2\log^{17} T \tag{0.7}$$

of D.R. Heath-Brown [11]. The original proof by Heath-Brown was based on Atkinson's formula. The details of the alternative approach can be found in [13], § 8.3.

The formula of [16] is useful only if the Dirichlet polynomial to be transformed is fairly short and the numbers $t(2\pi M_i)^{-1}$ lie near to an integer r. But in applications to Dirichlet series it is desirable to be able to deal with "long" sums as well. It is not advisable to transform such a sum by a single formula; but a more practical representation will be obtained if the sum is first split up into segments which are individually transformed using an optimally chosen value of r for each of them. The set of possible values of r can be extended from the integers to the rational numbers if a summation formula of the Voronoi type, to be given in § 1.9, for for sums

$$\sum_{a\le n\le b}{}' b(n)e_k(hn)f(n), b(n) = d(n) \quad \text{or} \quad a(n),$$

is applied. The transformation formula for Dirichlet polynomiala are deduced in § 4.1 as consequences of the theorems of Chapter 3 concerning
5 the transformation of more general exponential sums

$$\sum_{M_1\le m\le M_2} b(m)g(m)e(f(m)) \tag{0.8}$$

or their smoothed versions.

An interesting problem is estimating long exponential sums of the type (0.8). A result of this kind will be given in § 4.3, but only under rather restrictive assumptions on the function f, for we have to suppose that $f'(x) \approx Bx^{\alpha}$. It is of course possible that comparable, or at least nontrivial, estimates can be obtained in concrete cases without this assumption, making use of the special properties of the function f.

In view of the analogy between $\zeta^2(s)$ and $\varphi(s)$, a mean value result corresponding to Heath-Brown's estimate (0.7) should be an estimate for the sixth moment of $\varphi(\kappa/2+it)$. However, the proofs of (0.7) given in [11] and [13] utilize special properties of the function $\zeta^2(s)$ and cannot be immediately carried over to $\varphi(s)$. An alternative approach will be presented in § 4.4, giving not only (0.7) (up to the logarithmic factor), but also its analogue

(0.9)
$$\int_0^T |\varphi(\kappa/2 + it)|^6 \, dt \ll T^{2+\epsilon}.$$

This implies the estimate

(0.10)
$$|\varphi(\kappa/2 + it)| \ll t^{1/3+\epsilon} \quad \text{for} \quad t \geq 1,$$

which is not new but neverthless essentially the best known presently. In fact, (0.10) is a corollary of the mean value theorem

(0.11)
$$\int_0^T |\varphi(\kappa/2 + it)|^2 \, dt = (C_\circ \log T + C_1)T + o((T \log T)^{2/3})$$

of A. Good [9]. The estimate (0.10) can also proved directly in a rela- 6
tively simple way, as will be shown in § 4.2.

The plan of these notes is as follows. Chapters 1 and 2 contain the necessary tools - summation formulae of the Voronoi type and theorems on exponential integrals - which are combined in Chapter 3 to yield general transformation formulae for exponential sums involving $d(n)$ or $a(n)$. Chapter 4, the contents of which were briefly outlined above, is devoted to specializations and applications of the results of the preceding chapter. Most of the material in Chapters 3 and 4 is new and appears here the first time in print.

An attempt is made to keep the presentation selfcontained, with an adequate amount of detail. The necessary prerequisites include, beside standard complex function theory, hardly anything but familiarity with some well-known properties of the following functions: the Riemann and Hurwitz zeta functions, the gamma function, Bessel functions, and cusp forms together with their associated Dirichlet series. The method of van der Corput is occasionally used, but only in its simplest form.

As we pointed out, the theory of transformations of exponential sums to be presented in these notes can be viewed as a continuation or extension of some fundamental ideas underlying van der Coroput's

method. A similarity though admittedly of a more formal nature can also
be found with the circle method and the large sieve method, namely a
7 judicious choice of a system of rational numbers at the outset. In short,
our principal goal is to analyse what can be said about Dirichlet series and related Dirichlet polynomials or exponential sums by appealing only to the functional equation of the allied Dirichlet series involving the exponential factors $e(nr)$ and making only minimal use of the actual structure or properties of the individual coefficients of the Dirichlet series in question.

Chapter 1

Summation Formulae

THERE IS AN extensive literature on various summation formulae of **8**
the Voronoi type and on different ways to prove such results (see e.g. the series of papers by B.C. Berndt [3] and his survey article [4]). We are going to need such identities for the sums

$$\sum_{a \le n \le b}{}' b(n)e(nr)f(n),$$

where $0 < a < b$, $f \in C^1[a,b]$, $r = h/k$, and $b(n) = d(n)$ or $a(n)$. The case $f(x) = 1$ is actually the important one, for the generalization is easily made by partial summation. So the basic problem is to prove identities for the sums $D(x,r)$ and $A(x,r)$ (see Notation for definitions). In view of their importance and interest, we found it expedient to derive these identities from scratch, with a minimum of background and effort.

Our argument proceeds via Riesz means $D_a(x,r)$ and $A_a(x,r)$ where $a \ge 0$ is an integer. We follow A.L. Dixon and W.L. Ferrar [6] with some simplifications. First, in [6] the more general case when a is not necessarily an integer was discussed, and this leads to complications since the final result can be formulated in terms of ordinary Bessel functions only if a is an integer. Secondly, it turned out that for $a = 0$ the case $x \in \mathbb{Z}$, which requires a lengthy separate treatment in [6], can actually be reduced to the case $x \notin \mathbb{Z}$ in a fairly simple way.

To get started with the proofs of the main results of this chapter, we **9** need information on the Dirichlet series $E(s, r)$ and $\varphi(s, r)$, in particular their analytic continuations and functional equations. The necessary facts are provided in §§ 1.1 and 1.2.

Bessel functions emerge in the proofs of the summation formulae when certain complex integrals involving the gamma function are calculated. We could refer here to Watson [29] or Titchmarsh [26], but for convenience, in § 1.4 , we calculate these integrals directly by the theorem of residues.

In practice, it is useful to have besides the identities also approximate and mean value results on $D(x, r)$ and $A(x, r)$, to be given in § 1.5.

Identities for $D_a(x, r)$ and $A_a(x, r)$ are proved in §§ 1.6–1.8, first for $a \geq 1$ and then for $a = 0$. The general summation formulae are finally deduced in § 1.9.

1.1 The Function $E(s, r)$

The function

$$E(s, r) = \sum_{n=1}^{\infty} d(n) e(nr) n^{-s} (\sigma > 1) \tag{1.1.1}$$

where $r = h/k$, was investigated by T. Estermann [8], who proved the results of the following lemma. Our proofs are somewhat different in details, for we are making systematic use of the Hurwitz zeta-function $\zeta(s, a)$.

Lemma 1.1. *The function $E(s, h/k)$ can be continued analytically to a meromorphic function, which is holomorphic in the whole complex plane up to a double pole at $s = 1$, satisfies the functional equation*

$$\begin{aligned} E(s, h/k) = {} & 2(2\pi)^{2s-2} \Gamma^2(1 - s) k^{1-2s} \times \\ & \times \{E(1 - s, \bar{h}/k) - \cos(\pi s) E(1 - s, \bar{h}/k)\}, \end{aligned} \tag{1.1.2}$$

10 *and has at $s = 1$ the Laurent expansion*

$$E(s, h/k) = k^{-1}(s - 1)^{-2} + k^{-1}(2\gamma - 2\log k)(s - 1)^{-1} + \cdots \tag{1.1.3}$$

Also,

$$E(0, h/k) \ll k \log 2k. \tag{1.1.4}$$

Proof. The Dirichlet series (1.1.1) converges absolutely and thus defines a holomorphic function in the half-plane $\sigma > 1$. The function $E(s, h/k)$ can be expressed in terms of the Hurwitz zeta-function

$$\zeta(s, a) = \sum_{n=0}^{\infty} (n + a)^{-s} \quad (\sigma > 1, 0 < a \leq 1).$$

Indeed, for $\sigma > 1$ we have

$$\begin{aligned} E(s, h/k) &= \sum_{m,n=1}^{\infty} e_k(mnh)(mn)^{-s} \\ &= \sum_{\alpha,\beta=1}^{k} e_k(\alpha\beta h) \sum_{\substack{m \equiv \alpha \pmod k \\ n \equiv \beta \pmod k}} (mn)^{-s} \\ &= \sum_{\alpha,\beta=1}^{k} e_k(\alpha\beta h) \sum_{\mu,\nu=0}^{\infty} ((\alpha + \mu k)(\beta + \nu k))^{-s}, \end{aligned}$$

so that

$$E(s, h/k) = k^{-2s} \sum_{\alpha,\beta=1}^{k} e_k(\alpha\beta h)\zeta(s, \alpha/k)\zeta(s, \beta/k). \tag{1.1.5}$$

This holds, in the first place, for $\sigma > 1$, but since $\zeta(s, a)$ can be analytically continued to a meromorphic function which has a simple pole 11
with residue 1 at $s = 1$ as its only singularity (see [27], p. 37), the equation (1.1.5) gives an analytic continuation of $E(s, h/k)$ to a meromorphic function. Moreover, its only possible pole, of order at most 2, is $s = 1$.

To study the behaviour of $E(s, h/k)$ near $s = 1$, let us compare it with the function

$$k^{-2s}\zeta(s) \sum_{\alpha,\beta=1}^{k} e_k(\alpha\beta h)\zeta(s, \beta/k) = k^{1-2s}\zeta^2(s).$$

The difference of these functions is by (1.1.5) equal to

$$k^{-2s}\sum_{\alpha=1}^{k}\left(\sum_{\beta=1}^{k}e_k(\alpha\beta h)\zeta(s,\beta/k)\right)(\zeta(s,\alpha/k)-\zeta(s)). \tag{1.1.6}$$

Here the factor $\zeta(s,\alpha/k)-\zeta(s)$ is holomorphic at $s=1$ for all α, and vanishes for $\alpha=k$. Since the sum with respect to β is also holomorphic at $s=1$ for $\alpha\neq k$, the function (1.1.6) is holomorphic at $s=1$. Accordingly, the functions $E(s,h/k)$ and $k^{1-2s}\zeta^2(s)$ have the same principal part at $s=1$. Because

$$\zeta(s)=\frac{1}{s-1}+\cdots,$$

this principal part is that given in (1.1.3).

To prove the functional equation (1.1.2), we utilize the formula ([27], equation (2.17.3))

$$\zeta(s,a)=2(2\pi)^{s-1}\Gamma(1-s)\sum_{m=1}^{\infty}\sin(\frac{1}{2}\pi s+2\pi ma)m^{s-1}(\sigma<0). \tag{1.1.7}$$

Then the equation (1.1.5) becomes

$$\begin{aligned}E(s,h/k)&=-(2\pi)^{2s-2}\Gamma^2(1-s)k^{-2s}\times\\&\times\sum_{\alpha,\beta=1}^{k}e_k(\alpha\beta h)\sum_{m,n=1}^{\infty}\{e^{\pi is}e_k(m\alpha+n\beta)+e^{-\pi is}e_k(-m\alpha-n\beta)\\&-e_k(m\alpha-n\beta)-e_k(-m\alpha+n\beta)\}(mn)^{s-1}\quad(\sigma<0).\end{aligned}$$

12 Note that

$$\sum_{\alpha=1}^{k}e_k(\alpha\beta h\mp m\alpha)=\begin{cases}k & \text{if } \beta\equiv\pm m\bar{h} \pmod k,\\ 0 & \text{otherwise}\end{cases}$$

The functional equation (1.1.2) now follows, first for $\sigma<0$, but by analytic continuation elsewhere also.

For a proof of (1.1.4), we derive for $E(0,h/k)$ an expression in a closed form. By (1.1.5),

$$E(0,h/k)=\sum_{\alpha,\beta=1}^{k} e_k(\alpha\beta h)\zeta(0,\alpha/k)\zeta(0,\beta/k). \tag{1.1.8}$$

If $0<a<1$, then the series in (1.1.7) converges uniformly and thus defines a continuous function for all real $s\leq 0$. Hence, by continuity, (1.1.7) remains valid also for $s=0$ in this case. It follows that

$$\zeta(0,a)=\pi^{-1}\sum_{m=1}^{\infty}\sin(2\pi ma)m^{-1}.$$

But the series on the right equals $\pi(1/2-a)$ for $0<a<1$, whence

$$\zeta(0,a)=1/2-a. \tag{1.1.9}$$

Since $\zeta(0,1)=\zeta(0)=-1/2$, this holds for $a=1$ as well. Now, by (1.1.8) and (1.1.9)

$$E(0,h/k)=\sum_{\alpha,\beta=1}^{k} e_k(\alpha\beta h)(1/2-\alpha/k)(1/2-\beta/k).$$

From this it follows easily that 13

$$E(0,h/k)=-\frac{3}{4}k+k^{-2}\sum_{\alpha,\beta=1}^{k} e_k(\alpha\beta h)\alpha\beta. \tag{1.1.10}$$

To estimate the double sum on the right, observe that if $1\leq\alpha\leq k-1$ and β runs over an arbitrary interval, then

$$\left|\sum_{\beta} e_k(\alpha\beta h)\right| \ll \|\ \alpha h/k\ \|^{-1}\ .$$

Thus, by partial summation,

$$\left|\sum_{\alpha=1}^{k-1}\sum_{\beta=1}^{k} e_k(\alpha\beta h)\alpha\beta\right| \ll k^2\sum_{\alpha=1}^{k-1}\|\ \alpha h/k\ \|^{-1}$$

$$\ll k^2 \sum_{1\le\alpha\le k/2} k/\alpha \ll k^3 \log k,$$

and (1.1.4) follows from (1.1.10). □

1.2 The Function $\varphi(s, r)$

Let H be the upper half-plane $\operatorname{Im}\tau > 0$. The mappings

$$\tau \to \frac{a\tau + b}{c\tau + d}$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is an integral matrix of determinant 1, take H onto itself and constitute the (full) **modular group**. A function f which is holomorphic in H and not identically zero, is a **cusp form** of **weight** k for the modular group if

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau), \tau \in H \tag{1.2.1}$$

for all mappings of the modular group, and moreover

$$\lim_{\operatorname{Im}\tau\to\infty} f(\tau) = 0. \tag{1.2.2}$$

14 It is well-known that k is an even integer at least 12, and that the dimension of the vector space of cusp forms of weight k is $[k/12]$ if $k \not\equiv 2 \pmod{12}$ and $[k/12] - 1$ if $k \equiv 2 \pmod{12}$ (see [1], §§ 6.3 and 6.5).

A special case of (1.2.1) is $f(\tau + 1) = f(\tau)$. Hence, by periodicity, f has a Fourier series, which by (1.2.2) is necessarily of the form

$$f(\tau) = \sum_{n=1}^{\infty} a(n) e(n\tau). \tag{1.2.3}$$

The numbers $a(n)$ are called the **Fourier coefficients** of the cusp form f. The case $k = 12$ is of particular interest, for then $a(n) = \tau(n)$, the Ramanujan function defined by

$$\sum_{n=1}^{\propto} \tau(n) x^n = x \prod_{m=1}^{\infty} (1 - x^m)^{24} \; (|x| < 1).$$

We are going to need some information on the order of magnitude of the Fourier coefficients $a(n)$. For most purposes, the classical mean value theorem

$$\sum_{n\leq x} |a(n)|^2 = Ax^k + o(x^{k-2/5}) \tag{1.2.4}$$

of R.A. Rankin [24] suffices, though sometimes it will be convenient of necessary to refer to the estimate

$$|a(n)| \leq n^{(k-1)/2} d(n). \tag{1.2.5}$$

This was known as the Ramanujan-Petersson conjecture, untill it became a theorem after having been proved by P. Deligne [5]. In (1.2.5), it should be understood that f is a normalized eigenform (i.e. $a(1) = 1$) of all Hecke operators $T(n)$, but this is not an essential restriction, for a basis of the vector space of cusp forms of a given weight can be **15**
constructed of such forms.

Now (1.2.4) implies that the estimate (1.2.5), and even more, is true in a mean sense, and since we shall be dealing with expressions involving $a(n)$ for many values of n, it will be usually enough to know the order of $a(n)$ on the average.

It follows easily from (1.2.4) that the Dirichlet series

$$\varphi(s) = \sum_{n=1}^{\infty} a(n) n^{-s},$$

and, more generally, the series

$$\varphi(s, r) = \sum_{n=1}^{\infty} a(n) e(nr) n^{-s},$$

where $r = h/k$, converges absolutely and defines a holomorphic function in the half-plane $\sigma > (k + 1)/2$. It was shown by J.R. Wilton [30], in the case $a(n) = \tau(n)$, that $\varphi(s, r)$ can be continued analytically to an entire function satisfying a functional equation of the Riemann type. But his argument applies as such also in the general case, and the result is as follows.

Lemma 1.2. *The function $\varphi(s, h/k)$ can be continued analytically to an entire function satisfying the functional equation*

$$(k/2\pi)^s\Gamma(s)\varphi(s, h/k) = (-1)^{k/2}(k/2\pi)^{k-s}\Gamma(k-s)\varphi(k-s, -\overline{h}/k). \tag{1.2.6}$$

Proof. Let

$$\tau = \frac{h}{k} + \frac{iz}{k}, \quad \tau' = -\frac{\overline{h}}{k} + \frac{i}{zk},$$

16 where $Re\, z > 0$. Then $\tau, \tau' \in H$, and we show first that

$$f(\tau') = (-1)^{k/2} z^k f(\tau). \tag{1.2.7}$$

The points τ and τ' are equivalent under the modular group, for putting $a = \overline{h}, b = (1 - h\overline{h})/k, c = -k$, and $d = h$, we have $ad - bc = 1$ and

$$\frac{a\tau + b}{c\tau + d} = \tau'.$$

Also,

$$c\tau + d = -iz,$$

so that (1.2.7) is a consequence of the relation (1.2.1).

Now let $\sigma > (k+1)/2$. Then we have

$$\begin{aligned}(k/2\pi)^s\Gamma(s)\varphi(s, h/k) &= \sum_{n=1}^{\infty} a(n)e_k(nh)\int_0^{\infty} x^{s-1}e^{-2\pi nx/k}\,dx \\ &= \int_0^{\infty} x^{s-1} f\left(\frac{h}{k} + \frac{ix}{k}\right)dx.\end{aligned}$$

Here the integral over (0,1) can be written by (1.2.7) as

$$(-1)^{k/2}\int_0^1 x^{s-1-k} f\left(-\frac{\overline{h}}{k} + \frac{i}{xk}\right)dx = (-1)^{k/2}\int_1^{\infty} x^{k-1-s} f\left(-\frac{\overline{h}}{k} + \frac{ix}{k}\right)dx.$$

Hence

$$(k/2\pi)^s\Gamma(s)\varphi(s,h/k) \tag{1.2.8}$$
$$= \int\limits_1^\infty \left\{ x^{s-1} f\left(\frac{h}{k} + \frac{ix}{k}\right) + (-1)^{k/2} x^{k-1-s} f\left(-\frac{\bar{h}}{k} + \frac{ix}{k}\right)\right\} dx.$$

But the integral on the right defines an entire function of s, for, by (1.2.3), the function $f(\tau)$ decays exponentially as $\operatorname{Im}\tau$ tends to infinity. Thus (1.2.8) gives an analytic continuation of $\varphi(s,h/k)$ to an entire function. Moreover, it is immediately seen that the right hand side remains **17**
invariant under the transformation $h/k \to -\bar{h}/k$, $s \to k-s$ if $k/2$ is even, and changes its sign if $k/2$ is odd. Thus the functional equation (1.2.6) holds in any case. □

REMARK. *The special case $k = 1$ of* (1.2.6) *amounts to Hecke's functional equation*

$$(2\pi)^{-s}\Gamma(s)\varphi(s) = (-1)^{k/2}(2\pi)^{s-k}\Gamma(k-s)\varphi(k-s).$$

1.3 Asymptotic Formulae for the Gamma Function and Bessel Functions

The special functions that will occur in this text are the gamma function $\Gamma(s)$ and the Bessel functions $J_n(z), Y_n(z), K_n(z)$ of nonnegative integral order n. By definition,

$$J_n(z) = \sum_{k=0}^\infty \frac{(-1)^k (z/2)^{2k+n}}{k!(n+k)!}, \tag{1.3.1}$$

$$Y_n(z) = -\pi^{-1}\sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!}(z/2)^{2k-n} \tag{1.3.2}$$
$$+\pi^{-1}\sum_{k=0}^\infty \frac{(-1)^k (z/2)^{2k+n}}{k!(n+k)!}(2\log(z/2) - \psi(k+1) - \psi(k+n+1)),$$

and

$$(1.3.3) \qquad K_n(z) = \frac{1}{2}\sum_{k=0}^{n-1} \frac{(-1)^k(n-k-1)!}{k!}(z/2)^{2k-n}$$
$$+\frac{1}{2}(-1)^{n-1}\sum_{k=0}^{\infty} \frac{(z-2)^{2k+n}}{k!(n+k)!} 92\log(z/2) - \psi(k+1) - \psi(k+n+1)),$$

where

$$\psi(z) = \frac{\Gamma'}{\Gamma}(z).$$

In particular,

$$\psi(1) = -\gamma, \psi(n+1) = -\gamma + \sum_{k=1}^{n} k^{-1}, n = 1, 2, \ldots$$

18 Repeated use will be made of Stirling's formula for $\Gamma(s)$ and of the asymptotic formulae for Bessel functions. Therefore we recall these well-konwn results here for the convenience of further reference.

The following version of Stirling's formula is precise enough for our purposes.

Lemma 1.3. *Let $\delta < \pi$ be a fixed positive number. Then*

$$(1.3.4) \qquad \Gamma(s) = \sqrt{2\pi}\exp\{s - 1/2)\log s - s\}(1 + o(|s|^{-1}))$$

in the sector $|\arg s| \leq \pi - \delta, |s| \geq 1$. Also, in any fixed strip $A_1 \leq \sigma \leq A_2$ we have for $t \geq 1$

$$(1.3.5) \quad \Gamma(s) = \sqrt{2\pi}t^{s-1/2}\exp(-\frac{1}{2}\pi t - it + \frac{1}{2}\pi(\sigma - 1/2)i)(1 + o(t^{-1})),$$

and

$$(1.3.6) \qquad |\Gamma(s)| = \sqrt{2\pi}t^{\sigma-1/2}e^{-(\pi/2)t}(1 + 0(t^{-1})).$$

The asymptotic formulae for the functions $J_n(z), Y_n(z)$, and $K_n(z)$ can be derived from the analogous results for Hankel functions

$$(1.3.7) \qquad H_n^{(j)}(z) = J_n(z) + (-1)^{j-1}iY_n(z), j = 1, 2.$$

The variable z is here restricted to the slit complex plane $z \neq 0, |\arg z| < \pi$. Obviously,

$$J_n(z) = \frac{1}{2}\left(H_n^{(1)}(z) + H_n^{(2)}(z)\right), \tag{1.3.8}$$

$$Y_n(z) = \frac{1}{2i}\left(H_n^{(1)}(z) - H_n^{(2)}(z)\right). \tag{1.3.9}$$

The function $K_n(z)$ can also be written in terms of Hankel functions, for (see [29], p. 78)

$$K_n(z) = \frac{\pi}{2} i^{n+1} H_n^{(1)}(iz) \quad \text{for} \quad -\pi < \arg z < \pi/2, \tag{1.3.10}$$

$$K_n(z) = \frac{\pi}{2} i^{-n+1} H_n^{(2)}(-iz) \quad \text{for} \quad \frac{\pi}{2} < \arg z < \pi. \tag{1.3.11}$$

19

The asymptotic formulae for Hankel functions are usually derived from appropriate integral representations, and then the asymptotic behaviour of J_n, Y_n, and K_n can be determined by the relations (1.3.8) - (1.3.11) (see [29], §§ 7.2, 7.21 and 7.23). The results are as follows.

Lemma 1.4. *Let $\delta_1 < \pi$ and δ_2 be fixed positive numbers. Then in the sector*

$$|argz| \leq \pi - \delta_1, |z| \geq \delta_2 \tag{1.3.12}$$

we have

$$H_n^{(j)}(z) = (2/\pi z)^{1/2} \exp\left((-1)^{j-1} i\left(z - \frac{1}{2}n\pi - \frac{1}{4}\pi\right)\right)(1 + g_j(z)), \tag{1.3.13}$$

where the functions $g_j(z)$ are holomorphic in the slit complex plane $z \neq 0, |\arg z| < \pi$, and satisfy

$$|g_j(z)| \ll |z|^{-1} \tag{1.3.14}$$

in the sector (1.3.12). *Also, for real $x \geq \delta_2$,*

$$J_n(x) = (2/\pi x)^{1/2} \cos\left(x - \frac{1}{2}n\pi - \frac{1}{4}\pi\right) + o(x^{-3/2}), \tag{1.3.15}$$

$$Y_n(x) = (2/\pi x)^{1/2} \sin\left(x - \frac{1}{2}n\pi - \frac{1}{4}\pi\right) + o(x^{-3/2}), \tag{1.3.16}$$

and

$$K_n(x) = (\pi/2x)^{1/2} e^{-x}\left(1 + o(x^{-1})\right). \tag{1.3.17}$$

Strictly speaking, the functions g_j should actually be denoted by $g_{j,n}$,
20 say, because they depend on n as well, but for simplicity we dropped the index n, which will always be known from the context.

1.4 Evaluation of Some Complex Integrals

Let a be a nonnegative integer, $\sigma_1 \geq -a/2, \sigma_2 < -a, T > 0$, and let C_a be the contour joining the points $\sigma_1 - i\infty, \sigma_1 - Ti, \sigma_2 - Ti, \sigma_2 + Ti, \sigma_1 + Ti$, and $\sigma_1 + i\infty$ by straight lines. Let $X > 0, k$ a positive integer, and c a number such that

$$(k - a - 1)/2 \leq c < k.$$

In the next two sections we are going to need the values of the complex integrals

$$I_1 = \frac{1}{2\pi i} \int_{C_a} \Gamma^2(1 - s) X^s (s(s + 1) \ldots (s + a))^{-1}\, ds, \tag{1.4.1}$$

$$I_2 = \frac{1}{2\pi i} \int_{C_a} \Gamma^2(1 - s) \cos(\pi s) X^s (s(s + 1) \ldots (s + a))^{-1}\, ds, \tag{1.4.2}$$

and

$$I_3 = \frac{1}{2\pi i} \int_{(c)} \Gamma(k - s) \Gamma^{-1}(s) X^s (s(s + 1) \ldots (s + a))^{-1}\, ds. \tag{1.4.3}$$

Lemma 1.5. *We have*

$$I_1 = 2(-1)^{a+1} X^{(1-a)/2} K_{a+1}(2X^{1/2}), \tag{1.4.4}$$

$$I_2 = \pi X^{(1-a)/2} Y_{a+1}(2X^{1/2}), \tag{1.4.5}$$

and

$$I_3 = X^{(k-a)/2} J_{k+a}(2X^{1/2}). \tag{1.4.6}$$

Proof. For positive numbers T_1 and T_2 exceeding T, denote by $C_a(T_1, T_2)$ that part of C_a which lies in the strip $-T_1 \leq t \leq T_2$. The integrals I_1 and I_2 are understood as limits of the corresponding integrals over **21**
$C_a(T_1, T_2)$ as T_1 and T_2 tend to infinity independently. Similarly, I_3 is the limit of the integral over the line segment $[c - iT_1, c + iT_2]$.

Let N be a large positive integer, which is kept fixed for a moment. Denote by $\Gamma(T_1, T_2; N)$ the closed contour joining the points $N+1/2-iT_1$ and $N + 1/2 + iT_2$ with each other and with the initial and end point of $C_a(T_1, T_2)$, respectively, or with the points $c - iT_1$ and $c + iT_2$ in the case of I_3. Then, by the theorem of residues,

$$\frac{1}{2\pi i} \int\limits_{\Gamma} (\ldots)\, ds = \sum \mathrm{Res}, \tag{1.4.7}$$

where $(\ldots)$ means the integrand of the respective I_j, whose residues inside $\Gamma = \Gamma(T_1, T_2; N)$ are summed on the right.

By (1.3.6) and our assumptions on σ_1 and c, the integrals over those horizontal parts of $\Gamma(T_1, T_2; N)$ lying on the lines $t = -T_1$ and $t = T_2$ are seen to be $\ll (\log T_i)^{-1}$, $i = 1, 2$. Hence these integrals vanish in the limit as the T_i tend to infinity. Then the equation (1.4.7) becomes

$$-I_j + \frac{1}{2\pi i} \int\limits_{(N+1/2)} (\ldots)\, ds = \sum \mathrm{Res}\,. \tag{1.4.8}$$

Consider now the integrals over the line $\sigma = N + 1/2$.

By a repeated application of the formula $\Gamma(s) = s^{-1}\Gamma(s + 1)$, the Γ-factors in the integrands can be expressed in terms of $\Gamma(1/2 + it)$. Then, by some simple estimations, we find that the integrals in question vanish in the limit as N tends to infinity. Therefore (1.4.8) gives

$$I_J = -\sum_{k=0}^{a} \mathrm{Res}(\cdot\,, -k) - \sum_{k=1}^{\infty} \mathrm{Res}(\cdot\,, k),\ j = 1, 2, \tag{1.4.9}$$

$$I_3 = -\sum_{k=k}^{\infty} \operatorname{Res}(\cdot\,, k), \tag{1.4.10}$$

22 where the dot denotes the respective integrand.

