Lectures on
Stochastic Differential Equations
and Malliavin Calculus

By

S. Watanabe

Tata Institute of Fundamental Research
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1984

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Stochastic Differential Equations
and Malliavin Calculus

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under the

T.I.F.R.–I.I.Sc. Programme in Applications of Mathematics

Notes by

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Preface

These notes are based on six-week lectures given at T.I.F.R. Centre, Indian Institute of Science, Bangalore, during February to April, 1983. My main purpose in these lectures was to study solutions of stochastic differential equations as Wiener functionals and apply to them some infinite dimensional functional analysis. This idea was due to P. Malliavin. In the first part, I gave a calculus for Wiener functionals, which may be of some independent interest. In the second part, an application of this calculus to solutions of stochastic differential equations is given, the main results of which are due to Malliavin, Kusuoka and Stroock. I had no time to consider another approach due to Bismut, in which more applications to filtering theory and the regularity of boundary semigroups of diffusions are discussed.

I would like to thank M. Gopalan Nair and B. Rajeev for their efforts in completing these notes. Also I would like to express my gratitude to Professor K.G. Ramanathan and T.I.F.R. for giving me this opportunity to visit India.

S. Watanabe

Introduction

Let W_o^r be the space of all continuous functions $w = (w^k(t))_{k=1}^r$ from 1
$[o, T]$ to $\mathbb{R}^r$, which vanish at zero. Under the supremum norm, W_o^r is a Banach space. Let P be the r-dimensional Wiener measure on W_o^r. The pair (W_o^r, P) is usually called (r-dimensional) Wiener space.

Let A be a second order differential operator on $\mathbb{R}^d$ of the following form:

$$A = \frac{1}{2} \sum_{i,j=1}^{d} a^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{d} b^i(x) \frac{\partial}{\partial x^i} + c(x). \tag{0.1}$$

where $a^{ij}(x)) \geq 0$, i.e., non-negative definite and symmetric.

Now, let

$$a^{ij}(x) = \sum_{k=1}^{r} \sigma_k^i(x) \sigma_k^j(x)$$

and consider the stochastic differential equation

$$d\chi^i(t) = \sum_{k=1}^{r} \sigma_k^i(X(t)) dW^k(t) + b^i(X(t) dt, i = 1, 2, \ldots, d, \tag{0.2}$$
$$X(o) = x, x \epsilon \mathbb{R}^d.$$

We know if the coefficients are sufficiently smooth, a unique solution exists for the above SDE and a global solution exists if the coefficients have bounded derivative.

Let $X(t, x, w)$ be the solution of (0.2). Then $t \to X(t, x, w)$ is a sample path of A_o-diffusion process, where $A_o = A - c(x)$. The map $x \to X(t, x, w)$, for fixed t and w from $\mathbb{R}^d$ to $\mathbb{R}^d$ is a diffeomorphism (stochastic flow of diffeomorphisms), if the coefficient are sufficiently smooth. And the map $w \to X(t, x, w)$, for fixed t and x, is a Wiener functional, i.e., a measurable function from W^r_o to $\mathbb{R}^d$. 2

Consider the following integral on the Wiener space:

$$u(t,x) = E\left[\exp\left\{\int_o^T c(X(s,x,w))ds\right\}.f(X(t,x,w))\right] \tag{0.3}$$

where both f and c are smooth functions on $\mathbb{R}^d$ with polynomial growth order and $c(x) \leq M < \infty$. Then $u(t, x)$ satisfies

$$\begin{aligned} \frac{\partial u}{\partial t} &= Au \\ u|_{t=0} &= f \end{aligned} \tag{0.4}$$

and any solution of this initial value problem (0.4) with polynomial growth order coincides with $u(t, x)$ given by (0.3).

Suppose we take formally $f(x) = \delta_y(x)$, the Dirac δ-function at $y \epsilon \mathbb{R}^d$ and set

$$p(t,x,y) = E\left[\exp\left\{\int_o^t c(X(s,x,w))ds\right\}\delta_y(X(t,x,w))\right]; \tag{0.5}$$

then we would have

$$u(t,x) = \int_{\mathbb{R}^d} p(t,x,y)f(y)dy$$

and $p(t, x, y)$ would be the fundamental solution of (0.4). (0.5) is thus a formal expression for the fundamental solution of (0.4), often used intuitively, but $\delta_y(X(t, x, w))$ has no meaning as a Wiener functional. The purpose of these lectures is to give a correct mathematical meaning to
3 the formal expression $\delta_y(t, x, w))$ by using concepts like 'integration by parts on Wiener space', so that the existence and smoothness of the fundamental solution, or the transition probability density for (0.3), can be assured through (0.5). This is a way of presenting *Malliavin's calculus*, an infinite dimensional differential calculus, introduced by Malliavin with the purpose of applications to problems of partial differential equations like (0.4).

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Chapter 1

Calculus of Wiener Functionals

1.1 Abstract Wiener Space

Let W be a separable Banach space and let $B(W)$ be the Borel field, i.e., 5
topological σ-field. Let $\overset{*}{W}$ be the dual of W.

Definition 1.1. *A probability measure μ on $(W, B(W))$ is said to be a* Gaussian measure *if the following is satisfied:*

For every n and $\ell_1, \ell_2, \ldots, \ell_n$ in $\overset{*}{W}$, $\ell_1(W), \ell_2(W), \ldots, \ell_n(w)$, as random variables on $(W, B(W), \mu)$ are Gaussian distributed i.e., $\exists\ V = (v_{ij})_i^n$, $j = 1$ and $m \epsilon \mathbb{R}^n$ such that $(v_{ij}) \geq 0$ and symmetric and for every $c = (c_1, c_2, \ldots, c_n) \epsilon \mathbb{R}^n$,

$$\int\limits_W \exp\left\{\sum_{i=1}^{d} \sqrt{-1} c_i \ell_i(w)\right\} \mu(dw) = \exp\left\{\sqrt{-1} < m, c > -\frac{1}{2} < Vc, c >\right\}$$

where $< ., . >$ denotes the $\mathbb{R}^n$ -inner product.

We say that μ is a *mean zero Gaussian measure* if $m = 0$, or equivalently,

$$\int\limits_W \ell(w)\mu(dw) = 0 \quad \text{for every} \quad \ell \epsilon \overset{*}{W}.$$

Let $S(\mu)$ denote the support of μ. For Gaussian measure, $S(\mu)$ is a closed linear subspace of W and hence without loss of generality, we can assume $S(\mu) = W$ (otherwise, we can restrict the analysis to $S(\mu)$).

6 **Theorem 1.1.** *Given a mean zero Gaussian measure μ on $(W, B(W))$, there exists a unique separable Hilbert space $H \subset W$ such that the inclusion map i: $H \to W$ is continuous, $i(H)$ is dense in W and*

$$\int_W e^{\sqrt{-1}\ell(w)}\mu(dw) = e^{-\frac{1}{2}|\ell|_H^2} \tag{1.1}$$

where $|.|_H$ denotes the Hilbert space H-norm.

Remark 1. $H \subset W$ implies $\overset{*}{W} \subset H^* = H$ and for $h \epsilon H, \ell \epsilon \overset{*}{W}, \ell(h)$ is given by $\ell(h) =< \ell, h >_H$.

Remark 2. Condition (1.1) is equivalent to

$$\int_W \ell(w)\ell'(w)\mu(dw) =< \ell, \ell' >_H \text{ for every } \ell, \ell' \epsilon \overset{*}{W}. \tag{1.1)$'$ (}$$

Remark 3. The triple (W, H, μ) is called an *abstract Wiener space*.

Sketch of proof of Theorem 1.1: Uniqueness follows from the the fact that $H = \overline{W^*}^{|.|_H}$.

Existence: By definition of Gaussian measure, $\overset{*}{W} \subset L_2(\mu)$. Let $\tilde{H}$ be the completion of $\overset{*}{W}$ under L_2-norm. Let $j : \overset{*}{W} 3 \ell \to \ell(w) \epsilon \tilde{H}$; then j is one-one linear, continuous and has dense range. The continuity of j follows from the fact that (Fernique's theorem): there exists $\alpha > o$ such that

$$\int_W e^{\alpha \|w\|^2}\mu(dw) < \infty.$$

Now consider j^*, the dual map of j,

$$j^* : \tilde{H}^* = \tilde{H} \to W^{**} \supset W.$$

7 It can be shown that $j^*(\tilde{H}) \subset \omega$. Take $H = j^*(\tilde{H})$ and for $\overline{f}, \overline{h}$ in H, define

$$< \overline{f}, \overline{h} >=< f, h > \text{ where } \overline{f} = j^*(f), \overline{h} = j^*(h).$$

Example 1.1 (Wiener space)**.** Let $W = W_o^r$ and μ : r- dimensional Wiener measure.

$H = \{h = (h^i(t))_{i=1}^r \epsilon W_o^r : h^i(t)$ are absolutely continuous on $[o, T]$ with square integrable derivative $\dot{h}^i(t), 1 \leq i \leq r\}$

For $h = (h^i(t))_{i=1}^r, g = (g^i(t))_{i=1}^r$, define the inner product

$$< h, g >= \sum_{i=1}^{r} \int\limits_{o}^{T} \dot{h}^i(s) \dot{g}^i(s) ds.$$

Then H is a separable Hilbert space and (W, H, μ) is an abstract Wiener space which is called *r-dimensional Wiener space*.

Example 1.2. Let I be a compact interval in $\mathbb{R}^d$ and

$$K(x, y) = (k^{ij}(x, y))_{i,j=1}^r$$

where $k^{ij}(x, y) \epsilon C^{2m}(I \times I)$, and satisfies the following conditions:

(i) $k^{ij}(x, y) = k^{ij}(y, x) \ \forall \ x, y \epsilon I \ 1 \leq i, j \leq r$.

(ii) For any $c_{ik} \epsilon \mathbb{R}$, $i = 1, w, \cdots, r$, $k = 1, 2, \ldots, n$, $n \epsilon N$, $\sum\limits_{k,\ell=1}^{n} \sum\limits_{i,j=1}^{r} k^{ij}$ $(x_k, x_\ell) c_{ik} c_{j\ell} \geq 0, \ \forall \ x_k \epsilon I, \ k = 1, 2, \ldots, n$.

(iii) for $|\alpha| = m$, there exists $o < \delta \leq 1$ and $c > o$ such that

$$\sum_{i=1}^{r} \left[k^{(\alpha)ii}(x, x) + k^{(\alpha)ii}(y, y) - 2k^{(\alpha)ii}(x, y) \right] \leq c|x - y|^{2\delta}$$

where $$k^{(\alpha)ij}(xy) = D_x^\alpha D_y^\alpha k^{ij}(x, y).$$ 8

(As usual, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ is a multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_d$ and

$$D_x^\alpha = \frac{\partial|\alpha|}{\partial^\alpha x_1^{\alpha_1} \cdots \partial^\alpha x_d^{\alpha_d}}.$$

Now, for $f \epsilon C^m(I \to R^r)$, $f = (f^1, f^1, \ldots, f^r)$, define

$$||f||_{m,\epsilon} = \sum_{i=1}^{r} \sum_{|\alpha| \leq m} ||D^\alpha f^i||_\epsilon,$$

where

$$||f^i||_\epsilon = \max_{x \epsilon I} |f^i(x)| + \sup_{\substack{x \neq y \\ x,y \epsilon I}} \frac{|f^i(x) - f^i(y)|}{|x-y|^\epsilon}$$

Let

$$C^{m,\epsilon}(I \to \mathbb{R}^r) = \{w \epsilon C^m(I \to \mathbb{R}^r) : ||w||_{m,\epsilon} < \infty\}.$$

$W = (C^{m,\epsilon}, ||.||_{m,\epsilon})$ is a Banach space.

Fact. *For any $\epsilon, o \leq \epsilon < \delta, \exists$ a mean zero Gaussian measure on W such that*

$$\int_W w^i(x) w^j(x) \mu(dw) = k^{ij}(x,y) i, j = 1, 2, \ldots, r.$$

Then by theorem 1.1 it follows that there exists a Hilbert space $H \subset W$ such that (W, H, μ) is an abstract Wiener space. In this case, H is the reproducing kernel Hilbert space associated with the kernel K, which is defined as follows:

9 For $x = (x_1, x_2, \ldots, x_n)$, $x_k \epsilon I$, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$,

$$\lambda_k = \left(\lambda_k^i\right)_{i=1}^r \epsilon \mathbb{R}^r, \text{ define } W_{[x,\lambda]}(y) = (W^i_{[x,\lambda]}(y))_{i=1}^r$$

by

$$W^i_{[x,\lambda]}(y) = \sum_{j=1}^{r} \sum_{k=1}^{n} k^{ij}(y, x_k).\lambda_k^i,$$

and let

$$S = \Big\{ W_{[x,\lambda]} : x = (x_1, x_2, \ldots, x_n), x_k \epsilon I, \lambda = (\lambda_1, \ldots, \lambda_n), \lambda_k = (\lambda_k^i)_{r=1}^r \epsilon \mathbb{R}^r \text{ and } n \epsilon \mathbb{N} \Big\}.$$

For $W_{[x,\lambda]}, W_{y,\nu} \epsilon S$, when $x = (x_1, x_2, \ldots, x_{n_1})$, $\lambda = (\lambda_1, \ldots, \lambda_{n_1})$, $y = (y_1, \ldots, y_{n_2})$, $\nu = (\nu, \ldots, \nu_{n_2})$, define the inner product by

$$< W_{[x,\lambda]}, W_{x,\nu} >= \sum_{k=1}^{n_1} \sum_{\ell=1}^{n_2} \sum_{i,j=1}^{r} k^{ij}(x_k, y_\ell)\lambda_k^i \nu_\ell^j;$$

then $(S, < .,. >)$ is an inner product space and the reproducing kernel Hilbert space H is the completion of S under this inner product.

1.2 Einstein-Uhlenbeck Operators and Semigroups

Let (W, H, μ) be an abstract Wiener space and $(S, B(S))$ a measurable space. A map $x : W \to S$ is called an *S-valued Wiener functional,* if it is $B(W)|B(S)$-measurable. Two S-valued Wiener functionals x, y are said to be equal and denoted by $x = y$ if $x(w) = y(w)\, a.a.w\,(\mu)$. For the moment, we consider mainly the case $S = \mathbb{R}$.

Notation $L_p = L_p(W, B(W), \mu), 1 \leq p < \infty.$ **10**

Definition 1.2. *$F : W \to \mathbb{R}$ is a polynomial, if $\exists$ $n \epsilon \mathbb{N}$ and $\ell_1, \ell_2, \ldots, \ell_n \epsilon \overset{*}{W}$ and $p(x_1, \ldots, x_n)$, a real* polynomial *in n variables such that*

$$F(w) = p(\ell_1(w), \ell_2(w), \ldots, \ell_n(w)) \; \forall \; w \epsilon W.$$

In this expression of F, we can always assume that $\{\ell_i\}_{i=1}^n$ is an ONS in the sense defined below. We define degree (F) = degree (P) which is clearly independent of the choice of $\{\ell_i\}$. We denote by $\mathcal{P}$ the set of such polynomial and by $\mathcal{P}_n$ the set of polynomial of degree $\leq n$.

Fact. $\mathcal{P} \subset L_p, 1 \leq p < \infty$ *and the inclusion is dense*

Definition 1.3. *A finite or infinite collection $\{\ell_i\}$ of elements in $\overset{*}{W}$ is said to be an* orthonormal system (ONS) *if $< \ell_i, \ell_j >_H = \delta_{ij}$. It is said to be an* orthonormal basis (ONB) *if it is an ONS and $L(\ell_1, \ell_2, \ldots)^{|.|_H} = H$, where $L(\ell_1, \ell_2, \ldots)$ is the linear span of $(\ell_1, \ell_2, \ldots)$.*

Decomposition of L_2: We now represent L_2 as an infinite direct sum of subspaces and this decomposition is called the *Wiener-Chaos decomposition* or the *Wiener-Ito decomposition*.

Let $C_o = \{$ constants $\}$

Suppose $C_o, C_1, \ldots, C_{n-1}$ are defined. Then we define C_n as follows:

$$C_n = \bar{\mathcal{P}}_n^{\| \ \|L_2} \ominus [C_o \oplus C_1 \oplus \cdots \oplus C_{n-1}]$$

11 i.e., C_n is the orthogonal complement of $C_o \oplus \cdots \oplus C_{n-1}$ in $\bar{\mathcal{P}}_n\| \ \|L_2$. Since $\mathcal{P}$ is dense in L_2, it follows that

$$L_2 = C_o \oplus C_1 \cdots \oplus C_n \oplus \cdots$$

Hermite Polynomials: The Hermite polynomials are defined as

$$H_n(x) = \frac{(-1)^n}{n!} e^{x^2/2} \frac{d^n}{dx^n}(e^{-x^2/2}), n = 0, 1, 2, \ldots$$

They have the following properties:

1. $H_o(x) = 1$
2. $\sum\limits_{n=o}^{\infty} t^n H_n(x) = e^{-(t^2/2)+tx}$
3. $\dfrac{d}{dx} H_n(x) = H_{n-1}(x)$
4. $\int\limits_{\mathbb{R}} H_n(x) H_m(x) \dfrac{1}{\sqrt{(2\pi)}} e^{-x^2/2} dx = \dfrac{1}{n!}\delta_{n,m}$.

Let $\Lambda = \{a = (a_1, a_2, \ldots) | \, a_i \epsilon z^+, a_i = 0$ expect for a finite numbers of $i's\}$.

For $a\epsilon\Lambda$, $a! \triangleq \prod\limits_i (a_i!)$, $|a| \triangleq \sum\limits_i a_i$. Let us fix an $ONB(\ell_1, \ell_2, \ldots)$ in $\overset{*}{W}$. Then for $a\epsilon\Lambda$, we define

$$H_a(w) \triangleq \prod_{i=1}^{\infty} H_{a_i}(\ell_i(w)).$$

Since $H_o(x) \equiv 1$ and $a_i = 0$ expect for a finite number of $i's$, the above product is well defined. We note that $H_a(.)\epsilon\mathcal{P}_n$ if $|a| \leq n$.

12 **Proposition 1.2.** (i) $\left\{\sqrt{a!}H_a(w) : a\epsilon\Lambda\right\}$ *is an ONB in* L_2.

(ii) $\left\{\sqrt{a!}H_a(w) : a\epsilon\Lambda, |a| = n\right\}$ *is an ONB in* C_n.

Proof. Since $\{\ell_i\}$ is an ONB in $\overset{*}{W}$, $\{\ell_i(w)\}$ are $N(0.1)$, $i.i.d.$ random variables on W. Therefore,

$$\begin{aligned}
\int\limits_W H_a(w)H_b(w)\mu(dw) &= \prod_{i=1}^{\infty}\int\limits_W H_{a_i}(\ell_i(w))H_{b_i}(\ell_i(w))\mu(dw)\\
&= \prod_{i=1}^{\infty}\int\limits_{\mathbb{R}} H_{a_i}(x)H_{b_i}(x)\frac{1}{\sqrt{(2\pi)}}e^{-x^2/2}dx\\
&= \prod_i \frac{1}{a_i!}\delta_{a_i,b_i} = \frac{1}{a!}\delta_{a,b}.
\end{aligned}$$

Since $\mathcal{P}$ is dense in L_2, the system $\left\{\sqrt{a!}H_a(w); a\epsilon\Lambda\right\}$ is complete in L_2. □

Let J_n denote the orthogonal projection from L_2 to C_n. Then for $F\epsilon L_2$, we have $F = \sum\limits_n J_nF$. In particular, if $F\epsilon\mathcal{P}$, then the above sum is finite and $J_nF\epsilon\mathcal{P}$, $\forall\, n$.

Definition 1.4. *The function* $F : W \to \mathbb{R}$ *is said to be a smooth functional, if* $\exists\, n\epsilon\mathbb{N}, \ell_1\ell_2,\dots,\ell_n\epsilon\overset{*}{W}$, *and* $f\epsilon C^\infty(\mathbb{R}^n)$, *with polynomial growth order of all derivatives of* f, *such that*

$$F(w) = f(\ell_1(w), \ell_2(w), \dots \ell_n(w))\ \forall\ w\epsilon W.$$

We denote by S the class of all smooth functionals on W.

Definition 1.5. *For* $F(w)\epsilon S$ *and* $t \geq o$, *We define* $(T_tF)(w)$ *as follows:* 13

$$(T_tF)(w) \triangleq \int\limits_W F(e^{-t}w + \sqrt{(1 - e^{-2t})}u)\mu(du) \tag{1.2}$$

Note (i): If $F\epsilon S$ is given by

$$F(w) = f(\ell_1(w), \ldots \ell_n(w)), \; f\epsilon C^\infty(\mathbb{R}^n)$$

for some ONS $\{\ell_1, \ell_2, \ldots \ell_n\} \subset \overset{*}{W}$, then

$$(T_t F)(w) = \int\limits_{\mathbb{R}^n} f(e^{-t}\xi + \sqrt{(1-e^{-2t})}\eta) \frac{1}{(\sqrt{2\pi})^n} e^{-(|\eta|^2)/2} d\eta \tag{1.3}$$

where $\xi = (\ell_1(w), \ldots, \ell_n(w))\epsilon\mathbb{R}^n$.

Note (ii): The above definition can be also be used to define $T_t F$ when $F\epsilon L_p$.

Properties of $T_t F$:

(i) $F\epsilon S \Rightarrow T_t F\epsilon\mathcal{S}$

(ii) $F\epsilon\mathcal{P} \Rightarrow T_t F\epsilon\mathcal{P}$

(iii) For $f, G\epsilon\mathcal{S}$

$$\int\limits_W (T_t F)(w) G(w)\mu(dw) = \int\limits_W F(w)(T_t G)(w)\mu(dw)$$

(iv) $T_{t+s}F(w) = T_t(T_s F)(w)$

(v) If $F\epsilon\mathcal{S}$, $F = \sum\limits_n J_n F$, then

$$T_t F = \sum_n e^{-nt}(J_n F)$$

(vi) T_t is a contraction on L_p, $1 \leq p < \infty$.

14 *Proof.* (i) and (ii) are trivial and (iii) and (iv) follow easily from (v). Hence we prove only (v) and (vi). □

Proof of (v): Let $\ell \in \overset{*}{W}$ and

$$F(w) = E^{\sqrt{-1}}\ell(w) + \frac{1}{2}|\ell|_H^2.$$

Then

$$T_t F(w) = \int_W \exp\left[\sqrt{-1}e^{-t}\ell(w) + \sqrt{-1}\sqrt{(1-e^{-2t})}\ell(u) + \frac{1}{2}|\ell|_H^2\right]\mu(du)$$
$$= e^{\sqrt{-1}}e^{-t}\ell(w) + \frac{1}{2}|\ell|_H^2 \int_W e^{\sqrt{-1}}\sqrt{(1-e^{-2t})}\ell(u)_{\mu(du)}$$
$$= e^{\sqrt{-1}}e^{-t}\ell(w) + \frac{1}{2}e^{e^{2t}|\ell|_H^2}.$$

Let

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N) \in \mathbb{R}^{\vee}, N \in \mathbb{N}$$
$$\ell = \lambda_1\ell_1 + \cdots + \lambda_N\ell_N, \{\ell_i\}_{i=1}^N \text{ an ONS }.$$

Let

$$F(w) = e^{\sqrt{-1}}\ell(w) + \frac{1}{2}|\ell|_H^2.$$

Then

$$F(w) = \prod_{i=1}^N e^{\sqrt{-1}}\lambda_i\ell_i(w) - \frac{1}{2}(\sqrt{-1}\lambda_i)^2$$
$$= \sum_{m_1,\ldots,m_N=0}^{\infty} (\sqrt{-1}\lambda_1)^{m_1}\cdots(\sqrt{-1}\lambda_N)^{m_N} \times H_{m_1}(\ell_1(w))\cdots H_{m_N}(\ell_N(w)).$$

Applying T_t to both sides of the above equation, we have

$$e^{\sqrt{-1}}e^{-t}\ell(w) + \frac{1}{2}e^{2t}|\ell|_H^2 = T_tF(w) = \sum_{m_1,\ldots m_N=0}^{\infty} (\sqrt{-1}\lambda_1)^{m_1}\ldots(\sqrt{-1}\lambda_N)^{m_N}$$
$$\times T_t\left(\prod_{i=1}^{m_N} H_{m_i}(\ell_i(.))\right)(w).$$

15

Hence

$$T_t\left(\prod_{i=1}^{N} H_{m_i}(\ell_i(.))\right)(w) = \prod_{i=1}^{N} e^{-tm_i} H_{m_i}(\ell_i(w))$$
$$= e^{-t}\sum_{i=1}^{N} m_i \prod_{i=1}^{N} H_{m_i}(\ell_i)(w))$$

implies

$$(T_t H_a)(w) = e^{-|a|t} H_a(w).$$

If $P\epsilon P$, then $F = \sum_n J_n F$ where $J_n F \epsilon C_n$. Then since

$$\left\{\sqrt{a!}H_a(w) : a\epsilon\wedge, |a| = n\right\}$$

is an ONB for C_n, we finally have

$$(T_t F)(w) = \sum_n e^{-nt}(J_n F)(w).$$

Proof of (vi): Let $P_t(w, du)$ denote the image measure $\mu \circ \phi_{t,w}^{-1}$ of the map $\phi_{t,w} : W \to W$

$$\phi_{t,w}(u) = e^{-t} w + \sqrt{(}1 - e^{-2t})u.$$

Then

$$(T_t F)(w) = \int P_t(w, du) F(u), F\epsilon L_p.$$

16 First let F be a bounded Borel function on W. Then $F \in L_p$ and

$$\|T_t F\|_{L_p}^{p} = \left\{\int_W |\int_W P_t(w, du) F(u)|_\mu^P (dw)\right\}$$
$$\leq \left\{\int_W |\int_W P_t(w, du) F(u)|_\mu^P (dw)\right\}$$
$$=< 1, T_t(|F|^P) >_{L_2}$$

$$=< 1, |F|^P >_{L_2} \ (\because T_t 1 = 1)$$
$$= ||F||_P^p.$$

Hence $||T_t F||_{L_p} \leq ||F||_{L_p}$ holds for any bounded Borel function F. In the general case, for any $F \in L_p$, we choose F_n, bounded Borel functions, such that $F_n \rightarrow F$ in L_p. Then

$$||T_t F_n||_{L_p} \leq ||F_n||_{L_p} \ \forall\ n,$$
$$=> ||T_t F||_{L_p} \leq ||F||_{L_p}.$$

Actually T_t has a stronger contraction known as *hyper-contractivity*:

Theorem 1.3 (Nelson)**.** *Let* $1 \leq p < \infty, t > 0$ *and* $q(t) = e^{2t}(p-1)+1 > p$. *Then for* $F \epsilon L_{q(t)}$,

$$||T_t F||_{q(t)} \leq ||F||_p.$$

Remark. The semigroup $\{T_t : t \geq o\}$ is called the *Ornstein - Uhlenbeck Semigroup*.

