Lectures on Three-Dimensional Elasticity

By

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Tata Institute of Fundamental Research
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Three-Dimensional Elasticity

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These Lecture Notes are dedicated to
Professor K.G. Ramanathan

Avant-Propos

When studying any physical problem in Applied Mathematics, three essential stage are involved. 1

1. Modelling: An appropriate mathematical model, based on the physics or the engineering of the sitution, must be found. Usualluy these models are given **a pariori** by the physicists or the engineers themselves. However, mathematicians can also play an important role in this process especially considering the increasing emphasis on non - linear models of physical problems.

2. Mathematical study of the model: A model usally involves a set of ordinary' or partial differential equations or an (energy) functional to be minimized. One of the first tasks is to find a suitable functional space in which to study the problem. Then comes the study of existence and uniqueness or non -uniqueness of solutions. An important feature of linear theories is the existence of unique solutions depending continuousslly on the data (Hadamard's definition of well - posed problems). But with non-linear problems, non-uniqueness is a prevealent phenomenon. For instance, bifuracation of solutions is of special interest.

3. Numerical analysis of the model: By this is meant the description of, and the mathematical analysis of, approximation schemes, which **can** be run on a computer in a 'reasonable' time to get 'reasonably accurate' answers.

In the following set of lectures the first two of the above aspects will be studied with reference to the theory of elasticity in three dimensions.

2 In the first chapter a non-linear system of partial differential equations will be established as a mathematical model of elasticity. The non-linearity will appear in the highest order terms and this is an important source of difficulties. An energy functional will be established and it will be seen that the equations of equilibrium can be obtained as the Euler equations starting from the energy functional.

Existence results will be studied in the second chapter. The two important tools will be the use of the implicit function theorem and the theory of J. BALL.

Contents

Chapter 1

Description of Three - Dimensional Elasticity

THIS CHAPTER WILL be divided into four sections. In the first sec- **3**
tion some preliminaries on deformations in $\mathbb{R}^3$ will be discussed; the second will be devoted to the equations of eqilibrium and the third to constitutive equations. These together will give rise to the boundary value problem which will serve as the model for three - dimensional elasticity. The last section will describe the energy functional and the associated Euler equations will be seen to give the equations of equilinrium and the constitutive equations.

1.1 Geometrical Preliminaries

Let $\Omega \subset \mathbb{R}^3$ be a bounded open set. Let $\mathfrak{B}_R = \bar{\Omega}$, the closure of Ω in $\mathbb{R}^3$, stand for the *reference configuration*. (The subsript R will always stand for the reference configuration.) Let X_R be a generic point in $\mathfrak{B}_R$. If $\{e_1, e_2, e_3\}$ is the standard orthonormal basis for $\mathbb{R}^3$,

$$OX_R = X_{R_i} e_i \tag{1.1-1}$$

where OX_R stands for the position vector of X_R. (In the above relation **4**
and in all that follows, the summation convention for repeated indices will always be adopted.)

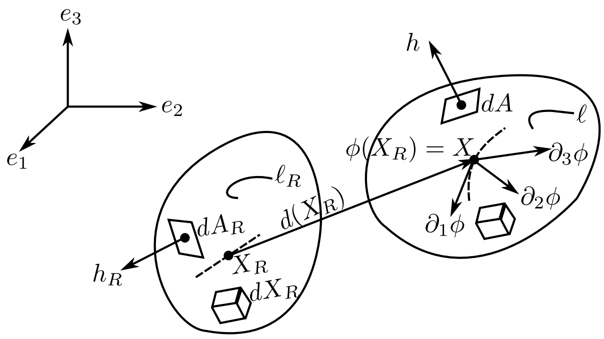

Figure 1.1.1:

Let $\phi : \mathfrak{B}_R \to \mathbb{R}^3$ be a sufficiently regular mapping. It is said to be a *deformation* if

(1.1-2) $$\det(\nabla\phi) > 0$$

where $\nabla\phi$ is called the *deformation gradient* and is a matrix given by

$$\nabla\phi \begin{vmatrix} \frac{\partial\phi_1}{\partial X_{R_1}} & \frac{\partial\phi_1}{\partial X_{R_2}} & \frac{\partial\phi_1}{\partial X_{R_3}} \\ \frac{\partial\phi_2}{\partial X_{R_1}} & \frac{\partial\phi_2}{\partial X_{R_2}} & \frac{\partial\phi_2}{\partial X_{R_3}} \\ \frac{\partial\phi_3}{\partial X_{R_1}} & \frac{\partial\phi_3}{\partial X_{R_2}} & \frac{\partial\phi_3}{\partial X_{R_3}} \end{vmatrix}$$

ϕ_i being the components of ϕ.

5 **Remark 1.1.1.** From (1.1-2) it follows that ϕ is locally one - one, though it may not be globally so.

The image set $\mathfrak{B} = \phi(\mathfrak{B}_R)$ is called the *deformed configuration*. Note that the mapping ϕ can be written as

(1.1-3) $$\phi = Id + u$$

and the mapping $u : \mathfrak{B}_R \to \mathbb{R}^3$ is called the *displacment*. It is also seen that

(1.1-4) $$\nabla\phi = I + \nabla u$$

where I is the identity matrix and ∇u is the *displacement gradient*.

The deformation gradient defines the deformation at $X = \phi(X_R)$ up to first order. If $dt\ e_1$ is a line segment parallel to e_1 at X_R, it is transformed into a curve at X whose tangent is $dt\ \partial_1\phi$, where $\partial_1\phi$ is the first colums vector of $\nabla\phi$. The magnitude dt is now 'streched' by dt $|\partial_1\phi|$, where $|.|$ stands for the Euclidean norm. The three vectors $\partial_1\phi$, $\partial_2\phi$, $\partial_3\phi$ are independent and, owing to the relation (1.1-2), preserve the orientation of $\{e_1, e_2, e_3\}$.

It will now be seen how volume, area and line elements are transformed under the defomation ϕ.

(i) *Volume elements:* The change from a volume element dX_R to dX in the deformed confirguration comes from the familiar change of variable formula in interation theory:

$$dx = \det(\nabla\phi(X_R))dX_R. \tag{1.1-5}$$

(ii) *Surface elements:* If dA_R is a surface element on $\mathscr{B}_R$ deformed onto a surface elemment dA on $\mathfrak{B}$, then

$$dA = \det(\nabla\phi(X_R))|(\nabla\phi(X_R))^{-t} n_R| dA_R \tag{1.1-6}$$

where n_R is the unit outer normal. (If F is any matrix, F^T stands **6**
for its transpose, F^{-1} for its inverse and $F^{-T} = (F^{-1})^T$).

The formula (1.1-6) will now be proved . This needs some preliminaries. Let $\mathbb{M}^3$ stand for the set of all 3×3 matrices. A tensor will be understood simply to be an element of $\mathbb{M}^3$.

Let $T : \mathfrak{B} \to \mathbb{M}^3$ be a tensor field. Then its *divergence* (assuming T to be smooth snough) is defined by

$$DIVT = \frac{\partial T_{ij}}{\partial X_j} e_i. \tag{1.1-7}$$

Thus each component of $DIV\ T$ is the divergence (in the usual sence) of the corresponding *row* vector of T. By a standard application of Green's formula it follows that

$$\int_{\mathfrak{B}} DIVT dX = \left(\int_{\mathfrak{B}} \frac{\partial T_{ij}}{\partial X_j} dX \right) e_i = \left(\int_{\partial\mathfrak{B}} T_{ij} n_j dA \right) e_j = \int_{\partial\mathfrak{B}} T_n dA \tag{1.1-8}$$

where n is the units outer normal to $\mathfrak{B}$. In the same vein $DIV_R(T_R)$ on tensor fields on $\mathfrak{B}_R$ can be defined and the analogue of (1.1-8) can be obtained.

Let $T : \mathfrak{B} \to \mathbb{M}^3$ be a tensor field. Its*Piola Transform* is a tensor field $T_R : \mathfrak{B}_R \to \mathbb{M}^3$ given by

$$T_R(X_R) = \det(\nabla\phi(X_R))T(X)(\nabla\phi(X_R))^{-T} \tag{1.1-9}$$

were $X = \phi(X_R)$.

This is a very useful transformation. The following theorem will establish the formula (1.1-6).

7 **Theorem 1.1.1.** *(i)* $T_R(X_R)n_R dA_R = T(X)ndA$

(ii) $\det(V\phi(X_R))(V\phi(X_R))^{-T}n_R dA_R = ndA$

(iii) $\det(\nabla\phi(X_R))|(\nabla\phi(X_R))^{-T}n_R|dA_R = dA.$

Proof. It can be shown that (cf. Exercise 1.1-1). □

$$DIV_R T_R(X_R) = \det(V\phi(X_R))DIVT(X) \tag{1.1-10}$$

If v_R is any arbitrary volume in $\mathfrak{B}_R$ and $\vartheta = \phi(v_R)$, then

$$\begin{aligned}\int_{\partial v_R} T_R(X_R)n_R dA_R &= \int_{\partial v_R} DIV_R T_R(X_R)dX_R \\ &= \int_{v_R} \det(\nabla\phi(X_R))DIVT(X_R))dX_R \\ &= \int_{v_R} DIVT(X)dX = \int_{\partial v} T(X)ndA\end{aligned}$$

which, as v was arbritrary, proves (i). The assertion (ii) follows by setting $T = I$. This is a vector relation and taking the Euclidean norm on both sides gives (iii).

Remark 1.1.2. The matrix $\det(\nabla\phi)(\nabla\phi)^{-T} = (\operatorname{adj}\nabla\phi)^{-T}$ is the matrix of cofactors of $\nabla\phi$.

(iii) Line elements: If ϕ is smooth enough, $\phi(X_R + \delta X_R) - \phi(X_R) = \nabla\phi(X_R)\delta X_R + o(\delta X_R)$.

Thus

$$|\phi(X_R+\delta X_R)-\phi(X_R)|^2 = \delta X_R^T\nabla\phi(X_R)^T\nabla\phi(X_R)\delta X_R+o(|\delta X_R|^2) \tag{1.1-11}$$

which gives the change in length. The matrix

$$C = \nabla\phi^T\nabla\phi \tag{1.1-12}$$

is called the *(right) Cauchy - Green strain tensor* and will play an im- 8
portant role in the theory. It is used to compute the length of an arc. If $f(I)$ is a curve ℓ_R in $\mathfrak{B}_R$, where $I \subset \mathbb{R}$ is an interval, and $\ell = \phi(\ell_R)$ is its image in $\mathfrak{B}$, then the length of ℓ is given by

$$\int_I |(\phi o f)'(t)|dt = \int_I \sqrt{c_{ij}(f(t))f_i'(t)f_j'(t)}dt$$

where C_{ij} are the components of the matrix C defined above.

Remark 1.1.3. The matrix

$$B = \nabla\phi\nabla\phi^T \tag{1.1-13}$$

called the *(left) Cauchy-Green strain tensor* will be introduced later and will play an important role in the constitutive equations.

Remark 1.1.4. The change in volume depends on a scalar $\det\nabla\phi$. The change is surface elements depends on a matrix, $(\operatorname{adj}\nabla\phi)$ and the change in line elements on a matrix, $C = \nabla\phi^T\nabla\phi$. All these will figure in the integral representing the energy (cf. Sect. 2.6).

To conclude this section, it will now be examined to what extent the strain tensor C is a measure of the deformation. The word 'deformation' can be interpreted in two ways - first the formal sense as defined earlier in this section; secondly, in an intuitive way which can be described as follows. If ϕ were merely to consist of a translation and then a rotation about a point in space, while it is a deformation in the strict sense, yet distances between points are not altered. So intuitively the body has not been 'deformed', Such a transformation is called a rigid deformation.

Thus, ϕ is said to be a *rigid deformation* if

$$\phi(X_R) = a + Q(OX_R), \tag{1.1-14}$$

9 where a $\in \mathbb{R}^3$ and Q is an orthogonal matrix whose determinant is +1.

The vector a above represents a translation and the matrix Q a rotation. The following notation will used for various classes of matrices:

$$\begin{aligned}
\mathbb{M}^3_+ &= \{F \in \mathbb{M}^3 |\det(F) > 0\}\\
\mathbb{O}^3 &= \{F \in \mathbb{M}^3 | F^T F = FF^T = I\}\\
\mathbb{O}^3_+ &= \{F \in \mathbb{O}^3 |\det(F) = +1\}\\
\mathbb{S}^3 &= \{F \in \mathbb{M}^3 | F^T = F\}\\
\mathbb{S}^3_> &= \{F \in \mathbb{S}^3 | F \text{is positive definite}\}.
\end{aligned}$$

Thus $Q \in \mathbb{O}^3_+$. Observe that if ϕ is rigid then $C = Q^T Q = I$. In fact, under suitable hypotheses, the converse is also true.

Theorem 1.1.2. *Let Ω be an open connected subset of $\mathbb{R}^3$. Let $\phi \in C^1(\Omega; \mathbb{R}^3)$ such that for all $x \in \Omega$,*

$$\nabla\phi(x)^T \nabla\phi(x) = I \tag{1.1-15}$$

Then, there exists a vector $a \in \mathbb{R}^3$ and a matrix $Q \in \mathbb{O}^3$ such that, for all $x \in \Omega$

$$\phi(x) = a + Q(0x). \tag{1.1-16}$$

Proof. Cf. Exercise 1.1-2 □

Theorem 1.1.3. *Let Ω be an open connected subset of $\mathbb{R}^3$ and let $\phi, \psi \in C^1(\Omega; \mathbb{R}^3)$ such that for all $x \in \Omega$*

$$\nabla\phi(x)^T \nabla\psi(x) = \nabla\psi(x)^T \nabla\phi(x). \tag{1.1-17}$$

10 *Assume further that* ψ *is one - one and that* $\det(\nabla\psi(x)) \neq 0$ *for all* $x \in \Omega$*. Then there exists* $a \in \mathbb{R}^3$ *and* $Q \in O^3$ *such that for all* $x \in \Omega$

$$\phi(x) = a + Q\psi(x). \tag{1.1-18}$$

Proof. Consider the mapping $\theta = \phi o \psi^{-1}$ on $\psi(\Omega)$. Clearly $\psi(\Omega)$ is connected. Also, under the given conditions, it is open by the theorem of invariance of domain. Further, $\theta \in C^1(\psi(\Omega); \mathbb{R}^3)$. Now from (1.1-17) it follows that θ satisfies (1.1-15) and so the previous theorem applies to θ and the result follows. □

Thus if two deformations have the same strain tensor then, upto a rigid deformation, they are the same. Thus C 'measures' the 'deformation' upto a rigid transformation. Naturally, a measure of the deviation from a rigid deformation is obtained from $C - I$. *The Green-St Venant strain tensor*, E, is defined by the relation

$$C - I = 2E \tag{1.1-19}$$

In terms of the displacement gradient,

$$I + 2E = C = \nabla\phi^T \nabla\phi = I + \nabla u^T + \nabla u + \nabla u^T \nabla u$$

or, componentwise,

$$E_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i + \partial_i u_m \partial_j u_m) \tag{1.1-20}$$

where ∂_i stands for $\dfrac{\partial}{\partial X_{R_i}}$

Exercises

1.1-1 Prove the *Piola identity* 11

$$DIV_R(det(\nabla\phi(X_R))(\nabla\phi((X_R))^{-T}) = 0.$$

Deduce the relation (1.1-10) from this.

1.1-2. Prove Theorem 1.1.2 (Hint: First show that at least locally, ϕ is an isometry; then show $\nabla\phi$ is locally contant and use the connectendness of Ω.)

1.1-3. Let $\phi : \mathbb{R}^n \to \mathbb{R}^n, n \geq 2$, be continuous. Assume that there exists $\ell > 0$ such that for all $x, y \in \mathbb{R}^n$ with $|x - y| = \ell, |\phi(x) - \phi(y)| = \ell$. Show that ϕ is an isometry, i.e. there exists $a \in \mathbb{R}^n$ and $Q \in \mathbb{O}^n$ such that for all $x \in \mathbb{R}^n$

$$\phi(x) = a + Qx.$$

1.1-4. Given a tensor field $\Gamma : \Omega \to \mathbb{S}^3$, find necessary and sufficient conditions such that there exists a mapping $\phi : \Omega \to \mathbb{R}^n$ with

$$\Gamma = \nabla\phi^T \nabla\phi$$

1.2 Euilibrium Equations

The equilibrium equations give the relationship between the given forces acting on a body and the state of "stress" (to be defined below) which results as a consequence of these forces.

Let the mass density as $X \in \mathfrak{B}$ be given by $\rho(X)$ while that at $X_R \in \mathfrak{B}_R$ is given by $\rho_R(X_R)$. The applied forces in $\mathfrak{B}$ are of two kinds.

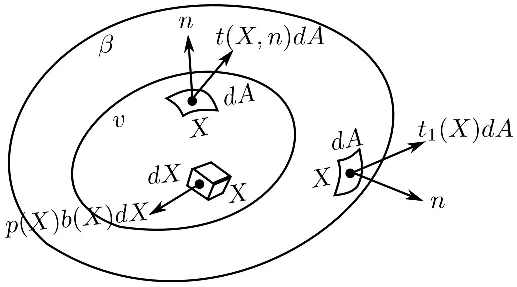

Figure 1.2.1:

12

(i) *Body (*or *volumic) forces:* $b : \mathfrak{B} \to \mathbb{R}^3$. The elementary force on a volume element dX will thus be $\rho(X)b(X)dX$. An example of a body force is gravity and in this case $b = (o, o, -g)$.

(ii) *Applied surface forces:* $t_1 : \partial\mathfrak{B}_1 \to \mathbb{R}^3$, where $\partial\mathfrak{B}_1$ is a portion of the boundary $\partial\mathfrak{B}$. If dA is a surface element, the applied force on it will be $t_1 dA$. An example of a surface force is a pressure load where $t_1 = -pn$, $p \in \mathbb{R}$, n the normal to dA.

A *system of forces* in $\mathfrak{B}$ consists of *body forces* (identical to (i) above) and *surface forces* $t : \mathfrak{B} \times \Sigma_1 \to \mathbb{R}^3$ where Σ_1 is the unit sphere in $\mathbb{R}^3$, i.e.,

$$\Sigma_1 = \{x \in \mathbb{R}^3 | |x| = 1\}.$$

If ϑ is any subvolume of $\mathfrak{B}$, dA a surface element of $\partial\vartheta$ and n the normal to it, the surface force $t(X, n)dA$ acts in it. Note that this is independent of ϑ, i.e. if ϑ_1 were another subvolume and dA lay on ∂v_1 with the same n as normal, the force acting on it will remain as $t(X, n)dA$. **13**
Further if $dA \subset \partial\mathfrak{B}$ and n were also normal to $\partial\mathfrak{B}$, it is required that

$$t(X, n) = t_1(X). \tag{1.2-1}$$

The vector $t(X, n)$ is called the *Cauchy stress vector*.

The following axiom is the basis of Continuum Mechanics in general, and consequently of the theory of elasticity in particular.

AXIOM OF STATIC EQUILIBRIUM. Let $\mathfrak{B}$ *be a deformed configuration in static equilibrium. There exists a system of forces such that for any subdomain* $\vartheta \subset \mathfrak{B}$*,the corresponding system of forces is equivalent to zero (in the sense of torsors). Thus*

$$\int_{\vartheta} \rho(X)\, b(X)\, dX \;+\; \int_{\partial\vartheta} t(X, n)\, dA = o. \tag{1.2-2}$$

$$\int_{\vartheta} OX \Lambda \rho(X) b(X) dX + \int_{\partial\vartheta} OX \Lambda t(x, n) dA = o. \tag{1.2-3}$$

The wedge Λ stands for the usual cross product of vectors in $\mathbb{R}^3$. The following notation will be useful in manipulating cross products.

For indices i, j, k taking values 1, 2, 3 the tensor of rank 3, ϵ_{ijk}, is defined by

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if}(i, j, k)\text{is an even permutation of}(1,2,3), \\ -1 & \text{if it is an odd permutation of}(1,2,3), \\ 0 & \text{otherwise}. \end{cases} \tag{1.2-4}$$

Then for vector $a, b\epsilon\mathbb{R}^3$,

$$a\Lambda b = \epsilon_{ijk} a_j b_k e_i. \tag{1.2-5}$$

14 The following consequence of the axiom of static equilibrium is of paramount importance.

Theorem 1.2.1 (Cauchy's Theorem). *Let* $\rho\epsilon C^\circ(\mathfrak{B}\ ;\ \mathbb{R})$, $b\epsilon C^\circ(\mathfrak{B};\mathbb{R}^3)$, $t(.,n)\ \epsilon C^1(\mathfrak{B};\mathbb{R}^3)$ *and* $t(X,.)\epsilon C^\circ(\sum_1;\mathbb{R}^3)$. *Then there exists a tensor field* $T\epsilon C^1\ (\mathfrak{B};\ \mathbb{M}^3)$ *such that*

$$t(X,n) = T(X)n, \text{ for all } X\epsilon\mathfrak{B}, n\epsilon\Sigma_1, \tag{1.2-6}$$

$$DIVT(X) + \rho(X)\, b(X) = 0, \text{ for all } X\epsilon\mathfrak{B}, \tag{1.2-7}$$

$$T(X) = T^T(X), \text{ for all } X\epsilon\mathfrak{B}. \tag{1.2-8}$$

Proof. Let X_0 be any point in $\mathfrak{B}$. Consider a tetrahedron ϑ with vertices X_0, V_1, V_2, V_3 as shown in Fig. 1.2.2. □

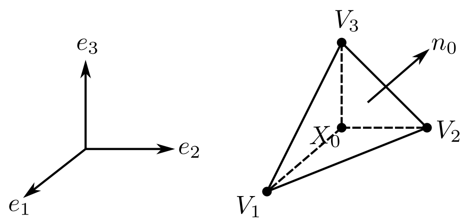

Figure 1.2.2:

Let $n_0\epsilon \sum_1$ be the normal to the plane $V_1V_2V_3$ and keep $n_\circ$ fixed to begin with. Let the distance of $X_\circ$ to the plane be δ, let $S_i(i = 1,2,3)$ be the surface opposite the vertex $V_i(i = 1, 2, 3)$ and S the surface opposite X_0. Since ρ, b are continuous on $\mathfrak{B}$, they are bounded. Thus by (1.2.2),

$$|\int_{\partial\vartheta} t(x, n)dA| \leq KVol(\vartheta)$$

K being a constant independent of δ. Since $\mathrm{Vol}(\vartheta) = K_1\delta^3$, $A(\delta) =$ Area **15**
of $S = K_2\delta^2, K_1, K_2$ being independent of δ, it follows that

(1.2-9) $$\lim_{\delta\to O} \frac{1}{A(\delta)} \int_{\partial\vartheta} t(X, n)dA = 0$$

Now,

$$\lim_{\delta\to O} \frac{1}{A(\delta)} \int_{S} t(X, n)\, dA = t(X_O, n_O)$$

and

$$\lim \delta \to 0 \frac{1}{A(\delta)} \int_{S_i} t(X, n)dA = (n_O.e_i)t(X_O, -e_i)$$

using the continuity of the given functions. Hence by (1.2-9),

(1.2-10) $$t(X_\circ, n_\circ) = -(n_\circ.e_i)t(X_\circ, -e_\circ).$$

If $n_\circ \to e_j$, again by continuity of t it follows that

$$t(X_\circ, e_j) = -t(X_\circ, -e_j).$$

Thus, on substituting this in (1.2-10),

(1.2-11) $$t(X_O, n_O) = t(X_O, e_j)n_j.$$

Setting

$$t(X_\circ, e_j) = T_{ij}(X_\circ)e_i$$

the equation (1.2-6) follows. The smoothness of T results from that of t w.r.t. X.

Using (1.2-6) in (1.2-2), for any volume ϑ,

$$\begin{aligned} 0 &= \int_{\vartheta} \rho(X)b(X)dX + \int_{\partial\vartheta} T(X)ndA \\ &= \int_{\vartheta} \rho(X)b(X) + DIV(T)dX, \end{aligned}$$

16 from which (1.2-7) follows as ϑ was aribitrary.

Finally by (1.2-3) and (1.2-5)

$$\begin{aligned} 0 &= \int_{L} \epsilon_{ijk}X_j\rho(X)b_k(X)dX + \int_{\partial\vartheta} \epsilon_{ijk}X_jT_{k\ell}n_\ell dA \\ &= -\int_{\vartheta} \epsilon_{ijk}X_j\frac{\partial T_{k\ell}}{\partial X_\ell}dX + \int_{\partial\vartheta} \epsilon_{ijk}X_jT_{k\ell}n_\ell dA \\ &= \int_{\vartheta} \epsilon_{ijk}\frac{\partial X_j}{\partial X_\ell}T_{k\ell}dX = \int_{\vartheta} \epsilon_{i\ell k}T_{k\ell}, \end{aligned}$$

using (1.2-7). Since ϑ was arbitrary,

$$\epsilon_{i\ell k}T_{k\ell} = 0$$

which is just a restatement of (1.2-8).

Remark 1.2.1. Given a tensor field $T : \mathfrak{B} \to \mathbb{M}^3$ satisfying (1.2-7) and (1.2-8), the vector field $t(X, n) = T(X)n$ satisfies (1.2-2) and (1.2-3).

The tensor $T(X)$ obtained in the above theorem is called the *Cauchy stress tensor* at the point $X \epsilon \mathfrak{B}$.

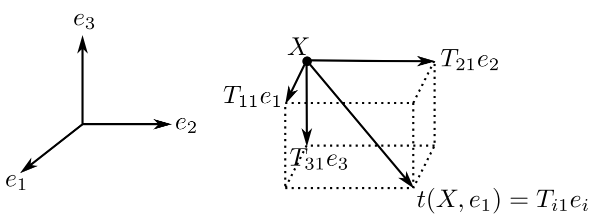

Figure 1.2.3:

Remark 1.2.2. The components of T can be interpreted as follows. If 17
an element dA has normal e_1 then the Cauchy stress vector acting on it, $t(X, e_1)$ has components T_{11}, T_{21} and T_{31} and so on.

The Cauchy stress tensor thus satisfies a boundary value problem:

$$\left.\begin{aligned} DIVT + \rho b &= 0 \\ T &= T^T \end{aligned}\right\} \text{in } \mathfrak{B}$$
$$T_n = t_1 \text{ on } \partial\mathfrak{B}_1$$

Let $u.v$ stand for the usual scalar product in $\mathbb{R}^3$, i.e. $v.v = u_i v_i$. If $A, B \epsilon \mathbb{M}^3$, denote

(1.2-12) $$A : B = A_{ij} B_{ij} = \mathrm{tr}(AB^T).$$

This is an inner product in $\mathbb{M}^3$ with the associated norm

(1.2-13) $$||A|| = \sqrt{A_{ij} A_{ij}}.$$

Using Green's formula, a variational form of the boundary value problem can be obtained.

If T is a tensor field and Θ is a vector field on $\mathfrak{B}$ then

$$\begin{aligned} \int_{\mathfrak{B}} DIV\ T.\Theta dX &= \int_{\mathfrak{B}} \frac{\partial T_{ij}}{\partial X_j} \Theta_i dX \\ &= -\int_{\mathfrak{B}} T_{ij} \frac{\partial \Theta_i}{\partial X_j} dX + \int_{\partial\mathfrak{B}} T_{ij} \Theta_i n_j dA \\ &= -\int_{\mathfrak{B}} T : GRAD\Theta dX + \int_{\partial\mathfrak{B}} Tn.\Theta dA. \end{aligned}$$

In particular, if T is a solution of the above boundary value problem 18
and if Θ vanishes on $\partial\mathfrak{B}_0 = \partial\mathfrak{B} \backslash \partial\mathfrak{B}_1$, then Green's formula above gives

$$0 = \int_{\mathfrak{B}} (DIVT + \rho b).\Theta dX$$

$$= \int_{\mathfrak{B}} (-T : GRAD\textcircled{H} + \rho b.\textcircled{H})dX + \int_{\partial\mathfrak{B}_1} t_1.\textcircled{H} dA.$$

Conversely if the above relation is satisfied for all $\textcircled{H}$ vanishing on $\partial\mathfrak{B}_0$ then T is a solution of the boundary value problem. Thus

Theorem 1.2.2. *The following are equivalent:*

(i) $\begin{cases} DIVT + \rho b & = 0, in\mathfrak{B} \\ \quad Tn & = t_1 \ in\ \partial\mathfrak{B}_1 \end{cases}$

(ii) *For all* $\textcircled{H} : \mathfrak{B} \to \mathbb{R}^3$, $\textcircled{H}$ *vanishing on* $\partial\mathfrak{B}_O$,

$$\text{(1.2-14)} \qquad \int_{\mathfrak{B}} T : GRAD\textcircled{H} dX = \int \rho b.\textcircled{H} dX + \int_{\partial\mathfrak{B}_1} t_1.\textcircled{H} dA.$$

The equations (1.2-14) form the so-called *variantional formulation* of the boundary value problelm (i). In Mechanics, it is also known as the *Principle of Virtual Work in the deformed configuration.*

The equations of equilibrium were established in the *Eulerian variable, X*, in the deformed configuration. However, this is of no use for computation as the deformation ϕ is unknown. So, the equations must be written in the reference configuration, which is a *fixed domain* given *a priori*, in terms of the *Lagrangian variable*, X_R. In doing this, it is desirable to retain as much of the *divergence form* of the equations as possible so that a similar variational formulation can be obtained in the reference configuration. It is here that the merit of the Piola transform is seem.

19 The Piola transform of the Cauchy stress tensor T, called the *first Piola-Kirchhoff stress tensor*, is denoted by T_R. Thus surya

$$T_R = \det(\nabla\phi)T(\nabla\phi)^{-T}.$$

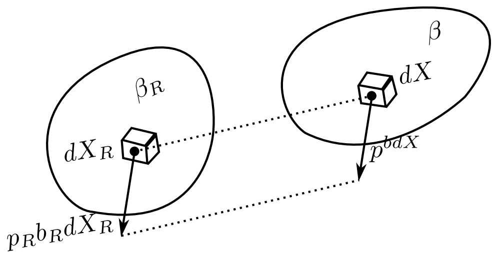

Figure 1.2.4:

By the principle of conservation of mass, it is known that

$$\rho_R(X_R)dX_R = \rho(X)dX.$$

By defining

(1.2-15) $$b_R(X_R) = b(X), \text{or} b_R = bo\phi$$

it follows that

$$\rho_R b_R dX_R = \rho b dX.$$

Note that, *a priori*, b_R depends on ϕ.

Remark 1.2.3. Since it is known that $dX = \det(\nabla\phi)dX_R$, it follows that

(1.2-16) $$\rho(X) = \frac{\rho_R(X_R)}{\det \nabla\phi(X_R)}.$$

Since the density at any point (in either configuration) has to be **20**
finite and positive, this, if not any other, is a necessary reason for a deformation to satisfy. $\det(\nabla\phi) \neq 0$.