Consider the integral I_1 first. Obviously

$$\operatorname{Res}(\cdot\,, -h) = (-1)^h h!((a-h)!)^{-1} X^{-h} \quad \text{for} \quad h = 0, 1, \ldots, a.$$

The sum of these can be written, on putting $k = a - h$, as

$$(-1)^a (X^{1/2})^{1-a} \sum_{k=0}^{a} (-1)^k (a-k)!(k!)^{-1} (2X^{1/2}/2)^{2k-a-1}. \tag{1.4.11}$$

The integrand has double poles at $s = 1, 2, \ldots,$ and the residue at k can be calculated, multiplying (for $s = k + \delta$) the expansions

$$\begin{aligned}
\Gamma^2(1-s) &= \delta^{-2}\Gamma^2(1-\delta)(k-1+\delta)^{-2}(k-2+\delta)^{-2}\ldots(1+\delta)^{-2}\\
&= \delta^{-2}((k-1)!)^{-2}(1-2\psi(k)\delta+\ldots),\\
(s(s+1)\ldots(s+a))^{-1} &= (k-1)!((k+a)!)^{-1}\\
&\qquad (1-(\psi(k+a+1)-\psi(k))\delta+\cdots),
\end{aligned}$$

and

$$X^s = X^k(1 + \delta \log X + \cdots).$$

We obtain

$$\operatorname{Res}(\cdot\,, k) = X^k((k+a)!(k-1)!)^{-1}(\log X - \psi(k+a+1) - \psi(k)), k = 1, 2, \ldots$$

Hence, also taking into account (1.4.11) and (1.3.3), we may write the sum of residues as

$$\begin{aligned}
2(-1)^a (X^{1/2})^{1-a} \Bigg\{ &\frac{1}{2} \sum_{k=0}^{(a+1)-1} (-1)^k ((a+1)-k-1)!(k!)^{-1} (2X^{1/2}/2)^{2k-(a+1)}\\
&+ \frac{1}{2}(-1)^{(a+1)-1} \sum_{k=0}^{\infty} (k!(k+(a+1))!)^{-1} (2X^{1/2}/2)^{2k+(a+1)}.\\
\cdot\,(2\log(2X^{1/2}/2) - \psi(k+1) - \psi(k+(a+1)+1))\Bigg\} &= 2(-1)^a X^{(1-a)/2} K_{a+1}(2X^{1/2})\cdot
\end{aligned}$$

Now (1.4.4) follows from (1.4.9).

23 The residues of the integrands of I_1 and I_2 at $s = k$ differ only by the sign $(-1)^k$. The series of residues of the integrand of I_2 can be written in terms of the function Y_{a+1}, and the assertion (1.4.5) follows by a calculation similar to that above.

Finally, the residue of the integrand of I_3 at $h \geq k$ is

$$(-1)^{h-k} X^h ((h+a)!(h-k)!)^{-1},$$

and putting $k = h - k$ the sum of these terms can be arranged so as to give

$$-X^{(k-a)/2} J_{k+a}(2X^{1/2}).$$

$\square$

This proves (1.4.6).

1.5 Approximate Formula and Mean Value Estimates for $D(x, r)$ and $A(x, r)$

Our object in this section is to derive approximate formulae of the Voronoi type for the exponential sums

$$D(x, r) = {\sum_{n \leq x}}' d(n) e(nr)$$

and

$$A(x, r) = {\sum_{n \leq x}}' a(n) e(nr),$$

and to apply these to the pointwise and mean square estimation of $D(x, r)$ and $A(x, r)$. As before, $r = h/k$ is a rational number.

A model of a result like this is the following classical formul for $D(x) = D(x, 1)$:

$$D(x) = (\log x + 2\gamma - 1)x \tag{1.5.1}$$
$$+(\pi\sqrt{2})^{-1} x^{\frac{1}{4}} \sum_{n \leq N} d(n) n^{-3/4} \cos(4\pi\sqrt{nx} - \pi/4) + o(x^{1/2+\epsilon_N - 1/2}),$$

where $x \geq 1$ and $1 \leq N \ll x$ (see [27], p. 269). The corresponding 24
formula for $D(x, r)$ will be of the form

$$(1.5.2)\quad D(x, h/k) = k^{-1}(\log x + 2\gamma - 1 - 2\log k)x + E(0, h/k) + \Delta(x, h/k),$$

where $\Delta(x, h/k)$ is an error term.

The next theorem reveals an analogy between $\Delta(x, r)$ and $A(x, r)$.

THEOREM 1.1. *For $x \geq 1, k \leq x$, and $1 \leq N \ll x$ the equation* (1.5.2) *holds with*

(1.5.3)

$$\Delta(x, h/k) = (\pi\sqrt{2})^{-1}k^{1/2}x^{1/4}\sum_{n\leq N} d(n)e_k(-n\bar{h})n^{-3/4}\cos(4\pi\sqrt{nx}/k - \pi/4) + 0(kx^{\frac{1}{2}+\epsilon_N-\frac{1}{2}}).$$

Also,

(1.5.4)

$$A(x, h/k) = (\pi\sqrt{2})^{-1}k^{\frac{1}{2}}x^{-\frac{1}{4}+\frac{k}{2}}\sum_{n\leq N} a(n)e_k(-n\bar{h})n^{-1/4-k/2}\cos\left(\frac{4\pi\sqrt{nx}}{k} - \frac{\pi}{4}\right) + 0(kx^{k/2+\epsilon_N-1/2}).$$

Proof. Consider first the formula (1.5.3). We follow the argument of proof of (1.5.1) in [27], pp. 266–269, with minor modifications.

Let δ be a small positive number which will be kept fixed throughout the proof. By Perron's formula,

$$(1.5.5)\qquad D(x, r) = \frac{1}{2\pi i}\int\limits_{1+\delta-iT}^{1+\delta+iT} E(s, r)x^s s^{-1}\,ds + 0(x^{1+\delta}T^{-1}),$$

where $r = h/k$ and T is a parameter such that

$$(1.5.6)\qquad 1 \leq T \ll k^{-1}x.$$

25 As a preliminary for the next step, which consists of moving the integration in (1.5.5) to the line $\sigma = -\delta$, we need an estimate for $E(s, r)$ in the strip $-\delta \leq \sigma \leq 1 + \delta$ for $|t| \geq 1$.

The auxiliary function

$$\left(\frac{s-1}{s-2}\right)^2 E(s,r) \tag{1.5.7}$$

is holomorphic in the strip $-\delta \leq \sigma \leq 1+\delta$, and in the part where $|t| \geq 1$ it is of the same order of magnitude as $E(s,r)$. This function is bounded on the line $\sigma = 1+\delta$, and on the line $\sigma = -\delta$ it is

$$\ll (k(|t|+1))^{1+2\delta}$$

by the functional equation (1.1.2) and the estimate (1.3.6) of the gamma function. The convexity principle now gives an estimate for the function (1.5.7), and as a consequence we obtain

$$|E(s,r)| \ll (k|t|)^{1-\sigma+\delta} \quad \text{for} \quad -\delta \leq \sigma \leq 1+\delta, |t| \geq 1. \tag{1.5.8}$$

Let C be the rectangular contour with vertices $1+\delta \pm iT$ and $-\delta \pm iT$. By the theorem of residues, we have

$$\frac{1}{2\pi i}\int_C E(s,r)x^s s^{-1}\,ds = k^{-1}(\log x + 2\gamma - 1 - 2\log k)x + E(0,r), \tag{1.5.9}$$

where the expansion (1.1.3) has been used in the calculation of the residue at $s = 1$.

The integrals over the horizontal parts fo C are $\ll x^{1+\delta}T^{-1}$ by (1.5.8) and (1.5.6). Hence (1.5.2), (1.5.5), and (1.5.9) give together

$$\Delta(x,r) = \frac{1}{2\pi i}\int_{\delta-iT}^{-\delta+iT} E(s,r)x^s s^{-1}\,ds + o(x^{1+\delta}T^{-1}). \tag{1.5.10}$$

The functional equation (1.1.2) for $E(s,r)$ is now applied. The term involving $E(1-s,\bar{h}/k)$ decreases rapidly as $|t|$ increases and it will be 26
estimated as an error term. Then for $\sigma = -\delta$, we obtain

$$E(s,r) = -2(2\pi)^{2s-2}\Gamma^2(1-s)k^{1-2s}\cos(\pi s)\sum_{n=1}^{\infty} d(n)e_k(-n\bar{h})n^{s-1}$$

$$+o((k(|t|+1))^{1+2\delta}e^{-\pi|t|}).$$

The contribution of the error term to the integral in (1.5.10) is

$$\ll k^{1+2\delta}x^{-\delta} \ll kx^{\delta} \ll x^{1+\delta}T^{-1}.$$

Thus we have

$$\Delta(x,r) = -\frac{1}{2}\pi^{-2}k\sum_{n=1}^{\infty} d(n)n^{-1}e_k(-n\bar{h})j_n + o(x^{1+\delta}T^{-1}), \tag{1.5.11}$$

where

$$j_n = \frac{1}{2\pi i}\int_{-\delta-iT}^{-\delta+iT} \Gamma^2(1-s)\cos(\pi s)(4\pi^2 nxk^{-2})^s s^{-1}\, ds. \tag{1.5.12}$$

At this stage we fix the parameter T, putting

$$T^2k^2(4\pi^2x)^{-1} = N + 1/2, \tag{1.5.13}$$

where N is an integer such that $1 \le N \ll x$. It is immediately seen that $T \ll k^{-1}x$. In order that the condition (1.5.6) be satisfied, we should also have $T \ge 1$, which presupposes that $N \gg k^2x^{-1}$. We may assume this, for otherwise the assertion (1.5.3) holds for trivial reasons. Indeed, if $1 \le N \ll k^2x^{-1}$, then (1.5.3) is implied by the estimate $\Delta(x,h/k) \ll x^{1+\epsilon}$, which is definitely true by (1.5.2) and (1.1.4).

Next we dispose of the tail $n > N$ of the series in (1.5.11). The integral j_n splits into three parts, in which t runs respectively over the intervals $[-T,-1], [-1,1]$, and $[1,T]$. The second integral is clearly
27 $\ll k^{2\delta}n^{-\delta}x^{-\delta}$, and these terms contribute $\ll kx^{\delta}$. The first and third integrals are similar; consider the third one, say j_n'.

By (1.3.5) we have for $-\delta \le \sigma \le \delta$ and $t \ge 1$

$$\begin{aligned}&\Gamma^2(1-s)\cos(\pi s)(4\pi^2 nxk^{-2})^s s^{-1} \\ &= A(\sigma)t^{-2\sigma}(4\pi^2 nxk^{-2})^{\sigma}e^{iF(t)}(1+O(t^{-1})),\end{aligned} \tag{1.5.14}$$

where $A(\sigma)$ is bounded and

$$F(t) = -2t\log t + 2t + t\log(4\pi^2 nxk^{-2}). \tag{1.5.15}$$

Thus

$$j'_n = Ak^{2\delta}n^{-\delta}x^{-\delta}\left(\int\limits_1^T t^{2\delta}e^{iF(t)}\,dt + o(T^{2\delta})\right). \tag{1.5.16}$$

The last integral is estimated by the following elementary lemma ([27], Lemma 4.3) on exponential integrals. □

Lemma 1.6. *Let $F(x)$ and $G(x)$ be real functions in the interval $[a, b]$ where $G(x)$ is continuous and $F(x)$ continuously differentiable. Suppose that $G(x)/F'(x)$ is monotonic and $|F'(x)/G(x)| \geq m > 0$. Then*

$$|\int\limits_a^b G(x)e^{iF(x)}\,dx| \leq 4/m. \tag{1.5.17}$$

Now by (1.5.15) and (1.5.13) we have

$$F'(t) = \log(4\pi^2 nxk^{-2}t^{-2}) \geq \log\left(\frac{n}{N+1/2}\right) \tag{1.5.18}$$

for $1 \leq t \leq T$, whence by (1.5.16) and (1.5.17)

$$j'_n \ll k^{2\delta}n^{-\delta}T^{2\delta}x^{-\delta}\left(\left(\log\left(\frac{n}{N+1/2}\right)\right)^{-1} + 1\right).$$

Thus

$$\sum_{n\geq 2N} d(n)n^{-1}|j'_n| \ll N^{\delta} \ll x^{\delta},$$

and
$$\sum_{N<n\leq 2N} d(n)n^{-1}|j'_n| \ll \sum_{1\leq m\leq N} d(N+m)m^{-1} \ll x^{\delta}.$$

Accordingly, in (1.5.11) the tail $n > N$ of the series can be omitted with **28** an error $\ll kx^\delta$, and taking into account the choice (1.5.13) of T, we obtain

$$\Delta(x,r) = -\frac{1}{2}\pi^{-2}k\sum_{n\le N} d(n)n^{-1}e_k(-n\bar{h})j_n + o(kx^{\frac{1}{2}+\delta}N^{\frac{1}{2}}). \tag{1.5.19}$$

The remaining integrals j_n will be calculated approximately by Lemma 1.5, and to this end we extend the path of integration in (1.5.12) to the infinite broken line through the points $\delta - i\infty, \delta - iT, -\delta - iT, -\delta - iT, -\delta + iT, \delta + iT$ and $\delta + i\infty$, estimating the consequent error when the j_n in (1.5.19) are replaced by the new integrals.

First, by (1.5.14) and (1.5.13),

$$\begin{aligned}\sum_{n\le N} d(n)n^{-1}\left|\int\limits_{-\delta+iT}^{\delta+iT}(\cdots)\right| &\ll \sum_{n\le N} d(n)n^{-1}\int\limits_{-\delta}^{\delta}(n/N)^\sigma\, d\sigma \\ &\ll N^\delta\sum_{n\le N} d(n)n^{-1-\delta} \ll x^\delta,\end{aligned}$$

where $(\cdots)$ means the integrand of j_n. The same estimate holds for the integrals over the line segment $[-\delta - iT, \delta - iT]$.

Next, by (1.5.14), (1.5.18), and Lemma 1.6, we have

$$\begin{aligned}\sum_{n\le N} d(n)n^{-1}\left|\int\limits_{\delta+iT}^{\delta+i\infty}(\cdots)\right| &\ll (k^{-2}x)^\delta\sum_{n\le N} d(n)n^{-1+\delta} \\ &\quad \left|\int\limits_{T}^{\infty} t^{-2\delta}\left(e^{iF(t)} + o(t^{-1})\right)dt\right| \\ &\ll (k^{-2}T^{-2}x)^\delta\sum_{n\le N} d(n)n^{-1+\delta}\left(\left(\log\left(\frac{N+1/2}{n}\right)\right)^{-1} + 1\right) \\ &\ll \sum_{n\le N/2} d(n)n^{-1} + \sum_{N/2<n\le N} d(n)(N+1/2-n)^{-1} \ll x^\delta,\end{aligned}$$

29 and similarly for the integrals over $[\delta - i\infty, \delta - iT]$.

These estimations show that (1.5.19) remains valid if the j_n are replaced by the modified integrals, which are of the type I_2 in (1.4.2) for $a = 0$ and $X = 4\pi^2 nxk^{-2}$, and thus equal to

$$2\pi^2(nx)^{1/2}k^{-1}Y_1(4\pi\sqrt{nx}/k)$$

by (1.4.5). The assertion (1.5.3) now follows when Y_1 is replaced by its expression (1.3.16) (which holds trivially for $n = 1$ with the error term $0(x^{-1})$ even in the interval (o, δ_2)).

The proof of (1.5.4) is quite similar. The starting point is the equation

$$A(x,r) = \frac{1}{2\pi i}\int\limits_{(k+1)/2+\delta-iT}^{(k+1)/2+\delta+iT} \varphi(s,r)x^s s^{-1}\,ds + o(x^{(k+1)/2+\delta}T^{-1}),$$

where $1 \leq T \ll k^{-1}x$. It should be noted that Deligne's estimate (1.2.5) is needed here; otherwise the error term would be bigger.

The integration is next shifted to the line segment $[(k-1)/2 - \delta - iT, (k-1)/2 - \delta + iT]$ with arguments as in the proof of (1.5.10), except that now there are no residue terms. Applying the functional equation (1.2.6) of $\varphi(s,r)$, we obtain

$$A(x,r) = (-1)^{k/2}(k/2\pi)^k \sum_{n=1}^{\infty} a(n)n^{-k}e_k(-\bar{h}n)\times$$
$$\times\frac{1}{2\pi i}\int\limits_{(k-1)/2-\delta-iT}^{(k-1)/2-\delta+iT} \Gamma(k-s)\Gamma(s)^{-1}(4\pi^2nxk^{-2})^s s^{-1}\,ds + o(x^{(k+1)/2+\delta}T^{-1}).$$

The parameter T is chosen as in (1.5.13) again. Next it is shown, as before, that the tail $n > N$ of the above series can be omitted, and that in the remaining terms the integration can be shifted to the whole **30**
line $\sigma = (k-1)/2 + \delta$. The new integrals are evaluated in terms of the function J_k using (1.4.6). Finally J_k is approximated by (1.3.15) (which holds trivially with the error term $o(x^{-1/2})$ even in the interval $(0, \delta_2)$) to give the formula (1.5.4). The proof of the theorem is now complete.

Choosing $N = k^{2/3}x^{1/3}$ and estimating the sums on the right of (1.5.3) and (1.5.4) by absolute values, one obtains the following estimates for $\Delta(x, r)$ and $A(x, r)$,

COROLLARY. *For $x \geq 1$ and $k \leq x$ we have*

$$\Delta(x, h/k) \ll k^{2/3}x^{1/3+\epsilon}, \tag{1.5.20}$$

$$A(x, h/k) \ll k^{2/3}x^{k/2-1/6+\epsilon}. \tag{1.5.21}$$

As another application of Theorem 1.1 we deduce mean value results for $\Delta(x, r)$ and $A(x, r)$.

THEOREM 1.2. *For $X \geq 1$ we have*

(1.5.22)
$$\int\limits_1^X |\Delta(x, h/k)|^2\, dx = c_1kX^{3/2} + o(k^2X^{1+\epsilon}) + o(k^{3/2}X^{5/4+\epsilon}),$$

(1.5.23)
$$\text{and} \int\limits_1^X |A(x, h/k)|^2\, dx = c_2(k)kX^{k+1/2} + o(k^2X^{k+\epsilon}) + o(k^{3/2}X^{k+1/4+\epsilon}),$$

where

$$c_1 = (6\pi^2)^{-1} \sum_{n=1}^{\infty} d^2(n)n^{-3/2}$$

and

$$c_2(k) = \left((4k+2)\pi^2\right)^{-1} \sum_{n=1}^{\infty} |a(n)|^2 n^{-k-1/2}.$$

31 *Proof.* The proofs of these assertions are very similar; so it suffices to consider the verification of (1.5.22) as an example. We are actually going to prove the formula

$$\int\limits_X^{2X} |\Delta(x, h/k)|^2\, dx = c_1k\left((2X)^{3/2} - X^{3/2}\right) \tag{1.5.24}$$

$$+o\left(k^2X^{1+\epsilon}\right) + 0\left(k^{3/2}X^{5/4+\epsilon}\right)$$

for $k \leq X$, and for $X \ll k$ we estimate trivially

$$\int_1^X |\Delta(x, h/k)|^2\, dx \ll k^2X^{1+\epsilon} \tag{1.5.25}$$

noting that $\Delta(x, h/k) \ll k \log 2k$ for $x \ll k$ by (1.5.2) and (1.1.4). Clearly (1.5.22) follows from (1.5.24) and (1.5.25).

Turning to the proof of (1.5.24), let $X \leq x \leq 2X$, and choose $N = X$ in the formula (1.5.3), which we write as

$$\Delta(x, h/k) = S(x, h/k) + 0(kx^\epsilon).$$

We are going to prove that

$$\int_X^{2X} |S(x, h/k)|^2\, dx = c_1k\left((2X)^{3/2} - X^{3/2}\right) + o\left(k^2X^{1+\epsilon}\right), \tag{1.5.26}$$

which implies (1.5.24) by Cauchy's inequality.

Squaring out $|S(x, h/k)|^2$ and integrating term by term, we find that

$$\int_X^{2X} |S(x, h/k)|^2\, dx = S_\circ + o(k(|S_1| + |S_2|)), \tag{1.5.27}$$

where

$$S_\circ = (4\pi^2)^{-1}k \sum_{n\leq X} d^2(n)n^{-3/2} \int_X^{2X} X^{1/2}\, dx,$$

$$S_1 = \sum_{\substack{m,n\leq X\\ m\neq n}} d(m)d(n)(mn)^{-3/4} \int_X^{2X} x^{1/2}e(2(\sqrt{m} - \sqrt{n})\sqrt{x}/k)\, dx,$$

$$S_2 = \sum_{m,n\leq X} d(m)d(n)(mn)^{-3/4} \int_X^{2X} x^{1/2}e\left(2\left(\sqrt{m} + \sqrt{n}\right)\sqrt{x}/k\right) dx.$$

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The sum $S_\circ$ gives the leading term in (1.5.26), for

$$S_\circ = c_1 k\left((2X)^{3/2} - X^{3/2}\right) + o\left(kX^{1+\epsilon}\right).$$

Further, by Lemma 1.6,

$$\begin{aligned} S_1 &\ll kX \sum_{\substack{m,n\le X \\ m<n}} d(m)d(n)(mn)^{-3/4}\left(\sqrt{n} - \sqrt{m}\right)^{-1} \\ &\ll kX \sum_{\substack{m,n\le X \\ m<n}} d(m)d(n)m^{-3/4}n^{-1/4}(n-m)^{-1} \\ &\ll kX^{1+\epsilon/2} \sum_{m\le X} m^{-1} \ll kX^{1+\epsilon}, \end{aligned}$$

and similarly for S_2. Hence (1.5.26) follows from (1.5.27), and the proof of (1.5.22) is complete. □

COROLLARY. *For $k \ll X^{1/2-\epsilon}$ and $X \to \infty$ we have*

(1.5.28) $$\int_1^X |\Delta(x, h/k)|^2\, dx \sim c_1 k X^{3/2},$$

(1.5.29) $$\int_1^X |A(x, h/k)|^2\, dx \sim c_2(k) k X^{k+1/2}.$$

It is seen that for $k \ll x^{1/2-\epsilon}$ the typical order of $|\Delta(x, h/k)|$ is $k^{1/2}x^{1/4}$, and that of $|A(x, h/k)|$ is $k^{1/2}x^{k/2-1/4}$. This suggests the following

CONJECTURE. *For $x \geq 1$ and $k \ll x^{1/2}$*

(1.5.30) $$|\Delta(x, h/k)| \ll k^{1/2}x^{1/4+\epsilon}$$

(1.5.31) $$|A(x, h/k)| \ll k^{1/2}x^{k/2-1/4+\epsilon}.$$

33 Note that (1.5.30) is a generalization of the old conjecture

$$|\Delta(x)| \ll x^{1/4+\epsilon}$$

in Dirichlet's divisor problem.

1.6 Identities for $D_a(x,r)$ and $A_a(x,r)$

The Case $a \geq 1$.

As generalizations of the sum functions $D(x,r)$ and $A(x,r)$, define the Riesz means

$$D_a(x,r) = \frac{1}{a!} \sideset{}{'}\sum_{n \leq x} d(n) e(nr)(x-n)^a \tag{1.6.1}$$

and

$$A_a(x,r) = \frac{1}{a!} \sideset{}{'}\sum_{n \leq x} a(n) e(nr)(x-n)^a, \tag{1.6.2}$$

where a is a nonnegative integer. Thus $D_\circ(x,r) = D(x,r)$ and $A_\circ(x,r) = A(x,r)$. Actually, for our later purposes, only the case $a = 0$ will be of relevance, but just in order to be able to deal with this somewhat delicate case by an induction from a to $a-1$, we shall need identities for $D_a(x,r)$ and $A_a(x,r)$ as well. These are contained in the following theorem.

THEOREM 1.3. *Let $a \geq 0$ be an integer. Then for $x > 0$ we have*

$$\begin{aligned} D_a(x,h/k) = {} & \frac{x^{1+a}}{(1+a)!k} \left(\log x + 2\gamma - 2\log k - \sum_{n=1}^{a+1} \frac{1}{n} \right) \\ & + \sum_{n=0}^{a} \frac{(-1)^n}{n!(a-n)!} E(-n, h/k) x^{a-n} + \Delta_a(x,h/k), \end{aligned} \tag{1.6.3}$$

where

(1.6.4)

$$\begin{aligned} & \Delta_a(x,h/k) = -(k/2\pi)^a x^{(1+a)/2} \sum_{n=1}^{\infty} d(n) n^{-(1+a)/2} \times \\ & \times \left\{ e_k(-n\bar{h}) Y_{1+a}(4\pi\sqrt{nx}/k) + (-1)^a (2/\pi) e_k(n\bar{h}) K_{1+a}(4\pi\sqrt{nx}/k) \right\}. \end{aligned}$$

Also, 34

$$A_a(x,h/k) = (-1)^{k/2} (k/2\pi)^a x^{(k+a)/2} \times \tag{1.6.5}$$

$$\times \sum_{n=1}^{\infty} a(n) n^{-(k+a)/2} e_k(-n\bar{h}) J_{k+a}(4\pi\sqrt{nx}/k).$$

Proof. (the case $a \geq 1$). By a well-known summation formula (see [13], p. 487, equation (A.14)), we have for any $c > 1$

$$D_a(x, r) = \frac{1}{2\pi i} \int_{(c)} E(s, r) x^{s+a} (s(s+1)\cdots(s+a))^{-1}\, ds.$$

First let $a \geq 2$, and move the integration to the broken line C_a joining the points $-1/3 - i\infty, -1/3 - i, -(a+1/2) - i, -(a+1/2) + i, -1/3 + i$, and $-1/3 + i\infty$. The residues at $1, 0, -1, \ldots, -a$ give the initial terms in (1.6.3); the expansion (1.1.3) is used in the calculation of the residue at $s = 1$. Note also that the integrand is $\ll |t|^{-2\sigma - a}$ for $|t| \geq 1$ and σ bounded (the implied constant depends on k and x), so that the theorem of residues gives

$$\Delta_a(x, r) = \frac{1}{2\pi i} \int_{C_a} E(s, r) x^{s+a} (s(s+1)\cdots(s+a))^{-1}\, ds.$$

The function $E(s, h/k)$ is now expressed by the functional equation (1.1.2), and the resulting series can be integrated term by term by the last mentioned estimate. The new integrals are of the type I_1 and I_2 in the notation of §1.4, and (1.6.4) follows, for $a \geq 2$, when these integrals are evaluated by Lemma 1.5.

Next we differentiate both sides of (1.6.3) with respect to x. By the definition (1.6.1) we have for $a \geq 2$

(1.6.6) $$D'_a(x, r) = D_{a-1}(x, r),$$

and consequently by (1.6.3)

(1.6.7) $$\Delta'_a(x, r) = \Delta_{a-1}(x, r).$$

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The right hand side of (1.6.4) shares the same property, for its derivative equals the same expression with a replaced by $a-1$, Formally, this can be verified by differentiation term by term using the relations

$$(x^n K_n(x))' = -x^n K_{n-1}(x) \tag{1.6.8}$$

and

$$(x^n Y_n(x))' = x^n Y_{n-1}(x). \tag{1.6.9}$$

But by (1.3.16) and (1.3.17) the series in (1.6.4) converges absolutely for $a \geq 1$, and the convergence is uniform in any interval $[x_1, x_2] \subset (0, \infty)$, which justifies the differentiation term by term for $a \geq 2$. This argument proves (1.6.4) for $a = 1$ also.

The identity (1.6.5) is proved in the same way, starting from the formula

$$A_a(x,r) = \frac{1}{2\pi i} \int\limits_{(c)} \varphi(s,r) x^{s+a} (s(s+1)\cdots(s+a))^{-1}\, ds,$$

where $c > (k+1)/2$. For $a \geq 2$ the integration can be shifted to the line $\sigma = k/2 - 2/3$, where we use the functional equation (1.2.6) to rewrite $\varphi(s, h/k)$. This leads to integrals of the type I_3, which can be expressed in terms of the Bessel function J_{k+a} by (1.4.6). As a result, we obtain the assertion (1.6.5) for $a \geq 2$. The case $a = 1$ is deduced from this by differentiation as above, using the relation

$$(x^n J_n(x))' = x^n J_{n-1}(x) \tag{1.6.10}$$

and the asymptotic formula (1.3.15). We have now proved the theorem in the case $a \geq 1$, and the case $a = 0$ is postponed to § 1.8.

Estimating the series in (1.6.4) and (1.6.5) by absolute values, one **36**
obtains estimates for $\Delta_a(x,r)$ and $A_a(x,r)$. In the case $a = 1$, the result is as follows. □

COROLLARY. *For $x \gg k^2$ we have*

$$|\Delta_1(x, h/k)| \ll k^{3/2} x^{3/4} \tag{1.6.11}$$

and

$$|A_1(x,h/k)| \ll k^{3/2}x^{k/2+1/4}. \tag{1.6.12}$$

REMARK. *The error term* $\Delta_\circ(x,r)$ *coincides with* $\Delta(x,r)$, *defined in* (1.5.2). *The relations* (1.6.6) *and* (1.6.7) *remain valid also for* $a = 1$ *if* x *is not an integer. Thus, in particular,*

$$\Delta_1'(x,r) = \Delta(x,r) \quad \textit{for} \quad x > 0, x \notin \mathbb{Z}. \tag{1.6.13}$$

Together with (1.6.4) *for* $a = 1$, *this yields* (1.6.4) *for* $a = 0, x \notin \mathbb{Z}$ *as well, if the differentiation term by term of* (1.6.4) *for* $a = 1$ *can be justified. This step is not obvious but requires an analysis which is carried out in the next section. After that the remaining case* $a = 0, x \in \mathbb{Z}$ *is dealt with in § 1.8 by a limiting argument.*

In analogy with (1.6.13), *we have*

$$A_1'(x,r) = A(x,r) \quad \textit{for} \quad x > 0, x \notin \mathbb{Z}. \tag{1.6.14}$$

This relation, which follows immediately from the definition (1.6.2), *is the starting point in the proof of* (1.6.5) *for* $a = 0$.

1.7 Analysis of the Convergence of the Voronoi Series

In this section we are going to study the series (1.6.4) and (1.6.5) for
37 $a = 0$ as a preliminary for the proof of Theorem 1.3 for this remaining value of a. In virtue of the analogy between $d(n)$ and $a(n)$, we may restrict ourselves to the analysis of the first mentioned series. Thus, let us consider the series

(1.7.1)
$$x^{1/2}\sum_{n=1}^{\infty} d(n)n^{-1/2}\left\{e_k(-n\bar{h})Y_1(4\pi\sqrt{nx}/k) + (2/\pi)e_k(n\bar{h})K_1(4\pi\sqrt{nx}/k)\right\}.$$

For $k = 1$ this is - up to sign - Voronoi's expression for $\Delta(x)$, and the more general series (1.7.1) will also be called a **Voronoi series**.

From the point of view of convergence, the factor $x^{1/2}$ in front of the Voronoi series is of course irrelevant, but because we are going to consider x as a variable in the next section, we prefer keeping x explicit all the time.

Denote by $\sum(a,b;x)$ that part of the Voronoi series in which the summation is taken over the finite interval $[a,b]$. The following theorem gives an approximate formula for $\sum(a,b;x)$.