Some Consequence of the Hyper-Contractivity: **17**

1) $J_n : L_2 \rightarrow C_n$ is a bounded operator on $L_p, 1 < p < \infty$.

Proof. Let $p > 2$. Choose t such that $e^{2t}+1 = p$.Then by Nelson's theorem, we have

$$||T_t F||_p \leq ||F||_2.$$

In particular

$$||T_t J_n F||_p \leq ||J_n F||_2 \leq ||F||_2 \leq ||F||_p.$$

But

$$||T_t J_n F||_p = e^{-nt}||J_n F||_p;$$

hence

$$||J_n F||_p \leq e^{nt}||F||_p.$$

For $1 < p < 2$, Considering the dual map J_n^* of J_n and applying the previous case, we get

$$\|J_n^* F\|_p \leq e^{nt} \|F\|_p.$$

□

But, for $F \in P, J_n^* = J_n$. Hence, by denseness of p, the results follows.

2) Let $V_n = C_0 \oplus \ldots C_1 \oplus C_n (V_n$ are called *Wiener chaos of order n*). Then, for every $1 \leq p, q < \infty, \|.\|_p$ and $\|.\|_p$ are equivalent on V_n, i.e., for every $F \in V_n, \exists\, C_{p,q,n} > 0$ such that

$$\|F\|_q \leq C_{p,q,n} \|F\|_p.$$

18 In particular, for $F \in V_n, \|F\|_p < \infty, 1 < p < \infty$.

Proof. Easy and omitted. □

Definition 1.6 (Ornstein-Uhlenbeck Operator)**.** *We define the generator L of the semigroup T_t, which is called Ornstein-Uhlenbeck Operator, as follows:*

For $F \epsilon P$, define

$$L(F) = \frac{d}{dt} T_t F|_{t=0} = \sum_n (-n) J_n F.$$

Note that L maps polynomials into polynomials. L can also be extended, as an operator on L_P, as the infinitesimal generator of a contraction semigroup on L_P. The extension of L will be given in later sections. In particular, for L_2, let

$$D(L) = \left\{ F \epsilon L_2 : \sum_n \|J_n F\|_2^2 < \infty \right\}$$

and for $F \epsilon D(L)$, define

$$L(F) = \sum_n (-n) J_n F.$$

In it easily seen that L is a self-adjoint operator on L_2.

Definition 1.7 (Fréchet derivative)**.** *For $F\epsilon P$ and $w\epsilon W$, define*

$$DF(w)(u) = \frac{\partial F}{\partial t}(w+tu)|_{t=o}\ \forall\ u\ \epsilon\ W.$$

For each $w\epsilon W$, $DF(w)$, which is called the *Fréchet derivative of F at w*, is a continuous linear functional on W i.e.,

$DF(w)\epsilon \overset{*}{W}$. More precisely, $DF(w)$ is given as follows: 19

Let $\{\ell_i\}$ be an ONS is $\overset{*}{W}$ and $F = p(\ell_1(w), \ldots, \ell_n(w))$, then

$$DF(w)(u) = \sum_{i=1}^{n} \partial_i p(\ell_1(w), \ldots, \ell_n(w)).\ell_i(u),$$

which we can also write as

$$DF(w) = \sum_{i=1}^{n} \partial_i p(\ell_1(w), \ldots, \ell_n(w)).\ell_i.$$

For $F\epsilon P$, the Fréchet derivative at w of order $k > 1$ is defined as

$$D^k F(w)(u_1, u_2, \ldots, u_k) = \frac{\partial^k}{\partial t_1 .. \partial t_k} F(w + t_1 u_1 + \cdots + t_k u_k)|_{t_1 = .. = t_k = 0}$$
$$\text{for } u_i \epsilon W, 1 \leq i \leq k.$$

Explicitly, if $F(w) = p(\ell_1(w), \ldots, \ell_n(w)))$, then

$$D^k F(w) = \sum_{i_1=1}^{n} .. \sum_{i_k=1}^{n} \partial_{i_1}, \partial_{i_2} \cdots \partial_{i_k} P(\ell_1(w), \ell_2(w), \ldots, \ell_n(w))). \times \ell_{i_1}^{\ell} \otimes .. \otimes \ell_{i_k}$$

where

$$\ell_{i_1} \otimes .. \otimes \ell_{i_k}(u_1, u_2, \ldots, u_k) \overset{\Delta}{=} \ell_{i_1}(u_1), \ldots, \ell_{i_k}(u_k).$$

Note that for each w, $D^k F(w)\epsilon \underbrace{\overset{*}{W} \otimes \cdots \otimes \overset{*}{W}}_{k \text{ times}}$ where

$$\underbrace{\overset{*}{W} \otimes \cdots \otimes \overset{*}{W}}_{k \text{ times}} \overset{\Delta}{=} \left\{ V : \underbrace{W x \cdots x W}_{k \text{ times}} \to \mathbb{R} | V \text{ is multilinear and continuous} \right\}.$$

20

Definition 1.8 (Trace Operator)**.** *Let* $\{h_i\}$ *be an* ONB *in* H*. For* $V\epsilon\overset{*}{W}\otimes\overset{*}{W}$ *we define the trace of* V *with respect to* H*, denoted as* $trace_H V$ *by*

$$trace\ _H V = \sum_{i=1}^{\infty} V(h_i, h_i).$$

Note that the definition is independent of the choice of ONB and for $V\epsilon\overset{*}{W}\otimes\overset{*}{W}$, $\text{trace}_H V$ exists and $\text{trace}_H(.)$ is a continuous function on $\overset{*}{W}\otimes\overset{*}{W}$.

Remark. For $\ell_1, \ell_2\epsilon\overset{*}{W}$,

$$\begin{aligned}\text{trace }_H\ell_1\otimes\ell_2 &= \sum_i \ell_1(h_i)\ell_2(h_i) = \sum_i <\ell_1, h_i>_H<\ell_2, h_i>_H\\ &=<\ell_1, \ell_2>_H .\end{aligned}$$

Theorem 1.4. *If* $F\epsilon P$*, then*

$$LF(w) = trace_H D^2 F\ (w) - DF\ (w)\ (w), \quad for\ \ w\ \epsilon\ W. \tag{1.3}$$

Proof. Let $\{\ell_1, \ell_2, \ldots, \ell_n\}$ be an ONS in $\overset{*}{W}$ and

$$F(w) = p(\ell_1(w), \ell_2(w), \ldots, \ell_n(w)).$$ □

By the remark, we see that

$$\begin{aligned}\text{RHS of (1.3)} &= \sum_{i=1}^{n} \partial_i\partial_i p(\ell_1(w), \ldots, \ell_n(w))\\ &\qquad - \sum_{i=1}^{n} \partial_i p(\ell_1(w), \ldots, \ell_n(w)).\ell_i(w).\end{aligned}$$

Now let $\xi = (\ell_1(w), \ldots, \ell_n(w))$, then

$$\frac{d}{dt}T_t F(w) = \frac{d}{dt}\int_{\mathbb{R}^n} p(e^{-t}\xi + \sqrt{(1-e^{-2t}})n)(2\pi)^{-n/2} e^{\frac{-|\eta|^2}{2}} d\eta$$

$$= -\int\limits_{\mathbb{R}^n} \sum_{i=1}^{n} e^{-t}\xi_i \partial_i p(e^{-t}\xi + \sqrt{(1-e^{-2t})}n)(2\pi)^{-n/2} e^{\frac{-|\eta|^2}{2}} d\eta$$

$$+ \int\limits_{\mathbb{R}^n} \sum_{i=1}^{n} \partial_i p(e^{-t}\xi + \sqrt{(1-e^{-2t})}n) \frac{\eta_i e^{-2t}(2\pi)^{-n/2}}{\sqrt{(1-e^{-2t})}} e^{\frac{-|\eta|^2}{2}} d\eta$$

$$- \int\limits_{\mathbb{R}^n} \sum_{i=1}^{n} e^{-t}\xi_i \partial_i p(e^{-t}\xi + \sqrt{(1-e^{-2t})}n)(2\pi)^{-n/2} e^{\frac{-|\eta|^2}{2}} d\eta$$

$$- \int\limits_{\mathbb{R}^n} \sum_{i=1}^{n} \partial_i p(e^{-t}\xi + \sqrt{(1-e^{-2t})}n) \frac{e^{-2t}(2\pi)^{-n/2}}{\sqrt{(1-e^{-2t})}} \times \partial_i (e^{\frac{-|\eta|^2}{2}}) d\eta.$$

21

Integrating the second expression by parts, we get

$$\frac{d}{dt} T_t F(w) = -\sum_{i=1}^{n} \xi_i e^{-t} T_t(\partial_i p)(\xi) + \sum_{i=1}^{n} e^{-2t} T_t(\partial_i^2 p)\xi.$$

Hence we have

$$LF(w) = \lim_{t\to 0} \frac{d}{dt} T_t F(w) = RHS.$$

□

Definition 1.9 (Operator δ). *Let $P_{\overset{*}{W}}$ be the totality of functions $F(w)$: $W \to \overset{*}{W}$ which can be expressed in the form*

$$F(w) = \sum_{i=1}^{n} F_i(w)\ell_i$$

for some $n \epsilon \mathbb{N}$, $\ell_i \epsilon_i \overset{*}{W}$ and $F_i(w) \epsilon p$, $i = 1, 2, \ldots, n$. $F \epsilon P_{\overset{*}{W}}$ is called a $\overset{*}{W}$-valued polynomial. The linear operator $\delta : P_{\overset{*}{W}} \to P_W$ is defined as follows:

$$\text{Let } \ell_1, \ell_2, \ldots, \ell_n, \ell \,\epsilon\, \overset{*}{W} \text{ and}$$
$$F(w) = p(\ell_1(w), \ldots, \ell_n(w))\ell.$$

Define 22

$$\delta F(w) = \sum_{i=1}^{n} \partial_i p(\ell(w), \ldots, \ell_n(w)) < \ell_i, \ell >_H - p(\ell_1(w), \ldots, \ell_n(w))\ell(w)$$

and extend the definition to every $F \epsilon P_{W^*}$ by linearity.

Proposition 1.5. *(i) For every $F \epsilon p$, $\delta(DF) = LF$. More generally if $F_1, F_2 \epsilon p$, then*

$$\delta(F_1.DF_2) =< DF_1, DF_2 >_H + F_1.L(F_2). \tag{1.4}$$

(ii) (Formula for integration by parts)

In $F \epsilon P$ and $G \epsilon P_{W^*}$, then

$$\int_W < G, DF >_H (w)\mu(dw) = - \int_W \delta G(w) F(w) \mu(dw) \tag{1.5}$$

which says that $\delta = -D*$.

Proof. (i) follows easily from definitions. (ii) We may assume

$$G(w) = p(\ell_1(w), \ldots, \ell_n(w))\ell \qquad F(w) = q(\ell_1(w), \ldots, \ell_n(w))$$

where $\{\ell_i\}$ is ONS in $\overset{*}{W}$. Then

$$< G, DF >_H = \sum_{i=1}^{n} (\partial_i q) p < \ell_i, \ell >_H$$

$$\delta G.F = \sum_{i=1}^{n} (\partial_i p).q < \ell_i, \ell >_H - p.q\, \ell(w).$$ □

23 So we have to prove that

$$\int_{\mathbb{R}} \sum_{i=1} n(\partial_i q(\xi)).p(\xi) < \ell_i, \ell >_H e^{\frac{-|\xi|^2}{2}} d\xi$$

$$= - \int_{\mathbb{R}^n} \sum_{i=1}^{n} [(\partial_i p(\xi))q(\xi) < \ell_i, \ell >_H - p(\xi)q(\xi) < \ell_i, \ell > \xi_i] e^{\frac{-|\xi|^2}{2}} d\xi$$

which follows immediately by integrating the *LHS* by parts

Proposition 1.6 (Chain rule)**.** *Let $P(t_1, \ldots, t_n)$ be a polynomial and $F_i \epsilon P$, for $i = 1, 2, \ldots, n$. Let $F = P(F_1, F_2 \ldots F_n)\epsilon P$. Then*

$$DF(w) = \sum_{i=1}^{n} \partial_i P(F_1(w), F_2(w), \ldots, F_n(w)).DF_i(w)$$

and

$$LF(w) = \sum_{i,j=1}^{n} \partial_i \partial_j P(F_1(w), \ldots, F_n(w)). < DF_i DF_j >_H \\ + \sum_{i=1}^{n} \partial_i P(F_1(w) m \ldots, F_n(w)) \times LF_i(w).$$

Proof. Easy. □

1.3 Sobolev Spaces over the Wiener Space

Definition 1.10. *Let $F \epsilon p, 1 < p < \infty, -\infty < s < \infty$. Then*

$$||F||_{p,s} \overset{\Delta}{=} ||(I - L)^{s/2} F||_p$$

where

$$(I - L)^{s/2} F \overset{\Delta}{=} \sum_{n=0}^{\infty} (1 + n)^{s/2} J_n F \epsilon P.$$

Proposition 1.7. (i) *If $p \leq p'$ and $s \leq s'$, then* **24**

$$||F||_{p,s} \leq ||F||_{p's'} \ \forall \ F \epsilon p.$$

(ii) $\forall\ 1 < p < \infty, -\infty < s < \infty, ||.||_{p,s}$ *are compatible in the sense that if, for any $(p, s), (p', s')$ and $F_n \epsilon p, n = 0, 1, 2, \ldots, ||F_n||_{p,s} \to 0$ and $||F_n - F_m||_{p',s'} \to 0$ as $n, m \to \infty$, then $||F_n||_{p',s'} \to 0$ as $n \to \infty$*

Proof. (i) Since, for fixed $s, ||F||_{p,s} \leq ||F||_{p',s'}$ if $p' > p$, it is enough to prove

$$||F||_{p,s} \leq ||F||_{p,s'} \text{ for } s' \geq s.$$

To prove this, it is sufficient to show that for $\alpha > o$,

$$\|(I-L)^{-\alpha}F\|_p \leq \|F\|_p \ \forall\ F\epsilon P.$$

We know that $\|T_t F\|_p \leq \|F\|_p$. From the Wiener-Chaos representation for $T_t F$ and $(I-L)^{-\alpha}F$, we have

$$(I-L)^{-\alpha}F = \frac{1}{\Gamma(\alpha)}\int\limits_o^\infty e^{-t}t^{\alpha-1}T_t F dt.$$

Hence

$$\begin{aligned}\|(I-L)^{-\alpha}F\|_p &\leq \frac{1}{\Gamma(\alpha)}\int\limits_o^\infty e^{-t}t^{\alpha-1}\|T_t F\|_p dt\\ &\leq \|F\|_p\end{aligned}$$

which proves the result.

25 (ii) Let $G_n = (I-L)^{s'/2}F_n\epsilon\mathcal{P}$. Therefore $\|G_n - G_m\|_{p'} \to 0$ as $n, m \to \infty$. Therefore, $\exists\ G\epsilon L_p$, such that $\|G_n - G\|_{p'} \to 0$. But

$$\|F_n\|_{p,s} \to 0 \Rightarrow \|(I-L)^{1/2(s-s')}G_n\|_p \to 0.$$

Enough to show $G = 0$. Let $H\epsilon P$. Then $(I-L)^{1/2(s'-s)}H\epsilon P$. Noting that $P \subset L_q$ for every $1 < q < \infty$, we have

$$\begin{aligned}\int\limits_W G.Hd\mu &= \lim_{n\to\infty}\int\limits_W G_n H d\mu\\ &= \lim_{n\to\infty}\int\limits_W (I-L)^{1/2(s-s')}G_n(I-L)^{1/2(s'-s)}Hd\mu\\ &= 0.\end{aligned}$$

Since $\mathcal{P}$ is hence in $L_p\ \forall\ q, G = 0$.

□

Definition 1.11. *Let* $1 < p < \infty, -\infty < s < \infty$. *Define* $\mathbb{D}_{p,s}$ = *the completion of* $\mathcal{P}$ *by the norm* $\| \ \|_{p,s}$.

Fact. 1) $\mathbb{D}_{p,o} = L_p$.

2) $\mathbb{D}_{p',s'} \hookrightarrow \mathbb{D}_{p,s}$ *if* $p \leq p', s \leq s'$.

Hence we have the following inclusions:

Let $o < \alpha < \beta, o < p < q < \infty$. *Then*

$$\begin{array}{ccccccccc} \mathbb{D}_{p,\beta} & \hookrightarrow & \mathbb{D}_{p,\alpha} & \hookrightarrow & \mathbb{D}_{p,o} = L_p & \hookrightarrow & \mathbb{D}_{p,-\alpha} & \hookrightarrow & \mathbb{D}_{p,-\beta} \\ \circlearrowright & & \circlearrowright & & & & \circlearrowright & & \circlearrowright \\ \mathbb{D}_{q,\beta} & \hookrightarrow & \mathbb{D}_{q,\alpha} & \hookrightarrow & \mathbb{D}_{q,o} = L_q & \hookrightarrow & \mathbb{D}_{q,-\alpha} & \hookrightarrow & \mathbb{D}_{q,-\beta} \end{array}$$

3) *Dual of* $D_{p,s} \equiv D'_{p,s} = D_{q,-s}$ *where* $\dfrac{1}{p} + \dfrac{1}{q} = 1$, *under the standard* **26**
identification $(L_2)' = L_2$.

This follows from the following facts:

Let $A = (I - L)^{-s/2}$. Then the following maps are isometric isomorphisms:

$$A : L_p \to \mathbb{D}_{p,s}$$
$$A : \mathbb{D}_{q,-s} \to L_q$$

and hence

$$\overset{*}{A} : (\mathbb{D}_{p,s})' \to L_q$$

is also an isometric isomorphism if $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Also, from the relation

$$\int\limits_{w} F(w)G(w)\mu(dw) = \int\limits_{W} (I - L)^{s/2} F(w)(I - L)^{-s/2} G(w)\mu(dw),$$

it is easy to see that $\mathbb{D}_{q,-s} \subset (\mathbb{D}_{p,s})'$, isometrically.

Definition 1.12.

$$\mathbb{D}_\infty = \bigcap_{p,s} \mathbb{D}_{p,s}$$
$$\mathbb{D}_{-\infty} = U_{p,s}\mathbb{D}_{p,s}$$
$$(\text{Hence} \quad \mathbb{D}'_\infty = U\mathbb{D}'_{p,s} = \mathbb{D}_{-\infty}.)$$

Thus $\mathbb{D}_\infty$ is a complete countably normed space and $\mathbb{D}_{-\infty}$ is its dual.

Remark. Let $S(\mathbb{R}^d)$ be the Schwartz space of rapidly decreasing C^∞-
27 functions, $H_{p,s}$ the (classical) Sobolev space obtained by completing $S(\mathbb{R}^d)$ by the norm

$$\|f\|_{p,s} = \|(|x|^2 - \triangle)^{s/2} f\|_p,\ f \epsilon S(\mathbb{R}^d)$$

where $\triangle$ denotes the Laplacian. Then it is well-known that

$$\bigcap_{p,s} H_{p,s} = \ = \bigcap_{s} H_{2,s}$$
$$U_{p,s}\ H_{p,s} = \ = U_s\ H_{2,s}.$$

Thus every element in $\bigcap_{p,s} H_{p,s}$ has a continuous modification, actually *a* C^∞ - modification. But in our case, the analogous results are not true.

First, in our case, $\bigcap_s \mathbb{D}_{2,s} \neq \mathbb{D}_\infty$. Secondly, $\exists\ F \epsilon \mathbb{D}_\infty$ which has no continuous modification on W, as the following example shows.

Example 1.3. Let $W = W_o^2 = \left\{ w \epsilon C([0,1] \to \mathbb{R}^2), w(0) = 0 \right\} \mu = P \equiv$ 2 − dim. Wiener measure. Let, for $w = (w_1, w_2) \epsilon W$,

$$F(w) = \frac{1}{2} \left\{ \int_o^1 w_1(s) dw_2(s) - \int_o^1 w_2(s) dw_1(s) \right\}$$

(stochastic area of Levy) where the integrals are in the sense of Itô's stochastic integrals.

Then $F\epsilon C_2 \subset \mathbb{D}_\infty$. But F has no continuous modification: suppose $\exists\ \hat{F}(w)$, continuous and such that $\hat{F}(w) = F(w)\, a.a.\, w(p)$. Let

$$\hat{\hat{F}}(w) = \frac{1}{2}\left[\int_o^1 (w_1(s)\dot{w}_2(s) - w_2(s)\dot{w}_1(s))ds\right]$$

for $w\epsilon C_o^2([0,1] \to \mathbb{R}^2)$. Note that $\hat{\hat{F}}$ has no continuous extension to W_o^2. **28**
On the other hand, we have the following fact: For $\delta > o$,

$$P\left\{|F(w) - \hat{\hat{F}}(\phi)| < \delta \| \|w - \phi\| < \epsilon\right\} \to 1$$

as
$$\epsilon \downarrow o,\ \forall\ \phi\epsilon C_o^2([0,1] \to \mathbb{R}^2).$$

Hence

$$\hat{F} \equiv \hat{\hat{F}} \text{ on } C_o^2([0,1] \to \mathbb{R}^2),\ \text{a contradiction.}$$

Definition 1.13. *Let $F\epsilon P$. Then*

$$D^k F(w)\epsilon \underbrace{W^* \otimes \cdots \otimes W^*}_{K\ times}$$

and we define the Hilbert-Schmidt norm of $D^k F(w)$ as

$$|D^k F(w)|^2_{HS} = \sum_{i_1,\ldots,i_k=o}^{\infty} \left\{D^k F(w)\,[h_{i_1}, \ldots, h_{i_k}]\right\}^2$$

where $\{h_i\}_{i=1}^{\infty}$ is an ONB in H.

Remark. 1) The definition is independent of the ONB chosen.

2) If $k = 1$, then $|DF(w)|^2_{HS} = |DF[w]|^2_H$.

Theorem 1.8 (Meyer)**.** *For $1 < p < \infty, k\epsilon Z^+$, there exist $A_{p,k} > a_{p,k} > 0$ such that*

$$a_{p,k}\| |D^k F|_{HS}\|_p \leq \|F\|_{p,k} \leq A_{p,k}(\|F\|_p + \| |D^k F|_{HS}\|_p) \qquad (1.5)$$

for every $F\epsilon\mathcal{P}$.

Before proving this result, let us consider the analogous result in classical analysis, which can be stated as: 29

For $1 < p < \infty$, there exists $a_p > 0$ such that

$$a_p \|\frac{\partial^2 f}{\partial x_i \partial x_j}\|_p \leq \|\Delta f\|_p, \ \forall \ f \epsilon \mathcal{S}(\mathbb{R}^d), \tag{1.6}$$

where $\mathcal{S}(\mathbb{R}^d)$ denotes the Schwartz class of C^∞ - rapidly decreasing functions.

Proof of (1.6): Let $p = 2$, then

$$\begin{aligned} \|\frac{\partial^2 f}{\partial x_i \partial x_j}\|_2 &= \|\xi_i \xi_j \hat{f}(\xi)\|_2, \text{ where } \hat{f}(\xi) = \int_{\mathbb{R}^d} e^{\sqrt{-1}\xi . x} f(x) dx \\ &\leq C_p \||\xi|^2 \ \hat{f}(\xi)\|_2^2 \\ &= C_p \|\triangle f\|_2. \end{aligned}$$

For the general case, we need Calderon-Zygmund theory of singular integrals or Littlewood-Paley inequalities. We here consider the Littlewood-Paley inequalities.