Multiplying equation (1.2-7) by det $(\nabla\phi)$ on both sides, it follows that

(1.2-17) $$DIV_R T_R + \rho_R b_R = 0 \text{ in } \mathfrak{B}_R$$

Thus the divergence form is preserved. Note however that T_R is *not* symemetric. A *symmetric* tensor to T_R can be defined. It is the *second Piola-Kirchhoff stress tensor*, $\sum_R$, given by

(1.2-18) $$\sum_R = \det(\nabla\phi)(\nabla\phi)^{-1}T(\nabla\phi)^{-T}$$

It is related to T_R by

(1.2-19) $$\sum_R = (\nabla\phi)^{-1}T_R.$$

Remark 1.2.4. It is understandable that T_R is not symmetric as it belongs partly to the reference configuration and partly to the deformed configuration and symmetry does not make much sense in such a situation.

Now we turn to the transformation of the surface forces. The *first Piola-Kirchhoff stress vector* is defined so that

(1.2-20) $$t_R(X_R, n_R) = T_R(X_R)n_R.$$

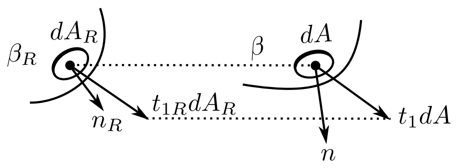

Figure 1.2.5:

21 Recall that $T_R(X_R)n_R dA_R = T(X)ndA$ and so

$$t_R(X_R, n_R)dA_R = t(X, n)dA.$$

If $\partial\mathfrak{B}_{1R}$ is the portion of $\partial\mathfrak{B}_R$ mapped by ϕ onto $\partial\mathfrak{B}_1$, define $t_{1R} : \partial\mathfrak{B}_{1R} \to \mathbb{R}^3$ by $t_{1R}dA_R = t_1 dA$. Again, *a priori*, $t_{1\mathbb{R}}$ depends on ϕ. Explicitly, by Theorem 1.1.1,

(1.2-21) $$t_{1R(X_R)} = \det(\nabla\phi(X_R))|(\nabla\phi(X_R))^{-T}n_R|t_1(\phi(X_R)).$$

The following result is easy to establish.

Theorem 1.2.3. *The equilibrium equations in the reference configuration are given by*

(1.2-22) $$DIV_R T_R + \rho_R b_R = o \text{ in } \mathfrak{B}_R$$

(1.2-23) $$(\nabla\phi)T_R^T = T_R(\nabla\phi)^T \ in \mathfrak{B}_R$$

(1.2-24) $$T_R n_R = t_{\mathbb{R}} on \partial\mathfrak{B}_{\mathbb{R}}.$$

Equivalently, in terms of $\sum_R$

(1.2-25) $$DIV_R(\nabla\phi \sum_R) + \rho_R b_R = 0 \ in \ \mathfrak{B}_R$$

(1.2-26) $$\sum_R = \sum_R^T \ in \ \mathfrak{B}_R$$

(1.2-27) $$\nabla\phi \sum_R n_R = t_{1\mathbb{R}} \ on \ \partial\mathfrak{B}_{1\mathbb{R}}.$$

Again, this is equivalent to the variational equations

(1.2-28) $$\int_{\mathfrak{B}_R} T_R : \nabla\theta dX_R = \int_{\mathfrak{B}_R} \rho_R b_R.\theta dX_R + \int_{\partial\mathfrak{B}_{1R}} t_{1R}.\theta dA_R$$

for all $\theta : \mathfrak{B}_R \to \mathbb{R}^3$ *vanishing on* $\partial\mathfrak{B}_{oR} = \partial\mathfrak{B}_R \backslash \partial\mathfrak{B}_{1\mathbb{R}}$.

Remark 1.2.5. Equations (1.2-28) go under the name of the principle of **22**
virtual work in the reference configuration.

To conclude this section, some classes of applied forces are considered. Recall that while ρ_R is completely known, b_R and $t_{1\mathbb{R}}$ depend in general on ϕ which is unknown.

A body force (resp. applied surfaces force) is a *dead load* if b_R (resp. t_{1R}) is a function of X_R only, independent of ϕ.

An example of a body force which is a dead load is gravity which is constant; $b = (o, o, -g)$. A trivial example of an applied surface force which is a dead load is $t_1 = 0$! The pressure is an example of an applied surface force which is *not* a dead load:

(1.2-29) $$t_1 = -pn$$

where $p > o$ indicates an inward directed force (pressure) and $p < 0$ indicates one which is directed outward (traction). Now

$$t_{1R} = -p \det(\nabla\phi)(\nabla\phi)^{-T} n_R \text{ on } \partial\mathfrak{B}_R$$

which clearly depends on ϕ!

A body force is said to be *conservative* if there exists a function $\beta : \mathbb{R}^3 \times \mathfrak{B}_R \to \beta(\phi, X_R)\epsilon\mathbb{R}$, such that

$$b_R(X_R) = \nabla_\phi \beta(\phi(X_R), X_R), \tag{1.2-30}$$

for all $X_R \epsilon \mathfrak{B}_R$ and all deformations ϕ. If which is the case then

$$\int_{\mathfrak{B}_R} \rho_R b_R.\theta dX_R = B(\phi)(\theta) \tag{1.2-31}$$

where

$$B(\psi) = \int_{\mathfrak{B}_R} \rho_R(X_R)\beta(\psi(X_R), X_R)dX_R. \tag{1.2-32}$$

23 *A body force which is a deal load is conservative,* $\beta(\Phi, X_R) = b_R(X_R).\Phi$.

An applied surface force is *conservative* if there exists a function $\tau_1 : \mathbb{R}^3 \times \partial\mathfrak{B}_{1R} \to \tau_1(\phi, X_R)\epsilon\mathcal{R}$ such that

$$t_{1R}(X_R) = \nabla_\Phi \tau_1(\phi(X_R), X_R). \tag{1.2-33}$$

Then again

$$\int_{\partial\mathfrak{B}_{1R}} t_{1R}.\theta dA_R = T_1'(\phi)(\theta) \tag{1.2-34}$$

where

$$T_1(\psi) = \int_{\partial\mathfrak{B}_{1R}} \tau_1(\psi(X_R), X_R)dA_R. \tag{1.2-35}$$

An applied surface force which is a deal load is conservative; $\tau_1(\phi, X_R) = t_{1\mathbb{R}}(X_R).\Phi$. A pressure load is conservative (Exercise 1.2-3).

Exercises

1.2-1 . (Da Silva's Theorem). Given any system of applied forces (with $\partial\mathfrak{B}_1 = \partial\mathfrak{B}$) show that there exists $Q\epsilon\mathbb{O}^3$ such that

$$\int_{\mathfrak{B}} \rho(X)OX\Lambda Qb(X)dX + \int_{\partial\mathfrak{B}} OX\Lambda Qt(X)dA = o$$
$$\int_{\mathfrak{B}} \rho(X)Q^T(OX)\Lambda b(X)dX + \int_{\partial\mathfrak{B}} Q^T(OX)\Lambda t(X)dA = o.$$

How many solutions exist?

1.2-2. Show that the fundamental axiom of static equilibrium is equivalent to

$$\int_{\vartheta} \rho(X)b(X).v(X)dX + \int_{\partial\vartheta} t(X,n).v(X)dX = 0$$

for every volume $\vartheta \subset \mathfrak{B}$ and for every *infinitesimal rigid dis-* **24**
placement v, i.e.,

$$v(x) = a + b\Lambda Ox, a, b\epsilon\mathbb{R}^3.$$

This is sometimes also called the principle of virtual work. 1.2−3.

1.2-3 Show that a pressure load is conservative.

1.3 Constitutive Equations

Given a body acted on by a system of forces, one's main objective is to compute the deformation ϕ which has 3 component functions. As a natural *intermediary*, the stress tensor T has come in which has 6 components (taking into account its symmetry). But so far, the boundary value problem obtained via the equilibrium equations has yielded only 3 equations (cf. (1.2-7)). Thus 6 more equations must be found.

From the physical point of view, observe that in obtaining the equilibrium equations, no property of the material under consideration has been used. Since different materials react differently to the same forces, obviously these equations alone cannot describe the response of the material.

Thus one is led to finding more equations to complete the system. A material is said to be *elastic* if there exists a mapping

$$\hat{T} : F \epsilon \mathbb{M}^3_+ \rightarrow \hat{T}(F) \epsilon \mathbb{S}^3$$

25 such that for any deformed configuration and any point $X = \phi(X_R)$,

$$T(X) = \hat{T}(\nabla\Phi(X_R)). \tag{1.3-1}$$

The map $\hat{T}$ is called the *response function* and (1.3-1) is called a *constitutive equation.*

Remark 1.3.1. The map $\hat{T}$ above does not depends explicitly on X_R. Such that a material is called *homogeneous.* If it were that

$$T(X) = \hat{T}(X_R, \nabla\Phi(X_R))$$

the material would be called a *non-homogeneous* elastic material.

If T_R is the Piola transform of T then it follows that

$$T_R = \det(\nabla\phi)\hat{T}(\nabla\phi)(\nabla\phi)^{-1} \stackrel{\text{def}}{=} \hat{T}_R(\nabla\phi) \tag{1.3-2}$$

which gives a reponse fucntion $\hat{T}_R : \mathbb{M}^3_+ \rightarrow \mathbb{M}^3$ for T_R. Similarly it is possible to write one for Σ_R in terms of a response function $\hat{\Sigma}_R : \mathbb{M}^3_+ \rightarrow \mathbb{S}^3$.

Theorem 1.3.1 (Polar Factorisation)**.** *Let F be an invertible matrix. Then there exist an orthogonal matrix R and symmetric, positive definite matrices U and V such that*

$$F = RU = VR. \tag{1.3-3}$$

Such a factorization is unique.

Proof. Cf. Exircise 1.3-1. ◻

Remark 1.3.1′. If $F \in \mathbb{M}^3_+$ then $R \in, \mathbb{O}^3_+$. If $G \in \mathbb{S}^3_>$ there exists a unique matrix $H \in \mathbb{S}^3_>$ such that $H^2 = G$. It is usual to write $H = G^{1/2}$. It can be
26 seen that $U = (F^T F)^{1/2}$ and $V = (FF^T)^{1/2}$, in the above theorem. Since $V = RUR^T$, U and V are similar. Then so are $B = FF^T$ and $C = F^T F$.

The constitutive equation (1.3-1) can be written componentwise as

$$T_{ll}(X) = \hat{T}_{ll}\left(\frac{\partial\phi_1}{\partial X_{R_1}}(X_R), \ldots, \frac{\partial\phi_3}{\partial X_{R_3}}(X_R)\right)$$

and so on. So knowing $\hat{T}$ is the same as knowing the functions $\hat{T}_{ij}$. However, the functions $\hat{T}_{ij}$ cannot be chosen arbitrarily. They must somehow reflect an *intrinsic* property of the material in equation, irrespective of the coordinate system chosen. This is the idea embodying the

AXIOM OF MATERIAL FRAME INDIFFERENCE. The Cauchy stress vector $t(X, n) = T(X)n$ should be independent of the particular basis in which the constitutive equation is expressed.

Theorem 1.3.2. *The following are equivalent.*

(i) *A response function $\hat{T} : \mathbb{M}^3_+ \to \mathbb{S}^3$ satisfies the axiom of material frame indifference.*

(ii) *For every $Q \in \mathbb{O}^3_+$ and for every $F \in \mathbb{M}^3_+$,*

$$\hat{T}(QF) = Q\hat{T}(F)Q^T. \tag{1.3-4}$$

(iii) *For every $F \in \mathbb{M}^3_+$ if $F = RU$ is its polar factorisation then*

$$\hat{T}(F) = R\hat{T}(U)R^T \tag{1.3-5}$$

(iv) *There exists a map $\tilde{\Sigma}_R : \mathbb{S}^3_> \to \mathbb{S}^3$ such that*

$$\tilde{\Sigma}_R(F) = \tilde{\Sigma}_R(F^T F) \tag{1.3-6}$$

for every $F \in \mathbb{M}^3_+$. 27

Proof. (i) ⇔ (ii) Instead of rotating the coordinate axes the same effect can be achived by rotating the deformed configuration. □

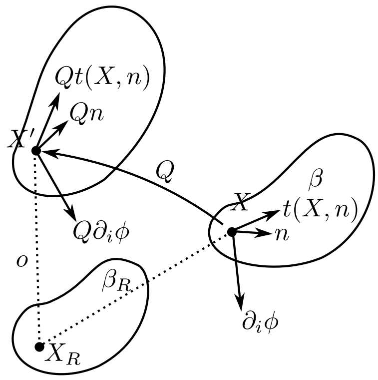

Figure 1.3.1:

Rotating $\mathcal{B}$ by a map $Q \in \mathbb{O}^3_+$, let X map into X'. The normal n at any goes to Qn and $t(X, n)$ goes to $Qt(X, n)$. Thus

$$t(X', Qn) = T'(X')Qn$$
$$t(X', Qn) = Qt(X, n) = QT(X)n.$$

28 Since n is arbitrary, it follows that

$$T'(X') = QT(x)Q^T \tag{1.3-7}$$

Thus

$$\hat{T}(Q\nabla\phi(X_R)) = Q\hat{T}(\nabla\phi(X_R))Q^T$$

for any $Q \in \mathbb{O}^3_+$ and any $F = \nabla\phi \in \mathbb{M}^3_+$. This shown that (i) $\Rightarrow$ (ii) Simply retracting the argument proves the converse.

(ii) $\Leftrightarrow$ (iii). If $F = RU$, then by (ii), since $R \in \mathbb{O}^3_+$

$$\hat{T}(RU) = R\hat{T}(U)R^T$$

which is (iii). Conversely, assuming (iii), if $F = RU$ then the polar factorization QF is $(QR)U$ for $Q \in \mathbb{O}^3_+$, as the factorization is unique. Thus

$$\hat{T}(QF) = QR\hat{T}(U)R^TQ^T = Q\hat{T}(F)Q^T.$$

(ii) $\Leftrightarrow$ (vi). Since $F = RU$ implies $U = (F^T F)^{1/2}$,

$$\begin{aligned}\hat{T}(F) &= R\hat{T}(U)R^T\\ &= FU^{-1}\hat{T}(U)U^{-1}F^T\\ &= F\tilde{S}(F^T F)F^T\end{aligned}$$

where $\tilde{S} : \mathbb{S}^3_> \to \mathbb{S}^3$. Conversely, if $\hat{T}(F)$ is of the above form, then if $F = RU$,

$$\hat{T}(U) = U\tilde{S}(U^2)U$$

and **29**

$$\begin{aligned}\hat{T}(F) &= F\tilde{S}(F^T F)F^T\\ &= F\tilde{S}(U^2)F^T\\ &= FU^{-1}\hat{T}(U)U^{-1}F^T\\ &= R\hat{T}(U)R^T.\end{aligned}$$

Now

$$\begin{aligned}\Sigma_R(F) &= \det(F)F^{-1}\hat{T}(F)F^{-T}\\ &= (\det(F^T F))^{1/2}\tilde{S}(F^T F) = \tilde{\Sigma}_R(F^T F).\end{aligned}$$

Remark 1.3.2. If one of the response functions, say Γ, can be written of either variables $F, F^T F = C, FF^T = B$ or E (where $C = I + 2E$), the following notation will be employed when the different dependences are expressed:

$$\Gamma = \hat{\Gamma}(F) = \tilde{\Gamma}(F^T F) = \overline{\Gamma}(FF^T) = \Gamma^*(E)$$

In the above theorem it has been proved that it is enough to know the action of $\hat{T}$ on a relatively small class of matrices like $\mathbb{S}^3_>$.

A material or response function is said to be *isotropic* if the Cauchy strees tensor (or vector) computed at a given point in the deformed configuration is the same if the same if the reference configuration is rotated by any rigid defomation.

While the axiom of material frame indifference is an *axiom* to verified by any response fucntion, isotropy is a *property* of a particular material. There can be materials which are non-isotropic; for instance, a body up of layers of different materials.

30 **Theorem 1.3.3.** *The following are equivalent.*

(i) *A response function* $\hat{T} : \mathbb{M}^3_+ \to \mathbb{S}^3$ *is isotropic.*

(ii) *For every* $F \in \mathbb{M}^3_+$ *and for every* $Q \in \mathbb{O}^3_+$,

(1.3-8) $$\hat{T}(F) = \hat{T}(FQ).$$

(iii) *There exists a map* $\overline{T} : \mathbb{S}^3_> \to \mathbb{S}^3$ *such that for every* $F \in \mathbb{M}^3_+$,

(1.3-9) $$\hat{T}(F) = \overline{T}(FF^T).$$

Proof. (i) $\Leftrightarrow$ (ii). Let $Q \in \mathbb{O}^3_+$. Rotate the reference configuration about a point $\overline{X}_R$ so that if $X_R \in \mathfrak{B}_R$ then

(□) $$\theta(X_R) = \overline{X}_R + Q^T(\overline{X}_R X_R).$$

Then

$$\phi^* = \phi o \theta^{-1}.$$

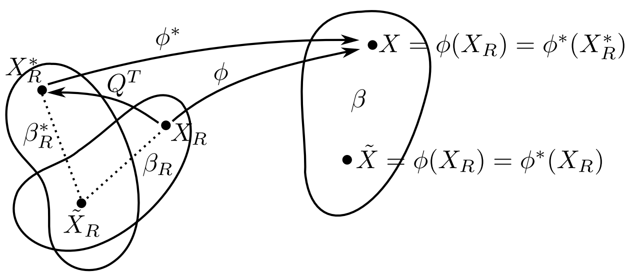

Figure 1.3.2:

The response function is isotropic if and only if

$$\hat{T}(\overline{X}) = \hat{T}(\nabla\phi(\tilde{X}_R)) = \hat{T}(\nabla\phi^*(\tilde{X}_R))$$

i.e.,
$$\tilde{T}(\nabla\phi(\tilde{X}_R)) = \hat{T}(\nabla\phi(\bar{X}_R)Q).$$

31 (ii) ⇔ (iii). Let $FF^T = GG^T, F, G \in \mathbb{M}^3_+$. Then $G^{-1}F \in \mathbb{O}^3_+$. Hence by (ii)
$$\hat{T}(G) = \hat{T}(G(G^{-1}F)) = \hat{T}(F).$$

So it is clear that $\hat{T}(F)$ depends only on FF^T. Conversely if $\hat{T}(F) = \bar{T}(FF^T)$ then for $Q \in \mathbb{O}^3_+$,
$$\hat{T}(FQ) = \bar{T}(FQQ^TF^T) = \bar{T}(FF^T) = \hat{T}(F).$$

□

Remark 1.3.3. By the axiom of material frame indifference, the constitutive equation could be expreesed in terms of a function of $C = F^TF$ and this involved rotating the *deformed*configuration $\mathfrak{B}$. By isotropy, the same could be expressed in terms of a function of $B = FF^T$ and this involved rotating the *reference* configuration $\mathfrak{B}_R$. Thus these two nations seem to be 'dual' ot each other.

Remark 1.3.4. For non-isotropic materials it can be shown that
$$\hat{T}(F) = \hat{T}(FQ)$$
for all $F \in \mathbb{M}^3_+$ but Q variying over a subgroup of $\mathbb{O}^3_+$.

In what follows, the material will allways be assumed to be isotropic.

Before proving a very powerful and elgent result on the structure of a reponse function which is isotrophic and material frame-indifferent, the following definition is needed.

Let $A \in \mathbb{M}^3$. Define $\imath_A$ to be the triple $(\imath_1(A), \imath_2(A), \imath_3(A))$ where $\imath_1(A), \imath_2(A)$ and $\imath_3(A)$ *are the principal invariants of* A, and

(1.3-10) $$\det(A - \lambda I) = -\lambda^3 + \imath_1(A)\lambda^2 - \imath_2(A)\lambda + \imath_3(A).$$

If $A = (a_{ij})$ and $\lambda_1, \lambda_2, \lambda_3$ are its eigenvalues, then

(1.3-11)
$$\imath_1(A) = a_{ii} = tr(A) = \lambda_1 + \lambda_2 + \lambda_3.$$

$$\iota_2(A) = \frac{1}{2}(a_{ii}a_{jj} - a_{jj}a_{ij}) = \frac{1}{2}((tr(A))^2 - tr(A^2))$$

(1.3-12) $= \text{tr}(\text{adj}\, A) = \lambda_1\lambda^2 + \lambda_2\lambda_3 + \lambda_3\lambda_1.$

(1.3-13)

$$\iota_3(A) = \det(A) = \frac{1}{6}((tr(A))^3 - 3tr(A)tr(A^2) + 2tr(A^3)) = \lambda_1\lambda_2\lambda_3.$$

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Further, if A is invertible,

(1.3-14) $$\iota_2(A) = (\det A)tr(A^{-1}).$$

The following theorem is one of the most important results in the theorey of elasticity.

Theorem 1.3.4 (Rivlin-Ericksen Theorem)**.** *A response function* $\hat{T} : \mathbb{M}^3_+ \to \mathbb{S}^3$ *is isotropic and material frame indifferent if, and only if, it is of the* $\hat{T}(F) = \overline{T}(FF^T)$ *where the mapping* $\overline{T} : \mathbb{S}^3_> \to \mathbb{S}^3$ *is of the form*

(1.3-15) $$\overline{T}(B) = \beta_o(\iota_B)I + \beta_1(\iota_B)B + \beta_2(\iota_B)B^2$$

for all $B \in \mathbb{S}^3_>$. *where* $\beta_o, \beta_1, \beta_2$ *are real valued functions.*

Proof. (i) Let $\hat{T} : \mathbb{M}^3_+ \to \mathbb{S}^3$ be material frame indifferent and isotropic. Then by isotropy $\hat{T}(F) = \overline{T}(FF^T)$ for some mapping $\overline{T} : \mathbb{S}^3_> \to \mathbb{S}^3$. Let $Q \in \mathbb{O}^3_+$ and $B \in \mathbb{S}^3_>$. On one hand , by isotrophy

$$\hat{T}(QB^{1/2}) = \overline{T}(QB^{1/2}B^{1/2}Q^T) = \overline{T}(QBQ^T).$$

On the other hand, by the material frame indifference,

$$\begin{aligned}\hat{T}(QB^{1/2}) &= Q\hat{T}(B^{1/2})Q^T\\ &= Q\overline{T}(B^{1/2}B^{1/2})Q^T = Q\overline{T}(B)Q.\end{aligned}$$

33

Thus $\overline{T}$ satisfies , for all $Q \in \mathbb{O}^3_+$, and $B \in \mathbb{S}^3_>$

(1.3-16) $$\overline{T}(QBQ^T) = Q\overline{T}(B)Q^T.$$

Conversly, let $\overline{T} : \mathbb{S}^3_> \to \mathbb{S}^3$ satisfy (1.3-16) and let $\hat{T}(F) = \overline{T}(F^T F)$. Then clearly, $\hat{T}$ is isotrophic. If $Q \in \mathbb{O}^3_+$, then

$$\begin{aligned}\hat{T}(QF) &= \overline{T}(QFF^T Q^T) = Q\overline{T}(FF^T)QT. \\ &= Q\hat{T}(F)Q^T\end{aligned}$$

and so $\hat{T}$ is material frame indefferent.

Thus it is now enough to check that a mapping $\overline{T} : \mathbb{S}^3_> \to \mathbb{S}^3$ satisfying (1.3-16) is of the form (1.3-15). (The converse is immediate to varify).

(ii) Let $\overline{T} : \mathbb{S}^3_> \to \mathbb{S}^3$ varify (1.3-16). It will now be shown that any matrix which diagonalizes $B \in \mathbb{S}^3_\geq$ also diagnalizes $\overline{T}(B)$, i.e., any eigenvector of B is an eigenvector of $\overline{T}(B)$.

Let $B \in \mathbb{S}^3_>$ and $Q \in \mathbb{O}^3_+$ (we can always assume that) such that

$$Q^T BQ = \text{diag }(\lambda_i)$$

where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalue of B. Define

$$Q_1 = \text{diag }(1, -1, -1), Q_2 = \text{diag }(-1, 1, -1), Q_3 = \text{diag }(-1, -1, 1).$$

Then $Q_k \in \mathbb{O}^3_+, k = 1, 2, 3.$

Also, **34**

$$Q_k^T Q^T BQQ_k = \text{diag } \lambda_i = Q^T BQ.$$

So,

$$\begin{aligned}Q_k^T Q^T \overline{T}(B)QQ_k &= \overline{T}(Q_k^T Q^T BQQ_k) \\ &= \overline{T}(Q^T BQ) \\ &= Q^T \overline{T}(B)Q.\end{aligned}$$

If $D = Q^T \overline{T}(B)Q$, then

$$Q_k^T DQ_k = D, k = 1, 2, 3.$$

If the diagonal entries of Q_k are $q_i^k (= 1 \text{ if } i = k, -1 \text{ if } i \neq k)$, then it follows that

$$D_{ij} = q_i^k D_{ij} q_j^k \text{ for all } 1 \leq i, j, k \leq 3.$$

Thus if $i = k \neq j$, then

$$D_{kj} = -D_{kj} \text{ or } D_{kj} = o.$$

Hence D is diagonal and this proves the claim.

(iii) It will now be a shown that if $\bar{T}$ satisfies (1.3-16) then, for all $B \in \mathbb{S}^3_>$,

$$\bar{T}(B) = b_o(B)I + b_1(B)B + b_2(B)B^2, \tag{1.3-17}$$

$b_\alpha, \alpha = 0, 1, 2$ being real valued functions on $\mathbb{S}^3_>$. □

35 **Case 1.** B has 3 distanct eigenvalues $\lambda_1, \lambda_2, \lambda_3$ with corresponding orthonormal eigenvectors p_1, p_2, p_3. Then

$$I = p_1 p_1^T + p_1 p_2^T + p_3 p_3^T \tag{1.3-18}$$

$$B = \lambda_1 p_1 p_1^T + \lambda_2 p_2 p_2^T + \lambda_3 p_3 p_3^T \tag{1.3-19}$$

$$B^2 = \lambda_1^2 p_1 p_1^T + \lambda_2^2 p_2 p_2^T + \lambda_3^2 p_3 p_3^T \tag{1.3-20}$$

Since the λ_i are distinct, the Vandermonde determinant

$$\det \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix}$$

is non-zero and so in $\mathbb{S}^3$, the span of $p_i p_i^T, i = 1, 2, 3$ is equal to that of I, B, B^2. But $\bar{T}(B)$, by step (ii) above, has the same eigenvectors as B. So

$$\bar{T}(B) = \mu_1 p_1 p_1^T + \mu_2 p_2 p_2^T + \mu_3 p_3 p_3^T \tag{1.3-21}$$

which implies that $\bar{T}(B) \in \text{span } \{I, B, B^2\}$.

Case 2. $\lambda_1 \neq \lambda_2 = \lambda_3$. Again one can write (1.3-18) and (1.3-19). Then the span of $p_1 p_1^T$ and $p_2 p_2^T + p_3 p_3^T$ is that of I and B. By step (ii), it can be seen that $\mu_2 = \mu_3$, since any non-zero vector spanned by p_2 and p_3 is also an eigenvector for $\bar{T}(B)$. Thus in (1.3-21)

$$\bar{T}(B) = \mu_1 p_1 p_1^T + \mu_2 (p_2 p_2^T + p_3 p_3^T)$$

which shows $\bar{T}(B) \in \text{span } (I, B)$.

36 **Case 3.** $\lambda_1 = \lambda_2 = \lambda_3$. In this case, one can similarly see that $B, \bar{T}(B)$ are both scalar multiples of I.

(iv) Case. 1. $\lambda_1, \lambda_2, \lambda_3$ are distinct eigenvalues of $B \in \mathbb{S}^3_>$, .
Let $Q \in \mathbb{O}^3_+$.

$$\begin{aligned}\bar{T}(QBQ^T) &= b_o(QBQ^T)I + b_1(QBQ^T)QBQ^T + b_2(QBQ^T)QB^2Q^T \\ &= Q(b_o(QBQ^T)I + b_1(QBQ^T)B + b_2(QBQ^T)B^2)Q^T\end{aligned}$$

But

$$\bar{T}(QBQ^T) = Q\bar{T}(B)Q^T = Q(b_o(B)I + b_1(B)B + b_2(B)B^2)Q^T.$$

Thus $b_\alpha : \mathbb{S}^3_> \to \mathbb{R}, \alpha = 0, 1, 2$ satisfy the functional identity

$$b_\alpha(QBQ^T) = b_\alpha(B) \tag{1.3-22}$$

for all $B \in \mathbb{S}^3_>$ and for all $Q \in \mathbb{O}^3_+$. Thus if Q diagnalizes B, it is seen that such a function b_α must be a function of the eigenvalues of B only. Now choosing $Q_i \in \mathbb{O}^3_+, i = 1, 2, 3$ as

$$Q_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, Q_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, Q_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

it is seen from (1.3-22) that b_α is a symmetric function of $\lambda_1, \lambda_2, \lambda_3$. i.e,. $b_\alpha(B) = \beta_\alpha(t_B)$.

This proves the theorem completely.

Theorem 1.3.5. *(a) Given $\mathfrak{B}_R$ and an isotropic material frame indifferent material, then in any deformed configuration $\mathfrak{B} = \phi(\mathfrak{B}_R)$, the Cauchy stress tensor is given by*

$$T(X) = \hat{T}(\nabla\phi(X_R)) = \bar{T}(\nabla\phi(X_R)\nabla\phi(X_R)^T), \tag{1.3-23}$$

37

$\bar{T} : \mathbb{S}^3_> \to \mathbb{S}^3$ *satisfying* (1.3-15).

(b) The second Piola-Kirchhoff stress tensor is given by

$$\Sigma_R(X_R) = \hat{\Sigma}_R(\nabla\phi(X_R)) = \tilde{\Sigma}_R(\nabla\phi(X_R)^T\nabla\phi(X_R)) \tag{1.3-24}$$

where $\tilde{\Sigma}_R : \mathbb{S}^3_> \to \mathbb{S}^3$ *satisfies, for all* $C \in \mathbb{S}^3_>$,

$$\tilde{\Sigma}_R(C) = \gamma_o(\iota_c)I + \gamma_1(\iota_c)C + \gamma_2(\iota_c)C^2. \tag{1.3-25}$$

Proof. Observe that

$$\begin{aligned}\hat{\Sigma}_R(F) &= \det(F)F^{-1}\hat{T}(F)F^{-T} \\ &= (\det(F^TF))^{1/2}F^{-1}\overline{T}(FF^T)F^{-T} \\ &= (\det C)^{1/2}F^{-1}[\beta_O(\iota_B)I + \beta_1(wr_B)B + B_2(\iota_B)B^2]F^{-T}\end{aligned}$$

where $C = F^TF, B = FF^T$. But these are similar. So $\iota_B = \iota_c$.

Further

$$\begin{aligned}F^{-1}F^{-T} &= C^{-1} \\ F^{-1}BF^{-T} &= I \\ F^{-1}B^2F^{-T} &= C.\end{aligned}$$

By the Cayley-Hamilton theorem,

$$-C^3 + \iota_1(C)C^2 - \iota_2(C)C + \iota_3(C)I = 0$$

or

$$C^{-1} = \frac{1}{\iota_3(C)}(C^2 - \iota_1(C)C + \iota_2(C)I)$$

38 where $\iota_3(C) = \det C \neq 0$. Thus it is clear from these considerations that Σ_R can be expressed in terms of C as in (1.3-24) - (1.3-25). □

It was seen in Section 1.1 that the Green-St Venant strain tensor E, given by $C = I + 2E$, 'measures' the actual deformation. If $\tilde{\Sigma}_R$ is sufficiently smooth it is possible to express it in terms of E. More precisely, the following result is true.