THEOREM 1.4. *Let $[x_1,x_2] \subset (0,\infty)$ be a fixed interval. Then uniformly for $x \in [x_1,x_2]$ and $2 \le a < b < \infty$ we have*

(1.7.2)
$$\sum(a,b;x) = Ax^{5/4}d(m)m^{-5/4}e_k(mh)\int\limits_{\sqrt{a}}^{\sqrt{b}} u^{-1}\sin(4\pi(\sqrt{m}-\sqrt{x})u/k)\,du$$
$$+o(a^{-14}\log a),$$

where m is the positive integer nearest to x (or any one of the two possibilities if $x > 1$ is half an odd integer), and A is a number depending only on k.

For the proof, we shall need the following elementary lemma.

Lemma 1.7. *Let $f \in C^2[a,b]$, where $0 < a < b$. Then* **38**

$$(1.7.3) \quad {\sum_{a\le n\le b}}' f(n)d(n)e_k(nh) = \int\limits_a^b (\Delta(t,h/k)f(t) - \Delta_1(t,h/k)f'(t))$$
$$+\int\limits_a^b \Delta_1(t,h/k)f''(t)\,dt + k^{-1}\int\limits_a^b (\log t + 2\gamma - 2\log k)f(t)\,dt.$$

Proof. According to (1.5.2), the sum under consideration is

$$\int\limits_a^b f(t)dD(t,h/k) = k^{-1}\int\limits_a^b f(t)(\log t + 2\gamma - 2\log k)\,dt + \int\limits_a^b f(t)d\Delta(t,h/k).$$

By repeated integrations by parts and using (1.6.13), we obtain

$$\begin{aligned}\int_a^b f(t)d\Delta(t,h/k) &= \int_a^b f(t)\Delta(t,h/k) - \int_a^b f'(t)\Delta(t,h/k)\,dt\\ &= \int_a^b (f(t)\Delta(t,h/k) - \Delta_1(t,h/k)f'(t))\\ &+ \int_a^b \Delta(t,h/k)f''(t)\,dt,\end{aligned}$$

and the formula (1.7.3) follows. □

Proof of Theorem 1.4. Because $h/k, x_1$, and x_2 will be fixed during the following discussion, we may ignore the dependence of constants on time.

First, by the asymptotic formulae (1.3.16) and (1.3.17) for Bessel functions, we have

(1.7.4)
$$\begin{aligned}\sum(a,b;x) = Ax^{1/4}\sum_{a\le n\le b} d(n)n^{-3/4}e_k(-n\bar{h})\cos(4\pi\sqrt{nx}/k - \pi/4)\\ + o(a^{-1/4}\log a).\end{aligned}$$

Lemma 1.7 is now applied to the sum here, with $-\bar{h}/k$ in place of h/k, and with

$$f(t) = x^{1/4}t^{-3/4}\cos(4\pi\sqrt{tx}/k - \pi/4).$$

39 The integrated terms in (1.7.3) are $\ll a^{-1/4}$, by (1.5.20) and (1.6.11). Also, by Lemma 1.6, the last term in (1.7.3) is $\ll a^{-1/4}\log a$. Thus it remains to consider the integral

(1.7.5)
$$\int_a^b \Delta_1(t,-\bar{h}/k)f''(t)\,dt.$$

In our case,

$$f''(t) = Ax^{5/4}t^{-7/4}\cos(4\pi\sqrt{tx}/k - \pi/4) + o(t^{-9/4}).$$

The contribution of the error term to (1.7.5) is $\ll a^{-1/2}$. Hence, in place of (1.7.5), it suffices to deal with the integral

$$x^{5/4}\int_a^b t^{-7/4}\Delta_1(t, -\overline{h}/k)\cos(4\pi\sqrt{tx}/k - \pi/4)\,dt. \tag{1.7.6}$$

For $\Delta_1(t, \overline{h}/k)$ we have the formula (1.6.4), which gives

$$\Delta_1(t, -\overline{h}/k) = At^{3/4}\sum_{n=1}^{\infty} d(n)n^{-5/4}e_k(nh)\cos(4\pi\sqrt{nt}/k + \pi/4) + o(t^{1/4}).$$

The contribution of the error term to (1.7.6) is $\ll a^{-1/2}$. Thus, the result of all the calculations so far is that

$$\sum(a, b; x) = Ax^{5/4}\sum_{n=1}^{\infty} d(n)n^{-5/4}e_k(nh)\times$$
$$\times\int_a^b t^{-1}\cos(4\pi\sqrt{tx}/k - \pi/4)\cos(4\pi\sqrt{nt}/k + \pi/4)\,dt + o(a^{-1/4}\log a).$$

Further, when the product of the cosines is written as the sum of two cosines, and the variable $u = \sqrt{t}$ is introduced, this equation takes the shape

$$\sum(a, b; x) = Ax^{5/4}\sum_{n=1}^{\infty} d(n)n^{-5/4}e_k(nh)\times$$
$$\times\left\{\int_{\sqrt{a}}^{\sqrt{b}} u^{-1}\cos(4\pi(\sqrt{n}+\sqrt{x})u/k)du - \int_{\sqrt{a}}^{\sqrt{b}} u^{-1}\sin(4\pi(\sqrt{n}-\sqrt{x})u/k)du\right\}$$
$$+ o(a^{-1/4}\log a).$$

By Lemma 1.6, the integrals here are $\ll a^{-1/2}|\sqrt{n}\pm\sqrt{x}|^{-1}$. Hence, **40**
if the integral standing on the right of (1.7.2) is singled out, the rest can be estimated uniformly as $\ll a^{-1/2}$. This completes the proof.

The problem on the nature of convergence of the Voronoi series is now reduced to the estimation of an elementary integral, and it is a simple matter to deduce the following

THEOREM 1.5. *The series* (1.7.1) *is boundedly convergent in any interval* $[x_1, x_2] \subset (0, \infty)$*, and uniformly convergent in any such interval free from integers. The same assertions hold for the series* (1.6.5) *for* $a = 0$*.*

Proof. The integral in (1.7.2) vanishes if $x = m$, and otherwise it tends to zero as a and b tend to infinity. Thus, in any case, the Voronoi series (1.7.1) converges. Moreover, if the interval $[x_1, x_2]$ contains no integer, then the integral in question is $\ll a^{-1/2}$ uniformly in this interval, where the Voronoi series is therefore uniformly convergent.

Finally, to prove the boundedness of the convergence in $[x_1, x_2]$, let x and m be as in Theorem 1.4, and put $x = m + \delta$, $c = \min(\sqrt{b}, \max(\sqrt{a}, 1/|\delta|))$. Then

$$\int\limits_{\sqrt{a}}^{\sqrt{b}} u^{-1}\sin(4\pi(\sqrt{m}-\sqrt{x})u/k)\,du = \int\limits_{\sqrt{a}}^{c} + \int\limits_{c}^{\sqrt{b}}$$
$$\ll \int\limits_{\sqrt{a}}^{c} |\sqrt{m}-\sqrt{x}|\,du + c^{-1}|\sqrt{m}-\sqrt{x}|^{-1} \ll 1.$$

Hence $\sum(a, b; x) \ll 1$ uniformly for all $0 < a < b$ and $x \in [x_1, x_2]$. □

1.8 Identities for $D(x, r)$ and $A(x, r)$

We are now in a position to prove Theorem 1.3 for $a = 0$. For conve-
41 nience of reference and because of the importance of this result, we state it separately as a theorem.

THEOREM 1.6. *For $x > 0$ we have*

$$D(x,h/k) = k^{-1}(\log x + 2\gamma - 1 - 2\log k)x + E(0,h/k) \tag{1.8.1}$$
$$-x^{1/2}\sum_{n=1}^{\infty} d(n)n^{-1/2}\left\{e_k(-n\bar{h})Y_1(4\pi\sqrt{nx}/k) + (2/\pi)e_k(n\bar{h})K_1(4\pi\sqrt{nx}/k)\right\}$$

and

$$A(x,h/k) = (-1)^{k/2}x^{k/2}\sum_{n=1}^{\infty} a(n)n^{-k/2}e_k(-n\bar{h})J_k(4\pi\sqrt{nx}/k). \tag{1.8.2}$$

Proof. Consider first the case when x is not an integer. Let $[x_1, x_2]$ be an interval containing x but no integer. Then the series on the right of (1.8.1) converges uniformly in this interval, by Theorem 1.5. Therefore the differentiation term by term of the identity (1.6.4) for $\Delta_1(x,h/k)$ is justified, which gives the formula (1.6.4) for $a = 0$, and thus also the formula (1.8.1) (see the remark in the end of § 1.6).

The case when $x = m$ is an integer will now be settled by Theorem 1.4 and the previous case. Let

$$S(x) = -x^{1/2}\sum_{n=1}^{\infty} d(n)n^{-1/2}$$
$$\left\{e_k(-n\bar{h})Y_1(4\pi\sqrt{nx}/k) + (2/\pi)e_k(n\bar{h})K_1(4\pi\sqrt{nx}/k)\right\}.$$

Then $S(x) = \Delta(x,h/k)$ if $x > 0$ is not an integer, and $S(m)$ is the value of $\Delta(m,h/k)$ asserted. We are going to show that

$$\frac{1}{2}\lim_{\delta\to o+}(D(m+\delta,h/k) + D(m-\delta,h/k)) \tag{1.8.3}$$
$$= k^{-1}(\log m + 2\gamma - 1 - 2\log k)m + E(0,h/k) + S(m).$$

Because $\frac{1}{2}(D(m+\delta,h/k) + D(m-\delta,h/k))$ equals $D(m,h/k)$ for all $\delta \in (0,1)$, this implies (1.8.1) for $x = m$.

First, the leading terms of the formula for $D(m \pm \delta, h/k)$, just proved, **42** give in the limit the leading terms on the right of (1.8.3). Therefore, it remains to prove that

$$\lim_{\delta\to 0+}(S(m+\delta)+S(m-\delta)-2S(m))=0. \tag{1.8.4}$$

Where

$$S(x)=S_1(x)+S_2(x)+S_3(x),$$

where the range of summation in the sums $S_i(x)$ is, respectively, $[1,\delta^{-1})$, $[\delta^{-1},\delta^{-3}]$, and (δ^{-3},∞). We estimate separately the quantities

$$\Delta_i(\delta)=S_i(m+\delta)+S_i(m-\delta)-2S_i(m).$$

Consider first $\Delta_1(\delta)$, writing

$$\Delta_1(\delta)=\sum_{n<\delta^{-1}} d(n)\alpha_n(\delta).$$

By the formulae

$$(x^{1/2}Y_1(4\pi\sqrt{nx}/k))'=2\pi(\sqrt{n}/k)Y_\circ(4\pi\sqrt{nx}/k), \tag{1.8.5}$$

$$(x^{1/2}K_1(4\pi\sqrt{nx}/k))'=-2\pi(\sqrt{n}/k)K_\circ(4\pi\sqrt{nx}/k), \tag{1.8.6}$$

which follow from (1.6.8) and (1.6.9), we find that $\alpha_n(\delta)\ll n^{-1/4}\delta$. Hence

$$\Delta_1(\delta)\ll\delta^{1/4}\log(1/\delta). \tag{1.8.7}$$

Next, by definition,

(1.8.8)

$$\Delta_2(\delta)=-\sum\left(\delta^{-1},\delta^{-3};m+\delta\right)-\sum\left(\delta^{-1},\delta^{-3};m-\delta\right)+2\sum\left(\delta^{-1},\delta^{-3};m\right).$$

To facilitate comparisons between the sums on the right, we write the factor $(m\pm\delta)^{5/4}$ in front of the formula (1.7.2) for $\sum(\delta^{-1},\delta^{-3};m\pm\delta)$
43 as $m^{5/4}+0(\delta)$. Then, by (1.8.8) and (1.7.2),

$$\Delta_2(\delta)\ll\Bigg|\int\limits_{\delta^{-1/2}}^{\delta^{-3/2}} u^{-1}\Big\{\sin(4\pi(\sqrt{m}-\sqrt{m+\delta})u/k)$$

$$+ \sin(4\pi(\sqrt{m} - \sqrt{m-\delta})u/k)\Big\}\, du\Bigg| + \delta^{1/4}\log(1/\delta).$$

The expression in the curly brackets is estimated as follows:

$$\begin{aligned}|\{\cdots\}| = |2\sin\left(2\pi\left(\sqrt{m+\delta} + \sqrt{m-\delta} - 2\sqrt{m}\right)u/k\right)\\ \cos\left(2\pi\left(\sqrt{m-\delta} - \sqrt{m-\delta}\right)u/k\right)|\\ \ll \delta^2 u.\end{aligned}$$

Hence

$$\Delta_2(\delta) \ll \delta^{1/4}\log(1/\delta). \tag{1.8.9}$$

Finally, by Theorem 1.4 and Lemma 1.6, we have for any $b > \delta^{-3}$

$$\sum(\delta^{-3}, b; m \pm \delta) \ll \delta^{3/2}\delta^{-1} + \delta^{3/4}\log(1/\delta) \ll \delta^{1/2},$$

and the same estimate holds also for $\sum(\delta^{-3}, b; m)$. Hence

$$\Delta_3(\delta) \ll \delta^{1/2}, \tag{1.8.10}$$

Now (1.8.7), (1.8.9), and (1.8.10) give together

$$S(m+\delta) + S(m-\delta) - 2S(m) \ll \delta^{1/4}\log(1/\delta),$$

and the assertion (1.8.4) follows. This completes the proof of (1.8.1), and (1.8.2) can be proved likewise. □

1.9 The Summation Formulae

We are now in a position to deduce the main results of this chapter, the summation formulae of the Voronoi type involving an exponential factor.

THEOREM 1.7. *Let* $0 < a < b$ *and* $f \in C^1[a,b]$. *Then* **44**

$$\sideset{}{'}\sum_{a \le n \le b} d(n)e_k(nh)f(n) = k^{-1}\int_a^b (\log x + 2\gamma - 2\log k)f(x)\,dx + k^{-1} \tag{1.9.1}$$

$$\sum_{n=1}^{\infty} d(n) \int_a^b \{-2\pi e_k(-n\bar{h}) Y_\circ(4\pi\sqrt{nx}/k) + 4e_k(n\bar{h}) K_\circ(4\pi\sqrt{nx}/k)\} f(x)\,dx$$

and

(1.9.2) $$\sideset{}{'}\sum_{a\le n\le b} a(n) e_k(nh) f(n)$$

$$= 2\pi k^{-1}(-1)^{k/2} \sum_{n=1}^{\infty} a(n) e_k(-n\bar{h}) n^{(k-1)/2} \int_a^b x^{(k-1)/2} J_{k-1}(4\pi\sqrt{nx}/k) f(x)\,dx.$$

The series in (1.9.1) *and* (1.9.2) *are boundedly convergent for a and b lying in any fixed interval* $[x_1, x_2] \subset (0, \infty)$.

Proof. We may suppose that $0 < a < 1$, for the general case then follows by subtraction. Accordingly, the sum in (1.9.1) is

$$\sideset{}{'}\sum_{n\le b} d(n) e_k(nh) f(n) = \int_a^b f(x) dD(x, h/k).$$

By an integration by parts, this becomes

(1.9.3) $$f(b) D(b, h/k) - \int_a^b f'(x) D(x, h/k)\,dx.$$

We substitute $D(x, h/k)$ from the identity (1.8.1), noting that the resulting series can be integrated term by term because of bounded convergence. Thus

$$\int_a^b f'(x) D(x, h/k)\,dx = \int_a^b f'(x)\Big\{k^{-1}(\log x + 2\gamma - 1$$

$$- 2\log k)x + E(0, h/k)\Big\}\,dx - \sum_{n=1}^{\infty} d(n) n^{-1/2} \int_a^b f'(x) x^{1/2}$$

$$\left\{e_k(-h\bar{h})Y_1(4\pi\sqrt{nx}/k) + (2/\pi)e_k(n\bar{h})K_1(4\pi\sqrt{nx}/k)\right\}\,dx.$$

This is transformed by another integration by parts, using also (1.8.5) and (1.8.6). The integrated terms then yield $f(b)D(b, h/k)$, again by (1.8.1), and the right hand side of the preceding equation becomes 45

$$f(b)D(b, h/k) - k^{-1}\int_a^b (\log x + 2\gamma - 2\log k)f(x)\,dx$$

$$+ 2\pi k^{-1}\sum_{n=1}^{\infty} d(n)\int_a^b \left\{e_k(-n\bar{h})Y_\circ(4\pi\sqrt{nx}/k) - (2/\pi)e_k(n\bar{h})K_\circ(4\pi\sqrt{nx}/k)\right\}$$
$$f(x)\,dx.$$

Substituting this into (1.9.3) we obtain the formula (1.9.1). It is also seen that the boundedness of the convergence of the series (1.9.1). □

The proof of (1.9.2) is analogously based on the identity (1.8.2) and the formula

$$(x^{k/2}J_k(4\pi\sqrt{nx}/k))' = 2\pi(\sqrt{n}/k)x^{(k-1)/2}J_{k-1}(4\pi\sqrt{nx}/k),$$

which follows from (1.6.10).

Notes

Our estimate (1.1.4) for $E(0, h/k)$ is stronger by a logarithm than the bound $E(0, h/k) \ll k\log^2 2k$ of Estermann [8].

The value $\zeta(0) = -1/2$ can also be deduced from (1.1.9) by observing that for fixed $s \neq 1$ the function $\zeta(s, a)$ is continuous in the interval $0 < a \leq 1$ (this follows e.g. from the loop integral representation (2.17.2) of $\zeta(s, a)$ in [27]).

The integrals I_1, I_2, and I_3 in § 1.4 can also be evaluated by the inversion formula for the Mellin transformation, using the Mellin transform pairs (see (7.9.11), (7.9.8), and (7.9.1) in [26])

$$\begin{aligned}
&x^{-\nu}K_\nu(x), \qquad && 2^{s-\nu-2}\Gamma\left(\frac{1}{2}s\right)\Gamma\left(\frac{1}{2}s-\nu\right),\\
&x^{-\nu}Y_\nu(x), \qquad && -2^{s-\nu-1}\pi^{-1}\Gamma\left(\frac{1}{2}s\right)\Gamma\left(\frac{1}{2}s-\nu\right)\cos\left(\left(\frac{1}{2}s-\nu\right)\pi\right),\\
&x^{-\nu}J_\nu(x), \qquad && 2^{s-\nu-1}\Gamma\left(\frac{1}{2}s\right)\,\Gamma\left(\nu-\frac{1}{2}s+1\right).
\end{aligned}$$

46

Theorems 1.1, 1.2, 1.6 and 1.7 (for sums involving $d(n)$) appeared in [18]. The error terms in Theorem 1.2 could be imporved. In fact, Tong [28] proved that

$$(*) \qquad \int\limits_2^X \Delta^2(x)\,dx = C_1X^{3/2} + o(X\log^5 X)$$

(for a simple proof, see Meurman [22]), and similarly it can be shown that (1.5.22) and (1.5.23) hold with error terms $o(k^2X\log^5 X)$ and $o(k^2X^k\log^5 X)$, respectively. An analogue of (*) for the error term $E(T)$ in (0.6) was obtained by Meurman in the above mentioned paper.

The general summation formulae of Berndt (see [3], in particular part V) cover (1.9.2) but not (1.9.1), because the functional equation (1.1.2) for $E(s,r)$ is not of the form required in Berndt's papers.

The novelty of the proof of Theorem 1.6 for integer values of x lies in the equation (1.8.3).

Analogues of the results in this and subsequent chapters can be proved for sums and Dirichlet series involving Fourier coefficients of **Maass waves**. H. Maass [21] introduced non-holomorphic cusp forms as auto-morphic functions in the upper half-plane H for the full modular group, which are eigenfunctions of the hyperbolic Laplacian $-y^2(\partial_x^2+\partial_y^2)$ and square integrable over the fundamental domain

$$\left\{z = x+yi \,\middle|\, -\frac{1}{2} \le x \le \frac{1}{2}, y>0, |z|\ge 1\right\}$$

47 with respect to the measure $y^{-2}\,dx\,dy$. Such functions, which are more-

over orthonormal with respect to the Petersson inner product, eigenfunctions of all Hecke operators T_n, and either even or odd as functions of x, are called Maass waves. A Maass wave f, which is associated with the eigenvalue $1/4 + r^2 (r \in \mathbb{R})$ of the hyperbolic Laplacian and an even function of x, can be expanded to a Fourier series of the form (see [20])

$$f(z) = f(x+yi) = \sum_{n=1}^{\infty} a(n) y^{1/2} K_{ir}(2\pi ny)\cos(2\pi nx).$$

It has been conjectured that $a(n) \ll n^{\epsilon}$, but this hypothesis-an analogue of (1.2.5) - is still unsettled. The weaker estimate $a(n) \ll n^{1/5+\epsilon}$ has been proved by J.-P. Serre.

As an analogue of the Dirichlet series $\varphi(s)$, one may define the L-function

$$L(s) = \sum_{n=1}^{\infty} a(n) n^{-s}.$$

This can be continued analytically to an entire function satisfying the functional equation (see [7])

$$\pi^{-s} L(s)\Gamma\left(\frac{s+ir}{2}\right)\Gamma\left(\frac{s-ir}{2}\right) = \pi^{s-1} L(1-s)\Gamma\left(\frac{1-s+ir}{2}\right)\Gamma\left(\frac{1-s-ir}{2}\right).$$

More generally, it can be proved that the function

$$L(s, h/k) = \sum_{n=1}^{\infty} a(n)\cos(2\pi nh/k) n^{-s}$$

has the functional equation

$$\begin{aligned}(k/\pi)^s L(s, h/k)\Gamma\left(\frac{s+ir}{2}\right)\Gamma\left(\frac{s-ir}{2}\right)\\ = (k/\pi)^{1-s} L(1-s, \bar{h}/k)\Gamma\left(\frac{1-s+ir}{2}\right)\Gamma\left(\frac{1-s-ir}{2}\right),\end{aligned}$$

which is an analogue (1.2.6). Results of this kind can be proved for **48**
"odd" Maass waves as well, and having the necessary functional equations at disposal, one may pursue the analogy between holomorphic and non-holomorphic cusp forms further.

Chapter 2

Exponential Integrals

49 AN INTEGRAL OF the type

$$\int_a^b g(x)e(f(x))\,dx$$

is called an **exponential integral**. The object of various "saddle-point theorems" is to give the value of such an integral approximately in terms of the possible **saddle point** $x_\circ \in (a,b)$ satisfying, by definition, the equation $f'(x_\circ) = 0$. Results of this kind can be found e.g. in [27], Chapter IV, and in [13], § 2.1.

For our purposes, the existing saddle-point theorems are some-times too crude. However, more precise results can be obtained for smoothed exponential integrals

$$\int \eta(x)g(x)e(f(x))\,dx,$$

where $\eta(x)$ is a suitable smooth weight function. The present chapter is devoted to such integrals. The main result of § 2.1 is a saddle-point theorem, and § 2.2 deals with the case when no saddle point exists.

2.1 A Saddle-Point Theorem for Smoothed Exponential Integrals

It will be convenient to single out a linear part from the function f, writing thus $f(x) + \alpha x$ in place of $f(x)$. Accordingly, our exponential integral reads

$$I = I(a,b) = \int_a^b g(x) e(f(x) + \alpha x)\, dx = \int_a^b h(x)\, dx, \tag{2.1.1}$$

say, where α is a real number.

For a given positive integer J and a given real number $U > 0$, we **50**
define the weight function $\eta_J(x)$ by the equation

$$\begin{aligned} I_J = I_J(a,b) &= U^{-J} \int_0^U du_1 \cdots \int_o^U du_J \int_{a+u}^{b-u} h(x)\, dx \\ &= \int_a^b \eta_J(x) h(x)\, dx, \end{aligned} \tag{2.1.2}$$

where $u = u_1 + \cdots + u_J$. We suppose that $JU < (b-a)/2$. Also, we define $I_\circ = I$, and interpret $\eta_\circ(x)$ as the characteristic function of the interval $[a,b]$. Clearly $0 < \eta_J(x) \leq 1$ for $x \in (a,b)$, and $\eta_J(X) = 1$ for $a + JU \leq x \leq b - JU$.

The following lemma gives an alternative expression for the integral I_J.

Lemma 2.1. *For any $c \in (a + JU, b - JU)$ we have*

$$\begin{aligned} I_J = (J!U^J)^{-1} \sum_{j=\circ}^{J} \binom{J}{j} (-1)^j \\ \left\{ \int_{a+jU}^{c} (x - a jU)^J h(x)\, dx \int_c^{b-jU} (b - jU - x)^J h(x)\, dx \right\}. \end{aligned} \tag{2.1.3}$$

Proof. The case $J = 0$ is trivial, and otherwise the assertion can be verified by induction using the recursion formula

$$I_J(a,b) = U^{-1} \int_{\circ}^{U} I_{J-1}(a+u_J, b-u_J)\, du_J. \tag{2.1.4}$$

For completeness we give some details of the calculations.

Supposing that (2.1.3) holds for the index $J-1$, we have by (2.1.4)

$$I_J = U^{-1}\left((J-1)!U^{J-1}\right)^{-1} \sum_{j=0}^{J-1} \binom{J-1}{j} (-1)^j \int_o^U \left\{ \int_{a+jU}^{c} (\max(x-a-jU- u_J, 0))^{J-1} h(x)\, dx + \int_c^{b-jU} (\max(b-jU-u_J-x, 0))^{J-1} h(x)\, dx \right\} du_J$$

$$= \left(J!U^J\right)^{-1} \sum_{j=0}^{J-1} \binom{J-1}{j} (-1)^{j+1} \left\{ \int_{a+(j+1)U}^{c} (x-a-(j+1)U)^J h(x)\, dx + \int_c^{b-(j+1)U} (b-(j+1)U-x)^J h(x)\, dx \right\} + \left(J!U^J\right)^{-1} \sum_{j=0}^{J-1} \binom{J-1}{j} (-1)^j \left\{ \int_{a+jU}^{c} (x-ajU)^J h(x)\, dx \int_c^{b-jU} (b-jU-x)^J h(x)\, dx \right\}$$

$$= \left(J!U^J\right)^{-1} \sum_{j=1}^{J-1} \left(\binom{J-1}{j-1} + \binom{J-1}{j} \right) (-1)^j \left\{ \int_{a+jU}^{c} (x-ajU)^J h(x)\, dx + \int_c^{b-jU} (b-jU-x)^J h(x)\, dx \right\}$$

$$+ \left(J!U^J\right)^{-1} \left\{ (-1)^J \left(\int_{a+JU}^{c} (x-a-JU)^J h(x)\, dx + \int_c^{b-JU} \right.\right.$$

$$(b - JU - x)^J h(x)\,dx + \int_a^c (x-a)^J h(x)\,dx + \int_c^b (b-x)^J h(x)\,dx \Bigg\},$$

which yields (2.1.3) for the index J. □ **51**

Remark. As a corollary of (2.1.3), we obtain the identity

$$\left(J!U^J\right)^{-1} \sum_{j=0}^{J} \binom{J}{j} (-1)^j (z - jU)^J = 1. \tag{2.1.5}$$

Indeed, this holds for $z = x - a$ with $a + JU \leq x \leq c$, since $\eta_J(x) = 1$ in this interval. Then, by analytic continuation, (2.1.5) holds for all complex z. Of course, (2.1.5) can also be verified directly in an elementary way.

Before going into formulations of the saddle-point theorems, it is convenient to list for future reference a number of conditions on the functions f and g. **52**

(i) $f(x)$ is real for $a \leq x \leq b$.

(ii) f and g are holomorphic in the domain

$$D = \left\{ z \,\middle|\, |z - x| < \mu(x) \quad \text{for some} \quad x \in [a, b] \right\},$$

where $\mu(x)$ is a positive function, which is continuous and piecewise continuously differentiable in the interval $[a, b]$.

(iii) There are positive functions $F(x)$ and $G(x)$ such that for $|z - x| < \mu(x)$ and $a \leq x \leq b$

$$|g(z)| \ll G(x),$$
$$\left| f'(z) \right| \ll F(x)\mu(x)^{-1}.$$

(iv) $f''(x) > 0$ and

$$f''(x) \gg F(x)\mu(x)^{-2}$$

for $a \leq x \leq b$.

(v) $\mu'(x) \ll 1$ for $a \leq x \leq b$ whenever $\mu'(x)$ exists.

(vi) $F(x) \gg 1$ for $a \leq x \leq b$.

Since $f'(x) + \alpha$ is monotonically increasing by (iv), it has at most one zero, say at $x_\circ$, in the interval (a, b). Whenever terms involving $x_\circ$ occur in the sequel, it should be understood that these terms are to be omitted if $x_\circ$ does not exist.

Remark. By Cauchy's integral formula for the derivatives of a holo-
53 morphic function, it follows from (ii) and (iii) that

$$\left|f^{(n)}(x)\right| \ll n! 2^n F(x) \mu(x)^{-n} \quad \text{for} \quad a \leq x \leq b, n = 1, 2, \ldots \tag{2.1.6}$$

Hence the conditions (iii) and (iv) together imply that

$$f''(x) \asymp F(x) \mu(x)^{-2} \quad \text{for} \quad a \leq x \leq b. \tag{2.1.7}$$

Next we state two saddle-point theorems. The former of these, due to F.V. Atkinson ([2], Lemma 1), deals with the integral I, and the latter is its generalization to I_J. Let

$$E_J(x) = G(x) \left(\left| f'(x) + \alpha \right| + f''(x)^{1/2} \right)^{-J-1}. \tag{2.1.8}$$

In the next theorem, and also later in this chapter, the unspecified constant A will be supposed to be positive.

