Lectures on
Stochastic Flows And Applications

By

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Notes by

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Preface

These notes are based on a series of lectures given at T.I.F.R. Centre, Bangalore during November and December 1985. The lectures consisted of two parts. In the first part, I presented basic properties of stochastic flows, specially of Brownian flows. Their relations with local characteristics and with stochastic differential equations were central problems. I intended to show the homeomorphism property of the flows without using stochastic differential equations. In the second part, as an application of the first part, I presented various limit theorems for stochastic flows. These include the following:

(a) Approximation theorems of stochastic differential equations and stochastic flows due to Bismut, Ikeda - Watanabe, Malliavin, Dowell et al.

(b) Limit theorems for driven processes due to Papanicolaou- Stroock-Varadhan.

(c) Limit theorems for stochastic ordinary differential equations due to Khasminskii, Papanicolau - Kohler, Kesten - Papanicolaou et al.

I intended to treat these limit theorems in a unified method.

I would like to thank M.K. Ghosh for his efforts in completing these notes. Also I wish to express my gratitude to Professor M.S. Raghunathan and *T.I.F.R*, for giving me this opportunity to visit India. Finally I would like to thank Ms. Shantha for her typing.

H.Kunita

Contents

Chapter 0

Introduction

Let us consider an ordinary differential equation on $\mathbb{R}^d$ **1**

$$\frac{dx}{dt} = f(x,t), \tag{0.1}$$

where $f(x,t)$ is continuous in (x,t) and is Lipschitz continuous in x. Denote by $\phi_{s,t}(x)$ the solution of the equation (0.1) starting from x at time s, i.e., with initial condition $x(s) = x$. It is a well known fact that $\phi_{s,t}(x)$ satisfies the following properties:

(a) $\phi_{s,t}(x)$ is continuous in s, t, x,

(b) $\phi_{t,u}(\phi_{s,t}(x)) = \phi_{s,u}(x)$ for any s, t, u and any x,

(c) $\phi_{s,s}(x) = x$ for any s,

(d) the map $\phi_{s,t} : \mathbb{R}^d \to \mathbb{R}^d$ is a homeomorphism for any s, t.

The map $\phi_{s,t}$ with above properties is called a flow of homeomorphisms.

A stochastic flow of homeomorphisms is an $\mathbb{R}^d$ valued random field $\phi_{s,t}(x,\omega)$, $o \leq s \leq t \leq T$, $x \in \mathbb{R}^d$ defined on a probability space $(\Omega F, P)$ such that, for almost all ω, it has the above mentioned properties $(a) \sim (d)$. In particular, if $\phi_{t_i,t_{i+1}}$, $i = 0, 1, \ldots, n-1$ are independent for any $0 \leq t_0 < t_1 < \ldots < t_n \leq T$, it is called a Brownian flow. An important class of Brownian flows is constructed by solving Ito's stochastic differential equation

$$dx(t) = \sum_{k=1}^{\tau} F_k(x,t)dB_k(t) + F_0(x,t)dt, \tag{0.2}$$

where $F_o(x,t), F_1(x,t), \ldots, F_r(x,t)$ are continuous in (x,t) and Lipschitz continuous in x, and $(B_1(t), \cdots, B_r(t))$ is a standard Brownian motion. Let $\phi_{s,t}(x,\omega)$ be the solution of the equation under the initial condition
2 $x(s) = x$. Then, taking a suitable modification, it defines a Brownian flow. This fact has been established by many authors, e.g. Elworthy [7], Malliavin [8], Bismut [3], Ikeda - Watanabe [13], Kunita [18] . However, not all stochastic flows can be constructed by the above method. In fact, we need an infinite number of independent Brownian motions $B_1(t), B_2(t), \ldots$ and functions $F_O(x,t), F_1(x,t), \ldots,$ or equivalently a Brownian with values in $C = C(\mathbb{R}^d; \mathbb{R}^d)$, i.e., a C- Brownian motion. Here, a continuous $\mathbb{R}^d$ valued random field $X(s,t), x \in \mathbb{R}^d, 0 \leq t \leq T$, is called a C- Brownian motion if $X(x,t_{i+1}) - X(x,t_i), i = 0,1,\ldots,n-1$ are independent for any $0 \leq t_0 < t_1 \ldots < t_n \leq T$. Then under a mild condition any Brownian flow can be obtained by a stochastic differential equation of the form

$$DC(t) = X(x(t), dt). \tag{0.3}$$

This fact is due to Le Jan [23] and Baxendale [1]. See also Fujiwara - Kunita [8].

The first part of these lectures will be devoted to the study of the basic properties of stochastic flows including the above facts. In Chapter 1 we shall characterize the Brownian flown by means of its local characteristics:

$$\begin{aligned} b(x,t) &= \lim \frac{1}{h}(E[\phi_{t,t+h}(x)] - x) \qquad (0.4)\\ a(x,y,t) &= \lim \frac{1}{h}(E[\phi_{t,t+h}(x) - x)(\phi_{t,t+h}(y) - y)^*], \end{aligned}$$

where $*$ is the transpose of the column vector $(.).b(x,t)$ and $a(x,y,t)$ will be referred to as infinitesimal mean and infinitesimal covariance respectively. It will be shown that these two objects determine the law of the Brownian flow.

In Chapter 2, we shall consider the stochastic differential equation based on C - valued Brownian motion or more generally C-valued continuous semimartingale $X(x, t)$, given in the form (0.3). It will be shown that the equation has a unique solution if its local characteristics are Lipschitz continuous and that the solution defines a stochastic flow. Con- **3**
versely, under some conditions the flow can be expressed as a solution of a suitable stochastic differential equation.

The second part of these notes will be devoted to various limit theorems concerning stochastic flows. We shall consider three types of limit theorems. The first is the approximation theorem for a stochastic differential equation given by (0.2). Let $v^\varepsilon(t) = (v_1^\varepsilon(t), \ldots, v_r^\varepsilon(t))$, $\varepsilon > 0$ be a piecewise smooth r dimensional process such that $B^\varepsilon(t) = \int_0^t v^\varepsilon(r)dr, \varepsilon > 0$ converges to a Brownian motion $B(t)$ as $\varepsilon \to 0$. Consider the stochastic ordinary differential equation

$$\frac{dx}{dt} = \sum_{k=1}^{r} F_k(x,t)v_k^\varepsilon(t) + F_o(x,t). \tag{0.5}$$

Let $\phi_{s,t}^\varepsilon(x)$ be the solution of (0.5) starting from x at time s. The question is whether the family of stochastic flows $\phi^\varepsilon, \varepsilon > 0$ converges weakly or strongly to a Brownian flow determined by the stochastic differential equation (0.2) (with some correction term). The problem has been considered by many authors in several special cases of $v^\varepsilon(t)$, e.g., Bismut [3], Dowell [6], Ikeda - Watanabe [14], Kunita [15], Malliavin [25].

The second limit theorem we shall be concerned with is that by Papanicolaou Stroock-Varadhan [29]. Consider the following system of stochastic differential equations

$$dx^\varepsilon(t) = \frac{1}{\varepsilon}F(x^\varepsilon(t), z^\varepsilon(t))dt + G(x^\varepsilon(t), z^\varepsilon(t))dt + \sum_{j=1}^{p} \sigma_j(x^\varepsilon(t), z^\varepsilon(t))d\beta_j(t)$$

$$dz^\epsilon(t) = \frac{1}{\varepsilon^2}\tilde{F}(x^\varepsilon(t), z^\varepsilon(t))dt + \frac{1}{\varepsilon}\sum_{j=1}^{q} \sigma_j(x^\varepsilon(t), z^\varepsilon(t))d\tilde{\beta}_j(t),$$

where $(\beta_1(t), \ldots, \beta_p(t))$ and $(\tilde{\beta}_1(t), \ldots, \tilde{\beta}_q(t))$ are independent Brownian motions, $F, G, \tilde{F}, \sigma, \tilde{\sigma}$ are bounded smooth functions with bounded

derivatives. The processes $z^{\varepsilon}(t), \varepsilon > 0$(called the driving processes) do
4 not converge. However, under suitable conditions on the coefficients, the processes $x^{\varepsilon}(t), \varepsilon > 0$(the driven processes) can converge weakly to a diffusion process.

The third one is concerned with the stochastic ordinary differential equation

$$\frac{dx}{dt} = \varepsilon f(x, t, \omega) + \varepsilon^2 g(x, t, \omega),$$

where f, g are random velocity fields satisfying suitable strong mixing conditions and $E[f] = 0$. The stochastic flows $\phi^{\varepsilon}_{s,t}$ determined by the above equation converge to the trivial flow $\phi^{0}_{s,t}(x) \equiv x$. Khasminskii [16], Papanicolaou - Kohlet [28], Borodin [5], Kesten- Papanicolaou [15] et al have shown that after changing the scale of time, the processes $\psi^{\varepsilon}_{t} = \phi^{\varepsilon}_{0,t/\varepsilon^2}, \varepsilon > 0$ converge weakly to a diffusion process.

In Chapter 3 we shall present a general limit theorem so that the above mentioned cases can be handled together.

Chapter 1

Brownian Flows

This chapter consists of four sections. In Section 1.1 we define stochas- **5**
tic flows of measurable maps. If a stochastic flow of measurable maps continuous in t, has independent increments, we call it a Brownian flow. We then describe its N- point motion and show that it is a Markov process. Finally we show that given a consistent family of transition probabilities, we can construct a stochastic flow with independent increments whose N-point motion will have the same family of transition probabilities. In Section 1.2, we introduce the notion of local characteristics of a Brownian flow which are essentially the infinitesimal mean and covariance of the flow. We then show the existence of a Brownian flow with prescribed local characteristics. In Section 1.3, we study Brownian flow of homeomorphisms. We show that if the local characteristics of a Brownian flow satisfy certain Lipschitz conditions, then it becomes a flow of homeomorphisms. In Section 1.4 we establish the diffeomorphism property of a Brownian flow under some smoothness Assumptions on its local characteristics.

1.1 Stochastic Flows with Independent Increments

Let (Ω, F, P) be a probability space. Let $T > 0$ be fixed. For $0 \leq s \leq t \leq T$, $x \in \mathbb{R}^d$, let $\phi_{s,t}(x, \omega)$ be an $\mathbb{R}^d$ valued random field such that for each fixed $s, t, s \leq t, \phi_{s,t}(., \omega)$ is a measurable map from $\mathbb{R}^d$ into $\mathbb{R}^d$. Let M

denote the totality of measurable maps from $\mathbb{R}^d$ into $\mathbb{R}^d$. Then $\phi_{s,t}$ can be regarded as an M - valued process.

Definition 1.1.1. *$\phi_{s,t}$ is called a stochastic flow of measurable maps if*

(i) *$\phi_{s,t}(x, .)$ is continuous in probability w.r.t (s, t, x),*

(ii) *$\phi_{s,s}$ = identity map a.s.for each S,*

(iii) *$\phi_{s,u} = \phi_{t,u} o \phi_{s,t}$ a.s. for each $s < t < u$.*

6 where o stands for the composition of maps. By a stochastic flow we will always mean a stochastic flow of measurable maps. Let $0 \leq t_0 < t_1 < \ldots < t_n \leq T$, $x_i \in \mathbb{R}^d, 0 \leq i \leq n-1$ be arbitrary. If $\phi_{t_i,t_{i+1}}(x_i), i = 0, 1, \ldots, n-1$ are independent random variables for any such $\{t_i, x_i\}$, then $\phi_{s,t}$ is called a stochastic flow with independent increments. Further, if $\phi_{s,t}(x, \omega)$ is continuous in t a.s. for each s, x then $\phi_{s,t}$ is called a Brownian flow (of measurable maps).

Let $\phi_{s,t}$ be a stochastic flow with independent increments. Let $x^{(N)} = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^{Nd}$ where each $x_i \in \mathbb{R}d$. Set $\phi_{s,t}(x^{(N)}) = (\phi_{s,t}(x_1), \phi_{s,t}(x_2), \ldots, \phi_{s,t}(x_N))$. For fixed $s, x^{(N)}, \phi_{s,t}(x^{(N)})$ is an $\mathbb{R}^{Nd}$ valued process. We claim that it is a Markov process with transition probability

$$P^{(N)}_{s,t}(\underline{x}^{(N)}, E) = P(\phi_{s,t}(\underline{x}^{(N)}) \in E). \tag{1.1.1}$$

Indeed, let

$$T^{(N)}_{s,t} f(\underline{x}^{(N)}) = \int f(\underline{y}^{(N)}) P^{(N)}_{s,t}(\underline{x}^{(N)}, d\underline{y}^{(N)}) \tag{1.1.2}$$

where $f : \mathbb{R}^{Nd} \to \mathbb{R}$ is a bounded measurable map. Let $F_{s,t} = \sigma(\phi_{u,v}(x) : s \leq u \leq v \leq t, x \in \mathbb{R}^d)$. Note that $\phi_{s,u} = \phi_{t,u} o \phi_{s,t}$ and the independent increment property of $\phi_{s,t}$ implies that $\phi_{t,u}$ is independent of $F_{s,t}$. Now

$$\begin{aligned} E\left[f\left(\phi_{s,u}(\underline{x}^{(N)}) | F_{s,t}\right)\right] &= E\left[f\left(\phi_{t,u}(\underline{y}^{(N)})\right)\right]_{y^{(N)}} = \phi_{s,t}(\underline{x}^{(N)}) \\ &= T^{(N)}_{t,u} f(\phi_{s,t}(\underline{x}^{(N)}). \end{aligned}$$

7 From the above property, we see that

$$T^{(N)}_{s,t} f = T^{(N)}_{s,t} o T^{(N)}_{t,u} f. \tag{1.1.3}$$

If f is bounded and continuous, then $T^{(N)}_{s,t} f((\underline{x}^{(N)})$ is a continuous function of $s, t, (\underline{x}^{(N)})$.

Remark 1.1.2. The family $\left\{P^{(N)}_{s,t}(\underline{x}^{(N)}, .)\right\}_{N=1,2,\ldots}$ defined by (1.1.1) of transition probabilities is consistent in the following sense. Suppose M and N are two positive integers and $N > M$. Let $1 \leq i_1 < i_2 < \cdots \leq N$ be positive integers, $\{x_{i_l} \cdots , x_{i_M}\}$ a subset of $\{x_1, \ldots, x_N\}$ and $E_1, \ldots, E_N$ Bo rel sets in $\mathbb{R}^d$ such that $E_k = \mathbb{R}^d$ if $K^M \notin \{i_1, \ldots, i_M\}$. Then

$$P^{(N)}_{s,t}(x_1, \ldots, x_N, E_1\, x \ldots xE_N) = P^{(M)}_{s,t}(x_{i_1}, \ldots, x_{i_M}, E_{i_1} x \ldots xE_{i_M}). \tag{1.1.4}$$

Proposition 1.1.3. *Let $\left\{P^{(N)}_{s,t}(\underline{x}^{(N)}, .) N = 1, 2, \ldots\right\}$ be a family of transition probabilities satisfying the consistency condition* (1.1.4)*. Assume that the corresponding $T^{(N)}_{s,t} f(\underline{x}^{(N)})$ is continuous in $(s, t, \underline{x}^{(N)})$ for any bounded continuous function f. Then there is a stochastic flow of measurable maps with independent increment whose N-point process has the transition probability $\left\{P^{(N)}_{s,t}(\underline{x}^{(N)}, .)\right\}$.*

Proof. Let $0 \leq t_1 < t_2 < \cdots < t_n \leq T$. Then by Kolmogorov consistency theorem, there exist n independent random fields $\xi_i, i = 1, 2, \cdots n$ s.t. the law of $(\xi_i(x_1)), \ldots, \xi_i(x_N))$ coincides with $P^{(N)}_{t_i,t_{i+1}}(\underline{x}^{(N)}, .)$ for any $\underline{x}^{(N)} \in \mathbb{R}^{Nd}$. Since $T^{(1)}_{s,t} f(x)$ is continuous *w.r.t* x for all bounded continuous f, we can pick a version of $\{\xi_i(x)\}$ s.t $x \to \xi_i(x)$ is measurable. For $t_i \leq t_j$, define

$$\xi_{t_i.,t_j} = \begin{cases} \text{identity if } \ i = j \\ \xi_{j-1} o \xi_{j-2} o \cdots o \xi_i \text{ if } i < j. \end{cases}$$

Denote the law of $\xi_{t_i,t_j}, t_i \leq t_j$ by $\{Q_{t_1 \cdots, t_n}$. Then $\{Q_{t_1,\ldots,t_n}(x)\}$ is a consistent family of probability measures as the parameters vary over $0 \leq t_1 < t_2 < \cdots < t_n \leq T, x \in \mathbb{R}^d$. Therefore, by Kolmogorov **8**
consistency theorem, there exists a random field $\phi_{s,t}(x, \omega)$ such that the joint law of $\{\phi_{t_i,t_j}(x_k), 1 \leq i \leq j \leq n, k = 1, \ldots, N\}$ coincides with $\{\xi_{t_i,t_j}(x_k), 1 \leq i \leq j \leq n, k = 1, 2, \ldots, N\}$. Again the continuity of

$T^{(1)}_{s,t} f(x)$ *w.r.t.* s, t, x allows us to pick a version such the map $s, t, x \to \phi_{s,t}(x, \omega)$ is measurable a.s This $\phi_{s,t}(x, \omega)$ clearly satisfies (i) and (ii) of Defn. 1.1.1 (iii) being an easy consequence of (1.1.3) above. Thus it is the required stochastic flow. □

Remark 1.1.4. In view of the above propositions there is a one to one correspondence between stochastic flows with independent increments and a consistent family of transition probabilities whose corresponding family of semigroups satisfies a certain continuity criterion.

1.2 Local Characteristics. Generator of N-Point Motion

Let $\phi_{s,t}(s, \omega)$ be a Brownian flow. We make the following assumptions:

Assumption 1. $\phi_{s,t}(x, .)$ *is square integrable and the following limits exist:*

(i) $\lim\limits_{h\to 0} \dfrac{1}{h} E[\phi_{t,t+h}(x) - x]$

Denote the above limit by $b(x, t)$. *Then* $b(x, t) = (b^1(x, t), \ldots, b^d(x, t))$ *is an* $\mathbb{R}^d$ *vector.*

(ii) $\lim\limits_{h\to 0} \dfrac{1}{h} E[(\phi_{t,t+h}(x) - x)(\phi_{t,t+h}(y) - y)^*]$

where x^* *stands for the transpose of* $x \in \mathbb{R}^d$. *Denote the above limit by* $a(x, y, t)$. *Then* $a(x, y, t) = (a_{ij}(x, y, t))$ *is a* $d \times d$ *matrix.*

The pair (a, b) *is called the local characteristics of the flow* $\phi_{s,t}$. *Clearly the matrix* $a(x, y, t) = (a_{ij}(x, y, t))$ *satisfies the following properties:*

9 *(i) Symmetry:* $a_{ij}(x, y, t) = a_{ij}(y, x, t)$ *for any* x, y, t

(ii) Nonnegative definiteness: $\sum\limits_{i,j,p,q} a_{ij}(x_p, x_q, t)\xi^i_p x j^i_q \geq 0$ *for any* $(x_1, \ldots, x_N)$ *and* $\xi_p = (\xi^1_p, \ldots \xi^d_p),\ p = 1, 2, \ldots, N.$

Assumption 2. *There exists a constant K independent of x, s, t such that*

$$|E[\phi_{s,t}(x) - x]| \leq K(1 + |x|)(t - s), \tag{1.2.1}$$
$$|E[\phi_{s,t}(x) - x)(\phi_{s,t}(y) - y)^*]| \leq K(1 + |x|)|(1 + |y|)(t - s). \tag{1.2.2}$$

Remark 1.2.1. Assumption 2 is technical in nature but it is not very restrictive. It will naturally be satisfied in most of the interesting cases.

Remark 1.2.2. It follows from $(A1)$ and $(A2)$ that

$$|b(x, t)| \leq K(1 + |x|), \tag{1.2.3}$$
$$|a(x, y, t)| \leq K(1 + |x|)(1 + |y|). \tag{1.2.4}$$

Note that the first norm is a usual vector norm and the second one is a matrix norm.

Let $M_{s,t}(x)$ be defined as follows:

$$M_{s,t}(x) = \phi_{s,t}(x) - x - \int_s^t b(\phi_{s,r}(x), r)dr. \tag{1.2.5}$$

Lemma 1.2.3. *For each $s, x, M_{s,t}(x), t \in [s, T]$ is a continuous L^2-martingale and*

$$< M^i_{s,t}(x), M^j_{s,t}(y) >= \int_s^t a_{ij}(\phi_{s,r}(x), \phi_{s,r}(y), r)dr \tag{1.2.6}$$

where $< ., . >$ stand for quadratic variation process.

Proof. Set
$$m_{s,t}(x) = E[\phi_{s,t}(x)]. \tag{1.2.7}$$

□

Then

$$\begin{aligned}\frac{\partial}{\partial t} m_{s,t}(x) &= \lim_{h \to 0} \frac{1}{h}[(m_{s,t+h}(x) m_{s,t}(x)] \\ &= \lim_{h \to 0} \frac{1}{h} E[(m_{t,t+h}(\phi_{s,t}(x)) - \phi_{s,t}(x))]. \end{aligned} \tag{1.2.8}$$

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Now
$$\frac{1}{h}|m_{t,t+h}(\phi_{s,t}(x)) - \phi_{s,t}(x)| \leq K(1 + |\phi_{s,t}(x)|)$$
Since $K(1 + |\phi_{s,t}(x)|)$ is integrable, we can change the order of lim and E in (1.2.8). Hence we get
$$\frac{\partial}{\partial t}|m_{s,t}(x) = E[b(\phi_{s,t}(x), t)] \text{for all} t \geq s.$$

Therefore
$$m_{s,t}(x) - x = \int_s^t E[b(\phi_{s,r}(x), r)]dr \tag{1.2.9}$$

Hence
$$E[M_{s,t}(x)] = 0.$$

Note that for $s < t < u$,
$$M_{s,u}(x) = M_{s,t}(x) + M_{t,u}(x)(\phi_{s,t}(x)).$$

Therefore
$$\begin{aligned} E[M_{s,u}(x)|F_{s,t}] &= M_{s,t}(x) + E[M_{t,u}(y)]_{y=\phi_{s,t}}(x) \\ &= M_{s,t}(x) \end{aligned}$$

This proves the first assertion. We will now establish (1.2.6). Define
$$V_{s,t}(x, y) = E[M_{s,t}(x)(M_{s,t}(y))^*]. \tag{1.2.10}$$

11 Then
$$V_{s,t+h}(x, y) - V_{s,t}(x, y) = E[(M_{s,t+h}(x) - M_{s,t}(x))(M_{s,t+h}(y) - M_{s,t}(y))^*].$$

Therefore
$$\begin{aligned} \frac{1}{h}[V_{s,t+h}(x, y) - V_{s,t}(x, y)] &= \frac{1}{h}E[(M_{t,t+h}(\phi_{s,t}(x))(M_{t,t+h}(\phi_{s,t}(y)))^*] \\ &= \frac{1}{h}E[(V_{t,t+h}(\phi_{s,t}(x), \phi_{s,t}(y))]. \end{aligned} \tag{1.2.11}$$

Letting $h \to 0$ in (1.2.11), we get

$$\frac{\partial}{\partial t} V_{s,t}(x, y) = E[a(\phi_{s,t}(x), \phi_{s,t}(y), t)].$$

Therefore

$$V_{s,t}(x, y) = \int_s^t E[a(\phi_{s,r}(x), \phi_{s,r}(y), r)]dr. \tag{1.2.12}$$

Set

$$N_{s,t}(x, y) = M_{s,t}(x)M_{s,t}(y)^* - \int_s^t a(\phi_{s,r}(x), \phi_{s,r}(y), r)\, dr. \tag{1.2.13}$$

Then (1.2.12) implies

$$E[N_{s,t}(x, y)] = 0.$$

Let $s < t < u$. Then a simple computation yields

$$\begin{aligned} N_{s,u}(x, y) &= N_{s,t}(x, y) + N_{t,u}(\phi_{s,t}(x), \phi_{s,t}(y)) \\ &\quad + M_{s,t}(x)M_{t,u}(\phi_{s,t}(y)) + M_{s,t}(y)M_{t,u}(\phi_{s,t}(x)). \end{aligned}$$

Hence

$$E[N_{s,u}(x, y)|F_{s,t}] = N_{s,t}(x, y).$$

Thus $N_{s,t}(x, y)$ is a martingale. This completes the proof of the lemma.

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Remark 1.2.4. $b(x, t)$ and $a(x, y, t)$ are often referred to as the infinitesimal mean and covariance respectively of the flow.

Let $x_k \in \mathbb{R}^d, k = 1, 2, \ldots, N, x_k = (x_k^1, \ldots, x_k^d)$. We define a differential operator $L_t^{(N)}$ as follows:

$$\begin{aligned} L_t^{(N)} f(\underline{x}^{(N)}) = \frac{1}{2} \sum_{i,j,k,\ell} a_{ij}(x_k, x_\ell, t) \frac{\partial^2 f}{\partial x_k^i \partial x_\ell^j}(\underline{x}^{(N)}) \\ + \sum_{i,k} b^i(x_k, t) \frac{\partial f}{\partial x_k^i}(\underline{x}^{(N)}). \end{aligned} \tag{1.2.14}$$

$L_t^{(N)}$ is elliptic operator which may be degenerate. It is the infinitesimal generator of $T_{s,t}^{(N)}$, which is the semigroup of N-point process. This fact will follow from the following theorem.

Theorem 1.2.5. *Let f be a C^2 function on $\mathbb{R}^{Nd}$ such that f and its derivatives have polynomial growth. Then the following holds for any $s, t, \underline{x}^{(N)}$:*

$$T_{s,t}^{(N)} f(\underline{x}^{(N)}) - f(\underline{x}^{(N)}) = \int_s^t T_{s,r}^{(N)} L_r^{(N)} f(\underline{x}^{(N)}) dr. \tag{1.2.15}$$

In particular

$$\lim_{h \to 0} \frac{1}{h} \left(T_{t,t+h}^{(N)} f - f \right) = L_t^{(N)} f. \tag{1.2.16}$$

Proof. The proof is essentially based on Ito's formula. It follows from Lemma 1.2.3 that for fixed (s, x), $\phi_{s,t}(x)$ is a continuous semimartingale with the following decomposition:

$$\phi_{s,t}(x) = x + M_{s,t}(x) + \int_s^t b(\phi_{s,r}(x), r) dr. \tag{1.2.17}$$

$\square$

By Ito's formula, we have

$$\begin{aligned} f(\phi_{s,t}(\underline{x}^{(N)})) - f(\underline{x}^{(N)}) - \int_s^t L_r^{(N)} f(\phi_{s,r}(\underline{x}^{(N)}))\, dr \\ = \sum_{k,i} \int_s^t \frac{\partial f}{\partial x_k^i} (\phi_{s,r}(\underline{x}^{(N)})) dM_{s,r}^i(x_k). \end{aligned} \tag{1.2.18}$$

13 Claim: $\phi_{s,t}(x)$ has finite moments of all orders. Granting the claim, the above is a zero mean martingale. Then taking expectation, we get

$$T_{s,t}^{(N)} f(\underline{x}^{(N)}) - f(\underline{x}^{(N)}) - \int_s^t T_{s,r}^{(N)} L_{s,r}^{(N)} f(\underline{x}^{(N)}) dr = 0,$$

which proves the theorem. So it remains to substantiate the claim which we do in the following lemma.

Lemma 1.2.6. *$\phi_{s,t}(x)$ has finite moments of any order. Further, for any p real and $\varepsilon > 0$, there is a positive constant $C = C(p, \varepsilon)$ such that*

$$E(\varepsilon + |\phi_{s,t}(x)|^2)^p \leq C(\varepsilon + |x|^2)^p \tag{1.2.19}$$

for any s, t, x.

Proof. Define $g(x) = \varepsilon + |x|^2$ and $f(x) = g(x)^P$. Let $L_r = L_r^{(1)}$. We shall apply Ito's formula to the 1− point process. We have

$$\begin{aligned} L_r f = 2pg(x)^{p-1} \sum_i b^i(x,t)x_i + pg(x)^{p-1} \sum_i a_{ii}(x,x,t) \\ + 2p(p-1)g(x)^{p-2} \sum_{i,j} a_{ij}(x,x,t)x_i x_j. \end{aligned}$$ □

Using the estimates (1.2.3) and (1.2.4), we have $|L_r f(x)| \leq C' f(x)$, where C' is a constant independent of x, r. Now for any n, define the stopping time

$$\tau_n = \tau_n(x,s) = \begin{cases} \inf\{t > s : |\phi_{s,t}(x)| \geq n\} \\ \infty \text{ if the above set is empty.} \end{cases}$$

Let the stopped process $\phi_{s,t\wedge\tau_n}$ be denoted by $\tilde{\phi}_{s,t}$. Then $f(\tilde{\phi}_{s,t}(x)) - \int_s^t L_r f \tilde{\phi}_{s,r} dr$ is a martingale. Therefore

$$\begin{aligned} E[f(\tilde{\phi}_{s,t}(x))] &= f(x) + \int_S^t E[L_r f(\tilde{\phi}_{s,r}(x))]dr \\ &\leq f(x) + C' \int_S^t E[f(\tilde{\phi}_{s,r}(x))]dr. \end{aligned}$$

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Hence by Gronwall's inequality, we have

$$E[f(\tilde{\phi}_{s,t}(x))] \leq f(x)e^{C'(t-s)}.$$

In the above inequality, the right hand side is independent of n; so letting $n \uparrow \infty$, we get

$$E[f(\phi_{s,t}(x))] \leq f(x)e^{C'(t-s)}.$$

Thus

$$E[\varepsilon + |\phi_{s,t}(x)|^2]^p \leq C(\varepsilon + |x|^2)^P.$$

Remark 1.2.7. (i) For each positive integer $k, 1 \leq k \leq N$, define

$$L_t^k = \frac{1}{2}\sum_{i,j} a_{ij}(x_k, x_k, t)\frac{\partial^2}{\partial x_k^i \partial x_k^j} + \sum_i b^i(x_k, t)\frac{\partial}{\partial x_k^i}. \tag{1.2.20}$$

L_t^k is the generator of $1-$point process. Then the following is easily verified:

$$L_t^{(N)} f = \sum_{k=1}^N L_t^{(N)} f + \sum_{k \neq \ell} \frac{1}{2}\sum_{i,j} a_{ij}(x_k, x_\ell, t)\frac{\partial^2}{\partial x_k^i \partial x_\ell^j}. \tag{1.2.21}$$

The second term in (1.2.21) could be regarded as the interaction between $\phi_{s,t}(x_k)$ and $\phi_{s,t}(x_\ell)$. Thus the generator of the $N-$point is the sum of the generators of the $1-$ point motions together with the cross - interaction. If no interaction exists then the second term will cease to exist and $\phi_{s,t}(x_1), \phi_{s,t}(x_2), \ldots$ would move independently.

(ii) The family of operators $\{L_t^{(N)}, N = 1, 2, \ldots\}$ is consistent in the
15 following sense: Let M, N be two positive integers and $N > M$. Let $1 \leq i_i < i_2 < \ldots < i_M \leq N, \{x_{1_1}, \ldots, x_{i_M}\}$ a subset of $\{x_1, \ldots, x_N\}$. Let f be a function of $\mathbb{R}^{Nd}$ depending only on $(x_{i_1}, \ldots, x_{i_M})$. Then

$$L_t^{(N)} f(x_1, \ldots, x_N) = L_t^{(M)} f(x_{1_1}, \ldots x_{i_M}), \tag{1.2.22}$$

which is an obvious consequence of the Remark 1.1.2. We conclude this section by showing the existence of a Brownian flow with given local characteristics.

Theorem 1.2.8. *Let $b(x,t)$ be an $\mathbb{R}^d$-valued bounded continuous function and $a(x,y,t)$ and $d \times d$-matrix valued bounded continuous function which is nonnegative definite and symmetric $(a_{ij}(x,y,t)) = (a_{ij}(y,x,t))$. Suppose a and b are twice spatially differentiable and their derivatives are bounded. Then there is a Brownian flow with local characteristics (a,b). Further, the law of the flow unique.*

Proof. We define $L_t^{(N)}$ as before. By Oleinik's theorem (Stroock - Varadhan [31]) there exists a unique $T_{s,t}^{(N)}$ such that

$$T_{s,t}^{(N)} f(\underline{x}^{(N)}) = f(\underline{x}^{(N)}) + \int_s^t T_{s,t}^{(N)} L_r^{(N)} f(\underline{x}^{(N)}) dr. \qquad \square$$

The consistency of the family $\left\{T_{s,t}^{(N)}, N = 1, 2, \cdots\right\}$ follows from that of $\{L_t^{(N)}, N = 1, 2, \dots\}$. By virtue of Proposition 1.1.3 there exists a stochastic flow with independent increments $\phi_{s,t}^{(x)}$ whose N-point process has the same semigroup as $T_{s,t}^{(N)}$. Since $T_{s,t}^{(N)}$ defines a diffusion semigroup, it follows that $\phi_{s,t}(x)$ is continuous in t a.s for each s, x. Hence $\phi_{s,t}$ is a Brownian flow.

Remark 1.2.9. Under the condition of the above theorem, the law of the Brownian flow is determined by the 2−points motions. Indeed, let $\left\{T_{s,t}^{(N)}, N = 1, 2, \cdots\right\}$ and $\left\{\tilde{T}_{s,t}^{(N)}, N = 1, 2, \cdots\right\}$ be consistent families of **16**
semigroups with generators $\left\{L_t^{(N)}, N = 1, 2, \dots\right\}$ and $\left\{\tilde{L}_t^{(N)}, N = 1, 2, \dots\right\}$ respectively. If $T_{s,t}^{(2)} = \tilde{T}_{s,t}^{(2)}$ then $T_{s,t}^{(N)}, \tilde{T}_{s,t}^{(N)}$ for any $N \geq 2$. This follows from the above theorem because the generator depends only on the local characteristics and $L_t^{(2)} = \tilde{L}_t^{(2)}$ implies $L_t^{(N)} = \tilde{L}_t^{(N)}, N = 1, 2, \dots$

1.3 Brownian Flow of Homeomorphisms

Definition 1.3.1. *Let $\phi_{s,t}(x)$ be a Braownian flow of measurable maps. Then $\phi_{s,t}(x)$ is said to be a Brownian flow of homeomorphisms if*

(i) $\phi_{s,t}(x)$ is continuous in (s, t, x) a.s

(ii) $\phi_{s,t}(., \omega) : \mathbb{R}^d \to \mathbb{R}^d$ is a homeomorphism for any $s < t$ a.s. Further if

(iii) $\phi_{s,t}(., \omega) : \mathbb{R}^d \to \mathbb{R}^d$ is a C^k-diffeomorphism for any $s < t$ a.s., then $\phi_{s,t}$ is called a Brownian flow of C^k-diffeomorphisms.

Apart from assumptions 1 and 2, we will impose a Lipschitz condition on the local characteristics (a, b).

Assumption 3. *There exists a constant L such that*

$$|b(x, t) - b(y, t) \leq L|x - y|, \tag{1.3.1}$$

$$|a(x, x, t) - 2a(x, y, t) + a(y, y, t)| \leq L|x - y|.{}^{2} \tag{1.3.2}$$

We will see in the next theorem that Assumption 3 makes a Brownian flow a flow of homeomorphisms.

Theorem 1.3.2. *Let $\phi_{s,t}$ be a Brownian flow satisfying $(A1) \sim (A3)$. Then it is a Brownian flow of homeomorphisms. More precisely there exists a version of $\phi_{s,t}$ which is a Brownian flow of homeomorphisms.*

17 The proof of the above theorem is based on several estimates which we will derive in the following lemmas. In these lemmas $(A1)$, $(A2)$, $(A3)$ will be assumed.