Theorem 1.3.6. *Let* $\mathfrak{B}_R$ *be the reference configuration of an isotropic, material frame indifferent elastic material. Assume that the functions* $\gamma_\alpha, a = 0, 1, 2$ *of* (1.3-25) *are differentiable at* $\iota_I = (3, 3, 1)$. *Then*

$$\Sigma_R = \tilde{\Sigma}_R(I + 2E) = -pI + (\lambda(trE)I + 2\mu E) + O(E), \tag{1.3-26}$$

where p, λ *and* μ *are constants.*

Proof. Using the relations

$$\begin{aligned} \operatorname{tr}(C) &= 3 + 2\operatorname{tr}(E) \\ \operatorname{tr}(C^2) &= 3 + 4\operatorname{tr}(E) + o(E) \\ \operatorname{tr}(C^3) &= 3 + 6\operatorname{tr}(E) + o(E) \end{aligned}$$

and the relations (1.3-11) - (1.3-13), it follows that

$$\begin{aligned} \iota_1(C) &= 3 + 2\operatorname{tr}(E) \\ \iota_2(C) &= 3 + 4\operatorname{tr}(E) + o(E) \\ \iota_3(C) &= 1 + 2\operatorname{tr}(E) + o(E) \end{aligned}$$

so that

$$\gamma(\iota_C) = \gamma(\iota_I) + (2\frac{\partial\gamma}{\partial\iota_I}(\iota_I) + 4\frac{\partial\gamma}{\partial\iota_2}(\iota_I) + 2\frac{\partial\gamma}{\partial\iota_3}(\iota_I))\operatorname{tr}(E) + O(E)$$

where $\gamma = (\gamma_O, \gamma_1, \gamma_2)$. This yields (1.3-26). In particular 39

(1.3-27) $$p = -(\gamma_1(\iota_I) + \gamma_1(\iota_1) + \gamma_2(\iota_1)).$$

(1.3-28) $$\lambda = \sum_{\alpha=0}^{2}\left(2\frac{\partial\gamma_\alpha}{\partial\iota_1}(\iota_1) + 4\frac{\partial\gamma_\alpha}{\partial\iota_2}(\iota_I) + 2\frac{\partial\gamma_\alpha}{\partial\iota_3}(\iota_I)\right).$$

(1.3-29) $$\mu = \gamma_1(\iota_1) + 2\gamma_2(\iota_1).$$

□

A reference configuration is a *natural state* if ‘there is no stress in it’, i.e., $p = 0$. In this case

(1.3-30) $$\Sigma_R = \Sigma_R^*(E) = \lambda\operatorname{tr}(E)I + 2\mu E + o(E)$$

and λ and μ are called *Lamé’s constants*. It is possible to obtain *a priori* some information on the nature of the Lamé’s constants.

Let $\mathfrak{B}_R$ be a natural state and have a ‘simple form’. Let $\phi^\in : \mathfrak{B}_R \to \mathbb{R}^3$ be of the form

(1.3-31) $$\phi^\in(X_R) = X_R + \in u(X_R) + o(\in; X_R),$$

where $\in > o$ is a small parameter, and where $\nabla u(X_R) = G$, a *constant* matrix. Such a deformation is a special case of the so-called *homogeneous deformations* where $\nabla\phi$ is a constant vector. Then

$$\begin{aligned} T^{\in}(X) &= \hat{T}(I + \in G + o(\in; X)) \\ &= \hat{T}(I + \in G) + 0(\in; X), X = \phi(X_R). \end{aligned}$$

Thus $DIVT^{\in}(X) = o(\in; X)$ i. e. to within first order in $\in$, *there can be no body force*: such deformations can only be produced by applied surface forces. Now it can be seen that

$$T^{\in}(X) = \in (\lambda(\operatorname{tr} G)I + \mu(G^T + G)) + o(\in; X) \tag{1.3-32}$$

40 using the fact that $\mathfrak{B}_R$ is a natural state. For particular $\phi^{\in}$ considered the corresponding $T^{\in}$ is of some simple form. This can be substituted in (1.3-32) and it is thus possible to obtain inequalities for λ and μ.

Experiment 1. Let $\mathcal{B}_R$ be a rectangular block. Choose

$$u^{\in}(X_R) \overset{\text{def}}{=} \in u(X_R) = \in \begin{bmatrix} X_{R_2} \\ 0 \\ 0 \end{bmatrix}. \tag{1.3-33}$$

Thus the body deforms as shown in figure 1.3.3.

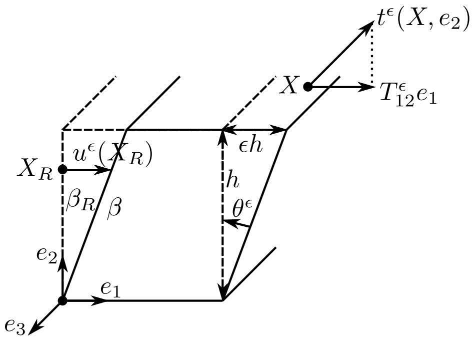

Figure 1.3.3:

Then it is logical to assume $T^{\in}_{12} = \in T_{12} + o(\in)$ where $T_{12} > o$. This follows from the interpretation of the components of the stress tensor (cf. Remark 1.2.2). Comparing this form with (1.3-32) if follows that

(1.3-34) $$\mu > 0.$$

41

Experiment 2. Let $\mathfrak{B}_R$ be a sphere which is contracted by means of a normal pressure. Thus

(1.3-35) $$u^{\in}(X_R) = \in \begin{bmatrix} -X_{R1} \\ -X_{R2} \\ -X_{R3} \end{bmatrix} + o(\in; X_R).$$

Thus

(1.3-36) $$T^{\in}(X) = -p \in I + o(\in; X), p > 0.$$

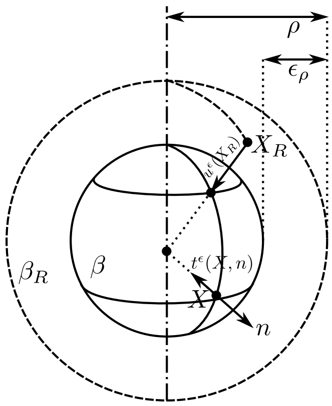

Figure 1.3.4:

It can then be shown that 42

(1.3-37) $$-p \in I = - \in (3\lambda + 2\mu)I + 0(\in)$$

from which it follows that

(1.3-38) $$3\lambda + 2\mu > 0.$$

Remark 1.3.5. This precludes *incompressible materials!* An example of an incompressible material is rubber.

Experiment 3. Let $\mathfrak{B}_R$ be a cylinder which is stretched as in figure 1.3.5.

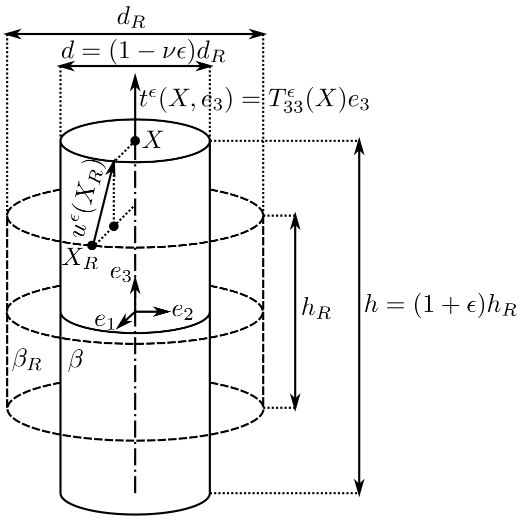

Figure 1.3.5:

43 Now

(1.3-39) $$u^\in(X_R) = \in \begin{bmatrix} -\nu X_{R1} \\ -\nu X_{R2} \\ X_{R3} \end{bmatrix} + 0(\in; X_R), \nu > 0$$

and

(1.3-40) $$T^\in(X) = \in \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & E \end{bmatrix} + 0(\in; X).$$

It can now be shown that

$$\nu = \frac{\lambda}{2(\lambda + \mu)}, E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \tag{1.3-41}$$

Since $\mu > 0$ and $3\lambda + 2\mu > 0$, it follows that $\lambda + \mu > 0$. Since $\nu > 0$ it follows that

$$\lambda > 0. \tag{1.3-42}$$

Thus $\lambda > o$ and $\mu > 0$. (This does not make sense for incompressible materials). The number ν is known as *Poisson's ratio* and E as *Young's modulus*. The Lame's constants can be expressed in terms of these quantities:

$$\lambda = \frac{E\nu}{(1 + \nu)(1 - 2\nu)}, \mu = \frac{E}{2(1 + \nu)}. \tag{1.3-43}$$

Thus $\lambda > O$ and $\mu > O$ is equivalent to

$$0 < \nu < 1/2, E > 0. \tag{1.3-44}$$

(For an incompressible material, $\nu = 1/2$).

An elastic material is said to be a *St Venant-Kirchhoff material* if 44

$$\Sigma_R^*(E) = \lambda(\mathrm{tr}\, E)I + 2\mu E. \tag{1.3-45}$$

It is also expressible in terms of C:

$$\tilde{\Sigma}_R(C) = \left\{\frac{\lambda}{2}(\iota_1(C) - 3) - \mu\right\} I + \mu C. \tag{1.3-46}$$

Then the Cauchy stress tensor can be written as

$$T = \bar{T}(B) = (\iota_3(B))^{1/2} \left\{\frac{\lambda}{2}(\iota_1(B) - 3) - \mu\right\} B + \mu(\iota_1(B))^{1/2} B^2. \tag{1.3-47}$$

Thus such a material is isotropic and material frame indifferent (cf. Theorem 1.3.4).

Remark 1.3.6. While the relation (1.3-45) between Σ_R and E is linear, as a function of u, E_R is *non-linear* since the dependence of E on u non-linear (cf. (1.1-20)).

The relation (1.3-45) can be written componentwise as follows:

$$\begin{aligned} \Sigma_{R_{ij}} &= \lambda E_{kk}\delta_{ij} + 2\mu E_{ij} \\ &\stackrel{\text{def}}{=} a_{ijk\ell}E_{k\ell} \end{aligned} \tag{1.3-48}$$

where the *elasticity coefficients* $a_{ijk\ell}$ are defined by

$$a_{ijk\ell} = \lambda\delta_{ij}\delta_{k\ell} + 2\mu\delta_{ik}\delta_{j\ell}. \tag{1.3-49}$$

The mapping $E \to \lambda(\operatorname{tr} E)I + 2\mu E$ is invertible if and only if $\mu(3\lambda + 2\mu) \neq o$ (and we know that $\mu(3\lambda + 2\mu) > O$ from above). Thus given Σ_R there corresponds a unique Σ. However, this is not always true in actual experiments for large deformations. This model can be expected to be acceptable only for small strains E.

45

Exercises

1.3-1. Given a matrix $A \in \mathbb{S}^3_>$ show that $A^{1/2}$ is uniquely defined in $\mathbb{S}^3_>$. If F is an invertible matrix and $F = RU = VS, U = (F^TF)^{1/2}$, $V = (FF^T)^{1/2}$ show that $R = FU^{-1}, S = V^{-1}F$ are orthogonal. Show also that $S = R$, thus proving theorem 1.3.1.

1.3-2. If $\mathfrak{B}_R$ is any reference configuration of an isotropic, material frame-indifferent material, explain why $\tilde{\Sigma}_R(I)$ is just a multiple of I as shown in Theorem 1.3.6.

1.3-3. Complete the details in the proof that $\lambda, \mu > O$ for a natural state. In particular, prove relations (1.3-32), (1.3-34), (1.3-37), and (1.3-41).

1.4 Hyperelasticity

If the constitutive equation is taken into account, the equilibrium equations in the reference configuration reduce to a system of three equations, for the three components of the deformation ϕ, along with boundary conditions:

(1.4-1) $$DIV_R \hat{T}_R(\nabla\phi) + \rho_R b_R = O \text{ in } \mathcal{B}_R$$

(1.4-2) $$\hat{T}_R(\nabla\phi) n_R = t_{1R} \text{ on } \partial\mathcal{B}_{1R}$$

(1.4-3) $$\phi = \phi_0 \text{ on } \partial\mathfrak{B}_{0R}.$$

This is equivalent to the variational equations

(1.4-4) $$\int_{\mathcal{B}_R} \hat{T}_R(\nabla\phi) : \nabla\theta dX_R = \int_{\mathcal{B}_R} \rho_R b_R.\theta dX_R + \int_{\partial\mathcal{B}_{1R}} t_{1R}.\theta dA_R$$

for all $\theta : \mathfrak{B}_R \to \mathbb{R}^3$, vanishing on $\partial\mathcal{B}_{oR}$.

It was seen in section 1.2 that if the body forces and applied surfaces **46**
were conservative, then (1.4-4) could be written in the form

(1.4-5) $$\int_{\mathfrak{B}_R} \hat{T}_R(\nabla\phi) : \nabla\phi dX_R = B'(\phi)\theta + T_1'(\phi)\theta$$

for real-valued functionals B and T_1 (cf. (1.2-32) and (1.2-35).

If it were possible to write

$$\int_{\mathcal{B}_R} \hat{T}_R(\nabla\phi) : \nabla\theta dX_R$$

as $W'(\phi)\theta$ for some functional W, then the problem (1.4-4) would reduce to finding the stationary points of the functional $W - (B + T_1)$.

Note that upto now, the equations which give the symmetry of $\Sigma_R = (\nabla\phi)^{-1} T_R$ have not been mentioned; it will be seen later (cf. Theorem 1.4.3) that for materials under consideration in this section, these equations will automatically be satisfied.

The above considerations lead to the following definition:

A homogeneous elastic material is said to be *hyperelastic* if there exists a differentiable function $\mathscr{W} : \mathbb{M}^3_+ \to \mathbb{R}$ such that

$$\hat{T}_R(F) = \frac{\partial \mathscr{W}}{\partial F}(F) \tag{1.4-6}$$

for all $F \in \mathbb{M}^3_+$, or componentwise,

$$\hat{T}_{R_{ij}}(F) = \frac{\partial \mathscr{W}}{\partial F_{ij}}(F). \tag{1.4-7}$$

47 A word on notation: the *Frechet derivtive* $\mathscr{W}'(F) : \mathbb{M}^3 \to \mathbb{R}$ is a continuous linear operator such that for F, and $F + G$ in $\mathbb{M}^3_+$,

$$\begin{aligned} \mathcal{W}(F+G) &= \mathcal{W}(F) + \mathcal{W}'(F)G + o(G) \\ &= \mathcal{W}(F) + \frac{\partial \mathcal{W}}{\partial F_{ij}}(F)G_{ij} + o(G). \end{aligned}$$

The term $\frac{\partial \mathcal{W}}{\partial F_{ij}}(F)G_{ij}$ will also be written as

$$\frac{\partial \mathcal{W}}{\partial F}(F) : G \overset{\text{def}}{=} \frac{\partial \mathcal{W}}{\partial F_{ij}}(F)G_{ij},$$

where the *matrix* $\frac{\partial \mathcal{W}}{\partial F}(F)$ has components $\frac{\partial \mathcal{W}}{\partial F_{ij}}(F)$.

Theorem 1.4.1. *Consider a homogeneous hyperlastio material acted on by body and applied surface forces which are conservative. Then the boundary value problem with respect to ϕ is formally equivalent to*

$$I'(\phi)\theta = 0 \tag{1.4-8}$$

for all $\theta : \mathfrak{B}_{\mathbb{R}} \to \mathbb{R}^3$, vanishing on $\partial\mathcal{B}_{oR}$ where, for all $\psi : \mathcal{B}_R \to \mathbb{R}^3$,

$$I(\psi) = \int_{\mathcal{B}_R} W(\nabla\psi(X_R))dX_R - (B(\psi) + T_1(\psi)). \tag{1.4-9}$$

Proof. Given $\psi : \mathfrak{B}_{\mathbb{R}} \to \mathbb{R}^3$ and $\mathcal{W} : \mathbb{M}^3_+ \to \mathbb{R}$, let

$$W(\psi) \overset{\text{def}}{=} \int\limits_{\mathcal{B}_R} W(\nabla\phi) dX_R.$$

Then given ψ and θ,

$$\begin{aligned} W(\psi + \theta) - W(\psi) &= \int\limits_{\mathcal{B}_R} (\mathcal{W}(\nabla(\Psi_\theta)(X_R)) - \mathcal{W}(\nabla\psi(X_R)))dx_R \\ &= \int\limits_{\mathcal{B}_R} \left[\frac{\partial \mathcal{W}}{\partial F}(\nabla\psi(X_R)) : \nabla\theta(X_R) + o(|v\theta(X_R)|; X_R) \right] dX_R \\ &= \int_{\mathcal{B}_R} \hat{T}_R(\nabla\psi) : \nabla\theta dX_R + o(\|\theta\|). \end{aligned}$$

Thus, at least formally, **48**

$$W'(\psi)\theta = \int_{\mathcal{R}} \hat{T}_R(\nabla\psi) : \nabla\theta dX_R \tag{1.4-10}$$

and the result follows. □

Remark 1.4.1. It must be verified in each circumstance thet W is Fréchet differentiable and that the right hand side of (1.4-10) does indeed give the Fréchet derivative. If the C^1-uniform norm is chosen for the space of differentiable vector functions on $\mathcal{B}_R$ and if the first partial derivatives of $\hat{T}_{R_{ij}}$ are Lipschitz Continuous it can be it can be seen that is indeed the case.

The functional W is called the *strain energy* and I is called the *total energy*. The function $\mathcal{W} : \mathbb{M}^3_+ \to \mathbb{R}$ is called the *stored energy function.*

Notice that the boundary value problerm is precisely the *Euler equations associated* to the total energy.

If ϕ_o on $\partial\mathfrak{B}_{oR}$ is extended to the whole of $\mathcal{B}_R$ and I_o defined by

$$I_o(\psi) = I(\psi + \phi_o)$$

then one looks for $\phi - \phi_o$ vanishing on $\partial\mathfrak{B}_{oR}$ such that

$$I'_o(\phi - \phi_o)\theta = o$$

49 for all $\theta : \mathfrak{B}_R \to \mathbb{R}^3$ vanishing on $\partial\mathfrak{B}_{oR}$. Thus particular solutions are those ϕ which satisfy

$$(1.4\text{-}11) \qquad I(\phi) = \begin{cases} \inf \\ \psi : \mathscr{B}_R \to \mathbb{R}^3 \\ \psi = \phi_0 \text{ on } \partial\mathscr{B}_{oR} \end{cases} I(\psi).$$

In the next chapter, it will be seen that the formulation in terms of the boundary value problem will be the basis for proving existence of solutions via the implicit function theorem while (1.4-11) will be the basis for the approach due to J. BALL.

A stored energy function $\mathcal{W} : \mathbb{M}^3_+ \to \mathbb{R}$ will be said to be material frame indifferent (resp isotropic) if $\hat{T}_R = \dfrac{\partial\mathcal{W}}{\partial F}$ is material frame imdifferent (resp. isotropic).

Now necessary and sufficient conditions for a stored energy function to be material frame indifferent or/and isotropic will be examined.

Theorem 1.4.2. *The stored energy function $\mathcal{W} : \mathbb{M}^3_+ \to \mathbb{R}$ is material frame indifferent if and only if, for all $F \epsilon \mathbb{M}^3_+$ and for all $Q \epsilon \mathbb{O}^3_+$*

$$(1.4\text{-}12) \qquad \mathcal{W}(QF) = \mathcal{W}(F).$$

Equivalently, it is material frame indifferent if and only if there exists a function $\mathcal{W} : \mathbb{S}^3_> \to \mathbb{R}$ such that for all $F \epsilon \mathbb{M}^3_+$

$$(1.4\text{-}13) \qquad \mathcal{W}(F) = \tilde{\mathcal{W}}(F^T F)$$

(cf. Equation (1.3-6)*).*

50 *Proof.* Since material frame indifference is equivalent to

$$\hat{T}(QF) = Q\hat{T}(F)Q^T$$

for all $F\epsilon\mathbb{M}^3_+$ and for all $Q\epsilon\mathbb{O}^3_+$, and since

$$\hat{T}_R(F) = \det F\hat{T}(F)F^{-T}$$

it follows that material frame indifference is equivalent to

$$\hat{T}_R(QF) = Q\hat{T}_R(F) \tag{1.4-14}$$

for all $F\epsilon\mathbb{M}^3_+$ and $Q\epsilon\mathbb{O}^3_+$, i.e.,

$$\frac{\partial\mathcal{W}}{\partial F}(QF) = Q\frac{\partial\mathcal{W}}{\partial F}(F) \tag{1.4-15}$$

in case of hyperelastic materials. Define

$$\mathcal{W}_Q(F) = \mathcal{W}(QF), Q\epsilon\mathbb{O}^3_+, F\epsilon\mathbb{M}^3_+. \tag{1.4-16}$$

Now if $F + G \,\epsilon\, \mathbb{M}^3_+$,

$$\begin{aligned}\mathcal{W}_Q(F+G) = \mathcal{W}(QF+QG) &= \frac{\partial\mathcal{W}}{\partial F}(QF) : QG + o(QG)\\ &= Q^T\frac{\partial\mathcal{W}}{\partial F}(QF) : G + o(G)\end{aligned}$$

where the relation $A : BC = B^T C$ has been used (cf. Remark 1.4.2).
Thus

$$\frac{\partial\mathcal{W}_Q}{\partial F}(F) = Q^T\frac{\partial\mathcal{W}}{\partial F}(QF).$$

Thus material frame indifference is equivalent to

$$\frac{\partial}{\partial F}(\mathcal{W}_Q(F) - \mathcal{W}(F)) = 0. \tag{1.4-17}$$

□

Now $\mathbb{M}^3_+$ is connected in $\mathbb{M}^3$(cf,. Exercise 1.4-2 and so the above is **51**
equivalent to)

$$\mathcal{W}(QF) = \mathcal{W}(F) + C(Q), \tag{1.4-18}$$

for all $F\epsilon\mathbb{M}^3_+, Q\epsilon\mathbb{O}^3_+$. Setting $F = I, Q, Q^2, \ldots$ successively, it follows that

$$\mathcal{W}(Q) = \mathcal{W}(I) + C(Q)$$
$$\mathcal{W}(Q^2) = \mathcal{W}(Q) + C(Q)$$
$$\ldots.$$

Thus fo any integer $p \geq 1$,

(1.4-19) $$\mathcal{W}(Q^P) = \mathcal{W}(I) + pC(Q).$$

Then

$$|\mathcal{W}(Q^P)| \geq p|C(Q)| - |\mathcal{W}(I)|.$$

Thus if $C(Q) \neq 0$, then $|\mathcal{W}(Q^P)| \to +\infty$ as $p \to \infty$. But the set $\mathbb{O}^3_+$ is compact in $\mathbb{M}^3_+$ and $\mathcal{W}$ is continuous since it is differentiable. Hence $C(Q) = 0$ and the first assertion is proved.

To prove the second equivalence, let $F = RU$ be the polar factorization of F (cf. Theorem 1.3.1). For $C\epsilon\mathbb{S}^3_>$, set

(1.4-20) $$\tilde{\mathcal{W}}(C) = \mathcal{W}(C^{1/2}).$$

Then

$$\mathcal{W}(F) = \mathcal{W}(RU) = \mathcal{W}(U) = \tilde{\mathcal{W}}(U^2) = \tilde{\mathcal{W}}(F^T F)$$

52 since $U^2 = F^T F$. Conversely, if (1.4-13) is true, then for $F\epsilon\mathbb{M}^3_+$ and $Q\epsilon\mathbb{O}^3_+$,

$$\mathcal{W}(QF) = \tilde{\mathcal{W}}(F^T Q^T QF) = \tilde{\mathcal{W}}(F^T F) = \mathcal{W}(F).$$

It can be show that (cf. Exercise 1.4-4) if $\mathcal{W}$ is differentiable, so is $\tilde{\mathcal{W}}$. Without loss of generality, it may be assumed that the matrix $\dfrac{\partial \tilde{\mathcal{W}}}{\partial C}$ is symetric. For

$$\tilde{\mathcal{W}}(C) = \tilde{\mathcal{W}}(\frac{C + C^T}{2}), C\epsilon\mathbb{S}^3_>.$$

Remark 1.4.2. The following identities involving the scalar product : in $\mathbb{M}^3$ are useful.

$$A : BC = \dot{\mathrm{tr}}(AC^T B^T) = \mathrm{tr}(B^{T^-} AC^T) = B^T A : C \tag{1.4-21}$$

(1.4-22)
$$A : BC = AC^T : B = B : AC^T = \mathrm{tr}(BCA^T) = \mathrm{tr}(CA^T B) = CA^T : B^T.$$

The identity (1.4-21) was used in the proof of the above theorem.

The following theorem says that in case of frame indifferent hyperelastic materials, the symmetry of the second Piola-Kirchhoff stress tensor is automatically verified.

Theorem 1.4.3. *Let the material be hyperelastic and material frame indifferent. Then*

$$\Sigma_R = \hat{\Sigma}_R(F) = \tilde{\Sigma}_R(F^T F) = 2\frac{\partial \tilde{\mathcal{W}}}{\partial C}(C), C = F^T F. \tag{1.4-23}$$

Thus the second Piola-Kirchhoff stress tensor is automatically symmetric. Conversely, if there exists a mapping $\tilde{\mathcal{W}} : \mathbb{S}^3_> \to \mathbb{R}$ *such that*

$$\hat{\Sigma}_R(F) = 2\frac{\partial \tilde{\mathcal{W}}}{\partial C}(F^T F) \tag{1.4-24}$$

then the material is hyperelastic with 53

$$\mathcal{W}(F) = \tilde{\mathcal{W}}(F^T F) \tag{1.4-25}$$

and consequently is material frame indifferent.

Proof. $\hat{\Sigma}_R(F) = F^{-1}\hat{T}_R(F) = F^{-1}\dfrac{\partial \bar{\mathcal{W}}}{\partial F}(F).$

Also $\mathcal{W}(F) = \tilde{\mathcal{W}}(F^T F)$. Now if $F, F + G \epsilon \mathbb{M}^3_+$,

$$\begin{aligned}
\mathcal{W}(F + G) - \mathcal{W}(F) &= \tilde{\mathcal{W}}(F^T F + F^T G + G^T F + G^T G) - \tilde{\mathcal{W}}(F^T F) \\
&= \frac{\partial \tilde{\mathcal{W}}}{\partial C}(F^T F) : (F^T G + G^T F) + o(G) \\
&= F\frac{\partial \tilde{\mathcal{W}}}{\partial C}(F^T F) : G + F(\frac{\partial \tilde{\mathcal{W}}}{\partial C}(F^T F))^T : G + o(G)
\end{aligned}$$

by (1.4-21) - (1.4-22). But $\dfrac{\partial \tilde{\mathcal{W}}}{\partial C}$ is symmetric. Thus

$$\mathcal{W}(F+G) - \mathcal{W}(F) = 2F(\frac{\partial \tilde{\mathcal{W}}}{\partial C}(F^T F)) : G + o(G).$$

54 Hence

$$\frac{\partial \mathcal{W}}{\partial F}(F) = 2F \frac{\partial \tilde{\mathcal{W}}}{\partial C}(F^T F)$$

or

$$\hat{\Sigma}_R(F) = F^{-1} \frac{\partial \mathcal{W}}{\partial F}(F) = 2 \frac{\partial \tilde{\mathcal{W}}}{\partial C}(F^T F).$$

Conversely, if $\mathcal{W}(F) = \tilde{\mathcal{W}}(F^T F)$ then consider the mapping $F \to F^T F$ from $\mathbb{M}^3_+$ into $\mathbb{S}^3_+$. One has

$$\mathcal{W}'(F)G = \tilde{\mathcal{W}}'(F^T F)(F^T G + G^T F)$$

or

$$\begin{aligned} \frac{\partial \mathcal{W}}{\partial F}(F) : G &= \frac{\partial \tilde{\mathcal{W}}}{\partial C}(F^T F) : (F^T G + G^T F) \\ &= 2F \frac{\partial \tilde{\mathcal{W}}}{\partial C}(F^T F) : G \qquad \text{as before.} \end{aligned}$$

Hence

$$\frac{\partial \mathcal{W}}{\partial F}(F) = F \hat{\Sigma}_R(F) = \hat{T}_R(F)$$

and the result follows. □

Now the effect of isotropy on a stored energy function can be similarly examined.

Theorem 1.4.4. *A stored energy function $\mathcal{W} : \mathbb{M}^3_+ \to \mathbb{R}$ is isotropic if, and only if, for every $F \epsilon \mathbb{M}^3_+$ and for every $Q \epsilon \mathbb{O}^3_+$,*

$$\mathcal{W}(F) = \mathcal{W}(FQ). \tag{1.4-26}$$

Proof. The argument runs along the same lines of that of Theorem 1.4.2 and is left as an exercise (cf. Exercise 1.4-5). □

Theorem 1.4.5. *A stored energy function* $\mathcal{W} : \mathbb{M}^3_+ \to \mathbb{R}$ *is material frame indiffernt and isotropic if, and only if, there exists a function*

$$\phi = (\,]_0, +\infty[\,)^3 \to \mathbb{R}$$

such that
$$\mathcal{W}(F) = \phi(\iota_{F^TF}) = \phi(\iota_{FF^T}) \qquad (1.4-27)$$

for every $F \epsilon \mathbb{M}^3_+$.

Proof. By the material frame indifference, there exists a function $\tilde{\mathcal{W}} : \mathbb{S}^3_> \to \mathbb{R}$ such that

$$\mathcal{W}(F) = \tilde{\mathcal{W}}(F^T F).$$

By the isotropy, if $Q \epsilon \mathbb{O}^3_+$, then

$$\mathcal{W}(F) = \mathcal{W}(FQ) = \tilde{\mathcal{W}}(Q^T F^T F Q).$$

55

Thus $\tilde{\mathcal{W}} : \mathbb{S}^3_> \to \mathbb{R}$ satisfies

$$\tilde{\mathcal{W}}(C) = \tilde{\mathcal{W}}(Q^T C Q)$$

for every $C \epsilon \mathbb{S}^3_>$ and for every $Q \epsilon \mathbb{O}^3_+$ (since for every $C \epsilon \mathbb{S}^3_>$ there corresponds $F = C^{1/2} \epsilon \mathbb{M}^3_+$ with $C = F^T F$). Now it was shown in the proof of the Rivlin-Ericksen Theorem (Theorem 1.3.4) that such a function must be a function of the principal invariants

Cnversely, if $\mathcal{W}(F) = \phi(1_{F^TF})$, let $Q \epsilon \mathbb{O}^3_+$.

Then

$$\iota_{(FQ)^T FQ} = \iota_{Q^T F^T FQ} = \iota_{F^T F}$$
$$\iota_{(QF)^T QF} = \iota_{F^T F}$$

and so

$$\mathcal{W}(F) = \mathcal{W}(QF) = \mathcal{W}(FQ)$$

and the thoerem is proved. ◻

The next result expresses the Piola-Kirchhoff stress tensors in terms of the function ϕ of the above theorem.