Consider the semigroups P_t and Q_t defined as follows:

$$P_t = e^{t\triangle},$$

i.e., $$(P_t f)\hat{}(\xi) = e^{-t|\xi|^2} \hat{f}(\xi), \ f \epsilon \mathcal{S}(\mathbb{R}^d)$$

and $$Q_t = e^{-t(-\triangle)^{1/2}}$$

i.e., $$(Q_t f)\hat{}(\xi) = e^{-t|\xi|} \hat{f}(\xi), \ f \epsilon \mathcal{S}(\mathbb{R}^d)$$

where $$\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{\sqrt{-1}\xi . x} f(x) dx.$$

30 The transition from P_t to Q_t is called *subordination of Bochner* and is given by

$$Q_t = \int_o^\infty P_s \mu_t(ds)$$

where μ_t is defined as

$$\int_o^\infty e^{-\lambda s}\mu_t(ds) = e^{-\sqrt{\lambda}t}.$$

Note that Q_t can also be expressed as

$$Q_t f(x) = \int_{\mathbb{R}^d} \frac{c_n t}{(t^2 + |x-y|^2)^{(d+1)/2}} f(y)dy$$

where $$c_n^{-1} = \int_{\mathbb{R}^d} \frac{1}{(1+|y|^2)^{(d+1)/2}} dy.$$

Now, we define *Littlewood-Paley functions* G_f and $G_f \rightarrow$, $f \epsilon \mathcal{S}(\mathbb{R}^d)$ as:

$$G_f(x) = \left[\int_o^\infty t \left\{ |\frac{\partial}{\partial t} Q_t f(x)|^2 + \sum_{i=1}^d Q_t f(x)|^2 \right\} dt \right]^{1/2}$$

and $$G_{f^\rightarrow}(x) = \left[\int_o^\infty \left\{ t|\frac{\partial}{\partial t} Q_t f(x)|^2 \right\} dt \right]^{1/2}.$$

Fact. (Littlewood-Paley Inequalities): *For* $1 < p < \infty, \exists\, o < a_p < A_p$ *such that*

$$a_p \|G_f(x)\|_p \leq \|f\|_p \leq A_p \|G_{f^\rightarrow}(x)\|_p, \ \forall\ f \epsilon \mathcal{S}(\mathbb{R}^d). \tag{1.7}$$

Define the operator R_j by **31**

$$(R_j f)\hat{}(\xi) = \frac{\xi_j}{|\xi|} \hat{f}(\xi)$$

R_j is called the *Riesz transformation*. In particular, when $d = 1$, it is called *Hilbert transform*. It is clear that

$$\frac{\partial^2}{\partial x_j \partial x_j} f(x) = R_i R_j \triangle f(x).$$

Fact. *For* $1 < p < \infty, \exists\, o < a_p < \infty$ *such that*

$$a_p \|R_j f\|_p \leq \|f\|_p. \tag{1.8}$$

Note that (1.6) follows from (1.8). Hence we prove (1.8). We have

$$\begin{aligned}(R_j Q_t f)\widehat{}(\xi) &= \frac{\xi_j}{|\xi|} e^{-t|\xi|} \hat{f}(\xi)\\ &= (Q_t R_j f)\widehat{}(\xi).\end{aligned}$$

Also
$$\sqrt{-1}\frac{\partial}{\partial t} R_j (Q_t f)(x) = \frac{\partial}{\partial x_j} Q_t f(x).$$

Hence we get
$$G_{\vec{R_j f}} \leq G_f,$$
which gives (1.8), by using (1.7). Now, we come to Meyer's theorem.

Proof of theorem 1.8.

Step 1. *Using the* $0 - U$ *semigroup* T_t, *we define* Q_t *by*

$$Q_t = \int_o^\infty T_s \mu_t(ds)$$

where
$$\int_o^\infty e^{-\lambda s} \mu_t(ds) = e^{-\sqrt{\lambda} t}.$$

32 Note that
$$Q_t = \sum_{n=o}^\infty e^{-\sqrt{n} t} J_n.$$

$F \epsilon \mathcal{P}$, we define G_F and ψ_F as follows:

$$G_F(w) = \left[\int_o^\infty t\left(\frac{\partial}{\partial t} Q_t F(w)\right)^2 dt\right]^{1/2}$$

and $$\psi_F(w) = \left[\int_0^\infty \left\{T_t(< DT_tF, DT_tF >_H^{1/2})(w)\right\}^2 dt\right]^{1/2}.$$

Then the following are true:
For $1 < p < \infty, \exists\, o < c_p < C_p < \infty$ such that

$$c_p\|F\|_p \leq \|G_F\|_p \leq C_p\|F\|_p,$$
$$c_p\|F\|_p \leq \|\psi_F\|_p \leq C_p\|F\|_p,\ \forall\ F\epsilon\mathcal{P} \text{ such that } J_oF = 0. \tag{1.9}$$

Proof. Omitted. □

Step 2 (An L_p-multiplier theorem)**.** *A linear operator $T_\phi : \mathcal{P} \to \mathcal{P}$ is said to be given by a multiplier $\phi = (\phi(n))$, if*

$$T_\phi F = \sum_{n=1}^{\infty} \phi(n) J_n F,\ \forall\ F\epsilon\mathcal{P}.$$

Note that the operators T_t, Q_t and L are given by the multipliers 33
$e^{nt}, e^{-\sqrt{n}t}$ and $(-n)$ respectively.

Fact. (Meyer-Shigekawa): *If $\phi(n) = \sum_{k=o}^{\infty} a_k \left(\frac{1}{n^\alpha}\right)^k$, $\alpha \geq o$ for $n \geq n_o$ for some n_o and $\sum_{k=o}^{\infty} |a_k| \left(\frac{1}{n_o^\alpha}\right)^k < \infty$, then $\exists\, c_p$ such that*

$$\|T_\phi F\|_p \leq c_p\|F\|_p,\ \forall\ F\epsilon\mathcal{P}. \tag{1.10}$$

Note that the hypothesis in the above fact is equivalent to: there exists $h(x)$ analytic, i.e., $h(x) = \sum a_k x^k$, near zero such that

$$\phi(n) = h\left(\frac{1}{n^\alpha}\right) \text{ for } n \geq n_o.$$

Proof of (1.10): First, we consider the case $\alpha = 1$. We have

$$T_\phi = \sum_{n=o}^{n_o-1} \phi(n) J_n + \sum_{n=n_o}^{\infty} \phi(n) J_n$$

$$= T_\phi^{(1)} + T_\phi^{(2)}.$$

We know that $T_\phi^{(1)}$ is L_p-bounded as a consequence of hyper contractivity, i.e.,

$$\|T^{(1)}F\|_p \leq c_p\|F\|_p.$$

Hence it is enough to show that

$$\|T_\phi^{(2)}F\| \leq c_p\|F\|_p.$$

Claim: $$\|T_t(I - J_o - J_1 - \cdots - J_{n_o-1})F\|_p \leq Ce^{-n_o t}\|F\|_p. \qquad (1.11)$$

Let $p > 2$. Choose t_o such that $p = e^{2t_o} + 1$. Then by Nelson's theorem,

$$\begin{aligned}
&\|T_{t_o}T_t(I - J_o - J_1 - \cdots - J_{n_o-1})F\|_p^2 \\
&\qquad \leq \|T_t(I - J_o - J_1 - \cdots - J_{n_o-1})F\|_2^2 \\
&\qquad = \|\sum_{n=n_o}^{\infty} e^{-nt}J_nF\|_2^2 \\
&\qquad = \sum_{n=n_o}^{\infty} e^{-2n_o t}\|J_nF\|_2^2 \\
&\qquad \leq e^{-2n_o t}\|F\|_p^2.
\end{aligned}$$

34 Therefore

$$\|T_t(I - J_o - J_1 - \cdots - J_{n_o-1})F\|_p \leq Ce^{-n_o t}\|F\|_p$$

where $C = e^{n_o t_o}$.

For $1 < p < 2$, the result (1.11) follows by duality. Define

$$R_{n_o} = \int_o^\infty T_t(I - J_o - J_1 - \cdots - J_{n_o-1})dt.$$

From (1.11), we get

$$\|R_{n_o}F\|_p \leq C\frac{1}{n_o}\|F\|_p$$

and it is clear that

$$\begin{aligned} R^2_{n_o}F &= \int\limits_o^\infty \int\limits_o^\infty T_t(I - J_o - J_1 - \cdots - J_{n_o-1})T_s(I - J_o - \cdots - J_{n_o-1})Fdtds \\ &= \int\limits_o^\infty \int\limits_o^\infty T_{t+s}(I - J_o - J_1 - \cdots - J_{n_o-1})Fdtds. \end{aligned}$$

Hence

$$\|R^2_{n_o}F\|_p \leq C.\frac{1}{n_o^2}\|F\|_p$$

and repeating this, we get

$$\|R^k_{n_o}F\|_p \leq C.\frac{1}{n_o^k}\|F\|_p.$$

Also, note that if $F \epsilon C_n, n \geq n_o$ **35**

$$\begin{aligned} R_{n_o}F &= \int\limits_o^\infty T_t J_n F dt \\ &= \frac{1}{n} J_n F \end{aligned}$$

and

$$R^k_{n_o}F = \frac{1}{n^k} J_n F.$$

Therefore

$$T^{(2)}_\phi F = \sum_{n=n_o}^\infty \sum_{k=o}^\infty a_k R^k_{n_o} J_n F = \sum_{k=1}^\infty a_k R^k_{n_o} F.$$

Hence

$$\|T^{(2)}_\phi F\|_p \leq U\left(\sum_k |a_k|\left(\frac{1}{n_o}\right)^k\right)\|F\|_p$$

which gives the result.

For the general case, i.e., $o < \alpha < 1$, define

$$Q_t^\alpha = \sum e^{-n^\alpha t} J_n F = \int\limits_o^\infty T_s \mu_t^{(\alpha)}(ds)$$

where

$$\int\limits_o^\infty e^{-\lambda s} \mu_t^{(\alpha)}(ds) = e^{-\lambda^\alpha t}.$$

As in the case $\alpha = 1$, write

$$T_\phi = T_\phi^{(1)} + T_\phi^{(2)}.$$

In this case also, we see that $T_\phi^{(1)}$ is L_p - bounded. Using (1.11),

$$\begin{aligned} \|Q_t^{(\alpha)}(I - J_o - J_1 - \cdots - J_{n_o-1})F\|_p \\ \leq C \int\limits_o^\infty \|F\|_p e^{-n_o s} \mu_t^{(\alpha)}(ds) \\ = C e^{-n_o^\alpha t} \|F\|_p. \end{aligned}$$

36 Define

$$R_{n_o} = \int\limits_o^\infty Q_t^{(\alpha)}(I - J_o - J_1 - \cdots - J_{n_o-1}) dt$$

and proceeding as in the case $\alpha = 1$, we get that $T_\phi^{(2)}$ is also L_p - bounded. Hence the proof of (1.10).

Remark. (Application of L_p - Multiplier Theorem)

Consider the semigroup $\{Q_t\}_{t \geq o}$. For $F \epsilon \mathcal{P}$, we have

$$Q_t F = \sum_{n=o}^\infty e^{-\sqrt{n} t} J_n F.$$

The generator C of this semigroup is given by

$$CF = \sum_{n=o}^{\infty}(-\sqrt{n})J_nF, F\epsilon\mathcal{P}.$$

If we define $|||.|||_{p,s}$ for $F\epsilon\mathcal{P}$ by

$$|||F|||_{p,s} = ||(I-C)^sF||_{\dot{p}}, 1 < p < \infty, -\infty < s < \infty$$

where $(I-C)^sF = \sum_{n=o}^{\infty}(I+\sqrt{n})^s J_nF$, then $||\ ||_{p,s}$ is equivalent to $|||.|||_{p,s}$, $\forall\ 1 < p < \infty, -\infty < s < \infty$. i.e., $\exists\ a_{p,s}, A_{p,s}, o < a_{p,s} < A_{p,s} < \infty \ni a_{p,s}|||F|||_{p,s} \leq ||F||_{p,s} \leq A_{p,s}|||F|||_{p,s}$.

Proof. Let $T_\phi F = \sum\limits_{n=o}^{\infty}\phi(n)J_nF, F\epsilon\mathcal{P}$, where

$$\begin{aligned}\phi(n) &= \left(\frac{1+\sqrt{n}}{\sqrt{1+n}}\right)^s, -\infty < s < \infty\\ &= h\left(\left(\frac{1}{n}\right)^{1/2}\right)\end{aligned}$$

with $h(x) = \left(\frac{1+x}{\sqrt{(1+x^2)}}\right)^s$ which is analytic near the origin. □

Note that $T_\phi^{-1} = T_{\phi^{-1}}$ where $\phi^{-1}(n) = \dfrac{1}{\phi(n)} = h^{-1}\left(\left(\dfrac{1}{n}\right)^{1/2}\right)$ with $h^{-1}(x) = \dfrac{1}{h(x)}$ also analytic near the origin. Thus both $T\phi$ and T_ϕ^{-1} are 37
bounded operators on L_p. Further,

$$(I-C)^sF = (I-L)^{s/2}T_\phi F = T_\phi(I-L)^{s/2}F$$

and
$$(I-L)^{s/2}F = T_\phi^{-1}(I-C)^sF = T_{\phi^{-1}}(I-C)^sF.$$

Hence our result follows easily from the fact that

$$||T_\phi F||_p \leq C_p||F||_p \text{ and } ||T_{\phi^{-1}}F||_p \leq C_p||F||_p.$$

To proceed further, we need the following inequality of Kchinchine.

Kchinchine's Inequality: Let (Ω, F, p) be a probability space. Let $\{\gamma_m(\omega)\}_{m=1}^{\infty}$ be a sequence of i.i.d. random variables on Ω with $P(\gamma_m = 1) = P(\gamma_m = -1) = 1/2$, i.e., $\{\gamma_m(\omega)\}$ is a coin tossing sequence.

a) If $\{a_m\}$ is a sequence of real numbers, then, $\forall\ 1 < p < \infty, \exists\, o < c_p < C_p < \infty$ independent of $\{a_m\}$ such that

$$c_p \left(\sum_{m=1}^{\infty} |a_m|^2 \right)^{p/2} \leq E\left(|\sum_{m=1}^{\infty} a_m \gamma_m(\omega)|^p \right) \leq C_p \left(\sum_{m=1}^{\infty} |a_m|^2 \right)^{p/2}. \tag{1.12}$$

b) If $\{a_{m,m'}\}$ is a (double) sequence of real numbers, then, $\forall\ 1 < p < \infty, \exists\ o < c_p < C_p < \infty$ independent of $\{a_{m,m'}\}$ such that

$$c_p \left(\sum_{m,m'}^{\infty} |a_{m,m'}|^2 \right)^{p/2} \leq E\left[\left\{ \sum_{m'=1}^{\infty} (\sum_{m'=1}^{\infty} a_{m',m}\, \gamma_m(\omega))^2 \right\}^{p/2} \right] \leq C_p \left(\sum_{m,m=1}^{\infty} |a^2_{m,m'}| \right)^{p/2}. \tag{1.13}$$

38 c) Let $((a_{mm'})) \geq o$ i.e., for any finite $m_1 < m_2 < \cdots < m_n$, the matrix $((a_{m_i m_j}))_{1 \leq i,j \leq n}$ is positive definite. Then, $\forall\ 1 < p < \infty, \exists\, o < C_p < c_p < \infty$ independence of $(a_{mm'})$ such that

$$c_p \left(\sum_i a_{ii} \right)^{p/2} \leq E\left[\left(\sum_{i,j} a_{ij} \gamma_i(\omega) \gamma_j(\omega) \right)^{p/2} \right] \leq C_p \left(\sum_i a_{ii} \right)^{p/2}. \tag{1.14}$$

Step 3. *(Extension of L-P inequalities to sequence of functionals).*
Let $F_n \in \mathcal{P}, n = 1, 2, \ldots$ with $J_o F_n = 0$. Then

$$\|\sqrt{\left(\sum_{n=1}^{\infty} (F_n)^2 \right)}\|_p \leq A'_p \|\sqrt{\left(\sum_{n=1}^{\infty} G^2_{F_n} \right)}\|_p,\ \forall\ 1 < p < \infty.$$

Proof. Let $\{\gamma_i(\omega)\}$ be a coin tossing sequence on a probability space (Ω, F, P). □

Let $\chi(\omega, w) = \sum_i \gamma(\omega) F_i(w), \omega \in \Omega_1, w \epsilon W$.

We first consider the case when $F_n \equiv 0$, $\forall\ n \geq N$. (Hence the above sum is finite). Then the general case can be obtained by a limiting argument. By Kchinchine's inequality, $\exists$ constants c_p, C_p independent of w such that

$$\begin{aligned} c_p \left(\sum_i F_i(W)^2 \right)^{p/2} &\leq E|X(\omega, w)|^p \\ &\leq C_p \left(\sum_i F_i(W)^2 \right)^{p/2} \quad \forall\ = w \epsilon W. \end{aligned}$$

Integrating w.r.t. μ, we get

$$\begin{aligned} c_p || \left(\sum_i F_i^2 \right)^{1/2} ||_p^p &\leq E \left\{ ||X(\omega, W)||_p^p \right\} \\ &\leq C_p || \left(\sum_i F_i^2 \right)^{1/2} ||_p^p. \end{aligned} \tag{1.15}$$

But by step 1, we have 39

$$||\chi(\omega, .)||_p \leq A_p ||G_X(\omega, .)||_p\ \forall\ \omega \epsilon \Omega. \tag{1.16}$$

Now

$$\begin{aligned} (G_{\rho(\omega,.)})^2 &= \left[\int_o^\infty t \left[\frac{d}{dt} Q_t \left(\sum_i \gamma_i(\omega) F_i(.) \right) \right]^2 dt \right] \\ &= \sum_{i,j} \gamma_i(\omega) \gamma_j(\omega) a_{ij}, \end{aligned}$$

where

$$a_{ij} = \int_o^\infty t \left(\frac{d}{dt} Q_t F_i \right) \left(\frac{d}{dt} Q_t F_j \right) dt.$$

Also

$$\sum_i a_{ij} = \sum_i \int_o^t t\left(\frac{d}{dt}Q_t F_i\right)^2 dt$$
$$= \sum_i G_{F_i}^2.$$

Then Kchinchine's inequality (c) implies

$$c_p\left(\sum_i G_{F_i}(W)^2\right)^{p/2} \leq E|G_{X(.,w)}|^p$$
$$\leq C_p\left(\sum_i G_{F_i}(W)^2\right)^{p/2}$$

where $o < c_p < C_p < \infty$.

Integrating over μ, we get

$$c_p\|\sqrt{\left(\sum_i G_{F_i}^2\right)}\|_p^p \leq E\|G_{X(.,.)}\|_p^p \leq C_p\|\sqrt{\left(\sum_i G_{F_i}^2\right)}\|_p^p. \tag{1.17}$$

(1.15), (1.16) and (1.17) together prove step 3.

Step 4 (Commutation relations involving D). *Let* $\{\ell_i\}_{i=1}^{\infty} \subset \overset{*}{W} \subset H, \{\ell_i\}$ *an ONB in H. Let* $D_iF = < DF, \ell_i >$, *for* $F \epsilon \mathcal{P}$. *Then* $D_iF \epsilon \mathcal{P}$, $\forall$ *i. Further,*

$$< DF, DF >_H = \sum_i (D_iF)^2 = |DF|_{HS}^2.$$

40

In fact,

$$|D^kF|_{HS}^2 = \sum_{i_1,\ldots,i_k} (D_{i_1}(D_{i_2}(\cdots\cdots(D_{i_k}(F))\cdots))^2.$$

Let

$$T_\phi = \sum_{n=o}^{\infty} \phi(n) J_n,$$

$$T_{\phi+} = \sum_{n=o}^{\infty} \phi(n+1) J_n.$$

Fact. $\forall\ i = 1, 2, \ldots, D_i T_\phi = T_\phi + D_i$.

Proof. We have seen that the set $\left\{\sqrt{a} H_a(w), a \epsilon A\right\}$ is an ONB in L_2. Therefore it suffices to prove

$$D_i T_\phi H_a = T_\phi + D_i H_a,\ \forall\ a \epsilon \Lambda. \qquad \square$$

If $a = (a_1, a_2, \ldots..)$ with $a_i > o$, then let $a(i) = (a_1, a_2, \ldots, a_{i-1}, a_i - 1, a_{i+1}, \ldots)$. From $H_a(w) = \prod_i H_{a_i}(\ell_i(w))$, it can be easily seen that

$$D_i H_a = \begin{cases} H_{a(i)} & \text{if } a_i > o \\ 0 & \text{if } a_i = o \end{cases}$$

Note that, if $|a| = n$,

$$T_\phi H_a = \phi(n) H_a \ (\because\ H_a \epsilon C_n)$$

implies

$$D_i T_\phi H_a = \phi(n) D_i H_a.$$

If $a_i > o$, then $D_i H_a = H_{a(i)}$ where $|a(i)| = n - 1$. Therefore 41

$$\begin{aligned} D_i T_\phi H_a &= \phi(n) H_{a(i)}. \\ &= T_{\phi+} H_{a(i)} = T_\phi + D_i H_a. \end{aligned}$$

If $a_i = 0$, this relation still holds since both sides are zero.

Corollary. *$T_t D_i F = e^t D_i T_t F,\ \forall\ i$ and hence*

$$Q_i D_i F = D_i \int_0^\infty \mu_t(ds) e^s T_s F,\ \forall\ i,\ \forall\ F \epsilon \mathcal{P}.$$

Step 5. *Now we use the previous steps to get the final conclusion.*

In the following c_p, C_p, a_p, A_p are all positive constants which may change in some cases, but which are all independent of the function F.

$1 < p < \infty$ is given and fixed. First we shall prove

$$c_p \| < DF, DF >_H^{1/2} \|_p \leq \|CF\|_p \leq C_p \| < DF, DF >_H^{1/2} \|_p \qquad (1.18)$$

where

$$C = \lim_{t \to o} \frac{Q_t - I}{t} \text{i.e.,} \quad Cf = \sum_n \left(-\sqrt{n}\right) J_n F.$$

From corollary of step 4, we have

$$T_t D_i F = e^t D_i T_t F, \ \forall \ F \epsilon \mathcal{P}.$$

$$T_t \left\{ \left(\sum_i f_i^2 \right)^{1/2} \right\} \geq \left[\sum_i (T_t f_i)^{1/2} \right], \ \forall \ f_i \epsilon \mathcal{P}$$

implies

$$T_t \left\{ \left(\sum_i (D_i F)^2 \right)^{1/2} \right\} \geq \left[\sum_i (T_t D_i F)^2 \right]^{1/2}$$

$$\geq e^t \left[\sum_i (D_t T_i F)^2 \right]^{1/2}$$

i.e $\quad T_t \sqrt{(< DF, DF >_H)} \geq e^t \sqrt{(< DT_t F, DT_t F >_H)}.$

42 Changing F by $T_t F$,

$$T_t(\sqrt{(< DT_t F, DT_t F >_H)}) \geq e^t \sqrt{(< DT_{2t} F, DT_{2t} F >_H)}.$$

Now

$$\psi_F \overset{\Delta}{=} \left[\int_o^\infty \{T_t(\sqrt{(< DT_t F, DT_t F >_H)})\}^2 \, dt \right]^{1/2}$$

$$\geq \left\{ \int_o^\infty e^{2t} < DT_{2t} F, DT_{2t} F >_H \, dt \right\}^{1/2}$$

$$= \text{ const.} \left\{ \int_o^\infty e^t < DT_t F, DT_t F >_H dt \right\}^{1/2} .$$

Therefore, by the Littlewood-Paley inequality (Step 1),

$$\|F\|_p \geq C_p \| \left\{ \int_o^\infty e^t < DT_t F, DT_t F >_H dt \right\}^{1/2} \|_p. \qquad (1.19)$$

Substituting $T_u F$ for F in (1.19),

$$e^{u/2} \|T_u F\|_p \geq C_p \| \left\{ \int_o^\infty e^s < DT_s F, DT_s F >_H ds \right\}^{1/2} \|_p.$$

Therefore

$$\begin{aligned}
\int_o^\infty e^u \|T_u F\|_p du &\geq C_p \int_o^\infty e^{u/2} \| \left\{ \int_u^\infty e^s < DT_s F, DT_s F >_H ds \right\}^{1/2} \|_p du \\
&\geq C_p \| \int_o^\infty e^{u/2} \left\{ \int_u^\infty e^s < DT_s F, DT_s F >_H ds \right\}^{1/2} du \|_p \\
&\geq C_p \| \left\{ \int_o^\infty ds \left[\int_o^\infty e^{u/2} T_{\{u \leq s\}} du \times e^{s/2} \sqrt{(< DT_s F, DT_s F >_H)} \right]^2 \right\}^{1/2} \|_p \\
&= C_p \| \left\{ \int_o^\infty \left[2(e^s - e^{s/2}) \sqrt{(< DT_s F, DT_s F >_H)} \right]^2 ds \right\}^{1/2} \|_p \\
&\geq 2C_p \| \left[\int_o^\infty e^{2s} < DT_s F, DT_s F >_H ds \right]^{1/2} \|_p \\
&\qquad - 2C_p \| \left[\int_o^\infty e^s < DT_s F, DT_s F >_H ds \right]^{1/2} \|_p.
\end{aligned}$$

Hence by (1.19),

$$\| \left[\int_o^\infty e^{2s} < DT_s F, DT_s F >_H ds \right]^{1/2} \|_p \leq d_p \|F\|_p + A_p \int_o^\infty e^u \|T_u F\|_p du.$$

By step 2, we know that if $\|(J_o + J_1)F\| = 0$, then

$$\|T_u F\|_p \leq C_p e^{-2u} \|F\|_p.$$

Therefore, if $(J_o + J_1)F = 0$,

$$\|F\|_p \geq C_p \| \left\{ \int_o^\infty e^{2s} < DT_sF, DT_sF >_H ds \right\}^{1/2} \|_p. \tag{1.20}$$

Suppose $F \epsilon \mathcal{P}$ satisfies $(J_o + J_1)F = 0$. By step 3,

$$\| < DF, DF >_H^{1/2} \|_p = \| \left\{ \sum_{i=1}^\infty (D_i F)^2 \right\}^{1/2} \|_p$$

$$\leq C_p \| \left\{ \sum_{i=1}^\infty (G_{D_i} F)^2 \right\}^{1/2} \|_p$$

$$= C_p \| \left\{ \sum_{i=1}^\infty \int_o^\infty t (\frac{d}{dt} Q_t D_i F)^2 dt \right\}^{1/2} \|_p. \tag{*}$$

By step 4,
$Q_t D_i F = D_i \tilde{Q}_t F$ where $\tilde{Q}_t F = \sum_n e^{-\surd(n-1)t} J_n$ implying

$$\frac{d}{dt} Q_t D_i F = D_i \left(\frac{d}{dt} \tilde{Q}_t \right) = D_i \tilde{Q}_t \text{ CRF}$$

where
$$RF = \sum_{n=1}^\infty \surd(1 - \frac{1}{n}) J_n F.$$

Hence

$$(*) \qquad = C_p \| \left\{ \int_o^\infty t < D\tilde{Q}_t CRF, D\tilde{Q}_t CRF >_H dt \right\}^{1/2} \|_p \tag{**}$$

44 since

$$\tilde{Q}_t = \int\limits_o^\infty \mu_t(ds) e^s T_s,$$

$$< D\tilde{Q}_t CRF, D\tilde{Q}_t CRF >_H^{1/2}$$

$$\leq \int\limits_o^\infty \mu_t(ds) e^s < DT_s CRF, DT_s CRF >_H^{1/2} ds$$

$$\leq \left[\int\limits_o^\infty \mu_t(ds) e^{2s} < DT_s CRF, DT_s CRF >_H ds \right]^{1/2}.$$

Since

$$\int\limits_o^\infty t\mu_t(ds)dt = ds \left(\text{follows from } \int\limits_o^\infty \int\limits_o^\infty te^{-\lambda s} \mu_t(ds)dt = \frac{1}{\lambda} \right),$$

we have

$$(**) \leq C_p \| \left\{ \int\limits_o^\infty e^{2s} < DT_s CRF, DT_s CRF >_H ds \right\}^{1/2} \|_p$$

$$\leq C_p \|CRF\|_p \leq C_p \|CF\|_p$$

(by (1.20) and since $RC = CR$ and $\|R\|_p < \infty$.)