Lemma 1.3.3. *For any real p, there is a positive constant $C = C(p)$ such that for any $\varepsilon > 0, x, y, \in \mathbb{R}^d$*

$$E(\varepsilon + |\phi_{s,t}(x) - \phi_{s,t}(y)|^2)^p \leq C(\varepsilon + |x-y|^2)^P. \tag{1.3.3}$$

Proof. Set $g(x,y) = \varepsilon + |x-y|^2$ and $f(x,y) = g(x,y)^p$. A simple computation yields

$$\begin{aligned} L_t^{(2)} f(x,y) = 2pg(x,y)^{p-1} &\left\{ \sum_i (b^i(x,t) - b^i(y,t))(x_i - y_i) \right\} \\ + pg(x,y)^{p-2} &\left\{ \sum_{i,j} (g(x,y))\delta_{ij} + 2(p-1)(x_i - y_i)(x_j - y_j)) \right. \\ &\left. \times (a_{ij}(x,x,t) - 2a_{ij}(x,y,t) + a_{ij}(y,y,t)) \right\}. \quad \square \end{aligned}$$

Using the estimates (1.2.3), (1.2.4) and (1.3.1), (1.3.2), we can find a constsnt C' such that

$$|L_t^{(2)} f(x,y)| \leq C' f(x,y).$$

Now, by Ito's formula,

$$E[f(\phi_{s,t}(x), \phi_{s,t}(y))] = f(x,y) + \int_s^t E[L_r^{(2)} f(\phi_{s,t}(x), \phi_{s,t}(y))]dr$$

$$\leq f(x,y) + C' \int_s^t E[f(\phi_{s,r}(x), \phi_{s,r}(y))]dr.$$

By Gronwall's inequality, we have

$$E[f(\phi_{s,t}(x), \phi_{s,t}(y))] \leq e^{C'(t-s)}_{f(x,y)}.$$

Hence there exists a constant C such that

$$E(\varepsilon + |\phi_{s,t}(x) - \phi_{s,t}(y)|^2)^p \leq C(\varepsilon + |x-y|^2)^p.$$

18

Lemma 1.3.4. *For any positive integer p there exists a constant $C = C(p)$ such that for any $x_o \in \mathbb{R}^d$, we have*

$$E|\phi_{s,t}(x_o) - x_o|^{2p} \leq C|t-s|^p(1 + |x_o|)^{2p}. \tag{1.3.4}$$

Proof. Fix a point $x_o \in \mathbb{R}^d$ and set $g(x) = |x - x_o|^2$, $f(x) = g(x)^p$. Then as before, using (1.2.3), (1.2.4), (1.3.1) and (1.3.2) we can find constants C_1, C_2, C_3 such that

$$|L_t f(x)| \leq C_1 f(x) + C_2 g(x)^{p-1/2}(1 + |x_o|) + C_3 g(x)^{p-1}(1 + |x_o|)^2. \quad \square$$

Therefore using Ito's formula, we have

$$\begin{aligned} E[f(\phi_{s,t}(x_o))] \leq\ & f(x_o) + C_1 \int_s^t E[f(\phi_{s,r}(x_o))]dr \\ & + C_2 \int_s^t E[g(\phi_{s,t}(x_o))^{p-1/2}]dr\,(1 + |x_o|) \\ & + C_3 \int_s^t E[g(\phi_{s,t}(x_o))^{p-1}]dr\,(1 + |x_o|)^2. \end{aligned}$$

Applying Gronwall's inequality, we have

$$E[f(\phi_{s,t}(x_o))] \leq C_4(1+|x_o|)\int\limits_s^t E[g(\phi_{s,r}(x_o))^{p-1/2}]dr + C_5(1+|x_o|)^2$$
$$\int\limits_s^t E[g(\phi_{s,r}(x_o))^{p-1}]dr. \quad (1.3.5)$$

Now for $p = \dfrac{1}{2}, 1$, the estimate (1.3.4) follows from ($A2$). Using (1.3.5), the same estimate follows for $p = 1\dfrac{1}{2}$. Again using (1.3.5), we get (1.3.4) for $p = 2$. Proceeding inductively, we conclude the result for any p.

Remark 1.3.5. If p is a positive integer there exists a constant $C = C(p)$ such that for any $x_o, y_o \in \mathbb{R}^d$

$$E|\phi_{s,t}(x_o) - x_o - (\phi_{s,t}(y_o) - y_o)|^{2p} \leq C|t-s|^p|x_o - y_o|^{2p}. \qquad (1.3.6)$$

19 Set $g(x,y) = |(x,x_o) - (y-y_o)|^2$, $f(x,y) = g(x,y)^p$. Then we can show that

$$|L_t^{(2)} f| \leq C_1 f + C_2 g^{p-\frac{1}{2}}|x_o - y_o| + C_3 g^{p-1}|x_o - y_o|^2.$$

The rest to the proof is similar to that of lamma 1.3.4.

Lemma 1.3.6. *Let p be a positive integer. Then there exists a constant $C = C(p)$ such that*

$$E|\phi_{s,t}(x) - \phi_{s',t'}(x')|^{2p}$$
$$\leq C\left\{|x-x'|^{2p} + (1+|x|+|x'|)^{2p}(|t-t'|^p + |s-s'|^p)\right\} \quad (1.3.7)$$

holds for any $s \leq t$, $s' \leq t'$ and $x, x' \varepsilon \mathbb{R}^d$.

Proof. We consider the case $s' \leq s \leq t \leq t'$ only. The other case will follow similarly. □

We split the proof in various steps

(a) $s = s'$ and $x = x'$.

$$\begin{aligned} E|\phi_{s,t}(x) - \phi_{s,t}(x)|^{2p} &= E|\phi_{s,t}(x) - \phi_{t,t'}, \phi_{s,t}(x))|^{2p} \\ &= \int E[|y - \phi_{t,t'}(y)|^{2p}]P(\phi_{s,t}(x)\varepsilon dy) \\ &\leq C_1|t - t'|^p \int (1 + |y|)^{2p} P(\phi_{s,t}(x)\varepsilon dy) \\ &\qquad\qquad \text{[using Lemma 1.3.4]} \\ &= C_1|t - t'|^p E(1 + \phi_{s,t}{}^{s,t}|)^{2p} \\ &\leq C_2|t - t'|^p (1 + |x|)^{2p} \end{aligned}$$

(b) $s = s', x \neq x'$.

$$\begin{aligned} E|\phi_{s,t}(x) - \phi_{s,t}(x')|^{2p} &\leq 2^{2p} \\ &\left\{E|\phi_{s,t}(x) - \phi_{s,t'}(x')|^{2p} + E|\phi_{s,t'}(x) - \phi_{s,t'}(x')|^{2p}\right\} \\ &\leq C_3 \left\{|t - t'|^p (1 + |x|)^{2p} + |x - x'|^{2p}\right\}. \\ &\qquad\qquad \text{[using lemma 1.3.3]} \end{aligned}$$

(c) $E|\phi_{s,t}(x) - \phi_{s',t'}(x')|^{2p} = E|\phi_{s,t}(x) - \phi_{s,t'}(\phi_{s',s}(x'))|^{2p}$

$$\begin{aligned} &= \int E|\phi_{s,t}(x) - \phi_{s,t'}(y)|^{2p}) P(\phi_{s',s}(x')\varepsilon dy) \\ &\leq C_3 \int \left\{|t - t'|^p (1 + |x|)^{2p} + |x - y|^{2p}\right\} P(\phi_{s',s}(x')\varepsilon dy) \\ &= C_3 \left\{|t - t'|^p (1 + |x|)^{2p} + E|x - \phi_{s',s} x'|^{2p}\right\} \\ &\leq C_3 \left\{|t - t'|^p (1 + |x|)^{2p}\right\} + C_4|x - x'|^{2p} + C_5|s - s'|^p (1 + |x'|)^{2p} \\ &\leq C_6 \left\{(|t - t'|^p + |s - s'|^p)(1 + |x| + |x'|)^{2p} + |x - x'|^{2p}\right\}. \end{aligned}$$

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We shall now state without proof a criterion for the continuity of random fields which is a generalization of the wellknown Kolmogorov's criterion for the continuity of stochastic processes.

Theorem 1.3.7 (Kolmogorov -Totoki)**.** *Let $\{x(\lambda), \lambda \in \Lambda\}$ be a random field where Λ is bounded rectangular in $\mathbb{R}^n$. Suppose there exist positive constants α, β, K such that*

$$E|X(\lambda) - X(\mu)|^{\alpha} \leq K|x - y|^{n+\beta}, \tag{1.3.8}$$

then $\{X(\lambda), \lambda \in, \Lambda\}$ has a continuous modification.

For the proof of the above theorem, see Kunita [18].

Proof of theorem 1.3.2. Let $p > 2(d + 2)$. Then by Theorem 1.3.7, $\phi_{s,t}(x)$ has a continuous modification. Therefore $\phi_{s,t}(., \omega) : \mathbb{R}^d \to \mathbb{R}^d$ is a continuous map for any $s < t$ a.s. For negative p we have

$$E[|\phi_{s,t}(x) - \phi_{s,t}(x)|^{2p}] \leq C|x - y|^{2p}$$
$$E[1 + |\phi_{s,t}(x)|^{2p}] \leq C(1 + |x|)^{2p}.$$

21 These two will imply that $\phi_{s,t}(., \omega)$ is a homeomorphism. The proof is exactly the same as in Kunita [18]. We omit the details.

1.4 Stochastic Flow of Diffeomorphisms

In this section, we will see that if the local characteristics (a, b) of a Brownian flow are smooth and their derivatives bounded then the flow becomes a flow of diffeomorphisms. We will make this precise in the next theorem. Before that, we will add a few words about notations. For a multiindex $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$, where $\alpha_i \in \mathbb{N}, i = 1, 2, \ldots, d, |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$

$$D^{\alpha} = D^{\alpha}_x = (\frac{\partial}{\partial x_1})^{\alpha_1} \ldots\ldots (\frac{\partial}{\partial x_d})^{\alpha_d}.$$

Theorem 1.4.1. *Let $\phi_{s,t}$ be a Brownian flow with local characteristics (a, b) satisfying $(A1), (A2), (A3)$. Assume further $a(x, y, t)$ and $b(x, t)$ are k-times continuously differentiable in x and y and $D^{\alpha}_x D^{\beta}_y a(x, y, t), D^{\alpha}_x$ $b(x, t), |\alpha| \leq k, |\beta| \leq k$ are bounded. Then $\phi_{s,t}(., \omega) : \mathbb{R}^d \to \mathbb{R}^d$ is C^{k-1} diffeomorphism for any $s < t$ a.s.*

We will prove the above theorem for $k = 2$. For higher k, the proof if similar. The proof is based on the following lemmas.

Lemma 1.4.2. *Let e be a unit vector in $\mathbb{R}, y(\neq 0)$ a real number. Set*

$$\eta_{s,t}(x, y) = \frac{1}{y}(\phi_{s,t}(x + ye) - \phi_{s,t}(x)).$$

Then, for any positive integer p, there exists a constant $C = C(p)$ such that

$$\begin{aligned} E|\eta_{s,t}(x, y) - \eta_{s',t'}(x', y')|^{2p} \leq C\Big\{&|x - x'|^{2p} + |y - y'|^{2p} \\ &+ (1 + |x| + |x'| + |y| + |y'|)^{2p} \times (|t - t'|^{p} + |s - s'|^{p})\Big\}. \quad (1.4.1) \end{aligned}$$

22

Lemma 1.4.3. *For $(x_1, x_2, x_3, x_4) \in \mathbb{R}^{4d}$, set*

$$g(x_1, x_2, x_3, x_4) = \frac{1}{y}(x_1 - x_2) - \frac{1}{y'},(x_3 - x_4)$$

where y, y' are fixed nonzero real number. Set $f = |g|^{2p}$, where p is a positive integer. Then there exist constants $C_i = C_i(p)$, $i = 1, 2$ such that

$$\begin{aligned} |L_t^{(4)} f(x_1, x_2, x_3, x_4)| &\leq C_1 f(x_1, x_2, x_3, x_4) \\ &+ C_2(|x_1 - x_3| + |x_2 - x_4|)^{2p}\left(1 + |\frac{1}{y'},(x_3 - x_4)|^{2p}\right). \quad (1.4.2) \end{aligned}$$

Proof of Lemma 1.4.3. We shall consider the case $d = 1$ for simplicity. The time t will be dropped from $a(x, y, t)$ and $b(x, t)$ since it is fixed. A simple computation yields

$$\begin{aligned} L_t^{(f)} f(x_1, x_2, x_3, x_4) = 2p &\left\{\frac{1}{y}(b(x_1) - b(x_2)) - \frac{1}{y'},(b(x_3) - b(x_4))\right\} \\ &|g(x_1, x_2, x_3, x_4)|^{2p-1} x \operatorname{sign} g(x_1, x_2, x_3, x_4) \\ + p(2p - 1)&\left[\frac{1}{y^2}\{a(x_1, x_2) - 2a(x_1, x_2) + a(x_2, x_2)\}\right] \\ - \frac{2}{yy'},&\{a(x_1, x_3) - a(x_1, x_4) + a(x_2, x_3) - a(x_2, x_4)\} \end{aligned}$$

$$+\frac{2}{y'^2}\{a(x_3,x_3)-2a(x_3,x_4)+a(x_4,x_4)\}]g(x_1,x_2,x_3,x_4)|^{2p-2}$$
$$= I_1 + I_2, \text{ say} \tag{1.4.3}$$

23 Using mean value theorem the first tern is written as

$$\frac{1}{2p}I_I = \left\{\left(\left(\int_0^1 b(x_1+\theta(x_2-x_1))d\theta\right)\frac{1}{y}(x_1-x_2)\right.\right.$$
$$\left.\left.-\left(\left(\int_0^1 b'(x_3+\theta(x_4-x_3))d\theta\right)\frac{1}{y'}(x_3-x_4)\right)\right)\right\}\times|g|^{2p-1}\text{ sign } g$$
$$=\left(\int_0^1 b'(x_1+\theta(x_2-x_1))d\theta\right)|g|^{2p}+\int_0^1\{b'(x_1+\theta(x_2-x_1))$$
$$-b'(x_3,\theta(x_4-x_3))\}\,d\theta x\frac{1}{y'}(x_3-x_4)|g|^{2p-1}\text{ sign } g.$$

Using mean value theorem once again, we can find positive constants C_3, C_4 such that

$$|i_1|\le C_3|g|^{2p}+C_4(|x_1-x_3|+|x_2-x_4|)|\frac{1}{y'}(x_3,x_4)||g|^{2p-1}.$$

Using the inequality $ab \le \dfrac{a^\alpha}{\alpha}+\dfrac{b^\beta}{\beta}$ where $\alpha,\beta\ge 1, \alpha^{-1}+\beta^{-1}=1$, we get

$$|I_1|\le C_5|g|^{2p}+C_6(|x_1-x_3|+|x_2-x_4|)^{2p}|\frac{1}{y'}(x_3,x_4)|^{2p}. \tag{1.4.4}$$

We next estimate I_2. Note the relation

$$a(x_i,x_k)-a(x_i,x_m)+a(x_j,x_k)-a(x_j,x_m)$$
$$=\int\int a''(x_i+\theta(a(x_j-x_i),x_k+\tau(x_m-x_k))\theta\mathscr{O}d\tau.(x_i-x_j)(x_k-x_m).$$

where $a''(x,y) = \dfrac{\partial^2}{\partial x \partial y} a(x,y)$.

Set

$$\begin{aligned} \xi_{ik}(\theta,\tau) &= a''((x_i + \theta(x_j - x_i), x_k + \tau(x_m - x_k)),\\ W_1 &= \frac{1}{y}(x_1 - x_2), W_3 = \frac{1}{y'}(x_3 - x_4). \end{aligned}$$

Then we have

$$\begin{aligned} \frac{I_2}{p(2p-1)} &= \left\{ \int_0^1 \int_0^1 (\xi_{11}(\theta,\tau)) W_1^2 - 2\xi_{13}(\theta,\tau) W_1 W_3 \right.\\ &\qquad\qquad \left. + \xi_{33}(\theta,\tau) W_3^2) d\theta d\tau \right\} |g|^{2p-2}\\ &= \left\{ \int_0^1 \int_0^1 \xi_{11}(\theta,\tau) d\theta d\tau \right\} |g|^{2p} + \left\{ \int_0^1 \int_0^1 (\xi_{11}(\theta,\tau)) \right.\\ &\qquad \left. - \xi_{13}(\theta,\tau) - \xi_{31}(\theta,\tau) - \xi_{33}(\theta,\tau)) d\theta d\tau \right\} |g|^{2p-2} |W_3|^2\\ &+ \left\{ \int_0^1 \int_0^1 (2\xi_{11}(\theta,\tau)) - \xi_{13}(\theta,\tau) - \xi_{31}(\theta,\tau) d\theta d\tau \right\} |g|^{2p-2} |g W_3|^2. \end{aligned}$$

Here, the relation $\int\int \xi_{13}(\theta,\tau) d\theta d\tau = \int\int \xi_{13}(\theta,\tau) d\theta d\tau$ in used. The first term in the above is bounded by $C_9 |g|^{2p}$. Again by using **24**
mean value theorem, the second term is bounded by $C_{10}(|x_1 - x_3| + |x_2 - x_4|)^2 |g|^{2p-2} |W_3|^2$. The third term is bounded by $C_{11}(|x_1 - x_3| + |x_2 - x_4|)^2 |g|^{2p-2} |W_3|$.Therefore, we have

$$\begin{aligned} |I_2| &\leq C_9 f + C_{10}(|x_1 - x_3| + |x_2 - x_4|)^2 |g|^{2p-2}\\ &\qquad + C_{11}(|x_1 - x_3| + |x_2 - x_4|) |g|^{2p-1} |\frac{1}{y'}(x_3 - x_4)|\\ &\leq C_{12} f + C_{13}(|x_1 - x_3| + |x_2 - x_4|)^{2p} \end{aligned}$$

$$+ C_{14}(|x_1 - x_3| + |x_2 - x_4|)^{2p}|\frac{1}{y'}(x_3 - x_4)|^{2p}. \qquad (1.4.5)$$

Finally, using the estimates (1.1.4) and (1.4.5) in (1.4.3), we obtain the desired result.

Proof of Lemma 1.4.2. In view of the notation of Lemma 1.4.3

$$f(\phi_{s,t}(x + ye), \phi_{s,t}(x), \phi_{s',t}(x' + y'e), \phi_{s',t}(x'))$$
$$= |\eta_{s,t}(x, y) - \eta_{s',t'}(x', y')|^{2p}.$$

We split the proof in two steps.

(a) $t = t'$, $s' < s$.

Applying Ito's formula, we get

$$E[f(\phi_{s,t}(x + ye), \phi_{s,t}(x), \phi_{s',t}(x' + y'e), \phi_{s',t}(x)))]$$
$$= E[f((x + ye, x, \phi_{s',s}(x' + y'e), \phi_{s,'s}(x')))]$$
$$+ E\left[\int_0^t L_r^{(4)} f(\phi_{s,r}(x, ye), \phi_{s,r}(x), \phi_{s',r}(x' + y'e), \phi_{s,r}(x'))dr\right].$$

25 Therefore using Lemma 1.4.3,

$$E[|\eta_{s,t}(x, y) - \eta_{s',t}(x', y')|^{2p}] \leq E[|e - \eta_{s',s}(x', y')|^{2p}]$$
$$+ C_1 \int_s^t E[\eta_{s,r}(x, y) - \eta_{s',r}(x', y')|^{2p}]dr$$
$$+ C_2 \int_s^t E[(|\phi_{s,r}(x + ye) - \phi_{s',r}(x', y'e)|$$
$$+ |\phi_{s,r}(x) - \phi_{s',r}(x')|)^{2p}.(1 + |\eta_{s',r}x', y')|)^{2p}]dr.$$

Consider the first term on the right hand side. Since

$$\eta_{s,'s}(x', y') - e = \frac{1}{y'}\{\phi_{s',s}(x' + y'e) - \phi_{s',s}(x') - y'e\},$$

using (1.3.6), we get

$$E|\eta_{s',s}(x', y') - e|^{2p} \leq C|s - s'|^p.$$

Applying Schwarz's inequality and (1.3.7), the third term can be dominated by $C'\{|x - x|^{2p} + |y - y|^{2p} + (1 + |x| + |x'| + |y + |y'|)^{2p}|s - s'|^p\}$. We then apply Gronwall's inequality to get the desired estimate.

(b) $s' < s < t' < t$.

Using the flow property, we have

$$\begin{aligned}
E|\eta_{s,t}(x,y) - \eta_{s,t}(x,y)|^{2p}] &= \int E\left[|\frac{1}{y}(\phi_{t',t}(z_1) - \phi_{t',t}(z_2) - z_1 + z_2)|^{2p}\right] \\
&\qquad \times P(\phi_{s,t'}(x + ye)\varepsilon dz_1', \phi_{s,t'}(x)\varepsilon dz_2) \\
&\leq C|t - t'|^p \frac{1}{|y|^{2p}} \int |z_1 - z_2|^{2p} P(\phi_{s,t'}(x + ye)\varepsilon dz_1, \phi_{s,t}(x)\varepsilon dz_2) \\
&= C|t - t'|^p \frac{1}{|y|^{2p}} E|\phi_{s,t'}(x + ye) - \phi_{s,t'}(x)|^{2p} \\
&\leq C'|t - t'|^P.
\end{aligned}$$

Now combining this estimate and the estimate in (a), we get the required inequality. **26**

Proof of theorem 1.4.1. Applying Theorem 1.3.7, $\eta_{s,t}(x, y)$ has a continuous extension at $y = 0$, i.e

$$\lim_{y\to 0} \frac{1}{y}(\phi_{s,t}(x + ye_i) - \phi_{s,t}(x)) = \frac{\partial}{\partial x_i}\phi_{s,t}(x)$$

exists for all s, t, x and for each $i = 1, 2, \ldots$, where $e_i = (0, \ldots, 1, 0, \ldots, 0)1$, being at the ith place, and is continuous in (s, t, x). Hence $\phi_{s,t}(x)$ is continuously differentiable.

Claim: $\phi_{s,t}$is a diffeomorphism.

In view of the fact that $\phi_{s,t}$ is a homeomorphism, it suffices to show that the Jacobain matrix $\partial\phi_{s,t}(x)$ is nonsingular. Consider the map $\mathbb{R}^d \times \mathbb{R}^{d^2} \to \mathbb{R}^d \times \mathbb{R}^{d^2}$

$$(x, z_1, \ldots, z_d) \to (\phi_{s,t}(x), \partial\phi_{s,t}(x)z_1, \ldots, \partial\phi_{s,t}(x)z_d). \tag{1.4.6}$$

We claim that the above is a Brownian flow. Indeed for $s < t < u$

$$\partial\phi_{s,u}(x) = \partial\phi_{t,u}(\phi_{s,t}(x))\partial\phi_{s,t}(x),$$

whence the flow property follows. Therefore (1.4.6) defines a Brownian flow of homeomorphisms. Thus the map $\partial\phi_{s,t}(x)$ is 1-1 and therefore $\partial\phi_{s,t}(x)$ is nonsingular.

Chapter 2

Stochastic Flows and Stochastic Differential Equations

This Chapter deals with the interplay between stochastic flows and stochastic differential equation. In section 2.1 we study non-Brownian stochastic flows.Under certain assumptions we establish the homeomorphism and diffeomorphism properties of such flows. In section 3.2 we define $C = C((R)^d; \mathbb{R}^d)$- valued semimartingales and their local characteristics. At the end of the this section we obtain a representation result for a C-valued Brownian motion. In section 2.3 we define the stochastic integrals of progressively measurable processes with respect to C-semimartingales, which is essentially a generalization of the usual stochastic integrals. In section 2.4 we introduce the concept of the solution of a stochastic differential equation in this setup and then show that the solution defines a stochastic flow. In the next section we take up the converse problem and obtain the representation of a stochastic flow by an SDE. In section 2.6 we discuss the inverse flows and backward infinitesimal generators. The Chapter ends with an appendix where we describe generalized Ito formula, Stratonovich integrals and Stratonovich stochastic differential equations. **27**

2.1 Non-Brownian Stochastic Flows

Let (Ω, F) be a standard measurable space and P a probability measure defined on it. Let $\phi_{s,t}(x,\omega)$ be a stochastic flow of measurable maps which is continuous in t. Let F_t denote the (right) continuous natural filtration of the flow i.e $F_t = \bigcap_{\varepsilon>0} \sigma(\phi_{u,v}(.) : 0 \leq u \leq v \leq t+\varepsilon)$. As in Chapter 1 we will make three assumptions on the flow.

Assumption 1. *$\phi_{s,t}x,.$ is square integrable and the following limits exit:*

$$\lim_{h\to 0} \frac{1}{h} E[\phi_{t,t+h}(x) - x)|F_t](\omega) = b(x,t,\omega), \textit{say, a,s for each } t$$

$$\lim_{h\to 0} \frac{1}{h} E[(\phi_{t,t+h}(x) - x)(\phi_{t,t+h}(y) - y)^*|F_t]\omega = a(x,y,t,\omega)$$

say, a.s. for each t.

28

In the above expressions the conditional expectation is computed with respect to a version of the regular conditional distribution which does exist by the stipulation on the measurable space. We further assume that $b(x,t,\omega)$ and $a(x,y,t,\omega)$ are jointly measurable and progressively measurable for each x (respectively x and y).

Definition 2.1.1. *The pair (a,b) is called the local characteristics of the flow $\phi_{s,t}$.*

Remark 2.1.2. If $\phi_{s,t}$ is a Brownian flow then $\phi_{t,t+h}(x)$ is independent of F_t and therefore $a(x,y,t)$ and $b(x,t)$ do not depend on ω. So in that case a and b coincide with the local characteristics of the Brownian flow. We show later that if the local characteristics a, b do not depend on ω then $\phi_{s,t}$ is a Brownian flow.

Assumption 2. *There exists a positive constant K (independent of ω) such that*

$$|E[\phi_s, t(x) - x|F_s]| \leq K(| + |x|)|t-s|, \tag{2.1.1}$$

$$|E[(\phi_{s,t}(x) - x)(\phi_{s,t}(y) - y)^*|F_s]| \leq K(1+|x|)(1+|y|)|t-s|. \tag{2.1.2}$$

These two inequalities imply

$$|b(x,t,\omega)| \leq K(1+|x|), \tag{2.1.3}$$
$$|a(x,y,t,\omega)| \leq K(1+|x|)(1+|y|). \tag{2.1.4}$$

Lemma 2.1.3. *For each s, x*

$$M_{s,t}(x) = \phi_{s,t}(x) - x - \int_s^t b(\phi_{s,r}(x), r)dr, t \geq s \tag{2.1.5}$$

is an L^2-martingale and **29**

$$< M^i_{s,t}(x), M^j_{s,t}(y) >= \int_s^t a_{ij}(\phi_{s,r}(x), \phi_{s,r}(y), r)dr. \tag{2.1.6}$$

Proof. The proof is the same as that of lemma 1.2.3. Set

$$m_{s,t}x, \omega = E[\phi_{s,t}(x)|F_s](\omega). \qquad \square$$

For $s < t < u$, we have

$$\begin{aligned} E[\phi_{s,t}(x)|F_s] &= \int \phi_{s,u}(x,\omega')p_t(\omega, d\omega') \\ &= \int \phi_{t,u}(\phi_{s,t}(x,\omega'),\omega')p_t(\omega, d\omega') \end{aligned}$$

where $p_t(\omega, d\omega')$ is a regular conditional distribution given F_t. Now for fixed s, t, ω

$$p_t(\omega, \{\omega' : \phi_{s,t}(x,\omega') = \phi_{s,t}(x,\omega)\}) = 1 \quad a.a.\omega.$$

Therefore

$$\begin{aligned} E[\phi_{s,u}(x)|F_t] &= \int \phi_{t,u}(\phi_{s,t}(x,\omega))p_t(\omega, d\omega') \\ &= m_{t,u}(\phi_{s,t}(x,\omega),\omega). \end{aligned}$$

Hence

$$m_{s,u}(x,\omega) = E[m_{t,u}(\phi_{s,t}(x))|F_s]$$

So

$$\frac{1}{h}[m_{s,t+h}(x,\omega) - m_{s,t}(x,\omega)] = \frac{1}{h}E[m_{t,t+h}(\phi_{s,t}(x)) - \phi_{s,t}(x)|F_s].$$

Letting $h \to 0$, we get

$$\frac{\partial}{\partial t} m_{s,t}(x,\omega) = E[b(\phi_{s,t}(x),t)|F_s].$$

30 Thus

$$m_{s,t}(x) - x = E\left[\int_s^t b(\phi_{s,r}(x),r)dr|F_s\right].$$

Then proceeding as in Lemma 1.2.3 we conclude that $M_{s,t}$ is a martingale. For the second assertion set

$$V_{s,t}(x,y,\omega) = E[M_{s,t}(x)M_{s,t}(y)^*|F_s].$$

We can show similarly that

$$V_{s,t}(x,y,\omega) = E\left[\int_s^t a(\phi_{s,r}(x),\phi_{s,r}(y),r)dr|F_2\right](\omega).$$

Again proceeding the same way we prove that

$$M_{s,t}(x)M_{s,t}(y)^* - \int_s^t a(\phi_{s,r}(x),\phi_{s,r}(y),r)dr$$

is a martingale. The proof is complete.

$$\text{Let} \quad \underline{x}^{(N)} = (x_1,\ldots,x_N) \in \mathbb{R}^{Nd}.$$

Define

$$(L_t^{(N)} f)(\underline{x}^{(N)},\omega) = \frac{1}{2}\sum_{i,j,k,\ell} a_{ij}(x_k,x_\ell,t,\omega)\frac{\partial^2 f}{\partial x_k^i \partial x_\ell^j}(\underline{x}^{(N)}) + \sum_{k,i} b^i(x_k,t,\omega)\frac{\partial f}{\partial x_k^i}(\underline{x}^{(N)}). \quad (2.1.7)$$

in this case $L_t^{(N)}$ is a random differential operator.

Theorem 2.1.4. *$\phi_{s,t}(x)$ has finite moments of any order and if f is a C^2-function, f and its derivatives are of polynomial growth, then*

$$f(\phi_{s,t}(x_1),\ldots,\phi_{s,t}(x_N)) - \int_s^t (L_r^{(N)} f)(\phi_{s,r}(x_1),\ldots,\phi_{s,r}(x_N))dr \quad (2.1.8)$$

is a martingale.

The proof is similar to that of Theorem 1.2.5 and hence it is omitted. **31**

Now we make our assumption which is essentially a Lipschitz condition on the local characteristics.

Assumption 3. *There exists a constant L (independent of t, ω) such that*

$$|b(x,t,\omega) - b(y,t,\omega)| \leq L|x-y|, \quad (2.1.8)$$

$$|a(x,x,t,\omega) - 2a(x,y,t,\omega)| + a(y,y,t,\omega)| < L|x-y|^2. \quad (2.1.9)$$

Theorem 2.1.5. *Let $\phi_{s,t}$ be a continuous stochastic flow of measurable maps satisfying $(A1) \sim (A3)$. Then it admits a modification which is a stochastic flow of homeomorphisms.*

Proof. The proof of this theorem also goes along the lines of Theorem 1.3.2. Using Ito's formula and Gronwall's inequality the following estimates can be derived:

(i) For any real p and $\varepsilon > 0$ there exists a constant $C = C(p) > 0$ such that for any t, x

$$E[(\varepsilon + |\phi_{s,t}(x))|^2)^p|F_s] \geq C(\varepsilon + |x|^2)^p a.s. \quad (2.1.10)$$

(ii) For real p and $\varepsilon > 0$ there exists $C = C(p)$ such that

$$E[(\varepsilon + |\phi_{s,t}(x) - \phi_{s,t}(y)|^2)^p|F_s] \geq C(\varepsilon + |x-y|^2)^p \quad (2.1.11)$$

holds for all t, x a.s

(iii) For any positive integer p there exists $C = C(p)$ such that

$$E[|\phi_{s,t}(x_o) - x_o|^{2p}|F_s] \leq C|t-s|^p(1 + |x_o|^2)^p \quad (2.1.12)$$

holds for any $x_o \in \mathbb{R}$ a.s

Using (2.1.10), (2.1.11), (2.1.12) we can show that **32**

$$E[|\phi_{s,t}(x)-\phi_{s',t'}(x')|^{2p}] \leq C(|x-x'|^{2p}+(1+|x|+|x'|)^{2p}.(|t-t'|^p+|s-s'|^p)) \tag{2.1.13}$$

Indeed, consider the case $s = s', x = x', t < t'$.

$$\begin{aligned} E\left[|\phi_{s,t}(x) - \phi_{s,t'}(x)|^{2p}\right] &= E\left[|\phi_{s,t'}(x) - \phi_{t,t'}(\phi_{s,t}(x))|^{2p}\right] \\ &= E\left\{E[|\phi_{s,t}(x) - \phi_{t,t'}(\phi_{s,t}(x))|^{2p}|F_t]\right\} \\ &= E\left\{E[|y - \phi_{t,t'}(y)|^{2p}|F_t]_{y=\phi_{s,t}(x)}\right\} \\ &\leq C|t-t'|^p E\left\{1 + |\phi_{s,t}(x)|^2)^p\right\} \\ &\leq C'|t-t'|^p(1+|x|^2)^p. \end{aligned}$$

The rest of the proof is exactly similar to that of Theorem 1.3.2. We therefore omit the details. □

Remark 2.1.6. Assuming suitable smoothness conditions on (a, b) and boundedness of their derivatives we can establish the diffeomorphism property of the flow exactly the same way as we did in Chapter 1.

2.2 Vector Valued Semimartingales

Let (Ω, F) be a standard measurable space and P a probability measure defined on it. Let there be given a filtration $\{F_t\}, o \leq t \leq T$. Let $X(x, t\omega), x \in \mathbb{R}^d, t \in [O, T]$ be a sample continuous $\mathbb{R}^d$-valued random field. We assume that for each x, t it is F_t-measurable. Let $C = C(\mathbb{R}^d; \mathbb{R}^d)$ endowed with compact uniform topology.

Definition 2.2.1. *$X(x, t)$ is called a C-valued martingale if it is an $\mathbb{R}^d$-valued martingale for each x. It is called a C^k-valued martingale if*
33 *$D_x^\alpha X(x, t)$ is a C-valued martingale for any $|\alpha| \leq k$. Let $X(x, t)$ be a continuous random field which admits the following decomposition.*

$$X(x, t) = X(x, o) + Y(x, t) + V(x, t)$$

where $Y(x,t)$ is a C-valued martingale, $V(x,t)$ a process of bounded variation for each x and $Y(x,o) = 0, V(x,0) = 0$. Then $X(x,t)$ is called a C-valued semimartingale. If $Y(x,t)$ is a C^k-martingale and $D_x^\alpha V(x,t), |\alpha| \leq k$ is of bounded variation then it is called a C^k-semimartingale.

Next we will define the local characteristics of an $\mathbb{R}^d$-valued semimartingale. Here we shall dispense with the continuity condition. Let $X(x,t)$ be an $\mathbb{R}^d$-valued semimartingale with parameter $x \in \mathbb{R}^d$ such that

$$X(x,t) = Y(x,t) + V(x,t)$$

where for each x, $Y(x,t)$ is a martingale and $V(x,t)$ a process of bounded variation. Assume that there exist d-vector valued process $\beta(x,t,\omega)$ with parameter x and $d\times d$-matrix valued process $\alpha(x,y,t,\omega)$ with parameters x,y such that β and α are progressively measurable w.r.t. $\{F_t\}$ and

$$\begin{aligned} V(x,t) &= \int_o^t \beta(x,r)dr, \\ < Y(x,t), Y(y,t)^* > &= \int_o^t \alpha(x,y,r)dr. \end{aligned}$$

Then the pair (α,β) is called the local characteristics of the semimartingale $X(x,t)$.

Remark 2.2.2. The above definition of local characteristics differs from that of a flow. Indeed, if $\phi_{s,t}$ is a stochastic flow as described in section 2.1, then

$$\phi_{s,t}(x) = x + M_{s,t}(x) + \int_s^t b(\phi_{s,r}(x),r)dr.$$

Here $\phi_{s,t}(x)$ is a semimartingale, $M_{s,t}(x)$ its martingale part and **34**
$\int_s^t b(\phi_{s,r}(x),r)dr$ is the bounded variation part. In this case

$$\begin{aligned} \beta(x,t) &= b(\phi_{s,t}(x),t), \\ \alpha(x,y,t) &= a(\phi_{s,t}(x),\phi_{s,t}(y),t). \end{aligned}$$

We shall always assume that α and β are integrable,

i.e., $$E[\int_o^t |\alpha(x,y,r)|dr] < \infty, \quad E[\int_o^t |\beta(x,r)|dr] < \infty.$$

Thus Y becomes an L^2- martingale.

Proposition 2.2.3. *Let $\{Y(x,t), x \in \mathbb{R}^d\}$ be a family of continuous $\mathbb{R}^d$-valued semimartingales with local characteristics (α, β). Suppose that α and β are Lipschitz continuous and $|\alpha(|(0,0,t)|, |\beta(0,t)|$ are bounded, then $Y(x,t)$ has a modification as continuous C-semimartingale. Further if α and β are k-times differentiable in each x, y and there exists K such that*

$$|D_x^\alpha D_y^\beta \alpha(x,y,t)| \le K, |D_x^\beta(x,t)| \le K, |\alpha| \le k, |\beta| \le k,$$

then $Y(x,t)$ has a modification as C^{k-1}-semimartingale.