Theorem 2.1. *Suppose that the conditions (i) - (v) are satisfied, and let I be defined as in* (2.1.1). *Then*

$$\begin{aligned} I = {} & g(x_\circ) f''(x_\circ)^{-1/2} e(f(x_\circ) + \alpha x_\circ + 1/8) \\ & + o\left(\int\limits_a^b G(x) \exp(-A|\alpha|\mu(x) - AF(x))\, dx \right) \\ & + o\left(G(x_\circ)\, \mu(x_\circ)\, F(x_\circ)^{-3/2} \right) + o\left(E_\circ(a)\right) + o\left(E_\circ(b)\right). \end{aligned} \tag{2.1.9}$$

Theorem 2.2. *Let $U > 0$, J a fixed nonnegative integer, $JU < (b-a)/2$, and suppose that the conditions (i)-(vi) are satisfied. Suppose also that*

$$U \gg \delta(x_\circ)\mu(x_\circ)F(x_\circ)^{-1/2}, \tag{2.1.10}$$

where $\delta(x)$ is the characteristic function of the union of the intervals $(a, a+JU)$ and $(b-JU, b)$. Let I_J be defined as in (2.1.2). *Then*

$$\begin{aligned} I_J &= \xi_J(x_\circ)g(x_\circ)f''(x_\circ)^{-1/2}e\left(f(x_\circ)+\alpha x_\circ + 1/8\right) \\ &\quad + o\left(\int\limits_a^b (1+(\mu(x)/U)^J)G(x)\exp(-A|\alpha|\mu(x) - AF(x)\,dx\right) \\ &\quad + o\left(\left(1+\delta(x_\circ)F(x_\circ)^{1/2}\right)G(x_\circ)\mu(x_\circ)F(x_\circ)^{-3/2}\right) \\ &\quad + o\left(U^{-J}\sum_{j=0}^{J}(E_J(a+jU)+E_J(b-jU))\right), \end{aligned} \tag{2.1.11}$$

where **54**

(2.1.12)

$$\xi_J(x_\circ) = 1 \quad \textit{for} \quad a + JU < x_c < b - JU,$$

(2.1.13)

$$\xi_J(x-\circ) = \left(J!U^J\right)^{-1}\sum_{j=0}^{j_1}\binom{J}{j}(-1)^j\sum_{\circ\le\nu\le J/2}c_\nu f''(x_\circ)^{-\nu}(x_\circ - a - jU)^{J-2\nu}$$

for $a < x_\circ \le a + JU$ with j_1 the largest integer such that $a + j_1U < x_\circ$,
(2.1.14)

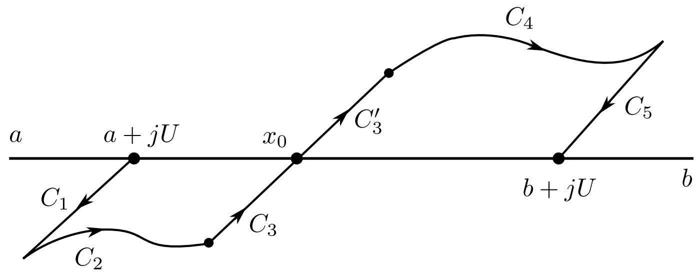
$$\xi_J(x_\circ) = (J!U^J)^{-1}\sum_{j=0}^{j_2}\binom{J}{j}(-1)^j\sum_{\circ\le\nu\le J/2}c_\nu f''(x_\circ)^{-\nu}(b - jU - x_\circ)^{J-2\nu}$$

for $b - JU \le x_\circ < b$ with j_2 the largest integer such that $b - j_2U > x_\circ$. The c_ν are numerical constants.

Proof. We follow the argument of Atkinson [2] with some modifications caused by the smoothing. There are four cases as regards the saddle point $x_\circ$: 1) $a + JU < x_\circ < b - JU$, 2) $x_\circ$ does not exist, 3) $a < x_\circ \le a + JU$, 4) $b - JU \le x_\circ < b$. Accordingly the proof will be in four parts.

1) Suppose first that $a + JU < x_\circ < b - JU$. Put $\lambda(x) = \beta\mu(x)$, where β is a small positive constant. Choose $c = x_\circ$ in the expression (2.1.3) for I_J.

   The intervals of integration in (2.1.3) are replaced by the paths shown in the figure. Here C_1, C_3, C_3', and C_5 are, respectively,



55

   the line segments $[a+jU, a+jU-(1+i)\lambda(a+jU)]$, $[x_o-(1+i)\lambda(x_o), x_o]$, $[x_o, x_o + (1_i)\lambda(x_o)]$, and $[b - jU + (1+i)\lambda(b - jU), b - jU]$. The curve C_2 is defined by $z = x - (1+i)\lambda(x), a + jU \leq x \leq x_o$, and analogously C_4 is defined by $z = x + (1+i)\lambda(x), x_o \leq x \leq b - jU$. By the holomorphicity assumption (ii) and Cauchy's integral theorem, we have

   (2.1.15)

$$I_J = \left(J!U^J\right)^{-1} \sum_{j=0}^{J} \binom{J}{j} (-1)^j \left\{ \int\limits_{C_1+C_2+C_3} (z - a - jU)^J h(z)\, dz \right.$$

$$\left. + \int\limits_{C_3'+C_4+C_5} (b - jU - z)^J h(z)\, dz \right\}.$$

   To estimate the modulus of $h(z)$, we need an upper bound for $Re\{2\pi i(f(z) + \alpha z)\}$. Let $z = x + (1+i)y$, where $a \leq x \leq b$ and $|y| \leq \lambda(x)$. By Taylor's theorem,
   (2.1.16)

$$f(z) + \alpha z = f(x) + \alpha x + (f'(x) + \alpha)(1+i)y + i f''(x) y^2 + \theta(x, y),$$

where
$$\theta(x, y) = \sum_{n=3}^{\infty} \left(f^{(n)}(x)/n!\right)((1 + i)y)^n.$$

By (2.1.6), we have
$$|\theta(x, y)| \ll F(x)|y|^3 \mu(x)^{-3},$$

so that by (iv) 56
$$|\theta(x, y)| \leq \frac{1}{2} y^2 f''(x)$$

if β is supposed to be sufficiently small. Then (2.1.16) gives

$$Re\{2\pi i(f(z) + \alpha z)\} \leq -2\pi(f'(x) + \alpha)y - \pi f''(x)y^2 \tag{2.1.17}$$

for $a \leq x \leq b$ and $|y| \leq \lambda(x)$.

Consider, in particular, the case $y = sgn(f'(x) + \alpha)\lambda(x)$, which occurs in the estimation of the integrals over C_2 and C_4. The right hand side of (2.1.17) is now at most
$$-A|f'(x) + \alpha|\mu(x) - AF(x).$$

In the cases $|\alpha| \geq 2|f'(x)|$ and $|\alpha| < 2|f'(x)|$ this is
$$\leq -A|\alpha|\mu(x) - AF(x)$$

and
$$\leq -AF(x) \leq -A|\alpha|\mu(x) - AF(x),$$

respectively. Hence for $z \in C_2 \cup C_4$

$$|h(z)| \ll G(x)\exp(-A|\alpha|\mu(x) - AF(x)). \tag{2.1.18}$$

The paths C_i for $i = 1, 2, 4$ and 5 depend on j, so that for clarity we denote them by $C_i(j)$. Let us first estimate the contribution of the integrals over the $C_2(j)$ and $C_4(j)$ to I_J. By the identity (2.1.5), the integrands in (2.1.15) combine to give simply $h(z)$ on $C_2(j) \cup C_3 \cup C_3' \cup C_4(j)$, hence in particular on $C_2(j) \cup C_4(j)$. Thus, by (2.1.18) and the assumption (v), viz. $\mu'(x) \ll 1$, the integrals in (2.1.15) restricted to $C_2(j)$ and $C_4(j)$ contribute 57

$$\ll \int_{a+JU}^{b-JU} G(x)\exp(-A|\alpha|\mu(x) - AF(x))\,dx.$$

Integrals over the other parts of the $C_2(j)$ and $C_4(j)$ are estimated similarly, but noting that the function in front of $h(z)$ is now $\ll 1+(\mu(x)/U)^J$. In this way it is seen that the integrals over the $C_2(j)$ and $C_4(j)$ give together at most the first error term in (2.1.11).

Next we turn to the integrals over the $C_1(j)$ and $C_5(j)$. By (2.1.17) we have

$$\begin{aligned}\int_{C_1(j)} (z-a-jU)^J h(z)\,dz \ll G(a+jU)\int_{\circ}^{\infty} y^J \exp(-2\pi|f'(a+jU)\\ + \alpha|y - \pi f''(a+jU)y^2)\,dy\\ \ll E_J(a+jU),\end{aligned}$$

and similarly for the integrals over the $C_5(j)$. Hence these integrals contribute the last error term in (2.1.11).

Finally, as was noted above, the integrals over $C_3 + C_3'$ give together the integral

$$K = (1+i)\int_{-\lambda(x_\circ)}^{\lambda(x_\circ)} h(x_\circ + (1+i)y)\,dy. \tag{2.1.19}$$

Applying Taylor's theorem and similar arguments as in the proof of (2.1.17), we find that for $|y| \le \lambda(x_\circ)$

$$g(x_o + (1+i)y) = g(x_o) + g'(x_o)(1+i)y + o\left(G(x_o)\mu(x_o)^{-2}y^2\right), \tag{2.1.20}$$

or, more crudely,

$$g(x_\circ + (1+i)y) = g(x_\circ) + o\left(G(x_\circ)\mu(x_\circ)^{-1}|y|\right), \tag{2.1.21}$$

and analogously

(2.1.22)

$$f(x_\circ + (1+i)y) + \alpha(x_\circ + (1+i)y) = f(x_\circ) + \alpha x_\circ$$
$$+ i f''(x_\circ) y^2 + \frac{1}{6} f'''(x_\circ) \quad (1+i)^3 y^3 + o\left(F(x_\circ)\mu(x_\circ)^{-4} y^4\right)$$

(2.1.23)

$$= f(x_\circ) + \alpha x_\circ + i f''(x_\circ) y^2 + o\left(F(x_\circ)\mu(x_\circ)^{-3} |y|^3\right).$$

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Let

(2.1.24) $$v = \lambda(x_\circ) F(x_\circ)^{-1/3},$$

and write $K = K_1 + K_2 + K_3$, where the integrals K_1, K_2, and K_3 are taken over the intervals $[-\lambda(x_\circ), -v], [-v, v]$, and $[v, \lambda(x_\circ)]$, respectively.

First, by (2.1.17) we have

$$\begin{aligned} K_1 + K_3 &\ll G(x_\circ) \int\limits_v^\infty \exp\left(-\pi f''(x_\circ) y^2\right) dy \\ &\ll G(x_\circ) v^{-1} f''(x_\circ)^{-1} \exp\left(-\pi v^2 f''(x_\circ)\right) \\ &\ll G(x_\circ)\mu(x_\circ) F(x_\circ)^{-2/3} \exp\left(-A F(x_\circ)^{1/3}\right), \end{aligned}$$

whence by (vi)

(2.1.25) $$K_1 + K_3 \ll G(x_\circ)\mu(x_\circ) F(x_\circ)^{-3/2}.$$

The integral K_2, which will give the saddle-point term is evaluated by applying (2.1.20) and (2.1.22). The latter implies that for $|y| \leq v$

(2.1.26) $$\begin{aligned} &e\left(f\left(x_\circ + (1+i)\right) + \alpha\left(x_\circ + (1+i)y\right)\right) \\ &= e\left(f(x_\circ) + \alpha x_\circ\right) \exp\left(-2\pi f''(x_\circ) y^2\right) \times \end{aligned}$$

$$\times \left\{ 1 + \frac{1}{3}\pi i f''(x_\circ)\,(1+i)^3 y^3 + o\left(F(x_\circ)\mu(x_\circ)^{-4}y^4\right) \right.$$
$$\left. + o\left(F(x_\circ)^2\mu(x_\circ)^{-6}y^6\right) \right\};$$

note that the last two terms in (2.1.22) are $\ll 1$ by the choice
59 (2.1.24) of v. When this equation is multiplied by (2.1.20) and the product is integrated over the interval $[-v, v]$, the integrals of those explicit terms involving odd powers of y vanish, and we end up with

(2.1.27)
$$K_2 = (1+i)g(x_\circ)e\left(f(x_\circ) + \alpha x_\circ\right) \int\limits_{-v}^{v} \exp\left(-2\pi f''(x_\circ)y^2\right) dy$$
$$+o\left(G(x_\circ) \int\limits_{-v}^{v} \exp\left(-2\pi f''(x_\circ)y^2\right) \left(\mu(x_\circ)^{-2}y^2 + F(x_\circ)\mu(x_\circ)^{-4}y^4 + \right.\right.$$
$$\left.\left. F(x_\circ)^2\mu(x_\circ)^{-6}y^6\right) dy\right).$$

In the main term, the integration can be extended to the whole line with an error $\ll G(x_\circ)\mu(x_\circ)F(x_\circ)^{-3/2}$, and since

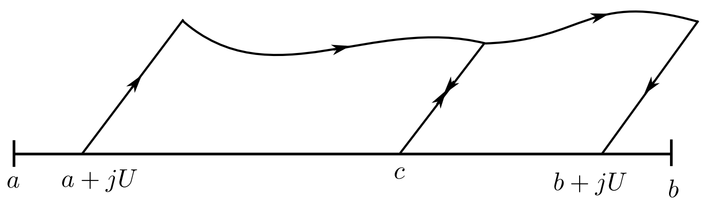
$$\int\limits_{-\infty}^{\infty} \exp\left(-cy^2\right) dy = (\pi/c)^{1/2} (c > 0), \tag{2.1.28}$$

the leading term in (2.1.27) gives the leading term in (2.1.11) with $\xi(x_\circ) = 1$, in accordance with (2.1.12). Further, as a generalization of (2.1.28), we have

$$\int\limits_{-\infty}^{\infty} \exp\left(-cy^2\right) y^{2\nu}\, dy = d_\nu c^{-\nu-1/2} (c > 0, \nu \geq 0) \tag{2.1.29}$$

where the d_ν are certain numerical constants, and by using this the error terms in (2.1.27) are seen to be $\ll G(x_\circ)\mu(x_\circ)F(x_\circ)^{-3/2}$.

2) Suppose next that $x_\circ$ does not exist. Then $f'(x) + \alpha$ is of the same sign, say positive, in the whole interval (a, b). Let c be a point in the interval $(a + JU, b - JU)$, write I_J as in (2.1.3), and transform the integrals over the intervals $[a + jU, c]$ and $[c, b - jU]$ using the contours shown in the figure, where the curvilinear part is defined by $z = x + (1 + i)\lambda(x)$ with $a + jU \leq x \leq b - jU$. Observe that the integrals over the segment $[c, c + (1 + i)\lambda(c)]$ cancel, by (2.1.5). Integrals over the other parts of the contours are estimated as in the preceding case, and these contribute the first and last error 60
term in (2.1.11).



If $f'(x) + \alpha$ is negative, then an analogous contour is used in the lower half-plane.

3) Consider now the case $a < x_\circ \leq a + JU$. Again choose $c \in (a + JU, b - JU)$. In (2.1.3) the integrals over $[c, b - jU]$ are written as in the preceding case, and likewise the integrals over $[a + jU, c]$ for $j > j_1$, in which case the saddle point $x_\circ$ does not lie in the (open) interval of integration. On the other hand, for $j \leq j_1$ the contour is of a shape similar to the first case. Only the last mentioned integrals require a separate treatment; the others give error terms as before.

A new complication is that the sum over $j \leq j_1$ of the integrals over the line segment $L = [x_\circ - (1 + i)\lambda(x_\circ), x_\circ + (1 + i)\lambda(x_\circ)]$ cannot be written as an integral of $h(z)$, but the integrals have to be evaluated separately. Other parts of the contours do not present any new difficulties.

Thus, consider the integral 61

$$K = U^{-J} \int_L (z - a - jU)^J h(z)\, dz. \tag{2.1.30}$$

This is of the same type as the integral K in (2.1.19) - with $g(z)$ replaced by $U^{-J}(z - a - jU)^J g(z)$ - so that in principle it would be possible to apply the result of the previous discussion as such. But then the function $G(x)$ would have to be replaced by $U^{-J}\mu(x)^J G(x)$, which may become large if J is large. Therefore we modify the argument in order to prevent the error term from becoming impracticably large.

But the first step in the treatment of the integral K is as before. Namely, let v be as in (2.1.24), put $z = x_\circ + (1+i)y$, and let K_1 and K_3 be the integrals with respect to y over the intervals $[-\lambda(x_\circ), -v]$ and $[v, \lambda(x_\circ)]$. Then $K_1 + K_3$ can be estimated as before, except that the extra factor $1 + (v/U)^J$ has to be inserted. But since $(v/U)^J \ll F(x_\circ)^{J/6}$ by (2.1.10) and (2.1.24), the estimate (2.1.25) remains valid even for the new integrals K_1 and K_3.

The new integral K_2, which represents the main part of K, is now

$$K_2 = (1+i)U^{-J} \int_{-v}^{v} (x_\circ - a - jU + (1+i)y)^J h(x_\circ + (1+i)y)\, dy.$$

For the function $h(x_\circ + (1+i)y)$ we are going to use a somewhat cruder approximation than before. By (2.1.21) and (2.1.23) we have

$$\begin{aligned} & h(x_\circ + (1+i)y) \\ &= \Big\{ g(x_\circ) e(f(x_\circ) + \alpha x_\circ) + o\Big(G(x_\circ)\mu(x_\circ)^{-1}|y|\Big) \\ &\quad + o\Big(F(x_\circ)G(x_\circ)\mu(x_\circ)^{-3}|y|^3\Big)\Big\} \times \exp\Big(-2\pi f''(x_\circ)y^2\Big). \end{aligned} \tag{2.1.31}$$

62 Since by (2.1.29), (2.1.10), and (iv)

$$U^{-J} \int_{-v}^{v} \Big(U^J + |y|^J\Big) |y|^\nu \exp\Big(-2\pi f''(x_\circ)y^2\Big)\, dy$$

$$\ll f''(x_\circ)^{-(\nu+1)/2} + U^{-J} f''(x_\circ)^{-(\nu+J+1)/2}$$
$$\ll \mu(x_\circ)^{\nu+1} F(x_\circ)^{-(\nu+1)/2}$$

the contribution of the error terms in (2.1.31) to K_2 is $\ll G(x_\circ)$ $\mu(x_\circ)\ F(x_\circ)^{-1}$. Hence

$$\begin{aligned}
K_2 = {} & \sqrt{2} g(x_\circ) e(f(x_\circ) + \alpha x_\circ + 1/8) U^{-J} \int_{-\nu}^{\nu} (x_\circ - a - jU + (1+i)y)^J \\
& \exp\left(-2\pi f''(x_\circ) y^2\right) dy + o\left(G(x_\circ)\mu(x_\circ)F(x_\circ)^{-1}\right) \\
= {} & \sqrt{2} g(x_\circ) e\,(f(x_\circ) + \alpha x_\circ + 1/8)\, U^{-J} \sum_{0 \le \nu \le J/2} (x_\circ - a - jU)^{J-2\nu} \times \\
& \times (1+i)^{2\nu} \binom{J}{2\nu} \int_{-\nu}^{\nu} y^{2\nu} \exp\left(-2\pi f''(x_\circ) y^2\right) dy \\
& + o\left(G(x_\circ)\mu(x_\circ)F(x_\circ)^{-1}\right).
\end{aligned}$$

As before, the integrals here can be extended to the whole real line with a negligible error. Then, evaluating the new integrals by (2.1.29) we find that with

$$c_\nu = 2^{-\nu} \pi^{-\nu-1/2} (1+i)^{2\nu} \binom{J}{2\nu} d_\nu$$

and with $\xi(x_\circ)$ as in (2.1.13), the resulting expression for I_J is as in (2.1.11).

4) The remaining case $b - jU \le x_\circ < b$ is analogous to the preceding one.

□

Remark. If f satisfies the conditions of Theorem 2.1 and 2.2 except that $f''(x)$ is negative in the interval $[a, b]$, then the results hold with the 63
minor modifications that in the main term the factor $e(f(x_\circ) + \alpha x_\circ + 1/8)$ is to be replaced by $e(f(x_\circ) + \alpha x_\circ - 1/8)$, and $|f''(x_\circ)|$ should stand in place of $f''(x_\circ)$.

2.2 Smoothed Exponential Integrals without a Saddle Point

Theorem 2.2 covers also the case of exponential integrals I_J without a saddle point. However, in applications, the condition (iv) on f'' may not be fulfilled. Nevertheless, if the assumption of f' is strengthened, then no assumption on f'' is needed. The next theorem is a result of this kind.

Theorem 2.3. *Suppose that the functions f and g satisfy the conditions (i) and (ii) in the preceding section, with $\mu(x) = \mu$, a constant. Suppose also that*

$$|g(z)| \ll G \quad \text{for} \quad z \in D, \tag{2.2.1}$$

$$|f'(x)| \asymp M \quad \text{for} \quad a \leq x \leq b, \tag{2.2.2}$$

and

$$|f'(z)| \ll M \quad \text{for} \quad z \in D. \tag{2.2.3}$$

Let I_J be as in (2.1.2) *with $\alpha = 0$ and $0 < JU < (b-a)/2$. Then*

$$I_J \ll U^{-J} G M^{-J-1} + \left(\mu^J U^{1-J} + b - a\right) G e^{-AM_\mu}. \tag{2.2.4}$$

Proof. By (2.2.2), the function $f'(x)$ cannot change its sign in the interval $[a, b]$. Suppose, to be specific, that $f'(x)$ is positive. By (2.2.3) and Cauchy's integral formula we have

$$|f^{(k)}(x)| \ll k! M (\mu/2)^{-k+1} \quad \text{for} \quad k = 1, 2, \ldots \quad \text{and} \quad a \leq x \leq b.$$

Then, by (2.2.2) and Taylor's theorem, it is seen that

$$Re(2\pi i f(z)) < -AMy \tag{2.2.5}$$

64 for $z = x + yi, a \leq x \leq b$, and $0 \leq y \leq \beta\mu = \lambda$, where β is a sufficiently small positive constant. $\square$

Now, for a proof of (2.2.4), the integral I_J is written as in (2.1.3), where the intervals $[a+jU, c]$ and $[c, b-jU]$ are deformed to rectangular contours respectively with vertices $a+jU, a+jU+i\lambda, c+i\lambda$, and c, or $c, c+i\lambda, b-jU+i\lambda$, and $b-jU$. Then (2.2.4) follows easily by (2.2.1), (2.2.5) and (2.1.5).

Notes

In the saddle-point lemma of Atkinson (Lemma 1 in [2]), the assumptions on the functions F and μ are weaker than those in Theorem 2.2, for the conditions (v) and (vi) are missing. Actually we posed these just for simplicity. On the other hand, one of the conditions in [2] is stronger than ours, for in place of (iv) there is an upper bound for $f''(z)^{-1}$ for $z \in D$. However, in the proof this is needed only on the real interval $[a, b]$, in which case it coincides with (iv).

The complications that arose in Theorem 2.2 when $x_\circ$ lies near a or b seem inevitable, for then the integrand is almost stationary near a or b, and consequently there is not so much to be gained by smoothing.

The case $J = 1$ of Theorem 2.3 is Lemma 2 in [16] and Lemma 2.3 in [13]. Our proof is not a direct generalization of that in [16] which turned out to become somewhat tedious for general J.

Theorems 2.2 and 2.3 may be useful in problems in which the stan- **65**
dard results (corresponding to $J = 0$) on exponential integrals are not accurate enough. An example of such an application is the improvement of the error terms in the approximate functional equations for $\zeta^2(s)$ and $\varphi(s)$ in [19].

The parameters U and J, which determine the smoothing, can be chosen differently at a and b. Such a version of Theorem 2.2 is given in [19], and the proof is practically the same. The corresponding smoothed integrals is of the type

$$U^{-J}V^{-K}\int\limits_{\circ}^{U} du_1 \cdots \int\limits_{\circ}^{U} du_J \int\limits_{\circ}^{V} dv_1 \cdots \int\limits_{\circ}^{V} dv_K \int\limits_{a+u}^{b-v} h(x)\,dx,$$

where $u = u_1 + \cdots + u_J$ and $v = v_1 + \cdots + v_K$. For $U = V$ and $J = K$, this amounts to the integral I_J in (2.1.2).

Chapter 3

Transformation Formulae for Exponential Sums

THE BASIC RESULTS of these notes, formulae relating exponential sums 66

$$\sum_{M_1 \le m \le M_2} b(m)g(m)e(f(m) = d(m) \quad \text{or} \quad a(m),$$

or their smoothed versions, to other exponential sums involving the same $b(m)$, are established in this chapter by combining the summation formulae of Chapter 1 with the theorems of Chapter 2 on exponential integrals. The theorems in [16] and [17] concerning Dirichlet polynomials (which will be discussed in § 4.1) were the first examples of such results. As will be seen, the methods of these papers work even in the present more general context without any extra effort.

3.1 Transformation of Exponential Sums

To begin with, we derive a transformation formula for the above mentioned sum with $b(m) = d(m)$. The proof is modelled on that of Theorem 1 in [16].

In the following theorems, $\delta_1, \delta_2, \ldots$ denote positive constants which may be supposed to be arbitrarily small. Further, put $L = \log M_1$ for short.

Theorem 3.1. *Let $2 \leq M_1 < M_2 \leq 2M_1$, and let f and g be holomorphic functions in the domain*

(3.1.1) $$D = \{z \,|z - x| < cM_1 \quad \text{for some} \quad x \in [M_1, M_2]\},$$

67 *where c is a positive constant. Suppose that $f(x)$ is real for $M_1 \leq x \leq M_2$. Suppose also that, for some positive numbers F and G,*

(3.1.2) $$|g(z)| \ll G,$$

(3.1.3) $$|f'(z)| \ll FM_1^{-1}$$

for $z \in D$, and that

(3.1.4) $$(0 <) f''(x) \gg FM_1^{-2} \quad \text{for} \quad M_1 \leq x \leq M_2.$$

Let $r = h/k$ be a rational number such that

(3.1.5) $$1 \leq k \ll M_1^{1/2-\delta_1},$$

(3.1.6) $$|r| \asymp FM_1^{-1}$$

and

(3.1.7) $$f'(M(r)) = r$$

for a certain number $M(r) \in (M_1, M_2)$. Write

$$M_j = M(r) + (-1)^j m_j, \; j = 1, 2.$$

Suppose that $m_1 \asymp m_2$, and that

(3.1.8) $$M_1^{\delta_2} \max\left(M_1 F^{-1/2}, |hk|\right) \ll m_1 \ll M_1^{1-\delta_3}.$$

Define for $j = 1, 2$

(3.1.9) $$p_{j,n}(x) = f(x) - rx + (-1)^{j-1}\left(2\sqrt{nx}/k - 1/8\right),$$

(3.1.10) $$n_j = \left(r - f'\left(M_j\right)\right)^2 k^2 M_j,$$

and for $n < n_j$ let $x_{j,n}$ be the (unique) zero of $p'_{j,n}(x)$ in the interval (M_1, M_2). Then

$$
\begin{aligned}
(3.1.11)\qquad & \sum_{M_1\le m\le M_2} d(m)g(m)e(f(m)) \\
&= k^{-1}\left(\log M(r) + 2\gamma - 2\log k\right) g(M(r))\, f''(M(r))^{-1/2} e(f(M(r)) \\
&\quad -rM(r) + 1/8) + +i2^{-1/2}k^{-1/2}\sum_{j=1}^{2}(-1)^{j-1} \\
&\quad \sum_{n<n_j} d(n)e_k\left(-n\bar{h}\right) n^{-1/4}x_{j,n}^{-1/4} g\left(x_{j,n}\right) p''_{j,n}\left(x_{j,n}\right)^{-1/2}\times \\
&\quad \times e\left(p_{j,n}(x_{j,n}) + 1/8\right) + o\left(FGh^{-2}km_1^{-1}L\right) + o\left(G(|h|)^{1/2}m_1^{1/2}L^2\right) + \\
&\quad + o\left(F^{1/2}G|h|^{-3/4}k^{5/4}m_1^{-1/4}L\right).
\end{aligned}
$$

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Proof. Suppose, to be specific, that $r > 0$, and thus $h > 0$. The proof is similar for $r < 0$.

The assertion (3.1.11) should be understood as an asymptotic result, in which M_1 and M_2 are large. Then the numbers F and n_j are also large. In fact,

$$
F \gg M_1^{1/2+\delta_1} \tag{3.1.12}
$$

and

$$
n_j \gg hkM_1^{2\delta_2}. \tag{3.1.13}
$$

For a proof of (3.1.12), note that by (3.1.6) and (3.1.5)

$$
F \gg M_1 r \ge k^{-1}M_1 \gg M_1^{1/2+\delta_1}.
$$

Consider next the order of n_j. By (3.1.1) and the holomorphicity of f in the domain (3.1.1) we have $f''(x) \ll FM_1^{-2}$, which implies together with (3.1.4) that

$$
f''(x) \asymp FM_1^{-2} \quad \text{for} \quad M_1 \le x \le M_2. \tag{3.1.14}
$$

Thus, by (3.1.7),

$$\left|r - f'\left(M_j\right)\right| \asymp m_j F M_1^{-2}, \tag{3.1.15}$$

so that by (3.1.10) and (3.1.6) we have

$$n_j \asymp F^2 k^2 m_j^2 M_1^{-3} \asymp F^{-1} h^3 k^{-1} m_j^2. \tag{3.1.16}$$

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This gives (3.1.13) owing to the estimates $m_j \gg M_1^{1+\delta_2} F^{-1/2}$ and $F \ll M_1 r$. Also, it follows that $n_j \ll M_1^A$, for $h \ll M_1$ and $m_j \ll M_1$ by (3.1.8).

The numbers n_j are determined by the condition

$$p'_{j,n_j}\left(M_j\right) = 0. \tag{3.1.17}$$

Then clearly

$$(-1)^j p'_{j,n}\left(M_j\right) > 0 \quad \text{for} \quad n < n_j.$$

On the other hand, by (3.1.7)

$$(-1)^j p'_{j,n}(M(r)) = -n^{1/2} M(r)^{-1/2} k^{-1} < 0$$

for all positive n. Consequently, for $n < n_j$ there is a zero $x_{j,n}$ of $p'_{j,n}(x)$ in the interval (M_1, M_2), and moreover $x_{1,n} \in (M_1, M(r)), x_{2,n} \in (M(r), M_2)$. Also, it is clear that $p'_{j,n}$ has no zero in the interval (M_1, M_2) if $n \geq n_j$.

To prove the uniqueness of $x_{j,n}$, we show that $p''_{j,n}$ is positive and thus $p'_{j,n}$ is increasing in the interval $[M_1, M_2]$. In fact,

$$p''_{j,n}(x) \asymp F M_1^{-2} \quad \text{for} \quad M_1 \leq x \leq M_2, n \leq 2n_j, \tag{3.1.18}$$

at least if M_1 is supposed to be sufficiently large. For by definition

$$p''_{j,n}(x) = f''(x) + (-1)^j \frac{1}{2} n^{1/2} x^{-3/2} k^{-1},$$

where by (3.1.16) and (3.1.8)

$$n^{1/2} x^{-3/2} k^{-1} \ll F m_1 M_1^{-3} \ll F M_1^{-2-\delta_3},$$

70 so that by (3.1.14) the term $f''(x)$ dominates.