Hence we have obtained

$$\| < DF, DF >_H \|_p \leq C_p \|CF\|_p \text{ if } (J_o + J_1)F = 0.$$

For $F \epsilon C_o \oplus C_1$, it is easy to verify directly that

$$\| < DF, DF >_H^{1/2} \|_p \leq C_p \|CF\|_p.$$

Hence we have proved

$$\| < DF, DF >_H^{1/2} \|_p \leq C_p \|CF\|_p, \ \forall \ F \epsilon \mathcal{P}. \tag{1.21}$$

The converse inequality of (1.21) can be proved by the following duality arguments: we have for $F, G \epsilon \mathcal{P}$,

$$
\begin{aligned}
\Big|\int_W CF.G d\mu\Big| &= \Big|\int CF(I-J_o)G d\mu\Big| \left(\because \int_W CF d\mu = 0\right)\\
&= \Big|\int_W CF.C\tilde{G} d\mu\Big| \left[\tilde{G} = C^{-1}(I-J_o)G\right]\\
&= \Big|\int C^2 F.\tilde{G} d\mu\Big| = \Big|\int < LF, \tilde{G} > d\mu\Big|\\
&= \Big|\int < DF, \tilde{G} >_H d\mu\Big| \left[\because < DF, \tilde{G} >_H \right.\\
&\left. = \frac{1}{2}\left\{L(F\tilde{G}) - LF.\tilde{G} - F.L\tilde{G}\right\} \text{ and } \int_W LF = 0 \ \forall\ F\epsilon\mathcal{P}\right]\\
&\leq \int |DF|_H |D\tilde{G}|_H d\mu\\
&\leq \|\,|DF|_H\|_p \|\,|D\tilde{G}|_H\|_q \left(\frac{1}{p} + \frac{1}{q} = 1\right)\\
&\leq C_q \|\,|DF|_H\|_p \|\,|C\tilde{G}\|_q \ \text{ by (1.21)}\\
&= C_q \|\,|DF|_H\|_p \|(I-J_o)G\|_q\\
&\leq a_q \|\,|DF|_H\|_p \|G\|_q.
\end{aligned}
$$

45 Hence taking the supremum w.r.t. $\|G\|_q \leq 1$, we have $\|CF\|_p \leq a_p \|\,|DF|_H\|_p$. The proof of (1.18) is complete.

Now we shall prove that

$$\|\,|D^k F|_{HS}\,\|_p \leq C_p \|C^k F\|_p \ \forall\ F\epsilon\mathcal{P} \tag{1.22}$$

$$\|\,|D^k F|_{HS}\,\|_p \leq C'_p \|C^k F\|_p \ \forall\ F\epsilon\mathcal{P} \ \text{ if } \ (J_o + J_1 + \cdots J_{k-1})F = 0 \tag{1.23}$$

Then, since

$$C_p \|(I-C)^s F\|_p \leq C'_p \|(I-L)^{s/2} F\|_p \leq C''_p \|(I-C)^s F\|_p$$

and $$a_p\|C^kF\|_p \leq \|(I-C)^kF\|_p + \|F\|_p,$$

Theorem 1.8 follows at once.

Proof of (1.22): (By induction). Suppose (1.22) holds for $1, 2, \ldots k$. Let 46
$\{\gamma_m(w)\}_{m\epsilon N^k}$ be coin tossing sequence indexed by $m = (i_1, i_2, \ldots, i_k)\epsilon\mathbb{N}^k$ on some probability space (Ω, F, P). Let $D_m = D_{i_1}D_{i_2}\cdots D_{i_k}$. Then

$$|D^kF|^2_{HS} = \sum_{m\epsilon\mathbb{N}^k}\{D_mF\}^2.$$

Set

$$X(\omega) = \sum_{m\epsilon\mathbb{N}^k}\gamma_m(\omega)D_mF.$$

Then

$$D_i\chi(\omega) = \sum_{m\epsilon\mathbb{N}^k}\gamma_m(\omega)D_iD_mF$$

and $$C\chi(\omega) = \sum_{m\epsilon\mathbb{N}^k}\gamma_m(\omega)CD_mF.$$

we know that, by (i),

$$\|\sqrt{\left(\sum_{i=1}^{\infty}|D_iX(\omega)|^2\right)}\|_p \leq C_p\|CX(\omega)\|_p \ \forall\ \omega.$$

Therefore

$$E\left\{\|\sqrt{\left(\sum_{i=1}^{\infty}|D_iX(\omega)|^2\right)}\|_p^p\right\} \leq C_pE\|CX(\omega)\|_p^p. \tag{1.24}$$

Therefore, by step 3,

$$\begin{aligned} E\left\{\|\sum_i(D_iX(\omega))^2\|_p^p\right\} &\geq a_p\|\sqrt{\left(\sum_{i,m}(D_iD_mF)^2\right)}\|_p^p \\ &= a_p\|\,|D^{k+1}F|_{HS}\|_p^p. \end{aligned} \tag{1.25}$$

On the other hand, by step 3,

$$E\|CX(\omega)\|_p^p = E\|\sum_{m\epsilon\mathbb{N}^k}\gamma_m(\omega)(CD_mF)\|_p^p$$
$$\leq C_p\|\left\{\sum_{m\epsilon\mathbb{N}^k}(CD_mF)^2\right\}^{1/2}\|_p^p$$
$$= C_p\|\left\{\sum_{m\epsilon\mathbb{N}^k}(D_mCR_kF)^2\right\}^{1/2}\|_p^p$$
$$(\text{ by step 4, where})R_kF = \sum_{n=k}^{\infty}\sqrt{(1-\frac{k}{n})}J_nF$$
$$= C_p\|\,|D^kCR_kF|_{HS}\|_p^p$$
$$\leq A_p\|C^{k+1}R_kF\|_p^p \text{ (by induction hypothesis)}$$
$$\leq A'_p\|C^{k+1}F\|_p^p\ (\because \|R_k\|_p \leq a_p \text{ by step 2}).$$

47 This together with (1.24) and (1.25) proves that

$$\|\,|D^{k+1}F|_{HS}\|_p \leq C_p\|C^{k+1}F\|_p$$

i.e., (1.22) holds for $k+1$ and the proof of (1.22) is complete. (1.23) can be proved in a similar manner.

Corollary to Theorem 1.8. *Let $F\epsilon D_{p,k}$, $1 < p < \infty$, $k\epsilon\mathbb{Z}^+$; then $D^\ell F\epsilon L_2(W \to H^{\otimes\ell})$ are defined for $\ell = 0, 1, \ldots k$, where*

$$H^{\otimes\ell} = \underbrace{H\otimes\cdots\otimes H}_{\ell-times}$$

is the Hilbert space of all continuous ℓ-multilinear forms on $\underbrace{H\otimes\cdots\otimes H}_{\ell-times}$ with Hilbert-Schmidt norm. Note that $H^{\otimes o} = \mathbb{R}$ and $H^{\otimes 1} = H$.

48 *Proof.* For $F\epsilon D_{p,k}$, $\exists\ F_n\epsilon\mathcal{P} \ni \|F_n - F\|_{p,k} \to 0$ which implies $\{F_n\}$ is Cauchy in $\mathbb{D}_{p,k}$. Hence using Meyer's theorem, we get

$$\|\,|D^\ell F_n - D^\ell F_m|_{HS}\| \leq C\|F_n - F_m\|_{p,k} \to 0$$

which gives the result. □

Recall that if $F \epsilon \mathcal{P}_{\overset{*}{W}}$ then
$F(w) = \sum\limits_{i=1}^{n} F_i(w)\ell_i$ for some $n, \ell_i \epsilon \overset{*}{W}$ and $F_i \epsilon \mathcal{P}$.
For

$$F(W) \sum_{i=1}^{n} F_i(w)\ell_i \epsilon \mathcal{P}_{\overset{*}{w}},$$

define
$$LF(w) = \sum_{i=1}^{n} LF_i(w)\ell_i$$

and
$$(1 - L)^{s/2} F(w) = \sum_{i=1}^{n} (1 - L)^{s/2} F_i(w)\ell_i.$$

For $1 < p < \infty$ and $-\infty, s < \infty$, define the norms $\|.\|^H_{p,s}$ on $\mathcal{P}_{\overset{*}{W}}$ by

$$\|F\|^H_{p,s} = \| \, |(I - L)^{s/2} F_i(w)|_H \|_p.$$

Let $\mathbb{D}^H p, s$ denote completion of $\mathcal{P}_{\overset{*}{W}}$ *w.r.t.* the norm $\|.\|^H_{p,s}$. It is clear that $\mathbb{D}^H_{p,s} \subset L_p(W \to H)$ for $s \geq 0$ and in fact $\mathbb{D}^H_{p,o} = L_p(W \to H)$.

Proposition 1.9. *The operator* $D : \mathcal{P} \to \mathcal{P}_{\overset{*}{W}}$ *can be extended as a continuous operator from* $\mathbb{D}_{p,s+1}$ *to* $\mathbb{D}^H_{p,s}$ *for every* $1 < p < \infty, -\infty < s < \infty$.

Proof. Let $\{\ell_i\} \subset \overset{*}{W}$ be a ONB in H and $F \epsilon \mathcal{P}$. Now **49**

$$|(I - L)^{s/2} DF|_H = \left(\sum_{i=1}^{\infty} \left[(I - L)^{s/2} D_i F \right]^2 \right)^{1/2}. \qquad \square$$

Using step 4 above, we get

$$\begin{aligned} |(I - L)^{s/2} DF|_H &= \left(\sum_{i=1}^{\infty} \left\{ D_i R (I - L)^{s/2} F \right\}^2 \right)^{1/2} \quad \text{where } R = \sum_{i=1}^{\infty} \left(\frac{n}{n+1} \right)^{s/2} J_n \\ &= |DR(I - L)^{s/2} F|_H. \end{aligned}$$

Therefore

$$\begin{aligned}
\| |(I-L)^{s/2} DF|_H \|_p &= \| |DRI-L)^{s/2} F|_H \|_p \\
&\leq C_p \| R(I-L)^{(s+1)/2} F\|_p \text{ (by Meyer's theorem)} \\
&\leq C_p' \|(I-L)^{(s+1)/2} F\|_p \text{ (by } L_p \text{ multiplier theorem)} \\
&= C_p' \|F\|_{p,s+1}.
\end{aligned}$$

i.e.,
$$\|DF\|^H_{p,s} \leq C_p' \|F\|_{p,s+1}$$

from which the result follows by a limiting argument. □

From the above proposition, it follows that we can define the dual map D^* of D, as a continuous operator

$$D^* : (\mathbb{D}^H_{p,s})' \to (\mathbb{D}_{p,s+1})'$$

i.e.,
$$D^* : \mathbb{D}^H_{p,s+1} \to \mathbb{D}_{p,s}, 1 < p < \infty, -\infty < s < \infty.$$

50 And we know that for $F \epsilon \mathcal{P}$, $D^* F = -\delta F$. Hence we have the following corollary.

Corollary. *$\delta : \mathcal{P}_{\overset{*}{W}} \to \mathcal{P}$ can be extended as a continuous operator from $\mathbb{D}^H_{P,s+1} \to \mathbb{D}_{P,s}$ for every $1 < p < \infty$, $-\infty < s < \infty$.*

Proposition 1.10. *Let $F \epsilon \mathbb{D}_{P,k}, G \epsilon \mathbb{D}_{q,k} (\mathbb{D}^H_{q,k})$ for $k \epsilon Z^+, 1 < p, q < \infty$ and let $1 < r < \infty$, such that $\dfrac{1}{p} + \dfrac{1}{q} = \dfrac{1}{r}$. Then $FG \epsilon \mathbb{D}_{r,k}$ (resp. $\mathbb{D}^H_{r,k}$) and $\exists\, C_{p,q,k} > 0$ such that*

$$\|FG\|_{r,k} \leq C_{p,q,k} \|F\|_{p,k} \|G\|_{q,k}$$
$$(\text{ resp. } \|FG\|^H_{r,k} \leq C_{p,q,k} \|F\|_{p,k} \|G\|^H_{q,k}).$$

Proof. Let $F, G \epsilon \mathcal{P}$; then we have

$$D(FG) = F.DG + G.DF$$

Therefore
$$|D[FG]|_H \leq |F||DG|_H + |G||DF|_H.$$

Similarly

$$D^2FG = FD^2G + 2DF \otimes DG + G.D^2F$$

and $$|D^2FG|_{HS} \le |F||D^2G|_{HS} + 2|DF|_H|DG|_H + |G||D^2F|_{HS}. \qquad \square$$

In this way, we obtain for every $k = 1, 2, \ldots,$

$$\sum_{\ell=o}^{k} |D^l(FG)|_{HS} \le C_k \left(\sum_{\ell=o}^{k} |D^l F|_{HS} \right) \left(\sum_{\ell=o}^{k} |D^\ell G|_{HS} \right).$$

$\square$

Applying Hölder's inequality, we get

$$\| \sum_{\ell=o}^{k} |D^\ell(FG)|_{HS} \|_r \le C_k \| \sum_{\ell=o}^{k} |D^\ell F|_{HS} \|_p . \| \sum_{\ell=o}^{k} |D^\ell G|_{HS} \|_q .$$

Then the result follows by using Meyer's theorem. And the case $G\epsilon\mathbb{D}^H_{q,k}$ **51**
follows by similar arguments.

Corollary. (i) $\mathbb{D}_\infty$ *is an algebra and the map*

$$\mathbb{D}_\infty \times \mathbb{D}_\infty \ni (F, G) \to FG\epsilon\mathbb{D}_\infty$$

is continuous.

(ii) *If* $F\epsilon\mathbb{D}_\infty, G\epsilon\mathbb{D}^H_\infty = \bigcap_{p,s} \mathbb{D}^H_{p,s}$, *then* $FG\epsilon\mathbb{D}^H_\infty$ *and the map* $(F, G) \to FG$ *is continuous.*

Hence we see that $\mathbb{D}_\infty$ is a nice space in the sense that

$$L : \mathbb{D}_\infty \to \mathbb{D}_\infty \text{ is continuous}$$
$$D : \mathbb{D}_\infty \to \mathbb{D}^H_\infty \text{ is continuous}$$
$$\delta : \mathbb{D}^H_\infty \to \mathbb{D}_\infty \text{ is continuous .}$$

Proposition 1.11. (i) *Suppose* $f\epsilon C^\infty(\mathbb{R}^n)$, *tempered and* $F_1, F_2, \ldots, F_n\ \epsilon\mathbb{D}_\infty$; *then* $F = f(F_1, F_2, \ldots, F_n)\epsilon\mathbb{D}_\infty$ *and*

(a) $DF = \sum_{i=1}^{n} \partial_i f(F_1, F_2, \ldots, F_n).DF_i$

(b) $LF = \sum_{i,j=1}^{n} \partial_i \partial_j f(F_1, F_2, \ldots, F_n) < DF_i, DF_j >_H$

$$+ \sum_{i=1}^{n} \partial_i f(F_1, F_2, \ldots, F_n).L(F_i).$$

(ii) *For* $F, G \epsilon \mathbb{D}_\infty$,

$$< DF, DG >_H = \frac{1}{2} \{L(FG) - LF.G - F.LG\}$$

and hence $< DF, DG >_H \, \epsilon \mathbb{D}_\infty.$

52 (iii) *If* $F, G, J \epsilon \mathbb{D}_\infty$, *then*

$$< D < DF, DF >_H, DJ >_H = < D^2F, DG \otimes DJ >_{HS}$$
$$+ < D^2G, DF \otimes DJ >_{HS} \; .$$

(iv) *If* $F \epsilon \mathbb{D}_\infty, G \epsilon \mathbb{D}_\infty^H$, *then*

$$\delta(FG) = < DF, G >_H + F.\delta G.$$

In particular, if $F, G \epsilon \mathbb{D}_\infty$ then

$$\delta(F.DG) = < DF, DG >_H + F.LG.$$

These formulas are easily proved first for polynomials and then generalized as above by standard limiting arguments.

1.4 Composites of Wiener Functionals and Schwartz Distributions

For $F = (F^1, F^2, \ldots, F^d) : W \to \mathbb{R}^d$, we state two conditions which we shall refer to frequently.

$$F^i \epsilon \mathbb{D}_\infty, i = 1, 2, \ldots d \qquad \text{(A.1)}$$

Setting

$$\sigma^{ij} =< DF^i, DF^j >_H \epsilon \mathbb{D}_\infty, \int (\det \sigma)^{-p}(w) d\mu(w) < \infty \ \forall\ 1 < p < \infty. \tag{A.2}$$

We note that $((\sigma_{ij})) \geq 0$.

Lemma 1. *Let $F : W \to \mathbb{R}^d$ satisfy (A.1) and (A.2). Then $\gamma = \sigma^{-1} \epsilon \mathbb{D}_\infty$ and*

$$D\gamma^{ij} = -\sum_{k,\ell=1}^{d} \gamma^{ik}\gamma^{j\ell} D\sigma^{k\ell}.$$

Proof. Let $\epsilon > 0$. Let

$$\sigma_\epsilon^{ij}(w) = \sigma^{ij}(w) + \epsilon\delta_{ij} > 0 \qquad \text{(i.e., positive definite).} \quad \square$$

Then it can be easily seen that if $\gamma_\epsilon = \sigma_\epsilon^{-1}$, then $\exists\ f \epsilon C^\infty(\mathbb{R}^{d^2}) \ni$ 53
$\gamma_\epsilon^{ij}(w) = f(\sigma_\epsilon^{ij}(w))$.

Then by proposition (1.11), since $\sigma_\epsilon^{ij} \epsilon \mathbb{D}_\infty, \gamma_\epsilon^{ij} \epsilon \mathbb{D}_\infty$. Further, it follows from the dominated convergence theorem that $\gamma_\epsilon^{ij} \to \gamma^{ij}$ in $L_p\ \forall\ 1 < p < \infty$.

Next we show that $D^k \gamma^{ij} \epsilon L_p(W \to H^{\otimes k})\ \forall\ 1 < p < \infty$. Hence, by Meyer's theorem, $\gamma \epsilon \mathbb{D}_{p,k}\ \forall\ 1 < p < \infty$ and $\forall\ k \epsilon Z^+$ implying $\gamma \epsilon \mathbb{D}_\infty$. We have

$$\sum_j \gamma_\epsilon^{ij} \sigma_\epsilon^{ik} = \delta^{ik}.$$

Therefore

$$\sum_j \gamma_\epsilon^{ij} D\sigma_\epsilon^{jk} + \sum_j \sigma_\epsilon^{jk} D\gamma_\epsilon^{ij} = 0$$

implies

$$D\gamma_\epsilon^{ij} = -\sum_{k,l=1}^{d} \gamma_\epsilon^{ik}\gamma_\epsilon^{jl} D\sigma_\epsilon^{kl}.$$

Similarly, we get

$$D^k \gamma_\epsilon^{ij} = -\sum \gamma_\epsilon . \gamma_\epsilon \cdots \gamma_\epsilon D^{m_1}\sigma_\epsilon \otimes \cdots \otimes D^{m_k}\sigma_\epsilon$$

where $m_1 + \cdots + m_k = k$ and we have omitted superscripts in $\sigma^{ij}_\epsilon, \gamma^{kl}_\epsilon$ etc. for simplicity. Therefore, since

$$\gamma^{ij}_\epsilon \to \gamma^{ij} \text{ in } L_p,$$
$$D^k\gamma^{ij}_\epsilon \to \sum \gamma.\gamma \cdots \gamma D^{m_1}\sigma \otimes \ldots \otimes D^{m_k}\sigma$$

in $$L_p(w \to H^{\otimes k}), \ \forall\ 1 < p < \infty$$

implies

$$D^k\gamma^{ij} \sum \gamma.\gamma \cdots \gamma D^{m_1}\sigma \otimes \cdots \otimes D^{m_k}\sigma \epsilon L_p(W \to H^{\otimes k}). \forall\ 1 < p < \infty.$$

54 **Lemma 2.** *Let $F : W \to \mathbb{R}^d$ satisfy (A.1) and (A.2).*

1) *Then, $\forall\ G\epsilon\mathbb{D}_\infty$ and $\forall\ i = 1, 2, \ldots d.\exists\ l_i(G)\epsilon\mathbb{D}_\infty$ which depends linearly on G and satisfies*

$$\int_W (\partial_i\phi_o F).G\mu(dw) = \int_W \phi_o F).l_i(G)d\mu, \tag{1.26}$$

$\forall\ \epsilon S(\mathbb{R}^d)$*. Furthermore, for any* $1 \le r < q < \infty$,

$$\sup_{\|G\|_{q,1} \le 1} \|l_i(G)\|_r < \infty. \tag{1.27}$$

Hence (1.26) and (1.27) hold for every $G\epsilon\mathbb{D}_{q,1}$.

2) *Similarly, for any $G\epsilon\mathbb{D}_\infty$, and $1 \le i_1, i_2, \ldots i_k \le d, k\epsilon\mathbb{N}, \exists\ l_{i_1\ldots i_k}$ $(G)\epsilon\mathbb{D}_\infty$ which depends linearly on G $\ni$*

$$\int_W (\partial_{i_1} \ldots \partial_{i_1}\phi o F).G d\mu = \int_W \phi o F l_{i_1} \ldots_{\cdot i_k} (G) d\mu,\ \forall \quad \phi\epsilon S(\mathbb{R}^d) \tag{1.26$'$}$$

and for $1 \le r < q < \infty$,

$$\sup_{\|G\|_{q,k} \le 1} \|l_{i_1\ldots i_k}(G)\|_r < \infty. \tag{1.27$'$}$$

Hence again (1.26)$'$ *and* (1.27)$'$ *hold for every* $G\epsilon\mathbb{D}_{q,k}$.

Proof. Note that $\phi o F \epsilon \mathbb{D}_\infty$ and

$$D(\phi o F) = \sum_{i=1}^{d} \partial_i \phi o F \cdot DF^i.$$

Therefore

$$< D(\phi o F), DF^j >_H = \sum_{i=1}^{d} \partial_i \phi o F \cdot \sigma^{ij}$$

and $$\partial_i \phi o F = \sum_{j=1}^{d} < D\phi o F, DF^j >_H \gamma^{ij}. \qquad \square$$

$\square$

Hence 55

$$\int_W \partial_i \phi o F.G d\mu = \sum_{j=1}^{d} \int_W < D\phi o F, \gamma^{ij} G DF^j >_H d\mu$$
$$= -\sum_{j=1}^{d} \int_W (\phi o F) \delta(\gamma^{ij} G DF^j) d\mu$$

Let

$$\begin{aligned}
\ell_i(G) &= -\sum_{j=1}^{d} \delta(\gamma^{ij} G DF^j) \\
&= -\sum_{j=1}^{d} [< D(\gamma^{ij} G), DF^j >_H + \gamma^{ij} G.LF^j] \\
&= -\sum_{j=1}^{d} \left[\left\{ -\sum_{k,\ell=1}^{d} G\gamma^{ik}\gamma^{j\ell} < D\sigma^{k\ell}, DF^j > + \gamma^{ij} < DG, DF^j >_H \right\} \right. \\
&\qquad \left. + \gamma^{ij} GLF^j \right].
\end{aligned}$$

Therefore

$$|\ell_i(G)| \leq \sum_{j=1}^{d}\left[\left\{\sum_{k,\ell=1}^{d}|\gamma^{ik}\gamma^{j\ell}||D\sigma^{k\ell}|_H.|G||DF^j|_H\right\}\right.$$
$$\left. + |\gamma^{ij}||DF^j|_H|DG|_H + |\gamma^{ij}||LF^j|.|G|\right].$$

Hence if p is such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, then

$$||l_i(G)||_r \leq \sum_{j=1}^{d}\left[\left\{\sum_{k,\ell=1}^{d}||\gamma^{ik}\gamma^{j\ell}||DF^j|_H|D\sigma^{k\ell}|_H||_p.||G||_q\right\}\right.$$
$$\left. +|| |\gamma^{ij}||DF^j|_H||_p|| |DG|_H||_q + || |\gamma^{ij}||LF^j| ||_p.||G||_q\right].$$

Now taking supremum over $||G||_q + || |DG|_H||_q \leq 1$, we get (1.27).
2) The proof is similar to that of (1) and we note that

$$\ell_{i_1\ldots i_k}(G) = \ell_{i_k}[\ldots[\ell_{i_2}[\ell_{i_1}(G)]]\ldots].$$

56 Let $\phi\epsilon S = S(\mathbb{R}^d)$, $-\infty < k < \infty$, where k is an integer. Let

$$||\phi||_{T_{2K}} = ||(1 + |x|^2 - \Delta)^k\phi||_\infty$$

where
$$||f||_\infty = \sup_{x\epsilon\mathbb{R}^d}|f(x)|.$$

Let

$$\bar{S}^{||.||_{T_{2k}}} = T_{2k}.$$

Facts. (1) $S \subset \ldots \subset T_{2k} \subset \ldots \subset T_2 \subset T_o = \{f\ cont.,\ f \to 0\ as\ |x| \to \infty\}$

$$\subset T_{-2} \subset \ldots T_{-2k}.$$

(2) $\bigcap_k T_k = S$

(3) $\bigcup\limits_{k} T_k = S'$.