Proof. Using Burkholder's inequality, one can show that

$$E[|Y(x,t) - Y(x',t')|^p] \le C(|t-t'|^{p/2} + |x-x'|^p).$$

Now using Kolmogorov's theorem one can complete the proof. □

Definition 2.2.4. *Let $X(x,t)$ be a C-valued process continuous in t. It is called a C-Brownian motion if for $0 \le t_0 < t_1 < \cdots < t_n \le T, X_0, X_{t_{i+1}} - X_{t_i} = 0, 1, \ldots, n-1$, are independent.*

35 **Remark 2.2.5.** For $(x_1, \ldots, x_N) \in \mathbb{R}^{Nd}, (X_t(x_1), \ldots, X_t(x_\mathbb{N}))$ is an $\mathbb{R}^{ND}$-valued Brownian motion in the usual sense, i.e., it is a Gaussian process with independent increments.

Proposition 2.2.6. *Let $X(x,t)$ be a continuous C-valued semimartingale with local characteristics (α, β). If α, β do not depend on ω then $X(x,t)$ is a C-Brownian motion*

Proof. This is a straightforward implication of Levy's characterization of Brownian motion. □

Example 2.2.7. Examples of C-Brownian motions.

Let $(B_t^1, \ldots, B_t^r)$ be an r-dimensional standard B.M. Let $F_0(x,t)$, $F_1(x,t)$, $\ldots, F_r(x,t)$ be $\mathbb{R}^d$-valued functions Lipschitz continuous in x. Let

$$X(x,t) = \int_o^t F_o(x,s)ds + \sum_{k=1}^r \int_o^t F_k(x,s)dB_s^k. \tag{2.2.1}$$

Then $X(x,t)$ is a $C - B.m.$ with $\beta(x,t) = F_0(x,t)$ and $\alpha(x,y,t) = \sum\limits_{k=1}^{r} F_k(x,t)F_k(y,t)^*$. Next consider an infinite sequence of independent one-dimensional $B.m's \{B_t^k\}_{k=1}^{\infty}$. Let $\{F_k(x,t)\}_{k=o}^{\infty}$ be a sequence of $\mathbb{R}^d$-valued functions such that there exists $L > 0$ satisfying

$$\sum_{k=o}^{\infty} |F_k(x,t) - F_k(y,t)|^2 \leq L|x-y|^2, \sum_{k=o}^{\infty} |F_k(x,t)|^2 \leq L(1+|x|^2.$$

Then

$$X(x,t) = \int_o^t F_o(x,r)dr + \sum_{k=1}^{\infty} \int_o^t F_k(x,r)dB_r^k \tag{2.2.2}$$

converges and is a $C-B.m.$ In this case $\beta(x,t)$ and $\alpha(x,y,t) = \sum\limits_{k=1}^{\infty} F_k(x,t)$ $F_k(y,t)^*$.

We will see in the next proposition that if the local characteristics of a C-B.m. satisfy suitable condition then it is of the form (2.2.2). **36**

Proposition 2.2.8. *Let $X(x,t)$ be a C-B.m. with local characteristics (α,β) and $X(x,0) = 0$. Assume α,β are Lipschitz continuous. Then there exist an infinite sequence of independent standard B.m.'s $\{B_t^k\}_{k=1}$ and functions $F_k(x,t), k = 0,1,2,\ldots$ such that*

$$X(x,t) = \int_o^t F_o(x,r)dr + \sum_{k=1}^{\infty} \int_o^t F_k(x,r)dB_r^k.$$

Proof. We will only consider the homogeneous case, i.e., when the local characteristics do not depend on t. Let $F_0(x) = E[X(x,1)]$, then $E[X(x,t)] = tF_0(x)$. Set

$$Y(x,t) = X(x,t) - tF_o(x) \qquad \square$$

Then $Y(x,t)$ is a zero-mean $C - B.m.$ Let $\{x_n\}$ be a dense subset of $\mathbb{R}^d$. Consider the sequence $\{Y^i(x_k,t)\}_{\substack{k=1,2,\ldots \\ i=1,2,\ldots,d}}$. By Schmidt's orthogonalization procedure we can find a sequence $\{B_t^k\}_{k=1,2,\ldots}$ of orthogonal

(=Independent) Brownian motions such that the linear span of $\{B_t^k, k = 1, 2, \ldots\}$ is equal to that of $\{Y^i(x_k, t), k = 1, 2, \ldots, i = 1, \ldots, d\}$. Set $F_k(x) = E[Y(x, 1)B_1^k]$. Then

$$F_k(x)t = E[Y(x, t)B_t^k],$$
$$Y(x, t) = F_o(x)t + \sum_{k=1}^{\infty} F_k(x)B_t^k.$$

Also it is easily seen that $\alpha(x, y) = \sum_{k=1}^{\infty} F_k(x)F_k(y)^*$.

2.3 Stochastic Integrals

37 Let $Y(x, t)$ be a C-martimgale such that the characteristic $\alpha(x, y, t, \omega)$ is integrable and continuous in (x, y). Further we make the following assumption:

Assumption 4. $\int_o^t \sup_{|x|,|y|\leq K} |\alpha(x, y, t, \omega)|dt < \infty$ *for any K and t.*

Let $f_t(\omega)$ be a progressively measurable $\mathbb{R}^d$-valued process such that

$$\int_o^t |\alpha(f_s, f_s, s)|ds < \infty \text{ a.s. for each } t. \tag{2.3.1}$$

Our endeavour here is to define the stochastic integral $\int_0^t Y(f_r, dr)$. This would be a natural generalization of the usual stochastic integral in the sense that if $Y(x, t) = xY_t$, Y_t an L^2-martingale then $\int_o^t Y(f_r, dr) = \int_o^t f_r dY_r$.

Case a f_t is continuous in t.

For a positive integer N, define the following stopping time.

$$\tau_N(\omega) = \begin{cases} \inf\Big\{ t \in [0, T] : \sup\limits_{0\leq r\leq t} |Y(f_r, t)| > N \quad \text{or} \\ \qquad\qquad \int_o^t \sup\limits_{0\leq r\leq s} |\alpha(f_r, f_r, s)|ds > N \Big\} \\ \infty, \text{if the above set is empty.} \end{cases}$$

Then $\tau_N \uparrow \infty$ as $N \uparrow \infty$. Set $Y_N(x,t) = Y(x, t\Lambda\tau_N)$. Then $Y_N(x,t)$ is a C-martingale with local characteristic $\alpha_N(x,y,t) = \alpha(x,y,t)I_{\{\tau_N>t\}}$. Let $\Delta = \{0 = t_0 < t_1 < \cdots < \tau_n = T\}$ be a partition of the interval [0,T]. For any $t \in [0,T]$ define

$$L_t^{N\Delta}(f) = \sum_{i=0}^{n-1}\{Y_N(f_{t_i\Lambda t}, t_{i+1}\Lambda t) - Y_N(f_{t_i\Lambda t}, t_i\Lambda t)\}. \tag{2.3.2}$$

Then $L_t^{N\Lambda}(f)$ is an $\mathbb{R}^d$-valued L^2-martingale, since for $t = t_k$, $s = t_i$, $k > i$,

$$\begin{aligned} &E\left[L_t^{N\Delta}(f) - L_s^{N\Delta}(f)|F_s\right] \\ &\qquad = \sum_{j=i}^{k-1} E\left[E\left[Y_N(f_{t_j}, t_{j+1}) - Y_N(f_{t_j}, t_j)|F_{t_j}\right]|F_s\right] = 0. \end{aligned}$$

Also **38**

$$\begin{aligned} &E\left[L_t^{N\Delta}(f) - L_s^{N\Delta}(f)(L_t^{N\Delta}(f) - L_s^{N\Delta}(f))^*|F_s\right] \\ &= \sum_{j=i}^{k-1} E[E[(Y_N(f_{t_j}, t_{j+1}) - Y_N^1(f_{t_j}, t_j))(Y_N(f_{t_j}, t_{j+1}) \\ &\qquad - Y_N(f_{t_j}, t_j))^*|F_{t_j}]|F_s] \\ &= \sum_{j=i}^{k-1} E[E[\int_{t_j}^{t_{j+1}} \alpha_N(f_{t_j}, f_{t_j}, r)dr|F_{t_j}]|F_s] \\ &= E[\int_s^t \alpha_N(f_r^\Delta, f_r^\Delta, r)dr|F_s], \end{aligned}$$

where $f_r^\Delta = f_{t_k}$ if $t_k \leq r < t_{k+1}$. Therefore

$$< L_t^{N\Delta}(f), L_t^{N\Delta}(f)^* >= \int_o^t \alpha_N(f_r^\Delta, f_r^\Delta, r)dr. \tag{2.3.3}$$

Now let $\{\Delta_n, n = 1, 2, \ldots\}$ be a sequence of partitions of $[0,T]$ such that $|\Delta_n| \to o$. Consider the corresponding sequence of L^2-martingales $\{L_t^{N\Delta_n}(f), n = 1, 2, \ldots\}$. As before it can be verified that

$$< L_t^{N\Delta_n}(f), L_t^{N\Delta_m}(f)^* >= \int_o^t \alpha_N(f_r^{\Delta_n}, f^{\Delta_m}, r)dr.$$

Therefore

$$
\begin{aligned}
&< L_t^{N\Delta_n}(f), -L_t^{N\Delta_m}(f), (L_t^{N\Delta_n}(f), -L_t^{N\Delta_m}(f))^* > \\
&\quad = \int_o^t \left[\alpha_N\left(f_r^{\Delta_n}, f_r^{\Delta_n}, r\right) - \alpha_N\left(f_r^{\Delta_n}, f_r^{\Delta_m}, r\right) - \alpha_N\left(f_r^{\Delta_m}, f_r^{\Delta_n}, r\right)\right. \\
&\qquad \left. +\alpha_N\left(f_r^{\Delta_m}, f_r^{\Delta_m}, r\right)\right] dr \to 0 \text{ as } m, n \to \infty \text{ a.s.}
\end{aligned}
$$

39 Now in view of (A4), the above also converges in L^1-sense. We then define

$$\lim_{n\to\infty} L_t^{N\Delta_n}(f) = L_t^N(f) = \int_o^t Y_N(f_r, dr). \tag{2.3.4}$$

Thus $L_t^N(f)$ is an L^2-martingale. For $N > M$ it is easy to see that $L_t^N(f)$ $= L_t^M(f)$ if $t < \tau_M(\leq \tau_N)$. Define

$$L_t(f) = L_t^N(f) \text{ if } t < \tau_N. \tag{2.3.5}$$

Then $L_t(f)$ is a continuous local martingale. We write

$$L_t(f) = \int_o^t Y(f_s, ds).$$

Case b. ***f_t is progressively measurable and bounded.***

In this case set $f_t^N = \dfrac{1}{N}\int_{t-1/N}^t f_s ds$. Then $\{f_t^N\}$ is uniformly bounded continuous process and it converges to f_t a.s. w.r.t. $dt \otimes dP$. Now since

$$
\begin{aligned}
&< L_t\left(f^N\right) - L_t\left(f^M\right), \left(L_t\left(f^N\right) - L_t\left(f^M\right)\right)^* > \\
&\quad = \int_o^t \left[\alpha\left(f_\nu^N, f_\nu^N, \nu\right) - \alpha\left(f_\nu^N, f_\nu^M, \nu\right) - \alpha\left(f_\nu^M, f_\nu^N, \nu\right)\right. \\
&\qquad \left. +\alpha\left(f_\nu^M, f_\nu^M, \nu\right)\right] d\nu \to 0 \text{ a.s.}
\end{aligned}
$$

therefore $\{L_t(f^N), N = 1, 2, \ldots\}$converges uniformly in t in probability (see Kunita [18], Thm 3.1). Hence we set

$$L_t(f) = \lim_{N\to\infty} L_t\left(f^N\right) = \int_o^t Y\left(f_r, dr\right).$$

Case c.

In the general case when f_t is only progressively measurable, write $f^N = (f \Lambda N)V(-N)$. then $\{L_t(f^N), N = 1, 2, \ldots\}$ converges uniformly in t in probability. Set

$$L_t(f) = \lim_{N \to \infty} L_t(f^N) = \int_o^t Y(f_r, dr).$$

Proposition 2.3.1. *Let f_t and g_t be progressively measurable processes satisfying (2.3.1). Then* 40

$$< \int_o^t Y(f_r, dr), \int_o^t Y(g_r, dr) = \int_o^t \alpha(f_s, g_s, s) ds. \tag{2.3.6}$$

Proof. It is straightforward.

Let $Y(x, t)$ and $Z(x, t)$ be continuous C-martingales with local characteristics α^Y and α^Z respectively satisfying $(A4)$. Let f_t and g_t be progressively measurable processes such that $\int_o^t |\alpha^Y(f_s, f_s, s)| ds < \infty$, $\int_o^t |\alpha^Z(g_s, g_s, s)| ds < \infty$ for all t a.s. Then $\int\limits_o^t Y(f_r, dr)$ and $\int\limits_o^t Z(g_r, dr)$ make sense. We are interested in computing

$$< \int_o^t Y(f_r, dr), (\int_s^t Z(g_r, dr))^* > .$$

□

Lemma 2.3.2. *There exists a random field $\alpha^{YZ}(y, z, t, \omega)$ which is measurable and is continuous in y, z such that*

$$< Y(y, t), Z(z, t)^* >= \int_o^t \alpha^{YZ}(y, z, r, \omega) dr. \tag{2.3.7}$$

Further

$$|\alpha^{YZ}_{ij}(y, z)| \leq \alpha^Y_{ii}(y)^{1/2}, \alpha^Z_{jj}(z)^{1/2}, \alpha^Y_{ii}(y) = \alpha^Y_{ii}(y, y), \tag{2.3.8}$$

and

$$\begin{aligned} |\alpha^{YZ}_{ij}(y, z) - \alpha^{YZ}_{ij}(y', z')| \leq \alpha^Y_{ii}(y)^{1/2} \left\{ \alpha^Z_{jj}(z) - 2\alpha^Z_{jj}(z, z') + \alpha^Z_{jj}(z') \right\}^{1/2} \\ + \alpha^Z_{jj}(z')^{1/2} \left\{ \alpha^Y_{ii}(y) - 2\alpha^Y_{ii}(y, y') + \alpha^Y_{ii}(y') \right\}^{1/2}. \end{aligned} \tag{2.3.9}$$

Definition 2.3.3. *$\alpha^{YZ}(y,z,t)$ is called the joint local charcteristic of Y and Z.*

Proof of Lemma 2.3.2. It is well known that for any fixed $y, z < Y(y,t)$,
41 $Z(z,t)^* >$ has a density function $\alpha^{YZ}(y,z,t)$ w.r.t. dt. Further

$$
\begin{aligned}
&| < Y^i(y,t), Z^j(z,t) > - < Y^i(y,s), Z^j(z,s) > | \\
&\quad \leq (| < Y^i(y,t) > - < Y^i(y,s) >)^{1/2} (< Z^j(z,t) > - < Z^j(z,s) >)^{1/2}.
\end{aligned}
$$

Therefore

$$\frac{1}{t-s} \left| \int_s^t \alpha_{ij}^{YZ}(y,z,r)dr \right| \leq \frac{1}{t-s} \left(\int_s^t \alpha_{ii}^{Y}(y,r)dr \right)^{1/2} \left(\int_s^t \alpha_{jj}^{Z}(z,r)dr \right)^{1/2}.$$

Letting $t \to s$, we get

$$|\alpha_{ij}^{YZ}(y,z,s)| \leq \alpha_{ii}^{Y}(y,s)^{1/2} \alpha_{jj}^{Z}(z,s)^{1/2} \text{ for all } y,z, \text{ a.a.s.}$$

The second inequality can be proved similarly. Also (2.3.9) implies that $\alpha^{YZ}(y,z,t)$ is continuous in y, z.

Theorem 2.3.4. *We have*

$$< \int_s^t Y(f_r, dr), \left(\int_s^t Z(g_r, dr) \right)^* >= \int_s^t \alpha^{YZ}(f_r, g_r, r)dr. \tag{2.3.11}$$

The proof is simple.

2.4 Construction of stochastic Flows by Stochastic Differential Equations

Let

$$X(x,t) = Y(x,t) + \int_o^t \beta(x,r)dr \tag{2.4.1}$$

be a continuous C-semimartingale whose local characteristics α, β are continuous in x, y and satisfy

$$\int_o^t \sup_{|x|,|y| \leq K} |\alpha(x,y,r,\omega)| dr < \infty, \int_o^t \sup_{|x| \leq K} |\beta(x,r)| dr < \infty \qquad (A4)'$$

42 for any $K > 0$ and any t.

Let f_t be a progressively measurable process satisfing (2.3.1) and $\int_o^t |\beta(f_r, r)| dr < \infty$. We define

$$\int_o^t X(f_r, dr) = \int_o^t \beta(f_r, r) dr + \int_o^t Y(f_r, dr). \tag{2.4.2}$$

Definition 2.4.1. *A continuous F_t-adapted $\mathbb{R}^d$-valued process ϕ_t is said to be a solution of the stochastic differential equation*

$$d\phi_t = X(\phi_t, dt) \tag{2.4.3}$$

starting from x at time $s(t \geq s)$ if

$$\phi_t = x + \int_s^t X(\phi_t r, dr) \quad \textit{for all} \quad t \geq s. \tag{2.4.4}$$

Example 2.4.2. Let us consider the Example 2.2.7, i.e.,

$$X(x, t) = \int_o^t F_o(x, s) ds + \sum_{k=1}^{r} \int_o^t F_k(x, s) dB_s^k.$$

In this case the SDE reduces to the usual one, viz.

$$d\phi_t = F_o(\phi_t, t) dt + \sum_{k=1}^{r} F_k(\phi_t, t) dB_t^k.$$

Theorem 2.4.3. *Assume that the local characteristics α, β of the semimartingale $X(x, t)$ satisfy Lipschitz continuity and have linear growth, then the SDE(2.4.3) has a unique solution for any x and s. Further if $\phi_{s,t}(x)$ is the solution starting from x at time s then it has a modification which is a stochastic flow of homeomorphisms.*

Proof. The proof is based on the method of successive approximations. The steps are similar to those used in proving the existence and uniqueness of the solution of a usual SDE. Set for $t \geq s$

$$\phi^0 = x$$

$$\phi_t^1 = x + \int_s^t X(\phi_r^o, dr)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\phi_t^n = x + \int_s^t X(\phi_r^{n-1}, dr) = x + \int_s^t \beta(\phi_r^{n-1}, dr) + \int_s^t Y(\phi_r^{n-1}, dr)$$

43 $\square$

Therefore, we have

$$\begin{aligned}
&E\left\{\sup_{s\le r\le t} |\phi_r^{n+1} - \phi_r^n|^2\right\} \\
&\le 2\left\{E\left[\sup_{s<r\le t} |\int_s^t (\beta(\phi_r^n, r) - \beta(\phi_r^{n-1}, r))dr|^2\right]\right. \\
&\quad \left. +4E\left[|\int_s^t Y(\phi_r^n, dr) - \int_s^t Y(\phi_r^{n-1}, dr)|^2\right]\right\} \\
&\le 2\left\{E\left[\sup_{s\le r\le t} |\int_s^t (\beta(\phi_r^n, r) - \beta(\phi_r^{n-1}, r))dr|^2\right]\right. \\
&\quad \left. +4E\left[\int_s^t Tr\left\{\alpha(\phi_r^n, \phi_r^n) - 2\alpha(\phi_r^n, \phi_r^{n-1}) + \alpha(\phi_r^{n-1}, \phi_r^{n-1})\right\} dr\right]\right\} \\
&\le 2\left\{L^2 TE\left[\sup_{s\le r\le t} \int_s^t |\phi_r^n - \phi_r^{n-1}|^2 dr\right] + 4LE\left[\int_s^t |\phi_r^n - \phi_r^{n-1}|^2 dr\right]\right\}
\end{aligned}$$

(where L is the Lipschitz constant associated with the local characteristics)

$$\begin{aligned}
&\le 2L(LT, 4)\int_s^t E\left[\sup_{s\le r'\le r} |\phi_{r'}^n - \phi_{r'}^{n-1}\right] dr \\
&\le \{2L(L**4)\}^n \frac{1}{n!}(t-s)^n L(1+|x|)^2, \text{by induction.}
\end{aligned}$$

Hence

$$\sum_{n=1}^{\infty} E\left\{\sup_{s\le r\le t} |\phi_r^{n+1} - \phi_r^n|^2\right\}^{1/2} < \infty.$$

Therefore $\{\phi_t\}$ converges uniformly in t in L^2-sense. Let

$$\phi_t = \lim_{n\to\infty} \phi_t^n.$$

Then ϕ is a solution of the equation (2.4.3). The uniqueness result is similar. We denote the solution by $\phi_{s,t}(x)$. We shall now establish the homeomorphism property of the solution. Note that **44**

$$M_{s,t}(x) = \phi_{s,t}(x) - x - \int_s^t \beta(\phi_{s,r}(x), r)dr = \int_s^t Y(\phi_{s,r}(x), dr)$$

is an L^2-martingale and

$$< M_{s,t}(x), M_{s,t}(y)^* >= \int_s^t \alpha(\phi_{s,r}(x), \phi_{s,r}(y), r)dr).$$

Therefore we get

$$E|\phi_{s,t}(x) - \phi_{s,t}(y)|^p \leq C|x-y|^p,$$

whence it follows that $\phi_{s,t}(x)$ is continuous in x. Next define

$$\tilde{\phi}_{s,u}(x) = \begin{cases} \phi_{s,u}(x) & \text{if} \quad s < u < t \\ \phi_{t,u}(\phi_{s,t}(x)) & \text{if} \quad u > t. \end{cases}$$

Then $\tilde{\phi}_{s,u}(x)$ is also a solution of (2.4.3) starting from x at time s. Therefore $\tilde{\phi}_{s,u}(x) = \phi_{s,u}(x)$ a.s. Hence $\phi_{s,t}$ has the flow property. Therefore by Theorem 2.1.5 $\phi_{s,t}$ is a stochastic flow of homeomorphisms.

Definition 2.4.4. $F_{s,t} = \sigma(X(x,u) - X(x,v) : s \leq u, v \leq t)$. *Then clearly* $\phi_{s,t}(x)$ *is* $F_{s,t}$*-measurable.*

Corollary 2.4.5. *If* $X(x,t)$ *is a C-valued B.M. then* $\phi_{s,t}$ *is a Brownian flow.*

Proof. For any $0 \leq t_0 < t_1 < \ldots t_n \leq T, F_{t_i,t_{i+1}}, i = 0,1,\ldots,n-1$ are independent. Hence $\phi_{t_i,t_{i+1}}, i = 0,1,\ldots,n-1$ are independent. Thus $\phi_{s,t}$ is a Brownian flow. □

2.5 Representation of Stochastic Flows by SDES

In the previous section we have seen that the solution of an SDE de-
fines a stochastic flow of homeomorphisms. Here we discuss the con-
45 verse problem, *i.e.* given a stochastic flow can it be represented as the
stochastic integral *w.r.t.* some semimartingale? To make it precise let
$\phi_{s,t}$ be a stochastic flow satisfying $(A1) \sim (A3)$ with local characteristics
(a, b).

Problem To find a continuous C-semimartingale $X(x, t)$ such that

$$\phi_{s,t}(r) = x + \int_s^t X(\phi_{s,r}(x), dr). \tag{2.5.1}$$

The solution to this problem is very when $a \equiv 0$. Indeed,

$$M_{s,t}(x) = \phi_{s,t}(x) - x - \int_s^t b(\phi_{s,r}(x), r)dr = 0.$$

Therefore $\phi_{s,t}$ is the solution of the stochastic ordinary differential equation

$$\begin{aligned} \frac{dx}{dt} &= b(x, t, \omega) \\ x(s) &= x. \end{aligned}$$

The general case is dealt with in the following theorem.

Theorem 2.5.1. *Let $\phi_{s,t}$ be a stochastic flow satisfying $(A1) \sim (A3)$ with local characteristics (a, b). Then there exists a unique continuous C-semimartingale $X(x, t)$ satisfying $(A4)'$ such that the representation* (2.5.1) *holds. Furthermore the local characteristics of $X(x, t)$ coincide with a, b.*

Definition 2.5.2. *The semimartingale $X(x, t)$ associated with $\phi_{s,t}(x)$ is called the infinitesimal generator (or stochastic velocity field) of the flow $\phi_{s,t}$.*

Proof of theorem 2.5.1. Set $M(x, t) = M_{0,t}(x)$. Then $M(x, t)$ is a continuous C-martingale with local characteristic $a(\phi_{0,t}(x), \phi_{0,t}(y), t)$,i.e.,

$$< M(x, t), M(y, t)^{*} >= \int_{o}^{t} a(\phi_{o,r}(x), \phi_{o,r}(y), r)dr \tag{2.5.2}$$

Next define 46

$$Y(x, t) = \int_{o}^{t} M(\phi^{-1}_{o,r}(x), dr). \tag{2.5.3}$$

Then using Theorem 2.3.4, we have

$$< Y(x, t), Y(y, t)^{*} >= \int_{o}^{t} a(x, y, r)dr. \tag{2.5.4}$$

We will show that

$$< X(x, t) = Y(x, t) + \int_{o}^{t} b(x, r)\, dr. \tag{2.5.5}$$

is the required semimartingale. Define

$$\tilde{M}_{s,t}(x) = \int_{s}^{t} Y(\phi_{s,r}(x), dr). \tag{2.5.6}$$

Then by Theorem 2.3.4, we have

$$< \tilde{M}_{s,t}(x), \tilde{M}_{s,t}(y)^{*} >= \int_{s}^{t} a(\phi_{s,r}(x), \phi_{s,r}(y), r)dr). \tag{2.5.7}$$

Also

$$< Y(x, t) - Y(x, s), M_{s,t}(y)^{*} >= \int_{s}^{t} a(x, \phi_{s,r}(y), r)dr. \tag{2.5.8}$$

Combining (2.5.4) and (2.5.8) we get

$$< \tilde{M}_{s,t}(x), M_{s,t}(y)^{*} >= \int_{s}^{t} a(\phi_{s,r}(x), \phi_{s,r}(y), r)dr). \tag{2.5.9}$$

Therefore

$$< M_{s,t}(x) - \tilde{M}_{s,t}(x), (M_{s,t}(x) - \tilde{M}_{s,t}(x))^{*} >= 0.$$

Hence

$$M_{s,t}(x) = \tilde{M}_{s,t}(x).$$

Thus 47

$$\phi_{s,t}(x) - x - \int_s^t b(\phi_{s,r}(x), r)dr = \int_s^t Y(\phi_{s,r}(x), dr).$$

This completes the existence part. Now we will show the uniqueness of the representation.

Suppose there exists another continuous C-semimartingale $X'(x,t)$ satisfying $(A4)'$ such that

$$\phi_{s,t}(x) = x + \int_{s-}^t X'(\phi_{s,r}(x), dr) = x + \int_s^t X(\phi_{s,r}(x), dr).$$

Claim:$X = X'$

Indeed, we can write

$$X'(x,t) = \int_o^t b'(x,r)dr + Y'(x,t)$$

where Y is a continuous C-martingale. By the uniqueness of Doob-Mayer decomposition, we have

$$\int_s^t b(\phi_{s,r}(x), r)dr = \int_s^t b'(\phi_{s,r}(x), r)dr,$$

and
$$\int_s^t Y(\phi_{s,r}(x), dr) = \int_s^t Y'(\phi_{s,r}(x), dr).$$

Let $\alpha^Y, \alpha^{Y'}$ and $\alpha YY'$ denote the local characteristics of Y, Y' and the joint local characteristic of (Y, Y') respectively. We have for any $s < t, x \in \mathbb{R}^d$

$$\begin{aligned} < \left(\int_s^t Y(\phi_{s,r}(x), dr) - \int_s^t Y'(\phi_{s,r}(x), dr)\right), \\ \left(\int_s^t Y(\phi_{s,r}(x), dr) - \int_s^t Y'(\phi_{s,r}(x), dr)\right)^* > \\ = \int_s^t \Big[\alpha^Y(\phi_{s,r}(x), \phi_{s,r}(x), r) - \alpha^{YY'}(\phi_{s,r}(x), \phi_{s,r}(x), r) \end{aligned}$$

$$-\alpha^{YY'}(\phi_{s,r}(x), \phi_{s,r}(x), r) + \alpha^{Y'}(\phi_{s,r}(x), \phi_{s,r}(x), r)\Big]\, dr.$$

48 This implies

$$\alpha^{Y}(\phi_{s,t}(x), \phi_{s,t}(x), t) - \alpha^{YY'}(\phi_{s,t}(x), \phi_{s,t}(x), t)$$
$$- \alpha^{Y'Y}(\phi_{s,t}(x), \phi_{s,t}(x), t) + \alpha^{Y'}(\phi_{s,t}(x), \phi_{s,t}(x), t) = 0$$

for almost all t. Putting $x = \phi_{s,t}^{-1}(x)$, we get

$$\alpha^{Y}(x, x, t) - \alpha^{YY'}(x, x, t) - \alpha^{Y'Y}(x, x, t) + \alpha^{Y'}(x, x, t) = 0 \text{ for a.a.t.}$$

Hence

$$< Y(x, t) - Y'(x, t), (Y(x, t) - Y'(x, t))^{*} >= 0.$$

Therefore

$$Y(x, t) = Y'(x, t).$$

Corollary 2.5.3. *If $\phi_{s,t}$ is a Brownian flow then $X(x, t)$ is a C-B.M.*

Proof. If $\phi_{s,t}$ is a Brownian flow then the local characteristics a, b are deterministic which in turn implies that the local characteristic of Y is deterministic. Thus $Y(x, t)$ is a $C.B.m.$ with local characteristic a. Hence $X(x, t)$ is a C-B.m. □

In the light of representation of stochastic flows by $SDES$ we reconsider the existence of a Brownian flow with given local characteristics (a, b). It was assumed in Theorem 1.2.8 that a and b were twice spatially differentiable and their derivatives bounded. In the next theorem we will drop these conditions and still show the existence of a Brownian flow using Theorem 2.5.1.

Theorem 2.5.4. *Suppose we are given a pair of functions $(a(x, y, t)$, $b(x, t))$ such that*

(i) *$a(x, y, t) = (a_{ij}(x, y, t))_{i,j=1,\ldots,d}$ is symmetric and nonnegative definite and is continuous in (x, y, t) and Lipschitz continuous in x, y.* 49

(ii) *$b(x, t) = (b^{i}(x, t))_{i=1,\ldots,d}$ is continuous in x, t and Lipschitz continuous in x.*

Then there exists a unique Brownian flow with local characteristics (a, b) *satisfying* $(A1) \sim (A3)$*(of Chapter 1).*

Proof. We can find a Gaussian random field $X(x, t, \omega)$with independent increments such that the mean of $X(x, t, \omega)$is $\int_o^t b(x, r)dr$ and $Cov(X(x, t),\ X(y, s)) = \int_o^{t\Lambda s} a(x, y, r)dr$. Also $X(x, t)$ has no fixed discontinuity. Therefore $X(x, t)$ is a Brownian motion for each x. Since $a(x, y, t)$ is Lipschitz continuous in $x, y, X(x, t)$ has a modification such that it is continuous in (x, t). Therefore $X(x, t)$ is a $C - B.m.$ Now consider the SDE based on $X(x, t)$

$$d\phi_t = X(\phi_t, dt), \phi_s = x, t \geq s.$$

The solution of the SDE $\phi_{s,t}(x)$ is a Brownian flow of homeomorphisms with local characteristics (a, b). This completes the existence part. □

uniqueness. Let $\phi_{s,t}$ be a Brownian flow with local characteristics (a, b) satisfying $(A1) \sim (A3)$. Then there exists an infinitesimal generator of $\phi_{s,t}$, say $X(x, t)$, which is a $C - B.m.$ with local characteristics (a, b). The law of $X(x, t)$ is uniquely determined by (a, b). Hence the law $\phi_{s,t}$ is unique.

Remark 2.5.5. The law of a non-Brownian flow is not in general determined by its local characteristics. Similarly the law of a C- semimartingale is not determined by its local characteristics. To justify this we produce a counterexample. Let B_t^1, B_t^2 be two independent one-dimensional Brownian motions. Set

$$Y_t = \int_o^t B_s^1 dB_s^1, Z_t = \int_o^t B_s^1 dB_s^2.$$

Here Y_t and Z_t have the same local characteristics, viz. $(B_t^1)^2$, but the laws of Y_t and Z_t are different.

50

Remark 2.5.6. We know that 2-point process determines the law of a Brownian flow. But 1-point process does not determine it. In fact one can find several Brownian flows with the same 1-point process. Here we give an example which is due to Harris [9].

Example. Let $x \in \mathbb{R}^1$ and $c(x)$ a real, nonnegative definite function of class C_b^2 and $c(0) = 1$. Set $a(x, y) = c(x - y)$. Then $a(x, y)$ is symmetric and nonnegative definite and is Lipschitz continuous. Therefore there exists a stochastic flow of homeomorphisms $\phi_{s,t}$ with local characteristic a. Consider the one-point process $\phi_{s,t}(x)$ s, x fixed). This is a diffusion with infinitesimal generator $L = \frac{1}{2}\frac{d^2}{dx^2}$. Therefore $\phi_{s,t}(x)t \in [s, T]$, is a Brownian motion for any x. There may be many such $c(x)$. Simplest Brownian flow would be $\phi_{s,t}(x) = x + B_t(B_t$: one-dimensional B.M). In this case $c(x) \equiv 1$.

To end this section it would perhaps be of some relevance to discuss the following problem. We know that the infinitesimal generator of 1-point motion of a Brownian flow is an elliptic operator. Now given an elliptic operator

$$L_t = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x, t)\frac{\partial^2}{\partial x_j \partial x_j} + \sum_{i=1}^{d} b^i(x, t)\frac{\partial}{\partial x_i}$$

(where (a_{ij}) is nonnegative definite, symmetric and is continuous in x), does there exists a Brownian flow whose 1-point motion has the infinitesimal generator L_t? The problem is reduced to finding $(a_{ij}(x, y, t))$ which should be nonnegative definite and symmetric with some smoothness condition such that $a_{ij}(x, x, t) = a_{ij}(x, t)$.

If a_{ij} is independent of t and $a_{ij}(x)$ is C_b^2 then we can find such $a(x, y)$ (see Ikeda-Watanabe [13]). Indeed there exists a square root of $a = (a_{ij})$ i.e., $a(x) = \sigma(x)\sigma(x)^*$ where $\sigma(x)$ is Lipschitz continuous. We then define $a(x, y) = \sigma(x)\sigma(y)$ which is symmetric and nonnegative definite. **51**
Finally the solution $\phi_{s,t}$ of

$$d\phi_t = b(\phi_t)dt + \sigma(\phi_t)dB_t$$

is a Brownian flow with local characteristic $a(x, y)$. The solution to the problem in the general case is not known.

2.6 Inverse Flows and Infinitesimal Generator

Let $\phi_{s,t}$ be a stochastic flow of homeomorphisms generated by a continuous C-semimartingale $X(x,t)$ with local characteristics a,b which are Lipschitz continuous and have linear growth. In other words $\phi_{s,t}$ satisfies the following SDE

$$d\phi_{s,t}(x) = X(\phi_{s,t}(x), dt), \phi_{s,s}(x) = x. \tag{2.6.1}$$

Let $\Psi_{s,t} \equiv \phi_{s,t}^{-1}$, the inverse of $\phi_{s,t}$. Then obviously $\Psi_{s,t} o \Psi_{t,u} \equiv \Psi_{s,u}$ for $s < t < u$. $\Psi_{s,t}$ is called a backward flow. In this section, we take up the following problem.

Problem To find the backward infinitesimal generator of $\Psi_{s,t}$.

In other words we want to represent $\Psi_{s,t}$ in the form (2.6.1). To accomplish this we have to define backward semimartingales and backward integrals.

Let $X(x,t)$ be a continuous C-semimartingale given by

$$X(x,t) = Y(x,t) + \int_o^t b(x,r)dr \tag{2.6.2}$$

where $Y(x,t)$ is a continuous C-martingale. Set

$$G_{s,t} = \sigma(Y(.,u) - Y(.,v) : s \leq u, v \leq t).$$

For fixed s, $X(x,t) - X(x,s)$, $t \in [s,T]$ is a $G_{s,t}$-semimartingale but for
52 fixed t $X(x,s) - X(x,t)$, $s \in [0,t]$ is adapted to $G_{s,t}$ but need not be a semimartingale. So here we make such an assumption.