Theorem 1.4.6. *Given a function* $\phi : (]_0, +\infty[)^3 \to \mathbb{R}$ *and a stored energy function*

$$\mathcal{W}(F) = \phi(\iota_1(C), \iota_2(C), \iota_3(C)), C = F^T F,$$

then

$$\frac{1}{2}\hat{T}_R(F) = \frac{\partial\phi}{\partial\iota_1}F + \frac{\partial\phi}{\partial\iota_2}(\iota_1 I - FF^T F) + \frac{\partial\phi}{\partial\iota_3}\iota_3 F^{-T} \tag{1.4-28}$$

56 *where*

$$\iota_k = \iota_k(F^T F) \text{ and } \frac{\partial\phi}{\partial\iota_k} = \frac{\partial\phi}{\partial\iota_k}(\iota_{F^T F}), k = 1, 2, 3.$$

Furthere

$$\begin{aligned} \frac{1}{2}\tilde{\Sigma}_R(C) &= \frac{\partial\phi}{\partial\iota_1}I + \frac{\partial\phi}{\partial\iota_2}(\iota_1 I - C) + \frac{\partial\phi}{\partial\iota_3}\iota_3 C^{-1} \\ &= \left(\frac{\partial\phi}{\partial\iota_1} + \frac{\partial\phi}{\partial\iota_2}\iota_1 + \frac{\partial\phi}{\partial\iota_3}\iota_2\right) I \\ &\quad - \left(\frac{\partial\phi}{\partial\iota_2} + \frac{\partial\phi}{\partial\iota_3}\iota_1\right) C + \frac{\partial\phi}{\partial\iota_3}C^2. \end{aligned} \tag{1.4-29}$$

Proof. Let Γ be the map $\Gamma : \mathbb{M}^3_+ \to \mathbb{S}^3_>$ given by $\Gamma(F) = F^T F$. Now $\hat{T}_R(F) = \dfrac{\partial\mathcal{W}}{\partial F}(F)$ where

$$\frac{\partial\mathcal{W}}{\partial F}(F) : G = \frac{\partial\phi}{\partial\iota_k}(\iota_C)\Gamma'(F)G.$$

Now $\iota_1(C) = tr\ C$ and so

$$\begin{aligned} \iota_{1'}(C)D &= \mathrm{tr}(D), \\ \iota_3(C) &= \frac{1}{6}\left[3(\mathrm{tr}\, C)^3 - 3(\mathrm{tr}\, C)\,\mathrm{tr}(C^2) + 2\,\mathrm{tr}(C^3)\right] \end{aligned} \tag{1.4-30}$$

and so

$$\iota_3'(C)D = \frac{1}{6}\left[3(\mathrm{tr}\, C)^2(\mathrm{tr}\, D) - 3(\mathrm{tr}\, D)\,\mathrm{tr}(C^2) - 6(\mathrm{tr}\, C)\,\mathrm{tr}(CD) + 6\,\mathrm{tr}(C^2 D)\right]$$

$$
\begin{aligned}
&= \frac{1}{2}\left[(\operatorname{tr} C)^2 - \operatorname{tr}(C^2)\right]\operatorname{tr}(D) + \operatorname{tr}(C^2 D) - \operatorname{tr}(C)\operatorname{tr}(CD) \\
&= \imath_2(C)\operatorname{tr}(D) + \operatorname{tr}(C^2 D) - \operatorname{tr}(C)\operatorname{tr}(CD).
\end{aligned}
$$

Now

$$
\begin{aligned}
\operatorname{tr}(C^2 D) - \operatorname{tr}(C)\operatorname{tr}(CD) &= \operatorname{tr}((C^2 - \imath_1(C)C)D) \\
&= \operatorname{tr}(\imath_3(C)C^{-1}D - \imath_2 D
\end{aligned}
$$

using the Cayley-Hamilton theorem. Thus 57

(1.4-31) $$\imath_{3'}(C)D = \imath_3(C)\operatorname{tr}(C^{-1}D).$$

Finally

$$
\begin{aligned}
\imath_2(C) &= \frac{1}{2}\left[(\operatorname{tr} C)^2 - \operatorname{tr}(C^2)\right] \\
\imath_{2'}(C)D &= \operatorname{tr}(C)\operatorname{tr} D - \operatorname{tr}(CD) \\
&= \operatorname{tr}((\imath_1(C)I - C)D) \\
&= \operatorname{tr}((\imath_2(C)C^{-1} - \imath_3(C)C^2)D)
\end{aligned}
$$

again using the Cayley-Hamilton theorem. This may be again written as

(1.4-32) $$\imath_{2'}(C)D = \imath_3(C)\operatorname{tr}(C^{=1})\operatorname{tr}(C^{-1}D) - \imath_3(C)\operatorname{tr}(C^{-2}D).$$

Also, $\Gamma'(F)G = F^T G + G^T F$. Thus'

$$
\begin{aligned}
\frac{\partial \mathcal{W}}{\partial F}(F) : G = \frac{\partial \phi}{\partial_{\imath_1}} &\operatorname{tr}(F^T G + G^T F) \\
&+ \frac{\partial \phi}{\partial i_2} i_3 \ \operatorname{tr}(C^{-1})\operatorname{tr}(C^{-1}(F^T G + G^T F)) \\
&+ \frac{\partial \phi}{\partial i_2} i_3 \ \operatorname{tr}(C^{-2})(F^T G + G^T F)) \\
&+ \frac{\partial \phi}{\partial i_2} i_3 \ \operatorname{tr}(C^{-1})(F^T G + G^T F))
\end{aligned}
$$

Now, $\operatorname{tr}(F^T G + G^T F) = 2F : G$

$\operatorname{tr}(C^{-1}(F^T G + G^T F)) = C^{-T} : (F^T G + G^T F)$

$$
\begin{aligned}
&= F^{-1}F^{-T} : (F^TG + G^TF) \\
&= 2F^{-T} : G(\text{Using}(1.4\text{-}21) - -(1.4\text{-}22)) \\
\text{tr}(C^{-2}(F^TG + G^TF)) &= C^{-2T} : (F^TG + G^TF) \\
&= C^{-2} : (F^TG + G^TF) \\
&= F^{-1}F^{-T}F^{-1}F^{-T} : (F^TG + G^TF) \\
&= 2F^{-T}F^{-1}F^T : G
\end{aligned}
$$

58

Hence

$$
\begin{aligned}
\frac{1}{2}\frac{\partial \mathcal{W}}{\partial F}(F) : G = {} & \frac{\partial \Phi}{\partial \iota_1}\iota_3 \,\text{tr}(C^{-1})F^{-T} : G \\
& - \frac{\partial \Phi}{\partial \iota_2}\iota_3 F^{-T}F^{-1}F^{-T} : G + \frac{\partial \Phi}{\partial \iota_3}\iota_3 F^{-T} : G
\end{aligned}
$$

Now, consider

$$
\begin{aligned}
&\iota_3 \ \text{tr}\ (C^{-1})F^{-1} - \iota_3 F^{-T}F^{-1}F^{-T} \\
&\qquad = \iota_3 \left[\text{tr}(C^{-1})F^{-T}F^{-1} - F^{-T}F^{-1}F^{-T}F^{-1}\right] F \\
&\qquad = \iota_3 \left[\text{tr}(B^{-1})B^{-1} - B^{-2}\right] F, B = FF^T \\
&\qquad = (\iota_2 B^{-1} - \iota_3 B^{-2})F
\end{aligned}
$$

since B and C are similar. Again $\iota_k = \iota_k(B)$. Now by the Cayley-Hamilton theorem,

$$\iota_2 B^{-1} - \iota_3 B^{-2} = \iota_1 I - B = \iota_1 I - FF^T.$$

Combining all these relation (1.4-28) follows. To obtain (1.4-29) note that $\hat{\sum}_R(F) = F^{-1}\hat{T}_R(F)$. Hence

$$\frac{1}{2}\hat{\Sigma}_R(F) = \frac{\partial \Phi}{\partial \iota_1} I + \frac{\partial \Phi}{\partial \iota_2}(\iota_1 I - F^TF) + \frac{\partial \Phi}{\partial \iota_3}\iota_3 F^{-1}F^{-T}$$

which gives the first relation. To get the second, by the Cayley- Hamilton theorem,

$$\iota_3 C^{-1} = C^2 - \iota_1 C + \iota_2 I$$

59 and the result follows. ◻

Remark 1.4.3. Compare the last relation in (1.4-29) with the statement of Rivlin- Ericksen theorem (Theorem 1.3.4)

Theorem 1.4.7. *Consider a St venant- Kirohhoff material with*

$$\Sigma_R^*(E) = \lambda\, tr\,(E) + 2\mu E \tag{1.4-33}$$

It is hyperlastic with

$$\begin{aligned} \mathcal{W}*(E) &= \frac{\lambda}{2}(tr\ E)^2 + \mu\ \mathrm{tr}(E^2) \\ &= \frac{(\lambda+2\mu)}{8}(\iota_1-3)^2 + \mu(\iota_1-3) - \frac{\mu}{2}(\iota_2-3) = \Phi(\iota_C) \end{aligned} \tag{1.4-34}$$

where

$$\iota_k = \iota_k(C), k = 1, 2, 3.$$

Proof. Set

$$\tilde{\mathcal{W}}(C) = \mathcal{W}(I+2E) = \mathcal{W}*(E).$$

Now

$$\begin{aligned} \mathcal{W}*(E+H) &= \mathcal{W}*(E) + \lambda\ tr\ E\ tr\ H + 2\mu\ tr\ (EH) + o(H) \\ &= \mathcal{W}*(E) + (\lambda(\mathrm{tr}\, E)I + 2\mu E) : H + o(H). \end{aligned}$$

Hence

$$\frac{\partial \mathcal{W}*}{\partial E}(E) = \sum_R^*(E).$$

This implies that

$$\tilde{\sum}_R(C) = 2\frac{\partial \tilde{\mathcal{W}}}{\partial C}(C)$$

and hence the material is hyperelastic. The verification of the expression for Φ is left as an exercise to the reader. □

Remark 1.4.4. The above result gives another proof (cf. equation 60
(1.3-47)) that St Venat-Kirchhoff materials are isotropic and material frame indifferent.

Remark 1.4.5. Other examples of hyperelastic materials will be seen in Chapter 2 (Ogden's materials)

Theorem 1.4.8. *Let $\mathfrak{B}_R$ be a natural state of a material which is isotropic and material frame indiffrent. Then if $\mathcal{W} \epsilon C^1(\mathbb{M}^3_+; R)$*

$$\mathcal{W} * (E) = \frac{\lambda}{2}(\mathrm{tr}\, E)^2 + \mu\, \mathrm{tr}(E^2) + o(|E|^2). \tag{1.4-35}$$

Proof. Let

$$\begin{aligned} \mathcal{W} * (E) &= \frac{\lambda}{2}(\mathrm{tr}(E))^2 + \mu\, \mathrm{tr}(E^2) + \delta\mathcal{W} * (E) \\ \frac{\partial \mathcal{W}*}{\partial E}E &= \lambda(\mathrm{tr}\, E)I + 2\mu E + \frac{\partial \delta \mathcal{W}*}{\partial E}(E) \\ &= \Sigma^*_R(E) = \lambda\, \mathrm{tr}\, E) + 2\mu E + o(E). \end{aligned}$$

Thus,

$$\frac{\partial \delta \mathcal{W}*}{\partial E}(E) = o(E).$$

Since subtracting a constant in $\mathcal{W}*$ does not change the stress tensors, it can be assumed, without loss of generaliry, that $\delta\mathcal{W} * (o) = o$. Hence

$$\delta\mathcal{W}*)(E) = \int_0^1 \frac{\partial \delta \mathcal{W}*}{\partial E}(E) : dt = o(|E|^2).$$

□

61 ## Exercises

1.4-1 For a non-homogeneous hyperelastic material,

$$\hat{T}_R(X_R, F) = \frac{\partial \mathcal{W}}{\partial F}(X_R, F)$$

for every $X_R \epsilon \mathcal{B}_R$, and for every $F \epsilon \mathbb{M}^3_+$. Extend the analyisi of this section to such materials.

1.4-2 (i) Show that $\mathbb{M}^3_+$ is a connected subset of $\mathbb{M}^3$.

(ii) Show by an example that $\mathbb{M}^3_+$ is not convex.

(iii) Identify its convex hull in $\mathbb{M}^3$.

1.4-3 Assume that (cf. Proof of Theorem 1.4.2)

$$\mathcal{W}(QF) = \mathcal{W}(F) + C(Q)$$

For all $F\epsilon\mathbb{M}^3_+$ and Q $\epsilon\mathbb{O}^3_+$. Show that $C(Q) = o$ without using the continuity of $\mathcal{W}$.

1.4-4 (i) Show that $\mathbb{S}^3_>$ is an open subset of $\mathbb{S}^3$.

(ii) Show that if $\mathcal{W}$ is differentiable, so is $\mathcal{W}$.

1.4-5 Prove Theorem 1.4.4: show that isotropy is equivalent to

$$\hat{T}_R(FQ) = \hat{T}_R(F)Q \text{for all} F\epsilon\mathbb{M}^3_+$$

and $Q\epsilon\mathbb{O}^3_+$, which is in turn equivalent to

$$\frac{\partial\mathcal{W}}{\partial F}(F) = \frac{\partial\mathcal{W}}{\partial F}(FQ)Q^T = \frac{\partial\mathcal{W}}{\partial F}(F), \mathcal{W}(FQ).(FQ).$$

1.4-6 Check the second relation in equation (1.4-34).

1.4-7 Consider an elastic matrial with

$$\overline{T}(B) = B_o(\iota_B)I + B_1(\iota_B)B + B_2(\iota_B)B^2.$$

Find necessary and sufficient conditions on **62**

$$\beta_\alpha : (]o, +\infty[)^3 \to \mathbb{R}, \alpha = 0, 1, 2$$

for the material to be hyperelastic.

1.4-8 In Theorem 1.4.8, compute the terms of order 2 in $\sum^*_R(E)$ and terms of order 3 in $\mathcal{W}*(E)$. Explain the discrepancy in the number of terms obtained in each case.

Chapter 2

Some Mathematical Aspects of Three-Dimensional Elasticity

IN THIS CHAPTER, questions of existence of solutions to the boundary value of three-dimensional elasticity will be examined. In the first section, some general considerations about these boundary value problems will be mentioned. The problems will be classified with respect to boundary conditions. As good models of elasticity must preclude uniqueness of solution, several examples of non-uniqueness will be presented. **63**

The first tool for the study of existence of solutions is the implicit function theorem. As this requires a knowledge of the linearized problem, the second section will briefly present linear elasticity and the third section will prove existence, albeit for a very narrow class of boundary conditions. The fourth section will study incremental methods, whose analysis follows closely related lines.

The last two sections will present results on polyconvexity and existence of solutions to the problem of minimizing the energy using the approach of J. BALL. Though several types of boundary conditions can be studied here, the main drawback is the lack of regularity of solutions and so it is not known if the solutions satisfy the equilibrium equations.

2.1 General Considerations About the Boundary Value Problems of Three-Dimensional Elasticity

64 Given a respones function $\hat{T}_R : \mathbb{M}^3_+ \to \mathbb{M}^3$ satisfying

$$F(\hat{T}_R(F))^T = \hat{T}_R(F)F^T \tag{2.1-1}$$

for every $F\epsilon\mathbb{M}^3_+$ and given the denstiy $\rho_R : \mathcal{B}_R \to \mathbb{R}$ and dead loads $b_R : \mathcal{B}_R \to \mathbb{R}^3, \imath_{\mathbb{R}} : \partial\mathcal{B}_{1R} \to \mathbb{R}^3$, the boundary value problem arising out of the equilibrium equations amounts to finding a deformation Φ which satisfies

$$DIV_R\hat{T}_R(\nabla\phi) + \rho_R b_R = 0 \text{ in } \mathcal{B}_R \tag{2.1-2}$$

$$\hat{T}_R(\nabla\phi)n_{\mathbb{R}} = t_{1R} \text{ on } \partial\mathcal{B}_{1R} \tag{2.1-3}$$

$$\phi = \phi_o \text{ (given) on } \partial\mathbb{B}_{oR} \tag{2.1-4}$$

The boundary condition $\phi = \phi_o$ on $\partial\mathbb{B}_{oR}$ is called a *boundary condition of place*. The boundary condition (2.1-3) on $\partial\mathbb{B}_{1R}$ is called a *boundary condition of traction* (and this defnition implies it is a dead load).

If $\partial\mathcal{B}_{oR} = \phi$ the problem is a *pure traction* boundary value problem.If $\partial\mathcal{B}_{1R}$ the problem is a *pure displacement* problem. If both $\partial\mathcal{B}_{oR}$ and $\partial\mathcal{B}_{1R}$ have strictly positive dA_R- measure, then the problem is a *mixed displacement-tranction* problem.

Recall that

$$\hat{\sum}_R(\nabla\phi) = (\nabla\phi)^{-1}\hat{T}_R(\nabla\Phi)$$

where
$$\hat{\sum}_R : \mathbb{M}^3_+ \to \mathbb{S}^3$$

65 and the boundary value problem (2.1-2)–(2.1-4) can be rewritten in terms of this tensor. If the material is hyperelastic (cf. Section 1.4), then

$$\hat{T}_R(F) = \frac{\partial\mathcal{W}}{\partial F}(F) \tag{2.1-5}$$

for a stored energy function $\mathcal{W} : \mathbb{M}^3_+ \to \mathbb{R}$ and the problem is equivalent to finding the sationary points of an energy functional,

$$\text{(2.1-6)} \qquad I(\psi) = \int_{\mathcal{B}_R} \mathcal{W}(\nabla\psi)dX_R - \left(\int_{\mathcal{B}_R} \rho_R b_R.\psi dX_R + \int_{\partial\mathcal{B}_{1R}} t_{1R}.\psi dA_R \right)$$

The partial differentail equations (2.1-2) are nonliner with respect to ϕ since the mapping $\hat{T}_R : \mathbb{M}^3_+ \to \mathbb{M}^3$ is non-linear, and of the second order. The non-linearity occurs in the highest order terms and this is a source of difficulty. Another source of difficulty is that the solution ϕ must satisfy det $(\nabla\phi) > o$. Thus for instance in (2.1-6) ψ must vary over $\mathbb{M}^3_+$ which is clearly not a vector sapce; in fact it is not even a convex subset of $\mathbb{M}^3$ (cf. Exercise 1.4-2).

The boundary condition of traction could be replaced by the so-called *boundary condition of pressure* (which is not a dead load, but it is conservative). Again it is possible to have a *pure pressure* boundary value problem (for instance, a soap bubble or a submarine) or *mixed displacement-pressure* boundary value problems.

These boundary condition, though being the only ones to be considered here, are far from exhaustive. Other types of conditions are possible.

In practice one can have *unilateral conditions*. For instance, if the **66**
body must remain in above the plane spanned by e_1, e_2 the boundary condition is $X_3 \geq o$ or $\phi_3(X_R) \geq o$

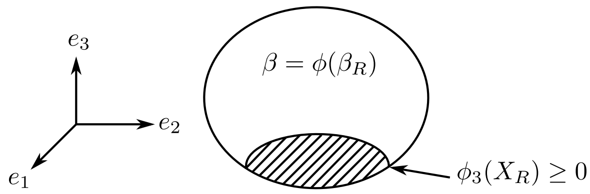

Figure 2.1.1:

It is also possible to have a mixture of displacement and stress boundary conditions. Consider $\mathcal{B}_R$ to be a plate with a pressure p compressing the lateral surface (fig 2.1.2) and given only by its *horizantal average*. Then the conditions are

(2.1-7) $$\begin{cases} u_1, u_2 \text{independent of} X_3 \\ u_3 = 0 \end{cases}$$

and

(2.1-8) $$\frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon} T(\nabla\phi(X_R)n\, dX_3 = -pn.$$

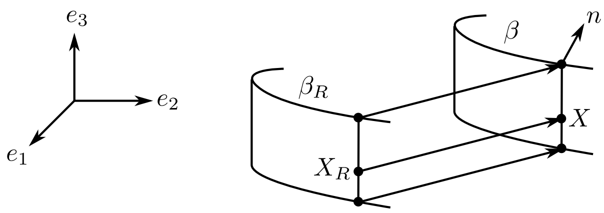

Figure 2.1.2:

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Apart from possibly the problem of bodies moving with constant velocity in a fluid, the pure traction problems are less common. The pure displacement problems are quite unrealistic.

In general, several deformed states are possible for the same system of forces, though they may not all be physically feasible or 'stable'. Nevertheless the mathematical model cannot recognize the feasible or stable ones. Hence a good model will always account for non-uniqueness of solutions. Several examlpes of non-uniqueness will now be given .

Example 2.1.1. A mixed displacement-traction problem. Consider a long circular cylinder fixed at either end. The body force is just its weight. On the lateral surface $t_{1R} = o$. Assume the body to be extremely pliable, and rotate one end by an angle of 2π and reglue it in its original position. Then a line parallel to the axis on the lateral surface

68 will deform into a curve and thus gives another solution other than the natural one, which will just be a slight bending of the cylinder under its weight. It is theoretically possible to rotate the face by $2k\pi$ for any positive integer k. Thus the model must account for an infinite number of solutions.

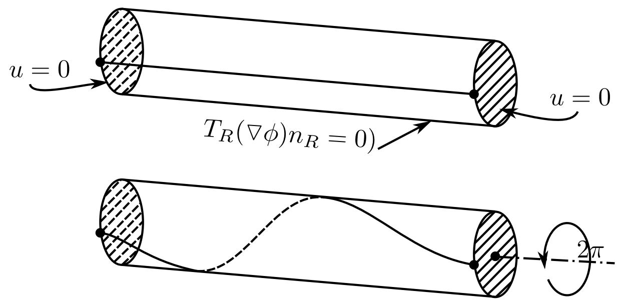

Figure 2.1.3:

Example 2.1.2 (F. JHON)**.** A pure displacement problem. Consider the body to be betwen two concentric spheres. Assume $u \equiv 0$ on both the inner and outer surfaces. Apart from the trivial solution, it is possible to have an infinite number of solutions by (theoretically) rotating the inner sphere about an axis by an angle of $2k\pi$ and re-glueing it to the body.

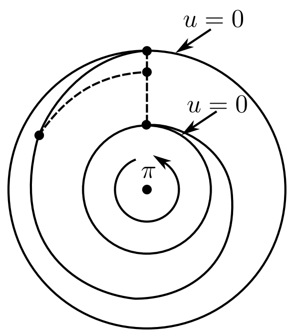

Figure 2.1.4:

69

Example 2.1.3 (C. ERICKSEN)**.** A pure traction problem.A rectangular lock is pulled normal to the upper and lower surfaces. Rotation of the configuration by π produces a (though urrealistic) solution where the body is comprssed!

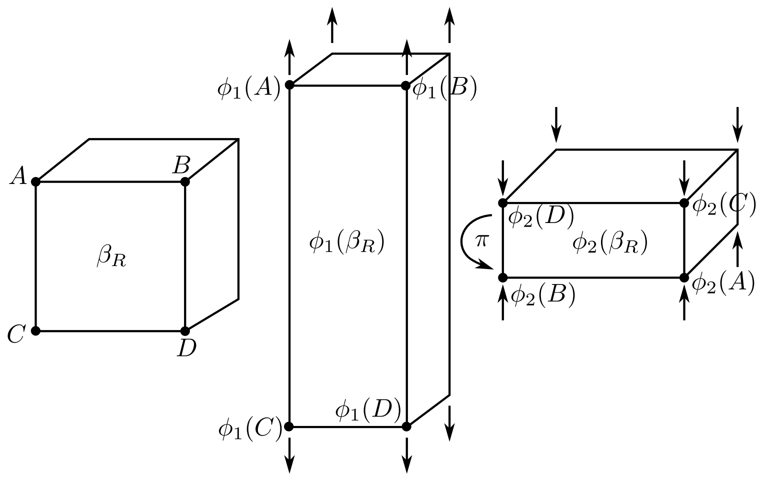

Figure 2.1.5:

70 **Example 2.1.4.** Consider a thin circular plate subjected to the boundary condition (2.1-7) - (2.1-8) with $p = \lambda p_1$. If $\lambda < 0$ (i.e the plate is pulled) or if $\lambda > o$ and small, $u \equiv o$ is the only possible solution. If λ exceeds a critical value, the plate can buckle upwards or downwards thus giving two additional solutions (cf. 2.1.6) This is a buckling phenomenon

Example 2.1.5. Eversion problems. A cut tennis ball (borrowed from a very good friend) can be made to exist in two diffrent states as shown in
71 fig 2.1.7. The everted state can be produced by pushing hard enough on the natural state.

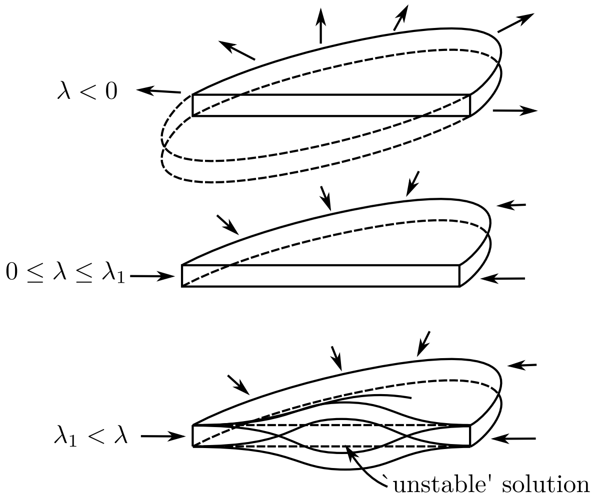

Figure 2.1.6:

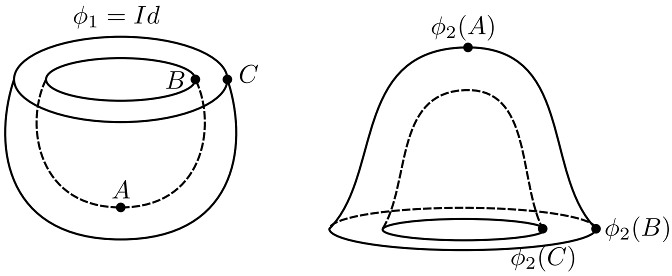

Figure 2.1.7:

It is possible to do the same thing to a tube. These are examples of multiple solutions to a pure traction problem.

Returning to the various restrictions on the model and its analysis, the first is the taking into account of properties like isotropy, hyperelasticity and the axiom of material frame indifference. These are fairly easy to handle. In sections 1.3 and 1.4 various necessary and sufficient conditions of the relevant functions were studied. The main restriction is that the solution ϕ must satisfy forces

$$\det(\nabla\phi) > o.$$

In using the implicit function theorem, this requirement is ignored at first and then shown to be satisfied for sufficiently small forces.

72 In the different approach of J. BALL, this is taken into account (a.e in $\mathcal{B}_R$) by imposing it on the set $\mathscr{U}$ of test functions over which the energy is minimized. This precludes the convexity of $\mathscr{U}$ which makes minimization more difficult than usual. In this approach, it will be nicely taken into account by imposing that $\mathcal{W}(F) \to +\infty$ when $\det(F) \to o^+$.

Even if $\det\nabla\phi > o$ everywhere on $\mathcal{B}_R$, it does not ensure that ϕ is a one-one mapping, a property natural to expect in a deformation.Thus if a body as in fig. 2.1.8(a) in contact with the horizontal plane is pushed along the two 'arms', it must take a shape as in fig 2.1.8(b). But the mathematical model will not preclude a situation as in fig 2.1.8(c) where the material penetrates itself, still keeping $\det\nabla\phi > 0$

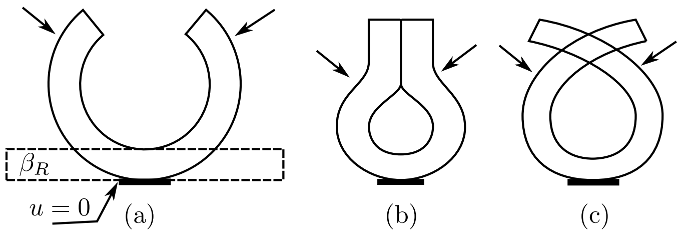

Figure 2.1.8:

73 In the case of incompressible materials, the energy is minimized over a set of function ψ (in a suitable function space) with

$$\det(\nabla\psi) = 1 \text{ a.e.}$$

Some of the noations used hitherto will be changed. $\mathfrak{B}_R$ will henceforth be denoted by $\bar{\Omega}, \Omega$ a bounded open subset of $\mathbb{R}^3$ and its boundary $\partial\mathfrak{B}_R$ by Γ. The portions $\partial\mathfrak{B}_{oR}and\partial\mathfrak{B}_{1R}$ will be denoted by Γ_o and Γ_1 respectively.

The generic point X_R will henceforth be labelled x and dX_R and dA_R will be changed to dx and da respectively. The derivatives $\dfrac{\partial}{\partial X_{Ri}}$ will be denoted by ∂_i and DIV_R by div. The normal n_R to $\partial\mathfrak{B}_R$ will now be given by $\nu = (\nu_i)$, the unit outer normal to Γ.

The tensors $\sum_R = (\sum_{R_{ij}})$ and $T_R = (T_{R_{ij}})$ will be denoted by (σ_{ij}) and (t_{ij}) respectively. The vectors $\rho_R b_R$ and t_{1R} will be denoted by $f = (f_i)$ and $g = (g_i)$ respectively. The symbols for $\phi, u, F = \nabla\phi, B = FF^T, C = F^T$ and $E = \dfrac{1}{2}(C - I)$ will remain unchanged.

Thus, for instance the equations of equilibrium im terms of $\sum_R$ read in the old natation as:

(2.1-9) $$\mathrm{DIV}_R\left(\nabla\phi \sum_R\right) + \sigma_R b_R = o \text{ in } \mathfrak{B}_R$$

(2.1-10) $$\nabla\phi \sum_R n_R = t_{1R} \text{ on } \partial\mathfrak{B}_{1R}$$

(2.1-11) $$\phi = \phi_0 \text{ on } \partial\mathfrak{B}_{0R}$$

These, when translated in to the new notations will read as **74**

(2.1-12) $$-\partial_j(\sigma_{kj}\partial_k\phi_i) = f_i \text{ in } \Omega$$

(2.1-13) $$\sigma_{kj}\partial_k\phi_i\nu_j = g_i \text{ on } \Gamma_1$$

(2.1-14) $$\phi_i = \phi_{0i} \text{ on } \Gamma_0$$

Exercises

2.1-1 . Assume that a pure traction problem has a solution ϕ. Show that

$$\int_{\mathfrak{B}_R} \rho_R b_R dX_R + \int_{\partial\mathfrak{B}_R} t_{1R} dA_R = 0$$

and $$\int_{\mathfrak{B}_R} \phi\Lambda\rho_R b_R dX_R + \int_{\partial\mathfrak{B}_R} \phi\Lambda t_{1R} dA_R = 0.$$

2.1-2 .Consider a hyperelastic incompressible material. In the constrained minimization problem

$$\inf_{\psi \epsilon \mathcal{U}} I(\psi)$$

where $$\mathcal{U} = \{\psi \mid \det(\nabla\psi) = 1 \text{a.e.}\},$$

show by a formal computation that the Lagrange multiplier is the pressure.