Theorem 1.12. *Let $F : W \to \mathbb{R}^d$ satisfy (A.1) and (A.2). Let $\phi \epsilon S \Leftrightarrow \phi o F \epsilon \mathbb{D}_\infty$). Then, $\forall\ k \epsilon \mathbb{N}$ and $\forall\ 1 < p < \infty, \exists\ C_{k,p} > 0$ such that $||\phi o F||_{p,-2k} \leq C_{p,k} ||\phi||_{T_{-2k}}$ for all $\phi \epsilon S$.*

Proof. Let $\psi = (1 + |x|^2 - \Delta)^{-k} \phi \epsilon S$. Then for $G \epsilon \mathbb{D}_\infty, \exists\ \eta_{2k}(G) \epsilon \mathbb{D}_\infty$ such that

$$\int\limits_W \left[(1 + |x|^2 - \Delta)^k \psi o F\right].G\mu(dw) = \int\limits_W \psi o F\,[\eta 2k(G)]\,\mu(dw)$$

i.e., $$\int\limits_W \phi o F.G d\mu = \int\limits_W (1 + |x|^2 - \Delta)^{-k} \phi o F.\eta_{2k}(G) d\mu. \qquad \square$$

Therefore 57

$$|\int\limits_W \phi o F.G d\mu| \leq ||\phi||_{T_{-2k}} ||\eta_{2k}(G)||_1.$$

Let

$$K = \sup_{||G||_{q,2k} \leq 1} ||\eta_{2k}(G)||_1 < \infty,$$

which follows easily from Lemma 2. Note that $\eta_{2k}(G)$ has a similar expression as $\ell_{i_1 \ldots i_k}(G)$ only with some more polynomials of F multiplied.

Then taking supremum over $||G||_{q,2k} \leq 1$ in the above inequality, we get

$$||\phi_o F||_{p,-2k} \leq K.||\phi||_{T_{-2k}}.$$

Since we can take any q such that $\frac{1}{r} = 1 < \frac{1}{q} < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, $p(1 < p < \infty)$ can also be chosen arbitrarily.

Corollary . *We can uniquely extend $\phi \epsilon S(\mathbb{R}^d) \to \phi o F \epsilon \mathbb{D}_\infty$ as a continuous linear mapping $T \epsilon T_{-2k} \to T(F) \epsilon \mathbb{D}_{p,-2k}$ for every $k \epsilon Z^+$ and $1 < p < \infty$.*

Indeed, the extension is given as follows:

$T \epsilon T_{-2k}$ implies $\exists\ \phi_n S(\mathbb{R}^d)$ such that $||\phi_n - T||_{T_{-2k}} \to 0$ which implies $\{\phi_n\}$ is Cauchy in T_{-2k} and hence, by Theorem 1.12, $\{\phi_n o F\}$ is Cauchy in $\mathbb{D}_{p,-2k}, 1 < p < \infty$ and hence we let $T(F) = \lim_{n\to\infty} \phi noF$, limit being taken $w.r.t.$ the norm $\| \quad \|_{p,-2k}$. Note that $T(F)$ is uniquely determined.

58 **Definition 1.14.** *$T(F)$ is called the* composite *of $T\epsilon T_{-2k}$ and F satisfying (A.1) and (A.2). Note that, since k is arbitrary, we have defined the composite $T(F)$ for every $T\epsilon S'(\mathbb{R}^d)$ as an element in $\mathbb{D}_{-\infty}$.*

Proposition 1.13. *If $T = f\epsilon\hat{C}(\mathbb{R}^d) = T_o \subset S'(\mathbb{R}^d)$, then $f(F) = foF$; the usual composite of f and F.*

Proof. $T\epsilon T_o$ implies there exists $\phi_n\epsilon S$ such that

$$\|\phi_n - f\|_{T_o} \to 0.$$

Obviously, we get $\|\phi_n oF - foF\|_p \to 0$ for $1 < p < \infty$. Hence the result follows by definition of $f(F)$. □

1.5 The Smoothness of Probability Laws

Lemma 1. *Let δ_y be the Dirac δ- function at $y\epsilon\mathbb{R}^d$.*

(i) *$\delta_y\epsilon T_{-2m}$ if and only if $m > \dfrac{d}{2}$.*

(ii) *if $m > \dfrac{d}{2}$, then the map $y\epsilon\mathbb{R}^d \to \delta_y\epsilon T_{-2m}$ is continuous.*

(iii) *if $m = \left[\dfrac{d}{2}\right] + 1, k\epsilon Z^+$, then $y\epsilon\mathbb{R}^d \to \delta_y\epsilon T_{-2m-2k}$ is $2k$ times continuously differentiable.*

Equivalently,

$$y\epsilon\mathbb{R}^d \to D^\alpha\delta_y\epsilon T_{-2m-2k}, \alpha\epsilon\mathbb{N}^d|\alpha| \leq 2k$$

is continuous.

59 *Proof.* Omitted. □

Corollary. *Let F satisfy (A.1) and (A.2) and $m = \left[\frac{d}{2}\right] + 1, k \epsilon Z^+$; then $y \to \delta_y(F) \epsilon \mathbb{D}_{p,-2m-2k}$ is $2k$ times continuously differentiable for every $1 < p < \infty$. In particular, we have the following:*

For every $G \epsilon \mathbb{D}_{q,2m+2k}$

$$< \delta_y(F), G > \epsilon C^{2k}(\mathbb{R}^d), \text{ where } < \delta_y(F), G >$$

denote the canonical bilinear form which we may write roughly as E^μ $(\delta_y(F).G)$.

Lemma 2. *Let $m = \left[\frac{d}{2}\right] + 1$ and $1 < q < \infty$. If $f \epsilon C(\mathbb{R}^d)$ with compact support, then*

$$\int\limits_{\mathbb{R}} f(y) < \delta_y F, G > dy = E^\mu(f o F.G)$$

for every $G \epsilon \mathbb{D}_{q,2m}$.

Proof. Let

$$i = (i_1, i_2, \ldots i_d), \Delta_i^{(n)} = \left[\frac{i_1}{2^n}, \frac{i_1 + 1}{2^n}\right] \times \ldots \times \left[\frac{i_d}{2^n}, \frac{i_d + 1}{2^n}\right]$$

and $$x_i^{(n)} = \left(\frac{i_1}{2^n}, \frac{i_2}{2^n}, \ldots \frac{i_d}{2^n}\right) \quad \text{where} \quad i_k \epsilon Z.$$ ◻

Note that $|\Delta_i^{(n)}| = \left(\frac{1}{2^n}\right)^d$, where $|.|$ denote the Lebesgue measure. For $f \epsilon C(\mathbb{R}^d)$ with compact support, we have

$$\sum_i f\left(x_i^{(n)}\right) |\Delta_i^{(n)}| \delta_{x_i(n)} \to \int\limits_{\mathbb{R}^d} f(x) \delta_x dx = f.$$

Note that the above integral is T_{-2m}-valued and the integration is in the sense of Bochner and hence the convergence is in T_{-2m}. Therefore, we have **60**

$$\sum_i f\left(x_i^{(n)}\right) |\Delta_i^{(n)}| \delta_{x_i}(F) \to f o F \text{ in } \mathbb{D}_{p,2m}$$

for $1 < p < \infty$. In particular,

$$< \sum_i f\left(x_i^{(n)}\right)|\Delta_i^{(n)}|\delta_{x_i}(F), G > \to E(foF.G) \text{ for every } G\epsilon\mathbb{D}_{q,-2m}.$$

But

$$< \sum_i f\left(x_i^{(n)}\right)|\Delta_i^{(n)}|\delta_{x_i}(F), G > \to \int\limits_{\mathbb{R}^d} f(x) < \delta_x F, G > dx;.$$

hence the result. □

Theorem 1.14. *Let $F = (F^1, F^2, \ldots, F^d)$ satisfy the conditions (A.1) and (A.2). Let $m = \left[\frac{d}{2}\right] + 1, k\epsilon Z^+$ and $1 < q < \infty$. Set, for every $G\epsilon\mathbb{D}_{q,2m+2k}$*

$$\mu_G^F(dx) = E^\mu(G(w) : F(w)\epsilon dx).$$

Then $\mu_G^F(x)$ has a density $P_G^F(x)\epsilon C^{2k}(\mathbb{R}^d)$ and $P_G^F(x) =< \delta_x(F), G >$.

Proof. Easily follows from Lemma 1 and Lemma 2. □

Remark. By the above theorem, we see that if G

$$G\epsilon\mathbb{D}_{q,\infty} = \bigcap_{k=o}^{\infty} \mathbb{D}_{q,k} 1 < q < \infty,$$

then $\mu_G^F(dx)$ has a C^∞- density. Further, if $G \equiv 1\epsilon\mathbb{D}_\infty$, then the probability law of F:

$$\mu_1^F(dx) = \mu\{w : F(x)\epsilon dx\}$$

61 has a C^∞-density. But we have

$$\mu_G^F(dx) = E^\mu(G|F = x)\mu_1^G(dx).$$

Hence

$$p_G^F(x) = E^\mu(G|F = x)p_1^F(x).$$

Chapter 2

Applications to Stochastic Differential Equations

2.1 Solutions of Stochastic Differential Equations as Wiener Functionals

From now on, we choose, as our basic abstract Wiener space (W, H, μ), 62
the following r-dimensional Wiener space (cf. Ex. 1.1).

Let

$$W = W_o^r = \{w \epsilon C[0, T] \to \mathbb{R}^r), w(0) = 0\}$$

$\mu = P$, the r-dimensional Wiener measure.

$$H = \Bigg\{ h \epsilon W_o^r ; h = (h^\alpha(t))_{\alpha=1}^r,$$

h^α absolutely continuous and

$$\int\limits_o^T \dot{h}^\alpha(S)^2 ds < \infty, \alpha = 1, 2 \ldots r \Bigg\}.$$

We define an inner product in H as follows:

$$< h, h' >_H = \sum_{\alpha=1}^{r} \int_{o}^{T} \dot{h}^\alpha(t)\dot{h}'^\alpha(t)dt, h', h\epsilon H.$$

With this inner product, $H \subset W$ is a Hilbert space. Further $\overset{*}{W} \subset H^* = H \subset W$ is given as follows:

$$\overset{*}{W} = \left\{ \ell \epsilon H : \ell = (\ell^\alpha(t))_{\alpha=1}^{r}, \ell^\alpha(t) = \int_{o}^{t} \dot{\ell}^\alpha(t)ds \right\}$$

and $\dot{\ell}^\alpha$ is a right continuous function of bounded variation on $[0, T]$ such that $\dot{\ell}^\alpha(T) = 0, \alpha = 1, \ldots r\}$.

63 If $\ell\epsilon\overset{*}{W}, w\epsilon W$, then

$$\ell(w) = -\sum_{\alpha=1}^{r} \int_{o}^{T} w^\alpha(t) d\dot{\ell}_\alpha(t)$$

and for $\ell\epsilon\overset{*}{W}, h\epsilon H$,

$$\begin{aligned} \ell(h) &= -\sum_{\alpha=1}^{r} \int_{o}^{T} h^\alpha(t) d\dot{\ell}_\alpha(t) \\ &= \sum_{\alpha=1}^{r} \int_{o}^{T} \dot{h}^\alpha(t)\dot{\ell}_\alpha(t)dt =< h, \ell >_H -. \end{aligned}$$

Let $B_t(W_o^r)$ = the completion of the σ-algebras on W_o^r generated by $(w^\alpha(s)), 0 \leq s \leq t$.

Stochastic Integrals: Let $\phi_\alpha(t, w)$ be jointly measurable in $(t, w), B_t$ adapted and

$$\int_{o}^{T} \phi_\alpha(t, w)dt < \infty \text{ a.s.}$$

Then it is well known that the stochastic integral

$$\int_{o}^{t} \phi_{\alpha}(s,w)dW_{s}^{\alpha}, (W_{t}^{\alpha}(w) = w^{\alpha}(t), \alpha = 1, 2, \ldots, r)$$

is a continuous local martingale.

Itô process: A continuous B_t-adapted process of the form

$$\xi_t = \xi_o + \sum_{\alpha=1}^{r} \int_{o}^{t} \phi_{\alpha}(s,w)dW_{s}^{\alpha} + \int_{0}^{t} \phi_{o}(s,w)ds$$

where

i) $\phi_{\alpha}(t,w)$ is B_t-adapted, jointly measurable with

$$\int_{o}^{T} \phi_{\alpha}^{2}(t,w)dt < \infty \text{ a.s.}$$

ii) $\phi_{o}(t,w)$ is B_t-adapted, jointly measurable with

$$\int_{o}^{T} |\phi_{0}(s,w)|ds < \infty \text{ a.s.}$$

is called an Itô process. **64**

Straton ovitch Integral: Let $\phi_{\alpha}(t,w)$ be an Itô process. Then ϕ_{α} is of the form

$$\phi_{\alpha}(t,w) = \phi_{\alpha}(o,w) + \sum_{\beta=1}^{r} \int_{o}^{t} \Xi_{\alpha,\beta}(s,w)dW_{s}^{\beta} + \int_{o}^{t} \Xi_{\alpha,o}(s,w)ds.$$

Then the Stratonovitch integral of ϕ_{α} w.r.t W^{α}, denoted by

$$\int_{o}^{t} \phi_{\alpha}(s,w)odW_{s}^{\alpha}$$

is defined as follows:

$$\int_o^t \phi_\alpha(s,w) o dW_s^\alpha \triangleq \int_o^t \phi_\alpha(s,w) dW_s^\alpha + \frac{1}{2}\int_o^t \Xi_{\alpha,\alpha}(s,w)ds.$$

Itô Formula: Let $\xi_t = (\xi_t', \ldots, \xi_t^d)$ be a d-dimensional Itô process,

i.e., $$\xi_t^i = \xi_o^i + \sum_{\alpha=1}^{\gamma} \int_o^t \phi_\alpha^i(s,w) dW_s^\alpha + \int_o^t \phi_o^i(s,w)ds, 1 \le i \le d.$$

1) Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a C^2 function. Then $f(\xi_t)$ is an Itô process we have the Itô formula:

$$\begin{aligned} f(\xi_t) = f(\xi_o) &+ \sum_{i=1}^{d}\sum_{\alpha=1}^{r}\int_o^t \partial_i f(\xi_s)\phi_\alpha^i(s,w)dW_s^\alpha \\ &+ \sum_{i=1}^{d}\int_o^t \partial_i f(\xi_s)\phi_o^i(s,w)ds \\ &+ \frac{1}{2}\sum_{\alpha=1}^{r}\sum_{i,j=1}^{d}\int_o^t \partial_{i,j}^2 f(\xi_s)(\phi_i^\alpha \phi_j^\alpha)(s,w)ds \end{aligned}$$

65 2) Suppose further that $\phi_\alpha^i(t,w), 1 \le i \le d, 1 \le \alpha \le r$ are Itô processes and set

$$\eta_t^i = \eta_o^i + \sum_{\alpha=1}^{r}\int_o^t \partial_o^i(s,w) o s W_s^\alpha + \int_o^t \phi_o^i(s,w)ds, 1 \le i \le d.$$

Then, if $f : \mathbb{R}^d \to \mathbb{R}$ is C^3, we have

$$f(\eta_t) - f(\eta_o) = \sum_{i=1}^{d}\sum_{\alpha=1}^{r}\int_o^t \partial_i f(\eta_s)\phi_\alpha^i(s,w) o dW_s^\alpha$$

$$+\sum_{\alpha=1}^{r}\int_{o}^{t}\partial_i f(\eta_s)\phi_o^i(s,w)ds.$$

Stochastic Differential Equations: Let $\sigma^i_\alpha(x), b^i(x)$ be functions of $\mathbb{R}^d$ for $i = 1, 2, \ldots d, \alpha = 1, \ldots r$ satisfying the following assumptions:

i) $\sigma^i_\alpha, b^i \epsilon C^\infty(\mathbb{R}^d \to \mathbb{R})\ \forall\ i = 1, \ldots d, \alpha = 1, \ldots r.$

ii) $\forall\ k\epsilon N, \partial_{i_1}\partial_{i_2}\cdots\partial_{i_k}\sigma^i_\alpha, \partial_{i_1}\ldots\partial_{i_k}b^i$

are bounded on $\mathbb{R}^d$.

Then

$$|\sigma^i_\alpha(x)| \leq K(1 + |x|),\ \forall\ i = 1, \ldots d, \alpha = 1, \ldots r,$$
$$|b^i(x)| \leq K(1 + |x|),\ \forall\ i = 1, \ldots d.$$

Consider the following SDE,

$$dX_t = \sigma_\alpha(X_t)dW^\alpha_t + b(X_t)dt,$$
$$X_o = x\epsilon\mathbb{R}^d \tag{2.1}$$

which is equivalent to saying

$$X^i_t = x^i + \sum_{\alpha=1}^{r}\int_o^t \sigma^i_\alpha(X_s)dW^\alpha_s + \int_o^t b^i(X_s)ds, i = 1, \ldots, d.$$

Then the following are true: There exists a unique solution $X_t =$ **66**
$X(t, x, w) = X^1_t, \ldots X^d_t$ of (2.1) such that

1) $(t, x) \to X(t, x, w)$ is continuous $(a.a.w)$.

2) $\forall\ t \geq 0, x \to X(t, x, w)$ is a diffeomorphism on $\mathbb{R}^d(a.a.w)$.

3) $\forall\ t \geq 0, x\epsilon\mathbb{R}^d, X(t, x, .)\epsilon L^p\ \forall\ 1 < p < \infty$.

Theorem 2.1. *Let* $t > 0, x\epsilon\mathbb{R}^d$ *be fixed. Then*

$$X^i_t = X^i(t, x, w)\epsilon\mathbb{D}_\infty,\ \forall\ i = 1, \ldots, d.$$

To find an expression for $< DX^i_t, DX^j_t >_{H'}$ let

$$Y_t = ((Y^i_j(t))), Y^i_j(t) = \frac{\partial X^i(t,x,w)}{\partial x^j}.$$

Let also

$$(\partial\sigma_\alpha)^i_j = \frac{\partial\sigma^j_\alpha(x)}{\partial x^j}; (\partial b)^i_j = \frac{\partial b^i}{\partial x^j}(x).$$

Then it can be shown that Y_t is given by the following SDE:

$$\begin{aligned} dY_t &= \partial\sigma_\alpha(X_t).Y_t dW^\alpha_t + \partial b(X_t).Y_t dt \\ Y_o &= I \end{aligned} \tag{2.2}$$

i. e.
$$\begin{aligned} Y^i_j(t) = \delta^i_j &+ \sum_{\alpha=1}^{r}\sum_{k=1}^{d}\int_o^t (\partial_k\sigma^i_\alpha)(X_s)Y^k_j(t)dW^\alpha_s \\ &+ \sum_{k=1}^{d}\int_o^t (\partial_k b^i)(X_s)Y^k_j(s)ds, i,j = 1,\ldots,d. \end{aligned}$$

Fact. $Y_t \epsilon L_p$ *i.e.*, $(\sum^d_{i,j=1}(Y^i_j(s))^2)^{1/2}\epsilon L_p \ \forall\ 1 < p < \infty$.

Also by considering the SDE

$$dZ_t = -Z_t.\partial\sigma_\alpha(X_t)dW^\alpha_s - Z_t[\partial b(X_t) - \sum_\alpha(\partial\sigma_\alpha.\partial\sigma_\alpha)(X_t)]dt \tag{2.3}$$

$$Z_o = I$$

67 and using Itô's formula, we can easily see that $d(Z_tY_t) = 0 \Rightarrow Z_tY_t \equiv I$

i.e., $$Z_t = Y^{-1}_t \text{ exists , } \forall\ t.$$

Fact. $Y^{-1}_t \epsilon L_p$

i.e., $$\left(\sum_{i,j=1}^{d}\left((Y^{-1}(t))^i_j\right)^2\right)^{1/2} \epsilon L_p \ \forall\ 1 < p < \infty,$$

since $Z_t \epsilon L_p$.

Theorem 2.2. *For every $t, 0 < t < T$ and $i, j = 1, \ldots d$,*

$$< DX_t^i, DX_t^j >= \sum_{\alpha=1}^{r} \int_o^t (Y_t Y_s^{-1} \sigma_\alpha(X_s))^i (Y_t Y_s^{-1} \sigma_\alpha(X_s))^j ds$$

where $$(Y_t Y_s^{-1} \sigma_\alpha(X_s))^i = \sum_{k,j} Y_k^i(t)(Y^{-1})_j^k(s) \sigma_\alpha^j(X_s).$$

Remark. The S.D.E (2.1) is given in the Stratonovitch form as

$$\begin{aligned} dX_t &= \sigma_\alpha(X_t) o dW_t^\alpha + \tilde{b}(X_t)dt \\ X_o &= x \end{aligned} \tag{2.1$'$}$$

where

$$\tilde{b}^i(x) = b^i(x) - \frac{1}{2} \sum_{k=1}^{d} \sum_{\alpha=1}^{r} \partial_k \sigma_\alpha^i(x) \sigma_\alpha^k(x)$$

and correspondingly, (2.2) and (2.3) are given equivalently as

$$dY_t = \partial\sigma_\alpha(X_t) Y_t o dW_t^\alpha + \partial\tilde{b}(X_t)dt \tag{2.2$'$}$$

$$dZ_t = -Z_t \partial\sigma_\alpha(X_t) o dW_t^\alpha + Z_t \partial\tilde{b}(X_t)dt. \tag{2.3$'$}$$

For the proof if theorem 2.1 and theorem 2.2, we need the following: **68**

Lemma 1. *Let X_t be the solution of (2.1) and $a_t = (a_t^i)$ be a continuous B_t adapted process. Suppose that $\xi_t = (\xi_t^i)$ satisfies*

$$\begin{aligned} d\xi_t &= \sum_{\alpha=1}^{r} \partial\sigma_\alpha(X_t) \xi_t dW_t^\alpha + \partial b(X_t) \xi_t dt + a_t dt \\ \xi_o &= 0. \end{aligned} \tag{2.4}$$

Then

$$\xi_t = \int_o^t Y_t Y^{-1} a_s ds = Y_t \int_o^t Y_s^{-1} a_s ds,$$

where Y_t is the solution of (2.2).

Proof. It is enough to verify that $\xi_t = \int\limits_o^t Y_t Y_s^{-1} a_s ds$ satisfies (2.4). Now

$$\begin{aligned} d\xi_t &= d(\int\limits_o^t Y_t Y^{-1} a_s ds) \\ &= dY_t. \int\limits_o^t Y_s^{-1} a_s ds + Y_t Y_t^{-1} a_t dt \\ &= dY_t \int\limits_o^t Y_s^{-1} a_s ds + a_t dt. \end{aligned}$$ □

Using (2.2), we get

$$\begin{aligned} d\xi_t &= (\partial\sigma_\alpha(X_t).Y_t dW_t^\alpha + \partial b(X_t)Y_t dt) \int\limits_o^t Y_s^{-1} a_s ds + a_t dt \\ &= \partial\sigma_\alpha(X_t)\xi_t dW_t^\alpha + \partial b(X_t)\xi_t dt + a_t dt; \end{aligned}$$

hence the lemma is proved. □

69 **Formal Calculations:**

By definitions,

$$DX_t^i[h] = \frac{\partial}{\partial\epsilon} X^i(t, x, w + \epsilon h)|_{\epsilon=o'} h\epsilon H.$$

But

$$\begin{aligned} X^i(t, x, w + \epsilon h) = x + \sum_\alpha \int\limits_o^t \sigma_\alpha^i(X(s, x, w + \epsilon h))d(W_s^\alpha + \epsilon h_s^\alpha) \\ + \int\limits_o^t b^i(X(s, x, w + \epsilon h))ds \end{aligned}$$

Hence

$$DX_t^i[h] = \sum_{\alpha=1}^{r}\sum_{k=1}^{d}\int_o^t \partial_k\sigma_\alpha^i(X_s)DX_s^k[h]dW_s^\alpha$$
$$+ \sum_{\alpha=1}^{r}\int_o^t \sigma_\alpha^i(X_s)dh_s^\alpha$$
$$+ \sum_{k=1}^{d}\int_o^t \partial_k b^i(X_s)DX_s^k[h]ds.$$

This is same as (2.4) with

$$a_s^i = \sum_{\alpha=1}^{r}\sigma_\alpha^i(X_s)\dot{h}_s^\alpha.$$

Hence formally we have

$$DX_t^i[h] = \sum_{\alpha=1}^{r}\int_o^t \left[Y_tY_s^{-1}\sigma_\alpha(X_s)\right]^i \dot{h}_s^\alpha ds.$$

Now, let for $i = 1, 2, \ldots d$,

$$\begin{aligned} \dot{\eta}_t^i, \alpha^{(s)} &= [Y_tY_s^{-1}\sigma_\alpha(X_s)]^i \qquad &\text{if } s \leq t \\ &= 0 &\text{if } s > t. \end{aligned}$$

For fixed $s, 0 \leq s \leq t \leq T, \dot{\eta}_t^{i,\alpha}, (s)$ satisfies the following:

$$\dot{\eta}_t^{i,\alpha}(s) = \sum_j \int_s^t \partial_j\sigma_\alpha^i(X_u)\dot{\eta}_u^{j,\alpha}(s)dW_u^\alpha$$
$$+ \sum_j \int_s^t \partial_j b^i(X_u)\dot{\eta}_u^{j,\alpha}(s)du + \sigma_\alpha^i(X_s). \qquad (2.5)$$

Note that this is same as (2.2) with initial condition $\sigma^i_\alpha(X_s)$. Now

$$DX^i_t[h] = < \eta^i_t, h >_H = \sum_\alpha \int_o^T \dot{\eta}^{i,\alpha}_t(s)\dot{h}^\alpha(s)ds$$

where
$$\eta^{i,\alpha}_t(s) = \int_o^s \dot{\eta}^{i,\alpha}_t(u)du \epsilon H.$$

Hence

$$< DX^i_t, DX^j_t >_H = \sum_{\alpha=1}^r \int_o^t [Y_t Y_s^{-1}\sigma_\alpha(X_s)]^i [Y_t Y_s^{-1}\sigma_\alpha(X_s)]^j ds.$$

A rigorous proof is given by using approximating arguments. Let

$$\phi_n(s) = \frac{k}{2^n}, \text{ if } \frac{k}{2^n} \leq s < \frac{k+1}{2^n}, n = 1, 2 \ldots$$

and
$$\psi_n(s) = \frac{k+1}{2^n}, \text{if} \frac{k}{2^n} < s \leq \frac{k+1}{2^n}, n = 0, 1, 2 \ldots$$

Using ϕ_n and ψ_n, we write the corresponding approximating equations of (2.1), (2.2), (2.5) as

$$dX^{(n)}_t = \sigma_\alpha\left(X^{(n)}_{\phi_n(t)}\right)dW^\alpha_t + b\left(X^{(n)}_{\phi_n(t)}\right)dt \tag{2.1)a$$
$$X^{(n)}_o = x$$
$$dY^{(n)}_t = \partial\sigma_\alpha\left(X^{(n)}_{\phi_n(t)}\right)Y^{(n)}_{\phi_n(t)}dW^\alpha_t + \partial b\left(X^{(n)}_{\phi_n(t)}\right)Y^{(n)}_{\phi_n(t)}dt \tag{2.2)a$$
$$Y^{(n)}_o = I.$$

$$\dot{\eta}^{i,\alpha,(n)}_t(s) = \sum_\alpha \sum_j \int_{\psi_n(S)\Lambda t}^t \partial_j \alpha^j_\alpha\left(X^{(n)}_{\Phi_n(u)}\right)\eta^{j,\dot{\alpha},(n)}_{\Phi_n(u)}(s)dW^\alpha_u$$
$$+ \sum_j \int_{\psi_n(S)\Lambda t}^t \partial_j b^i\left(X^{(n)}_{\Phi_n(u)}\right)\eta^{j,\dot{\alpha},(n)}_{\Phi_n(u)}(s)du + \sigma^i_\alpha\left(X^{(n)}_{\phi_n(s)}\right). \tag{2.5)a$$

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It is easily seen that (2.1)a has a unique solution $X_t^{(n)} \epsilon \mathcal{S}$: the space of smooth functionals, and $\partial X_t^{(n)} = Y_t^{(n)}$.