(A5) For any fixed t, $Y(.,s) - Y(.,t)$, $s \in [0,t]$, is a backward martingale adapted to $G_{s,t}$, *i.e.* for $s < u < t$

$$E[Y(x,s) - Y(x,t)|G_{u,t}] = Y(x,u) - Y(x,t).$$

Remark 2.6.1. Under (A.5) $X(x,t)$ has the same local characteristics (a,b).

Example 2.6.2. (1) Let $Y(x,t)$ be a $C-B.m.$ and $b(x,t)$ independent of Y. Then

$$X(x,t) = Y(x,t) + \int_o^t b(x,r)dr$$

is a backward semimartingale.

(2) Let $Y(x,z,t)$ be a $C-B.m.$ with parameter $z \in S$ and $z(t)$ S-valued stochastic process independent of Y. Then

$$\tilde{Y}(x,t) \equiv \int_o^t Y(x,z(r),dr)$$

is a backward integral as follows;

$$\int_s^t X(f_r, d\hat{r}) = \lim_{|\Delta|\to 0} \sum_{k=0}^{n-1} \{X(f_{t_{k+1}}, t_{k+1}) - X(f_{t_{k+1}}, t_k)\} \qquad (2.6.3)$$

where $\Delta = \{s = t_0 < t_1 < \ldots < t_n = t\}$

Theorem 2.6.3. *Let $\phi_{s,t}$ be a stochastic flow of homeomorphisms generated by a continuous C-semimartingale $X(x,t)$ satisfying (A.5) with local characteristics a, b satisfying the Lipschitz continuity and linear growth properties. Suppose that*

$$d(x,t) = \sum_j \frac{\partial}{\partial x_j} a_j(x,y,t)|_{y=x}$$

exists and is of linear growth, then $\psi_{s,t} = \Phi_{s,t}^{-1}$ is a continuous backward **53**
stochastic flow generated by $-X(x,t) + \int_0^t d(x,r)dr$, i.e.,

$$\psi_{s,t}(y) = y - \int_s^t X(\psi_{r,t}(y), d\hat{r}) + \int_s^t d(\psi_{r,t}(y), r)dr. \qquad (2.6.4)$$

Proof. We have

$$\Phi_{s,t}(x) = x + \int_s^t X(\Phi_{s,r}(x), dr).$$

□

Putting $x = \psi_{s,t}(y)$, we have

$$y = \psi_{s,t}(y) + \int_s^t X(\Phi_{s,r}(x), dr)|_{x=\psi_{s,t}(y)}. \tag{2.6.5}$$

We shall now compute $\int_s^t X(\Phi_{s,r}(x), dr)|_{x=\psi_{s,t}(y)}$. Let $\Delta = \{s = t_0, < t_1 < \cdots < t_n = t\}$. Then

$$\begin{aligned}
&\int_s^t X(\Phi_{s,r}(x), dr)|_{x=\psi_{s,t}(y)} \\
&\qquad = \lim_{|\Delta|\to 0} \sum_{k=0}^{n-1} \{X(\Phi_{s,t_k}(x), t_{k+1}) - X(\Phi_{s,t_k}(x), t_k)\}\,|_{x=\psi_{s,t}(y)} \\
&\qquad = \lim_{|\Delta|\to 0} \sum_{k=0}^{n-1} \{X(\psi_{t_k,t}(y), t_{k+1} - X(\psi_{t_k,t}(y), t_k))\}
\end{aligned} \tag{2.6.6}$$

Now

$$\begin{aligned}
&\sum_{k=0}^{n-1} \{X(\psi_{t_k,t}(y), t_{k+1}) - X(\psi_{t_k,t}(y), t_k)\} \\
&\quad = \sum_{k=0}^{n-1} [X(\psi_{t_{k+1},t}(y), t_{k+1}) - X(\psi_{t_{k+1},t}(y), t_k)] \\
&\qquad - \sum_{k=0}^{n-1} [\{X(\psi_{t_{k+1},t}(y), t_{k+1}) - X(\psi_{t_k,t}(y), t_{k+1})\} \\
&\qquad\qquad - \{X(\psi_{t_{k+1},t}(y), t_k) - X(\psi_{t_k,t}(y), t_k)\}] \\
&\quad = I_1^\Delta + I_2^\Delta, \quad say.
\end{aligned}$$

54 By definition

$$I_1^\Delta \to \int_s^t X(\psi_{r,t}(y), \hat{d}r).$$

Using mean value theorem, we have

$$I_2^\Delta = I_2^\Delta(y) = -\sum_k \sum_i \left\{\frac{\partial}{\partial x_i} X(\psi_{t_k,t}(y), t_{k+1}) - \frac{\partial}{\partial x_i} X(\psi_{t_k,t}(y), t_k)\right\}$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad\qquad\qquad \times (\psi^i_{t_{k+1},t}(y) - \psi^i_{t_k,t}(y)) \\
&- \frac{1}{2}\sum_{i,j,k} \xi^{\Delta}_{i,j,k}(\psi^i_{t_{k+1},t}(y) - \psi^i_{t_k,t}(y))(\psi^j_{t_{k+1},t}(y) - \psi^j_{t_k,t}(y)) \\
&= J^{\Delta}_1 + J^{\Delta}_2, \quad \text{say.}
\end{aligned}
$$

where $\xi^{\Delta}_{i,j,k}$ is a random variable given by

$$
\xi^{\Delta}_{i,j,k} = \frac{\partial^2}{\partial x_j \partial x_j} X(\eta_k, t_{k+1}) - \frac{\partial^2}{\partial x_j \partial x_j} X(\zeta_k, t_K)
$$

where $|\eta_k - \psi_{t_k,t}| \leq |\psi_{t_{k+1,t}} - \psi_{t_k,t}|, |\zeta_k - \psi_{t_k,t}| \leq |\psi_{t_{k+1,t}} - \psi_{t_{k,t}}|$. Hence $\sup\limits_k |\xi^{\Delta}_{i,j,k}| \to 0$ as $|\Delta| \to 0$. Thus $J^{\Delta}_2 \to 0$ as $|\Delta| \to 0$ and

$$
\begin{aligned}
J^{\Delta}_1(\Phi_{s,t}(x)) &= -\sum_i \sum_k \left\{ \frac{\partial}{\partial x_i} X(\Phi_{s,t_k}(x), t_{k+1}) - \frac{\partial}{\partial x_i} X(\Phi_{s,t_k}(x), t_k) \right\}. \\
&\times (\Phi^i_{s,t_{k+1}}(x) - \Phi^i_{s,t_k}(x)) \\
&\xrightarrow[|\Delta|\to 0]{} -\sum_i < \int_s^t \frac{\partial}{\partial x_i} X(\Phi_{s,r}(x), dr), \Phi^i_{s,t}(x) - x^i > \\
&= -\sum_i < \int_s^t \frac{\partial}{\partial x_i} X(\Phi_{s,r}(x), dr), \int_s^t X^i(\Phi_{s,r}(x), dr) > \\
&= -\sum_i \frac{\partial}{\partial x_i} \int_s^t a_i(\Phi_{s,r}(x), \Phi_{s,r}(x), r) dr \\
&= -\int_s^t d(\Phi_{s,r}(x), r) dr.
\end{aligned}
$$

Therefore 55

$$
J^{\Delta}_1(y) \xrightarrow[|\Delta|\to 0]{} 0 - \int_s^t d(\Phi_{s,r}(x), r) dr|_{x=\psi_{s,t}(y)} = -\int_s^t d(\psi_{r,t}(y), r) dr.
$$

Hence

$$
I^{\Delta}_2 \xrightarrow[|\Delta|\to 0]{} -\int_s^t d(\psi_{r,t}(y), r) dr.
$$

Combining all these results, we get

$$y = \psi_{s,t}(y) + \int_s^t X(\psi_{r,t}(y), \hat{d}r) - \int_s^t d(\psi_{r,t}(y), r)dr.$$

2.7 Appendix

Generalized ITO Formula, Stratonovich Integral and Stratonovich Stochastic Differential Equations

Let $X(x,t)$ be a one-dimensional continuous random field. It is said to be a continuous C^k-process if it is k-times continuously differentiable in x a.s. and $D^\alpha X(x,t)$ is continuous in (x,t) a.s. for $|\alpha| \leq k$. It is called a continuous C^k-martingale if $D^\alpha X(x,t)$ is martingale for any $x, |\alpha| \leq k$ and a continuous C^k-process of bounded variation if $D^\alpha X(x,t)$ is a process bounded variation for each x and $|\alpha| \leq k$. Finally $X(x,t)$ is said to be a continuous C^k-semimartingale if $X(x,t) = Y(x,t) + V(x,t)$, where $Y(x,t)$ is a continuous C^k-martingale and $V(x,t)$ a continuous C^k-process of bounded variation. Let $(a(x,y,t), b(x,t))$ be the local characteristics of $X(x,t)$. We make the following assumptions. $(A4)'_k$ $a(x,y,t)$ and $b(x,t)$
56 are k-times continuously differentiable in x and y (respectively x), and for $|\alpha|, |\beta| \leq k$ and for any $K > 0$

$$\int_o^t \sup_{|x|,|y| \leq K} |D_x^\alpha D_y^\beta a(x,y,r)|dr < \infty \qquad a.s.$$

$$\int_o^t \sup_{|x| \leq K} |D_x^\alpha b(x,r)|dr < \infty \qquad a.s.$$

We shall now present a differential rule for the composition of two processes, which is a generalization of the well known Ito formula.

Theorem 2.7.1 (Generalized Ito Formula I)**.** *Let $F(x,t)$ be a one-dimensional C^2- process and a C^1-semimartingale with local characteristics satisfying $(A4)_1$ and X_t an $\mathbb{R}^d$-valued continuous semimartingale. Then*

$$F(X_t, t) = F(X_o, 0) + \int_o^t F(X_t, dr) + \sum_{i=1}^d \int_o^t \left(\frac{\partial}{\partial x_i} F\right)(X_r, r) dX_r^i$$

$$+\sum_{i=1}^{d} < \int_o^t \left(\frac{\partial}{\partial x_i}F\right)(X_r, dr), X_t^i >$$
$$+\frac{1}{2}\sum_{i,j=1}^{d}\int_o^t \frac{\partial^2 F}{\partial x_i \partial x_j}(X_r, r)d < X_r^i, X_r^j > . \tag{2.7.1}$$

Proof. For a partition $\Delta = \{0 = t_o < t_1 < \cdots < t_n = t\}$, set

$$\begin{aligned} F(X_t, t) - F(X_o, 0) &= \sum_{k=0}^{n-1}\{F(X_{t_k}, t_{k+1}) - F(X_{t_K}, t_k)\} \\ &\quad + \sum_{k=0}^{n-1}\{F(X_{t_{k+}}, t_{k+1}) - F(X_{t_K}, t_{k+})\} \\ &= I_1^\Delta + I_2^\Delta, say. \end{aligned}$$

□

We have by the definition of the stochastic integral

$$I_1^\Delta \xrightarrow[|\Delta|\to 0]{} \int_o^t F(X_r, dr).$$

The second term is written as **57**

$$\begin{aligned} I_2^\Delta &= \sum_{k=0}^{n-1}\sum_{i=1}^{d}\left\{\frac{\partial}{\partial x_i}F(X_{t_k}, t_{k+1}) - \frac{\partial}{\partial x_i}F(X_{t_k}, t_k)\right\}(X_{t_{k+1}}^i - X_{t_k}^i) \\ &\quad + \sum_{k=0}^{n-1}\sum_{i=1}^{d}\frac{\partial}{\partial x_i}F(X_{t_k}, t_k)(X_{t_{k+1}}^i - X_{t_k}^i) \\ &\quad + \frac{1}{2}\sum_{i,j=1}^{d}\sum_{k=0}^{n-1}\frac{\partial^2}{\partial x_i \partial x_j}F(\xi_k, t_{k+1})(X_{t_{k+1}}^i - X_{t_k}^i)(X_{t_{k+1}}^j - X_{t_k}^j) \\ &= J_1^\Delta + J_2^\Delta + J_3^\Delta, \end{aligned}$$

where $|\xi_k - X_{t_k}| \le |X_{t_{k+1}} - X_{t_k}|$. Set

$$L_s^\Delta = \sum_{k=0}^{n-1}\left\{\frac{\partial}{\partial x_i}F(X_{t_k \Lambda s}, t_{k+1}\Lambda s) - \frac{\partial}{\partial x_i}F(X_{t_k}\Lambda s)\right\}$$

and $$L_s = \int_o^s \frac{\partial}{\partial x_i} F(X_r, dr).$$

Then L_s^Δ converge to L_s uniformly in s in probability as $|\Delta| \to 0$. On the other hand,

$$\sum_{k=0}^{n-1} (L_{t_{k+1}} - L_{t_k})(X^i_{t_{k+1}} - X^i_{t_K}) \xrightarrow[|\Delta|\to 0]{} < L, X^i_t > \text{ in probability}.$$

These two facts imply

$$\sum_{k=0}^{n-1} (L^\Delta_{t_{k+1}} - L^\Delta_{t_k})(X^i_{t_{k+1}} - X^i_{t_k}) \xrightarrow[|\Delta|\to 0]{} < L, X^i >_t \text{ in probability}.$$

Hence

$$J_1^\Delta \xrightarrow[|\Delta|\to 0]{} \sum_{i=1}^d < \int_o^t \frac{\partial}{\partial x_i} F(X_r, dr), X^i_t > .$$

Also

$$J_2^\Delta \xrightarrow[|\Delta|\to 0]{} \sum_{i=1}^d \int_o^t \frac{\partial}{\partial x_i} F(X_r, dr), dX^i_r.$$

58 Set

$$\tilde{J}_3^\Delta = \frac{1}{2} \sum_{i,j=1}^d \sum_{k=0}^{n-1} \frac{\partial^2}{\partial x_i \partial x_j} F(X_{t_k}, t_k)(X^i_{t_{k+1}} - X^i_{t_k})(X^j_{t_{k+1}} - X^j_{t_k}).$$

Then

$$\tilde{J}_3^\Delta \xrightarrow[|\Delta|\to 0]{} \frac{1}{2} \sum_{i,j=1}^d < \int_o^t \frac{\partial^2}{\partial x_i \partial x_j} F(X_r, r), dX^i_r, X^j_t > .$$

Now set

$$\xi^\Delta_{ijk} = \frac{\partial^2}{\partial x_i \partial x_j} F(\xi_k, t_{k+1}) - \frac{\partial^2}{\partial x_i \partial x_j} F(X_{t_k}, t_k).$$

Then

$$\sup_{i,j,k} |\xi^\Delta_{ijk}| \to 0 \quad as \quad |\Delta| \to 0.$$

Note that

$$|J_3^\Delta - \tilde{J}_3^\Delta| \leq \frac{1}{2} \sum_{i,j} \sup_{i,j,k} |\xi_{ijk}^\Delta| \left\{ \sum_k (X_{t_{k+1}}^i - X_{t_k}^i)^2 \right\}^{\frac{1}{2}} \times \left\{ \sum_k (X_{t_{k+1}}^i - X_{t_k}^i)^2 \right\}^{\frac{1}{2}} .$$

Since $\sum_k (X_{t_{k+1}}^\ell - X_{t_k}^\ell)^2$ tends to a finite value in probability as $|\Delta| \to 0$, the above converges to zero in probability. Hence

$$J_3^\Delta \xrightarrow[|\Delta|\to 0]{} \frac{1}{2} \sum_{i,j} \int_o^t \frac{\partial^2}{\partial x_i \partial x_j} F(X_r, r)\, d < X^i, X^j >_r .$$

Combining all these results, the desired formula (2.7.1) follows.

Remark 2.7.2. If $F(x,t)$ is a deterministic function twice continuously differentiable in x and once continuously differentiable in t, then the last but one term in the right hand side of (2.7.1) vanishes. This corresponds to the well known Ito formula.

Stratonovich Integral

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Let $X(x,t)$ be a continuous C^2-valued process and a continuous C^1-semimartingale. Let f_t be an $\mathbb{R}^d$-valued continuous semimartingale. We shall define the Stratonovich Integral of f_t based on $X(x,t)$. For a partition $\Delta = \{0 = t_o < t_1 < \cdots < t_n = t\}$ set

$$K_t^\Delta = \sum_{k=0}^{n-1} \frac{1}{2} \{(X(f_{t_{k+1}}, t_{k+1}) - X(f_{t_{k+1}}, t_k)) + (X(f_{t_k}, t_{k+1}) - X(f_{t_k}, t_k))\} \tag{2.7.2}$$

Lemma 2.7.3. *Suppose that the local characteristics of X and $\frac{\partial}{\partial x_i} X$ satisfy $(A4)_1'$. Then $\lim_{|\Delta|\to 0} K_t^\Delta$ exists and*

$$\lim_{|\Delta|\to 0} K_t^\Delta = \int_o^t X(f_r, dr) + \frac{1}{2} \sum_j < \int_o^t \left(\frac{\partial}{\partial x_j} X \right) (f_r, dr), f_t^j > . \tag{2.7.3}$$

*Proof.*Rewrite K_t^Δ as

$$
\begin{aligned}
K_t^\Delta &= \sum_{k=0}^{n-1} \{X(f_k, t_{k+1}) - X(f_{t_k}, t_k)\} \\
&+ \frac{1}{2} \sum_{k=0}^{n-1} \{(X(f_{t_{k+1}}, t_{k+1}) - X(f_{t_k}, t_{k+1})) - (X(f_{t_{k+1}}, t_k) - X(f_{t_k}, t_k))\} \\
&= L_t^\Delta + M_t^\Delta, \quad \text{say} \qquad \square
\end{aligned}
$$

We have

$$L_t^\Delta \xrightarrow[|\Delta| \to 0]{} \int_o^t X(f_r, dr).$$

By the mean value theorem,

$$
\begin{aligned}
M_t^\Delta = \frac{1}{2} \sum_j \sum_k \left\{ \frac{\partial}{\partial x_j} X(f_{t_k}, t_{k+1}) - \frac{\partial}{\partial x_j} X(f_{t_k}, t_k) \right\} \\
\left(f^j_{t_k+1} - f^j_{t_k} \right) + \frac{1}{4} \sum_{i,j,k} \xi^\Delta_{ijk} \left(f^i_{t_k+1} - f^i_{t_k} \right) \left(f^j_{t_k+1} -^j_{t_k} \right).
\end{aligned}
$$

60 where $\sup_k |\xi^\Delta_{ijk}| \to 0$ as $|\Delta| \to 0$. Then as in the proof of generalized Ito formula we get

$$M_t^\Delta \xrightarrow[|\Delta| \to 0]{} \frac{1}{2} \sum_j < \int_o^t \left(\frac{\partial}{\partial x_j} X \right) (f_r, dr), f_t^j > .$$

The lemma is thus proved.

Definition 2.7.4. *The limit of K_t^Δ as $|\Delta| \to 0$ is called the Stratonovich Integral of f_t based on $X(x, t)$ and is written as $\int_o^t X(f_r, odr)$.*

Proposition 2.7.5. *Let $X(x, t)$ be a continuous C^2-process and a C^1-semimartingale. Let the local characteristics of X and $\frac{\partial}{\partial x_j} X$ satisfy $(A4)'_1$. Let f_t be an $\mathbb{R}^d$-valued continuous semimartingale. Then the*

Stratonovich integral is well defined and is related to Ito integral by

$$\int_o^t X(f_r, odr) = \int_o^t X(f_r, dr) + \frac{1}{2}\sum_{j=1}^{d} < \int_o^t \left(\frac{\partial}{\partial x_j}X\right)(f_r, dr),\ f_t^j > . \tag{2.7.4}$$

Theorem 2.7.6 (Generalized Ito Formula II)**.** *Let $F(x,t)$ be a continuous C^3-process and C^2-semimartingale with local characteristics (α, β). Suppose that these are twice continuously differentiable and both α, β and their derivatives satisfy $(A4)'_1$. Let X_t be a continuous semimartingale. Then we have*

$$F(X_t, t) - F(X_o, 0) = \int_o^t F(X_r, odr) + \sum_i \int_o^t \left(\frac{\partial}{\partial x_j}F\right)(X_r, r)odX_r^j. \tag{2.7.5}$$

Proof. Rewrite the right hand side of 2.7.5 using Ito integral. By the generalized Ito formula I, we have

$$\begin{aligned}
&\frac{\partial}{\partial x_j}F(X_t, t) - \frac{\partial}{\partial x_j}F(X_o, 0) \\
&\qquad = \int_o^t \frac{\partial}{\partial x_j}F(X_r, dr) + \sum_i \int_o^t \frac{\partial^2}{\partial x_i \partial x_j}F(X_r, r)dX_r^j \\
&\qquad\quad + \sum_i < \int_o^t \frac{\partial^2}{\partial x_i \partial x_j}F(X_r, dr),\ X_t^i > \\
&\qquad\quad + \frac{1}{2}\sum_{i,k} \frac{\partial^3}{\partial x_i \partial x_j \partial x_k}F(X_r, r)d < X_r^i, X_r^k > .
\end{aligned}$$

Therefore **61**

$$\begin{aligned}
\int_o^t \left(\frac{\partial}{\partial x_j}F\right)(X_r, r)odX_r^j &= \int_o^t \frac{\partial}{\partial x_j}F(X_r, r)dX_r^j + \frac{1}{2} < \frac{\partial}{\partial x_j}F(X_t, t), X_t^j > \\
&= \int_o^t \frac{\partial}{\partial x_j}F(X_r, r)dX_r^j + \frac{1}{2} < \int_o^t \frac{\partial}{\partial x_j}F(X_r, dr), X_t^j > \\
&\quad + \frac{1}{2}\sum_i \int_o^t \frac{\partial^2}{\partial x_i \partial x_j}F(X_r, r)d < X_r^i, X_r^j > .
\end{aligned} \tag{2.7.6}$$

On other hand

$$\int_o^t F(X_r, odr) = \int_o^t F(X_r, dr) + \frac{1}{2}\sum_i < \int_o^t \left(\frac{\partial}{\partial x_i}F\right)(X_r, dr), X_t^i > . \tag{2.7.7}$$

Finally combining (2.7.1), (2.7.6) and (2.7.7) we get (2.7.5). □

Stratonovich Stochastic Differential Equation

Let $X(x,t)$ be a continuous C^2-process and a C^1-semimartingale with local characteristics (a,b) satisfying $(A4)'_1$. A continuous $\mathbb{R}^D$-valued semimartingale Φ_t is called a solution of the Stratonovich Stochastic Differential equation

$$d\phi_t = X(\Phi_t, odt) \tag{2.7.8}$$

starting from x at time s if it satisfies

$$\begin{aligned}\phi_t &= X + \int_s^t X(\Phi_r, odr)\\ &= x + \int_s^t X(\phi_r, dr) + \frac{1}{2}\sum_j < \int_s^t \frac{\partial}{\partial x_j} X(\phi_r, dr), \phi_t^j > .\end{aligned} \tag{2.7.9}$$

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Note that

$$\begin{aligned}&\sum_j \frac{1}{2} < \int_s^t \frac{\partial}{\partial x_j} X^i(\phi_r, dr), \phi_t^j >\\ &\qquad = \sum_j \frac{1}{2} < \int_s^t \frac{\partial}{\partial x_j} X^i(\phi_r, dr), \int_s^t X^j(\phi_r, dr) >\\ &\qquad = \frac{1}{2}\sum_j \int_s^t \frac{\partial}{\partial x_j} a_{ij}(x,y,r)|_{x=y=\phi_r}\, dr.\end{aligned}$$

Therefore, setting

$$d^i(x,t) = \sum_j \frac{\partial}{\partial x_j} a_{ij}(x,y,t)|_{y=x}, \tag{2.7.10}$$

we see that if a solution ϕ_t of (2.7.8) exists then it satisfies

$$\phi_t = x + \int_s^t X(\phi_r, dr) + \frac{1}{2}\int_s^t d(\phi_r, r)\, dr. \tag{2.7.11}$$

Consequently we have the following theorem.

Theorem 2.7.7. *Let $X(x,t)$ be a continuous C^2-process and a C^1-semimartingale with local characteristics a, b which are continuously differentiable in x, y and their derivatives are bounded. Suppose further that $d(x,t)$ defined in* (2.7.10) *is Lipschitz continuous. Then the Stratonovich SDE* (2.7.8) *has a unique solution and it defines a stochastic flow of homeomorphisms. Furthermore if a, b, d are k-times continuously differentiable and their derivatives bounded then the solution defines a stochastic flow of C^{k-1} diffeomorphisms.*

Chapter 3

Limit Theorems for Stochastic Flows

This Chapter is devoted to the study of limit theorems for stochastic flows. In section 3.1, we introduce the notion of weak and strong convergence of stochastic flows. In section 3.2 we discuss the convergence of random ordinary differential equations to a diffusion process. We state a theorem in this regard and elucidate it with various examples. In section 3.3 we state the main limit theorem. The proof of the theorem is very long. We develop it in the subsequent sections. In section 3.4 we discuss the tightness of $(N + M)$-point processes and in the next section the weak convergence of $(N + M)$-point process is dealt with. In section 3.6 we describe the tightness of Sobolev space-valued processes. We conclude the proof of the main limit theorem in section 3.7. In section 3.8 we complete the proof of the approximation theorem stated in section 3.2. In the next two sections we treat the ergodic and mixing cases. Finally, conclude the chapter with tightness and weak convergence of inverse flows. **63**

3.1 Weak and Strong Convergence of Stochastic Flows

Suppose we are given a family of filtrations $\{F_t^{\varepsilon}\}_{\varepsilon>0}$ and $X_t^{\varepsilon} = X^{\varepsilon}(x,t)$ a continuous C-semimartingale adapted to F_t^{ε} with local characteristics having "nice" properties. Let $\phi_{s,t}^{\varepsilon}$ be the stochastic flow generated by X_t^{ε}, i.e.,

$$d\,\phi_{s,t}(x) = X^{\varepsilon}\,(\phi_{s,t}(x), dt), t \geq s$$
$$\phi_{s,s}(x) = x.$$

We write $\phi_t^{\varepsilon} = \phi_{o,t}^{\varepsilon}(x)$. Our aim is to study the convergence of $(\phi_t^{\varepsilon}, X_t^{\varepsilon})$ as stochastic flows. We will introduce three notions of con-
64 vergence, viz, strong convergence, weak convergence and convergence as diffusion processes. Of these, the weak convergence plays the most importance role and we discuss it in detail. Before giving the definitions of various convergence we shall introduce some function spaces.

Let $C^m = c^m(\mathbb{R}^d; \mathbb{R}^d)$. Let $f \in C^m$ and N a positive integer. Then

$$\|f\|_{m,N} = \sum_{|\alpha|\leq\ m} \sup_{|x|\leq N} |D^{\alpha}\ f(x)|, N = 1, 2, \ldots\ldots\ldots$$

defines a family of seminorms and with this family of seminorms C^m becomes a complete separable space. Let $W_m = C([0,t]; C^m)$ be the set of all continuous maps from $[0,T]$ to C^m. For $\phi, X \in W_m$ let ϕ_t, X_t denote their values at $t \in [0,T]$. Define

$$|||\phi|||_{m,N} = \sup_{t\ \in\ [0,T]} \|\phi_t\|_{m,N}, N = 1, 2, \ldots\ldots.$$

The above family of seminorms makes W_m a complete separable space. Let $W_m^2 = W_m \times W_m$ and let $B(W_m^2)$ denote its topological Borel σ-field.

Assume that the local characteristics $a^{\in}, b^{\in}$ of the flow are $(m+1)$-times continuously spatially differentiable and the derivatives are bounded. Then in view of Remark 2.1.6 $(\phi^{\varepsilon}(.,\omega), X^{\varepsilon}\,(.,\omega)) \in W_m^2\ a.s$ In other words, it is a W_m^2-valued random variable. Let

$$P^{(\varepsilon)}\,(A) = P(\omega : (\phi_{.}^{\varepsilon}\,(\omega), X_{.}^{\varepsilon}(\omega)) \in A), A \in B(W_m^2)).$$

Definition 3.1.1. *Let $P^{(0)}$ be a probability measure on W_m^2. The family $(\phi_t^\varepsilon, X_t^\varepsilon), \varepsilon > 0$ is said to converge weakly to $P^{(0)}$ as stochastic flows if the family $\{P^{(\varepsilon)}, \varepsilon > 0\}$ converges to $P^{(0)}$ weakly.*

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Definition 3.1.2. *Let (ϕ_t, X_t) be pair of stochastic flow of C^m-diffeomorphism and continuous C^m-semimartingale. Then $(\phi_t^\varepsilon, X_t^\varepsilon)$ is said to converge strongly to (ϕ_t, X_t) as stochastic flows if $|||\phi^\varepsilon - \phi|||_{m,N}$ and $|||X^\varepsilon - X|||_{m,N}$ converge to 0 in probability for any $N = 1, 2, \ldots\ldots$.*

Remark 3.1.3. We shall show later that if $(\phi_t^\varepsilon, X_t^\varepsilon)$ converges weakly to (ϕ_t, X_t), i.e., $P^{(\varepsilon)}$ converges weakly to the joint law of $(\phi_t, X_t), |||X^\varepsilon - X|||_{m,N} \to 0$ in probability and some other conditions are satisfied then $|||\phi^\varepsilon - \phi|||_{m,N} \to 0$ and therefore $(\phi_t^\varepsilon, X_t^\varepsilon)$ converges to (ϕ_t, X_t) strongly.

Definition 3.1.4. *Let $x^{(N)} = (x_1, \ldots, x_N) \in \mathbb{R}^{Nd}, y^{(M)} = (y_1, \ldots, y_M) \in \mathbb{R}^{Md}$. Consider the $(N+M)$-point process $(\phi_t^\varepsilon\ (x^{(N)}), X_t^\varepsilon(y^{(M)}))$. Let $V_N = C([0,T]; \mathbb{R}^{Nd})$, $V_M = C([0,T]; \mathbb{R}^{Md})$. On the measurable space $(V_N X V_M$, $B(V_N\ X\ V_M))$ we define the law of the $(N+M)$-point process as follows:*

$$Q^{(\varepsilon)}_{(x^{(N)}, y^{(M)})}(A) = P\{\omega : (\phi_t^\varepsilon(x^{(N)}); X_t^\varepsilon(y^{(M)}) \in A\}, A \in B(V_N\ X\ V_M).$$

If the law of every $(N+M)$-point process converges weakly, then we say that the flow converges as diffusion process. Obviously, if $(\phi_t^\varepsilon, X_t^\varepsilon)$ converges weakly as stochastic flows, then the law of $(N+M)$-point process converges weakly.

Proposition 3.1.5. *The family of laws $\{P^{(\varepsilon)}\}_{\varepsilon>0}$ on $(W_m^2, B(W_m^2))$ converges weakly if we following two conditions are satisfied:*

(i) $\left\{P^{(\varepsilon)}\right\}_{\varepsilon>0}$ *is tight, i.e., for any $\delta > 0$ there exists a compact subset K_δ of W_m^2 such that $P^{(\varepsilon)}(K_\delta) > 1 - \delta$ for any $\varepsilon > 0$.*

(ii) $\left\{Q^{(\varepsilon)}_{(x^{(N)}, y^{(M)})}\right\}_{\varepsilon>0}$ *converges weakly for any $x^{(N)}, y^{(M)}, M, N = 1, 2, \ldots$.*

The proof is based on standard arguments. See Billingsley [2]. **66**

Sometimes it is convenient to consider $(\phi_t^\varepsilon, X_t^\varepsilon)$ as a process with values in Sobolev spaces. So we will now introduce a few Sobolev spaces. For a positive integer N let B_N denote the ball in $\mathbb{R}^d$ with centre at the origin and of radius N. Let $p > 1$. Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a function such that $D^\alpha f \in L^p(B_N)$ for all α such that $|\alpha| \le m$. For such functions we define the following seminorm

$$\|f\|_{m,p,N} = \left(\sum_{|\alpha| \le m} \int_{B_N} |D^\alpha f(x)|^p \, dx \right)^{1/p},$$

where the derivatives are in the sense of distributions. We define

$$H_{m,p}^{loc} = \left\{ f : \mathbb{R}^d \to \mathbb{R}^d | \, \|f\|_{m,p,N} < \infty \text{ for any } N \right\}.$$

The family of seminorms $\{\| \cdot \|_{m,p,N}, N = 1, 2, \ldots\}$ makes $H_{m,p}^{loc}$ a complete separable space. Let $W_{m,p} = C([0, T]; H_{m,p}^{loc})$. For $\phi \in W_{m,p}$ define

$$|||\phi|||_{m,p,N} = \sup_{t \in [0,T]} \|\phi\|_{m,p,N}, N = 1, 2, \ldots \ldots .$$

With this family of seminorms, $W_{m,p}$ is a complete separable space. For $p = \infty$ we write W_m, instead of $W_{m,\infty}$. Let $W_{m,p}^2 = W_{m,p} \ X \ W_{m,p}$. We usually suppose $p > d$. In such cases we have

$$W_{m+1,p}^2 \subset W_m^2 \subset W_{m,p}^2 \subset W_{m-1}^2.$$

These inclusions are consequences of the well known Sobolev imbedding theorem. Indeed, $H_{m+1,p}^{loc} \subset C^m \subset H_{m,p}^{loc}$. We shall now define the weak topology of $W_{m,p}$.

Definition 3.1.6. *Let $\langle ., . \rangle_N$ denote the canonical bilinear form on $H_{m,p}^{loc}$ restricted to B_N. We say that $\{\phi^n, n = 1, 2, \ldots\} \in W_{m,p}$ converges weakly to $\phi \in W_{m,p}$ if $\langle \phi_t^n, f \rangle_N$ converges to $\langle \phi_t, f \rangle_N$ uniformly in t for any $f \in (H_{m,p}^{loc})$. The space $W_{m,p}$ equipped with the weak topology is a complete separable space.*

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Remark 3.1.7. Let A be a bounded subset of $H_{m,p}^{loc}$, i.e., for any positive integer N there exists a constant K_N such that $\sup_{f \in A} \|f\|_{m,p,N} \le K_N$. Then A is relatively compact in $H_{m,p}^{loc}$ w.r.t. the weak topology.

We are now in a position to describe a criterion under which the family of measures $\{P^{\varepsilon}\}_{\varepsilon>0}$ is tight.

Proposition 3.1.8. *Let $\{P^{(\varepsilon)}\}_{\varepsilon>0}$ be a family of probability measures on $W^2_{m,p}$. Suppose for any positive integer N there exist positive integer N there exist positive numbers α, β, K such that*

(i) $E^{(\varepsilon)}[||\phi_t||^{\alpha}_{m,P,N} + ||X_t||^{\alpha}_{m,P,N}] \leq K,$

(ii) $E^{(\varepsilon)}[||\phi_t - \phi_s||^{\alpha}_{m,P,N} + ||X_t - X_s||^{\alpha}_{m,P,N}] \leq K|t-s|^{1+\beta}$

hold for any $t, s \in [0,T]$ and for any $\varepsilon > 0$. Then $\{P^{(\varepsilon)}\}_{\varepsilon>0}$ is tight in $W^2_{m,p}$ w.r.t. the weak topology.

The proof is similar to that of Kolmogorov's criteria for tightness and is therefore omitted.

Remark 3.1.9. The above Proposition is not true for $p = \infty$, ie.

$$E^{(\varepsilon)}\left[||\phi_t||^{\alpha}_{m,N} + ||X_t||^{\alpha}_{m,N}\right] \leq K \text{ for all } t \in [0,T]$$

and
$$E^{(\varepsilon)}\left[||\phi_t - \phi_s||^{\alpha}_{m,N} + ||X_t - X_s||^{\alpha}_{m,N}\right] \leq K|t-s|^{1+\beta},$$

for all $t, s \in [0,T]$, do not imply the tightness of $\{P^{(\varepsilon)}\}_{\varepsilon>0}$ in W^2_m.

If $\{P^{(\varepsilon)}\}$ is defined as a family of probability measures on W^2_m, then it can be extended to $W^2_{m,p}$ as follows: Consider the class of sets

$$\left\{A \cap W^2_m : A \in B(W^2_{m,p})\right\} = B(W^2_{m,p})|_{W^2_m} \subset B(W^2_m).$$

Define **68**

$$P^{\varepsilon}_{m,P}(A) = P^{(\varepsilon)}_m(A \cap W^2_m), A \in B(W^2_{m,P}).$$

Similarly P^{ε}_m can be extended to P^{ε}_{m-1} on W^2_{m-1}.