After these preliminaries we may go into the proof of the formula (3.1.11). Denote by 5 the sum under consideration. Actually it is easier to deal with the smoothed sum

$$S' = U^{-1}\int_{\circ}^{U} S(u)\,du, \tag{3.1.19}$$

where

$$S(u) = \sum_{M_1+u\le m\le M_2-u} d(m)g(m)e(f(m)). \tag{3.1.20}$$

The parameter U will be chosen later in an optimal way; presently we suppose only that

$$M_1^{\delta_4} \ll U \le \frac{1}{2}\min(m_1, m_2). \tag{3.1.21}$$

Since

$$\sum_{x\le n\le x+y} d(n) \ll y\log x \quad \text{for} \quad x^{\epsilon} \ll y \ll x, \tag{3.1.22}$$

(see [25]), we have

$$S - S' \ll GUL. \tag{3.1.23}$$

The summation formula (1.9.1) is now applied to the sum $S(u)$, which is first written as

$$S(u) = \sum_{a\le m\le b} d(m)g(m)e(f(m) - mr)e(mr),$$

with $a = M_1+u, b = M_2-u$. We may assume that neither of the numbers a and b is an integer, for the value of $S(u)$ for the finitely many other values of u is irrelevant in the integral (3.1.19). Then by (1.9.1)

(3.1.24)

$$S(u) = k^{-1}\int_{a}^{b}(\log x + 2\gamma - 2\log k)g(x)e(f(x) - rx)\,dx$$

$$+ k^{-1}\sum_{n=1}^{\infty} d(n)\int_a^b \left\{-2\pi e_k\left(-n\bar{h}\right)Y_\circ\left(4\pi\sqrt{nx}/k\right) + 4e_k\left(n\bar{h}\right)\right.$$

$$\left.K_\circ\left(4\pi\sqrt{nx}/k\right)\right\} g(x)e(f(x) - rx)\,dx$$

$$= k^{-1}\left\{I_\circ + \sum_{n=1}^{\infty} d(n)\left(e_k\left(-n\bar{h}\right)I_n + e_k\left(n\bar{h}\right)i_n\right)\right\},$$

71 say.

The integrals i_n are very small and quite negligible. Indeed, by (3.1.5) we have $\sqrt{nM_1}/k \gg \sqrt{n}M_1^{\delta_1}$, so that by (1.3.17)

$$k^{-1}\sum_{n=1}^{\infty} d(n)\,|i_n| \ll k^{-1}GM_1\sum_{n=1}^{\infty} d(n)\exp\left(-A\sqrt{n}M_1^{\delta_1}\right) \ll G\exp\left(-AM_1^{\delta_1}\right). \tag{3.1.25}$$

Consider next the integral $I_\circ$. We apply Theorem 2.1 with $\alpha = -r$ and $\mu(x)$ a constant function $\asymp M_1$. The assumptions of Theorem 2.1 are satisfied in virtue of the conditions of our theorem. By (3.1.7), the saddle point is $M(r)$. Hence the saddle-point term for $k^{-1}I_\circ$ equals the leading term in (3.1.11).

The first error term in (2.1.9) is

$$\ll LGM_1\exp(-AF)$$

which is negligible by (3.1.12).

The last two error terms contribute

$$\ll GL\left\{\left(\left|f'(a) - r\right| + F^{1/2}M_1^{-1}\right)^{-1} + \left(\left|f'(b) - r\right| + F^{1/2}M_1^{-1}\right)^{-1}\right\}. \tag{3.1.26}$$

For same reasons as in (3.1.15), we have

$$\left|f'(a) - r\right| \asymp Fm_1M_1^{-2},$$

and likewise for $|f'(b) - r|$. Hence the expression (3.1.26) is $\ll F^{-1}$
72 $Gm_1^{-1}M_1^2L$, which is further $\ll FGm_1^{-1}r^{-2}L$ by (3.1.6). The second error

term in (2.1.9), viz. $o(GM_1F^{-3/2}L)$, can be absorbed into this, for

$$M_1F^{-3/2} \ll r^{-1} \ll FM_1^{-1}r^{-2} \ll Fm_1^{-1}r^{-2}.$$

Hence the error terms for $k^{-1}I_\circ$ give together $o(FGh^{-2}km_1^{-1}L)$, which is the first error term in (3.1.11).

We are now left with the integrals

$$I_n = -2\pi \int_a^b Y_\circ\left(4\pi\sqrt{nx}/k\right) g(x)e(f(x) - rx)\,dx. \tag{3.1.27}$$

By (1.3.9), the function $Y_\circ$ can be written in terms of Hankel functions as

$$Y_\circ(z) = \frac{1}{2i}\left(H_\circ^{(1)}(z) - H_\circ^{(2)}(z)\right), \tag{3.1.28}$$

where by (1.3.13)

$$H_\circ^{(j)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \exp\left((-1)^{j-1} i\left(z - \frac{1}{4}\pi\right)\right)\left(1 + g_j(z)\right). \tag{3.1.29}$$

The functions $g_j(z)$ are holomorphic in the half-plane $Re\,z > 0$, and by (1.3.14)

$$\left|g_j(z)\right| \ll |z|^{-1} \quad \text{for} \quad |z| \geq 1, Re\,z > 0. \tag{3.1.30}$$

By (3.1.27) - (3.1.29) we may write

$$I_n = I_n^{(1)} - I_n^{(2)}, \tag{3.1.31}$$

where

(3.1.32)

$$I_n^{(j)} = i2^{-1/2}k^{1/2}n^{-1/4} \int_a^b x^{-1/4} g(x)\left(1 + g_j\left(4\pi\sqrt{nx}/k\right)\right) e\left(p_{j,n}(x)\right)\,dx.$$

For $n \leq 2n_j$ we apply Theorem 2.1 to $I_n^{(j)}$, again with $\alpha = -r$. The function 73

$$f(x) + (-1)^{j-1}\left(2\sqrt{nx}/k - 1/8\right)$$

now stands for the function f, and moreover $\mu(x) \asymp M_1$ and $F(x) = F$. The conditions of the theorem are satisfied, in particular the validity of the condition (iv) on f'' follows from (3.1.18), and the condition (iii) on f' can be checked by (3.1.3) and (3.1.16). The number $x_{j,n}$ is, by definition, the saddle point for $I_n^{(j)}$, and it lies in the interval (M_1, M_2) if and only if $n < n_j$. However, in $I_n^{(j)}$ the interval of integration is $[a, b] = [M_1 + u, M_2 - u]$, and $x_{j,n} \in (a, b)$ if and only if $n < n_j(u)$, where

$$n_j(u) = \left(r - f'\left(M_j + (-1)^{j-1}u\right)\right)^2 k^2 \left(M_j + (-1)^{j-1}u\right) \tag{3.1.33}$$

in analogy with (3.1.10). But for simplicity we count the saddle-point terms for all $n < n_j$, and the number of superfluous terms is then

$$\ll 1 + n_j - n_j(U) \ll 1 + F^2k^2m_1M_1^{-3}U. \tag{3.1.34}$$

The saddle-point term for $k^{-1}I_n^{(j)}$ is

$$\begin{aligned} &i2^{-1/2}k^{-1/2}n^{-1/4}x_{j,n}^{-1/4}g\left(x_{j,n}\right)\left(1 + g_j\left(4\pi\sqrt{nx_{j,n}}/k\right)\right)\times \\ &\times p''_{j,n}\left(x_{j,n}\right)^{-1/2} e\left(p_{j,n}\left(x_{j,n}\right) + 1/8\right). \end{aligned} \tag{3.1.35}$$

Multiplied by $(-1)^{j-1}d(n)e_k(-n\bar{h})$, these agree, up to $g_j(\ldots)$, with the individual terms of the sums on the right of (3.1.11). The effect of the omission of $g_j(\ldots)$ is by (3.1.30), (3.1.16), (3.1.18), and (3.1.5)

$$\begin{aligned} &\ll F^{-1/2}Gk^{1/2}M_1^{1/4}\sum_{n \ll M_1} d(n)n^{-3/4} \\ &\ll Gkm_1^{1/2}M_1^{-1/2}L \ll Gm_1^{1/2}L, \end{aligned}$$

74 which can be absorbed into the second error term in (3.1.11).

The extra saddle-point terms, counted in (3.1.34), contribute at most

$$\ll \left(1 + F^2k^2m_1M_1^{-3}U\right)F^{-1/2}Gk^{-1/2}M_1^{3/4+\epsilon}n_1^{-1/4},$$

which, by (3.1.16) and (3.1.6), is

$$\ll F^{1/2}Gh^{-3/2}k^{1/2}m_1^{-1/2}M_1^{\epsilon} + F^{-1/2}Gh^{3/2}k^{-1/2}m_1^{1/2}M_1^{\epsilon}U. \tag{3.1.36}$$

Now, allowing for these error terms, we have the same saddle-point terms given in (3.1.11), for all sums $S(u)$, and hence by (3.1.19) for S', too.

Consider now the error terms when Theorem 2.1 is applied to $I_n^{(j)}$ for $n \leq 2n_j$. The first error term in (2.1.9) is clearly negligible. Further, the contribution of the error terms involving $x_\circ$ to $S(u)$ is

$$\ll F^{-3/2}Gk^{-1/2}M_1^{3/4}\sum_{n\ll n_1} d(n)n^{-1/4},$$

which, by (3.1.16), (3.1.8) and (3.1.5), is

$$\ll Gkm_1^{3/2}M_1^{-3/2}L \ll Gkm_1^{1/2}M_1^{-1/2}L \ll Gm_1^{1/2}.$$

This is smaller than the second error term in (3.1.11).

The last two error terms are similar, so it suffices to consider $o(E_\circ(a))$ as an example. By (2.1.8) and (3.1.32), this error term for $k^{-1}I_n^{(j)}$ is

$$\ll Gk^{-1/2}M_1^{-1/4}n^{-1/4}\left(\left|p'_{j,n}(a)\right| + p''_{j,n}(a)^{1/2}\right)^{-1}.$$

Consider the case $j = 1$; the case $j = 2$ is less critical since $|p'_{2,n}(a)|$ cannot be small. Now $p'_{1,n_1(u)}(a) = 0$ and $p''_{1,n}(a) \asymp F^{-1}r^2$, so it is easily seen that **75**

$$\left(\left|p'_{1,n}(a)\right| + p''_{1,n}(a)^{1/2}\right)^{-1} \ll \begin{cases} F^{1/2}r^{-1} & \text{for} \quad |n-n_1(u)| \ll F^{-1/2}h^2m_1, \\ kM_1^{1/2}n_1^{1/2}\,|n-n_1(u)|^{-1} & \text{otherwise} \end{cases}$$

Note that by (3.1.8) and (3.1.6)

$$F^{-1/2}h^2m_1 \gg F^{-1}h^2M_1^{1+\delta_2} \gg hkM_1^{\delta_2}.$$

Hence, by (3.1.22), the mean value of $d(n)$ in the interval $|n - n_1(u)| \ll F^{-1/2}h^2m_1$ can be estimated as $o(L)$. It is now easily seen that the contribution to $S(u)$ of the error terms in question is

$$\ll Gh^{1/2}k^{1/2}m_1^{1/2}L^2,$$

which is the second error term in (3.1.11).

The smoothing device was introduced with the integrals $I_n^{(j)}$ for $n > 2n_j$ in mind. By (3.1.19), (3.1.24), (3.1.31), and (3.1.32), their contribution to S' is equal to

$$i2^{-1/2}k^{-1/2}\sum_{j=1}^{2}(-1)^{j-1}\sum_{n>2n_j} d(n)e_k\left(-n\bar{h}\right)n^{-1/4}\times \tag{3.1.37}$$
$$\times\int_{M_1}^{M_2}\eta_1(x)x^{-1/4}g(x)\left(1+g_j\left(4\pi\sqrt{nx}/k\right)\right)e\left(p_{j,n}(x)\right)\,dx,$$

where $\eta_1(x)$ is a weight function in the sense of Chapter 2, with $J = 1$ and U being the other smoothing parameter. The series in (3.1.24) is boundedly convergent with respect to u, by Theorem 1.7, so that it can be integrated term by term.

The smoothed exponential integrals in (3.1.37) are estimated by
76 Theorem 2.3, where $p_{j,n}(z)$ stands for $f(z)$, and $\mu \asymp m_1$. To begin with, we have to check that the conditions of this theorem are satisfied. We have
$$p'_{j,n}(z) = f'(z) - r + (-1)^{j-1}n^{1/2}z^{-1/2}k^{-1}.$$

Let $n > 2n_j$, and let z lie in the domain D, say $D_\circ$, of Theorem 2.3. Then by (3.1.16)
$$\left|n^{1/2}z^{-1/2}k^{-1}\right| \gg m_1FM_1^{-2}.$$

On the other hand, since $f'(M(r)) - r = 0$ and $|f''(z)| \ll FM_1^{-2}$ for $z \in D_\circ$ by (3.1.3) and Cauchy's integral formula, we also have

$$|f'(z) - r| \ll m_1FM_1^{-2}.$$

Thus, the condition (2.2.3) holds with

$$M = k^{-1}M_1^{-1/2}n^{1/2}. \tag{3.1.38}$$

Further, to verify the condition (2.2.2), compare $p'_{j,n}(x)$ with $p'_{j,n_j}(x)$, using (3.1.17) and the fact that $p'_{j,n_j}(x)$ is increasing in the interval $[M_1, M_2]$.

We may now apply the estimate (2.2.4) in (3.1.37). The second term on the right of (2.2.4) is exponentially small, for by (3.1.38), (3.1.16), and (3.1.8)

$$\begin{aligned} M\mu \gg k^{-1}M_1^{-1/2}n^{1/2}m_1 &\gg (n/n_1)^{1/2}\,Fm_1^2M_1^{-2} \\ &\gg (n/n_1)^{1/2}\,M_1^{2\delta_2}. \end{aligned}$$

Hence these terms are negligible.

The contribution of the terms $U^{-1}GM^{-2}$ in (2.2.4) to (3.1.37) is

$$\begin{aligned} &\ll Gk^{3/2}M_1^{3/4}U^{-1}\sum_{n\gg n_1} d(n)n^{-5/4} \\ &\ll Gk^{3/2}M_1^{3/4}n_1^{-1/4}U^{-1}L \\ &\ll GF^{-1/2}kM_1^{3/2}m_1^{-1/2}U^{-1}L \\ &\ll GFh^{-3/2}k^{5/2}m_1^{-1/2}U^{-1}L. \end{aligned}$$

77

Combining this with (3.1.23) and (3.1.36), we find that (3.1.11) holds, up to the additional error terms

$$\begin{aligned} &\ll GUL + F^{1/2}Gh^{-3/2}k^{1/2}m_1^{-1/2}M_1^{\epsilon} \\ &+ F^{-1/2}Gh^{3/2}k^{-1/2}m_1^{1/2}M_1^{\epsilon}U + GFh^{-3/2}k^{5/2}m_1^{-1/2}U^{-1}L. \end{aligned} \tag{3.1.39}$$

Here the second term is superseded by the last term in (3.1.11). Further, the first and last term coincide with the last term in (3.1.11) if we choose

$$U = F^{1/2}h^{-3/4}k^{5/4}m_1^{-1/4}. \tag{3.1.40}$$

Then, by (3.1.8), the third term in (3.1.39) is

$$\ll Gh^{3/4}k^{3/4}m_1^{1/4}M_1 \ll G(hk)^{1/2}m_1^{1/2}M_1^{\epsilon-\delta_2/4},$$

which can be absorbed into the second error term in (3.1.11).

It should still be verified that the number U in (3.1.40) satisfies the condition (3.1.21). By (3.1.8) and (3.1.6) we have

$$Um_1^{-1} \ll U\left(M_1^{1+\delta_2}F^{-1/2}\right)^{-1} \ll (hk)^{1/4}m_1^{-1/4}M_1^{-\delta_2} \ll M_1^{-\delta_2}$$

and also, in the other direction,

$$\begin{aligned} U &\gg F^{1/2}h^{-3/4}k^{5/4}M_1^{-1/4+\delta_3/4} \\ &\gg M_1^{1/4+\delta_3/4}h^{-1/4}k^{3/4} \gg M_1^{\delta_3/4} \end{aligned}$$

78 Hence (3.1.21) holds, and the proof of the theorem is complete.

The next theorem is an analogue of Theorem 3.1 for exponential sums involving the Fourier coefficients $a(n)$ of a cusp form of weight κ. The proof is omitted, because the argument is practically the same; the summation formula (1.9.2) is just applied in place of (1.9.1). Note that in (1.9.2) there is nothing is correspond to the first term in (1.9.1), and consequently in the transformation formula there are no counterparts for the first explicit term and the first error term in (3.1.11). □

Theorem 3.2. *Suppose that the assumptions of Theorem 3.1 are satisfied. Then*

$$\begin{aligned} (3.1.41) \qquad & \sum_{M_1 \le m \le M_2} a(m)g(m)e(f(m)) \\ &= i2^{-1/2}k^{-1/2}\sum_{j=1}^{2}(-1)^{j-1}\sum_{n<n_j} a(n)e_k\left(-n\bar{h}\right)n^{\kappa/2+1/4}\times \\ &\times x_{j,n}^{\kappa/2-3/4}g\left(x_{j,n}\right)p''_{j,n}\left(x_{j,n}\right)^{-1/2}e\left(p_{j,n}\left(x_{j,n}\right)+1/8\right) \\ &+ o\left(G\left(|h|k\right)^{1/2}M_1^{(\kappa-1)/2}m_1^{1/2}L^2\right) \\ &+ o\left(F^{1/2}G|h|^{-3/4}k^{5/4}M_1^{(\kappa-1)/2}m_1^{-1/4}L\right). \end{aligned}$$

3.2 Transformation of Smoothed Exponential Sums

We now give analogues of Theorem 3.1 and 3.2 for smoothed exponential sums provided with weights of the type $\eta_J(n)$. We have to pay for the better error terms in these new formulae by allowing certain weights to appear in the transformed sums as well.

Theorem 3.3. *Suppose that the assumptions of Theorem 3.1 are satisfied. Let*

$$(3.2.1) \qquad U \gg F^{-1/2}M_1^{1+\delta_4} \asymp F^{1/2}r^{-1}M_1^{\delta_4},$$

and let J be a fixed positive integer exceeding a certain bound (which depends on δ_4). Write for $j = 1, 2$ 79

$$M'_j = M_j + (-1)^{j-1} JU = M(r) + (-1)^j m'_j,$$

and suppose that $m'_j \asymp m_j$. Let n_j be as in (3.1.10)*, and define analogously*

$$n'_j = \left(r - f'\left(M'_j\right)\right)^2 k^2 M'_j. \tag{3.2.2}$$

Then, defining the weight function $\eta_J(x)$ in the interval $[M_1, M_2]$ as in (2.1.2)*, we have*

$$\begin{aligned}\sum_{M_1 \le m \le M_2} &\eta_J(m) d(m) g(m) e(f(m)) \\ = k^{-1}&(\log M(r) + 2\gamma - 2\log k)\, g(M(r)) f''(M(r))^{-1/2} \\ &\qquad e(f(M(r)) - rM(r) + 1/8) \\ &+ i2^{-1/2} k^{-1/2} \sum_{j=1}^{2} (-1)^{j-1} \sum_{n < n_j} w_j(n) d(n) e_k\left(-n\bar{h}\right) n^{-1/4} x_{j,n}^{-1/4} \times \\ &\times \left(x_{j,n}\right) p''_{j,n}\left(x_{j,n}\right)^{-1/2} e\left(p_{j,n}\left(x_{j,n}\right) + 1/8\right) \\ &\qquad + o\left(F^{-1} G |h|^{3/2} k^{-1/2} m_1^{1/2} UL\right),\end{aligned} \tag{3.2.3}$$

where

$$w_j(n) = 1 \quad \textit{for} \quad n < n'_j, \tag{3.2.4}$$

$$w_j(n) \ll 1 \quad \textit{for} \quad n < n_j, \tag{3.2.5}$$

$w_j(y)$ and $w'_j(y)$ are piecewise continuous functions in the interval (n'_j, n_j) with at most $J - 1$ discontinuities, and

$$w'_j(y) \ll \left(n_j - n'_j\right)^{-1} \quad \textit{for} \quad n'_j < y < n_j \tag{3.2.6}$$

whenever $w'_j(y)$ exists.

Proof. We follow the argument of the proof of Theorem 3.1, using however Theorem 2.2 in place of Theorem 2.1. Denoting by S_J the smoothed sum under consideration, we have by (1.9.1), as in (3.1.24), **80**

(3.2.7)

$$\begin{aligned} S_J &= k^{-1}\int\limits_{M_1}^{M_2}(\log x+2\gamma-2\log k)\,\eta_J(x)g(x)e(f(x)-rx)\,dx \\ &\quad + k^{-1}\sum_{n=1}^{\infty}d(n)\int\limits_{M_1}^{M_2}\Big\{-2\pi e_k\left(-n\bar{h}\right)Y_\circ\left(4\pi\sqrt{nx}/k\right)+4e_k\left(n\bar{h}\right)K_\circ \\ &\qquad\qquad \left(4\pi\sqrt{nx}/k\right)\Big\}\,\eta_J(x)g(x)e(f(x)-rx)\,dx \\ &= k^{-1}\left\{I_\circ+\sum_{n=1}^{\infty}d(n)(e_k(-n\bar{h})I_n+e_k(n\bar{h})i_n)\right\}. \end{aligned}$$

As in the proof of Theorem 3.1, the integrals i_n are negligible.

Consider next the integral $I_\circ$. We apply Theorem 2.2 choosing $\mu(x)\asymp M_1$ again. The saddle-point is $M(r)$, as before, and the saddle-point term for $I_\circ$ is the same as in the proof of Theorem 3.1. The first error term in (2.1.11) is exponentially small. The error terms involving E_J are also negligible if J is taken sufficiently large (depending on δ_4), since

$$U^{-1}f''(x)^{-1/2}\ll M^{-\delta_4}\quad\text{for}\quad M_1\le x\le M_2.$$

The contribution of the error term $o(G(x_\circ)\mu(x_\circ)F(x_\circ)^{-3/2})$ to $k^{-1}I_\circ$ is by (3.2.1) and (3.1.8)

$$\ll k^{-1}F^{-3/2}GM_1L\ll F^{-1}GUL\ll F^{-1}Gh^{-1/2}m_1^{1/2}U,$$

which does not exceed the error term in (3.2.3).

Turning to the integrals I_n, we write as in (3.1.31) and (3.1.32)

$$I_n=I_n^{(1)}-I_n^{(2)},$$

81 where

(3.2.8)

$$I_n^{(j)} = i2^{-1/2}k^{1/2}n^{-1/4}\int\limits_{M_1}^{M_2}\eta_J(x)^{-1/4}g(x)\left(1+g_j\left(4\pi\sqrt{nx}/k\right)\right)e\left(p_{j,n}(x)\right)\,dx.$$

Let first $n > 2n_j$. As in the proof of Theorem 3.1, we may apply Theorem 2.3 with $\mu \asymp m_1$ and M as in (3.1.38). Observe that by (3.2.1), (3.1.16), and (3.1.8)

$$U^{-1}M^{-1} \ll (n_1/n)^{1/2}\,F^{-1/2}m_1^{-1}M_1^{1-\delta_4} \ll (n_1/n)^{1/2}\,M_1^{-\delta_2-\delta_4}$$

whence we we may make the term $U^{-J}GM^{-J-1}$ in (2.2.4) negligibly small by taking J large enough. As before, the second term in (2.2.4) is also negligible.

The terms for $n \leq 2n_j$ are dealt with by Theorem 2.2. The saddle point terms occur again for $n < n_j$, and they are of the same shape as in (3.1.35) except that there is the additional factor

$$w_j(n) = \xi\left(x_{j,n}\right). \tag{3.2.9}$$

The property (3.2.4) of $w_j(n)$ is immediate by (2.1.12), for $M_1 + JU < x_{j,n} < M_2 - JU$ if and only if $n < n'_j$. Further, (3.2.5) follows from (2.1.12) - (2.1.14) by (3.2.1) and (3.1.18). To prove the property (3.2.6), consider

$$w_j(y) = \xi\left(x_{j,y}\right)$$

as a function of the continuous variable in the interval (n'_j, n_j). Here $x_{j,y}$ is the unique zero of $p'_{j,y}(x)$ in the interval $(M_1, M_1 + JU)$ for $j = 1$, and in the interval $(M_2 - JU, M_2)$ for $j = 2$. Thus $z = x_{j,y}$ satisfies

$$f'(z) - r + (-1)^{j-1}y^{1/2}z^{-1/2}k^{-1} = 0.$$

Hence, by implicit differentiation, **82**

$$p''_{j,y}(z)\frac{dx_{j,y}}{dy} + \frac{1}{2}(-1)^{j-1}y^{-1/2}z^{-1/2}k^{-1} = 0,$$

which implies that

$$\left|\frac{dx_{j,y}}{dy}\right| \asymp F^{-1}k^{-1}M_1^{3/2}n_1^{-1/2} \asymp m_1 n_1^{-1} \quad \text{for} \quad n'_j < y < n_j. \tag{3.2.10}$$

The function $\xi_J(x)$ in (2.1.13) and (2.1.14) is continuously differentiable except at the points $a + JU$ and $b - jU$, $j = 1, \ldots, J$, where terms appear or disappear. By differentiation, noting that $f'''(x) \ll FM_1^{-3}$, it is easy to verify that

$$\xi_J'(x) \ll U^{-1} \tag{3.2.11}$$

elsewhere in the intervals $(a, a+JU)$ and $(b-JU, b)$. By (3.1.10), (3.2.2), and (3.1.14) we have

$$\left(n_j - n_j'\right) n_j^{-1} \asymp m_1^{-1} U. \tag{3.2.12}$$

Now (3.2.6) follows from (3.2.10) - (3.2.12) at those points y for which $x_{j,y}$ is not of the form $a + jU$ or $b - jU$ with $1 \leq j \leq J - 1$.

As in the proof of Theorem 3.1, we may omit $g_j(\ldots)$ in the saddle-point terms with an admissible error

$$Gkm_1^{1/2} M_1^{-1/2} L \ll F^{-1} G h^{3/2} k^{-1/2} m_1^{1/2} U,$$

and after that these terms coincide with those in (3.2.3).

Consider finally the error terms in (2.1.11) for $I_n^{(j)}$. The first of these is clearly negligible. Also, for the same reason as in the case of $I_\circ$, the error terms involving E_J can be omitted if J is chosen sufficiently large.

83 Finally, the second error term in (2.1.11) for $k^{-1} I_n^{(j)}$ is

$$\ll F^{-3/2} G k^{-1/2} M_1^{3/4} n_1^{-1/4} \quad \text{for} \quad n < n_j'$$

and

$$\ll F^{-1} G k^{-1/2} M_1^{3/4} n_1^{-1/4} \quad \text{for} \quad n_j' \leq n < n_j.$$

The contribution of these to S_J is

$$\ll F^{-3/2} G k^{-1/2} M_1^{3/4} n_1^{3/4} L + F^{-1} G k^{-1/2} M_1^{3/4} n_1^{-1/4} \left(n_j - n_j'\right) L. \tag{3.2.13}$$

Here we estimated the mean value of $d(n)$ in the interval $[n_j', n_j)$ by $o(L)$, which is possible, by (3.1.22), for by (3.2.12), (3.1.16), (3.1.8), (3.2.1), and (3.1.6) we have

$$n_j - n_j' \ll m_1^{-1} n_1 U \ll F^2 k^2 m_1 M_1^{-3} U$$

$$\ll F^2k^2\left(M_1^{1+\delta_2}F^{-1/2}\right)M_1^{-3}\left(F^{-1/2}M_1^{1+\delta_4}\right)\asymp hkM_1^{\delta_2+\delta_4}.$$

By (3.2.12), the second term in (3.2.13) is

$$\asymp F^{-1}Gk^{-1/2}m_1^{-1}M_1^{3/4}n_1^{3/4}UL,$$

which is of the same order as the error term in (3.2.3) and dominates the first term, since

$$m_1^{-1}U\gg\left(F^{-1/2}M_1\right)M_1^{-1}=F^{-1/2}.$$

The proof of the theorem is now complete.

The analogue of the preceding theorem for exponential sums involving Fourier coefficients $a(n)$ is as follows. The proof is similar and can be omitted. □

Theorem 3.4. *With the assumptions of Theorem 3.3, we have* 84

$$\begin{aligned}(3.2.14)\qquad &\sum_{M_1\le m\le M_2}\eta_J(m)a(m)g(m)e(f(m))\\ &=i2^{-1/2}k^{-1/2}\sum_{j=1}^{2}(-1)^{j-1}\sum_{n<n_j}w_j(n)a(n)e_k\left(-n\overline{h}\right)n^{-\kappa/2+1/4}\\ &\quad\times x_{j,n}^{\kappa/2-3/4}g\left(x-j,n\right)p''_{j,n}\left(x_{j,n}\right)^{-1/2}e\left(p_{j,n}\left(x_{j,n}\right)+1/8\right)\\ &\quad+o\left(F^{-1}G|h|^{3/2}k^{-1/2}M_1^{(\kappa-1)/2}m_1^{1/2}UL\right).\end{aligned}$$

Remark. In practice it is of advantage to choose U as small as the condition (3.2.1) permits, i.e.

$$U\asymp F^{1/2+\epsilon}r^{-1}. \tag{3.2.15}$$

Then the error term in (3.2.3) is

$$o\left(F^{-1/2+\epsilon}G\left(|h|k\right)^{1/2}m_1^{1/2}\right) \tag{3.2.16}$$

and that in (3.2.14) is

$$o\left(F^{-1/2+\epsilon}G\left(|h|k\right)^{1/2}M_1^{(\kappa-1)/2}m_1^{1/2}\right). \tag{3.2.17}$$

Chapter 4

Applications

85 THE THEOREMS OF the preceding chapter show that the short exponential sums in quesion depend on the rational approximation of $f'(n)$ in the interval of summation. But in long sums the value of $f'(n)$ may vary too much to be approximated accurately by a single rational number, and therefore it is necessary to split up the sum into shorter segments such that in each segment $f'(n)$ lies near to a certain fraction r. By suitable averaging arguments, it is possible to add these short sums - in a transformed shape - in a non-trivial way. Variations on this theme are given in §§ 4.2 - 4.4. But as a preliminary for §§ 4.2 and 4.4, we first work out in § 4.1 the transformation formulae of Chapter 3 in the special case of Dirichlet polynomials related to $\zeta^2(s)$ and $\varphi(s)$.