2.2 The Linearized System of Elasticity

75 Consider the boundary value problem (2.1-12)-(2.1-14). In terms of the displacement u it can be rewritten as

$$-\partial_j(\sigma_{ij} + \sigma_{kj}\partial_k u_i) = f_i \text{ in } \Omega, \tag{2.2-1}$$

$$(\sigma_{ij} + \sigma_{kj}\partial_k u_i)\nu_j = g_i \text{ on } \Gamma_1, \tag{2.2-2}$$

$$u_i = u_{oi} \text{ on } \Gamma_o, \tag{2.2-3}$$

with the constitutive equation

$$\sigma_{ij} = \sigma^*_{ij}(E(u)) = \lambda E_{kk}(u)\delta_{ij} + 2\mu E_{ij}(u) + o(E) \tag{2.2-4}$$

where

$$E(u) = \frac{1}{2}\left(\nabla u^T + \nabla u + \nabla u^T \nabla u\right) \tag{2.2-5}$$

If u were defined in a suitable function space, whose functions vanish on Γ_0, then symbolically one can write

$$A(u) = \begin{bmatrix} f \\ g \end{bmatrix} \tag{2.2-6}$$

The linearised system of elasticity will then be formally defined as (assuming A is differentiable)

$$A'(0)u = \begin{bmatrix} f \\ g \end{bmatrix}$$

This can be derived as follows. The linearized strain tensor is

$$\epsilon(u) = \frac{1}{2}\left(\nabla u^T + \nabla u\right). \tag{2.2-7}$$

Then σ will in turn be linearized as

$$\sigma_{ij} = \lambda \epsilon_{kk}(u)\delta_{ij} + 2\mu\epsilon_{ij}(u). \tag{2.2-8}$$

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Substituting this in (2.2-1)-(2.2-3) and keeping only the first order terms, the linearized syatem elasticity turns out to be

$$-\partial_j \sigma_{ij} = f_i \text{ in } \Omega \tag{2.2-9}$$

$$\sigma_{ij} v_j = g_i \text{ on } \Gamma_1 \tag{2.2-10}$$

$$u_i = u_{oi} \text{ on } \Gamma_0 \tag{2.2-11}$$

where σ is given by (2.2-8). Note that such a syatem cannot be a model for elasticity (of Exercise 2.2-1) but only approximation of a model.

Remark 2.2.1. If the equations were written in terms of t_{ij} and then linearized, the same linearized system of elasticity would have been obtained. This is because $T_R = (I + \nabla u) \sum\limits_R$ and on linearizing this realtion only the part coming from $I \sum\limits_R = \sum\limits_R$ will be retained.

Before the existence and regularity of solution to the linearized system of elasticity can be studied the following notations for the Sobolev spaces will be needed.

Let $m \geq 0$ be an integer and $1 \leq p \leq +\infty$. Then

$$W^{m,p}(\Omega) = \{v \epsilon L^p(\Omega) \mid \partial^\alpha v \epsilon L^p(\Omega) \text{for all } \mid \alpha \mid \leq m\} \tag{2.2-12}$$

where α is a multi-index and $\partial^\alpha v$ is the corresponding partial derivative (in the sense of distribution). This space is a Banach space with the norm

$$\| v \|_{m,p.\Omega} = \left(\int\limits_\Omega \sum_{|\alpha| \leq m} \mid \partial^\alpha v \mid^p dx \right)^{1/p} \tag{2.2-13}$$

(with the standrad modification if $p = +\infty$). The semi-norm $|\ .\ |_{m,p,\Omega}$ is defined by 77

$$\| v \|_{m,p.\Omega} = \left(\int\limits_{\Omega} \sum_{|\alpha|=m} |\ \partial^{\alpha} v\ |^{p}\ dx \right)^{1/p} \tag{2.2-14}$$

If $m = 0, W^{0,p}(\Omega) = L^{p}(\Omega)$and $|\ .\ |_{0,p,\Omega}$ is the usual $L^{p}(\Omega)$- norm. If $\mathscr{D}(\Omega)$ is the space of C^{∞} functions with compact support in Ω, its closure in $W^{m,p}(\Omega)$ will be denoted by $W_{o}^{m,p}(\Omega)$.

If $p = 2$, it is customary to write $H^{m}(\Omega)$and$H_{0}^{m}(\Omega)$ instead of W^{m}, $2(\Omega)$ and $W_{0}^{m,2}(\Omega)$ respectively. The norms and semi-norms in this case will be written as $\| \cdot \|_{m,\Omega}$ respectively $| \cdot |_{m,\Omega}$ is the $L^{2}(\Omega)$- norm.

By Poincarés inequality, $| \cdot |_{m,p,\Omega}$ is a norm on $W_{0}^{m.p}(\Omega)$ and is equivalent to $\|\ .\ \|_{m,p\Omega}$, for$1 \leq p < \infty$.

In case of vector valued or tensor valued functions, the symbols $W^{m,p}(\Omega)$, $\mathbb{L}^{P}(\Omega)$ will be used to denote that each component is in $W^{m,p}$ (Ω) or $L^{p}(\Omega)$ respectively. However the symbols for the norms and semi-norms will not be altered.

The following result is fundamental.

Theorem 2.2.1 (Korn's Inequality)**.** *Let* Γ *be smooth enough. Then*

$$\{v = (v_{i}) \epsilon \mathbb{C}^{2}(\Omega) \mid \epsilon_{ij}(v) \epsilon L^{2}(\Omega), 1 \leq i, j \leq 3\} = \mathbb{H}^{1}(\Omega) \tag{2.2-15}$$

78 *Consequently, there exists constans* $C_{1} > 0 and C_{2} > 0$ *such that*
(2.2-16)

$$C_{1} \| v \|_{1,\Omega} \leq (|\ v\ |_{0,\Omega}^{2} + |\ \epsilon(v)\ |_{0,\Omega}^{2})^{1/2} \leq C_{2} \| v \|_{1,\Omega}\ for\ all v \epsilon H^{1}(\Omega).$$

Proof. See DUVAUT and LIONS [1972] or NITSCHE [1981]. The main difficulty is in proving (2.2-15). Since the second inquality of (2.2-16) is obvious, the first follows from (2.2-15) and the closed graph theorem. □

A consequence of the above result is

Theorem 2.2.2. *Let* $\mathbb{V}$ *be defined by*

$$\mathbb{V} = \{v \epsilon \mathbb{H}^{1}(\Omega) \mid v = 0\ on\ \Gamma_{0}\} \tag{2.2-17}$$

where the da-measure of Γ_0 *is strictly positive. Then the semi-norm* $|\, \epsilon(v) \,|_{0,\Omega}$ *is a norm on V equivalent to the norm* $\|\, .\, \|_{1,\Omega}$.

Proof. Cf. Exercise 2.2-2. □

Assume now that $u = 0$ on Γ_0. Let V be as in (2.2-17). Multiplying (2.2-9) by a function $v\epsilon\mathbb{V}$, integration by parts using Green's formula, and using (2.2-10), (2.2-11) and the symmetry of σ, the following variational formulation of the problem (2.2-9) - (2.2-11) can be obtained.

Find $u\epsilon\mathbb{V}$ such that, for all $v\epsilon\mathbb{V}$,

$$a(u, v) = L(v) \tag{2.2-18}$$

where

$$a(u, v) = \int\limits_{\Omega} (\lambda\epsilon_{kk}(u)\epsilon_{\ell\ell}(v) + 2\mu\epsilon_{ij}(u)\epsilon_{ij}(v))\, dx \tag{2.2-19}$$

and

$$L(v) = \int\limits_{\Omega} f_i v_i dX + \int\limits_{\Gamma_1} g_i v_i\, da. \tag{2.2-20}$$

79

By a simple application of Schwarz's inequality, it follows that $a(.,.)$ is a continuous bilinear form on $\mathbb{H}^1(\Omega)$ and L is a continuous functional on $\mathbb{H}^1(\Omega)$ (and hence on $\mathbb{V}$ as well).

The following existence result holds.

Theorem 2.2.3. *Consider the variatiational formulation of the linearized syatem of elasticity,* (2.2-18)*, or, equivalently, the problem: Find* $u\epsilon\mathbb{V}$ *such that*

$$J(u) = \inf_{v\epsilon\mathbb{V}} J(v) \tag{2.2-21}$$

where

$$J(v) = \frac{1}{2} a(v, v) - L(v) \tag{2.2-22}$$

if $\lambda > 0$ *and* $\mu > 0$ *then the problem has a unique solution.*

Proof. Observe that by Theorem 2.2.2 for all $v \epsilon \mathbb{V}$,

$$a(v, v) \geq 2\mu \mid \epsilon(v) \mid^2_{0,\Omega} \geq \alpha \parallel v \parallel^2_{1,\Omega} \tag{2.2-23}$$

where $\alpha > o$ (using $\lambda > o$ and $\mu > o$). Thus $J : \mathbb{V} \to \mathbb{R}$ is a convex, and continuous functional. Hence it is weakly lower semi continuous. Let $\{u_k\}$ be a minimizing sequence in $\mathbb{V}$, i.e.

$$J(u_k) \to \inf_{v \epsilon \mathbb{V}} J(v) < +\infty.$$

Since J is coercive (i.e., $J(v) \to \infty$as $\parallel v \parallel \to \pm\infty$) it follows that $\{u_k\}$
80 is a bounded sequence and hence has a weakly convergent subsequence. Let u be the weakly limit of the subsequence (again indexed by k for convenience). Then by the weak lower semi-continuity of J.

$$\inf_{v \epsilon \mathbb{V}} J(v) \leq J(u) \leq \lim_{k \to \infty} \inf J(u_k) = \inf_{v \in \mathbb{V}} J(v).$$

Thus J attains its mimimum at u. It is easy to see that equations (2.2-18) are simply equivalent to the equation $J'(u) = 0$. Hence the equivalencve of the two problems since J is convex and the existence of a solution.

If u_1 and u_2 are solutions in $\mathbb{V}$ then $a(u_1 - u_2, v) = 0$ for all $v \epsilon \mathbb{V}$. Setting $v = u_1 - u_2$ and using (2.2-23), it follows that $u_1 = u_2$, thus proving the uniqueness of the solution. □

Remark 2.2.2. The existence of a unique solution to (2.2-18) also follows directly from (2.2-23) by applying the *Lax-Milgram Lemma.*

Finally, let us state the result on the regularity of solutions to the above problem.

Theorem 2.2.4. *Suppose* Γ *is smooth enough and for some* $p \geq 2$, $f \epsilon \mathbb{C}^p(\Omega)$. *Assume* $\Gamma_1 = \phi$. *Then the solution* $u \epsilon \mathbb{V} = \mathbb{H}'_0(\Omega)$ *of the corresponding linearized system of elasticity is in the space* $\mathbb{V}^p(\Omega)$, *where*

$$\mathbb{V}^p(\Omega) = \{v \epsilon \mathbb{W}^{2,p}(\Omega) \mid v = 0 \text{ on } \Gamma\}. \tag{2.2-24}$$

Proof. The case $p = 2$ has been proved by NEČAS [1967]. If $p > 2$, the argument goes as follows. Let $A'(0) : \mathbb{V}^p(\Omega) \to \mathbb{C}^p(\Omega)$ represent the operator of the linearized system of elasticity. Then, if index $(A'(0)) \overset{\text{def}}{=} \dim(\ker A'(0)) - \dim(Coker A'(0))$, it was proved by GEYMONAT [1965] that for all $p, 1 < p < \infty$, the index was inde- **81**
pendent of p. Now, by the result of Necas above if $p = 2, A'(0)$ is onto and so $\dim(Coker(A'(o)) = 0$. By uniqueness of the solution, $\dim(ker(A'(0)) = 0$ for all $p \geq 2$. Hence the index is zero for all p and so $\dim(Coker(A'(o))) = 0$ for all $p \geq 2$, i.e., $A'(0)$ is onto for all $p \geq 2$, which proves the theorem. □

Remark 2.2.3. The above result does not follow from those of AGMON, DOUGLIS and NIRENBERG [1964]. Their results state that if $f \epsilon \mathbb{L}^p(\Omega)$ implise $u \epsilon \mathbb{W}^{2,p}(\Omega)$ then $f \epsilon \mathbb{W}^{m,p}(\Omega)$ implies $u \epsilon \mathbb{W}^{2+m,p}(\Omega)$ for our problem. The 'starting' regularity result ($m = 0$) needs be know a priori and Theorem 2.2.4 proves that in case of the linearized syatem of elasticity.

Caution! The $W^{2,p}$ regularity does not hold for the mixed displacement-traction linearized syatem of elasticity.

Exercises

2.2-1 Show that if $\epsilon(u) = \frac{1}{2}(\nabla u^T + \nabla u)$, then a constitutive equation of the form $\sigma = \sigma^*(\epsilon)$, with σ^* a linear function of ϵ, does not satisfy the axion of material frame indenfference.

2.2-2 Prove Theorem $2.2 - 2$. (Show first that $|\epsilon(v)|_{0\Omega})$ is a norm on $\mathbb{V}$. Prove the equivalence of norms by contradiction: assume there exists a sequence $v^k \epsilon \mathbb{V}$ with $||v^k||_{1,\Omega} = 1$ and $|\epsilon(v^k)|_{0,\Omega} \to 0$.

2.2-3 Consider the linearized system of elasticity in variational form with $\Gamma_0 = \phi$. Show that there exists a solution to the problem provided

$$\int_\Omega f\, dx = \int_\Gamma\ dx,$$

82 in the quotient space $H^1(\Omega)\mathbb{W}$, where

$$\mathbb{W} = \{v\epsilon\mathbb{H}^1(\Omega) \mid \epsilon(v) = 0\}.$$

Show also that

$$\mathbb{W} = \{v\epsilon|\mathbb{H}^1(\Omega)|v = a + b \wedge ox\}.$$

2.2-4 Extend the regularity result to the case $1 < p < 2$.

2.2-5 Show that if $\mu > 0$, there exists a $\lambda_0 < 0$ such that if $\lambda_0 < \lambda \leq 0$, the linear form $a(.,.)$ given by (2.2-19) is coercive.

2.3 Existence Theorems via Implicit Function Theorem

In this section, existence solutions to the *pure displacement problem* will be proved using the implict function theorem.

For simplicity, consider first a St Venabt-Kirchhoff marerial Recall that the constitutive equation in this case can be written as

$$\sigma_{ij} = a_{ijk\ell}\, E_{k\ell}(u) = \lambda E_{kk}(u)\delta_{ij} + 2\mu E_{ij}(u), \tag{2.3-1}$$

with $\lambda > 0$ and $\mu > 0$. Also

$$E_{ij}(u) = \epsilon_{ij}(u) + \frac{1}{2}\partial_i u_m \partial_j u_m, \tag{2.3-2}$$

where

$$\epsilon_{ij}(u) = \frac{1}{2}(\partial_i u_j + \partial_j u_i) \tag{2.3-3}$$

Then the boundary value problem (2.2-1)–(2.2-3) becomes ($\Gamma_1 = \phi$),

$$\begin{aligned} -\partial_j(a_{ijpq}\epsilon_{pq} + \frac{1}{2}a_{ajpq}\partial_p u_m \partial_q u_m + a_{kjpq}\partial_p u_q \partial_k u_i \\ + \frac{1}{2}a_{kjpq}\partial_p u_m \partial_q u_m \partial_k u_i) = f_i \text{ in } \Omega, \end{aligned} \tag{2.3-4}$$

83

(2.3-5) $$u = 0 \text{ on } \Gamma.$$

This can be written as

(2.3-6) $$A(u) = f \text{ in } \Omega,$$

(2.3-7) $$u = 0 \text{ on } \Gamma,$$

where $A(u) = (A_i(u))$ and

(2.3-8) $$A_i(u) = -\partial_j(a_{ijpq}\epsilon_{pq}(u) + \frac{1}{2}a_{ijpq}\partial_p u_m \partial_q u_m + a_{kjpq}\partial_p u_q \partial_k u_i + \frac{1}{2}a_{kjpq}\partial_p u_m \partial_q u_m \partial_k u_i)$$

The following existence result holds.

Theorem 2.3.1. *Assume that* Γ *is smooth enough. Then for each* $p > 3$ *there exist a neighbourhood* $\mathscr{F}^p$ *of* 0 *in* $\mathbb{L}^p\Omega$ *and a neighbourhood* $\mathcal{U}^p$ *of* 0 *in*

$$\mathbb{V}^p(\Omega) = \{v\epsilon\mathbb{W}^{2,p}(\Omega) | v = 0 \ \text{ on } \ \Gamma\}$$

such that for every $f\epsilon\mathscr{F}^p$ *the boundary value problem* (2.3-6)–(2.3-7) *has one, and only one, solution in* $\mathcal{U}^p$.

Proof. Since $\Omega \subset \mathbb{R}^3$, if $p > 3$ the inclusion

$$W^{1,p}(\Omega) \to C^0(\bar{\Omega})$$

is continuous. Further $W^{1,p}(\Omega)$ is an algebra (cf. ADAMS [1975]). Thus if $f, g\epsilon W^{1,p}(\Omega)$, $fg\epsilon W^{1,p}(\Omega)$ and

(2.3-9) $$||fg||_{1,p,\Omega} \leq C||f||_{1,p\Omega}||g||_{1,p.\Omega}$$

Hence $A : \mathbb{V}^p(\Omega) \subset \mathbb{W}^{2,p}(\Omega) \to \mathbb{C}^p(\Omega)$ is well-defined and is in- **84**
finitely Frechet differentiable. (In fact $D^4 A \equiv 0$). Since $A(0) = 0$, the conclusions of the theorem will stand proved if it is shown that

$$A'(0)\epsilon \text{I som}(\mathbb{V}^p(\Omega), \mathbb{L}^p(\Omega))$$

by virtue of the implicit function theorem. But the problem

(2.3-10) $$A'(0)u = f, u\epsilon\mathbb{V}^p(\Omega)$$

is none other that the linearized syatem of elasticity:

(2.3-11) $$\partial_j a_{ijpq}\epsilon_{pq}(u) = f_i \text{ in } \Omega$$

(2.3-12) $$u = 0 \text{ on } \Gamma$$

$A'(0)$ is continuous. It is one-one since the solution of the above system unique for $p \geq 2$. Also by the regularity theorem (cf. Theorem 2.2.4) it is onto as well. Hence by the closed graph theorem $A'(0)$ is an isomorphism form from $\mathbb{V}^p(\Omega)$ in to $\mathbb{L}^p(\Omega)$ and the theorem is proved. □

Remark 2.3.1. This proof breaks down in the case of the mixed-displacement traction problem because of the lack of $\mathbb{W}^{2,p}(\Omega)$ regularity of the associated linearized syatem.

85 **Remark 2.3.2.** One could think of solving the problem by defining A on $\mathbb{W}^{1,q}(\Omega)$ taking values in $\mathbb{W}^{1,q}(\Omega)$, thus avoiding the need of the regularity theorem which is not valid for other boundary conditions. Unfortunately, it has been proved by VALENT [1979] that on such spaces A is not Frechet differentiable.

Remark 2.3.3. If $a_{ijk\ell}$ were replaced by smooth functions $a_{ijk\ell}(x)$, the result is still true, thus extending the result to non-homogeneous materials.

In case of St Venant-Kirchhoff materials, it turned out that if $u \in \mathbb{W}^{2,P}(\Omega)$, then $E(u) \in \mathbb{W}^{1,p}(\Omega)$ and since σ^* was linear in $E, \sigma^*(E(u)) \in \mathbb{W}^{1,p}(\Omega)$. For more general constitutive equations given $\sigma^* : \mathbb{S}^3 \to \mathbb{S}^3$, it must first be proved that if $E \in \mathbb{W}^{1,P}(\Omega)$, then

$$\sigma^*(E)(x) \stackrel{\text{def}}{=} \sigma^*(E(x)), x \in \Omega$$

is indeed in $\mathbb{W}^{1,p}(\Omega)$. The following result answers this question. It is due to VALENT [1979].

Theorem 2.3.2. *Let $p > 3$. Given a tensor field $E \in \mathbb{W}^{1,q}(\Omega)$ and a mapping $\sigma^* \in C^1(\mathbb{M}^3, \mathbb{M}^3)$, the matrix valued function*

$$\sigma^*(E) : X \in \Omega \to \sigma^*(E(x))$$

is also in $\mathbb{W}^{1,q}(\Omega)$ and

$$\partial_q(\sigma^*_{ij}(E(x)) = \frac{\partial \sigma^* ij}{\partial E_{k\ell}}(E(x))\partial_q E_{k\ell}(x). \tag{2.3-13}$$

If σ^* is of class $C^{m+1}, m \geq 0$, them the mapping $\sigma^* : \mathbb{W}^{1,p}(\Omega) \to \mathbb{W}^{1,p}(\Omega)$ defined above is of class C^m and it is bounded in the sense

$$\sup_{\|E\|_{1,p,\Omega}} \leq r^{\|D^m \sigma^*(E)\| < +\infty} \tag{2.3-14}$$

for every $r > 0$. 86

Proof. **Step(i).** Let $\sigma^* \in C^1(\mathbb{M}^3, \mathbb{M}^3)$. Let $E \in \mathbb{W}^{1,p}(\Omega)$. Then the components of E are all continuous and so

$$\sigma^*(E(x)) \in C^o(\bar{\Omega}; \mathbb{M}^3) \hookrightarrow \mathbb{L}^P(\Omega).$$

Assume now that (2.3-13) has been proved. Then as

$$\frac{\partial \sigma^*_{ij}}{\partial E_{k\ell}}(E(x)) \in C^0(\bar{\Omega}) and \partial_q E_{k\ell}(x) \in L^p(\Omega).$$

it follows that $\partial_q \sigma^*(E) \in \mathbb{L}^P(\Omega)$. Hence $\sigma^*(E) \in \mathbb{W}^{1,p}(\Omega)$.

Now (2.3-13) will be proved. It must be shown that for any $\phi \in \mathscr{D}(\Omega)$,

$$\int_\Omega \sigma^*_{ij}(E(x))\partial_q \phi(x) dx = -\int_\Omega \frac{\partial \sigma^*_{ij}}{\partial E_{k\ell}}(E(x))\partial_q E_{k\ell}(x)\phi(x) dx \tag{2.3-15}$$

Let $\phi \in \mathscr{D}(\Omega)$ be fixed. If $E \in C^1(\bar{\Omega}; \mathbb{M}^3)$ then (2.3-15) follows by a direct application of Green's formula for smooth functions. But $C^1(\bar{\Omega}; \mathbb{M}^3)$ is dense in $\mathbb{W}^{1,p}(\Omega)$. Thus given $E \in \mathbb{W}^{1,p}(\Omega)$, let $E_n \in$

$C^1(\overline{\Omega}; \mathbb{M}^3)$ such that $E_n \to E$ in $\mathbb{W}^{1,p}(\Omega)$. The relation (2.3-15) is valid for each E_n.

Now $E_n \to E$ in $C^\circ(\overline{\Omega} : \mathbb{M}^3)$ as well, i.e. uniformly. Thus E_n, E are all uniformly bounded in Ω and so by Lebesuge's Dominated covergence theorem.

(2.3-16) $$\int\limits_\Omega \sigma^*_{ij}(E_n(x))\partial_q\phi(x)dx \to \int\limits_\Omega \sigma^*_{ij}(E(x))\partial_q\phi(x)dx.$$

Now.
$$\frac{\partial\sigma^*_{ij}}{\partial E_{E\ell}}(E_n(x))\phi(x) \to \frac{\partial\sigma^*_{ij}}{\partial E_k}(E(x))\phi(x)$$

87 uniformly and hence in $L^{p'}(\Omega), p'$ the conjugate exponent of p. Since $\partial_q(E_n)_{k\ell} \to \partial_q E_{k\ell}$ in $L^p(\Omega)$, it follows that

(2.3-17)
$$\int_\Omega \frac{\partial\sigma^*_{ij}}{\partial E_{k\ell}}(E_n(x))\partial_q(E_n)_{k\ell}(x)\phi/x) \to \int_\Omega \frac{\partial\sigma^*_{ij}}{\partial E_{k\ell}}(E(x))\partial_q E_{k\ell}(x)dx$$

and thus (2.3-15) is established for $E \in \mathbb{W}^{1,p}(\Omega)$.

Step(ii). It will be now shown that $\sigma : \mathbb{W}^{1,p}(\Omega) \to \mathbb{W}^{1,p}(\Omega)$ is continuous and bounded. Let $E_n \to E$ in $\mathbb{W}^{1,p}(\Omega)$. Then as before $E_n \to E$ in $C^o(\overline{\Omega}; \mathbb{W}^3)$ as well. Hence $\sigma^*_{ij}(E_n) \to \sigma^*_{ij}(E)$ uniformly and also in $L^p(\Omega)$. Similarly
$$\frac{\partial\sigma^*_{ij}}{\partial E_{k\ell}}(E^n) \to \frac{\partial\sigma^*_{ij}}{\partial E_{k\ell}}(E)$$
uniformly and $\partial_q(E_n)_{k\ell} \to \partial_q E_{k\ell}$ in $L^p(\Omega)$. 'Thus by (2.3-13),
$$\partial_q(\sigma^*_{ij}(E_n)) \to \partial_q(\sigma^*_{ij}(E)) \text{ in } L^p(\Omega),$$
thereby proving that $\sigma^*(E_n) \to \sigma^*(E)$ in $\mathbb{W}^{1,p}(\Omega)$. Thus the mapping is continuous.

If $\|E\|_{1,p,\Omega} \leq r$ then $|E|_{o,\infty,\Omega} \leq C(r)$. It then follows that $\sigma^*(E)$ is bounded uniformly and hence in $\mathbb{L}^p(\Omega)$ by a constant (which is a function r). Again $\dfrac{\partial\sigma^*_{ij}}{\partial E_{k\ell}}(E)$ is bounded uniformly by a constant and

$\partial_q E_{k\ell}$ is bounded in $L^p(\Omega)$ by a constant which depends only on r. These obsersvations lead us to the relation

(2.3-18)
$$\sup_{\|E\|_{1,p,\Omega}} \leq r^{\|\sigma^*(E)\|_{1,p,\Omega} < +\infty}$$

for every $r > 0$.

Step(iii). Let $\sigma^* \in C^2(\mathbb{M}^3; \mathbb{M}^3)$. It will be shown that $\sigma^* : \mathbb{W}^{1,p}(\Omega)$ is of class C^1 and that 88

(2.3-19)
$$D\sigma^*_{ij}(E)G = \frac{\partial \sigma^*_{ij}}{\partial E_{k\ell}}(E) G_{k\ell}.$$

for any $G \in \mathbb{W}^{1,p}(\Omega)$.

By step (i), as $\dfrac{\partial \sigma^*_{ij}}{\partial E_{k\ell}}$ is in C^1, it follows that for

$$E \in \mathbb{W}^{1,p}(\Omega), \frac{\partial \sigma^*_{ij}}{\partial E_{k\ell}}(E) \in \mathbb{W}^{1,p}(\Omega).$$

Since $\mathbb{W}^{1,p}(\Omega)$ is an algebra

$$\frac{\partial \sigma^*_{ij}}{\partial E_{k\ell}}(E) G_{k\ell} \in \mathbb{W}^{1,p}(\Omega)$$

and hence the right hand side of (2.3-19) defines a continuous linear operator on $\mathbb{W}^{1,p}(\Omega)$. To show that it does indeed define the Fréchet derivative , consider for $x \in \Omega$ fixed,

$$(\sigma *_{ij} (E+G) - \sigma^*_{ij}(E) - \frac{\partial \sigma^*_{ij}}{\partial E_{k\ell}}(E) G_{k\ell})(x)$$
$$= G_{k\ell}(x) \int_0^1 \left(\frac{\partial \sigma^*_{ij}}{\partial E_{k\ell}}(E+tG)(x) - \frac{\partial \sigma^*_{ij}}{\partial E_{k\ell}}(E)(x) \right) dt.$$

For $(E, G) \in \mathbb{M}^3 \times \mathbb{M}^3$, let

$$\in^{k\ell}_{ij} (E, G) \stackrel{\text{def}}{=} \int_0^1 \left(\frac{\partial \sigma *_{ij}}{\partial E_{k\ell}}(E+tG) - \frac{\partial \sigma *_{ij}}{\partial E_{k\ell}}(E) \right) dt.$$

The mapping $\in_{ij}^{k\ell}: \mathbb{M}^3 \times \mathbb{M}^3 \to \mathbb{R}$ defined in this fashion is of class C^1, since $\sigma*$ is now assumed to be of class C^2. Thus by the result of steps (i) and (ii), the associated mapping

$$\in_{ij}^{k\ell}: (E, G) \in (\mathbb{W}^{1,p}(\Omega) \times \mathbb{W}^{1,p}(\Omega)) \to \in_{ij}^{k\ell} (E, G) \in \mathbb{W}^{1,p}(\Omega)$$

is well- defined and continuouis, so that in particular , for a fixed $E \in \mathbb{W}^{1,p}(\Omega)$,

$$\in_{ij}^{k\ell} (E, G) \to \in_{ij} (E, 0) = 0$$

89 in $\mathbb{W}^{1,p}(\Omega)$ as $G \to 0$ in $\mathbb{W}^{1,p}(\Omega)$. Since

$$\sigma_{ij}^*(E+G) - \sigma_{ij}^*(E) - \frac{\partial \sigma_{ij}^*}{\partial E_{k\ell}}(E) G_{k\ell} = G_{kl} \in_{ij}^{k\ell} (E, G),$$

it has thus been proved that

$$D\sigma_{ij}^*(E) = \frac{\partial \sigma^* s_{ij}}{\partial E_{k\ell}}(E) G_{K\ell}.$$

The continutiy of $D\sigma_{ij}^*$ follows form that of the partial derivatives $\frac{\partial \sigma_{ij}^*}{\partial E_{k\ell}}$, which is again a consequence of step (ii).

Step (iv). To show the boundedness of $D\sigma * (E)$. Now,

$$\|D\sigma_{ij}^*(E)\| = \sup_{\|G\|_{1,p,\Omega} \leq 1} \|\frac{\partial \sigma_{ij}^*}{\partial E_{k\ell}}(E) G_{k\ell}\|_{1,p,\Omega}$$

which is readily seen to be bounded by a constant depending on r where $\|E\|_{1,p,\Omega} \leq r$. Thus it follows that

$$\sup_{\|E\|_{1,p,\Omega} \leq r} \|D\sigma * (E)\| < +\infty \qquad (2.3\text{-}20)$$

for every $r > 0$.