Proposition 3.1.10. *Suppose $m > 1, p > d$. Then $\{P^{(\varepsilon)}_{m-1}\}_{\varepsilon>0}$ is tight in W^2_{m-1} if $\{P^{\varepsilon}_{m,p}\}_{\varepsilon>0}$ is tight in $W^2_{m,p}$w.r.t. the weak topology.*

Proof. The proof follows from Kondraseev's theorem which states that any relatively compact set in $H^{loc}_{m,P}$ *w.r.t.* the weak topology is embedded in C^{m-1} as a relatively compact set. See Sobolev [30]. ◻

3.2 Approximation of Stochastic Differential Equations

Let $v^\varepsilon(t,\omega) = (v_1^\varepsilon(t,\omega), \ldots\ldots, v_r^\varepsilon(t,\omega))$ be an r-dimensional piecewise continuous stochastic process such that $E[v_i^\varepsilon(t)] = 0$ for all i. Let $F_k(x,t)$, $k = 0, 1, \ldots, r$ be continuous functions, C^∞ in x and the derivatives bounded.

Consider the following stochastic ordinary differential equation

$$\frac{dx}{dt} = \sum_{\ell=1}^{r} F_\ell(x,t) v_\ell^\varepsilon(t) + F_o(x,t). \tag{3.2.1}$$

Let $\phi_{s,t}^\varepsilon(x)$ denote the solution starting from x at time s. We now consider the following problem.

Problem. If $\{v^\varepsilon(t)\}_{\varepsilon>0}$ tends to a white noise or more precisely, if $B^\varepsilon(t) = \int\limits_o^t v^\varepsilon(s)ds$ tends to a Brownian motion $B_t = (B_1(t), \ldots, B_r(t))$ weakly or strongly then $\phi_t^\varepsilon (= \phi_{o,t}^\varepsilon)$ tends to a Brownian flow ϕ_t weakly or strongly and the limiting flow ϕ_t satisfies the following stochastic differential equation

$$d\phi_t = \sum_{\ell=1}^{r} F_\ell(\phi_t, t) o dB_\ell(t) + F_o(\phi_t, t)dt. \tag{3.2.2}$$

69 The solution of the above problem is not always in affirmative. In fact, we need some additional term in the right hand side of (3.2.2). To solve the problem in concrete terms we make the following assumptions: Let $G_t^\varepsilon = \sigma(v^\varepsilon(s) : 0 \leq s \leq t)$.

(A1)(a) $\int\limits_s^t |E[v_i^\varepsilon(r)|G_s^\varepsilon]|dr \to 0$ uniformly in s, t in L^2-sense.

(b) $E\left[\int\limits_{s}^{t} v_i^\varepsilon(\tau)d\tau \int\limits_j^\tau v_j^\varepsilon(\sigma)d\sigma | G_s^\varepsilon\right] \to \int\limits_s^t \nu_{ij}(r)dr$ unifomly in s, t, where ν_{ij} is a deterministic measurable function.

(c) there exist $\gamma > 1$, $K > 0$ such that

$$E\left[| \int_s^t |E[v_i^s(r)|G_s^\varepsilon]|dr|^\gamma |v_j^\varepsilon(s)|^\gamma \right] \leq K.$$

Remarks 3.2.1. (i) (a) and (b) roughly show that $\{B^\varepsilon(t)\}$ converges to a zeromean, martingale with quadratic variation $\int_s^t (v_{ij}(r) + v_{ji}(r))\, dr$. Hence $\{B^\varepsilon(t)\}_{\varepsilon>0}$ converges to a Brownian motion with mean 0 and covariance $\int_s^t (v_{ij}(r) + v_{ji}(r))\, dr$.

(ii) (C') ensures the tightness of the law pf $(B^\varepsilon(t), \phi_T^\varepsilon(x))$. Since $V^\varepsilon(t)$ converges to a white noise its moment tends to infinity. The condition (c) shows that the rates of divergence of the moment of $v^\varepsilon(t)$ and convergence of $\int_s^t |E[v^\varepsilon(r)|G_s^\varepsilon]|dr$ are balanced. In fact, in all examples given later the fourth moment of $v^\varepsilon(t) = 0(1/\varepsilon^2)$ and the fourth moment of $\int_s^t |E[v^\varepsilon(r)|G_s^\varepsilon]|dr = 0(\varepsilon^2)$.

We shall now that state the main result of this section. Let $W_m = C([0,T]; C^m), V^r = C([0,T]; \mathbb{R}^r)$. Let $P_m^{(\varepsilon)}$ Senate the law of $(\phi^\varepsilon, B^\varepsilon)$ defined on $W_m \times V^r$.

Theorem 3.2.2. *Assume (A1). Then $\{P_m^{(\varepsilon)}\}_{\varepsilon>0}$ converges weakly for any $m \geq 0$. Further the limit $P_m^{(0)}$ has the following properties: (i) $B(t)$ is an r-dimensional Brownian motion with zero mean and covariance $\int_o^t (v_{ij}(r) + v_{ji}(r))dr$. (ii) ϕ_t satisfies*

$$d\phi_t = \sum_{\ell=1}^r F_\ell(\phi_t, t) o dB_\ell(t) + F_o(\phi_t, t)\, dt + \sum_{1\leq \ell \leq m \leq r} S_{\ell,m}[F_\ell, F_m](\phi_t, t)dt. \tag{3.2.3}$$

where odB stands for Stratonovich integral and $S_{\ell,m} = \frac{1}{2}(\nu_{\ell m} - \nu_{m\ell})$, **70**

$$[F_\ell, F_m]^k(x,t) = \sum_{i=1}^{d} F_\ell^i(x,t)\frac{\partial}{\partial x_i}F_m^k(x,t) - \sum_{i=1}^{d} F_m^i(x,t)\frac{\partial}{\partial x_i}F_\ell^k(x,t).$$

Further, if $\{B^\varepsilon(t)\}$ *converges to* $B(t)$ *strongly then* $\{\phi_t^\varepsilon\}$ *converges to* ϕ_t *strongly. We do not give the proof here. We will do it later in a more general setup. Approximation theorems for the solution of stochastic differential equations have been discussed by several authors: McShane [27]. Wong-Zakai [34], Stroock-Varadhan [32], Kunita [19], [20], [21]. Ikeda-Nakao-Yamato [12]. Malliavin [25], Bismut [3]. Dowell [6]. We shall wlucidate the theorem with the help of a few examples.*

Example 3.2.3. Polygonal approximation of Brownian motion

Let $B(t) = (B_1(t), \ldots, B_r(t))$ be an r-dimensional Brownian motion.Set

$$v_\ell^\varepsilon(t) = \frac{1}{\varepsilon}\delta_k^\varepsilon B_\ell \quad \text{if} \quad \in k \leq t < \varepsilon(k+1)$$

where $$\Delta_k^\varepsilon B_\ell = B_\ell(\varepsilon(k+1)) = B_\ell(\varepsilon k).$$

Then

$$B^\varepsilon(t) = \int_o^t v^\varepsilon(s)ds \to B(t) \text{ uniformly in } t.$$

All we have to do is to verify the assumptions in $(A1)$. Since $v^\varepsilon(t)$ and $v^\varepsilon(s)$ are independent if $|t-s| > \varepsilon$, we have

$$|\int_s^t |E[v_\ell^\varepsilon(u)|G_s^\varepsilon]|du| \leq |\Delta_k^\varepsilon B_\ell| \quad \text{if} \quad \varepsilon k \leq s < \varepsilon(k+1).$$

71 Also

$$\text{variance } (\Delta_k^\varepsilon B_\ell) = \varepsilon \to 0.$$

Therefore (a) is satisfied. Now $var(v_j^\varepsilon(s)) = \dfrac{1}{\varepsilon}$, therefore

$$E\left[|\int_s^t |E[v_i^\varepsilon(u)|G_s^\varepsilon]|du|^2 |v_j^\varepsilon(s)|^2\right] \leq \left(3\varepsilon^2.3\frac{1}{\varepsilon^2}\right)^{1/2} = 3,$$

which verifies (c). Also it is easy to verify (b) with $\nu_{ij}(u) = \dfrac{1}{2}\delta_{ij}$. Hence $S_{\ell,m} = 0$.

Example 3.2.4. McShane Here we consider the approximation of two dimensional Brownian motion by continuously differentiable function. For $t \in [0,1]$, let $\phi_1(t)$ and $\phi_2(t)$ be continuously differentiable functions such that $\phi_1(0) = \phi_2(0) = 0$ and $\phi_1(1) = \phi_2(1) = 1$. Let $B(t) = (B_1(t), B_2(t))$ be a two dimensional standard Brownian motion. Set

$$v_i^\varepsilon(t) = \begin{cases} \frac{1}{\varepsilon}\dot{\phi}_i((t-k\varepsilon)/\varepsilon)\Delta_k^\varepsilon B_i & \text{if} \quad \Delta_k^\varepsilon B_1 \Delta_k^\varepsilon B_2 \geq 0, \\ \frac{1}{\varepsilon}\dot{\phi}_{3-i}((t-k\varepsilon)/\varepsilon)\Delta_k^\varepsilon B_i & \text{if} \quad \Delta_k^\varepsilon B_1 \Delta_k^\varepsilon B_2 < 0, \end{cases}$$

if $k\varepsilon < t < (k+1)\varepsilon, k = 0, 1, \ldots$, Then

$$B^\varepsilon(t) = \int_o^t v^\varepsilon(s)ds \to B(t) \text{ uniformly in } t.$$

As in the previous example, we can verify (a) and (b). Also it can be shown that

$$E\left[\int_s^t v_i^\varepsilon(\tau)d\tau \int_s^\tau v_j^\varepsilon(\sigma)d\sigma|G_s^\varepsilon\right] \to \frac{1}{2}\delta_{ij} + \frac{1}{\pi}\int_o^t [\phi_i(s)\dot{\phi}_j(s) - \dot{\phi}_j(s)\phi_i(s)]ds.$$

Example 3.2.5. Mollifiers approximation (Malliavin)

Let $B(t) = (B_1(t), \ldots, B_r(t))$ be an r-dimensional standard Brownian motion. Let ϕ be a nonnegative $C^\infty-$ function whose support is contained in [0,1] and $\int_o^1 \phi(s)ds = 1$. Set

$$\phi_\varepsilon(t) = \frac{1}{\varepsilon}\phi(t/\varepsilon), \ \varepsilon > 0$$

and $$B^{\varepsilon}(t) = \int_{o}^{\infty} \phi_{\varepsilon}(s-t)B(s)ds = \int_{o}^{\varepsilon} \phi_{\varepsilon}(s)B(s+t)ds.$$

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Then
$$B^{\varepsilon}(t) \to B(t) \text{ uniformly in } t.$$

Set
$$v^{\varepsilon}(t) = -\int_{o}^{\infty} \dot{\phi}_{\varepsilon}(s-t)B(s)ds.$$

Then
$$\int_{s}^{t} |E[v^{\varepsilon}(u)|G_{s}^{\varepsilon}]|du = \int_{s}^{s+\varepsilon} |E[v^{\varepsilon}(u)|G_{s}^{\varepsilon}]|du,$$
since $v^{\varepsilon}(u)$ is independent of G_{s}^{ε} if $u > s+\varepsilon$. Also
$$var(v^{\varepsilon}(u)) = \frac{1}{3}\int_{o}^{1} \phi(s)^2 ds.$$

Therefore
$$var(\int_{s}^{s+\varepsilon} v^{\varepsilon}(u)du) \leq \varepsilon.$$

Now it is easy to verify (a), (b) and (c). For (b), $\nu_{ij} = \frac{1}{2}\delta_{ij}$.

Example 3.2.6. Approximation by Ornstein-Uhlenbech processes (Dowell)

Let $B(t)$ be an r-dimensional standard Brownian motion. Let $\{v^{\varepsilon}(t)\}$ ne given by

$dv^{\varepsilon}(t) = -\frac{1}{\varepsilon}v^{\varepsilon}(t)dt + \frac{1}{\varepsilon}dB(t)$
$v^{\varepsilon}(0)$ = a Gaussian random variable with mean zero and covariance $\frac{1}{2\varepsilon}(\delta_{ij})$.

Then $\{v^\varepsilon(t)\}$ is a stationary Gaussian process with mean zero and covariance $\dfrac{1}{2\varepsilon}(\delta_{(ij)})$. Also $v^\varepsilon(t)$ is given by

$$v^\varepsilon(t) = e^{-\frac{1}{\varepsilon}(t-s)} v^\varepsilon(s) + \frac{1}{\varepsilon}\int_s^t e^{-\frac{1}{\varepsilon}(t-u)} dB(u).$$

Therefore 73

$$B^\varepsilon(t) - B^\varepsilon(s) = \varepsilon(1 - e^{-\frac{1}{\varepsilon}(t-s)}) v^\varepsilon(s) + \int_s^t (1 - e^{-\frac{1}{\varepsilon}(t-u)}) dB(u).$$

Since the variance of the first term in the *r.h.s.* is $0(\varepsilon)$ and the integrand tends to 1 as $\varepsilon \to 0$, we have

$$B^\varepsilon(t) - B^\varepsilon(s) \to B(t) - B(s).$$

The assumptions in $(A1)$ can be verified.

$$\begin{aligned}\int_s^t |E[v_i^\varepsilon(r)dr|G_s^\varepsilon]|dr &= \int_s^t (e^{-\frac{1}{\varepsilon}(r-s)}dr)|v_i^\varepsilon(s)| \\ &= \varepsilon(1 - e^{-\frac{1}{\varepsilon}(t-s)})|v_i^\varepsilon(s)| \to 0 \ \text{ as } \ \varepsilon \to 0.\end{aligned}$$

Also

$$\begin{aligned}E\left[\int_s^t v_i^\varepsilon(\tau)d\tau \int_s^\tau v_j^\varepsilon(\sigma)d\sigma|G_s^\varepsilon\right] &= E\left[\int_s^t v_j^\varepsilon(\sigma)d\sigma \int_\sigma^t v_i^\varepsilon(\tau)d\tau|G_s^\varepsilon\right] \\ &= E\left[\int_s^t v_j^\varepsilon(\sigma)d\sigma \left\{\varepsilon\left(1 - e^{-\frac{1}{\varepsilon}(t-\sigma)}\right) v_i^\varepsilon(\sigma)\right\} |G_s^\varepsilon\right] \\ &= \varepsilon\int_s^t \left(1 - e^{-\frac{1}{\varepsilon}(t-\sigma)}\right) E\left[v_i^\varepsilon(\sigma)v_j^\varepsilon(\sigma)|G_s^\varepsilon\right] d\sigma\end{aligned}$$

$$= \varepsilon \int_s^t \left(1 - e^{-\frac{1}{\varepsilon}(t-\sigma)}\right)\left[e^{-\frac{2}{\varepsilon}(\sigma-s)} v_i^\varepsilon(s) v_j^\varepsilon(s)\right] + \frac{1}{2\varepsilon}\left[\left(1 - e^{-\frac{2}{\varepsilon}(\sigma-s)}\right)\delta_{ij}\right] d\sigma$$

$$\to \frac{1}{2}(t-s)\delta_{ij}.$$

3.3 The Main Limit Theorem

In this section, we will state the main limit theorem for stochastic flows and the proof of the theorem will be developed in the subsequent sections.

74 Let for each $\varepsilon > 0$ there be given a continuous C^{k-1}- semimartingale $X^\varepsilon(x,t)$ such that $X^\varepsilon(x,0) = 0$ and it is adapted to F_t^ε. Let $(a^\varepsilon(x,y,t), b^\varepsilon(x,t))$ be the local charactersistics of $X^\varepsilon(x,t)$. Suppose that a^ε and b^ε are Lipschitz continuous and a^ε is k-times continuously differentiable in x, t, b^ε is $(k+2)$- times continuously diffrentiable in x. Also assume that $b^\varepsilon(x,t)$ is F_o^ε- measurable. We can write $X^\varepsilon(x,t)$ as

$$X^\varepsilon(x,t) = Y^\varepsilon(x,t) + \int_o^t b^\varepsilon(x,t)\,dt \tag{3.3.1}$$

where $Y^\varepsilon(x,t)$ is a continuous C^{k-1} -martingale. Set

$$G_t^\varepsilon = \sigma(Y^\varepsilon(x,u), b^\varepsilon(x,u) : 0 \le u \le t, x \in \mathbb{R}^d).$$

Then $X^\varepsilon(x,t)$ is G_t^ε-adapted contonuous C^{k-1}-semimartingale. Now we introduce the following quantities:

$$\overline{b}^\varepsilon(x,t) = E[b^\varepsilon(x,t)], \tilde{b}^\varepsilon(x,t) = b^\varepsilon(x,t) - \overline{b}^\varepsilon(x,t),$$

$$A_{ij}^\varepsilon(x,y,t,s) = E\left[\int_s^t \tilde{b}_i^\varepsilon(x,r)dr|G_s^\varepsilon\right] b_j^\varepsilon(y,s),$$

$$c_i^\varepsilon(x,t,s) = \sum_j \frac{\partial}{\partial x_j} A_{ij}^\varepsilon(x,y,t,s)|_{y=x},$$

$$d_{ijk}^\varepsilon(x,y,t,s) = E\left[\int_s^t \tilde{b}_i^\varepsilon(x,r)dr|G_s^\varepsilon\right] a_{jk}^\varepsilon(y,y,s).$$

We next introduce two sets of assumptions. The first is concerned with the convergence and the second with moments essentially needed for tightness.

$(A2)_k$ There exist continuous functions $a = (a_{ij}(x, y, t))$, $\bar{b} = (\bar{b}^i(x, t))$, $c = (c^i(x, t))$, $A = (A_{ij}(x, y, t))$, which are k-times continuously differentiable in x, y or x, as the case may be, and satisfy the following properties: 75

(1) $$E\left[\sup_{|x|\le K} |E\left[\int_s^t a^{\varepsilon}_{ij}(x, y, r)dr|G^{\varepsilon}_s\right] - \int_s^t a_{ij}(x, y, r)dr|\right] \to 0 \text{ as } \varepsilon \to 0,$$

(2) $$\sup_{|x|\le K} |\int_s^t \bar{b}^{\varepsilon}(x, r)dr - \bar{b}\int_s^t (x, r)dr| \to 0,$$

(3) $$E\left[\sup_{|x|\le K} |E\left[\int_s^t D^{\alpha}_x \tilde{b}^{\varepsilon}(x, r)dr|G^{\varepsilon}_s\right]|\right] \to 0 \text{ for } |\alpha| \le k,$$

(4) $$E\left[\sup_{|x|\le K} |E\left[\int_s^t A^{\varepsilon}_{ij}(x, y, t, r)dr|G^{\varepsilon}_s\right] - \int_s^t A_{ij}(x, y, r)dr|\right] \to 0,$$

(5) $$E\left[\sup_{|x|\le K} |E\left[\int_s^t c^{\varepsilon}(x, t, r)dr|G^{\varepsilon}_s\right] - \int_s^t c(x, r)dr|\right] \to 0.$$

All the above convergences are uniform in t, s for any $K > 0$.

$(A3)_k$ For any $K > 0$ there exist constants $\gamma > 1$ and $L > 0$ such that for any x, y, t, s

(6) $$E\left[\sup_{|x|\le K, |y|\le K} |D^{\alpha}_x D^{\beta}_y a^{\varepsilon}_{ij}(x, y, t)|^{\gamma}\right] \le L \ \forall \varepsilon > 0,\ |\alpha| \le k, |\beta| \le k,$$

(7) $$\sup_{|x|\le K} |D^{\alpha}_x \bar{b}^{\varepsilon}(x, t)| \le L \text{ for all } \varepsilon > 0, |\alpha| \le k$$

(8) $$E\left[\sup_{|x|\le K, |y|\le K} |D^{\alpha}_x D^{\beta}_y A^{\varepsilon}_{ij}(x, y, t, s)|^{\gamma}\right] \le L \ \forall \varepsilon > 0,\ |\alpha| \le k+1, |\beta| \le k+1,$$

(9) $$E\left[\sup_{|x|\le K,\ |y|\le K} |D^{\alpha}_x D^{\beta}_y d^{\varepsilon}_{ijk}(x, y, t, s)|^{\gamma}\right] \le L \ \forall \varepsilon > 0,\ |\alpha| \le k + 2, |\beta| \le k.$$

Let $\phi_t^\varepsilon(x) = \phi_{o,t}^\varepsilon(x)$ be the stochastic flow generated by $X^\varepsilon(x,t)$. Let $P_m^{(\varepsilon)}$, $m \leq k-1$, be the law of $(\phi_t^\varepsilon, X_t^\varepsilon)$ defined on W_m^2.

Theorem 3.3.1. *Assume* $(A2)_k$ *and* $(A3)_k$ *for some* $k \geq 2$. *Then* $\{P_{k-2}^{(\varepsilon)}\}_{\varepsilon>0}$ *converges weakly to* $P^{(o)}$ *as stochastic flows. The limits measure* $P^{(o)}$ *satisfy the following properties:*

(a) $X(x,t)$ *is a* $C^{k-1} -$ *B.m. with local characteristics* $(a+\overline{a}, \overline{b})$ *where*
76 $\overline{a}_{ij}(x,y,t) = A_{ij}(x,y,t) + A_{ij}(y,x,t)$,

(b) ϕ_t *is a Brownian flow of* C^{k-1}*-diffeomorphisms genereted by*

$$X(x,y) + \int_o^t c(x,r)dr.$$

Further if X_t^ε *converges strongly then* ϕ_t^ε *also converge strongly.*

In the next section, we will prove the tightness of $(N+M)$-point processes under an additional assumption $(A4)$. In section 3.5, we will first prove the weak convergence assuming $(A4)$ and then we will drop the assumption $(A4)$ and prove the same in the general case. In section 3.6, we prove the tightness of Sobolev space valued processes and in section 3.7 we conclude the proof of the main theorem.

3.4 Tightness of (N+M)-Point Processes

Let $\underline{x}^{(N)} = (x_1, \ldots, x_N) \in \mathbb{R}^{Nd}, \underline{y}^{(M)} = (y_1, \ldots, y_M) \in \mathbb{R}^{Md}$. Consider the of the $(N+M)$-point process $(\phi_t^\varepsilon((\underline{x}^{(N)})), X^\varepsilon(\underline{y}^{(M)}, t))$. As before we denote the law of the $(N+M)$- point process by $Q^{(\varepsilon)}{}_{(\underline{x}^{(N)}, \underline{y}^{(M)})}$ which is defined on $V_N \times V_M$. For fixed $\underline{x}^{(N)}$ and $\underline{y}^{(M)}$ we shall drop the subscript from $Q^{(\varepsilon)}{}_{(\underline{x}^{(N)}, \underline{y}^{(M)})}$ and write simply $Q^{(\varepsilon)}$. We shall show the tightness of these laws $\{Q^\varepsilon\}_{\varepsilon>0}$ under the following condition:

(A4) There exists a $K > 0$ such that

$$a^\varepsilon(x,y,t,\omega) = 0 \quad \text{if} \quad |x| \geq K, |y| \geq K,$$

$$b^\varepsilon(x,t,\omega) = 0 \quad \text{if} \quad |x| \geq K.$$

Under the above condition $X^\varepsilon(x,t,\omega) = 0$ if $|x| > K$ and the associated flow $\phi^\varepsilon_t(x)$ satisfies $|\phi^\varepsilon_t(x)| \leq K$ if $|x| \leq K$ and $\phi^\varepsilon_t(x) = x$ if $|x| > K$. Let $\gamma = \gamma(K)$ be the positive constant as in $(A3)_k$.

Lemma 3.4.1. *For any $p \in [2, 2\gamma]$ there exists a positive constant $C = C(p)$ such that*

$$E[|X^\varepsilon_t(x) - X^\varepsilon_s(x)|^p] \leq C|t-s|^{2-\frac{2}{p}} \text{ for all } \varepsilon > 0 \text{ and for all } x. \quad (3.4.1)$$

Proof. We will consider the case for $d = 1$ only for implicity. We suppress x from $X^\varepsilon_t(x), a^\varepsilon(x,x,t)$ etc. For fixed x. We have 77

$$E\left[|Y^\varepsilon_s - T^\varepsilon_s|^p\right] = \frac{1}{p}p(p-1)E\left[\int_s^a a^\varepsilon(r)|Y^\varepsilon_r - Y^\varepsilon_s|^{p-2}dr\right]$$

$$\leq \frac{1}{2}p(p-1)L^{\frac{1}{\gamma}}\int_s^t E\left[|Y^\varepsilon_r - Y^\varepsilon_s|^{(p-2)\frac{\gamma}{\gamma-1}}\right]^{\frac{\gamma-1}{\gamma}} dr \qquad \text{(by (A.3)(6))}$$

$$\leq \frac{1}{2}p(p-1)L^{\frac{1}{\gamma}}\int_s^t E\left[|Y^\varepsilon_r - Y^\varepsilon_s|^p\right]^{\frac{p-2}{p}} dr \qquad (3.4.2)$$

since $(p-2)\dfrac{\gamma}{\gamma-1} < p$. Now for any $a > 0$ we have $a^{\frac{p-2}{p}} < 1 + a$. Therefore

$$E\left[|Y^\varepsilon_t - Y^\varepsilon_s|^p\right] \leq \frac{1}{2}p(p-1)L^{\frac{1}{\gamma}}\left\{(t-s) + \int_s^t E\left[|Y^\varepsilon_r - Y^\varepsilon_s|^p\right] dr\right\}. \quad \square$$

By Gronwall's inequality, we get $E[|Y^\varepsilon_t - Y^\varepsilon_S|^p] \leq c|t-s|$. Substituting this back to (3.4.2) we have

$$E[|Y^\varepsilon_t - Y^\varepsilon_S|^P] \leq c\int_s^t |r-s|^{\frac{p-2}{p}} dr \leq c'|t-s|^{2-\frac{2}{p}}. \qquad (3.4.3)$$

Next, note that

$$
\begin{aligned}
|\int_s^t b^\varepsilon(r)dr|^p &= p\int_s^t \overline{b^\varepsilon}(\tau)|\int_s^t b^\varepsilon(r)dr|^{p-1}\ \mathrm{sign}\ (\tau)d\tau \\
&\qquad + p(p_1)\int_s^t \tilde{b}^\varepsilon(\tau)\int_s^\tau b^\varepsilon(\sigma)|\int_s^\sigma b^\varepsilon(r)dr|^{p-2}d\sigma \\
&= I_1^\varepsilon + I_2^\varepsilon, \text{say}
\end{aligned}
$$

where sign $(\tau) = sign\int_s^\tau b^\varepsilon(r)dr$. We have

$$
E[|I_1^\varepsilon|] \leq pL\int_s^t E\left[|\int_s^\tau b^\varepsilon(r)dr|^{p-1}\right]d\tau \qquad \text{(by(A.3)(7))}
$$

$$
\begin{aligned}
E[I_2^\varepsilon]| &\leq p(p-1)|\int_s^t E\left[A^\varepsilon(t,\sigma)|\int_s^\sigma b^\varepsilon(r)dr|^{p-2}\right]d\sigma| \\
&\leq p(p_1)L^{\frac{1}{\gamma}}\int_s^t E\left[|\int_s^\tau b^\varepsilon(r)dr|^{(p-2)\frac{\gamma}{\gamma-1}}\right]^{\frac{\gamma-1}{\gamma}} d\sigma
\end{aligned}
$$

78 by (A.3) (8). These two imply

$$
E\left[|\int_s^t b^\varepsilon(r)dr|^p\right] \leq c\left\{\int_s^t\left(E\left[|\int_s^\sigma b^\varepsilon(r)dr|^{p-1}\right] + E\left[|\int_s^\sigma b^\varepsilon(r)dr|^p\right]^{\frac{p-2}{p}}\right)d\sigma\right\}.
$$

Then we obtain similarly as the above

$$
E\left[|\int_s^t b^\varepsilon(r)dr|^p\right] \leq C'|t-s|^{2-\frac{2}{p}}.
$$

Lemma 3.4.2. *For any* $p > 3(2 - \dfrac{1}{r})$ *there exists a positve constant* $C = C(p)$ *such that*

$$E[|\phi_t^\varepsilon(x) - \phi_s^\varepsilon(x)|^p] \leq C|t-s|^{2-\frac{1}{\gamma}}, \tag{3.4.4}$$

for all $\varepsilon > 0$ *and for all* x*.*

Proof. We prove the case for $d = 1$ only. In view of $(A4)$ (3.4.4) is obvious if $|x| > K$. So we assume $|x| \leq K$. We suppress x from $\phi_t^\varepsilon(x)$. We have

$$\phi_t^\varepsilon - \phi_s^\varepsilon = \int_s^t \overline{b}^\varepsilon(\phi_r^\varepsilon, r)dr + \int_s^t \tilde{b}^\varepsilon(\phi_r^\varepsilon, r)dr + \int_s^t Y^\varepsilon(\phi_r^\varepsilon, dr). \square \tag{3.4.5}$$

Using Ito's formula for $F(x) = |x|^P$ and writing sign $r =$ sign $(\phi_r^\varepsilon - \phi_s^\varepsilon)$, we get

$$\begin{aligned}
|\phi_t^\varepsilon - \phi_s^\varepsilon|^p = p &\int_s^t \overline{b}^\varepsilon(\phi_r^\varepsilon, r)|\phi_r^\varepsilon - \phi_s^\varepsilon|^{p-1} sign(r)dr \\
&+ p\int_s^t \tilde{b}^\varepsilon(\phi_r^\varepsilon, r)|\phi_r^\varepsilon - \phi_s^\varepsilon|^{p-1} \text{ sign } (r)dr \\
&+ p\int_s^t |\phi_r^\varepsilon - \phi_s^\varepsilon|^{p-1} sign(r)\gamma^\varepsilon(\phi_r^\varepsilon, dr) \\
&+ \frac{1}{2}p(p-1)\int_s^t a^\varepsilon(\phi_r^\varepsilon, \phi_r^\varepsilon, r)|\phi_r^\varepsilon - \phi_s^\varepsilon|^{p-2}dr \\
= I_1^\varepsilon + I_2^\varepsilon &+ I_3^\varepsilon + I_4^\varepsilon, \text{say.}
\end{aligned} \tag{3.4.6}$$

Note that $|\phi_r^\varepsilon| \leq K$. Then from $(A3)_k$

$$|E[I_1^\varepsilon]| \leq pL\int_s^t E[|\phi_r^\varepsilon - \phi_s^\varepsilon|^{P-1}]dr. \tag{3.4.7}$$

We have 79

$$E[I_3^\varepsilon] = 0.. \tag{3.4.8}$$

By $(A3)_k$, we have

$$E[I_4^\varepsilon] \leq C \int\limits_s^t E[\phi_r^\varepsilon - \phi_s^\varepsilon|^{(p-2)\frac{\gamma}{\gamma-1}}]^{\frac{\gamma-1}{\gamma}} dr. \tag{3.4.9}$$

We shall now calculate I_2^ε. By Ito's formula

$$\begin{aligned}
&\tilde{b}^\varepsilon(\phi_r^\varepsilon, r)|\phi_r^\varepsilon - \phi_s^\varepsilon|^{p-1} \text{ sign } (r)\\
&= \int\limits_s^t \left\{\frac{\partial}{\partial x} b^\varepsilon(\phi_\sigma^\varepsilon, r) b^\varepsilon(\phi_\sigma^\varepsilon, \sigma) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\tilde{b}^\varepsilon(\phi_\sigma^\varepsilon, r) a^\varepsilon(\phi_\sigma^\varepsilon, \phi_\sigma^\varepsilon, \sigma)\right\}\\
&\qquad\qquad \times |\phi_\sigma^\varepsilon - \phi_s^\varepsilon|^{p-1}\text{sign}(\sigma) d\sigma\\
&+ (p-1)\int\limits_s^r \left[\tilde{b}^\varepsilon(\phi_\sigma^\varepsilon, r) b^\varepsilon(\phi_\sigma^\varepsilon, \sigma) + \frac{\partial}{\partial x}(\tilde{b}^\varepsilon\phi_\sigma^\varepsilon, r) a^\varepsilon(\phi_\sigma^\varepsilon, \phi_\sigma^\varepsilon, \sigma)\right]\\
&\qquad\qquad \times |\phi_\sigma^\varepsilon - \phi_s^\varepsilon|^{p-2} d\sigma\\
&+ \frac{1}{2}(p-1)(p-2)\int\limits_s^t \tilde{b}^\varepsilon(\phi_\sigma^\varepsilon, r) a^\varepsilon(\phi_\sigma^\varepsilon, \phi_\sigma^\varepsilon, \sigma)|\phi_\sigma^\varepsilon - \phi_s^\varepsilon|^{p-3}\text{sign}(\sigma) d\sigma\\
&+ \text{a martingale with zero mean.}
\end{aligned} \tag{3.4.10}$$

Therefore

$$E\left[I_2^\varepsilon\right] = p\int\limits_s^t$$

$$\begin{aligned}
&E\left[(c^\varepsilon(\phi_\sigma^\varepsilon, t, \sigma) + \frac{1}{2}\frac{\partial^2}{\partial x^2} d^\varepsilon(\phi_\sigma^\varepsilon, \phi_\sigma^\varepsilon, t, \sigma))|\phi_\sigma^\varepsilon - \phi_s^\varepsilon|^{p-1}\text{sign}(\sigma)\right] d\sigma\\
&\quad + p(p-1)\int\limits_s^t E[\{A^\varepsilon(\phi_\sigma^\varepsilon, \phi_\sigma^\varepsilon, t, \sigma) + \frac{\partial}{\partial x} d^\varepsilon(\phi_\sigma^\varepsilon, \phi_\sigma^\varepsilon, t, \sigma)\}|\phi_\sigma^\varepsilon - \phi_s^\varepsilon|^{p-2}] d\sigma\\
&\quad + \frac{1}{2}p(p-1)(p-2)\int\limits_s^t E\left[d^\varepsilon(\phi_\sigma^\varepsilon, \phi_\sigma^\varepsilon, t\sigma)|\phi_\sigma^\varepsilon - \phi_s^\varepsilon|^{p-3}\text{sign}(\sigma)\right] d\sigma.
\end{aligned}$$

Applying Holder inequality and using the tightness assumption, we have **80**

$$E\left[I_2^\varepsilon\right]| \le 2pL^{1/\gamma}\int_s^t E\left[|\phi_\sigma^\varepsilon - \phi_s^\varepsilon|^{(p-1)\frac{\gamma}{\gamma-1}}\right]^{\frac{\gamma-1}{\gamma}} d\sigma$$

$$+ 2p(p-1)L^{1/\gamma}\int_s^t E\left[|\phi_\sigma^\varepsilon - \phi_s^\varepsilon|^{(p-2)\frac{\gamma}{\gamma-1}}\right]^{\frac{\gamma-1}{\gamma}} d\sigma$$

$$+ p(p-1)(p-2)L^{1/\gamma}\int_s^t E\left[|\phi_\sigma^\varepsilon - \phi_s^\varepsilon|^{(p-3)\frac{\gamma}{\gamma-1}}\right]^{\frac{\gamma-1}{\gamma}} d\sigma \quad (3.4.11)$$

Summing up these estimates, we obtain

$$E\left[|\phi_s^\varepsilon - \phi_s^\varepsilon|^P\right] \le C\int_s^t \left\{E\left[|\phi_r^\varepsilon - \phi_s^\varepsilon|^{p-1}\right] + E\left[|\phi_s^\varepsilon - \phi_s^\varepsilon|^{(p-1)\frac{\gamma}{\gamma-1}}\right]^{\frac{\gamma-1}{\gamma}}\right.$$

$$\left. + E\left[|\phi_r^\varepsilon - \phi_s^\varepsilon|^{(p-2)\frac{\gamma}{\gamma-1}}\right]^{\frac{\gamma-1}{\gamma}} + E\left[|\phi_r^\varepsilon - \phi_s^\varepsilon|^{(p-3)\frac{\gamma}{\gamma-1}}\right]^{\frac{\gamma-1}{\gamma}}\right\} dr. \quad (3.4.12)$$

Since $|\phi_t^\varepsilon - \phi_s^\varepsilon| \le 2K, E[|\phi_t^\varepsilon - \phi_s^\varepsilon|^p] \le C|t-s|$. Substituting this in (3.4.12), we obtain

$$E\left[|\phi_t^\varepsilon - \phi_s^\varepsilon|^p\right] \le C\int_s^t \left\{r-s| + |r-s|^{\frac{\gamma-1}{\gamma}}\right\} dr.$$

Hence

$$E[|\phi_t^\varepsilon - \phi_s^\varepsilon|^p] \le C|t-s|^{2-\frac{1}{\gamma}}.$$

The following proposition is clear from Lemmas 3.4.1, 3.4.2 and Kolmogorov's theorem

Proposition 3.4.3. *The family of measures* $\left\{Q^{(\varepsilon)}_{(\underline{x}^{(N)}\underline{y}^{(M)})}\right\}_{\varepsilon>0}$ *is tight for any* $\underline{x}^{(N)}$ *and* $\underline{y}^{(M)}$.