Further,

$$DX_t^{(n)}[h] = \sum_{\alpha} \int_o^t \dot{\eta}_t^{i,\alpha,(n)}(S)\dot{h}^{\alpha}(s)ds.$$

Then the theorem 2.2 follows from the approximating theorem.

Theorem 2.3. *Suppose, for* $x \epsilon \mathbb{R}^m$, $A(x) = (A_\alpha^j(x)) \epsilon \mathbb{R}^m \otimes \mathbb{R}^r$, $B(x) = (B^i(x)) \epsilon \mathbb{R}^m$ *satisfy*

$$\|A(x)\| + |B(x)| \leq K(1 + |x|),$$
$$\|A(x) - A(y)\| + |B(x) - B(y)| \leq K_N|x - y| \ \forall\ |x|, |y| \leq N.$$

Also,

(a) Suppose $\alpha_n(t), \alpha(t)$ be $\mathbb{R}^m$ -valued continuous B_t adapted processes such that, for some $2 \leq p < \infty$,

$$\text{Sup}_n\, E\left[\sup_{o \leq t \leq T} |\alpha_n(t)|^{p+1}\right] < \infty,$$
$$E\left[\sup_{o \leq t \leq T} |\alpha_n(t) - \alpha(t)|^p\right] \to 0 \text{ as } n \to \infty$$

and let, for $i = 1, \ldots, n$,

$$\xi^i(t) = \alpha^i(t) + \sum_{\alpha=1}^{r} \int_o^t A_\alpha^i(\xi(s))dW^\alpha(s) + \int_o^t B^i(\xi(s))ds$$

and

$$\xi^{i,(n)}(t) = \alpha_n^i(t) + \sum_{\alpha=1}^{r} \int_o^t A_\alpha^i(\xi^{(n)}(\Phi_n(s)))dW_s^\alpha + \int_o^t B^i(\xi^{(n)}(\Phi_n(s)))ds,$$

then **72**

$$E\left[\sup_{o \leq s \leq T} |\xi^{(n)}(s)|^p\right] < \infty \text{ and}$$

$$E\left[\sup_{o\le s\le T}|\xi^{(n)}(s)-\xi(s)|^p\right]\to 0 \text{ as } n\to\infty.$$

(b) Suppose $\alpha_{n,\nu}(t), \alpha_\nu(t), t\epsilon[\nu, T]$ are $\mathbb{R}^m$-valued continuous B_t- adapted processes such that, for some $2 \le p < \infty$,

$$\sup_n \sup_{o\le\nu\le T} E\left[\sup_{\nu\le t\le T}|\alpha_{n,\nu}(t)|^{p+1}\right]<\infty,$$

$$\sup_{o\le\nu\le T} E\left[\sup_{\nu\le t\le T}|\alpha_{n,\nu}(t)-\alpha_\nu(t)|^p\right]\to \text{ as } n\to\infty.$$

Let

$$\xi^i_\nu(t)=\alpha^i_\nu(t)+\sum_{\alpha=1}^r\int_\nu^t A^i_\alpha(\xi_\nu(s))dW^\alpha_s+\int_\nu^t B^i(\xi_\nu(s))ds$$

and

$$\xi^{i,(n)}_\nu(t)=\alpha^i_{n,\nu}(t)+\sum_{\alpha=1}^r\int_{\psi_n(\nu)\Lambda t} A^i_\alpha(\xi^{(n)}_\nu(\Phi_n(s)))dW^\alpha_s+\int_{\psi_n(\nu)\Lambda t} B^i(\xi^{(n)}_\nu(\Phi_n(s)))ds.$$

Then

$$E\left[\sup_{\nu\le s\le T}|\xi^{(n)}_\nu(s)|^P\right]<\infty$$

and
$$E\left[\sup_{\nu\le s\le T}|\xi^{(n)}_\nu(s)-\xi_\nu(s)|^p\right]\to 0$$

uniformly in ν as $n\to\infty$.

73 Let $X_t=(X^i_t)^d_{i=1}$ satisfy (2.1). Let $\sigma_t=((\sigma_{ij}(t)))$ where

$$\sigma_{ij}(t)=< DX^i_t, DX^j_t >_H .$$

The problem now is to prove condition A.2, i.e.,

$$(\det\sigma_t)^{-1}\epsilon L_p\ \forall\ 1<p<\infty.$$

Let Y_t satisfy (2.2). Then Y_t can be considered as an element of $GL(d,\mathbb{R})$- the group of real non-singular $d\times d$ matrices. Then $(X_t, Y_t)\in\mathbb{R}^d\times GL(d,\mathbb{R})$. Let $r_t=(X_t, Y_t)$, which is determine by (2.1) and (2.2).

Definition 2.1. *Let* $(a^i(x))_{i=1}^d$ *be smooth functions on* $\mathbb{R}^d$ *and* $L = \sum_{i=1}^d a^i(x)\frac{\partial}{\partial x^i}$, *the corresponding vector field on* $\mathbb{R}^d$. *Then for*

$$r = (x, e)\epsilon\mathbb{R}^d \times GL(d, \mathbb{R})$$

we define

$$f_L^i(r) \triangleq \sum_{j=1}^d (e^{-1})_j^i a^j(x) i = 1, 2, \ldots d$$

and

$$f_L(r) = (f_L^i(r))_{i=1}^d.$$

Let

$$L_\alpha(x) = \sum_{i=1}^d \sigma_\alpha^i(x)\frac{\partial}{\partial x^i} \alpha = 1, 2, \ldots, r.$$

$$L_o(x) = \sum_{i=1}^d \tilde{b}_i(x)\frac{\partial}{\partial x^i}$$

where

$$\overline{b}_i(x) = b^i - \frac{1}{2}\sum_k \sum_\alpha \partial_k \sigma_\alpha^i(x)\sigma_\alpha^k(x).$$

Proposition 2.4. *Let* 74

$$L = \sum_i a^i(x)\frac{\partial}{\partial x^i}$$

be any smooth vector field on $\mathbb{R}^d$. *Then. for i* $= 1, 2, \ldots, d$,

$$f_L^i(r_t) - f_L^i(r_0) = \sum_{\alpha=1}^r \int_0^t f_{[L_\alpha, L]}^i(r_s) o dW_s^\alpha + \int_o^t f_{[L_o, L]}^i(r_s) ds$$

$$= \sum_{\alpha=1}^r \int_o^t f_{[L_\alpha, L]}^i(r_s) dW_s^\alpha$$

$$+ \int_o^t f^i_{\{[L_o, L] + \frac{1}{2}\sum_{\alpha=1}^r [L_\alpha, [L_\alpha, L]]\}}(r_s) ds,$$

where $[L_1, L_2] = L_1 L_2 - L_2 L_1$ *is the commutator of* L_1 *and* L_2.

Proof. $f^i_L(r_t) = [Y_t^{-1} a(X_t)]^i$ and we know that

$$dY_t^{-1} = -Y_t^{-1}\partial\sigma_\alpha(X_t) o dW_t^\alpha - Y_t^{-1}\partial\tilde{b}(X_t)dt$$

and
$$da(X_t) = \partial a(X_t)\sigma_\alpha(X_t) o dW_t^\alpha + \partial a(X_t)\tilde{b}(X_t)dt$$

where
$$\partial a(X_t) = ((\frac{\partial a^i}{\partial x^j}(X_t))).$$

The proof now follows easily from the Itô formula. □

Remark. $f_{L_\alpha}(r_s) = Y_s^{-1}\sigma_\alpha(X_s)$. Therefore

$$\sigma_t^{ij} =< DX_t^i, DX_t^j >_H = \sum_{\alpha=1}^{r} \int_0^t [Y_t f_{L_\alpha}(r_s)]^i [Y_t f_{L_\alpha}(r_s)]^j ds.$$

Proposition 2.5. *Let*

$$\hat{\sigma}_t^{ij}(w) = \sum_{\alpha=1}^{r} \int_0^t f^i_{L_\alpha}(r_s) f^j_{L_\alpha}(r_s) ds.$$

Then

$$(\det \sigma_t)^{-1} \epsilon L_P,\ \forall\ 1 < P < \infty \text{ iff } (\det \hat{\sigma}_t)^{-1} \epsilon L_P\ \forall\ 1 < p < \infty.$$

75

Proof. $\sigma_t = Y_t \hat{\sigma}_t Y_t^*$ implies $\det \sigma_t = (\det Y_t)^2 (\det \hat{\sigma}_t)$.
We know that $\|Y_t\|, \|Y_t^{-1}\| \epsilon L_P\ \ \forall\ \ 1 < p < \infty$, where

$$\|\sigma\| = \left(\sum_{i,j} |\sigma_{ij}|^2\right)^{1/2}.$$

Hence, if $\lambda_i^2, i = 1, 2, \ldots, d$ are the eigenvalues of $Y_t Y_t^*$ then

$$(\det Y_t)^2 = \det Y_t Y_t^* = \lambda_1^2 \cdots \lambda_n^2$$

and

$$\|Y_t\|^2 = \sum_i < Y_t Y_t^* e_i, e_i >$$

$$= \lambda_1^2 + \cdots + \lambda_n^2$$

where $(e_i)_{i=1}^d$ is an orthonormal basis in $\mathbb{R}^d$. Therefore

$$(\det\ Y_t)^2 \leq \|Y_t\|^{2n}.$$

Similarly

$$(\det\ Y_t^{-1})^2 \leq \|Y_t^{-1}\|^{2n}.$$

Hence the result. □

2.2 Existence of moments for a class of Wiener Functionals

Proposition 2.6. *Let* $\eta > 0$ *be a random variable on* (Ω, F, P)*. If,* $\forall\ N = 2, 3, 4, \ldots, \exists$ *constants* $c_1, c_2, c_3 > 0$ *(independent of N) such that*

$$P\left[\eta < \frac{1}{N^{c_1}}\right] = P\left[\eta^{-1} > N^{C_1}\right] \leq e^{-c_2 N^{C_3}},$$

then $\quad E[\eta^{-P}] < \infty, \forall\ p > 1.$

Proof.

$$\begin{aligned} E\left[\eta^{-P}\right] &\leq 1 + \sum_{N=1}^{\infty} E\left[\eta^{-P} : N^{C_1} \leq \eta^{-1} \leq (N+1)^{C_1}\right] \\ &\leq 1 + 2^{C_1 P} + \sum_{N=2}^{\infty} (N+1)^{C_1 P} e^{-C_2 N^{C_3}} \\ &< \infty. \end{aligned}$$

□ **76**

Example 2.1. Let $0 < \bar{t} \leq T$. Let

$$\eta = \int_0^{\bar{t}} |w(s)|^\gamma ds;\ \gamma > 0.$$

Then we will prove that $E[\eta^{-P}] < \infty, \forall\ 1 < P < \infty$. To prove this, we need a few lemmas.

Lemma A. *Let P be the Wiener measure on* $C([o,T] \to \mathbb{R}^r)$*. Then,* $\forall\ \epsilon > 0, 0 < t \leq T \exists\ C_1, C_2 > 0$ *and independent of* ϵ *and t such that*

$$P\left[\sup_{0\leq s\leq t} |w(s)| < \epsilon\right] \leq C_1 e^{-\dfrac{tc_2}{\epsilon^2}}.$$

Proof. For $X \epsilon \mathbb{R}^r, |x| < 1$, let

$$u(t,x) = P\left[\max_{0\leq s\leq t} |w(s) + x| < 1\right].$$

□

Then it well known that

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\triangle u \text{ in } \{|x| \leq 1\} \\ u|_{t=0} &= 1 \\ u|_{|x|=1} &= 0. \end{aligned}$$

Therefore, if λ_n, ϕ_n are the eigenvalues and eigenfunctions for the corresponding eigenvalue problem, then

$$u(t,x) = \sum_n e^{-\lambda_n t}\phi_n(x) \int_{|y|\leq 1} \phi_n(y)dy.$$

77 Also since $\{w(s)\} \sim \left\{\epsilon w\left(\frac{s}{\epsilon^2}\right)\right\}$ for every $\epsilon > 0$,

$$\begin{aligned} P\left[\sup_{0\leq s\leq t} |w(s)| < \epsilon\right] &= P\left[\sup_{0\leq s\leq \frac{t}{\epsilon^2}} |w(s)| < 1\right] \\ &= u\left(\frac{t}{\epsilon^2}, 0\right) \sim \phi_1(0) \int\limits_{|Y|\leq 1} \phi_1(y)dy \times e^{-\frac{\lambda_1^t}{\epsilon^2}} \end{aligned}$$

Lemma B. *Let*

$$\xi(t) = \sum_{\alpha=1}^{r} \int\limits_0^t \phi_\alpha(s,w)dW_s^\alpha + \int\limits_0^t \psi(s,w)ds.$$

Let

$$\sum_{\alpha=1}^{r} |\phi_\alpha(s,w)|^2 \leq k, |\psi(s,w)| \leq k.$$

Then, $\forall\ a > 0$ and $0 < \epsilon < \dfrac{a}{2k}, \exists\ c > 0$, independent of a, ϵ, and k such that

$$P(\tau_a < \epsilon) \leq e^{-\frac{ca^2}{k\epsilon}},$$

where $$\tau_a = \inf\{t : |\xi(t)| > a\}.$$

Proof. We know that we can write

$$\xi(t) = B(A_1(t)) + A_2(t)$$

where

$$A_1(t) = \sum_{\alpha=1}^{r} \int_0^t |\phi_\alpha(s,w)|^2 ds,$$

$$A_2(t) = \int_0^t \psi(s,w) ds$$

and $B(t)$ is a 1-dimensional Brownian motion with $B(0) = 0$. ◻

Hence

$$\{|\xi(t)| > a\} \subset \left\{|B(A_1(t))| > \frac{a}{2}\right\} U \left\{|A_2(t)| > \frac{a}{2}\right\}.$$

Further $|A_1(t)| \leq kt\ i = 1, 2$, and if **78**

$$\sigma^B_{a/2} = \inf\left\{t : |B(t)| > \frac{a}{2}\right\},$$

then

$$\left\{|B(A_1(t))| > \frac{a}{2}\right\} \subset \left\{A_1(t) > \sigma^B_{a/2}\right\}$$
$$\subset \left\{kt > \sigma^B_{a/2}\right\}$$

$$\Rightarrow \tau_a \geq \frac{a}{2k}\Lambda\sigma^B_{a/2}/k \text{ a.s.}$$

Therefore, if

$$\begin{aligned}
0 < \epsilon &< \frac{a}{2k}, \\
P[\tau_a < \epsilon] &\leq P\left[\sigma^B_{a/2} < k\epsilon\right] \\
&\leq p\left[\max_{0\leq s\leq k\epsilon} |B(s)| > \frac{a}{2}\right] \\
&\leq 2P\left[\max_{0\leq s\leq k\epsilon} B(s) > \frac{a}{2}\right] \\
&= 2\sqrt{\left(\frac{2}{\pi k\epsilon}\right)}\int\limits_{a/2}^{\infty} e^{-(x^2/k\epsilon)}dx \\
&\leq e - c.(a^2/k\epsilon).
\end{aligned}$$

Ex. 2.1 (Solution): Let $\bar{t}$ be such $0 < \bar{t} \leq T$ and for $N = 2, 3, \ldots$, define

$$\sigma_{2/N}(w) = \inf\left\{t : |w(t)| \geq \frac{2}{N}\right\}$$

and

$$\sigma^N_1(w) = \sigma_{2/N}(w)\Lambda\frac{\bar{t}}{2}.$$

Let

$$W_1 = \left\{w : \sigma_{2/N}(w) < \frac{\bar{t}}{2}\right\},$$

then, by lemma A, we have $P(w^c_1) \leq e^{-c_1N^2}$, for some constant c_1 independent of N. We denote the shifted path of $w(t)$ as

$$w^+_s(t) = w(t+s).$$

79 Define

$$\tau_{1/N}(w) = \inf\left\{t : |w(t) - w(0)| \geq \frac{1}{N}\right\}$$

and let
$$W_2 = \left\{W : \tau_{1/N}(w^+_{\sigma_1^N}) \geq \frac{\bar{t}}{N^3}\right\}.$$

Note that if $w\epsilon W_1 \cap W_2$ then $\sigma_1^N = \sigma_{2/N}$. By strong Markov property of Brownian motion, we get

$$\begin{aligned} P(W_2^C) &= P\left(\tau_{1/N} < \frac{\bar{t}}{N^3}\right) \\ &\leq e^{-C_3 N} \quad \text{(by lemma B)}. \end{aligned}$$

Define
$$\sigma_2^N(w) = \sigma_1^N + \tau_{1/N}(w^+_{\sigma_1^N})\Lambda\frac{\bar{t}}{N^3}.$$

From the definition, it follows that on W_2,

$$\sigma_2^N = \sigma_1^N + \frac{\bar{t}}{N^3}.$$

Clearly, if $t\epsilon\left[\sigma_1^N, \sigma_2^N\right]$, then $|w(t)| \leq \dfrac{3}{N}$ and if $w\epsilon W_1 \cap W_2$, then $\dfrac{1}{N} \leq |w(t)| \leq \dfrac{3}{N}$. Hence we have. for $w\epsilon W_1 \cap W_2$,

$$\begin{aligned} n(w) = \int_0^{\bar{t}} |w(s)|^\gamma ds &\geq \int_{\sigma_1^N}^{\sigma_2^N} |w(t)|^\gamma dt \\ &\geq \frac{\bar{t}}{N^3}.\frac{1}{N^Y} = \frac{\bar{t}}{N^{3+y}}. \end{aligned}$$

Now
$$P(W_1^C U W_2^C) \leq e^{-C_4 N}$$

Hence
$$P\left(\eta < \frac{\bar{t}}{N^{3+y}}\right) \leq e^{-C_4 N}, N = 2, 3, \ldots$$

which gives, by proposition 2.4, that $E(\eta^{-p}) < \infty$ for every $p > 1$. **80**

Example 2.1(a): Let

$$\eta(w) = \int_0^{\bar{t}} e^{-\frac{1}{|w(s)|^\gamma}} ds, 0 < \bar{t} \leq T, \gamma > 0;$$

then $E(\eta^{-p}) < \infty$ for all $1 < p < \infty$ when $\gamma < 2$, and for $\gamma \geq 2$ there exists p such that $E[\eta^{-p}] = \infty$.

Proof. Exercise. □

Example 2.2. Let

$$\eta(w) = \int_0^{\bar{t}} \left[\int_0^t |w(s)|^\gamma dW(s) \right]^2 dt, \text{ for } 0 < \bar{t} \leq T$$

fixed, then $E[\eta^{-p}] < \infty$, for every $1 < p < \infty$.

Proof. In example 2.1, we have seen stopping times σ_1^N and σ_2^N satisfying; $0 \leq \sigma_1^N < \sigma_2^N \leq \bar{t}, \sigma_2^N - \sigma_1^N = \frac{\bar{t}}{N^3}$ and

$$|w(u)| \leq \frac{3}{N}, \text{ if } u \epsilon \left[\sigma_1^N, \sigma_2^N\right].$$

Now, let

$$W_1 = \left\{ \sigma_2^N - \sigma_1^N = \frac{\bar{t}}{N^3} \right\},$$

$$W_2 = \left\{ W : \int_{\sigma_1^N}^{\sigma_2^N} |w(u)|^{2\gamma} du > \frac{\bar{t}}{N^{2\gamma+3}} \right\}.$$

□

By lemma B,

$$P(W_1^C) \leq e^{-C_1 N^{C_2}}$$

and we have seen that $P(W_2^C) \leq e^{-C_1 N^{C_2}}$. Let

$$\theta(s) = \int\limits_0^s |w(u)|^{2\gamma} du.$$

Then by representation theorem for martingales, there exists one-dimensional Brownian $B(t)$ such that 81

$$\int\limits_0^t |w(s)|^\gamma dW_s = B(\theta(t)).$$

For $w \epsilon W_1 \cap W_2$,

$$\begin{aligned}
\eta &= \int\limits_0^{\bar{t}} |B(\theta(t))|^2 dt \geq \int\limits_{\sigma_1^N}^{\sigma_2^N} |B(\theta(t))|^2 dt \\
&= \int\limits_{\theta(\sigma_1^N)}^{\theta(\sigma_2^N)} |B(s)|^2 d\theta^{-1}(s) \text{ changing the variables } \theta(t) \to s
\end{aligned}$$

$$\begin{array}{ll|l}
& = \int_{\theta(\sigma_1^N)}^{\theta(\sigma_2^N)} \dfrac{|B(s)|^2}{|w(\theta^{-1}(s))|^{2\gamma} ds} & s = \int_o^s \dfrac{d(\theta(u))}{|w(u)|^{2\gamma}} \\
& \geq \int_{\theta(\sigma_1^N)}^{\theta(\sigma_2^N)} |B(s)|^2 \left(\dfrac{N}{3}\right)^{2\gamma} ds & \theta^{-1}(s) = \int_o^{\theta^{-1}(s)} \dfrac{d\theta(u)}{|w(u)|^{2\gamma}} \\
& \geq \left(\dfrac{N}{3}\right)^{2\gamma} \int_{\theta(\sigma_1^N)}^{\theta(\sigma_1^N) + \frac{\bar{t}}{N^{2\gamma+3}}} |B(s)|^2 ds & = \int_o^s \dfrac{du}{|w(\theta^{-1}(u))|^{2\gamma}}
\end{array}$$

$$\text{i.e.,} \quad \eta \geq \left(\frac{N}{3}\right)^{2\gamma} \int\limits_{\theta(\sigma_1^N)}^{\theta(\sigma_1^N) + \frac{\bar{t}}{N^{2\gamma+3}}} |B(s)|^2 ds. \tag{2.6}$$

To proceed further, we need the following lemma whose proof will be given later.

Let $I = [a, b]$ and for $f \epsilon L^2(I)$ define

$$\bar{f} = \frac{1}{b-a} \int\limits_I f(x) dx$$

and

$$V_I(f) = \frac{1}{b-a} \int\limits_I (f(x) - \bar{f}^2) dx$$

82 V_I has following properties:

(i) $V_I(f) \geq 0 \ \ \forall \ f \ \epsilon \ L^2(I)$

(ii) $V_I^{1/2}(f+g) \leq V_I^{1/2}(f) + V_I^{1/2}(g)$

(iii) $V_I(f) \leq \dfrac{1}{b-a} \int\limits_I (f(x) - k)^2 dx$ for any constant k.