4.1 Transformation Formulae for Dirichlet Polynomials

The general theorems of the preceding chapter are now applied to Dirichlet polynomials

$$S(M_1, M_2) = \sum_{M_1 \leq m \leq M_2} d(m) m^{-1/2-it}, \tag{4.1.1}$$

$$S_\varphi(M_1, M_2) = \sum_{M_1 \leq m \leq m_2} a(m) m^{-k/2-it}, \tag{4.1.2}$$

as well as to their smoothed variants

$$\tilde{S}\,(M_1, M_2) = \sum_{M_1 \le m \le M_2} \eta_J(m) d(m) m^{-1/2-it}, \tag{4.1.3}$$

$$\tilde{S}_\varphi\,(M_1, M_2) = \sum_{M_1 \le m \le M_2} \eta_J(m) a(m) m^{-k/2-it}, \tag{4.1.4}$$

where $\eta_J(x)$ is a weight function defined in (2.1.2). **86**

We shall suppose for simplicity that t is a sufficiently large positive number, and put $L = \log t$. The function $\chi(s)$ is as in the functional equation $\zeta(s) = \chi(s)\zeta(1-s)$, thus

$$\chi(s) = s^s \pi^{s-1} \sin\left(\frac{1}{2} s\pi\right) \Gamma(1-s).$$

If σ is bounded and t tends to infinity, then (see [27], p. 68)

$$\chi(s) = (2\pi/t)^{s-1/2}\, e^{i(t+\pi/4)} \left(1 + o\left(t^{-1}\right)\right). \tag{4.1.5}$$

Define also

$$\phi(x) = ar\sin h\left(x^{1/2}\right) + \left(x + x^2\right)^{1/2}. \tag{4.1.6}$$

As before, $\delta_1, \delta_2, \ldots$ will denote positive constants which may be supposed to be arbitrarily small.

Theorem 4.1. *Let $r = h/k$ be a rational number such that*

$$M_1 < \frac{t}{2\pi r} < M_2 \tag{4.1.7}$$

and

$$1 \le k \ll M_1^{1/2-\delta_1}. \tag{4.1.8}$$

Write

$$M_j = \frac{t}{2\pi r} + (-1)^j m_j, \tag{4.1.9}$$

and suppose also that $m_1 \asymp m_2$ *and*

(4.1.10) $$t^{\delta_2} \max\left(t^{1/2} r^{-1}, hk\right) \ll m_1 \ll M_1^{1-\delta_3}.$$

Let

(4.1.11) $$n_j = h^2 m_j^2 M_j^{-1}.$$

87 *Then*

(4.1.12)

$$\begin{aligned} S(M_1, M_2) = \Big\{&(hk)^{-1/2} \left(\log(t/2\pi) + 2\gamma - \log(hk)\right) \\ &+ \pi^{1/4}(2hkt)^{-1/4} \sum_{j=1}^{2} \sum_{n<n_j} d(n) e\left(n\left(\frac{\bar{h}}{k} - \frac{1}{2hk}\right)\right) n^{-1/4} \times \\ &\times \left(1 + \frac{\pi n}{2hkt}\right)^{-1/4} \exp\left(i(-1)^{j-1}\left(2t\phi\left(\frac{\pi n}{2hkt}\right) + \frac{\pi}{4}\right)\right)\Big\} r^{it} \\ &\chi\left(\frac{1}{2} + it\right) + o\left(h^{-3/2} k^{1/2} m_1^{-1} t^{1/2} L\right) + o\left(h m_1^{1/2} t^{-1/2} L^2\right) \\ &+ o\left(h^{-1/4} k^{3/4} m_1^{-1/4} L\right). \end{aligned}$$

Proof. We apply Theorem 3.1 with $-r$ in place of r, and with

(4.1.13) $$f(z) = -(t/2\pi)\log z,$$

and

(4.1.14) $$g(z) = z^{-1/2}.$$

Then the assumptions of the theorem are obviously satisfied with

(4.1.15) $$F = t,$$

(4.1.16) $$G = M_1^{-1/2} \asymp r^{1/2} t^{-1/2},$$

and

(4.1.17) $$M(-r) = \frac{t}{2\pi r}.$$

Then the number n_j in (3.1.10) equals the one in (4.1.11).

The leading term on the right of (3.1.11) is

$$(hk)^{-1/2}(\log(t/2\pi) + 2\gamma - \log(hk))r^{it}(2\pi/t)^{it}e^{i(t+\pi/4)}$$

which can also be written, by (4.1.5), as

$$(hk)^{-1/2}(\log(t/2\pi) + 2\gamma - \log(hk))r^{it}\chi\left(\frac{1}{2} + it\right)\left(1 + o\left(t^{-1}\right)\right). \tag{4.1.18}$$

The function $p_{j,n}(x)$ reads in the present case **88**

$$-(t/2\pi)\log x + rx + (-1)^{j-1}\left(2\sqrt{nx}/k - 1/8\right) \tag{4.1.19}$$

and the numbers $x_{j,n}$ are roots of the equation

$$p'_{j,n} = -\frac{t}{2\pi x} + r + (-1)^{j-1}n^{1/2}x^{-1/2}k^{-1} = 0, \tag{4.1.20}$$

and thus roots of the quadratic equation

$$x^2 - \left(\frac{t}{\pi r} + \frac{n}{h^2}\right)x + \left(\frac{t}{2\pi r}\right)^2 = 0. \tag{4.1.21}$$

Moreover, since $x_{1,n} < x_{2,n}$, we have

$$x_{j,n} = \frac{t}{2\pi r} + \frac{n}{2h^2} + \frac{(-1)^j}{h^2}\left(\frac{n^2}{4} + \frac{hknt}{2\pi}\right)^{1/2} \tag{4.1.22}$$

and

$$(t/2\pi r)^2 x_{j,n}^{-1} = \frac{t}{2\pi r} + \frac{n}{2h^2} - \frac{(-1)^j}{h^2}\left(\frac{n^2}{4} + \frac{hknt}{2\pi}\right)^{1/2}. \tag{4.1.23}$$

Next we show that

$$2^{-1/2}k^{-1/2}x_{j,n}^{-3/4}p''_{j,n}\left(x_{j,n}\right)^{-1/2} = \pi^{1/4}(2hkt)^{-1/4}\left(1 + \frac{\pi n}{2hkt}\right)^{-1/4}. \tag{4.1.24}$$

Indeed, by (4.1.19) we have

$$2kx_{j,n}^{3/2}p''_{j,n}\left(x_{j,n}\right) = \pi^{-1}ktx_{j,n}^{-1/2} + (-1)^j n^{1/2},$$

which by (4.1.20) and (4.1.23) is further equal to

$$(-1)^{j-1}h^2n^{-1/2}\left(2\left(\frac{t}{2\pi r}\right)^2 x_{j,n}^{-1} - \frac{t}{\pi r}\right) + (-1)^j n^{1/2}$$
$$= \pi^{-1/2}(2hkt)^{1/2}\left(1 + \frac{\pi n}{2hkt}\right)^{1/2}.$$

This proves (4.1.24).

89 To complete the calculation of the explicit terms in (3.1.11), we still have to work out $p_{j,n}(x_{j,n})$. Note that by (4.1.22) and (4.1.23)

$$\left(2\pi rt^{-1}x_{j,n}\right)(-1)^j = 1 + \frac{\pi n}{hkt} + \left(\left(\frac{\pi n}{hkt}\right)^2 + \frac{2\pi n}{hkt}\right)^{1/2}$$
$$= \left(\left(\frac{\pi n}{2hkt}\right)^{1/2} + \left(1 + \frac{\pi n}{2hkt}\right)^{1/2}\right)^2,$$

whence

$$\log\left(2\pi rt^{-1}x_{j,n}\right) = (-1)^j 2ar\sin h\left(\left(\frac{\pi n}{2hkt}\right)^{1/2}\right). \tag{4.1.25}$$

Also, by (4.1.20) and (4.1.22),

$$2\pi rx_{j,n} + 4\pi(-1)^{j-1}n^{1/2}x_{j,n}^{1/2}k^{-1}$$
$$= 2t - 2\pi rx_{j,n}$$
$$= t - \frac{\pi n}{hk} + (-1)^{j-1}2t\left(\frac{\pi n}{2hkt} + \left(\frac{\pi n}{2hkt}\right)^2\right)^{1/2}$$

Together with (4.1.19), (4.1.25), and (4.1.6), this gives

$$2\pi p_{j,n}\left(x_{j,n}\right) = (-1)^{j-1}\left(2t\phi\left(\frac{\pi n}{2hkt}\right) - \frac{\pi}{4}\right) - t\log(t(2\pi) + t\log r + t - \frac{\pi n}{hk}.$$

Hence, using (4.1.5) again, we have

$$i(-1)^{j-1}e\left(p_{j,n}\left(x_{j,n}\right) + 1/8\right) = \tag{4.1.26}$$
$$= e\left(-\frac{n}{2hk}\right)\exp\left(i(-1)^{j-1}\left(2t\phi\left(\frac{\pi n}{2hkt}\right) + \frac{\pi}{4}\right)\right)r^{it}$$

$$\chi\left(\frac{1}{2}+it\right)\left(1+0\left(t^{-1}\right)\right).$$

By (4.1.18), (4.1.24), and (4.1.26), we find that the explicit terms on the right of (3.1.11) coincide with those in (4.1.12), up to the factor $1+o(t^{-1})$. The correction $o(t^{-1})$ can be omitted with an error

$$\ll t^{-1}\left((hk)^{-1/2}L+(hkt)^{-1/4}n_1^{3/4}L\right),$$

which is 90

$$\ll t^{-1}\left((hk)^{-1/2}L+t^{-1}h^2k^{-1}m_1^{3/2}L\right)$$

$$\ll hm_1^{1/2}t^{-1}L \tag{4.1.27}$$

by (4.1.11), (4.1.7), and (4.1.10). This is clearly negligible in (4.1.12).

Finally, the error terms in (3.1.11) give those in (4.1.12) by (4.1.15) and (4.1.16).

An application of Theorem 3.2 yields an analogous result for $S_\varphi(M_1, M_2)$. □

Theorem 4.2. *Suppose that the conditions of Theorem 4.1 are satisfied. Then*

(4.1.28)

$$\begin{aligned} S_\varphi(M_1,M_2) &= \pi^{1/4}(2hkt)^{-1/4}\left\{\sum_{j=1}^{2}\sum_{n<n_j}a(n)e\left(n\left(\frac{\bar{h}}{k}-\frac{1}{2hk}\right)\right)\times\right.\\ &\times n^{1/4-k/2}\left(1+\frac{\pi n}{2hkt}\right)^{-1/4}\exp\left(i(-1)^{j-1}\left(2t\phi\left(\frac{\pi n}{2hkt}\right)+\frac{\pi}{4}\right)\right)\left.\vphantom{\sum_{j=1}^{2}}\right\}r^{it}\\ &\chi(1/2+it)+0\left(hm_1^{1/2}t^{-1/2}L^2\right)+0\left(h^{-1/4}k^{3/4}m_1^{-1/4}L\right).\end{aligned}$$

Turnign to smoothed Dirichlet polynomials, we first state a transformation formula for $\tilde{S}(M_1, M_2)$.

Theorem 4.3. *Suppose that the conditions of Theorem 4.1 are satisfied. Let*

$$U \gg r^{-1}t^{1/2+\delta_4} \tag{4.1.29}$$

and let J be a fixed positive integer exceeding a certain bound (which depends on δ_4). Write for $j = 1, 2$

$$M'_j = M_j + (-1)^{j-1} JU = \frac{t}{2\pi r} + (-1)^j m'_j,$$

91 *and suppose that $m_j \asymp m'_j$. Define*

$$n'_j = h^2 (m'_j)^2 \left(M'_j\right)^{-1}. \tag{4.1.30}$$

Then, defining the weight function $\eta_J(x)$ in the interval $[M_1, M_2]$ with the aid of the parameters U and J, we have

$$\begin{aligned}\tilde{S}\,(M_1, M_2) = \Big\{&(hk)^{-1/2}(\log(t/2\pi) + 2\gamma - \log(hk)) \\ &+\pi^{1/4}(2hkt)^{-1/4} \sum_{j=1}^{2} \sum_{n<n_j} w_j(n) d(n) e\left(n\left(\frac{\bar{h}}{k} - \frac{1}{2hk}\right)\right) n^{-1/4} \times \\ &\times \left(1 + \frac{\pi n}{2hkt}\right)^{-1/4} \exp\left(i(-1)^{j-1}\left(2t\phi\left(\frac{\pi n}{2hkt}\right) + \frac{\pi}{4}\right)\right)\Big\} r^{it}\chi(1/2 + it) \\ &+o\left(h^2 k^{-1} m_1^{1/2} t^{-3/2} UL\right),\end{aligned} \tag{4.1.31}$$

where

$$w_j(n) = 1 \quad \textit{for} \quad n < n'_j, \tag{4.1.32}$$

$$w_j(n) \ll 1 \quad \textit{for} \quad n < n_j, \tag{4.1.33}$$

$w_j(y)$ and $w'_j(y)$ are piecewise continuous in the interval (n'_j, n_j) with at most $J - 1$ discontinuities, and

$$w'_j(y) \ll \left(n_j - n'_j\right)^{-1} \quad \textit{for} \quad n'_j < y < n_j \tag{4.1.34}$$

whenever $w'_j(y)$ exists

Proof. We apply Theorem 3.3 to the sum $\tilde{S}\,(M_1, M_2)$ with f, g, F, G and r as in the proof of Theorem 4.1; in particular, $F = t$. Hence the condition (3.2.1) on U holds by (4.1.29). The other assumptions of the theorem are readily verified, and the explicit terms in (4.1.31) were already calculated in the proof of Theorem 4.1,up to the properties of the weight functions $w_j(y)$, which follow from (3.2.4) - (3.2.6).

The error term in (3.2.3) gives that in (4.1.31). It should also be
92 noted that as in the proof of Theorem 4.1 there is an extra error term caused by $o(t^{-1})$ in the formula (4.1.5) for $\chi(1/2+it)$. This error term is $\ll hm_1^{1/2}t^{-1}L$, as was seen in (4.1.27). By (4.1.29) this can be absorbed into the error term in (4.1.31), and the proof of the theorem is complete.

The analogue of the preceding theorem for $\tilde{S}_\varphi(M_1, M_2)$ reads as follows. □

Theorem 4.4. *With the assumptions of Theorem 4.3, we have*

(4.1.35)
$$\begin{aligned}\tilde{S}_\varphi(M_1, M_2) = \pi^{1/4}(2hkt)^{-1/4}&\left\{\sum_{j=1}^{2}\sum_{n<n_j} w_j(n)a(n)\times\right.\\ &\times e\left(n\left(\frac{\bar{h}}{k}-\frac{1}{2hk}\right)\right)n^{1/4-k/2}\left(1+\frac{\pi n}{2hkt}\right)^{-1/4}\times\\ &\left.\times \exp\left(i(-1)^{j-1}\left(2t\phi\left(\frac{\pi n}{2hkt}\right)+\frac{\pi}{4}\right)\right)\right\}r^{it}\chi(1/2+it)\\ &+o\left(h^2k^{-1}m_1^{1/2}t^{-3/2}UL\right).\end{aligned}$$

Remark 1. It is an easy corollary of Theorem 4.1 that
(4.1.36)
$$\left|\sum_{x_1\le n\le x_2} d(n)n^{-1/2-it}\right| \ll \log t \quad \text{for} \quad t\ge 2 \quad \text{and} \quad |x_i - t/2\pi| \ll t^{2/3}.$$

Remark 2. In Theorems 4.3 and 4.4 the error term is minimal when U is as small as possible, i.e.

$$U \asymp r^{-1}t^{1/2+\epsilon}. \tag{4.1.37}$$

The error term then becomes

$$0\left(hm_1^{1/2}t^{-1+\epsilon}\right). \tag{4.1.38}$$

This is significantly smaller than the error terms in Theorems 4.1 and 4.2. for example, if $r = 1$ and $m_1 = t^{3/4}$, then the error in Theorem 4.1 is $\ll t^{-1/8}L^2$, while (4.1.38) is just $\ll t^{-5/8+\epsilon}$. The lengths n_j of the transformed sums are about $t^{1/2}$, which is smaller than the length $\asymp t^{3/4}$
93 of the original sum. A trivial estimate of the right hand side of (4.1.12) is $\ll t^{1/8}L$, which is a trivial estimate of the original sum is $\ll t^{1/4}L$.

Thanks to good error terms, Theorems 4.3 and 4.4 are useful when a number of sums are dealt with and there is a danger of the accumulation of error terms.

Remark 3. With suitable modifications, the theorems of this section hold for negative values of t as well. In this case r (and thus also h) will be negative. Because $p''_{j,n}$ is now negative, our saddle point theorems take a slightly different form (see the remark in the end of § 2.1. When the calculations in the proof of Theorem 4.1 are carried out, then instead of (4.1.12) we obtain, for $t < 0$,

$$\begin{aligned} S\ (M_1, M_2) = \Big\{ & (|h|k)^{-1/2}(\log|t|/2\pi) + 2\gamma - \log(|h|k) \\ & + \pi^{1/4}(2hkt)^{-1/4} \sum_{j=1}^{2} \sum_{n<n_j} d(n) e\left(n\left(\frac{\overline{h}}{k} - \frac{1}{2hk}\right)\right) n^{-1/4} \times \\ & \times \left(1 + \frac{\pi n}{2hkt}\right)^{-1/4} \exp\left(i(-1)^{j-1}\left(2t\phi\left(\frac{\pi n}{2hkt}\right) - \frac{\pi}{4}\right)\right)\Big\} |r|^{it} \\ & \chi\left(\frac{1}{2} + it\right) + o\left(|h|^{-3/2}k^{1/2}m_1^{-1}|t|^{1/2}L\right) + o\left(|h|m_1^{1/2}|t|^{-1/2}L^2\right) \\ & + o\left(|h|^{-1/4}k^{3/4}m_1^{-1/4}L\right), \end{aligned}$$

and similar modifications have to be made in the other theorems. Of course, this formula can also be deduced from (4.1.12) simply by complex conjugation.

4.2 On the Order of $\varphi(k/2 + it)$

Dirichlet series are usually estimated by making use of their approximate functional equations. For $\zeta^2(s)$, this result is classical-due to

Hardy-Littlewood and Titchmarsh - and states that for $0 \leq \sigma \leq 1$ and $t \geq 10$

$$\zeta^2(s) = \sum_{n \leq x} d(n) n^{-s} + \chi^2(s) \sum_{n \leq y} d(n) n^{s-1} + o\left(x^{1/2-\sigma} \log t\right), \tag{4.2.1}$$

where $x \geq 1, y \geq 1$, and $xy = (t/2\pi)^2$. Analogously, for $\varphi(s)$ we have **94**

$$\varphi(s) = \sum_{n \leq x} a(n) n^{-s} + \psi(s) \sum_{n \leq y} a(n) n^{s-k} + o\left(x^{k/2-\sigma} \log t\right), \tag{4.2.2}$$

where

$$\psi(s) = (-1)^{k/2} (2\pi)^{2s-k} \Gamma(k-s)/\Gamma(s).$$

For proofs of (4.2.1) and (4.2.2), see e.g. [19].

The problem of the order of $\varphi(k/2+it)$ can thus be reduced to estimating sums

$$\sum_{n \leq x} a(n) n^{-k/2-it} \tag{4.2.3}$$

for $x \ll t$; we take here t positive, for the case when t is negative is much the same, as was seen in Remark 3 in the preceding section.

Estimating the sum (4.2.3) by absolute values, we obtain

$$|\varphi(k/2+it)| \ll t^{1/2} L,$$

which might be called a "trivial" estimate. If there is a certain amount of cancellation in this sum, then one has

$$|\varphi(k/2+it)| \ll t^{\alpha}, \tag{4.2.4}$$

where $\alpha < 1/2$. An analogous problem is estimating the order of $\zeta(1/2+it)$, and in virtue of the analogy between $\zeta^2(1/2+it)$ and $\varphi(k/2+it)$, one would expect that if

$$|\zeta(1/2+it)| \ll t^{c}, \tag{4.2.5}$$

then (4.2.4) holds with $\alpha = 2c$. (Recently it has been shown by E. Bombieri and H. Iwaniec that $c = 9/56 + epsilon$ is admissible). In particular, as an analogue of the Lindelöf hypothesis for the zeta-function, **95**
one may conjecture that (4.2.4) holds for all positive α. A counterpart of the classical exponent $c = 1/6$ would be $\alpha = 1/3$, for which (4.2.4) is indeed known to hold, up to an unimportant logarithmic factor. More precisely, Good [9] proved that

$$|\varphi(k/2 + it)| \ll t^{1/3}(\log t)^{5/6} \tag{4.2.6}$$

as a corollary of his mean value theorem (0.11). The proof of the latter, being based on the spectral theory of the hyperbolic Laplacian, is sophisticated and highly non-elementary.

A more elementary approach to $\varphi(k/2 + it)$ via the transformation formulae of the preceding section leads rather easily to an estimate which is essentially the same as (4.2.6).

Theorem 4.5. *We have*

$$|\varphi(k/2 + it)| \ll (|t| + 1)^{1/3+\epsilon}. \tag{4.2.7}$$

Proof. We shall show that for all large positive values of t and for all numbers M, M' with $1 \leq M < M' \leq t/2\pi$ and $M' \leq 2M$ we have

$$\left|S_\varphi(M, M')\right| = \left|\sum_{M\leq m\leq M'} a(m)m^{-k/2-it}\right| \ll t^{1/3+\epsilon}. \tag{4.2.8}$$

A similar estimate could be proved likewise for negative values of t, and the assertion (4.2.7) then follows from the approximate functional equation (4.2.2).

Let δ be a fixed positive number, which may be chosen arbitrarily small. for $M \leq t^{2/3+\delta}$ the inequality (4.2.8) is easily verified on estimating the sum by absolute values.

96 Let now

$$\begin{aligned} M_\circ &= t^{2/3+\delta}, \\ M_\circ &< M < M' \leq t/2\pi, \end{aligned} \tag{4.2.9}$$

(4.2.10) $$K=(M/M_\circ)^{1/2},$$

and consider the increasing sequence of reduced fractions $r=h/k$ with $1\leq k\leq K$, in other words the Farey sequence of order K.

The *mediant* of two consecutive fractions $r=h/k$ and $r'=h'/k'$ is

$$\rho=\frac{h+h'}{k+k'}$$

The basic well-known properties of the mediant are :$r<\rho<r'$, and

(4.2.11) $$\rho-r=\frac{1}{k(k+k')}\asymp\frac{1}{kK},r'-\rho=\frac{1}{k'(k+k')}\asymp\frac{1}{k'K}.$$

Subdivide now the interval $[M,M']$ by all the points

(4.2.12) $$M(\rho)=\frac{t}{2\pi\rho}$$

lying in this interval; here ρ runs over the mediants. Then the sum $S_\varphi(M,M')$ is accordingly split up into segments, the first and last one of which may be incomplete. Thus, the sum $S_\varphi(M,M')$ now becomes a sum of subsums of the type

(4.2.13) $$S_\varphi(M(\rho'),M(\rho)),$$

up to perhaps one or two incomplete sums. This sum is related to that fraction $r=h/k$ of our system which lies between ρ and ρ'. We are going to apply Theorem 4.2 to the sum (4.2.13). The numbers m_1 and m_2 in the theorem are now $M(r)-M(\rho')$ and $M(\rho)-M(r)$. Hence $m_1\asymp m_2$ by (4.2.12) and (4.2.11), which imply moreover that **97**

(4.2.14) $$m_j\asymp tr^{-2}(r-\rho)\asymp k^{-1}K^{-1}M^2t^{-1}\asymp k^{-1}M^{3/2}t^{-2/3+\delta/2}.$$

This gives further

(4.2.15) $$Mt^{-1/3+\delta}\ll m_j\ll Mt^{-1/6+\delta/2}.$$

It follows that the incomplete sums contribute

$$\ll M^{1/2}t^{-1/6+\delta}\ll t^{1/3+\delta},$$

which can be omitted.

Next we check the conditions of Theorem 4.2, i.e. the conditions (4.1.8) and (4.1.10) of Theorem 4.1. The validity of (4.1.8) is clear by (4.2.10) and (4.2.9). In (4.1.10), the upper bound for m_1 follows from (4.2.15). As to the lower bound, note that $m_1 \gg Mt^{-1/2+\delta} \asymp t^{1/2+\delta}r^{-1}$ by (4.2.15), and that

$$hk = rk^2 \ll \left(M^{-1}t\right) \left(MM_{\circ}^{-1}\right) \asymp t^{1/3-\delta} \ll m_j t^{-3\delta}.$$

The error terms in (4.1.28) can be estimated by (4.2.10), (4.2.12), and (4.2.14). The first of them is $\ll L^2$, and the second is smaller. The number of subsums is

$$\asymp \left(tM^{-1}\right) K^2 \asymp t^{1/3-\delta}.$$

Hence the contribution of the error terms is $\ll t^{1/3}$.

Next we turn to the main terms in (4.1.28). A useful observation will be that the numbers

$$n_j \asymp m_j^2 h^2 M^{-1}$$

are of the same order for all relevant r, namely

$$n_j \asymp t^{2/3+\delta}. \tag{4.2.16}$$

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This is easily seen by (4.2.14).

To simplify the expression in (4.1.28), we omit the factors

$$\left(1 + \frac{\pi n}{2hkt}\right)^{-1/4} = 1 + o\left(k^{-2}Mnt^{-2}\right),$$

which can be done with a negligible error $\ll 1$.

We now add up the expression in (4.1.28) for different fractions r. Putting

$$\tilde{a}(n) = a(n)n^{-(k-1)/2},$$

we end up with the problem of estimating the multiple sum

$$t^{-1/4}\left|\sum_{h,k}(hk)^{-1/4}(h/k)^{it}\sum_{n<n_j}\tilde{a}(n)n^{-1/4}e\left(n\left(\frac{\overline{h}}{k}-\frac{1}{2hk}\right)\right)\times\right. \tag{4.2.17}$$

$$\times \exp\left(i(-1)^{j-1}\left(2t\phi\left(\frac{\pi n}{2hkt}\right)+\pi/4\right)\right)\Bigg|.$$

As a matter of fact, the numbers n_j depend also on $r = h/k$, but this does not matter, for only the order of n_j will be of relevance.

For convenience we restrict in (4.2.17) the summations to the intervals $K_\circ \leq k \leq K'_\circ, N_\circ \leq n \leq N'_\circ$, where $K'_\circ \leq \min(2K_\circ, K)$, and $N'_\circ \leq 2N_\circ, N_\circ \ll t^{2/3+\delta}$. Also we take for j one of its two values, say $j = 1$. The whole sum is then a sum of $o(t^\delta)$ such sums.

It may happen that some of the n-sums are incomplete. In order to have formally complete sums, we replace $\tilde{a}(n)$ by $\tilde{a}(n)\delta(h,k;n)$, where

$$\delta(h,k;n) = \begin{cases} 1 & \text{for} \quad n < n_1(h,k), \\ 0 & \text{otherwise}; \end{cases}$$

the dependence of n_1 on h/k is here indicated by the notation $n_1(h,k)$. Then, changing in (4.2.17) the order of the summations with respect to **99**
n and the pairs h, k, followed by applications of Cauchy's inequality and Rankin's estimate (1.2.4), we obtain

$$\ll t^{-1/4}N_\circ^{1/4}\left\{\sum_n \left|\sum_{h,k}\delta(h,k;n)\ (hk)^{-1/4}(h/k)^{it}\times\right.\right.$$
$$\left.\left.\times\ e\left(n\left(\frac{\bar{h}}{k}-\frac{1}{2hk}\right)+(1/\pi)t\phi\left(\frac{\pi n}{2hkt}\right)\right)\right|^2\right\}^{1/2}.$$

Here the square is written out as a double sum with respect to h_1, k_1, and h_2, k_2, and the order of the summations is inverted. Then, since $N_\circ \ll t^{2/3+\delta}$ and

(4.2.18) $$hk \asymp K_\circ^2 M^{-1} t,$$

the preceding expression is

(4.2.19) $$\ll K_\circ^{-1/2}M^{1/4}t^{-1/3+\delta}\left(\sum_{h_1,k_1}\sum_{h_2,k_2}|s(h_1,k_1;h_2,k_2)|\right)^{1/2},$$

where

$$s(h_1,k_1;h_2,k_2)=\sum_{N_\circ\le n\le N'_\circ}\delta(h_1,k_1;n)\,\delta(h_2,k_2;n)\,e(f(n)) \tag{4.2.20}$$

with

$$f(x)=x\left(\frac{\bar{h}_1}{k_1}-\frac{1}{2h_1k_1}-\frac{\bar{h}_2}{k_2}+\frac{1}{2h_2k_2}\right)+(t/\pi)\left(\phi\left(\frac{\pi x}{2h_1k_1t}\right)-\phi\left(\frac{\pi x}{2h_2k_2t}\right)\right). \tag{4.2.21}$$

Thus $s(h_1,k_1;h_2,k_2)$ is a sum over a subinterval of $[N_\circ, N'_\circ]$. It will be estimated trivially for quadruples (h_1,k_1,h_2,k_2) such that $h_1k_1 = h_2k_2$, and otherwise by van der Corput's method, applying the following well-known lemma (see [27], Theorem 5.9). □

Lemma 4.1. *Let f be a twice differentiable function such that*

$$0<\lambda_2\le f''(x)\le h\lambda_2 \quad \textit{or} \quad \lambda_2\le -f''(x)\le h\lambda_2$$

100 *throughout the interval (a,b), and $b\ge a+1$. Then*

$$\sum_{a<n\le b+1} e(f(n)) \ll h(b-a)\lambda_2^{1/2}+\lambda_2^{-1/2}.$$

Now, by (4.1.6),

$$\phi''(x)=-\frac{1}{2}x^{-3/2}(1+x)^{-1/2},$$

so that for our function f in (4.2.21)

$$f''(x)=-2^{-3/2}\pi^{-1/2}t^{1/2}x^{-3/2}\left((h_1k_1)^{-1/2}\left(1+\frac{\pi x}{2h_1k_1t}\right)^{-1/2}\right.$$
$$\left.-(h_2k_2)^{-1/2}\left(1+\frac{\pi x}{2h_2k_2t}\right)^{-1/2}\right),$$

and accordingly

$$\lambda_2 \asymp N_\circ^{-3/2}t^{1/2}\left|(h_1k_1)^{-1/2}-(h_2k_2)^{-1/2}\right|$$

$$\asymp K_\circ^{-3} M^{3/2} N_\circ^{-3/2} t^{-1} \left|h_1k_1 - h_2k_2\right|,$$

where we used (4.2.18). Hence, by Lemma 4.1,

$$\begin{aligned} s\left(h_1,k_1;h_2,k_2\right) &\ll K_\circ^{-3/2} M^{3/4} N_\circ^{1/4} t^{-1/2} \left|h_1k_1 - h_2k_2\right|^{1/2} \\ &\quad + K_\circ^{3/2} M^{-3/4} N_\circ^{3/4} t^{1/2} \left|h_1k_1 - h_2k_2\right|^{-1/2} \end{aligned}$$

if $h_1k_1 \neq h_2k_2$. By (4.2.18)

$$\sum_{h_1k_1 \neq h_2k_2} \left|h_1k_1 - h_2k_2\right|^{1/2} \ll \left(K_\circ^2 M^{-1} t\right)^{5/2}$$

and

$$\sum_{h_1k_1 \neq h_2k_2} \left|h_1k_1 - h_2k_2\right|^{-1/2} \ll \left(K_\circ^2 M^{-1} t\right)^{3/2} t^{\delta}.$$

Thus

$$\begin{aligned} \sum_{h_1,k_1} \sum_{h_2,k_2} \left|s\left(h_1,k_1;h_2,k_2\right)\right| &\ll K_\circ^{7/2} M^{-7/4} N_\circ^{1/4} t^2 \\ &\quad + K_\circ^{9/2} M^{-9/4} N_\circ^{3/4} t^{2+\delta} + K_\circ^2 M^{-1} N_\circ t \\ &\ll K_\circ^{7/2} M^{-7/4} t^{13/6+\delta} \\ &\quad + K_\circ^{9/2} M^{-9/4} t^{5/2+2\delta} + K_\circ^2 M^{-1} t^{5/3+\delta}. \end{aligned}$$

Hence the expression (4.2.19) is **101**

$$\begin{aligned} &\ll K_\circ^{5/4} M^{-5/8} t^{3/4+2\delta} + K_\circ^{7/4} M^{-7/8} t^{11/12+2\delta} + K_\circ^{1/2} M^{-1/4} t^{1/2+2\delta} \\ &\ll t^{1/3+2\delta}, \end{aligned}$$

and the proof of the theorem is complete.