The assertions for $\sigma^* \in C^{m+1}(\mathbb{M}^3; \mathbb{M}^3)$ follow by iterating the above arguments. □

Let $u \in \mathbb{V}^p(\Omega)$. The $A : \mathbb{V}^p(\Omega) \to \mathbb{L}^p(\Omega)$ is defined by

$$A(u) = - \div \text{ div } ((I + \nabla u)\sigma * (E(u))). \tag{2.3-21}$$

That this indeed maps $\mathbb{W}^{2,p}(\Omega)$ into $\mathbb{L}^p(\Omega)$ is a consequence of Theorem 2.3.2. If $u \in \mathbb{W}^{2,p}(\Omega)$, then $\nabla u \in \mathbb{W}^{1,p}(\Omega) E(u) \in \mathbb{W}^{1,p}(\Omega)$ (since this space is an algebra). Now by the above mentioned theorem, $\sigma * (E(u)) \in \mathbb{W}^{1,p}(\Omega)$ and as it is an algebra, $(I + \nabla u)\sigma * (E(u))$ is in 90
$\mathbb{W}^{1,p}(\Omega)$ and its divergence is in $\mathbb{L}^p(\Omega)$. It is also as regular as the map $\sigma* : \mathbb{W}^{1,p}(\Omega) \to \mathbb{W}^{1,p}(\Omega)$ as the other mapping found in the map A are linear or bilinear.

The boundary value problem for the pure displacement problem reduces to: given $f \in \mathbb{L}^p(\Omega)$, find $u \in \mathbb{V}^p(\Omega)$ such that

$$A(u) = f. \tag{2.3-22}$$

Theorem 2.3.3. *Let* Γ *be smooth enough, (i) Let* $p > 3$ *and* $\sigma* \in C^2(\mathbb{M}^3, \mathbb{M}^3)$*. Then* A *map* $\mathbb{W}^{2,p}(\Omega)$ *into* $\mathbb{L}^p(\Omega)$ *and is of class* C^1*. If in addition*

$$\sigma^*(E) = \lambda \operatorname{tr}(E) I + 2\mu E + o(E) \tag{2.3-23}$$

with $\lambda > 0$ *and* $\mu > 0$*, then* $A'(o) \in Isom(\mathbb{V}^p(\Omega), \mathbb{L}^p(\Omega))$*.*

(ii) If $\sigma^* \in C^3(\mathbb{M}^3, \mathbb{M}^3)$ *and if* $A'(0) \in Isom\ (\mathbb{V}^p(\Omega), \mathbb{L}^p(\Omega))$*,* then there exists $\rho_0^P > 0$ such that for all $0 \le \rho < \rho_0^P$ and for all

$$v \in B_\rho^P \overset{\text{def}}{=} \{v \in \mathbb{V}^P(\Omega) |\, \|v\|_{2,P,\Omega \le \rho}\}$$

$A'(v) \in Isom\ (\mathbb{V}^P(\Omega), \mathbb{L}^P(\Omega))$*. Further*

$$\gamma_\rho^P \overset{\text{def}}{=} \sup_{v \in B_\rho^P} \|(A'(v))^{-1}\| < +\infty. \tag{2.3-24}$$

Also, the map $v \to (A'(v))^{-1}$ *is Lipschits continuous on* B_ρ^P *i.e.,*

$$L_\rho^P \overset{\text{def}}{=} \sup_{\substack{v,w \in B_\rho^P \\ v \neq w}} \frac{\|(A'(v))^{-1} - (A'(w))^{-1}\|}{\|v - w\|_{2,p,\Omega}} < +\infty. \tag{2.3-25}$$

Proof. (i) That A maps $\mathbb{W}^{2,p}(\Omega)$ in $\mathbb{L}^P(\Omega)$ and is of class C^1 follows from obsevations made above. A simple computation shows that

$$A'_i(o)v = -\partial_j(D\sigma^*_{ij}(o)\left(\frac{\nabla v|\nabla v^T}{2}\right) + \sigma^*_{kj}(o)\partial_k v_i).$$

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If σ^* is of the form (2.3-23), this reduces to

(2.3-26) $$A'_i(o)v = -\partial_j(\lambda \in_{kk} (v)\delta_{ij} + 2\mu \in_{ij} (v))$$

since $\sigma^*(o) = o$ and

$$D\sigma^*_{ij}(o)G = \lambda G_{kk}\delta_{ij} + 2_\mu G_{ij}$$

But (2.3-26) is just the linearized system of elasticty (cf. Section 2.2) and by Theorem 2.2.3 and 2.2.4, is an isomorphism as was shown in Theorem 2.3.1.

(ii) Let $\sigma^* \in C^3(\mathbb{M}^3, \mathbb{M}^3)$. Then $A \in C^2(\mathbb{W}^{2,p}(\Omega); \mathbb{L}^p(\Omega))$. Since all mappings occurring in A have bounded second derivatives,

$$M^p(\rho) \stackrel{\text{def}}{=} \sup_{||v||_{2,p,\Omega \leq \rho}} ||A''(v)|| < +\infty.$$

Note that $M^P(\rho)$ is a non-decreasing functing of ρ. Now,

(2.3-27) $$\sup_{||v||_{2,p,\Omega \leq \rho}} ||A'(v) - A'(o)| \leq \rho M^P(\rho).$$

If $v \in B^P_\rho$, then

(2.3-28) $$A'(v) = A'(o)[I + (A'(o))^{-1}(A'(v) - A'(o))]$$

But

(2.3-29) $$||(A'(o))^{-1}(A'(v) - A'(o))|| \leq \gamma^P_o \rho M^P(\rho)$$

where

(2.3-30) $$\gamma^P_0 \stackrel{\text{def}}{=} ||(A'(o))^{-1}||.$$

If $\rho < \rho_0^P$ where ρ_o^p is such that

$$\rho_o^P M^P(\rho_o^P) < (\gamma_o^P)^{-1} \tag{2.3-31}$$

92 then it follows from (2.3-28) and (2.3-29) that $A'(v)$ is an isomorphism. Further

$$\|(A'(v))^{-1}\| \leq \frac{\|A'(o)\|}{1 - \gamma_o^P \rho M^P(\rho)} = \gamma_\rho^P$$

where

$$\gamma_\rho^P \overset{\text{def}}{=} \frac{\gamma_o^P}{1 - \gamma_o^P \rho M^P(\rho)}. \tag{2.3-31}$$

Finally if $v, w \in B_\rho^P$, then

$$(A'(v))^{-1} - (A'(w))^{-1} = (A'(v))^{-1}(A'(w) - A'(v))(A'(w))^{-1}$$

and (2.3-25) follows with

$$L_\rho^P \overset{\text{def}}{=} M^P(\rho)(\gamma_\rho^P)^2. \tag{2.3-33}$$

□

The latter part of the above theorem will be needed in the study of incremental methods (cf. Section 2.4) . The former part leads directly to the following existence theorem.

Theorem 2.3.4. *Let Γ be smooth enough and $\sigma^* \in C^2(\mathbb{M}^3, \mathbb{M}^3)$. Let $\sigma^*(E)$ be as in (2.3-23) with $\lambda > 0$ and $\mu > 0$. Then for any $p > 3$, there exists neighbourhoods $\mathscr{K}^p$ of 0 in $\mathbb{L}^p(\Omega)$ and $\mathscr{U}^p$ of 0 in $\mathbb{V}^p(\Omega)$ such that for each $f \in \mathscr{K}^p$ there exists one and only solution $u \in \mathscr{U}^p$ to equation* (2.3-22).

Proof. By the previous theorem , $A'(o)$ is an isomorphism and the result follows, as in Theorem 2.3.1, from the implicit function theorem. □

Remark 2.3.4. Theorem 2.3.1 is contained in Theorem 2.3.4. 93

The following result compares the solution as guaranteed by the above theorem and the solution of the linearized problem. It will thus be seen that for *'small' forces the linearized system is indeed a good approximation of the original model.*

Theorem 2.3.5. *Let the assumptions of the previous theorem hold with* $\sigma^* \in C^3\mathbb{M}^3; \mathbb{M}^3$. *For* $f \in \mathscr{F}^P \subset \mathbb{L}^P(\Omega)$, *let* $u(f) \in \mathscr{U}^P \subset \mathbb{V}^P(\Omega)$ *denote the solution to the problem* (2.3-22). *Let* $u_{\ell in}(f) \in \mathbb{V}^P(\Omega)$ *denote the solution of the equation*

$$A'(o)u_{\ell in}(f) = f. \tag{2.3-34}$$

Then

$$||u(f) - u_{\ell in}(f)||_{2,p,\Omega} = 0(|f|^2_{0,p,\Omega}). \tag{2.3-35}$$

Proof. By the implicit function theorem, it follows that u is also differentialbe as a functions of f. Thus

$$A'(u(f))u'(f) = I \text{ in } \mathbb{L}^P(\Omega).$$

In particular, taking $f = 0$, it follows that

$$u'(o) = (A'(o))^{-1}. \tag{2.3-36}$$

Now

$$u(f) = u(o) + u'(o)f + o(|f|^2_{o,p,\Omega})$$

as A is of class C^2. But $u(o) = o$. The result now follows from (2.3-34) and (2.3-36). □

94 *A major open problem* in elasticity is to prove the existence of a solution *'close to zero'* when f is 'small' , for the *mixed displacement-traction problem*. One could then compare the solutions of $A(u) = f$ and $A'(o)u = f$.

It was remarked in the beginning of this chapter (cf. Section 2.1) that even if we solved the problem, the solution must further satisfy the condition $\det(\nabla\phi) > o$, and in additionm be one-one. Hitherto these criteria have been ignored. The following result assures that if f is 'small enough' then these conditions are met.

Theorem 2.3.6. *Let the assumptions of theorem 2.3.4 hold. Further let* Γ *be connected. Then if* $|f|_{o,p,\Omega}$ *is sufficiently small, the mapping* $\phi = Id + u$ *satisfies det* $(\nabla\phi) > o$ *and is one-one.*

Proof. If $|f|_{o,p,\Omega}$ is small, then $\|u(f)\|_{2,p,\Omega}$ is small. Since $W^{2,p}(\Omega) \hookrightarrow C^1(\overline{\Omega})$, it follows that $\|u\|_{1,\infty,\Omega}$ is small. Hence, as the determinant is a continuous function of components of a matrix. it follows that

$$\det(\nabla\phi)(x) = \det(I + \nabla u)(x) > 0, \quad \text{for all } x \in \overline{\Omega}$$

Since $\phi \in \mathbb{W}^{2,p}(\Omega)$, it can be extended to a function $\phi \in \mathbb{W}^{2,p}(\mathscr{O}) \hookrightarrow C^1(\mathscr{O})$, where $\mathscr{O} \supset \overline{\Omega}$ (cf. NEČAS [1967]). Now $\phi = Id$, which is one-one, on Γ. If follows from a result of DE LA VALLÉE POUSSIN or MEISTERS and OLECH [1963]((cf. Remark 2.3.4) below) that ϕ is one-one $\overline{\Omega}$. ◻

Remark 2.3.5. The result of MEISTERS and OLECH states that if $\phi \in C^1(\mathscr{O}; \mathbb{R}^n)$, $\mathscr{O} \subset \mathbb{R}^n$ an open subset, if K is a compact subset of $\mathscr{O}$ with ∂K connected and, finally if ϕ is such that $\det(\nabla\phi) > 0$ on K and ϕ is 95
one-one on ∂K, then ϕ is one-one on ∂K. This result can be strengthened by allowing $\det \nabla\phi(x) = o$ on a finite subset of $\overset{o}{k}$ and an infinite proper subset of ∂K.

Remark 2.3.6. It is not quite necessary to resort to the use of the fairly deep result of MEISTERS and OLECH. If $|f|_{o,p,\Omega}$ is small $\|u\|_{1,\infty,\Omega}$ small, so without loss of generality it can be assumed that $\|\nabla u(x)\| < 1$ for some matrix norm induced by a vector norm, for all $x \in \Omega$. Now *if* $\phi \in C^o(\overline{\Omega}) \cap C^1(\Omega)$ *and if* Ω *is convex*, then

$$\begin{aligned}
\|\phi(x_1) - \phi(x_2) - (x_1 - x_2)\| &= \|u(x_1) - u(x_1) - u(x_2)\| \\
&\leq \sup_{x \in]x_1, x_2[} \|\nabla u(x)\| \|x_1 - x_2\| \\
&< \|x_1 - x_2\|.
\end{aligned}$$

Thus if $x_1 \neq x_2$, then necessarily $\phi(x_1) \neq \phi(x_2)$.

If u is 'small' then the strain tensor

$$E = \frac{1}{2}(\nabla u^T + \nabla u + \nabla u^T \nabla u)$$

is also 'small'. An open prolem is to study under what sufficient conditions the converse is true.

Exercises

2.3-1 Prove the analoge of Theorem 2.3.4 with the condition $u = 0$ on Γ repalced by $u = u_0$ on Γ.

2.3-2 Prove the analouge of Theorem 2.3.4 using the function spaces $C^{m,\alpha}$ instead of the spaces $W^{m,p}(\Omega)$.

2.3-3 Prove the result of MEISTERS and OLECH using the topological degree. Show also that in Theorem 2.3.6 $\phi(\overline{\Omega} = \overline{\Omega})$.

96 **2.3-4** Give a counter example to the result of MEISERS and OLECH when ∂k is not connected.

2.3-5 If $\mu > 0$, show that there exists $\lambda_0 < o$ such that if $\lambda_0 < \lambda \leq 0$, the existence result of Theorem 2.3.4 still holds.

2.3-5 (LE DRET (1982)) Examine the existence of a solution to the pure displacement problem in the incompressible case, i.e., $\det(I + \nabla u) = 1$.

2.4 Convergence of Semi-Discrete Incremental Methods

Consider again the pure displacement problem:

$$- \operatorname{div}((I|\nabla u)\sigma^*(E(u)) = f \text{ in } \Omega \tag{2.4-1}$$

$$u = o \text{ on } \Gamma. \tag{2.4-2}$$

It was shown in Section 2.3 that for small forces f, the problem had at least one solution which was also small. Considering that approach via the implicit function theorem as a 'direct' approach to the existence theory, by contrast the incremental methods provide a ' constructive' approach to the same. Because of its constructive nature, it could be of use for numerical approximation of the solution.

The basic idea is the following. Let

$$0 \leq \lambda^o < \lambda^1 < \ldots < \lambda^n < \lambda^{n+1} < \ldots < \lambda^N = 1$$

be a partition of the interval $[0, 1]$. Let f be a given sufficiently small force in $\mathscr{F}^p$ (cf. Theorem 2.3.4). Let U^n be the solution of

$$A(U^n) = \lambda^n f, \tag{2.4-3}$$

U^n belonging to $\mathcal{U}^p$ since $\lambda^n f \in \mathscr{F}^p$ if $f \epsilon \mathscr{F}^p$ and $\mathscr{F}^p$ is a ball. Note 97
$U^o = o$. Let u^n be an approximation of U^n. The idea is to construct u^{n+1} knowing u^n, via a simpler problem, namely a linear problem. Now

$$A(U^{n+1}) - A(U^n) = (\lambda^{n+1} - \lambda^n) f.$$

If $A(U^{n+1})$ is expanded about $A(U^n)$,

$$A(U^{n+1}) = A(U^n) + A'(U^n)(U^{n+1} - U^n) + o(|U^{n+1} - U^n|).$$

This inspires the equations defining the approximations u^n. Thus one tries to solve the sequence of problems

$$A(u^n)(u^{n+1} - u^n) = (\lambda^{n+1} \lambda^n) f, 0 \leq n \leq N - 1, \tag{2.4-4}$$

$$u^o = o. \tag{2.4-5}$$

Of course, it is necessary that at each stage $A'(u^n)$ be an isomorphism from $\mathbb{V}^p(\Omega)$ onto $\mathbb{L}^p(\Omega)$.

The following simple, yet crucial, observation is basic to the analysis of the above method. The equations (2.4-4) - (2.4-5) can be rewritten as

$$\frac{u^{n+1} - u^n}{\lambda^{n+1} - \lambda^n} = (A'(u^n))^{-1} f \tag{2.4-6}$$

$$u^o = o \tag{2.4-7}$$

which is none other than Euler's method for approximating the ordinary differential equation

$$u'(\lambda) = (A'(u(\lambda)))^{-1} f, u(o) = o. \tag{2.4-8}$$

Theorem 2.4.1. *Let σ^* be of class $C^3(\mathbb{M}^3, \mathbb{M}^3)$. Let $p > 3$ and $o < \sigma \leq$* 98
ρ_o^p (cf. Theorem 2.3.3). Let $f \epsilon \mathbb{L}^p(\Omega)$ be such that

(2.4-9) $$|f|_{o,p,\Omega} \leq \rho(\gamma_\rho^p)^{-1}$$

Then the ordinary differential equation (2.4-8) for $o \leq \lambda \leq 1$ has a unique solution $\bar{u}(\lambda)$ in the ball B_ρ^p. Besides

(2.4-10) $$\bar{u}(\lambda) = u(\lambda f).$$

Proof. The existence of a unique solution of the ordinary differential equation is classical. It is converted into an integral equation and using the estimates of Theorem 2.3.3 regarding the uniform houndedness of $(A'(v))^{-1}$ and the Lipschitz continuity of the map $v \to (A'(v))^{-1}$ on B_ρ^p, the result follows by a use of the contraction mapping theorem.

Now,

$$u'(\lambda) = (A'(u(\lambda)))^{-1} f$$

or

$$(A'(u(\lambda)))u'(\lambda) = f$$

or, again

$$\frac{d}{d\lambda}(A(u(\lambda)) - \lambda f) = 0.$$

Thus

$$(A(u(\lambda)) = \lambda f + C,$$

and as $u(0) = 0$, $C = 0$. This proves the theorem. □

Remark 2.4.1. The equation $A(u) = f$ has been imbedded in a one parameter family of problems $(A(u(\lambda)) = \lambda f$, where $u(1) = u$. Knowing
99 a solution for one value of λ, here, $\lambda = 0, u(0) = 0$, one tries to go continuously to $\lambda = 1$. This is the basis of the so - called continuation methods. (cf. RHEINBOLDT [1974]).

Remark 2.4.2. The condition (2.4-9) makes precise the neighbourhood $\mathscr{F}^p$ of o in $\mathscr{L}^p(\Omega)$ for which a solution 'close to zero' was guaranteed by Theorem 2.3.4.

The following theorem proves the convergence of the incremental method desribed above when the 'mash size' $\max\limits_{0\leq n\leq N-1}(\lambda^{n+1}-\lambda^n)$ approaches zero. The assumption that the applied forces should be small enough is also corroborated by numerical evidence: Otherwise it is often observed that the approximate solutions 'blow up' for a certain critical value of the parameter, corresponding for example to a phenomenon of 'buckling'.

Theorem 2.4.2. *Let $\sigma^*\epsilon C^3(\mathbb{M}^3,\mathbb{M}^3)$. Let $p>3$ and $0<\rho<\rho_0^p$. If*

$$|f|_{o,p,\Omega}\leq\rho(\gamma_\rho^p)^{-1},$$

then given any partition

$$o=\lambda^\circ<\lambda^1<\ldots<\lambda^N=1$$

of $[0,1]$, Euler's method; Find $u^n, 0\leq n\leq N, u^n\epsilon B^p$, such that

$$A'(u^n)(u^{n+1}-u^n)=(\lambda^{n+1}-\lambda^n)f \tag{2.4-11}$$

with $u^\circ=o$, is well defined and

$$\max_{0\leq n\leq N}\|u^n-u(\lambda^n f)\|_{2,p,\Omega}\leq C\max_{0\leq n\leq N-1}(\lambda^{n+1}\lambda^n) \tag{2.4-12}$$

where

$$C=C(\rho,P,|f|_{0,P,\Omega})>0.$$

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Proof. The proof is classical and again relies on the uniform boundedness and the Lipschitz continuity of the map $V\to(A'(V))^{-1}$ on B_ρ^p.

If $\Delta\lambda=\max\limits_{0\leq n\leq N-1}(\lambda^{n+1}-\lambda^n)$, then

$$\|u^N-u(f)\|_{2,p,\Omega}=0(\Delta\lambda). \tag{2.4-13}$$

If σ^N and σ were defined by

$$\sigma^N=\sigma^*(E(u^N)) \tag{2.4-14}$$

and

(2.4-15) $$\sigma = \sigma^*(E(u(f)))$$

then it is easy to see that

(2.4-16) $$\|\sigma^N - \sigma\|_{1,p,\Omega} = 0(\Delta\lambda).$$

To conclude, two open problems will now be stated.

The first is to analyse the *fully discrete* incremental method by adding the effect of finite element methods.

Secondly, one can construct formally a semi - discrete (or fully discrete) incremental method for the mixed displacement-traction problem. If it can be shown that the approximants u^n exist uniquely and that they converge in some sense as $\Lambda\lambda \to 0$, this could provide a valuable existence theorem for this class of problems. □

101 **Exercises**

2.4-1 (DESTUYNDER AND GALBE (1978)). For a St Venant - Kirchhoff meterial show that the map $\lambda \to \tilde{u}(\lambda)$ (cf. equation (2.4-10)) is analytic in a neighbourhood of 0.

2.4-2 Apply Newton's method to the equation $A(u) = f, u\epsilon\mathbb{V}^p(\Omega)$ and study its convergence to a solution of the equation

2.5 An Existence Theorem for Minimizing Functionals and Outline of its Application to Nonlinear Elasticity

In this section, an existence theorem for minimizing a functional will be proved. The functional considered will resemble the total energy functional of elasticity described in Section 1.4. Unfornately, however, the energy functionals of elasticity will not satisfy all the hypotheses of the theorem. But it will provide an insight as to what properties of the functional are to be considered and how to modify the theorem to suit such functionals. This will be done in the next section.

Theorem 2.5.1. *Let n and ν be integers ≥ 1. Let $\Omega \subset \mathbb{R}^n$ be an open subset and let $a\epsilon\mathbb{R}$. (If ,meas $\Omega =_+ \infty$, assume $a = 0$). Let*

$$g : \Omega \times \mathbb{R}^\nu \to [a, +\infty]$$

be a mapping such that

$$g(x, .) : \mathbb{R}^\nu \to [a, +\infty]$$

is convex and continuous for almost all $x\epsilon\Omega$;,

$$g(., q) : \Omega \to [a, +\infty]$$

is measurable for all $q\epsilon\mathbb{R}^\nu$. Then, if $q_k \to q$ weakly $\mathbb{L}^1(\Omega)$

$$\int_\Omega g(x, q(x))dx \leq \liminf_{k\to\infty} \int_\Omega g(x, q_k(x))dx. \tag{2.5-1}$$

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In other words, the mapping

$$q \to \int_\Omega g(x, q(x))dx$$

is weakly lower semi - continuous on $\mathbb{L}^1(\Omega)$.

Proof. First of all, without loss generality, it can be assumed that $g \geq 0$. (If meas $\Omega < +\infty$, replace g by $g - a$ meas (Ω)). The continuity in q for almost all x and the measurability in x for all q implies that g is a Caratheodory function and so if $q(x)$ is measurable in x, so is $g(x, q(x))$. Since now $g > 0$, the integral

$$\int_\Omega g(x, q(x))dx$$

makes sense.

Let $q_n \to q$ in $\mathbb{L}^1(\Omega)$ strongly. Let $\{q_{n_k}\}$ be any subsequence such that the sequence

$$\int_\Omega q(x, q_{n_k}(x))dx$$

is convergent. Now there exists a further subsequence (again denoted by q_{n_k} for convenience) such tha $q_{n_k}(x) \to q(x)$ for almost all x. Hence

$$g(x, q_{n_k}(x)) \to g(x, q(x))$$

for almost all x. Thus by Fatou's lemma

$$\begin{aligned}\int_\Omega g(x, q(x))dx &\leq \liminf_{k\to\infty} \int_\Omega g(x, q_{n_k}(x))dx \\ &= \lim_{k\to\infty} \int_\Omega g(x, q_{n_k}(x))dx.\end{aligned}$$

103 Then as the subsequence $\{q_{n_k}\}$ was chosen arbitrarilv subject to the condition that the integrals converge the relation (2.5-1) follows, when $q_n \to q$ strongly in $\mathbb{L}^1(\Omega)$. Thus the functional

$$J(q) \stackrel{\text{def}}{=} \int_\Omega g(x, q(x))dx \tag{2.5-2}$$

is strongly lower semi - continuous. It is also easy to see that J is convex. Now if $\alpha \epsilon \mathbb{R}$, then

$$\{q \epsilon \mathbb{L}^1(\Omega) | J(c) \leq \alpha\}$$

is strongly closed and convex and hence weakly closed (*Mazur's Theorem*). Thus it follows that J is weakly lower semi - continuous which is equivalent to (2.5-1). □

Remark 2.5.1. If g is independent of x, then it suffices to assume that g is convex and continuous.

Remark 2.5.2. The above result conuld be applied as follows: If $g(x, q) \geq c + b|q|^P, b > 0, p > 1$ then a minimizing sequence will be bounded in $\mathbb{L}^p(\Omega)$. Since $p > 1$, a weakly convergent subsequence in $\mathbb{L}^p(\Omega)$ can be extracted. If Ω is bounded, this implies weak convergence in $\mathbb{L}^1(\Omega)$ and an application of the above result would show that at the limit, J attains its minimum.

Remark 2.5.3. If g did not take the value $+\infty$, the convexity in q also implies continuity. However, if g assumed the value $+\infty$ continuity no longer follows from convexity. The inclusion of the value $+\infty$ in the range of g is necessary for applications. It will be needed (as was mentioned in Section 2.1) that the energy tends to $+\infty$ as det (F) approches 0 through positive values.

Theorem 2.5.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $\mathscr{W} : \Omega \times \mathbb{R}^\nu \to [a, +\infty]$ be such that* 104

$$\mathcal{W}(x, .) : \mathbb{R}^\nu \to [a, +\infty]$$

is convex and continuous for almost all $x \epsilon \Omega$,

$$\mathcal{W}(., q) : \Omega \to [a, +\infty]$$

is measurable for all $q \epsilon \mathbb{R}^\nu$. Let there exist c, b, p such that

$$b > 0, p > 1, \text{ and } \mathcal{W}(x, q) \geq c + b|q|^p \tag{2.5-3}$$

for all $q \epsilon \mathbb{R}^\nu$ and almost all $x \epsilon \Omega$. Let $\ell : \mathcal{W}^{1,p}(\Omega) \to \mathbb{R}$ be a continuous linear functional. Let $\Gamma_0 \subset \Gamma$ be of strictly positive da - measure and let $\mathbb{U}$ be a weakly closed subset of

$$\mathbb{V} = \left\{ v \epsilon \mathbb{W}^{1,p}(\Omega) | v = 0 \text{ on } \Gamma_0 \right\}. \tag{2.5-4}$$

Define

$$I(v) = \int\limits_\Omega \mathcal{W}(x, \nabla v(x)) dx - \ell(v) \tag{2.5-5}$$

for $v \epsilon \mathbb{V}$. Assume

$$\inf_{v \epsilon \mathbb{U}} I(v) < +\infty.$$

Then the problem: Find $u \epsilon \mathbb{U}$ such that

$$I(u) = \inf_{v \epsilon \mathbb{U}} I(v) \tag{2.5-6}$$

has atleast one solution.

Proof. Let v^k be a minimizing sequence in $\mathbb{U}$, i.e.$v^k \epsilon \mathbb{U}$ and

$$I(v^k) \to \inf_{v \epsilon \mathbb{U}} I(v) < +\infty.$$

Now, 105

$$\begin{aligned} I(v) &\geq C\text{meas}(\Omega) + b|v|^p_{1,p,\Omega} - ||\ell|| ||v||_{1,p,\Omega} \\ &\geq C\text{meas}(\Omega) + b'|v|^{p-}_{1,p,\Omega} - ||\ell|| ||v||_{1,p,\Omega} \end{aligned}$$

where $b' > 0$, using Poincare's inequality. Since $p > 1$, it follows that $I(v) \to +\infty$ as $||v||_{1,p,\Omega} \to +\infty$. Hence

$$\sup_k ||v^k||_{1,p,\Omega} < +\infty.$$

As $p > 1$, $\mathbb{V}^{1,p}(\Omega)$ is reflexive and a weakly convergent subsequence can be extracted. Denoting this subsequence again by v^k, let $v^k \to u$ weakly in $\mathbb{W}^{1,p}(\Omega)$. Since $\mathbb{U}$ is weakly closed, $u \epsilon \mathbb{U}$.

Now clearly $\ell(v^k) \to \ell(u)$. Also $\nabla v^k \to \nabla u$ weakly in $\mathbb{L}^p(\Omega)$ and hence (Ω is bounded, in $\mathbb{L}^1(\Omega)$. Then by Theorem 2.5.1, it follows that

$$\inf_{v \epsilon \mathbb{U}} I(v) \leq I(u) \leq \liminf_{k \to \infty} I(v^k) = \inf_{v \epsilon \mathbb{U}} I(v).$$

□

Remark 2.5.4. The application of this result to the linearized elasticity system is easy (cf. Exercise 2.5-1. But unfortunately, it is not directly applicable to non - linear elasticity.

The assumptions to be satisfied are firstly, $p > 1$, which leads to the choice of the appropriate space $\mathbb{G}^{1,p}(\Omega)$, and, secondly, the convexity of the function $\mathcal{W}$.

If the fuctional I were strictly convex, this would imply uniquer-ess of solutions which is not physically acceptable (cf. Section 2.1).
106 However, *it is not even possible to have* $F \to \mathcal{W}(F)$ *convex in* three - demensional elasticity (cf. Exercise 2.5-4). Note that in the linearized elasticity system, it is true that $\mathcal{W}$ is convex, but then one can show (cf. Exerise 2.2-1) that a linear model also contradicts the axiom of material

frame indifference. Finally, to show $\mathbb{U}$ to be weakly closed in $\mathbb{W}^{1,p}(\Omega)$, it is usually shown that it is strongly closed and convex. But typically, $\mathbb{U}$ will be non - conver with constraints like det $(\nabla\psi) > 0$ or det $(\nabla\psi) = 1$

Thus to overcome these two difficulties, the notions of *polyconvexity* and *compactness by compensation* will be introduced in the next section.

Exercises

2.5-1 Show that in the linearized system of elasticity, unilateral conditions can also be taken into account (apply Theorem 2.5.2 with

$$\mathbb{U} = \{v \epsilon \mathbb{V} | u_3 \geq 0 \text{ on } \Gamma_0' \subset \Gamma - \Gamma_0\},$$

where

$$\mathbb{V} = \{v \epsilon \mathbb{H}^1(\Omega) | v = 0 \text{ on } \Gamma_0'\}, \text{ da meas } \Gamma_0 > 0).$$

2.5-2 For a St Venant-Kirchoff material,

$$\mathcal{W}(\nabla_v) = \frac{\lambda}{2} tr(E)^2 + \mu \, \text{tr}(E^2), \lambda > 0, \mu > 0$$

where

$$E = E(v) = \frac{1}{2}(\nabla v^T + \nabla v + \nabla v^T \nabla v).$$

Show that the corresponding energy is coercive on the space

$$\mathbb{V} = \left\{v \epsilon \mathbb{W}^{1,4}(\Omega); v = 0 \text{ on } \Gamma_o\right\}, \text{ da -meas } \Gamma_0 > 0.$$

2.5-3 If $E = \dfrac{1}{2}(F^T F - I)$, show in the above case that $F \to \mathcal{W}(F)$ is convex. **107**

2.5-4 Show that the convexity of the function $F \to \mathcal{W}(F)$ is physically unrealistic (cf. TRUESDELL and NOLL [1955]).