Lemma C . *Let $B(t)$ be any one-dimensional Brownian motion on $I = [0, a]$. Then the random variable $V_{[0,a]}(B)$ satisfies:*

$$P\left[V_{[0,a]}(B) < \epsilon^2\right] \leq \sqrt{2} e^{-\dfrac{a}{2^7 \epsilon^2}}, \ \textit{for every } \epsilon, a > 0.$$

From (2.6), using the property (iii) of V_I, we get

$$\eta \geq \left(\frac{N}{3}\right)^{2\gamma} V_{\left[\theta(\sigma_1^N), \theta(\sigma_1^N) + \bar{t}/(N^{2\gamma+3})\right]}(B) \frac{\bar{t}}{N^{2\gamma+3}}.$$

Now let

$$W_3 = \left\{ w : \frac{\bar{t}}{3^{2\gamma} N^3} V_{[\theta(\sigma_1^N), \theta(\sigma_1^N) + t/(N^{2\gamma+3})]}(B) > \frac{\bar{t}}{N^m} \right\}$$

Then by lemma C, we have, for sufficiently large m,

$$\begin{aligned} P(W_3^C) &\leq e^{-C_3 N^{(m-3)-(2\gamma+3)}} \\ &\leq e^{-C_3 N^{C_4}}. \end{aligned}$$

Hence on $W_1 \cap W_2 \cap W_3$, $n \geq \dfrac{\bar{t}}{N^m} \geq \dfrac{1}{N^{C_5}}$. Now

$$\begin{aligned} P((W_1 \cap W_2 \cap W_3)^c) &\leq P(W_1^c) + P(W_2^c) + P(W_3^c) \\ &\leq e^{-c_6^{N^{C_7}}}. \end{aligned}$$

Hence by proposition 2.4, it follows that

$$E[\eta^{-p}] < \infty, \ \forall \quad 1 < p < \infty.$$

Proof of Lemma C: Using the scaling property of Brownian motion, **83**
we have

$$aV_{[\circ,1]}(B) \sim V_{[\circ,a]}(B).$$

Therefore, it is enough to prove that

$$p\left[V_{[\circ,1]}(B) < \epsilon^2\right] \leq \sqrt{2}e^{-1(2^7\epsilon^2)}.$$

For $t\epsilon[\circ,1]$, we can write

$$B(t) = t\xi_0 + \sqrt{2}\sum_{k=1}^{\infty}\left[\xi_k\left\{\frac{\cos(2\pi kt) - 1}{2\pi k}\right\} + \eta_k\frac{\sin 2\pi kt}{2\pi k}\right]$$

where $\{\xi_k\}, \{\eta_k\}$ are $i.i.d.N(\circ,1)$ random variables. Therefore

$$B(t) - \int_0^1 B(s)ds = \left(t - \frac{1}{2}\right)\xi_0 + \sqrt{2}\sum_{k=1}^{\infty}\left[\xi_k\frac{\cos 2\pi kt}{2\pi k} + \eta_k\frac{\sin 2\pi kt}{2\pi k}\right].$$

Note that the functions $\left\{t - \frac{1}{2}, \sin 2\pi kt\right\}$ are orthogonal to $\{\cos 2\pi kt\}$ in $L^2[\circ,1]$. Therefore

$$V = V_{[\circ,1]}(B) \geq \sum_{k=1}^{\infty}\xi_k^2 \times \frac{1}{(2\pi k)^2}.$$

Hence

$$E(e^{-2z^2V}) \leq E\left(e^{-2z^2}\sum_k \xi_k^2/(2\pi k)^2\right)$$

$$= \prod_k E\left(e^{-Z^2\xi_k^2/2\pi^2k^2}\right)$$

$$= \prod_k \left(1 + \frac{Z^2}{\pi^2k^2}\right)^{-1/2} = \sqrt{\left(\frac{Z}{\sin h\, z}\right)}$$

$$\leq \sqrt{2}e^{-z/4}.$$

Therefore

$$P(V < \epsilon^2) \leq e^{2z^2\epsilon^2} E(e^{-2z^2v})$$

$$\leq \sqrt{2}e^{2z^2\epsilon^2 - \frac{Z}{4}}, \ \forall\ z.$$

84 Taking $z = \dfrac{1}{16\epsilon^2}$, we get

$$P(V_{[0,1]}(B) < \epsilon^2) \leq \sqrt{2}e^{-1/(2^7\epsilon^2)}.$$

Example 2.3. Let

$$\xi(t) = \xi(0) + \sum_{\alpha=1}^{\gamma} \int_0^t \xi_\alpha(s) dW_s^\alpha + \int_0^t \xi_0(s) ds$$

and suppose $\exists$ a sequence of stopping times σ_1^N, σ_2^N,

$N = 2, 3, \ldots$, such that $0 \leq \sigma_1^N \leq \sigma_2^N \leq \bar{t}$ and

(i) $\sigma_2^N - \sigma_1^N \leq \dfrac{\bar{t}}{N^3}$.

(ii) $\sum\limits_{\alpha=1}^{\gamma} |\xi_\alpha(s)|^2 + |\xi_0(s)| \leq c_1,\ \forall\ s\epsilon\left[\sigma_1^N, \sigma_2^N\right]$,

(iii) $P\left[\sigma_2^N - \sigma_1^N < \dfrac{\bar{t}}{N^3}\right] \leq e^{-c_2 N^{C_3}}$

(iv) $P\left[\int\limits_{\sigma_1^N}^{\sigma_{2^N}} |\xi(t)|^2 dt \leq \dfrac{1}{N_4^C}\right] \leq e^{-c_2 N^{c_3}}$

where $c_i > 0$, $i = 1, 2, 3, 4$ are all independent of N. Let

$$\eta(t) = \eta(0) + \int\limits_0^t \xi(s) ds$$

and
$$\eta = \int\limits_0^{\bar{t}} |\eta(s)|^2 ds (\geq \int\limits_{\sigma_1^N}^{\sigma_2^N} |\eta(s)|^2 ds).$$

Then $\eta^{-1} \epsilon L_P$, $\forall\ 1 < p < \infty$. This follows from the estimate $\exists\ c_5 > 0, c_6 > 0, c_7 > 0$ (all independent of N) such that

$$P\left[\int\limits_{\sigma_1^N}^{\sigma_2^N} |\eta(t)|^2 dt \leq \frac{1}{N^{C_5}}\right] \leq e^{-c_6 N^{c_7}}.$$

To prove this, we need a few lemmas:

Lemma D. *Let* 85

$$\xi(t) = \xi_0 + \sum_{\alpha=1}^{\gamma} \int\limits_0^t \xi_\alpha(s) dW_s^\alpha + \int\limits_0^t \xi_0(s) ds.$$

Let
$$\sup_{t_1 < s \leq t_2} \sum_\alpha |\xi_\alpha(s)|^2 + |\xi_0(s)| \leq c.$$

Then $\forall\ 0 < \gamma < \frac{1}{2}, \exists\ c_1 > 0, c_2 > 0$ *such that*

$$P\left[\sup_{s,t,\epsilon[t_1,t_2]} \frac{|\xi(t) - \xi(s)|}{|t - s|^\gamma} > N\right] \leq e^{-c_1 N^{c_2}}, N = 2, 3, \ldots$$

Proof. Since we can always write

$$\xi(t) = \xi(0) + B\left(\int\limits_0^t \sum_\alpha (s)^2 ds\right) + \int\limits_0^t \xi_0(s) ds$$

where $B(t)$ is a 1-dimensional Wiener process, it is enough to prove the Lemma when $\xi(t) = B(t)$. For $w \epsilon W_0^r$, let

$$||w||_\gamma = \sup_{s,t\epsilon[0,T]} \frac{|w(t) - w(s)|}{|t - s|^\gamma}.$$

□

Let

$$W_\gamma = \left\{ w \epsilon W_0^r : ||w||_\gamma < \infty \right\}.$$

Then $W_\gamma \subset W_0^r$ is a Banach space and if $0 < \gamma < 1/2$, using the Kolmogorov-Prohorov theorem, it can be shown that P can be considered as a probability measure on W_γ (cf. Ex. 1.2 with $k(t, s) = t \Lambda s$). Therefore by Fernique's theorem,

$$E(e^{\alpha ||w||_\gamma^2}) < \infty$$

for some $\alpha > 0 \Rightarrow E(e^{||w||_\gamma)<\infty}$. Therefore

$$P(||w||_\gamma > N) \leq e^{-N} E[e^{||w||_\gamma}]$$
$$\leq e^{-c_1 N^{c_2}}$$

86 **Lemma E.** *Let $f(s)$ be continuous on $[a, b]$ and let*

$$\frac{|f(t) - f(s)|}{|t - s|^{1/3}} \leq k$$

and
$$\int_a^b |f(t)|^2 dt > \epsilon^2 \text{ where } \epsilon^3 \leq 2^2 k^3 (b - a)^{5/2}.$$

Let

$$g(t) = g(a) + \int_a^t f(s) ds.$$

Then

$$(b - a) V_{[a,b]}(g) \geq \frac{1}{2^9.48} \frac{\epsilon^{11}}{k^9 (b - a)^{1+9/2}}.$$

Proof. $\exists \quad t_o \epsilon [a, b]$ such that $|f(t_o)| > \dfrac{\epsilon}{(b-a)^{1/2}}$.

Therefore $|f(s)| \geq |f(t_o)| - |f(t_o) - f(s)|$ implies

$$|f(s)| \geq \frac{\epsilon}{2(b-a)^{1/2}} \text{ if } |t_o - s| \leq \frac{\epsilon^3}{k^3 2^3 (b-a)^{3/2}}.$$

We denote by I the interval of length

$$|I| = \frac{\epsilon^3}{k^3 2^3 (b-a)^{3/2}}$$

which is contained in $[a, b]$ and is of the form $[t_o, t_o + |I|]$ or $[t_o - |I|, t_o]$. Such I exists, since

$$\frac{\epsilon^3}{k^3 2^3 (b-a)^{3/2}} \leq \frac{b-a}{2}.$$

Note that $f(s)$ has constant sign in I. Therefore

$$\begin{aligned}(b-a)V_{[a,b]}(g) &= \int\limits_a^b (g(s) - \overline{g})^2 ds \\ &\geq \int\limits_I (g(s) - \overline{g})^2 ds \\ &\geq \int\limits_I (g(s) - \overline{g}|_I)^2 ds.\end{aligned}$$

But we can always find $t_1 \epsilon I$ with $\overline{g}|I = g(t_1)$. Therefore 87

$$\begin{aligned}(b-a)V_{[a,b]}(g) &\geq \int\limits_I \left(\int\limits_{t_1}^s f(u) du \right)^2 ds \\ &\geq \frac{\epsilon^2}{4(b-a)} \int\limits_I (s - t_1)^2 ds \\ &\geq \frac{\epsilon^2}{4(b-a)} \int\limits_\alpha^\beta \left(s - \frac{\alpha+\beta}{2} \right)^2 ds \text{ where } I = (\alpha, \beta)\end{aligned}$$

$$= \frac{1}{48} \frac{\epsilon^2}{(b-a)} |I|^3.$$

□

Proof of ex. 2.3: Let

$$W_1 = \left\{ \sigma_2^N - \sigma_1^N = \frac{\bar{t}}{N^3} \right\}.$$

$$W_2 = \left\{ \sup_{s,t \epsilon [\sigma_1^N, \sigma_2^N]} \frac{|\xi(t) - \xi(s)|}{|t-s|^{1/3}} \leq N \right\}$$

$$W_3 = \left\{ \int_{\sigma_1^N}^{\sigma_2^N} |\xi(t)|^2 dt \geq \frac{1}{N^{c_4}} \right\}.$$

Then by Lemma D and assumptions (iii) and (iv), we get

$$P\left(w_1^c \cup w_2^c \cup W_3^c\right) \leq e^{-a_1 N^{a_2}}, a_1 > 0, a_2 > 0.$$

Hence, if $w \epsilon W_1 \cap W_2 \cap W_3$, by Lemma E, we can choose $c_5 > 0$ such that

$$\left(\sigma_1^N - \sigma_1^N\right) V_{[\sigma_1^N, \sigma_2^N]}(\eta) > \frac{1}{N^{c_5}}$$

and since

$$V_{[\sigma_1^N, \sigma_2^N]}(\eta) \leq \frac{1}{\sigma_2^N - \sigma_1^N} \int_{\sigma_1^N}^{\sigma_2^N} |\eta(t)|^2 dt$$

we have

$$P\left[\int_{\sigma_1^N}^{\sigma_2^N} |\eta(t)|^2 dt \leq \frac{1}{N^{c_5}} \right] \leq e^{-a_1 N^{a_2}}.$$

88 **Key Lemma:** Let $\eta(t) = \eta(0) + \Sigma_{\alpha=1}^r \int_o^t \eta_\alpha(s) dW_s^\alpha + \int_o^t \eta_o(s) ds$ where $\eta_o(t)$

is also an Itô process given by

$$\eta_o(t) = \eta_o(0) + \sum_{\beta=1}^{r} \int_o^t \eta_{o\beta}(s) dW_s^\beta + \int_o^t \eta_{oo}(s) ds.$$

Suppose we have sequences of stopping times $\{\sigma_1^N\}, \{\sigma_2^N\}$ such that $0 \leq \sigma_1^N < \sigma_2^N \leq \bar{t}$ for $0 < \bar{t} \leq T, N = 2, 3, \ldots$ and satisfying

(i) $\sigma_2^N - \sigma_1^N \leq \dfrac{\bar{t}}{N^3}$,

(ii) $P\left(\sigma_2^N - \sigma_1^N < \dfrac{\bar{t}}{N^3}\right) \leq e^{-c_1 N^{c_2}}, \exists$ for some $C_1, c_2 > 0$

(iii) $\exists\, c_3 > 0$ such that for a.a.w

$$|\eta(t)| + \sum_{\alpha=o}^{r} |\eta_\alpha(t)| + \sum_{\beta=o}^{r} |\eta_{o\beta}(t)| \leq c_3$$

for every $t \epsilon \left[\sigma_1^N, \sigma_2^N\right]$.

Then for any given $c_4 > o, \exists\, c_5, c_6, c_7 > o$ (which depend only on c_1, c_2, c_3, c_4) such that

$$P\left[\int_{\sigma_1^N}^{\sigma_2^N} |\eta(t)|^2 dt \leq \frac{1}{N^{c_5}}, \sum_{\alpha=o}^{r} \int_{\sigma_1^N}^{\sigma_2^N} |\eta_\alpha(t)|^2 dt > \frac{1}{N^{c_4}}\right]$$
$$\leq e^{-c_6 N^{c_7}}, N = 2, 3, \ldots.$$

Proof. For simplicity, we take $\bar{t} = 1$. Let

$$W_1 = \left[\sigma_2^N - \sigma_1^N = \frac{1}{N^3}\right]$$
$$W_2 = \left[\sup_{s,t\epsilon[\sigma_1^N, \sigma_2^N]} \frac{|\eta_o(t) - \eta_o(s)|}{|t-s|^{1/3}} \leq N\right]$$

then, by the hypothesis (ii), (iii) and Lemma D, $\exists$ constants $d_1, d_2 > 0$ 89
such that

$$P(W_1^c \cup W_2^c) \leq e^{-d_1 N^{d_2}}. \tag{2.7}$$

Now, by representation theorem, on $[\sigma_1^N, \sigma_2^N], \eta(t)$ can be written as

$$\eta(t) = \eta(\sigma_1^N) + B(A(t)) + g(t) \tag{2.8}$$

where

$$A(t) = \int_{\sigma_1^N}^{t} \sum_{\alpha=1}^{r} |\eta_\alpha(s)|^2 ds, g(t) = \int_{\sigma_1^N}^{t} \eta_o(s) ds$$

and $B(t)$ is one-dimensional Brownian motion with $B(0) = 0$. ◻

In Ex. 2.3, we obtained that, for every $a_1 > 0, \exists\, a_2 > 0$ such that

$$\left[V_{[\sigma_1^N, \sigma_2^N]}(g) \leq \frac{1}{N^{a_2}}\right] \subset W_1^c \cup W_2^c \cup \left[\int_{\sigma_1^N}^{\sigma_2^N} |\eta_o(t)|^2 dt < \frac{1}{2N^{a_1}}\right]. \tag{2.9}$$

Let

$$W_3 = \left[\sum_{\alpha=o}^{r} \int_{\sigma_1^N}^{\sigma_2^N} |\eta_\alpha(t)|^2 dt \geq \frac{1}{N^{c_4}}\right].$$

Choose a_3 such that $a_3 > c_4 + 1$, which implies

$$\frac{1}{2N^{c_4}} > \frac{1}{N^{a_3}}, N = 2, 3, \ldots$$

Therefore

$$W_3 \subset \left[\int_{\sigma_1^N}^{\sigma_2^N} |\eta_o(t)|^2 dt \geq \frac{1}{2N^{c_4}}\right] \cup \left[A\left(\sigma_2^N\right) \geq \frac{1}{2N^{c_4}}\right]$$

$$\subset W_{3,1} \cup W_{3,2}$$

where

$$W_{3,1} = \left[\int\limits_{\sigma_1^N}^{\sigma_2^N} |\eta_o(t)|^2 dt > \frac{1}{2N^{c_4}}, A\left(\sigma_2^N\right) < \frac{1}{N^{a_3}} \right]$$

$$W_{3,2} = \left[A\left(\sigma_2^N\right) \geq \frac{1}{N^{a_3}} \right]$$

90

In (2.9), taking $a_1 = c_4$, we get, $\exists\ a_2 > 0$ such that

$$\left[V_{[\sigma_1^N, \sigma_2^N]}(g) \leq \frac{1}{N^{a_2}}, \int\limits_{\sigma_1^N}^{\sigma_2^N} |\eta_o(t)|^2 dt > \frac{1}{2N^{c_4}} \right] \subset W_1^c \cup W_2^c.$$

So, in particular,

$$W_{3,1} \cap \left[V_{[\sigma_1^N, \sigma_2^N]}(g) \leq \frac{1}{N^{a_2}} \right] \subset W_1^c \cup W_2^c. \tag{2.10}$$

Let

$$W_4 = \left[\int\limits_{\sigma_1^N}^{\sigma_2^N} |\eta(t)|^2 dt < \frac{1}{N^{a_4}} \right],$$

where a_4 is some constant which will be chosen later. Then, for $w \epsilon W_4 \cap W_1$,

$$V_{[\sigma_1^N, \sigma_2^N]}(\eta) \leq \frac{1}{\left(\sigma_2^N - \sigma_1^N\right)} \int\limits_{\sigma_1^N}^{\sigma_2^N} |\eta(t)|^2 dt \leq \frac{N^3}{N^{a_4}}$$

i.e.

$$V_{[\sigma_1^N, \sigma_2^N]}(\eta) \leq \frac{1}{N^{a_s}} \text{ if } a_4 \geq a_5 + 3. \tag{2.11}$$

Let

$$W_5 = \left[\sup_{0 \leq u \leq 1/(N^{a_3})} |B(u)| \leq \frac{1}{N^{a_5}} \right]$$

then, by Lemma A,

$$P(W_5^c) \leq d_3 e^{-N}, \text{ if } a_3 > 2a_5 + 1. \tag{2.12}$$

91 Now, for $w \epsilon W_{3,1} \cap W_4 \cap W_1 \cap W_5$, by (2.8),

$$\begin{aligned} V^{1/2}_{[\sigma_1^N, \sigma_2^N]}(g) &\leq V^{1/2}_{\sigma_1^N, \sigma 2^N}(\eta) + V^{1/2}_{[\sigma_1^N, \sigma_2^N]}(B(A(t))) \\ &\leq \frac{1}{N^{a_5/2}} + \frac{1}{N^{a_5/2}} \\ &\quad \text{(by (2.11) and definition of} W_5 \text{and since, on} \\ &\qquad [\sigma_1^N, \sigma_2^N], 0 \leq A(t) \leq \frac{1}{N^{a_3}}) \\ &= \frac{2}{N^{a_5/2}}. \end{aligned}$$

Now choose a_5 such that $\frac{2}{N^{a_5/2}} \leq \frac{1}{N^{a_2}}$; then

$$V_{[\sigma_1^N, \sigma_2^N]}(g) \leq \frac{1}{N^{a_2}}.$$

Hence

$$W_{3,1} \cap W_4 \cap W_1 \cap W_5 \subset \left[V_{[\sigma_1^N, \sigma_2^N]}(g) \leq \frac{1}{N^{a_2}} \right]$$

which implies by (2.10) that

$$W_{3,1} \cap W_4 \cap W_1 \cap W_5 \subset W_1^c \cup W_2^c.$$

Therefore

$$W_{3,1} \cap W_4 \subset W_1^c \cup W_2^c \cup W_5^c.$$

So choosing $a_3 \geq c_4 + 1, a_3 > 2a_5 + 1, a_5 > 2(a_2 + 1)$ and $a_4 \geq a_5 + 3$, we can conclude form (2.7) and (2.12) that

$$P\left[W_1^c \cup W_2^c \cup W_5^c\right] \leq e^{-d_4 N^{d_5}}, \ \forall\ N = 2, 3, \ldots.$$

for some constants $d_4 > 0$ and $d_5 > 0$ and therefore

$$P\left[W_{3,1} \cap W_4\right] \leq e^{-d_4 N^{d_5}}, \ \forall\ N = 2, 3, \ldots. \tag{2.13}$$

92 Next we prove that $W_{3,2} \cap W_4$ is also contained in a set which is exponentially small, i.e.,

$$P(W_{3,2} \cap W_4) \leq^{-d_6 N^{d_7}}$$

for some $d_6 > 0, d_7 > 0$.

For $w \epsilon W_1$, we divide $\left[\sigma_1^N, \sigma_2^N\right] = \left[\sigma_1^N, \sigma_1^N + \frac{1}{N^3}\right]$ into N^m subintervals of the same length viz.

$$I_k = \left[\sigma_1^N + \frac{k}{N^{3+m}}, \sigma_1^N + \frac{k+1}{N^{3+m}}\right], k = 0, 1, \ldots N^m - 1.$$

Also, we choose $m > a_3$. Then

$$\int\limits_{I_k} |\eta(t)|^2 dt = \int\limits_{I_k} |\eta(\sigma_1^N) + B(A(t)) + g(t)|^2 dt \tag{2.14}$$

$$= \int\limits_{A(I_k)} |\eta(\sigma_1^N) + B(s) + g(A^{-1}(s))|^2 dA^{-1}(s)$$

$$\left(\text{where } A(I_k) = \left[A\left(\sigma_1^N + \frac{k}{N^{3+m}}\right), A\left(\sigma_1^N + \frac{k+1}{N^{3+m}}\right)\right]\right)$$

$$\geq \frac{1}{c} \int\limits_{A(I_k)} |\eta(\sigma_1^N) + B(s) + g(A^{-1}(s))|^2 ds$$

$$\left(\text{since } A(t) = \int\limits_{\sigma_1^N}^{t} a(s) ds \Rightarrow dA^{-1}(s) = \frac{ds}{a(A^{-1}(s))}\right.$$

$$\left.\text{and } a(s) = \sum_{\alpha=1}^{r} |\eta_\alpha(s)|^2 \leq c\right).$$

Let

$$J_k = \left[A(\sigma_1^N + \frac{k}{N^{3+m}}), A(\sigma_1^N + \frac{k}{N^{3+m}}) + \frac{1}{N^{a_3+m}}\right].$$

Note that $J_k' s$ are of constant length. Then

$$W_1 \cap \left[|A(I_k)| \geq \frac{1}{N^{a_3+m}}\right] \subset W_1 \cap [A(I_k) \supset J_k]$$

$$\subset W_1 \cap \left[\int\limits_{I_k} |\eta(t)|^2 dt \geq \frac{1}{c} \int\limits_{J_k} |\eta(\sigma_1^N) + B(s) + g(A^{-1}(s))|^2 ds \right] \text{ by 2.14}$$

$$\subset W_1 \cap \left[\int\limits_{I_k} |\eta(t)|^2 dt \geq \frac{|J_k|}{c} V_{J_k}(B(.) + \tilde{g}) \right] \text{ (where } \tilde{g} = g(A^{-1}))$$

$$\subset W_1 \cap \left[\int_{I_k} |\eta(t)|^2 dt \geq \frac{|J_k|}{c} \left(V_{j_k}^{1/2}(B) - V_{J_k}^{1/2}(\tilde{g}) \right)^2 \right]. \tag{2.15}$$

93 Since

$$g = \int\limits_o^t \eta_o(s) ds \text{ and } |\eta_o(s)| \leq c \text{ on } \left[\sigma_1^N, \sigma_2^N\right],$$

$$|\tilde{g}(t) - \tilde{g}(s)| \leq c|A^{-1}(t) - A^{-1}(s)|.$$

Therefore with

$$t_o = A\left(\sigma_1^N + \frac{k}{N^{3+m}}\right),$$

$$\begin{aligned}
V_{J_k}(\tilde{g}) &\leq \frac{1}{|J_k|} \int\limits_{J_k} |\tilde{g}(t) - \tilde{g}(t_o)|^2 dt \\
&\leq \frac{c^2}{|J_k|} \int\limits_{J_k} (A^{-1}(t) - A^{-1}(t_o))^2 ds \\
&\leq c^2 \left[A^{-1} \left\{ A\left(\sigma_1^N + \frac{k}{N^{3+m}}\right) + \frac{1}{N^{a_3+m}} \right\} - \left(\sigma_1^N + \frac{k}{N^{3+m}}\right) \right]^2 \\
&\leq c^2 [\sigma_1^N + \frac{k+1}{N^{3+m}} - (\sigma_1^N + \frac{k}{N^{3+m}})]2 (\text{ since } J_k \subset A(I_k)) \\
&= \frac{c^2}{N^{6+2m}}.
\end{aligned} \tag{2.16}$$

Hence

$$W_1 \cap \left[J_k^{1/2}(B) > \frac{2c}{N^{3+m}}, |A(I_k)| \geq \frac{1}{N^{a_3+m}} \right] \tag{2.17}$$

$$\subset W_1 \cap \left[\int_{I_k} |(\eta)|^2 dt \geq \left[c\frac{1}{N^{3+m}}\right]^2 \frac{N_{|J_k|}}{c}\right] \text{ by 2.15 and 2.16}$$
$$= W_1 \cap \left[\int_{I_k} |\eta(t)^2 dt \geq \frac{c}{N^{6+3m+a}3}\right].$$

94

Let

$$W_6 = \bigcap_{k=o}^{N^m-1} \left[V_{J_k}^{1/2}(B) \geq \frac{2c}{N^{3+m}}\right].$$

Since

$$A(\sigma_2^N) = \sum_{k=0}^{N^m-1} |A(I_k)|, w\epsilon W_1 \cap W_{3,2}$$
$$\Rightarrow \exists\, k \ni |A(I_k)| \geq \frac{1}{N^{a_3+m}}$$
$$\Rightarrow W_1 \cap W_{3,2} \subset \cup_{k=0}^{N^m-1} \left\{|A(I_k)| > \frac{1}{N^{a_3+m}}\right\}.$$

Therefore

$$W_1 \cap W_6 \cap W_{3,2} \subset \bigcup_{k=0}^{N^m-1} \left\{\left[|A(I_k)| > \frac{1}{N^{a_3+m}}\right], V_{J_k}^{1/2}(B) \geq \frac{2c}{N^{3+m}}\right\} \cap W_1$$
$$\subset \bigcup_{k=0}^{N^m-1} \left[\int_{I_k} |\eta(t)|^2 dt \geq \frac{c}{N^{6+3m+a_3}}\right] \cap W_1 \text{ by 2.17}$$
$$\subset \left[\int_{\sigma_1^N}^{\sigma_2^N} |\eta(t)|^2 dt \geq \frac{c}{N^{6+3m+a_3}}\right] \cap W_1. \tag{2.18}$$

Therefore, if we choose a_4 such that

$$\frac{1}{N^{a_4}} < \frac{c}{N^{6+3m+a_3}},\ \forall\, N = 2, 3, \ldots,$$

then (2.18) implies $W_1 \cap W_6 \cap W_{3,2} \cap W_4 = \phi$, which implies $W_{3,2} \cap W_4 \subset W_1^c \cup W_6^c$.

$$
\begin{aligned}
P(W_6^c) &\leq \sum_k P\left[V_{J_k}^{1/2}(B) < \frac{2c}{N^{3+m}}\right] \\
&\leq N^m e^{-d_8 |J_k| \backslash (2c \backslash (N^{3+m}))^2} \ \forall\ k \text{ (by Lemma (C))} \\
&= N^m e^{-d_9 N^{6+2m-a_3-m}} \\
&\leq N^m e^{d_9 N^6} \text{ (since } m > a_3) \\
&\leq e^{-d_{10} N^{d_{11}}} \qquad (2.19)
\end{aligned}
$$

95 Choosing $c_5 = a_4$, (2.13) and (2.19) give us the required result

2.3 Regularity of Transition Probabilities

We are now going to obtain a sufficient condition for (A.2) to be satisfied in the case of X_t which is the solution to (2.1).