3.5 Weak Convergence of (N+M)-Point processes

Let $\underline{x}^{(N)} = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^{Nd}$ and $\underline{y}^{(M)} = (y_1, \ldots, y_M) \in \mathbb{R}^{Md}$. As before let $Q^{(\varepsilon)}_{(\underline{x}^{(N)}\underline{y}^{(M)})}$ be the law of $\phi^\varepsilon_t(\underline{x}^{(N)}, X^\varepsilon_t(\underline{y}^M))$ defined on $V_N \times V_M$. In this section we discuss the weak convergence of $\left\{Q^{(\varepsilon)}_{(\underline{x}^{(N)},\,\underline{y}^{(M)})}\right\}_{\varepsilon>0}$.

81 **Theorem 3.5.1.** *Assume $(A_2)_k, (A3)_k$ for some $k \geq 2$ and $(A4)$. Then for each $\underline{x}^{(N)}$ and $\underline{y}^{(M)}$ the family* [1] $\left\{Q^{(\varepsilon)}_{(\underline{x}^{(N)},\,\underline{y}^{(M)})}\right\}_{\varepsilon>0}$ *converges weakly to a probalility measure $Q^{(0)}_{(\underline{x}^{(N)},\,\underline{y}^{(M)})}$. Further let $P^{(0)} = P^{(0)}_{k-1}$ be a probability measure on W_{k-1} satisfying (a) and (b) of Theorem 3.3.1. Then the law of $(\phi_t(\underline{x}^{(N)}), X_t(\underline{y}^{(M)}), P^{(0)})$ coincides with $Q^{(0)}_{(\underline{x}^{(N)},\,\underline{y}^{(M)})}$ for any $\underline{x}^{(N)}$, $\underline{y}^{(M)}$.*

The proof of the above theorem will be developed through several Lemmas. By Proposition 3.4.3 the family $\{Q^{(\varepsilon)}_{(\underline{x}^{(N)},\,\underline{y}^{(M)})}\}_{\varepsilon>0}$ is tight. Let $Q^{(0)}$ be an accumulation point of $\{Q^\varepsilon\}^{1)}_{\varepsilon>0}$ as $\varepsilon \to 0$. Therefore $\{Q^\varepsilon\}$ converges along a subsequence $\varepsilon_n \downarrow 0$. Let h be a bounded continuous function on $\mathbb{R}^{(N+M)\ell d}$, where ℓ is a positive integer. Let $(\phi, X) \in V_N \times V_M$. For $0 \leq s_1 < s_2 < \cdots < s_\ell \leq s$, define

$$\phi(\phi, X) = h(\phi_{s1}, \ldots \phi_{s\ell}, X_{s1}, \ldots, X_{s\ell}). \tag{3.5.1}$$

Then ϕ is a bouded continuous function on $V_N \times V_M$. Also set

$$\phi^\varepsilon(\omega) = h(\phi^\varepsilon_{s1}(\underline{x}^{(N)}, \omega), \ldots, \phi^\varepsilon_{s\ell}(\underline{x}^{(N)}, \omega), X^\varepsilon_{s1}(\underline{y}^{(M)}, \omega) \ldots, X^\varepsilon_{s\ell}(\underline{y}^{(M))}, \omega)) \tag{3.5.2}$$

ϕ^ε is a measurable function defined on (Ω, F, P), the basic probability space.

We have

$$E\left[\left(\int_s^t f(\phi^\varepsilon_r, r)dr\right)^{\phi^\varepsilon}\right] = E_{Q^{(\varepsilon)}}\left[\left(\int_s^t f(\phi_r, r)dr\right)^{\phi}\right] \tag{3.5.3}$$

for any function f for which the above makes sense.

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Lemma 3.5.2. *Let* $\{g^\varepsilon(x,t,\omega)\}_{\varepsilon>0}$ *be a family of* C^1 *valued processes satisfying*

(a) *for any* $K>0$ *there exist* $\gamma>1$ *and* $L>0$ *such that*

$$E\left[\sup_{|x|\le K}|D^\alpha g^\varepsilon(x,t)|^\gamma\right]\le L,\quad |\alpha|\le 1, \tag{3.5.4}$$

(b) *there exists a deterministic function* $g(x,t)$ *such that for any* $K>0$

$$E\left[\sup_{|x|\le K}|E[\int_s^t g^\varepsilon(x,r)dr|G_s^\varepsilon]-\int_s^t g(x,r)dr|\right]\to as\varepsilon\to 0. \tag{3.5.5}$$

Then

$$E\left[\left(\int_s^t g^{\varepsilon_n}(\phi_r^{\varepsilon_n},r)dr\right)\phi^{\varepsilon_\eta}\right]\xrightarrow[\varepsilon_n\to 0]{} E_{Q^{(0)}}\left[\left(\int_s^t g(\phi_r,r)dr\right)\phi\right], \tag{3.5.6}$$

where $\phi_r^{\varepsilon_n}=\phi_r^{\varepsilon_n}(\underline{x}^{(N)})$.

Proof. By the tigtness of $\{\phi_t^\varepsilon\}_{\varepsilon>0}$, for any $\theta,\eta>0$ there exists $\zeta>0$ such that the set

$$A^\varepsilon(\zeta,\eta)=\left\{\omega\sup_{|t-s|\le\zeta}|\phi_r^\varepsilon-\phi_s^\varepsilon|<\eta\right\} \tag{3.5.7}$$

satisfies

$$P(A^\varepsilon(\zeta,\eta))>1-\theta.\square \tag{3.5.8}$$

This is a consequence of the well known Arzela-Ascoli theorem. By (a), for any $\delta>0$ jthere exists $\eta>0$ such that

$$E\left[\sup_{\substack{|x-y|<\eta\\ |x|\le K,|y|\le K}}|g^\varepsilon(x,t)-g^\varepsilon(y,t)|\right]\le\delta. \tag{3.5.9}$$

[1] 1) We suppress $(\underline{x}^{(N)},\underline{y}^{(M)})$ from $Q^{(\varepsilon)}_{(\underline{x}^{(N)},\underline{y}^{(M)})}$.

This follows from mean value theorem. We fix η. Let Δ be a partition of $[s,t]$ given by $\Delta = \{s = t_o < t_1 \cdots < t_n = t\}, |\Delta| < \zeta$. We have

$$
\begin{aligned}
&|E\left(\left(\int_s^t g^\varepsilon(\phi^\varepsilon_r, r)dr - \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} g^\varepsilon(\phi^\varepsilon_{t_k}, r)dr\right)\phi^\varepsilon\right)| \\
&\quad \leq |E\left(\left(\int_s^t g^\varepsilon(\phi^\varepsilon_r, r)dr - \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} g^\varepsilon(\phi^\varepsilon_{t_k}, r)dr\right)\phi^\varepsilon\right) : (A^\varepsilon(\zeta,\eta))| \\
&\quad + E\left(\left(\int_s^t g^\varepsilon(\phi^\varepsilon_r, r)dr - \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} g^\varepsilon(\phi^\varepsilon_{t_k}, r)dr\right)\phi^\varepsilon\right) : (A^\varepsilon(\zeta,\eta)^c)| \\
&\quad \leq \delta(t-s)\|\phi\| + 2L^{\frac{1}{\gamma}}(t-s)\|\phi\|\theta \qquad (3.5.10)
\end{aligned}
$$

83 Again e have

$$
\begin{aligned}
&\left|E\left[\left(\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} g^\varepsilon(\phi^\varepsilon_{t_k}, r)dr\right)\phi^\varepsilon\right] - E_{Q^{(0)}}\left[\left(\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} g^\varepsilon(\phi^\varepsilon_{t_k}, r)dr\right)\phi\right]\right| \\
&\leq \left|\sum_{k=0}^{n-1}\left\{E\left[\left(E\left[\int_{t_k}^{t_{k+1}} g^\varepsilon(y,r)dr|G^\varepsilon_{t_K}\right]|_{y=\phi^\varepsilon_{t_k}}\right)\phi^\varepsilon\right]\right.\right. \\
&\qquad\qquad\qquad \left.\left. -E\left[\left(\int_{t_k}^{t_{k+1}} g(\phi^\varepsilon_{t_k}, r)dr\right)\phi^\varepsilon\right]\right\}\right| \\
&+\left|\sum_{k=0}^{n-1}\left\{E\left[\left(\int_{t_k}^{t_{k+1}} g(\phi^\varepsilon_{t_k}, r)dr\right)\phi^\varepsilon\right] - E_{Q^{(0)}}\left[\left(\int_{t_k}^{t_{k+1}} g(\phi_{t_k}, r)dr\right)\phi\right]\right\}\right| \\
&= I_1^\varepsilon + I_2^\varepsilon, \quad \text{say.} \qquad (3.5.11)
\end{aligned}
$$

Now $I_1^\varepsilon \to 0$ as $\varepsilon \to 0$ by (b) and $I_2^\varepsilon \to 0$ along a subsequence by the weak conver of $Q^{(\varepsilon)}$. Therefore

$$
\begin{aligned}
&\overline{\lim_{\varepsilon_n \to 0}}\left|E\left[\left(\int_s^t g^{\varepsilon_n}(\phi^{\varepsilon_n}_r, r)dr\right)\phi^{\varepsilon_n}\right] - E_{Q^{(0)}}\left[\left(\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} g(\phi_{t_k}, r)dr\right)\phi\right]\right| \\
&\qquad\qquad \leq \delta(t-s)\|\phi\| + 2L^{\frac{1}{\gamma}}(t-s)\|\phi\|\theta^{\frac{\gamma-1}{\gamma}}.
\end{aligned}
$$

Since δ, θ are arbitary, we let $\theta, \delta \to 0$ and conclude the assertion of the lemma.

We make the following convention as a definition.

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Definition 3.5.3. *If $E[f^{\varepsilon_n}\phi^{\varepsilon_n}] \to E_{Q^{(0)}}[f\phi]$ as $\varepsilon_n \to 0$ then we say that $f^{\varepsilon_n} \to f$ weakly.*

Lemma 3.5.4. *The following are L^2 martingales with respect to $Q^{(0)}$*

$$M_t(x) = \phi_t(x) - \int_0^t (\overline{b}(\phi_r(x), r) + c(\phi_r(x), r))dr \tag{3.5.12}$$

if $x \in \{x_1, \ldots, x_N\}$,

$$Y_t(y) = X_t(y) - \int_0^t \overline{b}(y, r)dr \tag{3.5.13}$$

if $y \in \{y_1, \ldots, y_M\}$,

Proof. We will consider the case $d = 1$ only. Take a subsequence $\varepsilon_n \downarrow 0$ such that $Q^{(\varepsilon_n)} \to Q^{(0)}$ weakly. We have

$$E\left[\left(\phi_t^{\varepsilon_n} - \phi_s^{\varepsilon_n} - \int_s^t \overline{b}^{\varepsilon_n}(\phi_r^{\varepsilon_n}, r)dr - \int_s^t \tilde{b}_r^{\varepsilon_n}(\phi_r^{\varepsilon_n}, r)dr\right)\phi^{\varepsilon_n}\right] = 0. \tag{3.5.14}$$

$\square$

Now $\phi_t^{\varepsilon_n} - \phi_s^{\varepsilon_n} \to \phi_t - \phi_s$ weakly. By $(A2)_k$ and the previous lemma

$$\int_s^t \overline{b}_r^{\varepsilon_n}(\phi_r^{\varepsilon_n}, r)dr \to \int_s^t \overline{b}(\phi_r, r)dr \quad \text{weakly.} \tag{3.5.15}$$

Using Ito's formula we have

$$\begin{aligned}\int_s^t \tilde{b}^{\varepsilon_n}(\phi_r^{\varepsilon_n}, r)dr = \int_s^t \tilde{b}^{\varepsilon_n}(\phi_s^{\varepsilon_n}, r)dr + \int_s^t dr& \\ \left(\int_s^r \frac{\partial}{\partial x}\tilde{b}^{\varepsilon_n}(\phi_\sigma^{\varepsilon_n}, r)\tilde{b}^{\varepsilon_n}(\phi_\sigma^{\varepsilon_n}, \sigma)d\sigma\right)& \\ + \frac{1}{2}\int_s^t dr\left(\int_s^r \frac{\partial^2}{\partial x^2}\tilde{b}^{\varepsilon_n}(\phi_\sigma^{\varepsilon_n}, r)a^{\varepsilon_n}(\phi_\sigma^{\varepsilon_n}, \phi_\sigma^{\varepsilon_n}, \sigma)d\sigma\right)&\end{aligned}$$

$$+ \text{ a martingale}.$$
$$= I_1^{\varepsilon_n} + I_2^{\varepsilon_n} + I_3^{\varepsilon_n} + I_4^{\varepsilon_n}, \text{say}. \tag{3.5.16}$$

Since $E\left[\int_s^t \tilde{b}^{\varepsilon_n}(z,r)dr|G_s^{\varepsilon_n}\right]_{z=\phi_s^{\varepsilon_n}} \to 0$, we have $I_1^{\varepsilon_n} \to 0$ weakly by lemma 3.5.2. Using $(A2)_k$, we obtain

$$E\left[I_2^{\varepsilon_n}|G_s^{\varepsilon_n}\right] = E\left[\int_s^t c^{\varepsilon_n}(\phi_\sigma^{\varepsilon_n}, t, \sigma)d\sigma|G_s^{\varepsilon_n}\right] \to \int_s^t C(\phi_r, r)dr \text{ weakly}.$$

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$$E\left[I_3^{\varepsilon_n}|G_s^{\varepsilon_n}\right] = E\left[\int_s^t \frac{\partial^2}{\partial x^2} d^{\varepsilon_n}(\phi_\sigma^{\varepsilon_n}, \phi_\sigma^{\varepsilon_n}, t, \sigma)d\sigma|G_s^{\varepsilon_n}\right] \to 0$$

weakly since $\dfrac{\partial^2}{\partial x^2} d^{\varepsilon_n} \to 0$. Also $I_4^{\varepsilon_n} \to 0$ weakly since it is a martingale. Combining all these result it follows that $M_t(s)$ is a martingale. The same procedure matatis mutandis shows that $Y_t(y)$ is a martingale.

In the next lemma, we shall compute the quadratic variations of these martigales.

Lemma 3.5.5. *With respect to $Q^{(0)}$ we have*

$$\text{(i)} < M_t(x), M_t(y)^* >= \int_0^t (a+\tilde{a})(\phi_r(x), \phi_r(y), r)dr, \tag{3.5.17}$$
$$\text{(ii)} < M_t(x), Y_t(y)^* >= \int_0^t (a+\tilde{a})(\phi_r(x), y, r)dr, \tag{3.5.18}$$
$$\text{(iii)} < Y_t(x), Y_t(y)^* >= \int_0^t (a+\tilde{a})(x, y, r)dr. \tag{3.5.19}$$

Proof. As before, we consider the case $d = 1$ only. Using Ito's formula, we have

$$\begin{aligned}\phi_t^\varepsilon(x)\phi_t^\varepsilon(y) - \phi_s^\varepsilon(y) &= \int_s^t \phi_r^\varepsilon(x)\overline{b}^\varepsilon(\phi_r^\varepsilon(y), r)dr + \int_s^t \phi_r^\varepsilon(y)\overline{b}^\varepsilon(\phi_r^\varepsilon(x), r)dr \\ &+ \int_s^t \phi_r^\varepsilon(x)\tilde{b}^\varepsilon(\phi_r^\varepsilon(y), r)dr + \int_s^t \phi_r^\varepsilon(x)\tilde{b}^\varepsilon(\phi_r^\varepsilon(x), r)dr \\ &+ \int_s^t a^\varepsilon(\phi_r^\varepsilon(x), \phi_r^\varepsilon(x), r)dr + \text{ a martingale}.\end{aligned} \tag{3.5.20}$$

□

Let $\varepsilon_n \downarrow 0$ be the same sequence as in Lemma 3.5.4 then

$$\phi_t^{\varepsilon_n}(x)\phi_t^{\varepsilon_n}(y) - \phi_s^{\varepsilon_n}(x)\phi_s^{\varepsilon_n}(y) \to \phi_t(x)\phi_t(y) - \phi_s(x)\phi_s(y) \text{ weakly,}$$
$$\int_s^t \phi_r^{\varepsilon_n}(x)\overline{b}^{\varepsilon_n}(\phi_r^{\varepsilon_n}(y), r)dr \to \int_s^t \phi_r(x)\overline{b}(\phi_r(y), y)dr \text{ weakly,}$$
$$\int_s^t a^{\varepsilon_n}(\phi_r^{\varepsilon_n}(x), \phi_r^{\varepsilon_n}(y), r)dr \to \int_s^t a(\phi_r(x), \phi_r(y), r)dr \text{ weakly.}$$

We next consider the third term in the right hand side of (3.5.2). We have **86**

$$\begin{aligned}
\phi_r^{\varepsilon_n}(x)\tilde{b}^{\varepsilon_n}(\phi_r^{\varepsilon_n}(y), r) &= \phi_s^{\varepsilon_n}(x)\tilde{b}^{\varepsilon_n}(\phi_s^{\varepsilon_n}(y), r)\\
&+ \int_s^r \phi_\sigma^{\varepsilon_n}(x)\frac{\partial}{\partial x}\tilde{b}^{\varepsilon_n}(\phi_\sigma^{\varepsilon_n}(y), r)b^{\varepsilon_n}(\phi_\sigma^{\varepsilon_n}(y), \sigma)d\sigma\\
&+ \int_s^r b^{\varepsilon_n}\phi_\sigma^{\varepsilon_n}(x), \sigma\tilde{b}^{\varepsilon_n}(\phi_\sigma^{\varepsilon_n}(y), r)d\sigma\\
&+ \int_s^r \frac{\partial}{\partial x}\tilde{b}^{\varepsilon_n}(\phi^{\varepsilon_n}(y), r)a^{\varepsilon_n}(\phi_\sigma^{\varepsilon_n}(x), \phi_\sigma^{\varepsilon_n}(y), \sigma)d\sigma\\
&+ \frac{1}{2}\int_s^r \phi_\sigma^{\varepsilon_n}(x)\frac{\partial^2}{\partial x^2}\tilde{b}^{\varepsilon_n}(\phi_\sigma^{\varepsilon_n}(y), r)a^{\varepsilon_n}(\phi_\sigma^{\varepsilon_n}(y), \phi_\sigma^{\varepsilon_n}(y), \sigma)d\sigma\\
&\qquad + \text{ a martingale}\\
&= I_1^{\varepsilon_n}(r) + I_2^{\varepsilon_n}(r) + I_3^{\varepsilon_n}(r) + I_4^{\varepsilon_n}(r) + I_5^{\varepsilon_n}(r) + I_6^{\varepsilon_n}(r). \quad (3.5.21)
\end{aligned}$$

Now $\int_s^t I_1^{\varepsilon_n}(r)dr \to 0$ weakly since $\phi_s^{\varepsilon_n}$ is $G_s^{\varepsilon_n}$- measurable. Since

$$E\left[\int_s^t I_2^{\varepsilon_n}(r)dr|G_s^{\varepsilon_n}\right] = E\left[\int_s^t \phi_\sigma^{\varepsilon_n}(x)c_\sigma^{\varepsilon_n}(\phi_\sigma^{\varepsilon_n}(y), t, \sigma)d\sigma|G_s^{\varepsilon_n}\right]$$

therefore $$\int_s^t I_3^{\varepsilon_n}(r)dr \to \int_s^t \phi_r(x)c(\phi_r(y), r)dr \text{ weakly.}$$

Also

$$\int_s^t I_3^{\varepsilon_n}(r)dr \to \int_s^t A(\phi_\sigma(x), \phi_\sigma(y), \sigma)d\sigma \text{ weakly,}$$

and $$\int_s^t I_k^{\varepsilon_n}(r)dr \to 0 \text{ weakly for } k = 4, 5, 6.$$

To sum up we have

$$
\begin{aligned}
(1)\phi_t(x)\phi_t(y) - \phi_s(x)\phi_s(y) - \int_s^t \phi_r(x)\overline{b}(\phi_r(y), r)dr \\
- \int_s^t \phi_r(y)\overline{b}(\phi_r(x), r)dr - \int_s^t a(\phi_r(x), \phi_r(y), r)dr \\
- \int_s^t \phi_r(x)c(\phi_r(y), r)dr - \int_s^t \phi_r(y)c(\phi_r(x), r)dr \\
- \int_s^t \tilde{a}(\phi_r(x), \phi_r(y), r)dr
\end{aligned}
$$

87 is a martingale with respect to $Q^{(0)}$, i.e

$$
\begin{aligned}
\phi_t(x)\phi_t(y) - \phi_s(x)\phi_s(y) - \int_s^t \phi_r(x)(\overline{b}(\phi_r(y), r) + c(\phi_r(y), r))dr \\
- \int_s^t \phi_r(y)(\overline{b}(\phi_r(x), r) + c(\phi_r(x), r))dr \\
- \int_s^t (a + \tilde{a})(\phi_r(x), \phi_r(y), r)dr
\end{aligned}
$$

is a $Q^{(0)}$-martingale.

On the other hand by Ito's formula we have

(2) $\phi_t(x)\phi_t(y) = \phi_s(x)\phi_s(y) + \int_s^t \phi_r(x)d\phi_r(y) + \int_s^t \phi_r(y)d\phi_r(x) + < \phi_t(x), \phi_t(y) >$.

Obviously

$$< \phi_t(x), \phi_t(y) >=< M_t(x), M_t(y) > .$$

Substituting (2) in (1) we find

$$
\begin{aligned}
\int \phi_r(x)dM_r(y) + \int_s^t \phi_r(y)dM_r(x) \\
- \int_s^t (a + \tilde{a})(\phi_r(x), \phi_r(y), r)dr + < M_t(x), M_t(y) >
\end{aligned}
$$

is a martingale. Hence

$$< M_t(x), M_t(y)^* >= \int_0^t (a + \tilde{a})(\phi_r(x), \phi_r(y), r)dr.$$

This proves (i). The relations (ii) and (iii) can be proved similarly.

Proof of the Theorem 3.5.1 under (A4) Let $\underline{x}_0^{(N)} = (x_10, \ldots, x_N0) \in \mathbb{R}^{Nd}$, $\underline{y}_0^{(M)} = (y_10, \ldots, x_M0) \in \mathbb{R}^{Md}$ be two fixed points. For $\underline{x} = (x_1, \ldots, x_N)\varepsilon\ \mathbb{R}^{Nd}$ and $\underline{y} = (y_1, \ldots, x_M) \in \mathbb{R}^{Md}$, define the following differential operator

$$
\begin{aligned}
L^{(N,M)}_{s,\underline{y}_o^{(M)}} f(\underline{x}, \underline{y}) = {} & \frac{1}{2} \sum_{p,q=1}^{N} \sum_{i,j=1}^{d} (a + \tilde{a})_{ij}(x_p, x_q, s) \frac{\partial^2 f(\underline{x}, \underline{y})}{\partial x_p^i \partial x_q^j} \\
& + \sum_{i,p} \left\{ \overline{b}^i(x_p, s) + c^i(x_p, s) \right\} \frac{\partial f}{\partial x_p^i}(x, y) \\
& + \frac{1}{2} \sum_{p,q} \sum_{i,j} (a + \tilde{a})_{ij}(y_{p^0}, y_{q^0}, s) \frac{\partial^2 i(\underline{x}, \underline{y})}{\partial y_p^i \partial y_q^j} \\
& + \sum_{i,p} \overline{b}^1(y_{p^0}, s) \frac{\partial f(\underline{x}, \underline{y})}{\partial y_p^i} \\
& + \frac{1}{2} \sum_{p,q} \sum_{i,j} (a, \tilde{a})_{ij}(x_p, y_{q^0}, s) \frac{\partial^2 f(\underline{x}, \underline{y})}{\partial x_p^i \partial y_q^j}
\end{aligned}
\tag{3.5.22}
$$

By Ito's formula, if is a C^2-function with bounded derivatives, then 88
using Lemmas 3.5.5 and 3.5.6, we find that the following is a martingale with respect to $Q^{(0)}$

$$
f(\phi_t, X_t) - \int_0^t L^{(N,M)}_{s,\underline{y}_o^{(M)}} f(\phi_s, X_s) ds. \tag{3.5.23}
$$

Therefore $Q^{()}$ is the solution of the martingale problem for $L^{(N,M)}_{s,y_0^{(M)}}$ and hence it is unique. Now $\left\{ Q^{(0)}_{(\underline{x}_0^N, \underline{y}_0^M)} \right\}_{(\underline{x}_0^M, \underline{y}_0^M)}$ is a consistent family of measures. Therefore there exists a unique probability measure $P^{(0)}$ on W^2_{k-1} such that the law of $(\phi_t(\underline{x}_0^{(N)}, X_t(\underline{y}_0^M))$ with respect to $P^{(0)}$ is $Q^{(0)}_{(\underline{x}_0^N, \underline{y}_0^M)}$. Also (3.5.17)–(3.5.19) with $x \in \{x_10, \ldots, x_N0\}$, $y \in \{y_10, \ldots, Y_M0\}$ is satisfied with respect to $P^{(0)}$. Hence $X_t(x)$ is a

$C^{k-1}-$ valued Brownian motion with local characteristics $(a+\tilde{a}, b)$. We claim that ϕ_t is generated $X_t(x) + \int_0^t c(x,r)dr$. Set

$$\tilde{M}_t(x) = \int_0^t Y(\Phi_s, ds).$$

Then

$$< \tilde{M}_t(x), \tilde{M}_t(y)^* >= \int_0^t (a+\tilde{a})(\phi_r(x), \phi_r(y), r)dr.$$

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$$< M_t(x) - \tilde{M}_t(x), (M_t(y) - \tilde{M}_t(y))^* >= 0.$$

This implies $M_t(x) = \tilde{M}_t(x)$. Thus ϕ_t is generated by $X_t(x) + \int\limits_o^t c(x,r)dr$.

Proof of the Theorem 3.5.1 (without (A4)) Let $P^{(0)}$ be a probability measure on W^2_{k-1} satisfying (a) and (b) of the theorem. Then the law of $(\phi_t(\underline{x}_o^{(N)}), X_t(\underline{y}_o^{(M)}), P^{(0)})$ is $Q^{(0)}_{(\underline{x}_o^{(N)}, \underline{y}_o^{(M)})}$. We claim that $Q^{(\varepsilon)} \xrightarrow[\varepsilon\to 0]{} Q^{(0)}$ weakly.

Step 1. We consider the truncated process for $K > 0$. Let $\psi_K : \mathbb{R}^d \to \mathbb{R}$ be a smooth function such that

$$\psi_K(x) = \begin{cases} 1 & \text{if} \quad |x| \leq /2, \\ 0 & \text{if} \quad |x| > K, \end{cases}$$

and $0 \leq \psi_K \leq 1$. Set $X^{\varepsilon,K}(x,t) = X^{\varepsilon}(x,t)\psi_K(x)$. Then the local characteristics of $X^{\varepsilon,K}$ are $a^{\varepsilon}(x,y,t)\psi_K(x)\phi_K(y)$ and $b^{\varepsilon}(x,t)\psi_K(x)$ which obviously satisfy (A4). Let $\phi_t^{\varepsilon,K}$ be the flow generated by $X^{\varepsilon,K}$. Denote the law of $(\phi_t^{\varepsilon,K}(\underline{x}_o^{(N)}), X_t^{\varepsilon,K}(\underline{y}_o^{(M)}))$ by $Q^{\varepsilon,K}$. Then $Q^{(\varepsilon,K)} \xrightarrow[\varepsilon\to 0]{} Q^{0,K}$ weakly. Let us compare $Q^{(0,K)}$ and $Q^{(0)}$. We may assume that $|\underline{x}_o^{(N)}| \leq \dfrac{K}{2}, |\underline{y}_o^{(M)}| \leq \dfrac{K}{2}$. Let $A \in B(V_N \times V_M)$.

Then clearly

$$Q^{(0)}\left(A \cap \left\{\phi : \|\phi\| < \frac{K}{2}\right\}\right) = Q^{(0,K)}\left(A \cap \left\{\phi \|\phi < \frac{K}{2}\right\}\right). \quad (3.5.24)$$

Step 2. In order to show the weak convergence of $\{Q^{(\varepsilon)}\}_{\varepsilon<0}$, it suffices to show that for any closed subset S of $V_N \times V_M$

$$\overline{\lim_{\varepsilon\to 0}} Q^{(\varepsilon)}(S) \leq Q^{(0)}(S).$$

For any $\delta > 0$, there exists a $K > 0$ such that $Q^{(0)}{}_{(G_K)} > 1-\delta$, **90**
where $G_K = \{\phi : ||\phi|| < K/2\}$.Then

$$Q^{(0,k)}(G_K) = Q^{(0)}(G_K) > 1-\delta, \text{ by (3.5.24)}$$

Since $Q^{(\varepsilon,K)} \to Q^{(0,K)}$ weakly, therefore

$$\underline{\lim_{\varepsilon\to 0}} Q^{(\varepsilon,K)}(G_K) \geq Q^{(0)}(G_K).$$

Hence there exists an $\varepsilon_o > 0$ such that for any $\varepsilon < \varepsilon_o$, we have $Q^{(\varepsilon,K)}(G_K) > 1-2\delta$. Therefore

$$\begin{aligned} Q^{(\varepsilon)}(S) &= Q^{(\varepsilon)}(S \cap G_K) + Q^{(\varepsilon)}(S \cap G_K^c) \\ &\leq Q^{(\varepsilon,K)}(S \cap G_K) + Q^{(\varepsilon,K)}(G_K^c) \\ &\leq Q^{(\varepsilon,K)}(S \cap \bar{G}_K) + 2\delta. \end{aligned}$$

Thus

$$\begin{aligned} \overline{\lim_{\varepsilon\to 0}} Q^{(\varepsilon)}(S) &\leq \overline{\lim_{\varepsilon\to 0}} Q^{(\varepsilon,K)}(S \cap \bar{G}_K) + 2\delta \\ &\leq Q^{(0,K)}(S \cap \bar{G}_K) + 2\delta \\ &\leq Q^{(0)}(S) + 2\delta \end{aligned}$$

Since δ is arbitrary

$$\overline{\lim_{\varepsilon\to 0}} Q^{(\varepsilon)}(S) \leq Q^{(0)}(S)$$

3.6 Tightness of Sobolev Space-valued Processes

Let $P^{(\varepsilon)}_{m,p}$ denote the law of $(\phi^{\varepsilon}_t, X^{\varepsilon}_t)$ on $W^2_m = (C([0,T]; H^{loc}_{m,p}))^2$ for $m \geq k-1$. We shall discuss the tightness of $\{P^{(\varepsilon)}_{m,p}\}_{\varepsilon>0}$. We begin by considering the case $m = 0, W^2_{0,p} = (C([0,T] : L^{loc}_p))^2$. For some $K > 0$ consider a truncated process $(\phi^{\varepsilon,K}_t, X^{\varepsilon,K}_t)$. Let the law of the truncated process be denoted by $P^{(\varepsilon,K)}_{0,p}$. Let $\tilde{\psi}_K : \mathbb{R}^d \to \mathbb{R}^d$ be a smooth function such that

$$\tilde{\psi}^{(x)}_K = \begin{cases} x & \text{if } |x| < K/2 \\ 0 & \text{if } |x| > K \end{cases}$$

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Set $\tilde{X}^{\varepsilon,K}_t = \tilde{\psi}_K(X^{\varepsilon,K}_t)$. Let the law of $(\phi^{\varepsilon,K}_t, \tilde{X}^{\varepsilon,K}_t)$ be denoted by $\tilde{P}^{(\varepsilon,K)}_{o,p}$. We have the following result about the tightness of $\left\{\tilde{P}^{(\varepsilon,K)}_{0,p}\right\}$.

Lemma 3.6.1. $\left\{\tilde{P}^{(\varepsilon,K)}_{0,p}\right\}_{\varepsilon>0}$ *is tight with respect to the weak topology of* $W^2_{0,p}$ *for any* $K > 0$ *and* $p > dV3\left(2 - \dfrac{1}{\gamma}\right)$

Proof. We have form Lemma 3.4.2

$$E\left[|\phi^{\varepsilon,K}_t(x) - \phi^{\varepsilon,K}_s(x)|^p\right] \leq L|t-s|^{2-\frac{1}{\gamma}} \quad \text{for all} \quad x \in \mathbb{R}^d$$

and

$$E\left[|\phi^{\varepsilon,K}_t(x)|^p\right] \leq L \quad \text{for all} \quad x \in \mathbb{R}^d$$

□

Integrating the above relations with respect to the Lebesgue measure on the ball $B_n = \{x : |x| \leq n\}$, we get

$$E\left[||\phi^{\varepsilon,K}_t - \phi^{\varepsilon,K}_s||^p_{0,p,n}\right] \leq L \text{ vol } (B_n)|t-s|^{2-\frac{1}{\gamma}}$$
$$E\left[||\phi^{\varepsilon,K}_t||^p_{0,p,n}\right] \leq L \text{ Vol } (B_n).$$

We can find similar estimates for $\tilde{X}^{\varepsilon,K}_t$. These estimates imply the tightness of $\{P^{(\tilde{\varepsilon},K)}{}_{0,p}\}_{\varepsilon>0}$ in the weak topology.

Let $p^{(0,K)}_{k-1}$ be a probability measure on W^2_{k-1}, with local characteristics $a(x,y,t)\psi_K(x)\psi_K(y)$ and $b(x,t)\psi_K(x)$, where $\psi_K : \mathbb{R}^d \to \mathbb{R}^1$ is the smooth function described in section 3.5. For $0 \leq t_1 < t_2 \cdots < t_N$ define

$$
\begin{aligned}
&\tilde{P}^{(0,K)}_{k-1}(X_{t_1} \in A_1, \ldots, X_{t_N} \in A_N, \phi_{t1} \in B_1, \ldots, \phi_{tN} \in B_N)\\
&\quad = P^{(0,K)}_{k-1}(\psi_K(X_{t_1}) \in A_1, \ldots, \psi_K(X_{t_N}) \in B_N, \phi_{t_1} \in B_1 \ldots, \phi_{t_N} \in B_N)
\end{aligned}
$$

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The measure $\tilde{P}^{(0,K)}_{k-1}$ on W^2_{k-1} can be extended to a measure $\tilde{P}^{(0,K)}_{0,p}$ on $W^2_{0,p}$.

Lemma 3.6.2. $\{\tilde{P}^{(\varepsilon,K)}_{0,p}\}_{\varepsilon>0}$ *converges weakly to* $\tilde{P}^{(0,K)}_{0,p}$ *with respect to the weak topology of* $W^2_{0,p}$.