2.5-5 For a St Venant-Kirchhoffmaterial, show that the solution u obtained via the implicit function theorem minimizes locally the energy in $\mathbb{W}^{1\infty}(\Omega)$ but not necessarily in $\mathbb{W}^{1,4}(\Omega)$.

2.6 J. BALL'S Polyconvexity and Existence Theorems in Three Dimensional Elasticity

In the last section, it was seen that the lack of convexity of the stored energy function was an obstacle to the application of the existence theorem (cf. Theorem 2.5.2). Now, an extension of the notion of convexity following J. BALL will be introduced.

Recall that if A is a matrix, then $\text{adj}(A)$ stands for the transpose of the matrix of cofactors of A. The following identity holds:

(2.6-1) $$A(\text{adj}\, A) = (\text{adj}\, A)A = \det(A)I.$$

Thus, if A is invertible

(2.6-2) $$\text{adj}(A) = \det(A)A^{-1}.$$

Also,

(2.6-3) $$\text{adj}(AB) = \text{adj}(B)\,\text{adj}(A).$$

and

(2.6-4) $$(\text{adj}\, A)^T = \text{adj}(A^T)$$

In the study of deformations (cf. Section 1.1), it was seen that lengths were modified by a function of $F(=\nabla\phi)$ via $C = F^T F$. Surface areas were changed in terms of $\text{adj}(F)$ and volume elements were altered by a factor of $\det(F)$. Since it is natural to expect that a stored energy function somehow takes these into account, it is reasonable to assume that

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(2.6-5) $$\mathcal{W}(F) = \mathcal{G}(F, \text{adj}(F), \det(F))$$

for all $F \epsilon \mathbb{M}^3_+$, where

$$\mathcal{G} : \mathbb{M}^3_+ \times \mathbb{M}^3_+ \times]0, \infty[\to \mathbb{R}$$

is a given function (since for $F \epsilon \mathbb{M}^3_+, \text{adj}(F) \epsilon \mathbb{M}^3_+$). While it not true that $\mathcal{W}$ as a function of F is convex (cf. Exercise 2.5-4), it is no longer impossible to expect $\mathcal{G}$ to be a convex function of its three arguments, F, adj(F) and det F (of course, the domain of definition of $\mathcal{G}$ is not a convex set, but that is easily handled as will be seen below). For instance, $\mathcal{W}(F) = detF$ is not a convex function but $\mathcal{G}(\delta) = \delta$ is convex!

Let V be a vector space and $U \subset V$ ary subset. Let $J : U \to \mathbb{R}$ be a function. It is said to be *convex* if there exists $\bar{J} : co(U) \to \mathbb{R}$, where $co(U)$ is the convex hull of U, such that

$$\bar{J}(v) = J(v) \tag{2.6-6}$$

for every $v \epsilon U$.

Let $\mathbb{U}$ be a non - empty suset of $\mathbb{M}^3$ and let

$$\mathcal{U} = \{(F, \text{adj}(F), detF) | F \epsilon \mathbb{U}\}. \tag{2.6-7}$$

Thus $\mathcal{U} \subset \mathbb{M}^3 \times \mathbb{M}^3 \times \mathbb{R}$. A function $\mathcal{W} : \mathcal{U} \to \mathbb{R}$ is said to be polyconvex if there exists a convex function $\mathcal{G} : \mathcal{U} \to \mathbb{R}$ such that (2.6-5) holds for every $F \epsilon \mathbb{U}$

If $\mathbb{U} = \mathbb{M}^3_+$ then $co(\mathbb{U}) = \mathbb{M}^3$ and $\mathcal{U} = \mathbb{M}^3_+ \times \mathbb{M}^3_+ \times]o, +\infty[$ while co **109**
$(\mathscr{U}) = \mathbb{M}^3 \times \mathbb{M}^3 \times]o, +\infty[$. Thus a stored energy function $\mathcal{W} : \mathbb{M}^3_+ \to \mathbb{R}$ is polyconvex if there exists a convex function

$$\mathcal{G} : \mathbb{M}^3 \times \mathbb{M}^3 \times]o, +\infty[\to \mathbb{R}$$

such that (2.6-5) holds for all $F \epsilon \mathbb{M}^3_+$.

The condition of polyconvexity on the stored energy function leads to a class of hyperelastic materials known as *OGDEN'S materials* (which we now define for the compressible case: for the incompressible case, see Exercise 2.6-8).

As a simplest possible example, let $a > o$, $b > o$ and $\Gamma :]o, +\infty[\to \mathbb{R}$ be a convex function. Define

$$\mathcal{W}(F) = a\|F\|^2 + b\|\text{adj}\, F\|^2 + \Gamma(\det F) \tag{2.6-8}$$

where

$$\|F\|^2 = tr(F^T F) = F : F. \tag{2.6-9}$$

Clearly as $||.||^2$ is a convex function, and as Γ is also convex, it follows that $\mathcal{W}$ defined by (2.6-8) is polyconvex with

$$\mathcal{G}(F, H, \delta) = a||F||^2 + b||H||^2 + \Gamma(\delta). \tag{2.6-10}$$

Remark 2.6.1. It will later be assumed that $\Gamma(\delta) \to +\infty$ as $\delta \to o^+$.

Let $F \epsilon \mathbb{M}^3_+$. Let $U = (F^T F)^{1/2}$ with eigenvalues v_1, v_2, v_3. These are called the *principal stretches* of F. Then it is easy to see that

$$||F||^2 = v_1^2 + v_2^2 + v_3^2 \tag{2.6-11}$$

$$\|\operatorname{adj} F\|^2 = v_1^2 v_2^2 + v_2^2 v_3^2 + v_3^2 v_1^2. \tag{2.6-12}$$

110 Hence (2.6-8) will now read as

$$\mathcal{W}(F) = a\left(v_1^2 + v_2^2 + v_3^2\right) + b\left(v_1^2 v_2^2 + v_2^2 v_3^2 + v_3^2 v_1^2\right) + \Gamma(v_1 v_2 v_3). \tag{2.6-13}$$

This can be generalized to get an *OGDEN Material* as follows. Let $\Gamma :]o, +\infty[\to \mathbb{R}$ be a convex function. Let $a_i > o, 1 \leq i \leq M, b_j > o, 1 \leq j \leq N$. Further let,

$$\begin{cases} 1 \leq \alpha_1 < \cdots < \alpha_M \\ 1 \leq \beta_1 < \cdots < \beta_N. \end{cases} \tag{2.6-14}$$

Now, if $C = F^T F$,

$$\begin{cases} \operatorname{tr}(C^{\alpha/2}) = v_1^\alpha + v_2^\alpha + v_3^\alpha \\ \operatorname{tr}((\operatorname{adj} C)^{\beta/2}) = (v_1 v_2)^\beta + (v_2 v_3)^\beta + (v_3 v_1)^\beta. \end{cases} \tag{2.6-15}$$

Now define for $F \epsilon \mathbb{M}^3_+$,

$$\mathcal{W}(F) = \sum_{i=1}^{M} a_i \operatorname{tr}(C^{\alpha_i/2}) + \sum_{j=1}^{N} b_j \operatorname{tr}(\operatorname{adj} C)^{\beta_i/2} + \Gamma(\det(F)). \tag{2.6-16}$$

Theorem 2.6.1. *(i) An Ogden's Material is polyconvex.*

(ii) It satisfies the following coerciveness inequality.

$$(2.6\text{-}17)\quad \mathcal{W}(F) \geq C_o + C_1\|F\|^{\alpha_M} + C_2\|\operatorname{adj}(F)\|^{\beta_N} + \Gamma(\det(F)).$$

Proof. Each summand in (2.6-16) is a symmetric function of the eigen-
111 values v_k(resp. $v_k v_{k+1}$) and is convex and non-decreasing with respect
to each variable on $]o, +\infty[^3$. Such a function is convex with respect to
F (resp. $\operatorname{adj} F$) and thus $\mathcal{W}$ is polyconvex. (cf. Exercise 2.6-1). □

To prove the coerciveness notice that

$$\begin{aligned}\operatorname{tr}(C^{\alpha/2}) &= v_1^\alpha + v_2^\alpha + v_3^\alpha \\ &\geq C(\alpha)((v_1^2 + v_2^2 + v_3^2)^{1/2})^\alpha \\ &= C(\alpha)\|F\|^\alpha\end{aligned}$$

for any $\alpha \geq 1$, by the equivalence of norms in $\mathbb{R}^3$. The inequality (2.6-17) follows directly from this observation.

The stored energy function of a St Venant-Kirchhoff material:

$$(2.6\text{-}18)\quad \mathcal{W}(F) = -\left(\frac{3\lambda + 2\mu}{4}\right)\operatorname{tr}(C) + \left(\frac{\lambda + 2\mu}{8}\right)\operatorname{tr}(C^2) + \frac{\lambda}{4}\operatorname{tr}(\operatorname{adj}(C)) + \left(\frac{6\mu + 9\lambda}{8}\right),$$

is not polyconvex (Exercise 2.6-2). This stems from the fact that the coefficient of $tr(C)$ is < 0.

Remark 2.6.2. In the incompressible case, $\det(F) = 1$. Thus $\mathcal{W}(F)$ has no dependence on $\det(F)$. Such Ogden's materials comprise the so-called *MOONEY-RIVLIN materials.*

To fix ideas, consider an Ogden's material described by

$$(2.6\text{-}19)\qquad \mathcal{W}(F) = \sum_{i=1}^{M} a_i \operatorname{tr}(C^{\alpha_i/2}) + \sum_{j=1}^{N} b_j (tr(\operatorname{adj} C)^{\beta_{j/2}}) + \Gamma(\det F)$$

with $a_1 > o, 1 \leq i \leq M, b_j > o, 1 \leq j \leq N$ and $\Gamma(\delta) \geq C\delta^r + d, c > o$.

Then by Theorem 2.6.1, the following coerciveness condition holds:

(2.6-20) $\mathcal{W}(F) \geq a + b(\|F\|^p + \|\operatorname{adj} F\|^q + (\det F)^r),\ p = \max_i \alpha_i,\ q = \max_j \times \beta_j.$

112 Since it is natural to desire that

$$\int_\Omega \mathcal{W}(F)dx < +\infty,$$

the set of admissible deformations will typically be of the form

(2.6-21) $\mathbb{U} = \left\{\psi \epsilon \mathbb{W}^{1,p}(\Omega) | \operatorname{adj}(\nabla\psi)\, \epsilon \mathbb{L}^q(\Omega), \det(\nabla\psi)\, \epsilon L^r(\Omega) \det (\nabla\psi) > o \text{ a.e., } \psi = \phi_o \text{ on } \Gamma_o\right\}.$

Such sets are 'highly' non-convex (cf. Exercise 2.6-3). It will not be shown that $\mathbb{U}$ is weakly closed (in fact that is not true in general) but it will be shown that weakly convergent subsequences can be extracted which suit the purpose of minimizing the energy.

In order to do this, the mappings $\phi \to \operatorname{adj}(\nabla\phi)$ and $\phi \to \det(\nabla\phi)$ have to be looked at more closely.

Counting indices modulo 3, the matrix $\operatorname{adj}(\nabla\phi)$ can be defined by

(2.6-22) $$(\operatorname{adj}(\nabla\phi))_{ij} = (\partial_{i+1}\phi_{j+1}\partial_{i+2}\phi_{j+2} - \partial_{i+2}\phi_{j+1}\partial_{i+1}\phi_{j+2}).$$

If $\phi \epsilon \mathbb{W}^{1,p}(\Omega), p \geq 2$, then it is easy to see that $\operatorname{adj}(\nabla\phi) \epsilon \mathbb{L}^{p/2}(\Omega)$. The mapping defined in this fashion between $\mathbb{W}^{1,P}(\Omega)$ and $\mathbb{L}^{P/2}(\Omega)$ is non-linear and continuous. We denote weak convergence by $\rightharpoonup$.

Theorem 2.6.2. *If* $\phi \epsilon \mathbb{W}^{1,P}(\Omega), p \geq 2$, *then* $\operatorname{adj}(\nabla\phi) \epsilon \mathbb{L}^{P/2}(\Omega)$ *Further*

(2.6-23) $$\left.\begin{array}{l} \phi^n \rightharpoonup \phi in \mathbb{W}^{1,P}(\Omega), p \geq 2 \\ \operatorname{adj}(\nabla\phi^n) \rightharpoonup H \text{ in } \mathbb{L}^g(\Omega), q \geq 1 \end{array}\right\} \text{ implies } H = \operatorname{adj}(\nabla\phi).$$

113 *Proof.* (1) First an alternative definition of $\operatorname{adj}(\nabla\phi)$ will be established in the sense of distributions. Let $\phi \epsilon C^\infty(\overline{\Omega})$. Then a simple computation yields

(2.6-24) $$(\operatorname{adj}(\nabla\phi))_{ij} = \partial_{i+2}(\phi_{j+2}\partial_{i+1}\phi_{j+1}) - \partial_{i+1}(\phi_{j+2}\partial_{i+2}\phi_{j+1})$$

(with no summation on i and j). If $\theta\epsilon\mathcal{D}(\Omega)$, then

$$\int_\Omega (\text{adj}(\nabla\phi))_{ij}\theta dx = -\int_\Omega \phi_{j+1}\partial_{j+2}\phi_{j+1}\partial_{i+2}\theta dx + \int_\Omega \phi_{j+2}\partial_{j+2}\phi_{j+1}\partial_{i+1}\theta dx \tag{2.6-25}$$

For fixed $\theta\epsilon\mathcal{D}(\Omega)$, a simple application of Holder's inequality shows that each term in (2.6-25) is a continuous function if $\phi\epsilon\mathbb{H}^1(\Omega)$ (where if $\phi\epsilon\mathbb{W}^{1,P}(\Omega)$, $p \geq 2$). Since $C^\infty(\overline{\Omega})$ is dense in each of these spaces, it follows that (2.6-25) is true for $\phi\epsilon\mathbb{W}^{1,P}(\Omega)$, $p \geq 2$. Thus (2.6-24) holds for $\phi\epsilon\mathbb{W}^{1,P}(\Omega)$ in the sense of distributions.

(ii) Let $\phi^n \rightharpoonup \phi$ in $\mathbb{W}^{1,P}(\Omega)$. Let $\theta = (\theta_{ij}), \theta_{ij}\epsilon\mathcal{D}(\Omega)$. Let $p^* = +\infty$ if $p \geq 3$ and be given by

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{3} \tag{2.6-26}$$

for $p < 3$. Then for $1 \leq q < p^*$,

$$\mathbb{W}^{1,P}(\Omega) \hookrightarrow \mathbb{L}^q(\Omega), \tag{2.6-27}$$

i.e., the above inclusion is compact. Now if $\chi\epsilon\mathcal{D}(\Omega)$ is fixed and $\psi^n \rightharpoonup \psi$ in $W^{1,P}(\Omega)$ and $\phi^n \rightharpoonup \phi$ in $W^{1,P}(\Omega)$, then

$$\psi^n \rightharpoonup \psi \text{ in } L^q(\Omega)$$
$$\partial_k\phi^n \rightarrow \partial_k\phi \text{in} L^q(\Omega).$$

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If further $\frac{1}{p} + \frac{1}{q} \leq 1$, it will then follow that

$$\int_\Omega \psi^n\partial_k\phi^n X dx \rightarrow \int_\Omega \psi\partial_k\phi X dx.$$

From this observation and from (2.6-25), it follows that

$$\int_\Omega (\text{adj}\,(\nabla\phi^n)) : \theta dx \rightarrow \int_\Omega (\text{adj}\,(\nabla\phi^n)) : \theta dx \tag{2.6-28}$$

provided that $q < p^*$ and $\frac{1}{q} + \frac{1}{p} \leq 1$. It is easy to see that this is equivalent to $p > 3/2$, which is satisfied anyway.

(iii) Let $\phi^n \rightharpoonup \phi$ in $\mathbb{W}^{1,P}(\Omega)$ and $\mathrm{adj}(\nabla\phi^n) \rightharpoonup H$ in $\mathbb{L}^q(\Omega)$. By (ii) above for any $\theta = (\theta_{ij}), \epsilon\mathcal{D}(\Omega)$, (2.6-28) holds and also

$$\int_\Omega (\mathrm{adj}\,(\nabla\phi^n)) : \theta dx \to \int_\Omega H : \theta dx.$$

Thus

$$\int_\Omega (\mathrm{adj}\,(\nabla\phi^n)) - H) : \theta = o \tag{2.6-29}$$

for all $\theta = (\theta_{ij}), \theta_{ij}\epsilon\mathcal{D}(\Omega)$ and since $(\mathrm{adj}(\nabla\phi^n)) - H)\epsilon\mathbb{L}^1(\Omega)$ and $\mathcal{D}(\Omega)$ is dense in $L^1(\Omega)$, it follows that

$$H = \mathrm{adj}\,(\nabla\phi)$$

thus proving the theorem. □

Remark 2.6.3. If p^* is as in (2.6-26), if $\phi_j\epsilon L^{p^*}(\Omega)$ and $\partial_i\phi_k\epsilon L^p(\Omega)$, and if $\frac{1}{p} + \frac{1}{p^*} \leq 1$ (which is equivalent to $p \geq 3/2$). the product $\phi_j\partial_i\phi_k$ belongs to $L^1(\Omega)$.

115 Now using (2.6-25), the adjugate of $\nabla\phi$ for $\phi\epsilon\mathbb{W}^{1,p}(\Omega)$ can be defined in the sense of distributions. This definition of the adjugate of $\nabla\phi$, extended to $p \geq 3/2$ is denoted by

$$\mathrm{Adj}\,(\nabla\phi)\,.$$

The step (ii) of the proof of the above theorem goes through for $p > 3/2$ as remarked in the proof itself. Hence for $p > 3/2$, if $\phi^n \rightharpoonup \phi$ in $\mathbb{W}^{1,P}(\Omega)$, then $Adj(\nabla\phi^n) \rightharpoonup (\nabla\phi)$ in the sense of distributions.

Remark 2.6.4. The above theorem implies that the set

$$\mathbb{U} = \left\{(\phi, H)\epsilon\mathbb{W}^{1,P}(\Omega) \times \mathbb{L}^q(\Omega) | H = \mathrm{adj}\,(\nabla\phi)\right\}$$

is weakly closed in $\mathbb{W}^{1,P}(\Omega) \times \mathbb{L}^q(\Omega)$ for $p \geq 2$, $q \geq 1$. *But the set*

$$\left\{\phi \epsilon \mathbb{W}^{1,P}(\Omega) | \operatorname{adj}(\nabla\phi) \epsilon \mathbb{L}^q(\Omega)\right\}$$

is not necessarily weakly closed in $\mathbb{W}^{1,P}(\Omega)$. (cf. Exercise 2.6-4).

Now consider the mapping $\phi \to \det(\nabla\phi)$. In the first place

(2.6-30) $$\det(\nabla\phi) = \sum_{\sigma\epsilon\mathcal{P}_3} \operatorname{sgn}(\sigma)\partial_1\phi_{\sigma(1)}\partial_2\phi_{\sigma(2)}\partial_3\phi_{\sigma(3)}$$

where $\mathcal{P}_3$ is the set of all permutations of (1,2,3). Now $\det(\nabla\phi)$ will be in $L^1(\Omega)$ if $\phi\epsilon\mathbb{W}^{1,P}(\Omega)$, $p \geq 3$. But one use the integrability of the adjugate and improve on this by noting that

(2.6-31) $$\det(\nabla\phi) = \partial_i\phi_1 (\operatorname{adj}(\nabla\phi))_{i1}$$

(summation with respect to i). Now, if $\partial_i\phi_1 \in L^p(\Omega)$ and $(\operatorname{adj}(\nabla\phi))_{i1} \in L^p(\Omega)$ and if $1/p + 1/q \leq 1$, then $\det(\nabla\phi)\epsilon L^1(\Omega)$. Thus (2.6-31) will be used to define $\det(\nabla\phi)$, for if $p < 3$, the formula (2.6-30) makes no sense.

Theorem 2.6.3. *Let* $p \geq 2, \phi\epsilon\mathbb{W}^{1,P}(\Omega)$ *such that* $\operatorname{adj}(\nabla\phi)\epsilon\mathbb{L}^{P'}(\Omega)$, *where* $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. *Then* $\det(\nabla\phi)$ *given by* (2.6-31) *is in* $L^1(\Omega)$. *Further* **116**

(2.6-32)
$$\left.\begin{aligned} &\phi^n \rightharpoonup \phi \mathit{in} \mathbb{W}^{1,P}(\Omega), p \geq 2 \\ &\operatorname{adj}(\nabla\phi^n) \rightharpoonup H \text{ in } \mathbb{L}^q(\Omega), \frac{1}{p} + \frac{1}{q} \leq 1 \\ &\det(\nabla\phi^n) \rightharpoonup \delta \text{ in } L^r(\Omega), r \geq 1 \end{aligned}\right\} \text{ implies } \begin{cases} H = \operatorname{adj}(\nabla\phi) \\ \delta = \det(\nabla\phi). \end{cases}$$

Proof. (i) The main difficulty in the proof is to give an alternative definition of the determinant in the sense of distributions. Let $\phi\epsilon C^\infty(\overline{\Omega})$. Then

$$\begin{aligned} \det(\nabla\phi) &= \partial_i\phi_1(\operatorname{adj}(\nabla\phi))_{i1} \\ &= \partial_i(\phi_1(\operatorname{adj}(\nabla\phi))_{i1}) \end{aligned}$$

using the Piola identity (cf. Exercise 1.1-1). So, for $\theta\epsilon\mathcal{D}(\Omega)$

$$\int_\Omega \det(\nabla\phi)\theta dx = -\int_\Omega \phi_1(\mathrm{adj}(\nabla\phi))_{i1}\partial_i\theta dx. \tag{2.6-33}$$

It will be shown that (2.6-33) is valid for $\phi \in \mathbb{W}^{1,P}(\Omega)$ with $\mathrm{adj}(\nabla\phi) \in \mathbb{L}^{P'}$, where $\frac{1}{p}+\frac{1}{p'}=1$.

Now by the Piola identity, for smooth ϕ, $\partial_i(\mathrm{adj}(\nabla\phi))_{i1} = o$ or for $\theta\epsilon\mathcal{D}(\Omega)$

$$\int_\Omega (\mathrm{adj}(\nabla\phi))_{i1}\partial_i\theta dx = o. \tag{2.6-34}$$

By the density of smooth functions in $\mathbb{W}^{1,P}(\Omega)$, it follows that (2.6-34) is true for all $\phi\epsilon\mathbb{W}^{1,P}(\Omega)$, $p \geq 2$.

For expository convenience, set $\phi = \phi_1$, $W_i = (\mathrm{adj}((\nabla\phi))_{i1}$. Then $\phi\epsilon w^{1,P}(\Omega)$ and $w_i\epsilon L^{P'}(\Omega)$, $\frac{1}{p}+\frac{1}{p'}=1$.

117 Let $\rho\epsilon\mathcal{D}(\mathbb{R}^3), \rho \geq o$ and $\int_{\mathbb{R}^3}\rho = 1$. Define

$$\rho_k(x) = k^3\rho(kx) \tag{2.6-35}$$

so that ρ_k has the same properties as ρ together with the property that supp (ρ_k) skrinks to zero as $k \to \infty$. Let w_i be extended by o outside Ω and define

$$(\rho_k * w_i)(x) = \int_{\mathbb{R}^3}\rho(k)(x-y)w_i(y)dy. \tag{2.6-36}$$

Then the function $\rho_k * w_i$ is smooth and converges to w_i in $L^{P'}(\Omega)$

Let $\theta\epsilon\mathcal{D}(\Omega)$ be fixed. Then there exists $k_o = k_o(\theta)$ such that the support of the map $y \to \rho_k(x-y)$ is contained in Ω for all $k \geq k_o$ and for all $x\epsilon\,\mathrm{supp}(\theta)$. then for $k \geq k_o$,

$$\mathrm{div}(\rho_k * w)(X) = \int_\Omega \frac{\partial}{\partial x_i}\rho_k(x-y)w_i(y)dy = o \tag{2.6-37}$$

using (2.6-34), for $x\epsilon\,\mathrm{supp}(\theta)$. Now if $\phi^k \epsilon C^\infty(\overline{\Omega})$ and $\phi^k \to \phi$ in $\mathbb{W}^{1,P}(\Omega)$, it follows that

$$-\int_\Omega \phi^k(\rho_k * w_i)\partial_i\theta dx = \int_\Omega \partial_i\phi^k(\rho_k * w_i)\theta dx + \int_{\mathrm{supp}(\theta)} \phi^k\partial_i(\rho_k * w_i)\theta dx.$$

By (2.6-37) the second integral on the right-hand side vanishes. Now passing to the limit in each of the integrals as $k \to \infty$,

$$-\int_\Omega \phi w_i\partial_i\theta dx = \int_\Omega \partial_i\phi w_i\theta dx.$$

from which (2.6-33) follows.

(ii) If $\phi^n \rightharpoonup \phi$ in $\mathbb{W}^{1,P}(\Omega)$ and $\mathrm{adj}(\nabla\phi^n) \rightharpoonup \mathrm{adj}(\nabla\phi)$in $\mathbb{L}^q(\Omega)$, with $p \geq 2$ and $\frac{1}{p}+\frac{1}{q} \leq 1$, then using the same type of compactness argument **118**
as in the proof of theorem 2.6.2, it can be shown that for $\theta\epsilon\mathcal{D}(\Omega)$,

$$\int_\Omega \det(\nabla\phi_n)\ \theta dx \to \int_\Omega \det(\nabla\phi)\ \theta dx.$$

(iii) Using the previous step, the conclusions of the theorem can be drawn exactly as in Theorem 2.6.2. ◻

Remark 2.6.5. As in the case of $\mathrm{adj}(\nabla\phi)$, if $p \geq 3/2$ the determinant can also be defined in $\mathcal{D}'(\Omega)$ using the fact that $\phi_1(\mathrm{adj}(\nabla\phi))_{i1}\epsilon L^1(\Omega)$. The distribution obtained is denoted by Det $(\nabla\phi)$.

Remark 2.6.6. The above theorem shows that the set

$$\left\{(\phi, H, \delta)\epsilon\mathbb{W}^{1,P}(\Omega) \times \mathbb{L}^q(\Omega) \times L^r(\Omega) | H = \mathrm{adj}\,(\nabla\phi)\,, \delta = \det\,(\nabla\phi)\right\}$$

is weakly closed in $\mathbb{W}^{1,P}(\Omega) \times L^r(\Omega)$. But the set

$$\left\{\phi\epsilon\mathbb{W}^{1,P}(\Omega) |\ \mathrm{adj}\,(\nabla\phi)\ \epsilon\mathbb{L}^q(\Omega), \det\,(\nabla\phi)\ \epsilon L^r\,(\Omega)\right\}$$

is not necessarily weakly closed in $\mathbb{W}^{1,P}(\Omega)$.

The following existence theorem can now be proved.

Theorem 2.6.4 (J. BALL). *Let* $\mathcal{W} : \mathbb{M}^3_+ \to \mathbb{R}$ *be a stored energy function, such that*

(i) (Polyconvexity) there exists $\mathcal{G} : \mathbb{M}^3 \times \mathbb{M}^3 \times]o, +\infty[\to \mathbb{R}$ *which is convex and such that for all* $F \epsilon \mathbb{M}^3_+$,

(2.6-38) $$\mathcal{W}(F) = \mathcal{G}(F, \operatorname{adj}(F), \det(F));$$

(ii) (Continuity at $+\infty$ *if* $F_n \to F$ *in* $\mathbb{M}^3_+, H_n \to H$ *in* $\mathbb{M}^3_+$ *and* $\delta_n \to o^+$, *then*

(2.6-39) $$\lim_{n\to\infty} \mathcal{G}(F_n, H_n, \delta_n) = +\infty;$$

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(iii) (Conerciveness) There exist $a \epsilon \mathbb{R}, b > o, p \geq 2, q \epsilon \mathbb{R}$ *with* $\dfrac{1}{p} + \dfrac{1}{q} \leq 1$, *and* $r > 1$, *and* $r > 1$, *such that and for all* $(F, H, \delta) \epsilon \mathbb{M}^3 \times \mathbb{M}^3 \times]o, +\infty[$,

(2.6-40) $$\mathcal{G}(F, H, \delta) \geq a + b(||F||^p + ||H||^q + \delta^r).$$

Let $\Omega \subset \mathbb{R}^3$ *be a bounded open subset with boundary* $\Gamma = \Gamma_o \cup \Gamma_1$ *where the da-measure of* Γ_o *is* > 0.

Let $f \epsilon \mathbb{L}^\rho(\Omega), g \epsilon \mathbb{L}^\sigma(\Gamma_1)$ *such that the maps*

$$\psi \epsilon \mathbb{W}^{1,P}(\Omega) \to \int_\Omega f \cdot \psi dx \text{ and } \psi \epsilon \mathbb{W}^{1,P}(\Omega) \to \int_{\Gamma_1} g.\psi da$$

are continuous.

Let $I : \mathbb{W}^{1,P}(\Omega) \to \mathbb{R}$ *be given by*

(2.6-41) $$I(\psi) = \int_\Omega \mathcal{W}(\nabla\psi)\, dx - \left(\int_\Omega f.\psi dx + \int_{\Gamma_1} g.\psi da \right)$$

and $\mathbb{U} \subset \mathbb{W}^{1,P}(\Omega)$*be defined by*

(2.6-42) $$\mathbb{U} = \left\{\psi\epsilon\mathbb{W}^{1,p}(\Omega) | \operatorname{adj}(\nabla\phi)\,\epsilon L^q(\Omega), \det(\nabla\psi)\,\epsilon L^r(\Omega), \det(\nabla\psi) > o \text{ a.e.}, \psi = \phi_o \text{ on } \Gamma_o\right\},$$

with $\phi_o\epsilon\mathbb{W}^{1,P}(\Omega)$*. Assume that* $\mathbb{U} = \phi$ *and that*

$$\inf_{\psi\epsilon\mathbb{U}} I(\psi) < +\infty$$

Then the problem: Find $\phi\epsilon\mathbb{U}$ *such that*

(2.6-43) $$I(\phi) = \inf_{\psi\epsilon\mathbb{U}} I(\psi)$$

has at least one solution.

Proof. **Step (i).** *Transformation of the problem.* Define $\bar{\mathcal{G}} : \mathbb{M}^3 \times \mathbb{M}^3 \times \mathbb{R} \to \mathbb{R}U\{+\infty\}$ by **120**

(2.6-44) $$\bar{\mathcal{G}}(F, H, \delta) = \begin{cases} \mathcal{G}(F, H, \delta) \text{ if } \delta > o \\ +\infty \text{ if } \delta \leq o. \end{cases}$$

Then $\bar{\mathcal{G}}$ is easily seen to be convex and continuous into $[a, +\infty]$.

Thus the functional

(2.6-45) $$\bar{I}(\psi) = \int_\Omega \bar{\mathcal{G}}\left(\nabla\psi, \operatorname{adj}(\nabla\psi), \det(\nabla\psi)\right) dx - \left(\int_\Omega f.\psi dx + \int_{\Gamma_1} g.\psi da\right)$$

is well-defined.