Remark. The preceding proof works for $\zeta^2(s)$ as well, and gives

$$\left|\zeta^2(1/2+it)\right| \ll t^{1/3+\epsilon}.$$

This is, of course, a known result, but the argument of the proof is new, though there is a van der Corput type estimate (Lemma 4.1) as an element in common with the classical method.

4.3 Estimation of "Long" Exponential Sums

The method of the preceding section is now carried over to more general exponential sums

$$\sum_{M \leq m \leq M'} b(m)g(m)e(f(m)), b(m) = d(m) \quad \text{or} \quad a(m), \tag{4.3.1}$$

which are "long" in the sense that the length of a sum may be of the order of M. "Short" sums of this type were transformed in Chapter 3 under rather general conditions. Thus the first steps of the proof of Theorem 4.5 –dissection of the sum and transformation of the subsums-can be repeated in the more general context of sums (4.3.1) without any new assumptions, as compared with those in Chapter 3. But it turned out to be difficult to gain a similar saving in the summation of the transformed sums without more specific assumptions on the function f. However, if we suppose that f' is approximately a power, the analogy with the
102 previous case of Dirichlet polynomials will be perfect. The result is as follows.

Theorem 4.6. *Let* $2 \leq M < M' \leq 2M$, *and let* f *be a holomorphic function in the domain*

$$D = \left\{ z \,\middle|\, |z - | < cM \quad \textit{for some} \quad x \in [M, M'] \right\}, \tag{4.3.2}$$

where c is a positive constant. Suppose that $f(x)$ *is real for* $M \leq x \leq M'$, *and that either*

$$f(z) = Bz^{\alpha}\left(1 + 0\left(F^{-1/3}\right)\right) \quad \textit{for} \quad z \in D \tag{4.3.3}$$

where $\alpha \neq 0, 1$ *is a fixed real number, and*

$$F = |B|M^{\alpha}, \tag{4.3.4}$$

or

$$f(z) = B \log z\left(1 + o\left(F^{-1/3}\right)\right) \quad \textit{for} \quad z \in D, \tag{4.3.5}$$

where

$$F = |B|. \tag{4.3.6}$$

Let $g \in C^1[M, M']$, and suppose that for $M \leq x \leq M'$

$$|g(x)| \ll G, |g'(x)| \ll G'. \tag{4.3.7}$$

Suppose also that

$$M^{3/4} \ll F \ll M^{3/2}. \tag{4.3.8}$$

Then

$$\left|\sum_{M \leq m \leq M'} b(m)g(m)e(f(m))\right| \ll (G + MG')M^{1/2}F^{1/3+\epsilon}, \tag{4.3.9}$$

where $b(m) = d(m)$ or $\tilde{a}(m)$.

Proof. We give the details of the proof for $b(m) = d(m)$ only; the other case is similar, even slightly simpler.

It suffices to prove the assertion for $g(x) = 1$, because the general case can be reduced to this by partial summation. **103**

The proof follows that of Theorem 4.5, which corresponds to the case $f(z) = -t \log z$. Then $F = t$, so that the condition (4.3.8) states $t^{2/3} \ll M \ll t^{4/3}$. We restricted ourselves to the case $t^{2/3} \ll M \ll t$, which sufficed for the proof of Theorem 4.5, but the method gives, in fact,

$$\left|\sum_{M \leq m \leq M'} \tilde{a}(m)m^{it}\right| \ll M^{1/2}t^{1/3+\epsilon} \quad \text{for} \quad t^{2/3} \ll M \ll t^{4/3}.$$

This follows, by the way, also from the previous case by a "reflection", using the approximate functional equation (4.2.2).

The analogy between the number t in Theorem 4.5 and the number F in the present theorem will prevail throughout the proof. Accordingly, we put

$$M_\circ = F^{2/3+\delta}$$

and define, as in (4.2.10),

$$K = (M/M_\circ)^{1/2}. \tag{4.3.10}$$

We may suppoe that $M \geq M_\circ$, for otherwise the assertion to be proved, viz.

$$\left|\sum_{M\leq m\leq M'} d(m)e(f(m))\right| \ll M^{1/2}F^{1/3+\epsilon}, \tag{4.3.11}$$

is trivial.

Consider the case when f is of the form (4.3.3); the case (4.3.5) is
104 analogous and can be dealt with by obvious modifications. We suppose that B is of a suitable sign, so that $f''(x)$ is positive.

The equation (4.3.3) can be formally differentiated once or twice to give correct results, for by Cauchy's integral formula we have

$$f'(z) = \alpha B z^{\alpha-1}\left(1 + o\left(F^{-1/3}\right)\right) \tag{4.3.12}$$

and

$$f''(z) = \alpha(\alpha-1)Bz^{\alpha-2}\left(1 + o\left(F^{-1/3}\right)\right) \tag{4.3.13}$$

for z lying in a region D' of the type (4.3.2) with c replaced by a smaller positive number. Hence, for $z \in D'$,

$$\left|f'(z)\right| \asymp FM^{-1}, \tag{4.3.14}$$

and

$$\left|f''(z)\right| \asymp FM^{-2}, \tag{4.3.15}$$

so that the parameter F plays here the same role as in Chapters 2 and 3.

The proof now proceeds as in the previous section. The set of the mediants ρ is constructed for the sequence of fractions $r = h/k$ with $1 \leq k \leq K$, and the interval $[M, M']$ is dissected by the points $M(\rho)$ such that

$$f'(M(\rho)) = \rho. \tag{4.3.16}$$

The numbers m_1 and m_2 then have formally the same expressions as before by (4.3.15) and (4.3.16)

$$m_j \asymp k^{-1}K^{-1}M^2F^{-1} \asymp k^{-1}M^{3/2}F^{-2/3+\delta/2} \tag{4.3.17}$$

in analogy with (4.2.14). This implies that **105**

$$MF^{-1/3+\delta} \ll m_j \ll \min\left(MF^{-1/6+\delta/2}, M^{1/2}F^{1/3+\delta/2}\right); \tag{4.3.18}$$

note that $k \gg F^{-1}M$ for $M \gg F$ by (4.3.16) and (4.3.14). The upper estimate in (4.3.18) shows that the possible incomplete sums in the dissection can be omitted.

The subsums are transformed by Theorem 3.1, the assumptions of which are readily verified as in the proof of Theorem 4.5.

Of the three error terms in (3.1.11), the second one is $\ll M^{1/2}\log^2 M$, and the others are smaller. Since the number of subsums is $\asymp F^{1/3-\delta}$, the contribution of these is $\ll M^{1/2}F^{1/3}$.

The leading term in (3.1.11) is $\ll F^{-1/2}k^{-1}M\log F$. For a given k, h takes $\ll FM^{-1}k$ values. Hence the contribution of the leading terms is

$$\ll F^{1/2}K\log F \ll M^{1/2}F^{1/6}.$$

Consider now the sums of length n_j in (3.1.11). By (3.1.16) and (4.3.17) we have (cf. (4.2.16))

$$n_j \asymp F^{2/3+\delta}. \tag{4.3.19}$$

For convenience, we restrict the triple sum with respect to j, n, and h/k by the conditions $j = 1, N_\circ \le n \le N'_\circ$, and $K_\circ \le k \le K'_\circ$, where $K_\circ \asymp K'_\circ$ and $N_\circ \asymp N'_\circ$. Denote the saddle point $x_{1,n}$ by $x(r,n)$ in order to indicate its dependence on r. Then, for given r, the sum with respect to n can be written as

$$\sum_n Q(r,n)d(n)e\left(f_r(n)\right), \tag{4.3.20}$$

where **106**

$$Q(r,n) = i2^{-1/2}k^{-1/2}n^{-1/4}x(r,n)^{-1/4}\left(f''(x(r,n))-\right. \tag{4.3.21}$$

$$-\frac{1}{2}k^{-1}n^{1/2}x(r,n)^{-3/2}\Bigg)^{-1/2}$$

and

$$f_r(n) = -n\bar{h}/k + f(x(r,n)) - rx(r,n) + 2k^{-1}n^{1/2}x(r,n)^{1/2}. \tag{4.3.22}$$

The range of summation in (4.3.20) is either the whole interval $[N_\circ, N'_\circ]$, or a subinterval of it if $n_1 \leq N'_\circ$. Since

$$|Q(r,n)| \asymp F^{-1/2}K_\circ^{-1/2}M^{3/4}N_\circ^{-1/4},$$

we may write

$$Q(r,n) = F^{-1/2}K_\circ^{-1/2}M^{3/4}N_\circ^{-1/4}q(r,n),$$

where $|q(r,n)| \asymp 1$. Then, using Cauchy's inequality as in the proof of Theorem 4.5, we obtain

$$\begin{aligned}\sum_r \sum_n Q(r,n)d(n)e\left(f_r(n)\right) & \\ \ll F^{-1/2+\delta}K_\circ^{-1/2}M^{3/4}N_\circ^{1/4}&\left(\sum_{r_1,r_2}|s\left(r_1,r_2\right)|\right)^{1/2},\end{aligned} \tag{4.3.23}$$

where

$$s\left(r_1,r_2\right) = \sum_n q\left(r_1,n\right)q\left(r_2,n\right)e\left(f_{r_1}(n) - f_{r_2}(n)\right).$$

The saddle point $x(r,n)$ is defined implicitly by the equation

$$f'(x(r,n)) - r + k^{-1}n^{1/2}x(r,n)^{-1/2} = 0. \tag{4.3.24}$$

Therefore, by the implicit function theorem,

(4.3.25)
$$\left|\frac{dx(r,n)}{dn}\right| \asymp \left(K_\circ^{-1}M^{-1/2}N_\circ^{-1/2}\right)\left(FM^{-2}\right)^{-1} \asymp F^{-1}K_\circ^{-1}M^{3/2}N_\circ^{-1/2}$$

107 Then it is easy to verify that

$$\left|\frac{dq(r,n)}{dn}\right| \ll N_\circ^{-1}F^{\delta/2};$$

the assumption $M \ll F^{4/3}$ is needed here. Consequently, if

$$\left|\sum_n e\left(f_{r_1}(n) - f_{r_2}(n)\right)\right| \le \sigma(r_1, r_2) \tag{4.3.26}$$

whenever n runs over a subinterval of $[N_\circ, N'_\circ]$, then by partial summation

$$|s(r_1, r_2)| \ll \sigma(r_1, r_2)\, F^{\delta/2}.$$

Thus, in order to prove that the left hand side of (4.3.23) is $\ll M^{1/2}F^{1/3+\circ(\delta)}$, it suffices to show that

$$\sum_{r_1, r_2} \sigma(r_1, r_2) \ll F^{5/3+o(\delta)} K_\circ M^{-1/2} N_\circ^{-1/2}. \tag{4.3.27}$$

With an application of Lemma 4.1 in mind, we derive bounds for $\frac{d^2}{dn^2}(f_{r_1}(n) - f_{r_2}(n))$, where n is again understood for a moment as a continuous variable. First, by (4.3.22) and (4.3.24),

$$\begin{aligned}\frac{d f_r(n)}{dn} &= -\bar{h}/k + \left(f'(x(r,n)) - r + k^{-1}n^{1/2}x(r,n)^{-1/2}\right)\frac{dx(r,n)}{dn} \\ &\quad + k^{-1}n^{-1/2}x(r,n)^{1/2} \\ &= -\bar{h}/k + k^{-1}n^{-1/2}x(r,n)^{1/2},\end{aligned}$$

and further

$$\frac{d^2 f_r(n)}{dn^2} = \frac{1}{2}k^{-1}n^{-1/2}x(r,n)^{-1/2}\frac{dx(r,n)}{dn} - \frac{1}{2}k^{-1}n^{-3/2}x(r,n)^{1/2}. \tag{4.3.28}$$

Here the first term, which is

$$\ll F^{-1}K_\circ^{-2}MN_\circ^{-1} \tag{4.3.29}$$

by (4.3.25), will be less significant.

The saddle point $x(r,n)$ is now approximated by the point $M(r)$, which is easier to determine. By definition

$$f'(M(r)) = r;$$

hence by (4.3.12) **108**

$$\alpha BM(r)^{\alpha-1} = r\left(1 + o\left(F^{-1/3}\right)\right),$$

which gives further, by (4.3.4),

$$M(r) = \left(\frac{M^{\alpha}}{|\alpha|F}\right)^{2\beta} |r|^{2\beta}\left(1 + o\left(F^{-1/3}\right)\right) \tag{4.3.30}$$

with

$$\beta = \frac{1}{2(\alpha-1)}.$$

But the difference of $M(r)$ and $x(r,n)$ is at most the maximum of m_1 and m_2, so that by (4.3.17)

$$\begin{aligned} x(r,n) &= M(r) + o\left(F^{-1}K_{\circ}^{-1}K^{-1}M^2\right) \\ &= M(r)\ \left(1 + o\left(F^{-1}K_{\circ}^{-1}K^{-1}M\right)\right). \end{aligned}$$

Hence by (4.3.30)

$$x(r,n) = \left(\frac{M^{\alpha}}{|\alpha|F}\right)^{2\beta} |r|^{2\beta}\left(1 + o\left(F^{-1}K_{\circ}^{-1}K^{-1}M\right)\right);$$

note that by (4.3.10)

$$F^{-1}K_{\circ}^{-1}K^{-1}M \geq F^{-1}K^{-2}M = F^{-1/3+\delta}.$$

So the second term in (4.3.28) is

$$-\frac{1}{2}\left(|\alpha|F\right)^{-\beta} k^{-1} M^{\alpha\beta} n^{-3/2}|r|^{\beta} + o\left(F^{-1}K_{\circ}^{-2}K^{-1}M^{3/2}N_{\circ}^{-3/2}\right).$$

The expression (4.3.29) can be absorbed into the error term here, for $N_{\circ} \ll K^{-2}M$ by (4.3.10) and (4.3.19). Hence (4.3.28) gives

$$\begin{aligned} &\frac{d^2}{dn^2}\left(f_{r_1}(n) - f_{r_2}(n)\right) \\ &\quad = \frac{1}{2}\left(|\alpha|F\right)^{-\beta} M^{\alpha\beta} n^{-3/2}\left(|h_2|^{\beta}k_2^{-1-\beta} - |h_1|^{\beta}k_1^{-1-\beta}\right) \\ &\qquad\qquad + o\left(F^{-1}K_{\circ}^{-2}K^{-1}M^{3/2}N_{\circ}^{-3/2}\right). \end{aligned} \tag{4.3.31}$$

109 By the following lemma, the differences $|h_2|^{\beta}k_2^{-1-\beta} - |h_1|^{\beta}k_1^{-1-\beta}$ are distributed as one would expect on statistical grounds. □

Lemma 4.2. *Let $H \geq 1, K \geq 1$, and $0 < \Delta \ll 1$. Let α and β be non-zero real numbers. Then the number of quadruples (h_1, k_1, h_2, k_2) such that*

$$H \leq h_i \leq 2H,\ K \leq k_i \leq 2K \tag{4.3.32}$$

and

$$\left|h_1^\alpha k_1^\beta - h_2^\alpha k_2^\beta\right| \leq \Delta H^\alpha K^\beta \tag{4.3.33}$$

is at most

$$\ll HK \log^2(2HK) + \Delta H^2 K^2, \tag{4.3.34}$$

where the implied constants depend on α and β.

We complete first the proof of Theorem 4.6, and that of Lemma 4.2 will be given afterwards.

In our case, the number of pairs (r_1, r_2) such that

$$\left||h_2|^\beta k_2^{-1-\beta} - |h_1|^\beta k_1^{-1-\beta}\right| \leq \Delta K_\circ^{-1}\left(FM^{-1}\right)^\beta \tag{4.3.35}$$

is at most

$$\ll FK_\circ^2 M^{-1} \log^2 F + \Delta F^2 K_\circ^4 M^{-2} \tag{4.3.36}$$

by Lemma 4.2. Let

$$\Delta_\circ = c_\circ F^{-1} K_\circ^{-1} K^{-1} M,$$

where $c_\circ$ is a certain positive constant. For those pairs (r_1, r_2) satisfying (4.3.35) with $\Delta = \Delta_\circ$ we estimate trivially $\sigma(r_1, r_2) \ll N_\circ$. Then, by (4.3.36), their contribution to the sum in (4.3.27) is

$$\ll FK_\circ^2 M^{-1} N_\circ \log^2 F \ll F^{5/3+2\delta} K_\circ M^{-1/2} N_\circ^{-1/2}.$$

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Let now $\Delta_\circ \leq \Delta \ll 1$, and consider those pairs (r_1, r_2) for which the expression on the left of (4.3.35) lies in the interval $(\Delta K_\circ^{-1}(FM^{-1})^\beta, 2\Delta K_\circ^{-1}(FM^{-1})^\beta]$. If $c_\circ$ is chosen sufficiently large, then the main term (of order $\asymp \Delta K_\circ^{-1} M^{1/2} N_\circ^{-3/2}$) on the right of (4.3.31) dominates the error term. Then by Lemma 4.1

$$\sigma(r_1, r_2) \ll \Delta^{1/2} K_\circ^{-1/2} M^{1/4} N_\circ^{1/4} + \Delta^{-1/2} K_\circ^{1/2} M^{-1/4} N_\circ^{3/4}.$$

The number of the pairs (r_1, r_2) in question is $\ll \Delta F^2 K_\circ^3 K M^{-2} \log^2 F$ by (4.3.36) and our choice of $\Delta_\circ$, so that they contribute

$$\begin{aligned}
&\ll \Delta^{3/2} F^{2+\delta} K_\circ^{5/2} K M^{-7/4} N_\circ^{1/4} + \Delta^{1/2} F^{2+\delta} K_\circ^{7/2} K M^{-9/4} N_\circ^{3/4} \\
&\ll F^{2+\delta} K_\circ M^{-1/2} N_\circ^{-1/2} \left(K^{5/2} M^{-5/4} N_\circ^{3/4} + K^{7/2} M^{-7/4} N_\circ^{5/4} \right) \\
&\ll F^{5/3+\delta} K_\circ M^{-1/2} N_\circ^{-1/2}.
\end{aligned}$$

The assertion (4.3.27) is now verified, and the proof of Theorem 4.6 is complete.

Proof of Lemma 4.2. To begin with, we estimate the number of quadruples satisfying, besides (4.3.32) and (4.3.33), also the conditions

$$(h_1, h_2) = (k_1, k_2) = 1. \tag{4.3.37}$$

By symmetry, we may suppose that $H \geq K$. The condition (4.3.33) can be written as

$$h_1^\alpha k_1^\beta = h_2^\alpha k_2^\beta (1 + o(\Delta)).$$

Raising both sides to the power α^{-1} and dividing by $h_2 k_1^{\beta/\alpha}$, we obtain

$$\left| \frac{h_1}{h_2} - \left(\frac{k_2}{k_1} \right)^{\beta/\alpha} \right| \ll \Delta.$$

111 for given k_1 and k_2, the number of fractions h_1/h_2 satisfying this, (4.3.32), and (4.3.37), is $\ll 1 + \Delta H^2$ by the theory of Farey fractions. Summation over the pairs k_1, k_2 in question gives

$$\ll K^2 + \Delta H^2 K^2 \ll HK + \Delta H^2 K^2.$$

Consider next quadruples satisfying (4.3.32) and (4.3.33) but instead of (4.3.37) the conditions

$$(h_1, h_2) = h, (k_1, k_2) = k$$

for certain fixed integers h and k. Then, writing $h_1 = hh'_i, k_i = kk'_i$, we find that the quadruples (h'_1, k'_1, h'_2, k'_2) satisfy the conditions (4.3.32), (4.3.33), and (4.3.37) with H and K replaced by H/h and K/k. Hence, as was just proved, the number of these quadruples is

$$\ll HK/hk + \Delta H^2 K^2 (hk)^{-2}.$$

Finally, summation with respect to h and k gives (4.3.34).

Example. To illustrate the scope of Theorem 4.6, let us consider the exponential sum

$$S = \sum_{M \leq m \leq M'} b(m) e\left(\frac{X}{m}\right) \tag{4.3.38}$$

where $b(m)$ is $d(m)$ or $\tilde{a}(m)$. By the theorem,

$$S \ll M^{1/6} X^{1/3+\epsilon} \quad \text{for} \quad M^{7/4} \ll X \ll M^{5/2}.$$

Thus, for $M \asymp \chi^{1/2}$, one has $S \ll M^{5/6+\epsilon}$. In the case $b(m) = d(m)$ it is also possible to interpret S as the double sum

$$\sum_{\substack{m,n \geq 1 \\ M \leq mn \leq M'}} e\left(\frac{X}{mn}\right).$$

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This can be reduced to ordinary exponential sums, fixing first m or n, but it can be also estimated by more sophisticated methods in the theory of multiple exponential sums. For instance, B.R. Srinivasan's theory of n-dimensional exponent pairs gives, for $M \asymp X^{1/2}$ and $b(m) = d(m)$,

$$S \ll M^{1-\ell_1+\ell_\circ}, \tag{4.3.39}$$

where $(\ell_\circ, \ell_1)$ is a two-dimensional exponent pair (see [13], § 2.4). Of the pairs mentioned in [13], the sharpest result is given by $(\frac{23}{250}, \frac{56}{250})$, namely (4.3.39) with the exponent $217/250 = 0.868$. The optimal exponent given by this method is $0.86695\ldots$ (see [10]). If a conjecture concerning one- and two-dimensional exponent pairs (Conjecture P in [10]) is true, then the exponent could be improved to $0.8290\ldots$, which is smaller than $5/6$. But in any case, for $b(m) = \tilde{a}(m)$ the sum S seems to be beyond the scope of ad hoc methods because of the complicated structure of the coefficients $a(m)$.

4.4 The Twelfth Moment of $\zeta(1/2+it)$ and Sixth Moment of $\varphi(k/2+it)$

In this last section, a unified approach to the mean value theorems 0.7 and 0.9 will be given.

Theorem 4.7. *For $T \geq 2$ we have*

$$\int\limits_{\circ}^{T} |\zeta(1/2+it)|^{12}\, dt \ll T^{2+\epsilon} \tag{4.4.1}$$

and

$$\int\limits_{\circ}^{T} |\varphi(k/2+it)|^{6}\, dt \ll T^{2+\epsilon}. \tag{4.4.2}$$

113 *Proof.* The proofs of these estimates are much similar, so it suffices to consider (4.4.2) as an example, with some comments on (4.4.1).

It is enough to prove that

$$\int\limits_{T}^{2T} |\varphi(k/2+it)|^{6}\, dt \ll T^{2+\epsilon}. \tag{4.4.3}$$

Actually we are going to prove a discrete variant of this, namely that

$$\sum_{\nu} |\varphi(k/2+it_\nu)|^{6} \ll T^{2+\epsilon} \tag{4.4.4}$$

whenever $\{t_\nu\}$ is a "well-spaced" system of numbers such that

$$T \leq t_\nu \leq 2T, \left|t_\mu - t_\nu\right| \geq 1 \quad \text{for} \quad \mu \neq \nu. \tag{4.4.5}$$

Obviously this implies (4.4.3). Again, (4.4.4) follows if it is proved that for any $V > 0$ and for any system $\{t_\nu\}, \nu = 1, \ldots, R$, satisfying besides (4.4.5) also the condition

$$|\varphi(k/2+it_\nu)| \geq V, \tag{4.4.6}$$

one has

$$R \ll T^{2+\epsilon} V^{-6}. \tag{4.4.7}$$

The last mentioned assertion is easily verified if

$$V \ll T^{1/4+\delta} \tag{4.4.8}$$

where δ again stands for a positive constant, which may be chosen as small as we please, and which will be kept fixed during the proof. Indeed, one may apply the discrete mean square estimate

$$\sum_{\nu} |(k/2 + it_\nu)|^2 \ll T^{1+\delta} \tag{4.4.9}$$

which is an analogue of the well-known discrete mean fourth power estimate for $|\zeta(1/2 + it)|$ (see [13], equation (8.26)), and can be proved in the same way. Now (4.4.9) and (4.4.6) together give **114**

$$R \ll T^{1+\delta} V^{-2}, \tag{4.4.10}$$

and thus $R \ll T^{2+5\delta} V^{-6}$ if also (4.4.8) holds.

Henceforth we may assume that

$$V \gg T^{1/4+\delta}. \tag{4.4.11}$$

Then by (4.4.10)

$$R \ll T^{1/2-\delta}. \tag{4.4.12}$$

Large values of $\varphi(s)$ on the critical line can be investigated in terms of large values of partial sums of its Dirichlet series, by the approximate functional equation (4.2.2). The partial sums will be decomposed as in the proof of Theorem 4.5. However, in order to have compatible decompositions for different values $t \in [T, 2T]$, we define the system of fractions $r = h/k$ in terms of T rather than in terms of t. As a matter of fact, the "order" K of the system will not be a constant, but it varies as a certain function $K(r)$ of r. More exactly, write

$$M(r, t) = \frac{t}{2\pi r}, \tag{4.4.13}$$

and letting R be the cardinality of the system $\{t_\nu\}$ satisfying (4.4.5), (4.4.6), and (4.4.11), define

$$K(r) = M(r, T)^{1/2} T^{-1/3} R^{-1/3}. \tag{4.4.14}$$

We now construct the (finite) set of all fractions $r = h/k \geq 1$ satisfying the conditions

$$k \leq K(r), \tag{4.4.15}$$

$$K(r) \geq T^{\delta}, \tag{4.4.16}$$

115 and arrange these into an increasing sequence.

This sequence determines the sequence $\rho_1 < \rho_2 < \cdots < \rho_P$ of the mediants, and we define moreover $\rho_\circ = \rho_1^{-1}$. We apply (4.2.2) for $\sigma = k/2$, choosing

$$x = x(t) = M(\rho_\circ, t), y = y(t) = (t/2\pi)^2 x^{-1}. \tag{4.4.17}$$

Then, if (4.4.6) and (4.4.11) hold, at least one of the sums of length $x(t_\nu)$ and $y(t_\nu)$ in (4.2.2) exceeds $V/3$ in absolute value. Let us suppose that for at least $R/2$ points t_ν we have

$$\left| \sum_{n \leq x(t_\nu)} \tilde{a}(n) n^{-1/2 - it_\nu} \right| \geq V/3; \tag{4.4.18}$$

the subsequent arguments would be analogous if the other sum were as large as often.

The sum in (4.4.18) is split up by the points $M(\rho_i, t_\nu)$ as in § 4.2. As to the set of points t_ν satisfying (4.4.18), there are now two alternatives: either

$$\left| \sum_{n \leq M(\rho_P, t_\nu)} \tilde{a}(n) n^{-1/2 - it_\nu} \right| \geq V/6 \tag{4.4.19}$$

for $\gg R$ points, or there are functions $M_1(t), M_2(t)$ of the type $M(\rho_i, t)$ such that $M_1(t) \asymp M_2(t)$ and

$$\left| S_\varphi \left(M_1(t_\nu), M_2(t_\nu) \right) \right| \gg VL^{-1}, \tag{4.4.20}$$

with $L = \log T$, for at least $\gg RL^{-1}$ points t_ν. We are going to derive an upper bound for R in each case.