Proof. Fix N, take $t_1 < t_2 < \cdots < t_N$. It follows from the weak convergence of the $(N + M)$- point process that

$$
\lim_{\varepsilon\to 0} E^{(\varepsilon,K)}_{k-1}\left[\phi_{ti}(x_1)\ldots\phi_{tN}(x_N)\right] E^{(0,K)}_{k-1}\left[\phi_{ti}(x_1)\ldots\phi_{tN}(x_N)\right]. \tag{3.6.1}
$$

Let $\eta_1(x), \ldots, \eta_N(x) \in L^q(B_n)$, where $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Multiply the above to the both sides of (3.6.1)) and integrate over B_n. Then we get

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} E^{(\varepsilon,K)}_{k-1}\left[(\phi_{t_1}, \eta_1)(\phi_{t_2}, \eta_2)\ldots(\phi_{t_N}\eta_N)\right]\\
&\qquad\qquad = E^{(0,K)}_{k-1}\left[(\phi_{t_1}\eta_1)\ldots(\phi_{t_N}\eta_N)\right]. \qquad (3.6.2)
\end{aligned}
$$

Similarly we can show that for $\alpha_1, \ldots, \alpha_N > 0$, β_1, ..., $\beta_M > 0$, $\zeta_1 \ldots, \zeta_M \in L^q(B_n)$

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} E^{(\varepsilon,K)}_{k-1}\left[(\phi_{t_1}, \eta_1)^{\alpha_1}\ldots(\phi_{t_N}, \eta_N)^{\alpha_N}(\tilde{\psi}_K(X_{t_1}), \zeta_1)^{\beta_1}\ldots(\tilde{\psi}_K(X_{t_M}).\zeta_M)^{\beta_M}\right]\\
&\quad = E^{(0,K)}_{k-1}[(\phi_{t_1}, \eta_1)^{\alpha_1}\ldots(\phi_{t_N}, \eta_N)^{\alpha_N}(\tilde{\phi}_K(X_{t_1}), \zeta_1)^{\beta_1}\ldots\\
&\qquad (\tilde{\psi}_K(X_{t_M}), \zeta_M)^{\beta_M}]\\
&\quad = E^{(0,K)}_{k-1}[(\phi_{t_1}, \eta_1)^{\alpha_1}\ldots(\phi_{t_N}, \eta_N)^{\alpha_N}(X_{t_1}, \zeta_1)^{\beta_1}\ldots(X_{t_M}, \zeta_M)^{\beta_M}]
\end{aligned}
$$

We can replace $\tilde{P}^{(\varepsilon,K)}_{k-1}$ etc. by $\tilde{P}^{(\varepsilon,K)}_{0,p}$ etc. and hence $\tilde{P}^{(\varepsilon,K)}_{0,P} \to \tilde{P}^{(0,K)}_{0,p}$ with respect to the weak topology. □

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Proposition 3.6.3. $\{P_{0,p}{}^{(\varepsilon)}\}_{\varepsilon>0}$ *converges weakly.The proof is similar to that of the weak convergence of* $\{Q^{(\varepsilon)}\}$ *and is therefore omitted.*

Theorem 3.6.4. *Assume* $(A2)_k, (A3)_k, k \geq 2$. *Then* $\{P_{k-2,p}{}^{(\varepsilon)}\}_{\varepsilon>0}$ *is tight with respect to weak topology of* $W^2_{k-2,p}$

Proof. Consider the system of equations for $D^\alpha \phi^\varepsilon_t, |\alpha| \leq k-2$. Set $D^\alpha \phi^\varepsilon_t = {}^\alpha\phi^\varepsilon_t$ and consider $\underline{\phi}^\varepsilon_t = ({}^\alpha\phi^\varepsilon_t |\alpha| \leq k-2)$,

$$d\phi^\varepsilon_t = X^\varepsilon(\phi^\varepsilon_t, dt)$$
$$d^\alpha\phi^\varepsilon_t = \sum_{i=1}^{d} \frac{\partial}{\partial x_i} X^\varepsilon(\phi^\varepsilon_t, dt)({}^\alpha\phi^\varepsilon_t)^i, |\alpha| = 1$$

etc. In vector notation

$$d\phi^\varepsilon_t = \underline{X}^\varepsilon(\underline{\phi}^\varepsilon_t, dt)$$

where $\underline{X}^\varepsilon(x,t) = (X^\varepsilon(x,t), \sum\limits_i \frac{\partial}{\partial x_i} X^\varepsilon(x,t)x^i_\beta, \ldots), \underline{x} = (x, x_\beta, \ldots)$. Then the law of $(\underline{\phi}^\varepsilon_t, \underline{X}^\varepsilon_t)$, viz.$\underline{P}_{0,p}{}^{(\varepsilon)}$ with respect to the weak topology which is equivalent to the tightness of $\{P^{(\varepsilon)}_{k-2,p}\}$. □

Remark 3.6.5. In view of Proposition 3.1.10, the family of measures-$\{P_{k-3}{}^{(\varepsilon)}\}_{\varepsilon>0}$ in tight in W^2_{k-3}

3.7 Proof of the Main Theorem

The weak convergence of $\{P_m{}^{(\varepsilon)}\}_{\varepsilon>0}(m \leq k-3)$ has already been proved. Here we shall prove the strong convergence of ϕ^ε_t under the assumption that X^ε_t converges strongly to X^o_t. We shall prove that $\phi^\varepsilon_t \to \phi^\circ_t$ strongly. Consider the stochastic differential equation

$$d\phi^o_t = X^o(\phi^o_t, dt) + c(\phi^o_t)dt, \phi^o_o = x \tag{3.7.1}$$

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The solution of (3.7.1) is denoted by $\phi^o_t(x)$. For $m \leq k-3$, let $\tilde{P}^{(\varepsilon)}_m$ denoted the law of $(\phi^\varepsilon_t, X^\varepsilon_t, \phi^o_t, X^o_t)$ defined on $W^2_m \times \tilde{W}^2_m$ (where $\tilde{W}^2_m$) is a replica of W^2_m). A typical element of $\tilde{W}^2_m$ will be denoted as $(\tilde{\phi}, \tilde{X})$. We show that $\{\tilde{P}_m{}^{(\varepsilon)}\}_{\varepsilon>0}$ converges weakly to some $\tilde{P}_m{}^{(0)}$. Since $\{\tilde{P}_m{}^{(\varepsilon)}\}_{\varepsilon>0}$ is tight, let $\tilde{P}_m{}^{(0)}$ be any limits point of $\{\tilde{P}_m{}^{(\varepsilon)}\}$. Then $\tilde{P}_m{}^{(0)}|_{W^2_m} = P_m{}^{(0)}$

($\equiv$ limit of ${P_m}^{(\varepsilon)}$). By the first assertion of the theorem ϕ_t is generated by $X_t + \int_o^t c(x,r)dr$. Since $X_t = \tilde{X}_t$ a.s. ${\tilde{P}_m}^{(0)}, \phi_t = \tilde{\phi}_t$ a.s ${\tilde{P}_m}^{(0)}$ by the uniqueness of the solution i.e., ${\tilde{P}_m}^{(0)}$ is supported in the diagonal set $\{(\phi, X, \tilde{\phi}, \tilde{X}) : \phi = \tilde{\phi}, X = \tilde{X}\}$. Let ρ_m be the metric on W_m^2, i.e., for $(\phi, \tilde{\phi}) \in C([0,T]; C^m) \times C([0,T]; C^m)$,

$$\rho_m(\phi,\psi) = \sum_N \frac{1}{2^N} \frac{|||\phi - \psi|||_{m,N}}{1 + |||\phi - \psi|||_{m,N}}$$

which is a bounded continuous function on $W_m^2 \times \tilde{W}_m^2$. Therefore

$$E\left[\rho_m(\phi^\varepsilon, \phi^o)\right] = {\tilde{E}_m}^{(\varepsilon)}\left[\rho_m(\phi, \tilde{\phi})\right] \to {E^{(0)}}_m \left[\rho_m(\phi, \tilde{\phi})\right] = 0,$$

since $\rho_m(\phi, \tilde{\phi}) = 0$ a.s. Hence $\phi^\varepsilon \to \phi^o$ strongly.

3.8 Proof of the Approximation Theorem

In this section, we shall discuss the proof of the approximation theorem for stochastic ordinary differential equations described in section 3.2 Recall

$$\frac{d\phi_t^\varepsilon}{dt} = \sum_{k=1}^r F_k(\phi_t^\varepsilon, t)v_k^\varepsilon(t) + F_o(\phi_t^\varepsilon, t). \tag{3.8.1}$$

We assume (A1). To prove the theorem we need the following lemma.

Lemma 3.8.1. *Assume* (*A*1). *Then for any continuous functions* $f(x,t)$ *and* $g(x,t)$ *on* $\mathbb{R}^d \times [0,T]$

$$E\left[\int_s^t f(x,\tau)v_i^\varepsilon(\tau)d\tau \int_s^t g(y,\sigma)v_j^\varepsilon(\sigma)d\sigma | G_s^\varepsilon\right] \to \int_s^t f(x,r)g(y,r)v_{ij}(r)\,dr \tag{3.8.2}$$

uniformly on compact sets.

Proof. Let $f(t)$ and $g(t)$ be bounded measurable functions.Then

$$E\left[\int_s^t f(\tau)v_i^\varepsilon(\tau)d\tau\int_s^t g(\sigma)v_j^\varepsilon(\sigma)d\sigma|G_s^\varepsilon\right]| \leq nK\|f\|\,\|g\|\,|t-s| \tag{3.8.3}$$

Indeed, the left hand side (3.8.3) is equal to

$$\left|E\left[\int_s^t\int_\sigma^t f(\tau)E[v_i^\varepsilon(\tau)|G_\sigma^\varepsilon]d\tau)g(\sigma)v_i^\varepsilon(\sigma)d\sigma|G_s^\varepsilon\right]\right|$$
$$\leq K^{\frac{1}{\gamma}}\|f\|\,\|g\|\,|t-s|,\ \text{[from (A1 (c))]}. \quad \square$$

So in view of (3.8.3) it is enough to prove the case when $f(x,t)$, $g(x,t)$ are step functions of t. Therefore assume that

$$f(x,t) = f(x,t_i), g(x,t) = g(x,t_i) \text{ for } t_i \leq t < t_{i+1}$$

Then

$$\begin{aligned}
&E\left[\int_s^t f(x,\tau)v_i^\varepsilon(\tau)d\tau\int_s^\tau g(y,\sigma)v_j^\varepsilon(\sigma)d\sigma|G_s^\varepsilon\right]\\
&= \sum_k f(x,t_k)g(y,t_k)E\left[\int_{t_k}^{t_{k+1}} v_i^\varepsilon(\tau)d\tau\int_{t_k}^{\tau} v_j^\varepsilon(\sigma)d\sigma|G_s^\varepsilon\right]\\
&+ \sum_k f(x,t_k)E\left[\int_{t_k}^{t_{k+1}} v_i^\varepsilon(\tau)d\tau\int_s^{t_k} g(y,\sigma)v_j^\varepsilon(\sigma)d\sigma|G_s^\varepsilon\right]\\
&= I_1^\varepsilon + I_2^\varepsilon, \quad \text{say.}
\end{aligned}$$

Now

$$I_1^\varepsilon \xrightarrow[\varepsilon\to 0]{} \sum_k f(x,t_k)g(y,t_k)\int_{t_k}^{t_{k-1}} \nu_{ij}(r)\,dr = \int_s^t f(x,t)g(y,r)\nu_{ij}(r)dr.$$

$$I_2^\varepsilon = \sum_k f(x,t_k)E\left[E\left[\int_{t_k}^{t_{k+1}} v_i^\varepsilon(\tau)d\tau|Gt_k^\varepsilon\right]\int_s^{t_k} g(y\sigma)v_j^\varepsilon(\sigma)d\sigma|G_s^\varepsilon\right] \xrightarrow[\varepsilon\to 0]{} 0,$$

since $$E\left[\int_{t_k}^{t_{k+1}} v_i^\varepsilon(\tau)d\tau|G_{t_k}^\varepsilon \to \text{ as } \varepsilon \to 0\right]$$

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Proof of the approximation theorem (Theorem 3.2.2)

In view of the main limit theorem, all we have to do is to verify $(A2)_k$ and $(A3)_k$. In this case, we have

$$b^\varepsilon(x,t) = \sum_{k=1}^{r} F_k(x,t)v_k^\varepsilon(t) + F_o(x,t),$$

$$\overline{b}^\varepsilon(x,t) = F_o(x,t), \tilde{b}^\varepsilon(x,t) = \sum_{k=1}^{r} F_k(x,t)v_k^\varepsilon(t),$$

$$A_{ij}^\varepsilon(x,y,t,r) = \sum_{k,\ell} E\left[\int_s^t F_\ell^i(x,\tau)v_\ell^\varepsilon(\tau)d\tau|G_r^\varepsilon\right] F_k^j(y,r)v_k^\varepsilon(r)$$

Therefore

$$E\left[\int_s^t A_{ij}^\varepsilon(x,y,t,r)dr|G_s^\varepsilon\right]$$
$$= \sum_{k,\ell} E\left[\int_s^t d\tau F_\ell^i(x,\tau)v_\ell^\varepsilon(\tau)\int_s^t F_k^j(y,\sigma)v_k^\varepsilon(\sigma)d\sigma|G_s^\varepsilon\right]$$
$$\underset{\varepsilon\to 0}{\longrightarrow} \sum_{k,\ell}\int_s^t F_\ell^i(x,r)F_k^j v_{ij}(r)dr.$$

$(A3)_k$ can be proved similarly. Let $\tilde{P}_m{}^{(\varepsilon)}$ denote the law of $(\phi_t^\varepsilon, X_t^\varepsilon, B_t^\varepsilon)$ defined on $W_m^2 \times V_r$. Then $\{\tilde{P}_m{}^{(\varepsilon)}\}_{\varepsilon>0}$ converges to $\tilde{P}_m^{(0)}$ weakly. This can be proved in the same manner as we did earlier. The limit

$$((\phi_t, X_t, B_t), \tilde{P}_m^{(0)})$$

satisfies:

(i) X_t is a C-Brownian motion with local characteristics $a_{ij}(x,y,t) = \sum\limits_{k,\ell} F^i_\ell(x,t)F^j_k(y,t)\overline{\nu}_{ij}(t)$, where $\overline{\nu}_{ij} = \nu_{ij} + \nu_{ij}, b(x,t) = F_o(x,t)$.

(ii) ϕ_r is generated by $X_t + \int\limits_o^t c(x,r)dr$, where

$$c^j(x,t) = \sum_{\ell,k,i} \frac{\partial}{\partial x_i} F^i_\ell(x,t)F^i_k(x,t)\nu_{k\ell}(t).$$

97 (iii) $B(t)$ is a Brownian motion with mean zero and variance $\int\limits_o^t \overline{\nu}_{ij}(r)dr$ and

$$< X_t(x), B_k(t) >= \sum_\ell \int_o^t F_\ell(x,r)\overline{\nu}_{\ell k}(r)dr$$

This can be proved the same way as was done in Lemma 3.5.6. Now set

$$\tilde{X}_t(x) = \sum_{k=1}^r \int\limits_o^t F_k(x,s)dB_k(s) + \int\limits_o^t F_o(x,s)ds.$$

Then it can be shown that $< X_t - \tilde{X}_t >= 0$ which implies $X_t = \tilde{X}_t$. Therefore, with respect to $P^{(0)}_m$

$$d\phi_t = \sum_{k=1}^r F_k(\phi_t,t)dB_k(t) + c(\phi_t,t)\,dt \tag{3.8.4}$$

We shall now change the Ito form to Stratonovich form.

$$\begin{aligned}\int_o^t F_k(\phi_s(x),s)odB_k(s) &= \int_o^t F_k(\phi_s(x),s)dB_k(s)\\ &+ \frac{1}{2}\sum_{\ell,i}\int_o^t \frac{\partial}{\partial x_i}F_x(\phi_s(x),s)\times F^i_\ell(\phi_s(x),s)\overline{\nu}_{k,\ell}(s)ds.\end{aligned} \tag{3.8.5}$$

Also

$$c^j(x,s) - \frac{1}{2}\sum_{k,\ell,i}\frac{\partial}{\partial x_i}F^i_k(x,s)F^j_\ell(x,s)\overline{\nu}_{k\ell}(s)$$
$$= \frac{1}{2}\sum_{1\le k\le \ell\le r}(\nu_{k,\ell}(s) - \nu_{\ell k}(s))[F_k, F_\ell]^j(x,s). \quad (3.8.6)$$

Combining all these results,we conclude the proof.

3.9 Ergodic Case

Let $z(t), t \geq 0$ be a time-homogeneous Markov process with state space S. Let $P_t(x, A)$ be the transition probability function. Assume $P_t(x, .)$ has a unique invariant probability measure μ and $z(t)$ is a stationary ergodic process such that

$$P_t(z(t) \in A) = \mu(A).$$

For $z \in S$, $x \in \mathbb{R}^d$, let $F(x,z)$ and $G(x,z)$ be $\mathbb{R}^d$-valued functions **98**
smooth in x and the derivation bounded and continuous in (x,z). Let $Y(x,z,t,\omega)$ and $Z(x,z,t,\omega)$ be continuous random fields such that for each fixed z, $Y(.,z,t)$ and $Z(.,z,t)$ are continuous C^∞-valued martingales with local characteristics $a^Y(x,y,z)$ and $a^Z(x,y,z)$ respectively. Assume $z(t)$ and $\{Y, Z\}$ are independent. Consider the following stochastic differential equation

$$d\phi^\varepsilon_t = \varepsilon F(\phi^\varepsilon_t, z(t))dt + \varepsilon^2 G(\phi^\varepsilon_t, z(t))\,dt$$
$$+ \sqrt{\varepsilon}Y(\phi^\varepsilon_t, z(t), dt) + \varepsilon Z(\phi^\varepsilon_t, z(t),\ dt). \quad (3.9.1)$$

Let $\phi^\varepsilon_t(x)$ denote the solution of (3.9.1) with the initial condition $\phi^\varepsilon_o(x) = x$. Then if $\varepsilon \to 0$, $\phi^\varepsilon_t \to$ trivial flow, i.e., $\phi^o_t(x) \equiv x$. We will see that if we change the scale of time then the limit flow becomes nontrivial. Set $\psi^\varepsilon_t = \phi^\varepsilon_{t/\varepsilon}$. Then (3.9.1) becomes

$$d\psi^\varepsilon_t = F(\psi^\varepsilon_t, z(t/\varepsilon))\,dt + \varepsilon G(\psi^\varepsilon_t, z(t/\varepsilon))\,dt$$

$$+ Y^\varepsilon(\psi_t^\varepsilon, z(t/\varepsilon), dt) + \sqrt{3} Z^\varepsilon(\psi_t^\varepsilon, z(t/\varepsilon), dt) \quad (3.9.2)$$

where

$$\left.\begin{aligned} Y^\varepsilon(x, z(t/\varepsilon), dt) &= \sqrt{\varepsilon} Y(x, z(t/\varepsilon), dt/\varepsilon) \\ Z^\varepsilon(X, z(t/\varepsilon), dt) &= \sqrt{\varepsilon} Z(x, z(t/\varepsilon), dt/\varepsilon). \end{aligned}\right\} \quad (3.9.3)$$

Let

$$G_t^\varepsilon = \sigma(z(r), Y(x, z, r), Z(x, z, r) : r \leq t/\varepsilon).$$

Set

$$X^\varepsilon(x, t) = \int_o^t F\left(x, z\left(\frac{r}{\varepsilon}\right)\right) dr + \varepsilon \int_o^t G\left(x, z\left(\frac{r}{\varepsilon}\right)\right) dr$$

$$+ \int_o^t Y^\varepsilon\left(x, z\left(\frac{r}{\varepsilon}\right), dr\right) + \sqrt{\varepsilon} \int_o^t Z^\varepsilon\left(x, z\left(\frac{r}{\varepsilon}\right), dr\right). \quad (3.9.4)$$

99 Claim: $(\psi_t^\varepsilon, X_t^\varepsilon)$ converges as stochastic flows.

Indeed, for any $k > 0$ $(A2)_k$ can be verified as follows:

$$E\left[\int_s^t D_x^\alpha F\left(x, z\left(\frac{r}{\varepsilon}\right)\right) dr | G_s^\varepsilon\right] = \varepsilon E\left[\int_{s/\varepsilon}^{t/\varepsilon} F\left(x, z\left(\frac{r}{\varepsilon}\right)\right) dr | G_s^\varepsilon\right]$$

$$\to (t-s) \int D_x^\alpha F(x, z) \mu(dz) \text{ in } L^1 \text{-sense} \quad (3.9.5)$$

by the Ergodic theorem

$$E\left[\int_s^t D_y^\beta D_x^\alpha a^Y\left(x, y, z\left(\frac{r}{\varepsilon}\right)\right) dr | G_s^\varepsilon\right] \to \int D_y^\beta D_x^\alpha a^Y(x, y, z) \mu(dz). \quad (3.9.6)$$

$(A3)_k$ is clear. Therefore $(\psi_t^\varepsilon, X_t^\varepsilon)$ converge weakly as stochastic flow. The limit is a pair of Brownian flow and C^∞-Brownian motion with local characteristics $\int_o^t a^Y(x, y, z)\mu(dz), \int F(x, y)\mu(dz)$.

Special Cases Consider the following stochastic ordinary differential equation

$$\frac{d\phi_t^\varepsilon}{dt} = \varepsilon F(\phi_t^\varepsilon, z(t)) + \varepsilon^2 G(\phi_t^\varepsilon, z(t)). \quad (3.9.7)$$

In this case $\psi_t^\varepsilon \to \psi_t$ such that ψ_t satisfies a deterministic equation

$$\frac{d\psi_t}{dt} = \bar{F}(\psi_t), \tag{3.9.8}$$

where

$$\bar{F} = \int F(x,z)\mu(dz). \tag{3.9.9}$$

This may be regarded as a low of large number for the flow ψ_t^ε.

Let $F(x,t)$ be a period function of t with period 1 and $z(t) = t$ on $T^1 = [0,1]$ (one dimensional torus), an ergodic process with invariant measure dt. Consider

$$\frac{dx}{dt} = F(x, \frac{t}{\varepsilon}). \tag{3.9.10}$$

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The solution of (3.9.10) $\phi_t^\varepsilon(x) \xrightarrow[\varepsilon\to 0]{\phi} {}^{0}_{t}$, where ϕ_t^o satisfies

$$\frac{d\phi_t^o}{dt} = \bar{F}(\phi_t^o), \tag{3.9.11}$$

where

$$\bar{F}(x) = \int_o^1 F(x,t)dt. \tag{3.9.12}$$

This is sometimes called up the averaging of the equation (3.9.10).

Let us next consider the case when

$$a^Y(x,y,z) = 0, \int F(x,z)\mu(dz) = 0 \tag{3.9.13}$$

In this case the limits of ψ_t^ε is also a trivial flow. So we have to change the time scale in a different way. Set $\tilde{\psi}_t^\varepsilon = \phi_{t/\varepsilon^2}$. Then $\tilde{\psi}_t^\varepsilon$ is generated by $\tilde{X}_t^\varepsilon$ where

$$\begin{aligned} X_t^\varepsilon(x) = \frac{1}{\varepsilon}\int_o^t F\left(x, z\left(\frac{r}{\varepsilon^2}\right)\right)dr + \int_o^t G\left(x, z\left(\frac{r}{\varepsilon^2}\right)\right)dr \\ + \int_o^t \tilde{Z}^\varepsilon\left(x, z\left(\frac{r}{\varepsilon^2}\right), dr\right), \end{aligned} \tag{3.9.14}$$

with

$$\tilde{Z}^{\varepsilon}\left(x, z\left(\frac{t}{\varepsilon^2}\right), t\right) = \varepsilon Z\left(x, z\left(t/\varepsilon^2\right), \frac{t}{\varepsilon^2}\right). \tag{3.9.15}$$

We now make an assumption regarding the existence of a recurrent potential. ($A5$) There exists a unique recurrent potential, viz.

$$\psi(z, A) = \lim_{t\to\infty} \int_o^t (P_r(z, A) - \mu(A))dr. \tag{3.9.16}$$

[uniform convergence $w.r.t.z$]

and

$$\psi(f)(z) = \int \psi(z, dz')f(z')$$

maps C_b^∞ into C_b^∞. It is clear that if $\int f(x)\mu(dx) = 0$ then

$$\psi(f)(z) = \lim_{t\to\infty} \int_o^t T_r f(z)dr. \tag{3.9.17}$$

101 where T_r is the semigroup corresponding to the transition function $P_t(x, .)$.

Theorem 3.9.1. *Assume* ($A5$). *Then* $(\tilde{\psi}_t^\varepsilon, \tilde{X}_t^\varepsilon)$ *converge to a Brownian flow of* C^∞*-diffeomorphism and* C^∞*-Brownian flow. The local characteristics are given by*

$$\overline{a}_{ij}(x, y) = \int a_{ij}^Z(x, y, z)\mu(dz) + \int \left\{\hat{F}^i(x, z)F^j(y, z) + \hat{F}^j(y, z)F^i(x, z)\right\}\mu(dz) \tag{3.9.18}$$

$$\overline{b}(x) = \int G(x, z)\mu(dz), \tag{3.9.19}$$

$$c^i(x) = \sum_{k=1}^d \int \frac{\partial \hat{F}^i}{\partial x_k}(x, z)F^k(x, z)\mu(dz) \tag{3.9.20}$$

where $\hat{F}(x, z) = \int \psi(z, dz')F(x, z')$. $\quad$ (3.9.21)

Proof. Let $\tilde{G}^{\varepsilon}_t = \sigma(Z(x,z,s), z(s) : s \leq t/\varepsilon^2)$. We verify $(A2)_k$ for any $k > 0$.

$$E\left[\int_s^t G\left(x, z\left(\frac{r}{\varepsilon}\right)\right) dr | \tilde{G}^{\varepsilon}_s\right] = \varepsilon^2 E\left[\int_{s/\varepsilon^2}^{t/\varepsilon} G(x, z(r)) dr | \tilde{G}^{\varepsilon}_s\right]$$
$$\to (t-s)\int G(x,z)\mu(dz) \text{in} L^1 \text{ -sense,}$$

by the Ergodic theorem

$$E\left[\int\limits_s^t a^Z\left(x, y, z\left(\frac{r}{\varepsilon^2}\right)\right) dr | \tilde{G}^{\varepsilon}_s\right] \to (t-s)\int a^Z(x,y,z)\mu(dz).$$

$$E\left[\frac{1}{\varepsilon}\int\limits_s^t D^{\alpha}_x F\left(x, z\left(\frac{r}{\varepsilon^2}\right)\right) dr | \tilde{G}^{\varepsilon}_s\right] = \varepsilon E\left[\int\limits_{s/\varepsilon^2}^{t/\varepsilon^2} D^{\alpha}_x F(x, z(r)) dr | \tilde{G}^{\varepsilon}_s\right]$$
$$= \varepsilon \int\limits_o^{(t-s)/\varepsilon^2} T_r(D^{\alpha} F)\left(x, z\left(\frac{r}{\varepsilon^2}\right)\right) dr.$$

([by Markov property])

[where $T_r F(x,z) = \int T_r(z, dz') F(x, z')$] $\xrightarrow[\varepsilon\to 0]{}$ by $(A5)$. **102**

$$E\left[\frac{1}{\varepsilon}\int\limits_s^t F^i(x, z(\tau/\varepsilon^2)) d\tau \int\limits_s^{\tau} F^j(y, z(\sigma/\varepsilon^2)) d\sigma | \tilde{G}^{\varepsilon}_s\right]$$
$$= \varepsilon^2 E\left[\int\limits_{s/\varepsilon^2}^{t/\varepsilon} F^j(y, z(\sigma)) d\sigma \int\limits^{t/\varepsilon^2} \sigma F^i(x, z(\tau)) dt | \tilde{G}^{\varepsilon}_s\right]$$
$$= \varepsilon^2 \int\limits_o^{(t-s)/\varepsilon^2} d\sigma\left(\int P_{\sigma}(z, dz') F^j(y, z')\right)\left(\int\limits_o^{t/\varepsilon^2-\sigma} T_{\tau}(F^i)(x, z') d\tau)|_{z=z(s/\varepsilon^2)}\right)$$
$$\xrightarrow[\varepsilon\to 0]{} (t-s)\int F^j(y, z')\hat{F}^i(x, z')\mu(dz').$$

[using Ergodic theorem and $(A5)$] □

Similarly

$$\sum_k \frac{1}{\varepsilon^2} E\left[\int_s^t \frac{\partial F^i}{\partial x_k}(x, z(\tau/\varepsilon^2))d\tau \int_s^\tau F^k(x, z(\sigma/\varepsilon^2))d\sigma | \tilde{G}_s^\varepsilon\right] \xrightarrow[\varepsilon\to 0]{} (t-s)c^i(x).$$

We next check $(A3)_k$.

$$\frac{1}{\varepsilon} E\left[\int_s^t D^\alpha F\left(x, z\left(\frac{r}{\varepsilon^2}\right)\right) dr | \tilde{G}_s^\varepsilon\right] \frac{1}{\varepsilon} D^\beta F(x, z(s/\varepsilon^2))$$
$$= \int_o^{(t-s)/\varepsilon^2} T_r(D^\alpha F)(x, z(s/\varepsilon^2)) dr D^\beta F(x, z(s/\varepsilon^2))$$

is bounded (independent of ε). This completes the proof

We shall next consider an example concerning the limit theorems studied by Papanicolaou-Stroock-Varadhan [29].

Example 3.9.2. Let $(x^\varepsilon(t), z^\varepsilon(t))$ be the diffusion process in $\mathbb{R}^d \times \mathbb{R}^d$ defined by the following system of stochastic differential equations:

$$dx^\varepsilon(t) = \frac{1}{\varepsilon} F(x^\varepsilon(t), z^\varepsilon(t))dt + G(x^\varepsilon(t), z^\varepsilon(t))dt + \sum_{j=1}^r \sigma_{.j}(x^\varepsilon(t), z^\varepsilon(t)) d\beta_j(t) \tag{3.9.22}$$

$$dz^\varepsilon(t) = \frac{1}{\varepsilon^2} \tilde{F}(x^\varepsilon(t), z^\varepsilon(t))dt + \frac{1}{\varepsilon} \sum_j \tilde{\sigma}_{.j}(x^\varepsilon(t), z^\varepsilon(t)) d\tilde{\beta}_j(t) \tag{3.9.23}$$

103 where $(\beta_1(t), \ldots, \beta_r(t))$ and $(\tilde{\beta}_1(t), \ldots, \tilde{\beta}_r(t))$ are r-dimensional Brownian motions independent of each other. Here $z^\varepsilon(t)$ is called the driving process and $x^\varepsilon(t)$ the driven process. As $\varepsilon \to 0$, the right hand side of (3.9.23) diverges and hence the system of solutions $z^\varepsilon(t)$, $\varepsilon > 0$ does not converge. On the other hand, the first component $x^\varepsilon(t)$ varies slowly

compared with $z^\varepsilon(t)$. Papanicolaou-Stroock-Varadhan have shown that under some conditions on the coefficients, $x^\varepsilon(t)$ converges weakly to a diffusion process. Let $(\phi_t^\varepsilon(x,z), \psi_t^\varepsilon(x,z))$ denote the solution of (3.9.22), (3.9.23) starting from (x,z) at time $t=0$. The pair defines a stochastic flow of diffeomorphisms, but the first component $\phi_t^\varepsilon(x,z) \equiv \phi_t^\varepsilon(.,z)$, z being fixed, does not in general. However, if $\tilde{F}(x,z) = \tilde{F}(z)$, $\tilde{\sigma}(x,z) = \tilde{\sigma}(z)$ then (3.9.23) defines a closed system, $\psi_t^\varepsilon(x,z)$ does not depend on x. In this case the mapping $\phi_t^\varepsilon(.,z) : \mathbb{R}^d \to \mathbb{R}^d$ becomes a stochastic flow for each z, generated by

$$X^\varepsilon(x,dt) = \frac{1}{\varepsilon} F(x, z^\varepsilon(t))dt + \sum_j \sigma_{.j}(x, z^\varepsilon(t)) d\beta_j(t). \tag{3.9.24}$$

Now the solution $z^\varepsilon(t)$ has the same law as $z^1(t/\varepsilon^2)$. Put

$$\tilde{X}^\varepsilon(x,t) = \int\limits_o^t \left[\frac{1}{\varepsilon} F(x, z^1(r/\varepsilon^2)) + G(x, z^1(r/\varepsilon^2))\right] dr$$
$$+ \sum_j \int_o^t \sigma_{.j}(x, z^1(r/\varepsilon^2)) d\beta_j(r). \tag{3.9.25}$$

Let $\tilde{\phi}_t^\varepsilon(x,z)$ be the flow generated by $\tilde{X}_t^\varepsilon$ where z is the initial value. Then the law of $(\tilde{\phi}_t(x,z), \tilde{X}_t^\varepsilon) \equiv$ the law of $(\phi_t^\varepsilon(x,z), X_t^\varepsilon)$. Therefore $(\tilde{\phi}_t^\varepsilon(x,z), \tilde{X}_t^\varepsilon)$ converges.

Remark 3.9.3. These convergence problems do not include the homogenization problem. In fact here our conditions are more stringent than those in homogenization. If we write $a^\varepsilon(x,y) = a(\frac{x}{\varepsilon}, \frac{y}{\varepsilon})$, $b^\varepsilon(x) = b(\frac{x}{\varepsilon})$, then for the convergence of stochastic flows associated with $a^\varepsilon, b^\varepsilon$ we **104**
need boundedness conditions on the derivatives of a^ε and b^ε. Such condition are not satisfied for the homogenization.

3.10 Mixing Case

Suppose we are given a filtration $\{G_{s,t}\}$, $0 \le s \le t < \infty$ such that $G_{s,t} \subset G_{s't'}$ if $s' \le s \le t \le t'$. For each $t > 0$ we define the strong mixing rate

$\beta(t)$ as follows

$$\beta(t) = \sup_{s} \sup_{A\varepsilon G_{o,s'},\, B\varepsilon G_{s+t,\infty}} |P(A \cap B) - P(A)P(B)|. \tag{3.10.1}$$

Not that if $G_{o,s}$ and $G_{s+t,\infty}$ are independent for any s, then $\beta(t) = 0$. Hence $\beta(t) \sim 0$ means $G_{o,s}$ and $G_{s+t,\infty}$ are close to being independent. In particular, if u is a $G_{t,\infty}$-measurable random variable such that $E[u] = 0$ then $E[u|G_{o,s}|] \sim 0$ if $\beta(t-s) \sim 0$. more precisely, we have the following lemma.

Lemma 3.10.1. *Let u be a $G_{t,\infty}$-measurable random variable such that $E[u] = 0$. Then for any $p, q > 1$ with $p^{-1} + q^{-1} < 1$, we have*

$$E\,[|E[u|G_{o,s}]|^r]^{1/r} \leq C\beta(t-s)^{1/p} E[|u|^q]^{1/q} \tag{3.10.2}$$

where $C = C(p, q)$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

Proof. (Ibragimov-linnik [11]) let v be a bounded $G_{o,s}$-measurable random variable. For any $p', q' > 1$ with $\frac{1}{p'} + \frac{1}{q'} = 1$, we have

$$\begin{aligned} E[|E[u|G_{o,s}]||v|] &\leq E[|E[u|G_{o,s}]|]^{1/p'} E[|E[u|G_{o,s}]|\ |v|^{q'}]^{1/q'} \\ &\leq E[uv_1]^{1/p'} E[|u|\ |v|^{q'}]^{1/q'}, \end{aligned} \tag{3.10.3}$$

105 where $v_1 = signE[u|G_{0,s}]$. Similarly we have for $p'', q'' > 1$ with $\frac{1}{p''} + \frac{1}{q''} = 1$

$$\begin{aligned} E[uv_1] &= E[u(v_1 - E(v_1))] \\ &\leq E[u_1(v_1 - E(v_1))]^{1/p''} E[|v_1 - E(v_1)|\ |u|^{q''}]^{1/q''}, \end{aligned}$$

where $u_1 =$ sign $(E[v_1|G_{t,\infty}] - E[v_1])$. Set $A = \{\omega : u_1 = 1\}$, $B = \{\omega : v_1 = 1\}$. Then

$$\begin{aligned} |E[u_1v_1] - E[u_1]E[v_1]| &\leq |P(A \cap B^c) - P(A^c \cap B) + P(A^c \cap B^c)| \\ &\quad + P(A)P(B^c) + P(A^c)P(B) - P(A)P(B) - P(A^c)P(B^c)| \end{aligned}$$

$$\leq \quad 4\beta(t-s).$$

Since $|v_1 - E[v_1]| \leq 2$, the above inequality implies

$$E[uv_1] \leq 2^{\frac{1}{q''}+\frac{2}{p''}}\beta(t-s)^{\frac{1}{p''}}E[|u|^{q''}]^{1/q''}.$$

Substituting the above in (3.10.3), we get

$$E[E[u|G_{o,s}]|\,|v|] \leq (2^{\frac{1}{q''}+\frac{2}{p''}})^{1/p'}\beta(t-s)^{\frac{1}{p'p''}}E[|u|^{q''}]^{1/q''}E[|v|^{q'p''}]^{\frac{1}{p'p''}}.$$

Set $q = q''$ and $p = p'p''$, then $q'p'' = r'$, where $\dfrac{1}{r'} + \dfrac{1}{r} = 1$ and complete the proof. □

We establish a similar estimate for C^1-valued random variable.

Lemma 3.10.2. *Let $u(x,\omega)$ be a C^1-valued random variable, $G_{t,\infty}$-measurable and $E[u(x)] = 0$ for any x. Then for any $p, q > 1$ with $p^{-1}+q^{-1} < 1$ and for any $K > 0$ there exists a constant $C = C(p,q,d,K)$ such that*

$$E\left[\sup_{|x|\leq K}|E\left[u(x)|G_{o,s}\right]^r\right]^{1/r} \leq C\beta(t-s)^{\frac{1}{p(d+1)}}E\left[\sum_{|\alpha|\leq 1}\sup_{|x|\leq K}|D^\alpha u(x)|^q\right]^{1/q} \tag{3.10.4}$$

where $\dfrac{1}{r} = \dfrac{1}{p} + \dfrac{1}{q}$.