We recall that

$$
\begin{aligned}
L_\alpha(x) &= \sum_{i=1}^d \sigma_\alpha^i(x) \frac{\partial}{\partial x^i}, \alpha = 1, 2 \ldots, r \\
L_o(x) &= \sum_{i=1}^d \tilde{b}i(x) \frac{\partial}{\partial x^i}
\end{aligned}
$$

where

$$
\tilde{d}^i(x) = b^i(x) - \frac{1}{2} \sum_{k,\alpha} \partial_k \sigma_\alpha^i(x) \sigma_\alpha^k(x).
$$

Let

$$
\begin{aligned}
\Sigma_0 &= \{L_1, L_2, \ldots, L_r\} \\
\Sigma_1 &= \{[L_\alpha, L] : L \epsilon \Sigma_o, \alpha = 0, 1, \ldots, r\} \\
\ldots & \quad \ldots \quad \ldots \quad \ldots \\
\Sigma_n &= \{[L_\alpha, L] : L \epsilon \Sigma_{n-1}, \alpha = 0, 1, \ldots, r\}.
\end{aligned}
$$

96 Therefore

$$L\epsilon \sum_n \Rightarrow \exists\ \alpha_o \epsilon \{1,2,\ldots r\}, \alpha_i \epsilon \{0,\ldots r\}, i = 1\ldots n$$

such that

$$L = [L_{\alpha_n}[\ldots[L_{\alpha_2}[L_{\alpha_1}, L_{\alpha_o}]]\ldots].$$

Let

$$(L_\alpha, L) := [L_\alpha, L], \alpha = 1, 2\ldots r$$
$$(L_0, L) := [L_o, L] + \frac{1}{2}\sum_{\beta=1}^{r}[L_\beta, [L_\beta, L]].$$

Then we have

$$f_L^i(r_t) - f_L^i(r_o) = \sum_{\alpha=1}^{r}\int_0^t f^i_{(L_\alpha, L)}(r_s)dW_s^\alpha + \int_o^t f^i_{(L_0, L)}(r_s)ds$$

where f_L^i, r_t etc. are as in proposition 2.3. Let

$$\Sigma'_o = \Sigma_o$$
$$\cdots\cdots$$
$$\Sigma'_n = \{(L_\alpha, L) : L\epsilon\Sigma'_{n-1}\};$$

then

$$L\epsilon \sum_n{}' \text{ implies}$$
$$L = (L_{\alpha_n}, (L_{\alpha_{n-1}} \cdots (L_{\alpha_1}, L_{\alpha_o}))\cdots)$$
$$= L_{\alpha_o}, \alpha_1\cdots\alpha_n$$

for some

$$\alpha_o\epsilon\{1, 2, \ldots, r\}, \alpha_i\epsilon\{0, \ldots, r\}, i = 1, \ldots, n.$$

Let

$$\hat{\Sigma}'_m = \Sigma'_o \cup \Sigma'_1 \cup \cdots \cup \Sigma'_m,$$
$$\hat{\Sigma}_m = \Sigma_o \cup \Sigma_1 \cup \cdots \cup \Sigma_m$$

It is easy to see that the following two statements are equivalent: 97

(i) at $x\epsilon R^d, \exists\ M$ and $A_1, A_2 \ldots, A_d \epsilon \hat{\Sigma}_{M'}$ such that $A_1(x), A_2(x) \ldots A_d(x)$ are linearly independent.

(ii) at $x\epsilon R^d, \exists\ M$ and $A_1, A_2 \ldots, A_d \epsilon \hat{\Sigma}_{M}$ such that $A_1(x), A_2(x) \ldots A_d(x)$ are linearly independent.

Theorem 2.7. *Suppose for $x\epsilon\mathbb{R}^d$, $\exists\ M > 0$ and $A_1, A_2, \ldots, A_d \epsilon \hat{\Sigma}_{M'}$ such that $A_1(x), A_2(x), \ldots, A_d(x)$ are independent. Then, for every $t > 0$,*

$$X_t = (X_1(t,x,w), X_2(t,x,w), \ldots, X_d(t,x,w)),$$

which is the solution of (2.1), satisfies (A.2) and hence the probability law of $\chi(t,x,w)$ has C^∞-density $p(t,x,y)$.

Remark 1. $p(t,x,y)$ is the fundamental solution of

$$\frac{\partial u}{\partial t} = \left[\frac{1}{2}\sum_{\alpha=1}^{r} L_\alpha^2 + L_o\right] u$$

$$u|_{t=o} = f$$

i.e., $$u(t,x) = \int_{\mathbb{R}^d} p(t,x,y) f(y) dy.$$

Remark 2. The general equation

$$\frac{\partial u}{\partial t} = \left[\frac{1}{2}\sum_{\alpha=1}^{r} L_\alpha^2 + L_o + c(.)\right] u, \quad \text{where } c\epsilon C_b^\infty(\mathbb{R}^d)$$

has also C^∞-fundamental solution and is given by

$$p(t,x,y) =< \Delta_y(X(t,x,w)), G(w) >$$

98 where

$$G(w) = e^{\int_o^t c(X(t,x,w))ds} \epsilon \mathbb{D}_\infty.$$

Remark 3. The hypothesis in the theorem 2.6 is equivalent to the following: For $x\epsilon\mathbb{D}^d$, $\exists\ M > 0$ such that

$$\inf_{\ell\epsilon S^{d-1}} \sum_{A\epsilon\hat{\Sigma}_{M'}} < A(x), \ell >^2 > 0 \tag{2.20}$$

where

$$S^{d-1} = \{l\epsilon\mathbb{D}^d : |\ell| = 1\}.$$

Proof of theorem 2.7. By (2.20), $\exists\ \epsilon_o > 0$ and bounded neighbourhood $U(x)$ of x in $\mathbb{R}^d$, $U(I_d)$ in $GL(d, \mathbb{R})$ such that

$$\inf_{\ell \epsilon S^{d-1}} \sum_{A \epsilon \hat{\sum}_{M'}} < (e^{-1}A)(y), \ell >^2 \geq \epsilon_o \tag{2.21}$$

for every $y \epsilon U(x)$ and $e \epsilon U(I_d)$. Let $l \epsilon S^{d-1}$ and A be any vector field. Define

$$f_A^{(l)}(r) =< f_A(r), \ell >,$$

(cf. definition 2.1) where $<, >$ is the inner product in $\mathbb{R}^d$; then we have the corresponding Itô formula as

$$f_A^{(\ell)}(r_t) - f_A^{(\ell)}(r_o) = \sum_{\alpha=1}^{r} \int_o^t f_{(L_\alpha, A)}^{(\ell)}(r_s) dW_s^\alpha + \int_o^t f_{(L_o, A)}^{(\ell)}(r_s) ds.$$

where $r_t = (Y_t, Y_t), X_t, Y_t$ being the solution of (2.1), (2.2) respectively. **99**

Recall that

$$\hat{O}_t^{ij} = \sum_{\alpha=1}^{r} \int_o^t f_{L_\alpha}^i(r_s) f_{L_\alpha}^j(r_s) ds$$

and by proposition 2.5, to prove the theorem, it is enough to prove that $(\det \hat{\Sigma}_t^{-1} \epsilon) \epsilon L_p$ for $1 < p < \infty$. Now

$$\begin{aligned} < \hat{\sigma}_t \ell, \ell > &= \sum_{i,j=1}^{d} \hat{\Sigma}_t^{i,j} \ell^i \ell^j, \ell = (\ell^1, \ell^2, \ldots, \ell^d) \\ &= \sum_{\alpha=1}^{r} \int_o^t [f_{L_\alpha}^{(\ell)}(r_s)]^2 ds. \end{aligned}$$

Let $A \epsilon \hat{\sum}_{M'}$. Note that $A \epsilon \hat{\sum}_{M'}$ implies $\exists\ n$, $0 \leq n \leq M$ and $\alpha_i \epsilon$ $\{0, 1, 2, \ldots, r\}, 0 \leq i \leq n, \alpha_o \neq 0$, such that

$$A = L_{\alpha_o, \alpha_1, \ldots, \alpha_n}.$$

Also note that the number of elements in $\hat{\sum}_{M'}$ is

$$\sum_{n=o}^{M} r(r+1)^n = k(M)(\text{ say }).$$

Define the stopping time σ by

$$\sigma = \inf\{t : (X_t, Y_t) \notin U(x) \times U(I_d)\}$$

By lemma B, for $\overline{t} > 0$, we have

$$P\left(\sigma < \frac{\overline{t}}{N3}\right) \leq e^{-c_1 N^3}.$$

Now in the Key lemma, set for $N = 2, 3, \ldots, \sigma_1^N = 0$ and

$$\sigma_2^N = \sigma \Lambda \frac{\overline{t}}{N3}.$$

100 Then the following are satisfied:

(i) $0 \leq \sigma_1^N < \sigma_2^N \leq \overline{t}, \sigma_2^N - \sigma_1^N \leq \frac{\overline{t}}{N3}$.

(ii) $P(\sigma_2^N - \sigma_1^N < \frac{\overline{t}}{N3}) \leq e^{c_1 N^3}$,

(iii) If we set

$$C = \sup_{l \epsilon S^{d-1}} \sup_{r \epsilon U(x) \times U(I_d)} \sum_{A \epsilon \hat{\Sigma}_{M'+1}} \left[f_A^{(l)}(r)\right]^2,$$

then for

$$t \epsilon \left[\sigma_1^N, \sigma_2^N\right], \sum_{A \epsilon \hat{\Sigma}_{M'+1}} \left[f_A^{(l)}(r)\right]^2 \leq C < \infty.$$

For

$$w \epsilon W_1 = \left\{\sigma_2^N - \sigma_1^N = \frac{\overline{t}}{N3}\right\},$$

by choice $U(x) \times U(I_d)$ and (2.21), we have

$$\inf_{|\ell|=1} \int\limits_{\sigma_1^N}^{\sigma_2^N} \sum_{A \epsilon \hat{\Sigma}_{M'}} [f_A^{(\ell)}(r_s)]^2 ds \geq \epsilon_o \frac{\overline{t}}{N^3} \tag{2.22}$$

Choose $\gamma > 0$ such that

$$\frac{1}{k(M)} \frac{\epsilon_o \overline{t}}{N^3} \geq \frac{1}{N^\gamma}.$$

For $A = L_{\alpha_o, \alpha_1, \ldots, \alpha_n} \epsilon \hat{\sum}_{M'}$ and $\ell \epsilon S^{d-1}$, define

$$W_k^{A,\ell} = \int\limits_{\sigma_1^N}^{\sigma_2^N} \left[f^{(\ell)} L_{\alpha_o, \alpha_1, \ldots, \alpha_{k-1}}(r_s) \right]^2 ds < \frac{1}{N^{C_{k-1}}},$$

$$\sum_{\alpha=0}^{r} \int\limits_{\sigma_1^N}^{\sigma_2^N} \left[f^{(\ell)} L_{\alpha_o, \alpha_1, \ldots, \alpha_{k-1} \alpha}(r_s) \right]^2 ds \geq \frac{1}{N^{C_k}}, k = 1, 2, 3, \ldots, n,$$

where $C_n, C_{n-1}, \ldots C_o$ are obtained applying Key Lemma successively **101**
as follows:

Let $C_n = \gamma > 0$. Then by Key Lemma, $\exists\, C_{n-1}, a_n, b_n$ such that

$$P(W_n^{A,\ell}) \leq e^{-a_n N^{b_n}}.$$

Now again by Key Lemma, for given $C_{n-1}, \exists\, C_{n-2}, a_{n-1}, b_{n-1}$ such that

$$P(W_{n-1}^{A,\ell}) \leq e^{-a_{n-1} N^{b_{n-1}}}.$$

And proceeding like this, we see that given $C_1, \exists\, C_o, a_1, b_1$ such that

$$P(W_1^{A,\ell}) \leq e^{-a_1 N^{b_1}}.$$

Hence we see that

$$P(W_n^{A,\ell}) \leq e^{-aN^b, k=1,2,\ldots,n,}$$

where $$a = \min\{a_i\}_{1 \leq i \leq n}, b = \min\{b_i\}_{1 \leq i \leq n}.$$

Note that $C_n, C_{n-1}, \ldots C_o$ and a, b are independent of ℓ since they depend only on γ, C and c_1. Let

$$W^{A,\ell} = \bigcup_{k=1}^{n} W_k^{A,\ell}. \text{ Then } P(W^{A,\ell}) \leq e^{-a' N^{b'}}$$

and

$$P(W(\ell)) \leq e^{-a'' N^{b''}} \text{ where } W(\ell) = \bigcup_{A \epsilon \hat{\Sigma}_{M'}} W^{A,\ell}. \tag{2.23}$$

102 From (2.22), for $w \epsilon W_1$, we get

$$\int_{\sigma_1^N}^{\sigma_2^N} \sum_{A \epsilon \hat{\Sigma}_{M'}} [f_A^{(l)}(r_s)]^2 ds \geq \frac{\epsilon_o}{N^3} \bar{t} \leq k(M) \frac{1}{N^\gamma}.$$

Hence $\exists\, A \epsilon \hat{\Sigma}_{M'}$ such that

$$\int_{\sigma_1^N}^{\sigma_2^N} \left[f_A^{(l)}(r_s) \right]^2 ds \geq \frac{1}{N^\gamma}.$$

Hence if $A = L_{\alpha_o, \alpha_1, \ldots, \alpha_n}$,

$$\sum_{\alpha=o}^{r} \int_{\sigma_1^N}^{\sigma_2^N} \left[f^{(\ell)}_{L_{\alpha_o, \alpha_1, \ldots, \alpha_{n-1} \alpha}}(r_s) \right]^2 ds \geq \frac{1}{N^\gamma}. \tag{2.24}$$

Now suppose $w \epsilon W_1 \cap W(l)^c$ which implies $w \notin W_k^{A,\ell}$ for every $A \epsilon \hat{\sum}_M$ and $k = 1, 2, \ldots, n$. Then by definition of $W_k^{A,\ell}$ and by (2.24), it follows that

$$\int_{\sigma_1^N}^{\sigma_2^N} \left[f^{(\ell)}_{L_{\alpha_o, \alpha_1, \ldots, \alpha_{n-1} \alpha}}(r_s) \right]^2 ds \geq \frac{1}{N^{C_{n-1}}}$$

and consequently

$$\sum_{\alpha=o}^{r} \int_{\sigma_1^N}^{\sigma_2^N} \left[f^{(\ell)}_{L_{\alpha_o,\alpha_1,\ldots,\alpha_{n-1},\alpha}}(r_s) \right]^2 ds \geq \frac{1}{N^{C_{n-1}}}. \tag{2.25}$$

And $w \notin W^{A,\ell}_{n-1}$ together with (2.25) gives

$$\int_{\sigma_1^N}^{\sigma_2^N} \left[f^{(\ell)}_{L_{\alpha_o,\alpha_1,\ldots,\alpha_{n-2},}}(r_s) \right]^2 ds \geq \frac{1}{N^{C_{n-2}}}.$$

Continuing like this, we get

$$\int_{\sigma_1^N}^{\sigma_2^N} \left[f^{(\ell)}_{L_{\alpha_o}}(r_s) \right]^2 ds \geq \frac{1}{N^{C_o}}.$$

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Now, let $\bar{c} = \max\left\{ C_o = C_o(A) : A \epsilon \hat{\sum}_{M'} \right\}$. Then we have $\sum_{\alpha=1}^{r} \int_{\sigma_1^N}^{\sigma_2^N} \left[f^{(\ell)}_{L_{\alpha_o}}(r_s) \right]^2 ds \geq \frac{1}{N^{\bar{c}}}$. Hence we have proved that for $\ell \epsilon s^{d-1}$ and $w \epsilon W_1 \cap W(\ell)^c$, $\exists\, \bar{c} > o$ (independent of ℓ) such that

$$\sum_{\alpha=1}^{r} \int_{\sigma_1^N}^{\sigma_2^N} \left[f^{(\ell)}_{L_\alpha}(r_s) \right]^2 ds \geq \frac{1}{N^{\bar{c}}}. \tag{2.26}$$

We have

$$\sigma^{ij}_{\bar{t}} = \sum_{\alpha=1}^{r} \int_{o}^{\bar{t}} f^{i}_{L_\alpha}(r) f^{j}_{L_\alpha}(r_s) ds.$$

Now let

$$q^i j = \sum_{\alpha=1}^{r} \int_{\sigma_1^N}^{\sigma_2^N} f^{i}_{L_\alpha}(r_s) f^{j}_{L_\alpha}(r_s) ds.$$

Note that

$$\sum_{\alpha=1}^{r}\int\limits_{\sigma_1^N}^{\sigma_2^N}\left[f^{(\ell)}_{L_\alpha}(r_s)\right]^2 ds=\sum_{i,j=1}^{d} q^{ij}\ell^i\ell^j = Q(\ell) \quad (\text{ say}).$$

Also, $\det_{\sigma\bar{t}} \geq \det q \geq \lambda_1^d$ where $\lambda_1 = \inf\limits_{|l|=1} Q(l)$, the smallest eigenvalue of q. Hence to prove that $\sigma_t^{-1}\epsilon L_p$,it is sufficient to prove that $\lambda_t^{-1}\epsilon L_p, \ \forall\ p$.

By definition of q^{ij}, we see that $\exists\ c'$ such that $|q^{ij}| \leq \frac{c'}{N^3}$. Therefore

$$|Q(\ell) - Q(l')| \leq \frac{c''}{N^3}|\ell - \ell'|. \tag{2.27}$$

Hence $\exists\ l_1, l_2, \ldots l_m$ such that

$$\bigcup_{k=1}^{m} B\left(\ell_k; \frac{N^3}{2c''}{}_{N^{\bar{c}}}\right) = S^{d-1},$$

where $B(x, s)$ denotes ball around x with radius s.

104 Also it can be seen that $m \leq c''' N^{\bar{c}-3)d}$. Then, $\ell \epsilon S^{d-1}$ implies $\exists\ \ell_k$ such that $|\ell - \ell_k| \leq \frac{N^3}{2c''}{}_N\bar{c}$. Hence by (2.27)

$$|Q(l) - Q(l_k)| \leq \frac{1}{2N^{\bar{c}}}.$$

But for $w\epsilon W_1 \cap (\cap W(\ell_k)^c), Q(\ell_k) \geq \frac{1}{2N^c}$. Hence for

$$w\epsilon W_1 \cap (\cap W(\ell_k)^c), Q(\ell) \geq \frac{1}{2N^{\bar{c}}}.$$

So

$$\inf_{|\ell|=1} Q(l) \leq \frac{1}{2N^{\bar{c}}} \text{ on } W_1\left(\bigcap_{k=1}^{m} W(\ell_k)^c\right)$$

i.e., $\lambda_1 \geq \frac{1}{2N^c}$ on $W_1 \bigcap \left(\bigcap_{k=1}^m W(l_k)^c\right)$.

But we have

$$P(W_1^c U W(\ell)) \leq e^{-\bar{a}N^{\bar{b}}}$$

and hence

$$P\left[W_1^C \bigcup \left(\bigcup_{k=1}^{m} W(l_k)\right)\right] \le c''' N^{(\bar{c}-3)d} e^{-\bar{a}N^{\bar{b}}}.$$

i.e.,

$$P\left[W_1^C \bigcup \left(\bigcup_{k=1}^{m} W(l_k)\right)\right] \le e^{-\bar{a}N^{\bar{b}_1}}$$

which gives the result.

A more general result is given below whose proof is similar to that of theorem 2.7.

Theorem 2.8. *Let* 105

$$U_M(x) = \inf_{|l|=1} \sum_{A\epsilon\hat{\Sigma}_{M'}} < A(x), \ell >^2 .$$

Suppose for $x\epsilon\mathbb{R}^d$, $\exists\ M > 0$ *and* $U(x)$, *neighbourhood of* x *such that for every* $\bar{t} > 0$

$$P\left[U_M(Xt) < \frac{1}{N} \text{ for all } t\epsilon[0, \bar{t}\Lambda\tau_{U(x)}]\right] = 0\left(\frac{1}{N^k}\right) \text{ as } N \to \infty \text{ for all } k > 0$$

$$(\text{ where } \tau_{U(x)} = \inf\{t : X_t \not\epsilon U(x)\}).$$

Then the same conclusion of theorem 2.7 holds.

NOTES ON REFERENCES

106 Malliavin calculus, a stochastic calculus of variation for Wiener functionals, has been introduced by Malliavin [7]. It has been applied to regularity problem of heat equations in Malliavin [8], Ikeda-Watanabe [3], Stroock [16], [17], [18]. The main material in Chapter 2 is an introduction to the recent result of Kusuoka and Stroock on this line. In Chapter 1, we develop the Malliavin calculus following the line developed by Shigekawa [13] and Meyer [10].

Chapter 1:

1.1. (a) For the theory of Gaussian measures on Banach spaces, Fernique's theorem and abstract Wiener spaces, cf Kuo [5].

(b) That the support of a Gaussian measure on Banach space is a linear space can be found in Itô [4].

(c) For the details of Ex. 1.2, cf. Baxendale [1].

1.2. (a) An interesting exposition on Ornstein Uhlenbeck semigroups and related topics can be found in Meyer [10].

(b) The hyper-contractivity of Ornstein Uhlenbeck semigroup (Theorem 1.3) was obtained by Nelson [11]. Cf. also Simon [14] and, for an interesting and simple probabilistic proof, Neveu [12].

107 (c) For the fact stated in Def. 1.8, we refer to Kuo [5].

1.3. (a) For a general theory of countably normed linear spaces and their duals, we refer to Gelfand-Silov [2].

(b) For Ex. 1.3, details can be found in Ikeda-Watanabe [3], Chap.VI, Sections 6 and 8. Cf. also Stroock [19].

(c) Littlewood-Paley inequalities for a class of symmetric diffusion semigroups have been obtained by Meyer [9] as an application of Burkholder's inequalities for martingales, which include the inequalities (1.7) and (1.9) as special cases. Cf. also Meyer [10]. An analytical approach to Littlewood-Paley theory can be seen in E.M. Stein [15]

(d) L_p multiplier theorem in Step 2 was given by Meyer. Proof here based on the hyper-contractivity is due to Shigekawa (in an unpublished note).

(e) The proof of Theorem 1.9 given here is based on the hand-written manuscript of Meyer distributed in the seminars at Paris and Kyoto, cf, also Meyer [10].

(f) The spaces of Sobolev-type for Wiener functionals were introduced by Shigekawa [13] and Stroock [16], cf. also [3]. By using the results of Meyer, they are more naturally and simply defined as we did in this lecture.

1.4. (a) The composite of Wiener functionals and Schwartz distributions was discussed in [21] for the purpose of justifying what is called "Donsker's δ - functions", cf. also Kuo [5], [6]. **108**

1.5. (a) The result on the regularity of probability laws was first obtained by Malliavin [8].

Chapter 2:

2.1. (a) For the general theory of stochastic calculus; stochastic integrals, Itô processes and SDE's we refer to Ikeda-Watanabe [3], Stroock [19] and Varadhan [20].

(b) For the proof of approximation theorem 2.3, we refer to [3], chapter V, Lemma 2.1.

2.2. The key lemma was first obtained, in a weaker form, by Malliavin [8]. Cf. also [3]. The Key lemma in this form is due to Kusuoka and Stroock (cf. [18]) where the idea in Ex. 2.3 plays an important role.

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