116 Consider first the former alternative. We apply the following large values theorem of M.N. Huxley for Dirichlet polynomials (for a proof, see [12] or [15]). □

Lemma 4.3. *Let N be a positive integer,*

$$f(s) = \sum_{n=N+1}^{2N} a_n n^{-s}, \tag{4.4.21}$$

and let $s_r = \sigma_r + it_r, r = 1, \ldots, R$, be a set of complex numbers such that $\sigma_r \geq 0, 1 \leq |t_r - t_{r'}| \leq T$ for $r \neq r'$, and

$$|f(s_r)| \geq V > 0.$$

Put

$$G = \sum_{n=N+1}^{2N} |a_n|^2 .$$

Then

$$R \ll \left(GNV^{-2} + TG^3NV^{-6}\right)(NT)^{\epsilon}. \tag{4.4.22}$$

This lemma cannot immediately be applied to the Dirichlet polynomial in (4.4.19), for it is not of the type (4.4.21), and the length of the sum depends moreover on t_ν. To avoid the latter difficulty, we express the Dirichlet polynomials in question by Perron's formula using the function

$$f(w) = \sum_{n \leq N} \tilde{a}(n) n^{-w}$$

with $N = M(\rho_P, 2T)$. Letting $y = N^{1/2}$ and $\alpha = 1/\log N$, we have

$$\sum_{n \leq M(\rho_P, t_\nu)} \tilde{a}(n) n^{-1/2 - it_\nu} = \frac{1}{2\pi i} \int_{\alpha - iY}^{\alpha + iY} f\left(1/2 + it_\nu + w\right)$$
$$M\left(\rho_P, t_\nu\right)^w w^{-1}\, dw + o\left(T^{\delta}\right).$$

Now, in view of (4.4.19), there is a number $X \in [1, Y]$ and numbers **117**
N_1, N_2 with $N_1 < N_2 \leq \max(2N_1, N)$ such that writing

$$f_\circ(w) = \sum_{N_1 \leq n \leq N_2} \tilde{a}(n) n^{-w}$$

we have

$$\int_{-X}^{X} |f_\circ(1/2 + \alpha + i(t_\nu + u))| \, du \gg V\chi L^{-2} \tag{4.4.23}$$

for at least $\gg RL^{-2}$ points t_ν. Next we select a sparse set of $R_\circ$ numbers t_ν with

$$R_\circ \ll 1 + RX^{-1}L^{-2} \tag{4.4.24}$$

such that (4.4.23) holds for these, and moreover $|t_\mu - t_\nu| \geq 3X$ for $\mu \neq \nu$. Further, by (4.4.23) and similar quantitative arguments as above, we conclude that there exist a number $w \gg VL^{-2}$, a subset of cardinality $\gg R_\circ L^{-1}$ of the set of the $R_\circ$ indices just selected, and for each ν in this subset a set $\gg VW^{-1}XL^{-3}$ points $u_{\nu,\mu} \in [-X, X]$ such that

$$W \leq \left| f_\circ\left(1/2 + \alpha + i\left(t_\nu + u_{\nu,\mu}\right)\right) \right| \leq 2W \tag{4.4.25}$$

and

$$\left| u_{\nu,\lambda} - u_{\nu,\mu} \right| \geq 1 \quad \text{for} \quad \lambda \neq \mu.$$

The system $t_\nu + u_{\nu,\mu}$ for all relevant pairs ν, μ is well-spaced in the sense that the mutual distance of these numbers is at least 1, and its cardinality is

$$\gg R_\circ VW^{-1}XL^{-4}.$$

On the other hand, its cardinality is by Lemma 4.3

$$\ll \left(N_1 W^{-2} + TN_1 W^{-6}\right) T^\delta \ll W^{-1}\left(NV^{-1} + TNV^{-5}\right) T^\delta L^{10}.$$

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These two estimates give together

$$R_\circ X \ll \left(NV^{-2} + TNV^{-6}\right) T^\delta L^{14}.$$

But $R_\circ X \gg RL^{-2}$ by (4.4.24), so finally

$$R \ll \left(NV^{-2} + TNV^{-6}\right) T^{2\delta} \ll NV^{-2}T^{2\delta} \tag{4.4.26}$$

by (4.4.11). This means that a direct application of Lemma 4.3 gives a correct result in the present case though the conditions of the lemma are not formally satisfied.

Since ρ_P was the last mediant, we have by (4.4.14), (4.4.16), and the definition of N

$$N \ll T^{2/3+2\delta} R^{2/3}.$$

Together with (4.4.26), this implies

$$R \ll T^{2/3+4\delta} R^{2/3} V^{-2},$$

whence

$$R \ll T^{2+12\delta} V^{-6} \tag{4.4.27}$$

We have now proved the desired estimate for R in the case that (4.4.19) holds for $\gg R$ indices ν.

Turning to the alternative (4.4.20), we write

$$S_\varphi\left(M_1(t), M_2(t)\right) = \sum_{i=i_1}^{i_2} S_\varphi\left(M\left(\rho_{i+1}, t\right), M\left(\rho_i, t\right)\right) \tag{4.4.28}$$

for $T \leq t \leq 2T$. The sums S_φ here are transformed by Theorem 4.2. That unique fraction $r = hk$ which lies between ρ_i and ρ_{i+1} will be used as the fraction r in the theorem. Write $M = M_1(T)$ and 119

$$K = M^{1/2} T^{-1/3} R^{-1/3}. \tag{4.4.29}$$

Then by (4.4.14) we have $K(r) \asymp K$ for those r related to the sums in (4.4.28). Since for two consecutive fractions r and r' of our system we have $r' - r \leq [K(r')]^{-1}$, it is easily seen that $K(r) - K(r') < 1$. Thus either r and r' are consecutive fractions in the Farey system of order $[K(r)]$, or exactly one fraction $r'' = h''/k''$ with $K(r') < k'' \leq K(r)$ of this system lies between them. Then, in any case, $|r - \rho_j| \asymp (kK)^{-1}$ for $j = i$ and $i + 1$, whence as in (4.2.14) we have

$$m_j \asymp k^{-1}K^{-1}M^2T^{-1} \asymp k^{-1}M^{3/2}T^{-2/3}R^{1/3}. \tag{4.4.30}$$

Hence $m_j \ll M^{1-\delta/3}$ by (4.4.12), so that the upper bound part of the condition (4.1.10) is satisfied. The other conditions of Theorem 4.2 are easily checked as in the proof of Theorem 4.5.

The error terms in Theorem 4.2 are now by (4.4.30) and (4.4.13) $o(k^{1/2}k^{-1/2}L^2)$ and $o(K^{3/4}K^{1/4}M^{-1/4}L)$, and the sum of these for different r is

$$\begin{aligned} &\ll k^2M^{-1}TL^2 + K^3TM^{-5/4}L \\ &\ll T^{1/3}R^{-2/3}L^2 + M^{1/4}R^{-1}L \ll T^{1/3}R^{-2/3}L^2. \end{aligned}$$

If

$$T^{1/3}R^{-2/3} \ll VT^{-\delta},$$

then these error terms can be omitted in (4.4.20). Otherwise

$$\begin{aligned} R &\ll T^{1/2+3\delta/2}V^{-3/2} \\ &\ll \left(T^{1/2+3\delta/2}V^{-3/2}\right)^4 = T^{2+6\delta}V^{-6} \end{aligned}$$

120 and we have nothing to prove. Hence, in any case, we may omit the error terms in (4.1.28).

Consider now the explicit terms in Theorem 4.2. For the numbers n_j we have by (4.1.11), (4.4.30), and (4.4.29)

$$n_j \asymp K^{-2}M \asymp T^{2/3}R^{2/3}. \tag{4.4.31}$$

Denote by $S_r(t)$ the explicit part of the right hand side of (4.1.28) for the sum related to the fraction r. Then by (4.4.20), (4.4.28), and the error estimate just made we have

$$\left|\sum_r S_r(t_\nu)\right| \gg VL^{-1} \tag{4.4.32}$$

for at least $\gg RL^{-1}$ numbers t_ν.

At this stage we make a brief digression to the proof of the estimate (4.4.1). So far everything we have done for $\varphi(s)$ goes through for $\zeta^2(s)$ as well, except that in Theorem 4.1 there is the leading explicit term and the first error term which have no counterpart in Theorem 4.2. The additional explicit term is $\asymp (hk)^{-1/2}L$, and the sum of these over the relevant fractions r is $\ll T^{1/6}L$, which can be omitted by (4.4.11). The additional error term in (4.1.12) is also negligible, for it is dominated in our case by the second one. So the analogy between the proofs of (4.4.1) and (4.4.2) prevails here, like also henceforth.

It will be convenient to restrict the fractions $r = h/k$ in (4.4.32) suitably. Suppose that $K_\circ \le k \le K'_\circ$, where $K_\circ \asymp K'_\circ$ and $K_\circ \ll K$, and suppose also that for two different fractions $r = h/k, r' = h'/k'$ in our **121**
system we have

$$\left| r - r' \right| \gg K_\circ^{-2} T^\delta \tag{4.4.33}$$

and

$$0 < \left| \frac{1}{hk} - \frac{1}{h'k'} \right| < (K'_\circ)^{-2} \tag{4.4.34}$$

An interval $[K_\circ, K'_\circ]$ and a set of fractions of this kind can be found such that

$$\left| \sum_r S_r(t_\nu) \right| \gg VT^{-2\delta} \tag{4.4.35}$$

for at least $R_1 \gg RT^{-2\delta}$ numbers t_ν. The sum over r here is restricted as indicated above.

Let

$$Z = K_\circ^2 M^{-1} T. \tag{4.4.36}$$

There exists a number R_2 such that those intervals $[T + pZ, T + (P + 1)Z]$ containing at least $R_2/2$ and at most $2R_2$ of the R_1 numbers t_ν contain together $\gg R_1 L^{-1}$ of these. Omit the other numbers t_ν, and suppose henceforth that the t_ν under consideration lie in these $\ll R_1R_2^{-1}$ intervals. Summing (4.4.35) with respect to those t_ν lying in the interval $[T + pZ, T + (p + 1)Z]$, we obtain by Cauchy's inequality

$$R_2VT^{-2\delta} \ll \left(R_2 \sum_\nu \left|\sum_r S_r(t_\nu)\right|^2\right)^{1/2}. \tag{4.4.37}$$

The following inequality of P.X. Gallagher (see [23], Lemma 1.4) is now applied to the sum over t_ν.

Lemma 4.4. *Let $T_\circ, T \geq \delta > 0$ be real numbers, and let A be a finite set in the interval $[T_\circ + \delta/2, T_\circ + T - \delta/2]$ such that $|a' - a| \geq \delta$ for any*
122 *two distinct numbers $a, a' \in A$. Let S be a continuous complex valued function in $[T_\circ, T_\circ + T]$ with continuous derivative in $(T_\circ, T_\circ + T)$. Then*

$$\sum_{a\epsilon A} |S(a)|^2 \leq \delta^{-1} \int\limits_{T_\circ}^{T_\circ+T} |S(t)|^2\, dt + \left(\int\limits_{T_\circ}^{T_\circ+T} |S(t)|^2\, dt\right)^{1/2} \left(\int\limits_{T_\circ}^{T_\circ+T} |S'(t)|^2\, dt\right)^{1/2}.$$

The lengths n_j of the sums in $S_r(t)$ depend linearly on t. However, the variation of n_j in the interval $T + pZ \leq t < T + (p + 1)Z$ is only $o(1)$, so that (4.4.35) and (4.4.37) remain valid if we redefine $S_r(t)$ taking n_j constant in this interval. Lemma 4.4 then gives

$$\begin{aligned}\sum_\nu \left|\sum_r S_r({}_\nu)\right|^2 &\ll \int\limits_\circ^Z \left|\sum_r S_r(T + pZ + u)\right|^2 du \\ &+ \left(\int\limits_\circ^Z \left|\sum_r S_r(T + pZ + u)\right|^2 du\right)^{1/2} \left(\int\limits_\circ^Z \left|\sum_r S'_r(T + pZ + u)\right|^2 du\right)^{1/2}.\end{aligned} \tag{4.4.38}$$

Let $\eta(u)$ be a weight function of the type $\eta_J(u)$ such that $JU = Z, \eta(u) = 1$ for $0 \leq u \leq Z, \eta(u) = 0$ for $u \notin (-Z, 2Z)$, and J is a large positive integer. Then

$$\int\limits_\circ^Z \left|\sum_r S_r(T + pZ + u)\right|^2 du \leq \int\limits_{-Z}^{2Z} \eta(u) \left|\sum_r S_r\right|^2 du \tag{4.4.39}$$

$$= \sum_{r,r'} \int\limits_{-Z}^{2Z} \eta(u) S_r \overline{S_{r'}}\, du.$$

We now dispose of the nondiagonal terms. Put $t(u) = T + pZ + u$. When the integral on the right of (4.4.39) is written as a sum of integrals, recalling the definition (4.1.28) of $S_r(t)$, a typical term is

$$\int\limits_{-Z}^{2Z} \eta(u) g(u) e(f(u))\, du, \tag{4.4.40}$$

where 123

$$\begin{aligned} g(u) = {} & \pi^{1/2} 2^{-1/2} (hkh'k')^{-1/4} \tilde{a}(n) \overline{\tilde{a}(n')}\, (nn')^{-1/4} \times \\ & \times e\left(n\left(\frac{\overline{h}}{k} - \frac{1}{2hk} \right) - n'\left(\frac{\overline{h'}}{k'} - \frac{1}{2h'k'} \right) \right) t(u)^{-1/2} \\ & \left(1 + \frac{\pi n}{2hkt(u)} \right)^{-1/4} \left(1 + \frac{\pi n}{2h'k't(u)} \right)^{-1/4}, \\ f(u) = {} & (-1)^{j-1} \left\{ (t(u)/\pi) \phi\left(\frac{\pi n}{2hkt(u)} \right) + 1/8 \right\} \\ & - (-1)^{j'-1} \left\{ (t(u)/\pi) \phi\left(\frac{\pi n'}{2h'k't(u)} \right) + 1/8 \right\} + (t(u)/\pi) \log(r/r'), \end{aligned}$$

$r = h/k, r' = h'/k'$, j' and j' are 1 or 2, and $n < n_j, n' < n_{j'}$. Now

$$\left| \frac{d}{du} \left(t(u) \log(r/r') \right) \right| = \left| \log(r/r') \right| \gg K_\circ^{-2} M T^{-1+\delta}$$

by (4.4.33), while by (4.1.6)

$$\left| \frac{d}{du} \left(t(u) \phi\left(\frac{\pi n}{2hkt(u)} \right) \right) \right| \asymp (hkT)^{-1/2} n^{1/2} \asymp K_\circ^{-1} M^{1/2} n^{1/2} T^{-1},$$

which is by (4.4.31)

$$\ll K_\circ^{-1} K^{-1} M T^{-1} \ll K_\circ^{-2} M T^{-1}.$$

Accordingly,

$$|f'(u)| \asymp |\log(r,/r')| \gg T^{\delta}Z^{-1}.$$

We may now apply Theorem 2.3 to the integral (4.4.40) with $\mu \asymp Z$, $M \asymp |\log(r,r')| \gg T^{\delta}Z^{-1}$, and $U \asymp Z$. If $J \asymp \delta^{-2}$ and δ is small, then this integral is negligible. A similar argument applies to the integral involving S_r' in (4.4.38). Consequently, it follows from (4.4.37) - (4.4.39) that

$$R_2VT^{-2\delta} \ll R_2^{1/2}\left\{\sum_r \int_{-Z}^{2Z} |S_r(t(u))|^2 \, du \right.$$
$$\left. + \left(\sum_r \int_{-Z}^{2Z} |S_r(t(u))|^2 \, du\right)^{1/2} \left(\sum_r \int_{-Z}^{2Z} \left|S_r'(t(u))\right|^2 \, du\right)^{1/2}\right\}^{1/2}.$$

124 Summing these inequalities with respect to the $\ll R_1R_2^{-1}$ values of p, we obtain by Cauchy's inequality

$$R_1L^{-1}VT^{-2\delta} \ll R_1^{1/2}\left\{\sum_{p,r} \int_{-Z}^{2Z} |S_r(T+pZ+u)|^2 \, du \right.$$
$$\left. + \left(\sum_{p,r} \int_{-Z}^{2Z} |S_r(T+pZ+u)|^2 \, du\right)^{1/2} \left(\sum_{p,r} \int_{-Z}^{2Z} \left|S_r'(T+pZ+u)\right|^2 \, du\right)^{1/2}\right\}^{1/2}.$$

For each p, the integrals here are expressed by the mean value theorem. Then by (4.4.36) this implies (recall that $RT^{-2\delta} \ll R_1 \leq R$)

$$RV \ll R^{1/2}K_{\circ}M^{-1/2}T^{1/2+4\delta}L\left\{\sum_{p,r} \left|S_r(t_p)\right|^2 \right. \tag{4.4.41}$$
$$\left. + \left(\sum_{p,r} \left|S_r(t_p)\right|^2\right)^{1/2} \left(\sum_{p,r} \left|S_r'(t_p')\right|^2\right)^{1/2}\right\}^{1/2},$$

where $\{t_p\}$ is a set of numbers in the interval $(T - Z, 2T + 2Z)$ such that

$$\left|t_p - t_{p'}\right| \geq Z \quad \text{for} \quad p \neq p', \tag{4.4.42}$$

and similarly for $\{t'_p\}$.

The rest of the proof will be devoted to the estimation of the double sums on the right of (4.4.41). For convenience we restrict in S_r and S'_r the summation to an interval $N \leq n \leq N'$, where $N \asymp N'$, and take $j = 1$. The notation S_r is still retained for these sums. The original sum can be written as a sum of $o(L)$ new sums. We are going to show that

$$\sum_{p,r} \left|S_r(t_p)\right|^2 \ll \left(K_\circ^{-2} K^{-2} M^2 T^{-1} + K_\circ^{-1} M^{1/2} R\right) T^{2\delta}. \tag{4.4.43}$$

It will be obvious that the argument of the proof of this gives the same estimate for the similar sum involving S'_r as well. Then the inequality (4.4.41) becomes **125**

$$RV \ll \left(K^{-1} M^{1/2} R^{1/2} + K_\circ^{1/2} M^{-1/4} T^{1/2} R\right) T^{6\delta} \ll R^{5/6} T^{1/3+6\delta};$$

recall the definition (4.4.29) of K. This gives

$$R \ll T^{2+36\delta} V^{-6},$$

as desired.

To prove the crucial inequality (4.4.43), we apply methods of Halász and van der Corput. The following abstract version of Halász's inequality is due to Bombieri (see [23], Lemma 1.5, or [13], p. 494).

Lemma 4.5. *If $\xi, \varphi_1, \ldots, \varphi_R$ are elements of an inner product space over the complex numbers, then*

$$\sum_{r=1}^{R} |(\xi, \varphi_r)|^2 \leq \| \xi \|^2 \max_{1 \leq r \leq R} \sum_{s=1}^{R} |(\varphi_r, \varphi_s)|.$$

Suppose that the numbers N and N' above are integers, and define the usual inner product for complex vectors $a = (a_N, \ldots, a_{N'}), b = (b_N, \ldots, b_{N'})$ as

$$(a, b) = \sum_{n=N}^{N'} a_n \overline{b}_n.$$

Define vectors

$$\xi = \left\{\overline{\tilde{a}(n)} n^{-1/4}\right\}_{n=N}^{N'},$$

$$\varphi_{p,r} = \left\{\left(1+\frac{\pi n}{2hkt_p}\right)^{-1/4} e\left(n\left(\frac{\overline{h}}{k}-\frac{1}{2hk}\right)+\left(t_p/\pi\right)\phi\left(\frac{\pi n}{2hkt_p}\right)\right)\right\}_{n=N}^{N'}$$

with the convention that if $n_1 < N'$, then in $\varphi_{p,r}$ the components for $n_1 \le n \le N'$ are understood as zeros. Then by (4.1.28) we have

$$\left|S_r(t_p)\right| \ll K_\circ^{-1/2} M^{1/4} T^{-1/2} \left|\left(\xi, \varphi_{p,r}\right)\right|.$$

126 Hence, by Lemma 4.5, there is a pair p', r' such that

$$\sum_{p,r} \left|S_r(t_p)\right|^2 \ll K_\circ^{-1} M^{1/2} N^{1/2} T^{-1+\delta} \sum_{p,r} \left|\left(\varphi_{p,r}, \varphi_{p',r'}\right)\right|. \tag{4.4.44}$$

If now

$$\sum_{p,r} \left|\left(\varphi_{p,r}, \varphi_{p',r'}\right)\right| \ll \left(K_\circ^{-1} K^{-1} M + K M^{-1/2} R T\right) T^\delta, \tag{4.4.45}$$

then (4.4.43) follows from (4.4.44); recall that $N \ll K^{-2}M$ by (4.4.31). Hence it remains to prove (4.4.45).

Let

$$f_{p,r}(x) = x\left(\frac{\overline{h}}{k}-\frac{1}{2hk}\right)+\left(t_p/\pi\right)\phi\left(\frac{\pi x}{2hkt_p}\right).$$

The estimation of $|(\varphi_{p,r}, \varphi_{p',r'})|$ can be reduced, by partial summation, to that of exponential sums

$$\sum_n e\left(f_{p,r}(n) - f_{p',r'}(n)\right).$$

Namely, if this sum is at most $\Delta(p, r)$ in absolute value whenever n runs over a subinterval of $[N, N']$, then

$$\left|\left(\varphi_{p,r}, \varphi_{p',r'}\right)\right| \ll \Delta(p, r).$$

So in place of (4.4.45) it suffices to show that

$$\sum_{p,r} \Delta(p,r) \ll \left(K_\circ^{-1} K^{-1} M + K M^{-1/2} R T\right) T^{\delta}. \tag{4.4.46}$$

The quantity $\Delta(p,r)$ will be estimated by van der Corput's method. To this end we need the first two derivatives of the function $f_{p,r}(x) - f_{p',r'}(x)$ in the interval $[N, N']$. By the definition (4.1.6) of the function $\phi(x)$ we have

$$\phi'(x) = \left(1 + x^{-1}\right)^{1/2},$$
$$\phi''(x) = -\frac{1}{2} x^{-3/2} (1+x)^{-1/2}.$$

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Then by a little calculation it is seen that

$$f'_{p,r}(x) - f'_{p',r'}(x) = \frac{\overline{h}}{k} - \frac{1}{2hk} - \frac{\overline{h'}}{k'} + \frac{1}{2h'k'} \tag{4.4.47}$$
$$+ B K_\circ^{-3} M^{3/2} N^{-1/2} \left| hkt_p^{-1} - h'k't_p^{-1} + \frac{1}{2}\pi x \left(t_p t_{p'}\right)^{-1} \left(\frac{hk}{h'k'} - \frac{h'k'}{hk}\right)\right|,$$

where $|B| \asymp 1$, and

$$\left| f''_{p,r}(x) - f''_{p',r'}(x) \right| \asymp K_\circ^{-3} M^{3/2} N^{-3/2} \left| hkt_p^{-1} - h'k't_{p'}^{-1} + \frac{1}{2}\pi x \left(t_p^{-2} - t_{p'}^{-2}\right)\right|. \tag{4.4.48}$$

We shall estimate $\Delta(p,r)$ either by Lemma 4.1, or by the following simple lemma (see [27], Lemmas 4.8 and 4.2). We denote by $\| \alpha \|$ the distance of α from the nearest integer.

Lemma 4.6. *Let $f \in C^1[a,b]$ be a real function with $f'(x)$ monotonic and $\| f'(x) \| \geq m > 0$. Then*

$$\sum_{a<n\leq b} e(f(n)) \ll m^{-1}.$$

Turning to the proof of (4.4.46), let us first consider the sum over the pairs p, r'. Trivially,

$$\Delta(p', r') \ll N \ll K^{-2}M \ll K_\circ^{-1}K^{-1}M.$$

If $p \neq p'$, then by (4.4.47)

$$\left|f'_{p,r'}(x) - f'_{p',r'}(x)\right| \asymp K_\circ^{-1}M^{1/2}N^{-1/2}T^{-1}\left|t_p - t_{p'}\right|.$$

128 We may apply Lemma 4.6 if

$$\left|t_p - t_{p'}\right| \ll K_\circ M^{-1/2}N^{1/2}T,$$

and the corresponding part of the sum (4.4.46) is

$$\ll K_\circ M^{-1/2}N^{1/2}T\sum_p \left|t_p - t_{p'}\right|^{-1} \ll K_\circ^{-1}K^{-1}MT^\delta;$$

recall (4.4.42), (4.4.36), and (4.4.31).

Ohterwise $\Delta(p, r)$ is estimated by Lemma 4.1. Now by (4.4.48)

$$\left|f''_{p,r'}(x) - f''_{p',r'}(x)\right| \asymp K_\circ^{-1}M^{1/2}N^{-3/2}T^{-1}\left|t_p - t_p\right| \gg N^{-1},$$

so that these values of p contribute

$$\begin{aligned}\ll \sum_p \left(N\left(K_\circ^{-1}M^{1/2}N^{-3/2}\right)^{1/2} + N^{1/2}\right) &\ll \left(K_\circ^{-1/2}M^{1/4}N^{1/4} + N^{1/2}\right)R \\ &\ll M^{1/2}R,\end{aligned}$$

which is clearly $\ll KM^{-1/2}RT$.

For the remaining pairs p, r in (4.4.46) we have $r \neq r'$. Let p be fixed for a moment. Then

$$\left|hkt_p^{-1} - h'k't_{p'}^{-1}\right| \gg T^{-1}, \tag{4.4.49}$$

save perhaps for one "exceptional" fraction $r = h/k$; note that by the assumption (4.4.34) no two different fractions in our system have the same value for hk. If (4.4.49) holds, then by (4.4.48)

$$\left|f''_{p,r}(x) - f''_{p',r'}(x)\right| \asymp K_\circ^{-3}M^{3/2}N^{-3/2}\left|hkt_p^{-1} - h'k't_{p'}^{-1}\right|.$$

Then, if r runs over the non-exceptional fractions,

$$\begin{aligned}\sum_r \left|f''_{p,r}(x) - f''_{p',r'}(x)\right|^{-1/2} &\ll K_\circ^{3/2} M^{-3/4} N^{3/4} T^{1/2} \sum_{m \ll K_\circ^2 M^{-1} T} m^{-1/2} \\ &\ll K_\circ^{5/2} M^{-5/4} N^{3/4} T \\ &\ll KM^{-1/2} T,\end{aligned}$$

and 129

$$\sum_r N \left|f''_{p,r}(x) - f''_{p',r'}(x)\right|^{1/2} \ll K_\circ^{3/2} M^{-3/4} N^{1/4} T \ll KM^{-1/2} T.$$

Hence by Lemma 4.1

(4.4.50) $$\sum_r \Delta(p,r) \ll KM^{-1/2} T.$$

Consider finally $\Delta(p,r)$ for the exceptional fraction. We shall need the auxiliary result that for any two different fractions h/k and h'/k' of our system we have

(4.4.51) $$\left\| \frac{\bar{h}}{k} - \frac{1}{2hk} - \frac{\overline{h'}}{k'} + \frac{1}{2h'k'} \right\| \gg K_\circ^{-2} M^2 T^{-2}.$$

For if $k \neq k'$, then the left hand side is $\gg K_\circ^{-2}$ by the condition (4.4.34), like also in the case $k = k'$ if $h \not\equiv h' \pmod{k}$. On the other hand, if $k = k'$ and $h \equiv h' \pmod{k}$, then $|h - h'| \gg K_\circ$, and the left hand side is

$$\left|\frac{1}{2hk} - \frac{1}{2h'k}\right| \gg (hh')^{-1} \gg K_\circ^{-2} M^2 T^{-2}.$$

Let r be the exceptional fraction (for given p), and suppose first that for a certain small constant c

(4.4.52) $$\left|hkt_p^{-1} - h'k't_{p'}^{-1}\right| \leq cK_\circ M^{1/2} N^{1/2} T^{-2}$$

in addition to the inequality

(4.4.53) $$\left|hkt_p^{-1} - h'k't_p^{-1}\right| \ll T^{-1}$$

which defines the exceptionality. Then, by (4.4.51), the first four terms in (4.4.47) dominate, and we have

$$\| f''_{p,r}(x) - f'_{p'r'}(x) \| \gg K_\circ^{-2} M^2 T^{-2}.$$

Hence by Lemma 4.6 130

$$\Delta(p,r) \ll K^2 M^{-2} T^2 \ll K M^{-3/2} T^{5/3} \ll K M^{-1/2} T,$$

since $K \ll M^{1/2} T^{-1/3}$ and $M \gg T^{2/3}$.

On the other hand, if (4.4.52) does not hold, then by (4.4.48) and (4.4.53)

$$K_\circ^{-3} M^{3/2} N^{-3/2} T^{-1} \gg \left| f''_{p,r}(x) - f''_{p',r'}(x) \right| \gg K_\circ^{-2} M^2 N^{-1} T^{-2}.$$

Hence by Lemma 4.1

$$\begin{aligned} \Delta(p,r) &\ll K_\circ^{-3/2} M^{3/4} N^{1/4} T^{-1/2} + K_\circ M^{-1} N^{1/2} T \\ &\ll M T^{-1/2} + M^{-1/2} T \ll M^{-1/2} T. \end{aligned}$$

Now we sum the last estimations and those in (4.4.50) with respect to p to obtain

$$\sum_{\substack{p,r \\ r \neq r'}} \Delta(p,r) \ll K M^{-1/2} R T.$$

Taking also into account the previous estimations in the case $r = r'$, we complete the proof of (4.4.46), and also that of Theorem 4.7.

Notes

Theorems 4.1 and 4.2 were proved in [16] for integral values of r. The results of § 4.1 as they stand were first worked out in [17].

In §§ 4.2 - 4.4 we managed (just!) to dispense with weighted versions of transformation formulae. The reason is that in all the problems touched upon relatively large values of Dirichlet polynomials and expo-
131 nential sums occurred, and therefore even the comparatively weak error

terms of the ordinary transformation formulae were not too large. But in a context involving also small or "expected" values of sums it becomes necessary to switch to smoothed sums in order to reduce error terms. A challenging application of this kind would be proving the mean value theorems

$$\int_{T}^{T+T^{2/3}} |\zeta(1/2+it)|^4 \; dt \ll T^{2/3+\epsilon}$$
$$\int_{T}^{T+T^{2/3}} |\varphi(k/2+it)|^2 \; dt \ll T^{2/3+\epsilon},$$

respectively due to H. Iwaniec [14] and A. Good [9] (a corollary of (0.11) in a unified way using methods of this chapter.

The estimate (4.4.2) for the sixth moment of $\varphi(k/2+it)$ actually gives the estimate for $\varphi(k/2+it)$ in Theorem 4.5 as a corollary, so that strictly speaking the latter theorem is superfluous. However, we found it expedient to work out the estimate of $\varphi(k/2+it)$ in a simple way in order to illustrate the basic ideas of the method, and also with the purpose of providing a model or starting point for the more elaborate proofs of Theorem 4.6 and 4.7, and perhaps for other applications to come.

The method of § 4.4 can probably be applied to give results to the effect that an exponential sum involving $d(n)$ or $a(n)$ which depends on a parameter X is "seldom" large as a function of X. A typical example is the sum (4.3.38). An analogue of Theorem 4.7 would be

$$\int_{X_1}^{X_2} \left| \sum_{M_1 \le m \le M_2} b(m) e\left(\frac{X}{m}\right) \right|^6 dx \ll X_1^{5/2+\epsilon}$$

for $M_1 \asymp M_2, X_1 \asymp X_2, X_1 \gg M_1^2$, and $b(m) = d(m)$ or $\tilde{a}(m)$.

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