Proof. Let N be any positive number. There exists $x_1, \ldots, x_n (n \leq (2N + 1)^d)$ such that $\{B(x_i; \dfrac{K}{N}), i = 1, \ldots, n\}$, where $B(x_i; \dfrac{K}{N})$ is the ball with centre x_i and of radius $\dfrac{K}{N}$, covers the cube $[-K, K]^d$. Then we have 106

$$\begin{aligned}\sup_{|x|\leq K}|E[u(x)|G_{o,s}]| &\leq \max_i \sup_{x\in B(x_i,\frac{K}{N})}|E[u(x)-u(x_i)|G_{o,s}]| \\ &\quad + \max|E[u(x_i)|G_{o,s}]| \\ &= I_1 + I_2, \text{say}. \end{aligned} \tag{3.10.5}$$

□

Using mean value theorem, we have

$$\begin{aligned} I_1 &\leq \frac{K}{N} \max_i \sum_{|\alpha|=1} \sup_{x \varepsilon B(x_i, \frac{K}{N})} |E[D^\alpha u(x)|G_{o,s}]| \\ &\leq \frac{K}{N} \sum_{|\alpha|=1} \sup_{|x|\leq K} |E[D^\alpha u(x)|G_{o,s}]|. \end{aligned}$$

Therefore

$$\begin{aligned} E\left[\sup_{|x|\leq K} |E[u(x)|G_{o,s}]|^r\right]^{1/r} &\leq \frac{K}{N} E\left[\sum_{|\alpha|=1} \sup_{|x|\leq K} |E[D^\alpha u(x)|G_{o,s}]|^r\right]^{1/r} \\ &\quad + C(2N+1)^d \beta(t-s)^{1/P} \max E[|u(x_i)|^q]^{\frac{1}{q}}. \end{aligned}$$

Set $N = \beta(t-s)^{-\frac{1}{p(d+1)}}$. Then $N^d \beta(t-s)^{1/p} = \beta(t-s)^{\frac{1^i}{p(d+1)}}$. Since $r < q$ the above is bounded by

$$(K + C3^d)\beta(t-s)^{\frac{1}{p(d+1)}} E\left[\sum_{|\alpha|\leq 1} \sup_{|x|\leq K} |D^\alpha u(x)|^q\right]^{1/q}.$$

This proves the lemma.

Lemma 3.10.3. *Let $u(x)$ (resp. $v(x)$) be a C^1-valued random variable which is $G_{u,u}$ (resp. $G_{t,t}$)-measurable where $t < u$. Suppose $E[u(x)] = 0$ for all x and set $w(x) = E[u(x)v(x)]$. Then for any p, q with $p^{-1} + 2q^{-1} < 1$ and for any $K > 0$, there is a positive constant $C = C(p, q, K)$ such that for $s < t$*

$$\begin{aligned} E\left[\sup_{|x|\leq K} |E[u(x)v(x) - w(x)|G_{o,s}]|^\delta\right]^{1/\delta} &\leq C\{\beta(u-t)\beta(t-s)\}^{\frac{1}{2p(d+1)}} \\ E\left[\sum_{|\alpha|\leq 1} \sup_{|x|\leq K} |D^\alpha u(x)|^{2q}\right]^{1/2q} &\times E\left[\sum_{|\alpha|\leq 1} \sup_{|x|\leq K} |D^\alpha v(x)|^{2q}\right]^{\frac{1}{2q}} \end{aligned} \tag{3.10.6}$$

107 *where $p^{-1} + 2q^{-1} = \delta^{-1}$.*

Proof. Lemma 3.10.2 with uv-w substituted for u. Then

$$
\begin{aligned}
&E\left[\sup_{|x|\le K}|E[u(x)v(x)-w(x)|G_{o,s}]|^r\right]^{1/r}\\
&\le C\beta.(t-s)^{\frac{1}{p(d+1)}}E\left[\sum_{|\alpha|\le 1}\sup_{|x|\le K}|D^\alpha(uv-w)|^q\right]^{1/q}\\
&\le C'\beta.(t-s)^{\frac{1}{p(d+1)}}E\left[\sum_{|\alpha|\le 1}\sup_{|x|\le K}|D^\alpha u(x)|^{2q}\right]^{1/2q}\times\\
&\qquad E\left[\sum_{|\alpha|\le 1}\sup_{|x|\le K}|D^\alpha v(x)|^{2q}\right]^{1/2q}, \qquad (3.10.7)
\end{aligned}
$$

where $p^{-1}+q^{-1}=r^{-1}$. Since $\delta<r, r$ can be replaced by δ in the left hand side of (3.10.7). Next note that $\delta^{-1}=r^{-1}+q^{-1}$. Then by Holder inequality

$$
\begin{aligned}
&E\left[\sup_{|x|\le K}|E[u(x)v(x)|G_{o,s}]|^\delta\right]^{1/\delta}\\
&=E\left[\sup_{|x|\le K}|E[E[u(x)|G_{o,s}]v(x)|G_{o,s}]|^\delta\right]^{1/\delta}\\
&\le E\left[\sup_{|x|\le K}|E[u(x)|G_{o,t}]|^r G_{o.s}\right]^{\frac{\delta}{r}}E\left[\sup_{|x|\le K}|E[v(x)|^q|G_{o,s}]^{\frac{\delta}{q}}\right]^{1/\delta}\\
&\le E\left[\sup_{|x|\le K}|E[u(x)|G_{o,t}]|^r\right]^{1/r}.E\left[\sup_{|x|\le K}|v(x)|^q\right]^{\frac{1}{q}}\\
&\le C\beta.(u-t)^{\frac{1}{p(d+1)}}E\left[\sum_{|\alpha|\le 1}\sup_{|x|\le K}|D^\alpha u(x)|^q\right]^{1/q}E\left[\sup_{|x|\le K}|v(x)|^q\right]^{1/q}.
\end{aligned}
$$

□

Finally we have by Lemma 3.10.1 108

$$|w(x)|\le C'\beta(u-t)^{1/p}E[|u(x)|^q]^{1/q}E[|v(x)|^r]^{1/r}$$

$$\leq C'\beta(u-t)^{1/p}E[|u(x)|^q]^{1/q}E[|v(x)|^q]^{1/q},$$

where $p^{-1}+q^{-1}+r^{-1}=1$ and $r<q$. Therefore the left hand of (3.10.6) is bounded by

$$C\beta(u-t)^{\frac{1}{p(d+1)}}E[\sum_{|\alpha|\leq 1}\sup_{|x|\leq K}|D^\alpha u(x)|^q]^{1/q}E[\sup_{|x|\leq K}|v(x)|^q]^{1/q}. \tag{3.10.8}$$

Hence the square of the left hand side of (3.10.6) is bounded by the the product of the right hand side 3.10.7 and (3.10.8). This proves the lemma.

Consider a stochastic defferential equation with a parameter $\varepsilon>0$

$$\begin{aligned} dx(t) &= \varepsilon F(x(t),t)dt+\varepsilon^2 G(x(t),t)dt+\varepsilon Y(x(t),dt) \\ x\ &\varepsilon\,\mathbb{R}^d, 0\leq t<\infty, \end{aligned} \tag{3.10.9}$$

where $F(x,t,\omega), G(x,t,\omega)$ are $\mathbb{R}^d$-valued random fields, F_o-measurable, continuous in (x,t) and k-times continuously differentiable in x, the first derivatives are bounded.

Further $E[F(x,t)]=0$ for any x, t. $Y(x,t), 0\leq t<\infty$, is a continuous C^{k-1}-martingale adapted to F_t with local characteristics $a(x,y,t,\omega)$ with properties similar to the above. The equation is similar to the one introduced in the previous section. In fact $F(x,z(t)), G(x,z(t))$ etc. of the previous section correspond to $F(x,t,\omega), G(x,t,\omega)$ etc. of this section. Let $\phi_t^\varepsilon(x)$ be the stochastic flow determined by the above stochastic differential equation. Then both ϕ_t^ε and $\phi_{t/\varepsilon}^\varepsilon$. We shall study the weak convergence of ψ_t^ε as $\varepsilon\to 0$ as stochastic flows under a mixing condition on the stochastic defferential equation. Such a limit theorem has extensively been studied in the case $Y\equiv 0$, i.e., in the case of stochastic or-
109 dinary differential equation see Khasminskii [17], Kohler-Papanicolaou [28], Kesten-Papanicolaou [15]. In those works the weak convergence on diffusion processes has been studied. Here we shall study the weak convergence as stochastic flows including the case $Y\not\equiv 0$. Let

$$G_{s,t}=\sigma(F(.,u),G(.,u)-Y(.,v), s\leq u,v\leq t)$$

and let $\beta(t)$ be the strong mixing rate associated with $G_{s,t}$. We shall introduce the assumptions:

$(A6)_k$ There are continuous functions $a = (a_{ij}(x,y,t), A = (A_{ij}(x,y,t)$, $b = (b^i(x,t)), c = (c^i(x,t))$ which are k-times continuously differentiable with respect to x, y or x, as the case may be, and the first derivatives are bounded such that the following are satisfied

$$|a_{ij}(x,y,t) - \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int\limits_t^{t+\varepsilon} E\left[a_{ij}\left(x, y, \frac{r}{\varepsilon^2}\right)\right] dr| = 0,$$

$$|A_{ij}(x,y,t) - \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^3} \int\limits_t^{t+\varepsilon} \int\limits_t^{\tau} E\left[F^i(x, \frac{\sigma}{\varepsilon^2}) F^j\left(y, \frac{\tau}{\varepsilon^2}\right)\right] \sigma d\tau| = 0,$$

$$|b^i(x,t) - \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int\limits_t^{t+\varepsilon} E\left[G^i\left(x, \frac{r}{\varepsilon^2}\right)\right] dr| = 0,$$

$$|(x,t) - \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^3} \int\limits_t^{t+\varepsilon} d\tau \int\limits_t^{\tau} d\sigma \sum_j E\left[F^j(x, \frac{\sigma}{\varepsilon^2}) \frac{\partial}{\partial x_j} F^j\left(x, \frac{\tau}{\varepsilon^2}\right)\right]| = 0,$$

uniformly on compact sets.

Let $p, q > 1$ be such that $p^{-1} + q^{-1} < 1$.

$(A7)_{k,p,q}$ (a) $\int\limits_o^\infty \beta(r)^{\frac{1}{2p(d+1)}} dr < \infty$,

(b) For any $K > 0$ the 2qth moments of $\sup\limits_{|x|\le K, |y|\le K} |D_x^\alpha D_y^\beta a(x,y,t)|$, $\sup\limits_{|x|\le K} |D^\alpha G(x,t)|$ are all bounded by a positive constant independent of t, $|\alpha| \le k$, $|\beta| \le K$.

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Remark 3.10.4. Note that here we are assuming the existence of infinitesimal limits whereas in the previous cases we have assumed the existence of global limits.

Theorem 3.10.5. *Assume* $(A6)_k$ *and* $(A7)_{k,p,q}$ *for some* p, q *such that* $p^{-1} + q^{-1} < 1$. *Then the conclusion of the main limit theorem is valid.*

We shall give the proof of the theorem in a more general setting.

Let $X_t^\varepsilon = X^\varepsilon(x,t), \varepsilon > 0$ *be a family of continuous* C^{k-1}*- semimartingales adapted to* $F_t^\varepsilon, \varepsilon > 0$ *with local characteristics* $a^\varepsilon(x,y,t,\omega)$, $b^\varepsilon(x,t,\omega)$*. We assume as before* a^ε *and* b^ε *are continuous in* t *and* k*-times continuously differentiable in* x, y *and* x *respectively. Set* $\overline{b}^\varepsilon = E[b^\varepsilon(x,t)]$ *and* $\tilde{b}^\varepsilon = b^\varepsilon - \overline{b}^\varepsilon$*. In place of* $(A6)_k$ *we assume:*

$(A8)_k$ *There are continuous functions* $a = (a_{ij}(x,y,t)), \overline{b} = (\overline{b^i}(x,t)), c = (c^i(x,t)), A = (A_{ij}(x,y,t))$ *which are k-times continuously differentiable with respect to* x, y *or* x *and the first derivatives are bounded and*

$$|a(x,y,t) - \frac{1}{\varepsilon}\int_t^{t+\varepsilon} E[a^\varepsilon(x,y,r)]dr| \to 0 \text{ as } \varepsilon \to 0,$$

$$|\overline{b}(x,t) - \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \overline{b}^\varepsilon(x,r)dr| \to 0 \text{ as } \varepsilon \to 0,$$

$$|A_{ij}(x,y,t) - \frac{1}{\varepsilon}\int_t^{t+\varepsilon} d\tau \int_t^{\tau} d\sigma E[\tilde{b}_i^\varepsilon(x,\tau) b_j^\varepsilon(y,\sigma)]| \to 0 \text{ as } \varepsilon \to 0,$$

$$|c^i(x,t) - \frac{1}{\varepsilon}\int_t^{t+\varepsilon} d_\tau \int_t^{\tau} d\sigma \sum_j E[b_j^\varepsilon(x,\sigma)\frac{\partial}{\partial x_j}\tilde{b}_i^\varepsilon(x,\tau)]| \to 0 \text{ as } \varepsilon \to 0,$$

111 *uniformly on compact sets. Set*

$$G_{s,t}^\varepsilon = \sigma(Y^\varepsilon(.,u) - Y^\varepsilon(.,v), b^\varepsilon(.,u), s \le u, v \le t).$$

Let $\beta^\varepsilon(t)$ *be the strong mixing rate associated with* $G_{s,t}^\varepsilon$*. We introduce an assumption so that as* $\varepsilon \to 0, \beta^\varepsilon(t) \to 0, E[|\tilde{b}^\varepsilon|r|] \to \infty$*, but the rates of the convergence and divergence are balanced. Let* $p, q > 1$ *be such that* $p^{-1} + 2q^{-1} < 1$*. We restrict the time interval to* $[0, T]$*.*

$\gamma : (A9)_{k,p,q}$ $a^\varepsilon = (a^\varepsilon(x,y,t))$ *and* $\overline{b}^\varepsilon = (\overline{b}^\varepsilon(x,t))$ *satisfy the same moments conditions as in* $(A7)_{k,p,q}$ *independently of* ε*. Furthermore the mixing rate* $\beta^\varepsilon(t)$ *satisfies: with* γ *such that* $p^{-1} + q^{-1} = \gamma^{-1}$*:*

(a) $\lim\limits_{\varepsilon\to 0}\int\limits_0^T \beta^\varepsilon(\tau)^{\frac{\gamma}{p(d+1)}}d\tau = 0, \int\limits_\varepsilon^T \beta^\varepsilon(\tau)^{\frac{\gamma}{p(d+1)}}d\tau = 0\left(\int\limits_0^T \beta^\varepsilon(\tau)^{\frac{\gamma}{p(d+1)}}d\tau\right)$

(b) *For each $K > 0$ there is a positive constant $L = L_K$ such that*
$\int\limits_0^T \beta^\varepsilon(\tau)^{\frac{\gamma}{p(d+1)}}d\tau . E\left[\sup\limits_{|x|\leq K}|D^\beta \tilde{b}^\varepsilon(x,t)|^{2q}\right]^{1/q} \leq L$ *for all* $\varepsilon > 0$.

Theorem 3.10.6. *Assume $(A8)_k$ and $(A9)_{k,p,q}$ for p,q such that $p^{-1} + 2q^{-1} < 1$. Then the conclusion of the main limit theorem is valid.*

Before describing the proof of the above Theorem let us point out that Theorem 3.10.5 can be deduced form Theorem 3.10.6. Indeed, set $b^\varepsilon(x,t) = \frac{1}{\varepsilon}F(x,t/\varepsilon^2) + G(x,t/\varepsilon^2), \gamma^\varepsilon(x.t) = \frac{1}{\sqrt{\varepsilon}}Y(x,t/\varepsilon^2)$. Then $(A8)_k$ immediately follows from $(A6)_k$. Set $G^\varepsilon_{s,t} = G_{\frac{s}{\varepsilon^2},\frac{t}{\varepsilon^2}}$.

Then we have $\beta^\varepsilon(r) = \beta(r/\varepsilon^2)$. Therefore

$$\int\limits_0^T \beta^\varepsilon(\tau)^{\frac{\gamma}{2p(d+1)}}d\tau < \varepsilon^2\int\limits_0^T \beta^\varepsilon(\tau)^{\frac{\gamma}{2p(d+1)}}d,$$

$$E\left[\sup_{|x|\leq K}|D^\beta\tilde{b}^\varepsilon(x,t)|^{2q}\right]^{1/q} = \frac{1}{\varepsilon^2}E\left[\sup_{|x|\leq K}|D^\beta F(x,t)|^{2q}\right]^{1/q}.$$

Hence $(A9)_{k,p,q}$ is satisfied. **112**

Proof of theorem 3.10.6. Set $D^\alpha\tilde{b}^\varepsilon(x,\tau) = u^\varepsilon(x,\tau)$. Then by Lemma 3.10.2, we have for γ such that $\gamma^{-1} = p^{-1} + q^{-1}$

$$E\left[\sup_{|x|\leq K}|\int\limits_t^s E[u^\varepsilon(x,\tau)|G^\varepsilon_{o,s}]|^\gamma\right]^{1/\gamma}$$
$$\leq C\left(\int\limits_t^s \beta^\varepsilon(\tau-s)^{\frac{\gamma}{p(d+1)}}d\tau\right)^{\frac{1}{\gamma}}\sup_\tau E\left[\sum_{|\beta|\leq 1}\sup_{|x|\leq K}|D^\beta u^\varepsilon(x,\tau)|^q\right]^{1/q}.$$

Now pick δ such that $p^{-1} + 2q^{-1} = \delta^{-1}$ and $\tilde{p} = r\delta^{-1}$. Then $\tilde{\delta q} = q$, where $\tilde{q}$ is the conjugate of $\tilde{p}$. Then by Holder inequality

$$
\begin{aligned}
& E\left[\sup_{|x|\le K} |\int_t^s E[u^\varepsilon(x,\tau)|G^\varepsilon_{o,s}]d\tau|u^\varepsilon(x,s)|^\delta\right]^{1/\delta} \\
& \le C\left(\int_t^s \beta^\varepsilon(\tau-s)^{\frac{\gamma}{p(d+1)}}d\tau\right)^{\frac{1}{\gamma}} \sup_\tau E\left[\sum_{|\beta|\le 1}\sup_{|x|\le K}|D^\beta u^\varepsilon(x,\tau)|^q\right]^{1/q} \times \\
& E\left[\sup_{|x|\le k}|u^\varepsilon(x,s)|^q\right]^{1/q} \\
& \le L
\end{aligned}
$$

This proves $(A3)_k$. We next prove $(A2)_k$. Set

$$\overline{a}^\varepsilon(x,y,r) = E[a^\varepsilon(x,y,r)].$$

Then

$$\int_s^t \overline{a}^\varepsilon(x,y,r)dr \to \int_s^t a(x,y,r)dr$$

follows immediately.

$$\sup_{|x|\le K} |\int_s^t E[\overline{a}^\varepsilon(x,y,r)|G^\varepsilon_{o,s}]dr| \to 0 \text{ in } L^1 - \text{ sense},$$

where $\tilde{a}^\varepsilon = a^\varepsilon - \overline{a}^\varepsilon$. Similar estimate implies

$$
\begin{aligned}
& E\left[\sup_{|x|\le K} |\int_t^s E[u^\varepsilon(x,\tau)|G^\varepsilon_{o,s}]d\tau|^r\right]^{\frac{1}{\gamma}} \\
& \le C\left(\int_t^s \beta^\varepsilon(\tau-s)^{\frac{\gamma}{p(d+1)}}d\tau\right)^{\frac{1}{\gamma}} E\left[\sum_{|\beta|\le 1}\sup_{|x|\le K}|D^\beta u^\varepsilon(x,\tau)|^q\right]^{1/q} \\
& \to 0 \text{ as } \varepsilon \to 0.
\end{aligned}
$$

113 Set $K^{\varepsilon}(\tau,\sigma,x,y) = \tilde{b}^{\varepsilon}(x,\tau)b^{\varepsilon}(y,\sigma)$ and $\overline{K}^{\varepsilon} = E[K^{\varepsilon}], \tilde{K}^{\varepsilon} = K^{\varepsilon} - \overline{K}^{\varepsilon}$
Then by $(A.8)_k$

$$\int_t^{t+\varepsilon} d\tau \int_t^{\tau} d^{\sigma} \overline{K}^{\varepsilon}_{ij}(\tau,\sigma,x,y) = \varepsilon A_{ij}(x,y,t) + o(t,\varepsilon) \tag{3.10.10}$$

where $\dfrac{o(t,\varepsilon)}{\varepsilon} \to 0$ uniformly in t as $\varepsilon \to 0$. By Lemma 3.10.1, we have

$$|\int_t^{t+\varepsilon} d\tau \int_s^t d\sigma \overline{K}^{\varepsilon}_{ij}(\tau,\sigma,x,y)| \leq \int_t^{t+\varepsilon} d\tau \int_s^t d\sigma \beta^{\varepsilon}(\tau-\sigma)^{\overline{\dfrac{1}{p(d+1)}}}$$
$$\sup_{\sigma} E[|b^{\varepsilon}_j(y,\sigma)|^{2q}]^{\frac{1}{2q}} \times \sup_{\tau} E[|\tilde{b}^{\varepsilon}_i(x,\tau) + |^{2q}]^{1/2q}. \tag{3.10.11}$$

Note that

$$\int_t^{t+\varepsilon} d\tau \int_s^t d\sigma \beta^{\varepsilon}(\tau-\sigma)^{\frac{1}{p(d+1)}} = o\left(\int_o^T \beta^{\varepsilon}(\tau)^{\overline{\dfrac{1}{p(d+1)}}} d\tau\right).$$

Therefore (3.10.11) converges to zero. Hence

$$\int_t^{t+\varepsilon} d\tau \int_s^{\tau} d\sigma \overline{K}^{\varepsilon}_{ij}(\tau,\sigma,x,y) = \varepsilon A_{ij}(x,y,t) + o(t,\varepsilon).$$

This proves

$$\lim_{\varepsilon\to 0} \int_s^t d\tau \int_s^{\tau} d\sigma \overline{K}^{\varepsilon}_{ij}(\tau,\sigma,x,y) = \int_s^t A_{ij}(x,y,\tau) d\tau.$$

On the other hand, by Lemma 3.10.3,

$$E\left[|E\left[\int_s^t d\tau \int_s^{\tau} d\sigma \tilde{K}^{\varepsilon}_{ij}(\tau,\sigma,x,y)|G^{\varepsilon}_{o,s}\right]|^{\delta}\right]^{1/\delta}$$
$$\leq C \sup_{\tau} E\left[|\tilde{b}^{\varepsilon}(x,\tau)|^{2q}\right]^{1/2q} E[|b^{\varepsilon}(y,\tau)|^{2q}]^{1/2q}$$
$$\times \left(\int_s^t d\tau \int_s^{\tau} d\sigma \beta^{\varepsilon}(\tau-\sigma)^{\frac{\gamma}{2p(d+1)}} \beta^{\varepsilon}(\sigma-s)^{\frac{\gamma}{2p(d+1)}}\right)^{\frac{1}{\gamma}}$$

$$\leq C \sup_{\tau} E\left[|\tilde{b}^{\varepsilon}(x,\tau)|^{2q}\right]^{1/2_q} E[|b^{\varepsilon}(y,\tau)|^{2q}]^{1/2q}$$

$$\left(\int_o^T \beta^{\varepsilon}(\tau)^{\frac{\gamma}{2p(d+1)}} d\tau\right)^{2/\gamma} \to 0 \text{ as } \varepsilon \to 0.$$

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These two computations yield

$$\lim_{\varepsilon\to 0} E\left[\int_s^t d\tau \int_s^{\tau} d\sigma K_{ij}^{\varepsilon}(\tau,\sigma,x,y)|G_{o,s}^{\varepsilon}\right] = \int_s^t A_{ij}(x,y,\tau)d^{\tau}.$$

This completes the proof.

3.11 Tightness and Weak Convergence of Inverse Flows

Let $X^{\varepsilon}(x,t), \varepsilon \to 0$ be a family of continuous C-semimartingales adapted to F_t^{ε} with local characteristics $(a^{\varepsilon}, b^{\varepsilon})$ satisfying Lipschitz continuity and linear growth properties. Assume that $X^{\varepsilon}(x,t)$ is a backward semimartingale, i.e., it satisfies $(A5)$ of Chapter 2. Set

$$\hat{X}^{\varepsilon}(x,t) = -X^{\varepsilon}(x,t) + \int_o^t d^{\varepsilon}(x,r)dr,$$

where
$$d_i^{\varepsilon}(x,t) = \sum_j \frac{\partial}{\partial x_j} a_{ij}(x,y,t)|_{y=x}.$$

It is a backward semimartingale with local characteristics $a^{\varepsilon}, -b^{\varepsilon} + d^{\varepsilon}$. We make the following assumption:

$\widehat{(A2)}_k$ The tightness condition $(A2)_k$ is satisfied for the backward semimartingale. Set $\Psi_{s,t}^{\varepsilon} = (\phi_{s,t}^{\varepsilon})^{-1}$ and $\Psi_t^{\varepsilon} = \Psi_{o,t}^{\varepsilon}$, where $\phi_{s,t}^{\varepsilon}$ is the flow generated by $X^{\varepsilon}(x,t)$. Then for each $t, \{\Psi_{s,t}^{\varepsilon}\}$, $s \in [0,T]$ is tight. We claim that for each $s, \{\Psi_{s,t}^{\varepsilon}\}$, $t \in [s,T]$ is tight.

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Theorem 3.11.1. *Let $k \geq 4$. The family of laws of Ψ_t^{ε} on $W_m, m \leq k-4$, is tight.*

Proof. Let $\phi_{s,t}^{\varepsilon,K}$ be the flow generated by the truncated process $X^{\varepsilon,K}(x,t)$ and let $\Psi_{s,t}^{\varepsilon,K} = (\phi_{s,t}^{\varepsilon,K})^{-1}$. We shall prove the tightness of $\Psi_t^{\varepsilon,K}$ in $W_{o,p}$. We suppress K. Let $s < t$. Then

$$\begin{aligned}\Psi_t^\varepsilon(x) - \Psi_s^\varepsilon(x) &= \Psi_s^\varepsilon o \Psi_{s,t}^\varepsilon(x) - \Psi_s^\varepsilon(x)\\ &= \partial\Psi_s^\varepsilon(\xi_s^\varepsilon(x))(\Psi_{s,t}^\varepsilon(x) - x),\end{aligned} \tag{3.11.1}$$

where $|\xi_s^\varepsilon(x)| \leq K$. For any $p \geq 3$, there is a positive constatnt C' such that

$$E|\Psi_{s,t}^\varepsilon(x) - x|^p \leq C'|t-s|^{2-\frac{3}{p}}.$$

□

In fact $\Psi_{s,t}^\varepsilon(x)$ satisfies the following backward stochastic differential equation

$$\Psi_{s,t}^\varepsilon(x) = x - \int_s^t X^\varepsilon(\Psi_{r,t}^\varepsilon(x), \hat{d}r) + \int_s^t d^\varepsilon(\Psi_{r,t}^\varepsilon(x), r)\hat{d}r.$$

Therefore arguing as in Lemma 3.4.2 we get the above estimate. On the other hand, the inverse of the Jacobian matrix $-\partial\phi_t^\varepsilon$ satisfies as in the case of a usual stochastic differential equation (see Ikeda-Watanabe [13])

$$(\partial\phi_t^\varepsilon)^{-1} = I - \int_o^t (\partial\phi_r^\varepsilon)^{-1} X^\varepsilon(\phi_r^\varepsilon, dr) - \int_o^t (\partial\phi_r)^{-1} G(\phi_r^\varepsilon(x), r)dr$$

where $G_{ij}(x,t) = \sum_k \frac{\partial}{\partial y_j}\frac{\partial}{\partial x_k} a_{ik}^\varepsilon(x,y,t)|_{y=x}$. Then we can show that $(\partial\phi_t^\varepsilon)^{-1}$ converges weakly in the same way as we did earlier. Therefore the associated laws are tight. Consequently, for any $\delta > 0$, there is a $C = C(\delta)$ such that

$$P\left(\sup_{\substack{|x|\leq K\\ t\in[0,T]}} |(\partial\phi_t^\varepsilon)^{-1}(x)| \leq C\right) > 1 - \delta.$$

Now, since $\partial\Psi^\varepsilon_t(x) = (\partial\phi^\varepsilon_t)^{-1}(\Psi^\varepsilon_t(x))$ and $|\Psi^\varepsilon_t(x)| \leq K$ if $|x| \leq K$, we get 116

$$P\left(\sup_{\substack{|x|\leq K\\ t\in[0,T]}} |\partial\Psi^\varepsilon_t(x)| \leq C\right) > 1-\delta.$$

Define

$$A_{\varepsilon,\delta} = \left\{\omega : \sup_{|x|\leq K, t\in[0,T]} |\partial\Psi^\varepsilon_t,(x)| \leq C\right\}.$$

Then $P(A_{\varepsilon,\delta} > 1-\delta$ for any ε. Therefore

$$\begin{aligned} E\left[|\Psi^\varepsilon_t(x) - \Psi^\varepsilon_s(x)|^P : A_{\varepsilon,\delta}\right] &\leq C^P E[|\ \Psi^\varepsilon_{s,t}(x) - x|^P : A_{\varepsilon,\delta}] \\ &\leq C^P.C'|t-s|^{2-\frac{3}{P}}. \end{aligned}$$

Therefore the measure $P^{(\varepsilon,\delta)}(.) = P(.|A_{\varepsilon,\delta})$ satisfies

$$E^{(\varepsilon,\delta)}[|\Psi^\varepsilon_t(x) - \Psi^\varepsilon_s(x)|^P] \leq C^P.C'(1-\delta)^{-1}|t-s|^{2-\frac{3}{P}}.$$

Hence the family of laws $\{P^{(\epsilon,\delta)}\}$ is tight for any $\delta > 0$. Since δ is arbitrary, we see that the family of laws of $\Psi^\varepsilon_t, \varepsilon > 0$ is tight. Now consider the nontruncated case. We see, as before, that the laws of Ψ^ε_t converge weakly with respect to the weak topology of $W_{o,p}$. Hence they are tight in the weak topology of $W_{o,p}$. We can prove the tightness of Ψ^ε_t with respect to the weak topology of $W_m, m \leq k-4$ in the same manner.

Remark 3.11.2. In the mixing case the tightness assumption is symmetric with respect to the forward and backward cases. Hence the tightness and weak convergence of inverse flows are always valid.

Remark 3.11.3. The limit of Ψ^ε_t is unique and it coincides with the inverse of the limit of ϕ^ε_t.

NOTES ON REFERENCES

Chapter 1

[1.1]The study of N-point processes was initiated by T.E.Harris [9] and P. Baxendale [1].

[1.2] The local characteristics of Brownian flow was introduced by Le Jan [23].

[1.3] and [1.4]. Most materials are taken from Kunita [18] and are adapted to this context.

Chapter 2

[2.3] Stochastic integral based on C-semimartingale is due to Le Jan [23] and Le Jan-Watanabe [24]. Also see Borkar [4] for related problems.

[2.5] The infinitesimal generators of stochastic flows were obtained by [23], [24] and Fujiwara-Kunita [8].

[2.6] This section is adapted from [18].

[2.7] Appendix: Generalized Ito formula presented here is an improvement of the same titled formula in [18]. Conditions imposed here are much simpler.

Chapter 3

[3.1] Ideas of using Sobolev spaces for the tightness of measures originated from Kushner [22]. See also Ikeda-Watanabe [13].

[3.4] Moment inequalities in this section are some modification of Kunita [20]. The method of introducing truncated process is due to Kestern-Papanicolaou [15].

[3.5] The weak convergence of $(N + M)$-point processes is suggested by [15]. Lemma 3.5.2 is originated from Khasminskii [16].

[3.6–3.9] The arguments are adapted from [20].

[3.10] Mixing lemmas given here are analogues of those in [15], **118**
although in [15] the forms are apparently different.

[3.11] The tightness of inverse flow is adapted from [20].

Bibliography

[1] P. Baxendale: Brownian motions in the diffeomorphisms Group I. Composito Mathematica **53** (1984), 19–50.

[2] P. Billingsley: Convergence of probability measures, John Wiley and Sons, New York, 1968.

[3] J. Bismut: Mechanique Aleatoire, Lecture Notes in Math. 866, Springer-Verlag, Berlin, Heidelberg, New York, 1981.

[4] V.S. Borkar: Evolutions of interacting particles in a Brownian medium, Stochastics, 1984, Vol 14, pp. 33–79.

[5] A.V.Borodin: A limit theorem for solutions of differential equations with random right hand side, Theory Probab. Appl. **22** (1977) 482–497.

[6] R.M.Dowell: Differentiable approximations to Brownian motion on manifolds, Ph.D. Thesis, University of Warwick, Mathematics Institute, Coventy, 1980.

[7] D. Elworthy: Stochastic dynamical systems and their flows, Stochastic analysis ed. by A. Friedman and M. Pinsky, 79–95, Academic Press, New York, 1978.

[8] T. Fujiwara-H. Kunita: Stochastic differential equations of jump type and *Lé* vy processes in diffeomorphisms group, Kyoto Math. J. **25** (1985), 71–106.

[9] T.E. Harris: Brownian motions on the diffeomorphisms of the plane, Ann. Prob. **9** (1981), 232–254.

[10] T.E.Harris: Coalescing and non-coalescing stochastic flows in $\mathbb{R}^1$, Stochastic Proc. Appl. **17** (1984), 187–210.

[11] J.A.Ibragimov - Yu. V. Linnik: Independent and stationary sequences of random variables, Groningen, Wolter-Noordhoff, 1971.

120 [12] N. Ikeda - S. Nakao - Y. Yamato: A class of approximations of Brownian motion, Publ. RIMS, Kyoto Univ. **13** (1977), 285–300.

[13] N. Ikeda - S. Watanabe: Stochastic differetial equations and diffusion processes, North-Holland/Kodansha, 1981.

[14] N. Ikeda - S. Watanabe, Stochastic flow of diffeomorphisms, Stochastic analysis and appl. edited by M.A. Pinsky, 179–198, (1984).

[15] H. Kesten - G.C. Papanicolaou: A limit theorem for turbulent diffusion, Commun. Math. Phys. **65** (1979), 97–128.

[16] R.Z.Khasminkii: Principle of averaging for parabolic and elliptic differential equations and for Markov processes with small diffusion, Theory Probab. Appl. **8** (1963), 1–21.

[17] R.Z. Khasminskii: A limit theorem for solutions of differential equations with random right hand sides, Theory Probab. Appl. **11** (1966), 390–406.

[18] H. Kunita: Stochastic differential equations and stochastic flow of diffeomorphisms, Lecture Notes in Math. 1099 (1984), 144–303.

[19] H. Kunita: On the convergence of solutions of stochastic ordinary differential equations as stochastic flows of diffeomorphisms, Osaka J. Math. **21** (1984), 883–991.

[20] H. Kunita: Convergence of stochastic flows connected with stochastic ordinary differential equations, stochastics, to appear.

[21] H. Kunita: Limit theorems for stochastic differential equations and stochastic flows of diffeomorphisms, Lecture Notes in Control and Information sciences, **78** (1985), Springer - Verlag.

[22] H. Kushner: An application of Sobolev imbedding theorems to criteria for the continuity of the processes with a vector parameter, The Annals of Math. Statist. **40**, 517–528 (1964).

[23] Y. Le Jan: Flots de diffusions dans IR^d, C.R. Acad. Sci Paris 294 **121**
(1982), Serie **1**, 697–699.

[24] Y. Le Jan - S. Watanabe: Stochastic flows of diffeomorphisms, Taniguchi Symp. S.A. Katata, 1982, 307–332.

[25] P. Malliavin: Stochastic calculus of variations and hypoelliptic operators, Proc. of Inter. Symp. SDE Kyoto, 1976, Kinokuniya, Tokyo, 1978.

[26] H. Matsumoto: Convergence of driven flows of diffeomorphisms, Stochastics.

[27] E.J.McShane: Stochastic calculus and stochastic models, AP, New York, 1974.

[28] G.C. Papanicolaou - W. Kohler: Asymptotic theory of mixing stochastic ordinary differential equations, Comm. Pure. Appl. Math. **27** (1974), 641–668.

[29] C.G.Papanicolaou - D.W. Stroock - S.R.S. Varadhan: Martingale approach to some limit theorems, 1976 Duke Turbulence Conf., Duke Univ. Math Series III, 1977.

[30] S.L.Sobolev: Application of functional analysis in mathematical physics, Amer. Math. Society (1963).

[31] D.W. Stroock: Topics in stochastic differential equations, Tata Institute of Fundamental Research, Bombay 1982.

[32] D.W. Stroock - S.R.S. Varadhan: Multidimensional diffusion processes, Springer - Verlag, Berlin, 1979.

[33] S. Watanabe: Stochastic flow of diffeomorphisms, Proc. fourth Japan U.S.S.R. Symp. Probab. Theory, Lecture Notes in Math. **1021** (1983), 690–698.

[34] E. Wong - M. Zakai: On the relation between ordinary and stochastic differntial equations, Inter. J. Engng. Sci **31** (1965), 213–229.