Let $\psi\epsilon\mathbb{W}^{1,P}(\Omega)$ with $\operatorname{adj}(\nabla\psi)\epsilon\mathbb{L}^q(\Omega)$ and $\det(\nabla\psi)\epsilon L^r(\Omega)$. If $\bar{I}(\psi) < \infty$, then it follows that $\det(\nabla\psi) > o$ a.e.

If $\psi\epsilon\mathbb{U}$ then $I(\psi) = \bar{I}(\psi)$. Thus the original problem is equivalent to minimizing $\bar{I}(\psi)$ over $\mathbb{U}$.

Step (ii). It can be shown that (cf. Exercise 2.6-5) for all $\psi \epsilon \mathbb{W}^{1,P}(\Omega)$ with $\psi = \phi_o$ on Γ_o,

(2.6-46) $$\int\limits_{\Omega} |\psi|^p dx \leq d \left[\int\limits_{\Omega} |\nabla\psi|^p dx + \left(\int\limits_{\Gamma_o} |\phi_o| da \right)^p \right]$$

where $d > 0$. Hence for $\psi \epsilon \mathbb{U}$,

$$\bar{I}(\psi) \geq a \text{ meas } \Omega + b' \int\limits_{\Omega} \|\nabla\psi\|^p dx + b \int\limits_{\Omega} \| \operatorname{adj}(\nabla\psi) \|^q dx + b \int\limits_{\Omega} (\det(\nabla\psi))^r - C\|\psi\|_{1,p,\Omega}$$

with $b' > o$, or

(2.6-47) $\bar{I}(\psi) \geq C_0 + C_1\|\psi\|^p_{1,p,\Omega} + C_2|\operatorname{adj}(\nabla\psi)|^q_{o,q,\Omega} + C_3|\det(\nabla\psi)|^r_{o,r,\Omega}$

121 with $C_1, C_2, C_3 > 0$. (cf. Remark 2.6.7)

Step (iii). Let $\phi^n \in \mathbb{U}$ be a minimizing sequence for I. From the coerciveness (2.6-47), it follows that ϕ^n, $\operatorname{adj}(\nabla\phi^n)$ and $\det(\nabla\phi^n)$ are bounded in $\mathbb{W}^{1,p}(\Omega)$, $\mathbb{L}^q(\Omega)$ and $L^r(\Omega)$ respectively. Since these spaces are reflexive, a subsequence ϕ^n can be found such that

$$\begin{aligned} &\phi^n \rightharpoonup \phi \ \mathbb{W}^{1,p}(\Omega)\\ &\operatorname{adj}(\nabla\phi^n) \rightharpoonup H \text{ in } \mathbb{L}^q(\Omega)\\ &\det(\nabla\phi^n) \rightharpoonup \delta \text{ in } L^r(\Omega). \end{aligned}$$

But by the previous theorem $H = \operatorname{adj}(\nabla\phi)$ and $\delta = \det(\nabla\phi)$. Thus by the convexity and contnuity of $\bar{\mathcal{G}}, \bar{I}$ is weakly lower semi-continuous (cf. Theorem 2.5.1) and so

$$\bar{I}(\phi) \leq \liminf_{n \to \infty} \bar{I}(\phi^n) < +\infty.$$

Hence $\det(\nabla\phi) > o$. It can be show that $\phi|_{\Gamma_o} = \phi_o$ (cf. Exercise 2.6-6) and so $\phi \in \mathbb{U}$ and it follows that $\bar{I}$ and hence I attains a minimum at ϕ. □

Remark 2.6.7. The coerciveness condition (2.6-47) can be obtained only on $\mathbb{U}$ and *not on the whole space*. It uses the fact that $\phi = \phi_o$ on Γ_o for all the functions under consideration.

Several comments on the above result are in order here. First of all, unlike the approach based on the implicit function theorem, the result **122**
is applicable to "all" forces *(not just "small" ones)* and to *all boundary conditions*. Of course, in case of the pure traction problem, the forces must satisfy certain compatibility conditions. It is also applicable to the *mixed displacement-pressure problem* (cf. Exercise 2.6-7).

A shortcoming of this approach is the *lack of* regularity of the solution. Here it is not know if the minimizing function satisfies the equilibrium equations *even in a weak sense*. Further even thought it is true that the solution satisfies $\det(\nabla\phi) > 0$ a. e. , additional conditios are needed to insure that ϕ is one-one (see BALL [1981c]).

It is possible to extend this approach to cover the incompressible case where $\det(\nabla\phi) = 1$. (cf. Exercise 2.6-8).

Consider a St Venant-Kirchhoff material. If the forces are 'small enough' it was shown that there exists a 'small' solution to the pure displacement problem (Theorem 2.3.1). However, in the pure displacement or mixed displacement traction problem, owing to the non-polyconvexity, it cannot be shown that the energy is minimized. An *open problem* is to prove existence of 'small' solutions for small forces the mixed problem such materials.

To conclude this section, it will now be examined how to choose a stored energy function given a compressible material. Consider a compressible material (steel, for instance!). Around a natural state it is known that the stress tensor Σ_R can be written as

$$\Sigma_R^*(E) = \lambda(\operatorname{tr} E)I + 2\mu E + o(E), \tag{2.6-48}$$

where E is the Green-St Venant strain tensor. The constants λ and μ are **123**
strictly positive and can be determined form experiments, albeit approximately. It has been shown that (cf. Section 1.4)

$$\mathcal{W}(F) = \frac{\lambda}{2}(\operatorname{tr}(E))^2 + \mu \operatorname{tr}(E^2) + o(|E|^2). \tag{2.6-49}$$

The foillowing theorem due to CIARLET and GEYMONAT [1982] says that it is possible to express such a material as a simple Ogden's material.

Theorem 2.6.5. *Given* $\lambda > 0$, $\mu > 0$, *it is possible to find* $a > 0$, $b > 0$ *and a function* $\Gamma :]0, +\infty[\to \mathbb{R}$ *which is convex satisfying*

$$\Gamma(\delta) \geq C\delta^2 + d, C > o \tag{2.6-50}$$

such that the corresponding stored energy function

$$\mathcal{W}(F) = a||F||^2 + b||\operatorname{adj} F||^2 + \Gamma(\det F) \tag{2.6-51}$$

agrees to $\lambda/2(\operatorname{tr}(E))^2 + \mu \operatorname{tr}(E^2)$ *upto* $o(|E|^2)$.

Proof. **(Sketch)**. Setting $C = F^T F = I + 2E$, then

$$\begin{aligned} ||F||^2 &= \operatorname{tr} C = \operatorname{tr}(I + 2E) \\ ||\operatorname{adj} F||^2 &= \operatorname{tr}(\operatorname{adj}(I + 2E)) \\ \det(F) &= \sqrt{(\det(I + 2E))}. \end{aligned}$$

Expanding these about I, it it follows that

$$\begin{aligned} \mathcal{W}(F) = 3a + 3b + \Gamma(1) + (2a + 4b + \Gamma'(1)) \operatorname{tr}(E) - (2b + \Gamma'(1)) \operatorname{tr}(E^2) \\ + (2b + \frac{1}{2}(\Gamma'(1) + \Gamma''(1)))(\operatorname{tr}(E))^2 + o(|E|^2). \end{aligned}$$

Comparing with (2.6-49), it follows that

$$2a + 4b + \Gamma' 1 = 0, \tag{2.6-52}$$

$$-(2b + \Gamma'(1)) = \mu. \tag{2.6-53}$$

$$2b + \frac{1}{2}(\Gamma'(1) + \Gamma''(1)) = \frac{\lambda}{2}. \tag{2.6-54}$$

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These equation must be solved such that $a > o, b > o$ and $\Gamma''(1) \leq o$ (Γ is convex). (By (2.6-52)) it follows that $\Gamma'(1) < o$). It is easy to see that any point $(\Gamma'(1)), \Gamma''(1))$ on the open line-segment shown in Fig. 2.6.1 give $a > o, b > o$ satisfying the above equations.

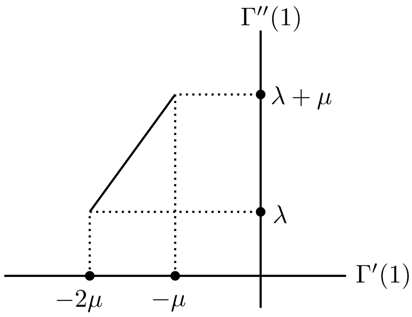

Figure 2.6.1:

Now to choose a convex function $\Gamma :]0, +\infty[\to \mathbb{R}$ satisfying (2.6-50). One can find $\alpha \geq 0, \beta > 0$ such that

(2.6-55) $$\Gamma(\delta) = \alpha\delta^2 - \beta \log \delta.$$

This function also is such that $\Gamma(\delta) \to +\infty$ as $\delta \to o^+$. □

It follows now that the associated minimization problem has at least one solution by J. BALL's theorem. Here

(2.6-56) $$\mathbb{U} = \{\psi \in \mathbb{H}^1(\Omega) | \operatorname{adj}(\nabla\psi) \in \mathbb{L}^2(\Omega), \det(\nabla\psi) \in L^2(\Omega), \det(\nabla\psi) > 0 \text{a.e. and } \psi = \phi_0 \text{ on } \Gamma_0\}$$

Remark 2.6.8. In (2.6-50), the term $C\delta^2$ could have been replaced by $C\delta^r$, for $r > 1$. The definition fo $\mathbb{U}$ would be modified accordingly. 125

Remark 2.6.9. It is also possible to choose $\mathcal{X}(F)$ in the form

(2.6-57) $$\mathscr{H}(F) = a_1||F||^2 + a_2||F||^4 + b||\operatorname{adj} F||^2 + \Gamma(\det F)$$

where $a_1 > 0, a_2 > 0, b > 0$ and Γ convex. (The St Venant-Kirchhoff stored energy function resembles this, only $a_1 < 0$.). In this case, it can be seen that the admissible range of values $(\Gamma'(1), \Gamma''(1))$ lies in the open triangle of Fig. 2.6.2. (Exercise 2.6-9).

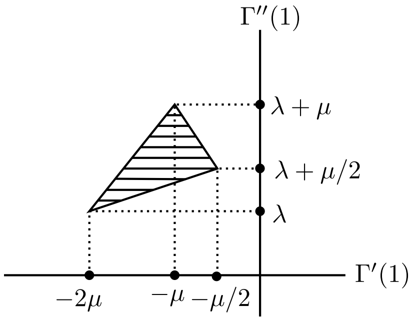

Figure 2.6.2:

Exercises

2.6-1 Let $\mathscr{H} : \mathbb{M}^3 \to \mathbb{R}$ be a function such that

$$\mathcal{H}(F) = \phi(v_1, v_2, v_3), F \in \mathbb{M}^3_+$$

where $v_i, i = 1, 2, 3$ are the principal stretches of F. If ϕ is a symetric function which is convex on $(]o, +\infty[)^3$ and non-decreasing in each variable, show that $\mathcal{W}$ is convex.

126 **2.6-2** Show that the stored energy function $\mathcal{W}$ for a St Venant-Kirchhoff matrial (cf. (2.6-18)) is not polyconvex.

2.6-3 Show that the set

$$\{\psi \in \mathbb{W}^{1,p}(\Omega) | \operatorname{adj}(\nabla\psi) \in \mathbb{L}^q(\Omega), q \geq 1\}$$

is not convex ($p \geq 2$).

2.6-4 If $p > 2$ and $q \leq p/2$ show that $\phi^n \rightharpoonup \phi$ in $\mathbb{W}^{1,p}(\Omega)$ implies $\operatorname{adj}(\nabla\phi^n) \rightharpoonup \operatorname{adj}(\nabla\phi)$ in $\mathbb{L}^q(\Omega)$.

2.6-5 Show that there exists a constant $d > 0$ such that for all $\psi \in \mathbb{W}^{1,p}(\Omega), \psi = \psi_0$ on Γ_0,

$$\int_\Omega |\psi|^p dx \leq d\left[\int_\Omega |\nabla\psi|^p dx + \left(\int_{\Gamma_o} |\phi_o| da\right)^p\right].$$

2.6-6 If $\phi^n \rightharpoonup \phi$ in $\mathbb{W}^{1,p}(\Omega)$ and $\phi^n = \phi_o$ on Γ_o, show that $\phi = \phi_o$ on Γ_o.

2.6-7 Apply J. BALL's theorem to the mixed diaplacement-pressure problem.

2.6-8 . Let $\tilde{\mathbb{U}}$ be defined by

$$\bar{\mathbb{U}} = \{(\phi, H) \in \mathbb{H}^1(\Omega) \times \mathbb{L}^2(\Omega) | H = \operatorname{adj}(\nabla\phi), \phi = \phi_o \text{ on } \Gamma_0, \\ \det(\nabla\phi) = 1 a.e\}$$

(incompressible case). Assume $\tilde{U} = \mathscr{D}$. (i) Show that $\tilde{U}$ is weakly closed in the product space $\mathbb{H}^1(\Omega) \times \mathbb{L}^2(\Omega)$. (ii) Consider

$$\mathcal{W}(F) = a||F||^2 + b||\operatorname{adj} F||^2, a > o, b > o,$$
$$I(\psi) = \int_\Omega \mathcal{W}(\nabla\psi) dx - (\int_\Omega f.\psi dx + \int_{\Gamma_1} g.\psi da).$$

Show that the problem: Find $\phi \in \mathbb{U}$ such that

$$\mathbb{U} = \{\phi \in \mathbb{H}^1(\Omega); \operatorname{adj}\nabla\phi \in \mathbb{L}^2(\Omega), \phi = \phi_o) \text{ on } \Gamma_0, \det\nabla\phi = 1 a.e.\}$$
$$I(\phi) = \inf_{\psi \in \mathbb{U}} I(\psi)$$

has at least one solution. (iii) If ϕ is smooth, show that the *Lagrange multiplier* arising out of equality constraint $\det(\nabla\phi) = 1$, is the pressure. (cf. Exercise 2.1-2). 127

2.6-9. Check the assertion made in Remark 2.6.9.

128 Bibliography, Comments and some Open Problems

No attempt has been made to give an exhaustive list of pertinent references.

The first chapter of these lecture notes gave a description of elasticity in three demensions. For further refrences, one may also consult GERMAIN [1972], GREEN and ZERNA [1968], GREEN and ADKINS [1970], GURTIN [1981a, 1981 b], MARSDEN and HUGHES [1978,1983], STOKER [1968], TRUESDELL and NOLL [1956], VALID [1977], WANG and TRUESDELL [1973], ERINGEN [1962] and WASHING [1975].

The second chapter discussed some methods for proving the existence of solution to the boundary value problem of non-linear elasticity and to the associated variational problem, in the case of hyperelastic materials.

For references about the linearized system of elasticity, see DUVAUT and LIONS [1972], FICHERA [1972] and GURTIN [1972].

The key result in proving existence via the *implicit function theorem* is the $W^{2,p}(\Omega)$-regularity of the linearized system of elasticity. The case $p = 2$ was proved by NECAS [1967] and the regularity for other p was proved by GEYMONAT [1965]. From this regularity result, (proved however only for the pure *displacement problem*) the existence theorem was independently proved by CIARLET and DESTUYNDER [1979b], MARSDEN and HUGHES [1978], VALENT [1979]. The basic idea, however, goes back to SPOPPELLI [1954] and VAN BUREN [1968]. The extension of this result to more general constutive equations was particularly studied by VALENT [1979]. See also VALENT [1978a, 1978b].

129 The necessity of the $W^{2,p}(\Omega)$-regularity of the linearized problem restricts the application of this method to *pure displacement problems*. It is also possible to treat the *pure traction problem*, which is more complicated owing to the compability conditions which the given forces must satisfy. For details see CHILLINGWORTH, MARSDEN and WAN [1982].

The increment method described in Section 2.4 is none other than Euler's method for approximating an appropriate differential equation in a Solbolev space. In other words, this method appears as an infinite demensional version of the so called *continuation by differentiation* approach as described, for instance, in RHEINBOLDT [1974].

To the best of the author's Knowledge, the convergence of increment methods for non-linear elasticity problems has been analysed so far only in some special cases, such as the one demensional model of a thin shallow spherical shell, by ANSELONE and MOORE [1966] or some finite dimensional structural problems by RHEINBOLDT [1981]. The results presented in these lecture can be found in BERNADOU, CIARLET and HU [1982].

For a decription of incremental methods in non-linear elasticity see MASON [1980], ODEN [1972] and WASHIZU [1975].

The variation approach is based on the famous article of BALL [1977]. In addition to the notion of *polycomvexity* another essential contribution of J. BALL is that one can pass to the weak limit in certain non-convex sets as was seen in Section 2.6. This idea of compactness by compensation was also developed by MURAT [1978, 1979] and TARTAR [1979]. See also AUBERT and TAHRAOUI [1982].

Other important refrences are BALL [1981a,1981b, 1981c], BALL, **130**
CURRIE and OLIVER [1981], BALL, KNOPS and MARSDEN [1978]. See also EKELAND and TEMAN [1974] for the general problem of minimizing functionals.

The notion of polyconvexity led to the definition of an *Ogden's material* (cf. OGDEN [1972]). A *St Venant-Kirchhoff material* is not Ogden's material and the existence of a solution to the corresponding mixed displacement-traction problem is open. In this connection see also ATTEIA and DEDIEU [1981] and DACOROGNA [1982a, 1982b]. For yet another approach, see ODEN [1979].

One of the drawbacks of J. BALL's approach is the lack of regularity of the solution and so one does not know if the solution thus obtained satisfies the *equilibrium equation* even in a weak sense. In this contex, see the results of LE TALLEC [1981] and LE TALLEC and ODEN [1981] for incompressible materials.

To conclude, we present a list of some of the open problems in nonlinear elasticity. Some of them have been mentioned in the text before.

1. Let $C = \nabla\phi^T\nabla\phi$. If $C - I$ is 'small', in a sense to be made precise, can it be said that ϕ is close to a rigid deformation? If some boundary conditions are imposed, can it be shown that ϕ is one-one?

   In this context cf. KOHN [1982], ALEXANDER and ANTHAN [1982], ANTMAN [1979].

131 2. The standard implicit function theorem approach fails for mixed problems. Could "hard" implicit function theorem like that of NASH and MOSER be used? In case of special domains like a thin plate, the singularities are know explicity. Could this be used, and the implicit function theorem used only on the "regular" part of the solution?

3. Study of incremental methods taking into account the finite element methods.

4. An incremental method can be *formally* written down for the mixed problem. If it can be shown to be convergent, this would provide an existence theorem for the mixed problem.

5. The minimization procedure of J BALL does not imply that the solution is small if the forces are small. How can one "distinguish" the expected small solution in this case? (In the case of the pure displacement problem, the solution via the implicit function theorem does not seem to be a local minimum of the energy in the "right" space).

6. A study *plasticity* has been taken up by TEMAN and STRANG-[1980a, 1980b]. They use the linear theory in the part corresponding to elasticity. Can one obtain better results by incorporating the non-linear theory, using J. BALL's approach?

7. A study of 'non local' constotutive equations. Here the comstitutive equation is of the form

$$T(x) = \int_{\mathbb{B}} \rho_k(x - y)\hat{T}(\nabla\phi(y))dy$$

   where ρ_k is a *mollifier.*

8. One of the *hardest* open problems of the study of the *evolution problem* which is a non-linear hyperbolic problem. The only available results are in the one-demensional case due to DIPERNA [1983]. See also HUGHES, KATO and MARSDEN [1976]. **132**

9. Plate theory. A plate can be thought of as a domain $\Omega^{\epsilon} = \omega \times]-\epsilon, +\epsilon[$, where $\omega \subset \mathbb{R}^2$ is a bounded open set and $\epsilon > o$ is a small parameter. (cf. Fig.1)

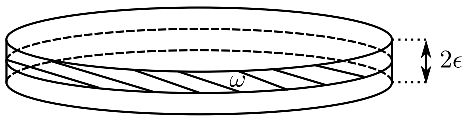

Figure 1

By methods of asymptotic expansions, the solution $(u^{\epsilon}, \sigma^{\epsilon})$ can be formally expanded as

$$(u^{\epsilon}, \sigma^{\epsilon}) = (u^0, \sigma^0) + (u^1, \sigma^1) + \cdots$$

where (u^0, σ^0) satisfies a well-known two-dimensional plate model. In the linearized theory CIARLET and DESTUYNDER [1980a], CIARLET and KESAVAN [1980], DESTUYNDER [1980] have studied the problems extensively. One can compare the three dimensional and two dimensional problems and show that (for example)

$$\frac{||u^{\epsilon}_{3d} - u^{\epsilon}_{2d}||_{1,\Omega^{\epsilon}}}{||u^{\epsilon}_{3d}||_{1,\Omega^{\epsilon}}} \to o \text{ as } \epsilon \to 0.$$

The problem is to *numerically* verify this. Computing by the finite element method, one gets $u^{\epsilon}_{3d,h}$ and $u^{\epsilon}_{2d,h}$ approximating u^{ϵ}_{3d} and u^{ϵ}_{2d} respectively. Since ϵ is small, unless h is of the same order, the linear **133** systems become very ill-conditioned. But if h is of the same order of ϵ, the solution $u^{\epsilon}_{3d,h}$ is not very accurate. Thus to find a better method of approximating these solutions and verify the convergence described above.

10. In the nonlinear case CIARLET [1980] (see also CIARLET and-DESTUYNDER [1979], CIARLET and RABIER [1980]) has shown that with certain boundary conditions the three dimensional plate model for a St Venant -Kirchhoff material is approximated (formally) by the well-known two-dimensional von Karman model. While the latter has a satisfactory existence theorey, the former has none. If at least for ϵ small enough it can be shown that the three dimensional problem has a solution converging to a given solution of the dimensional problem, an existance theorem for such special domains can be obtained.

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List of Notations

General Conventions: (1) Unless otherwise indicated, Latin indices take their values in the set $\{1, 2, 3\}$, and the repeated index convention for summation is systematically used in conjunction with this rule.

(2) If a quantity is denoted X in the deformed configuration, the corresponding quantity in the reference configuration is denoted X_R.

Vectors and Matrices

(e_i) : orthonormal basis in $\mathbb{R}^3$

$v = (v_i)$: vector v with components v_i

$A = (A_{ij})$: matrix A with elements A_{ij} (i : row index, j : column index)

$u \cdot v = u_i v_i$: Euclidean inner product

$|u| = \sqrt{u \cdot u}$: Euclidean vector norm

$$\mathscr{E}_{ijk} = \begin{cases} +1 \text{ if } (i, j, k) \text{ is an even permutation of } (1, 2, 3) \\ -1 \text{ if } (i, j, k) \text{ is an odd permutation of } (1, 2, 3) \\ 0 \text{ otherwise} \end{cases}$$

$u \Lambda v = \mathscr{E}_{ijk} u_j b_k e_i$: cross product in $\mathbb{R}^3$

$A : B = A_{ij} B_{ij} = \mathrm{tr}(AB^T)$: matrix inner product

$\|A\| = \sqrt{A : A}$: matrix norm associated with the matrix inner product

A^{-T} : $(A^{-1})^T$ (A^{-1}: inverse matrix; A^T: transposed matrix).

$adj\mathrm{A}$: adjugate of a matrix (transpose of the cofactor matrix)

$l_A = (l_1(A); l_2(A), l_3(A))$: set of the principal invariants of a matrix of order 3

$l_1(A) = a_{ii} = \mathrm{tr}(A)$

$l_2(A) = \frac{1}{2}(a_{ii}a_{jj} - a_{ij}a_{ij})$(= $\det A \,\mathrm{tr}\, A^{-1}$ if A is invertible)

$l_3(A) = \det A$

$\mathbb{M}^3$: set of all matrices of order 3
$\mathbb{M}^3_+ = \{F \in \mathbb{M}^3 | \det F > 0\}$
$\mathbb{O}^3 = \{F \in \mathbb{M}^3 | F^T F = F F^T = I\}$
$\mathbb{O}^3_+ = \mathbb{O}^3 \cap M^3_+ = \{F \in \mathbb{O}^3 | \det F = 1\}$
$\mathbb{S}^3 = \{F \in \mathbb{M}^3 | F = F^T\}$
$\mathbb{S}^3_> = \{F \in \mathbb{S}^3 | F$ is positive definite$\}$
$F = RU = VR$: polar factorization of an invertible matrix ($R \in \mathbb{O}^3; U, V \in \mathbb{S}^3_>$)
$C^{1/2}$: square root of a matrix $C \in \mathbb{S}^3_>$

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Functions and Function Spaces

Id : identity mapping
$v'(a)$: Fréchet derivative of the mapping v at the point a
$\partial^\alpha v = \dfrac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}, |\alpha| = \alpha_1 + \cdots + \alpha_n$ (multi-index notation for partial derivatives)
$\dfrac{\partial \mathscr{W}}{\partial F}(F) = \left(\dfrac{\partial \mathscr{W}}{\partial F_{ij}}(F)\right) \in \mathbb{M}^3$ (for a mapping $\mathscr{W} : \mathbb{M}^3 \to \mathbb{R}$)
$X \hookrightarrow Y$: the canonical injection from X into Y is continuous
$X \overset{c}{\hookrightarrow} Y$: the canonical injection from X into Y is compact
$\rightharpoonup$: weak convergence
$C^0(X, Y)$: set of all continuous mappings from X into Y
$C^m(X; Y)$: space of all m times continuously differentiable mappings from X into $Y (1 \le m \le \infty)$
$C^m(X) = C^m(X; \mathbb{R}), 0 \le m \le \infty.$
$W^{m,p}(\Omega) = \{v \in L^p(\Omega); \partial^\alpha v \in L^p(\Omega)$ for all $|\alpha| \le m\}$ $(W^{0,p}(\Omega) = L^p(\Omega))$

$H^m(\Omega) = W^{m,2}(\Omega)$

$\mathbb{L}^p(\Omega), \mathbb{W}^{m,p}(\Omega), \mathbb{H}^m(\Omega)$: corresponding spaces of vector-valued, or matrix-valued, functions

$|v|_{0,p,\Omega}$: norm of the space $L^p(\Omega)$, $1 \leq p \leq \infty$.

$\|v\|_{m,p,\Omega} = \{\int_\Omega \sum_{|\alpha|\leq m} |\partial^\alpha v|^p dx\}^{1/p}$: norm of the space $W^{m,p}(\Omega)$, $1 \leq p \leq \infty$

$\|v\|_{m,\infty,\Omega} = \max_{|\alpha|\leq m} |\partial^\alpha v|_{0,\infty,\Omega}$: norm of the space $W^{m,\infty}(\Omega)$

$|v|_{m,p,\Omega} = \{\int_\Omega \sum_{|\alpha|=m} |\partial^\alpha v|^p dx\}^{1/p}$, $1 \leq p \leq \infty$

$|v|_{m,\infty,\Omega} = \max_{|\alpha|=m} |\partial^\alpha v|_{0,\infty,\Omega}$

$\mathscr{D}(\Omega) = \{v \in \mathscr{C}^\infty(\Omega)$; supp v is a compact subset of $\Omega\}$

$W_0^{m,p}(\Omega)$: closure of $\mathscr{D}(\Omega)$ in $W^{m,p}(\Omega)$

$H_0^m(\Omega) = W_0^{m,2}(\Omega)$

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Miscellaneous

$[a, +\infty] = [a, +\infty[\cup\{+\infty\}, a \in R$

$f(x) = 0(x)$: $\lim_{\substack{x\to 0\\ x\neq 0}} \frac{|f(x)|}{\|x\|} = 0$

$c \circ U$: convex hull of U (smallest convex set containing U)

144 Notations in the Deformed Configuration

$\mathscr{B} = \phi(\mathscr{B}_R)$: deformed configuration

$X = \phi(X_R)$: generic point of $\mathscr{B}$

$\partial\mathscr{B}$: boundary of $\mathscr{B}$

$\partial\mathscr{B} = \partial\mathscr{B}_0 \cup \partial\mathscr{B}_1$: dA-measurable partition of $\partial\mathscr{B}$

n : unit outer normal along $\partial\mathscr{B}$

dX : volume element in $\mathscr{B}$

dA : surface element on $\partial\mathscr{B}$

$$\mathrm{GRAD}\,\Theta = \left(\frac{\partial \Theta_i}{\partial X_j}\right) \in \mathbb{M}^3 \text{ (for a mapping } \Theta : \mathscr{B} \to \mathbb{R}^3\text{)}$$

$\mathrm{DIV}\ \mathrm{T} = \dfrac{\partial T_{ij}}{\partial X_j} e_i \in \mathscr{R}^3$: divergence of a tensor field $T : \mathscr{B} \to \mathbb{M}^3$

$\partial(X) \in \mathbb{R}$: density per unit mass at $X \in \mathscr{B}$

$b(X) \in \mathbb{R}^3$: body force density per unit mass at $X \in \mathscr{B}$

$t_1(X) \in \mathbb{R}^3$: applied surface force density per unit area of $\partial\mathscr{B}$ at $X \in \partial\mathscr{B}$

$t(X, n), X \in \mathscr{B}, |n| = 1$: Cauchy stress vector in $\mathscr{B}$

$T(X)$: Cauchy stress tensor at $X \in \mathscr{B}$

$\hat{T}, \overline{T}$: response function for $T = \hat{T}(F) = \overline{T}(B)$, with $F \in \mathbb{M}^3_+$, $B = FF^T$.

$\mathscr{B}_R$, or $\overline{\Omega}$: reference configuration

X_R, or x : generic point of $\mathscr{B}_R$

$\partial\mathscr{B}_R$, or Γ : boundary of $\mathscr{B}_R$

$\partial\mathscr{B}_R = \partial\mathscr{B}_{0R} \cup \partial\mathscr{B}_{1R}$, or $\Gamma = \Gamma_0 \cup \Gamma_1$: dA_R-measurable partition of $\partial\mathscr{B}_R$

n_R, or ν : unit outer normal along $\partial\mathscr{B}_R$

dX_R, or dx : volume element in $\mathscr{B}_R$

dA_R, or da : surface element on $\partial\mathscr{B}_R$

$\phi, \psi : \mathscr{B}_R \to \mathbb{R}^3$: deformation of $\mathscr{B}_R$ (smooth maps with $\det \cdot > 0$)

$u, v : \mathscr{B}_R \to \mathbb{R}^3$: displacement ($\phi = \mathrm{Id} + u, \psi = \mathrm{Id} + v$)

$$\partial_i = \frac{\partial}{\partial X_{R_i}}$$

$\mathrm{DIV}_R T_R = \frac{\partial T_{Rij}}{\partial X_{R_j}} e_i \in \mathbb{R}^3$: divergence of a tensor field $T_R : \mathscr{B}_R \to \mathbb{M}^3$

$\nabla_\phi = \left(\frac{\partial \phi_i}{\partial X_{R_j}}\right) \in \mathbb{M}^3_+$: deformation gradient

$\nabla_u = \left(\frac{\partial u_i}{\partial X_{R_j}}\right) \in \mathbb{M}^3$: displacement gradient

$C = \nabla_\phi^T \nabla_\phi \in \mathbb{S}^3_>$: right Cauchy-Green strain tensor

$B = \nabla_\phi \nabla_\phi^T \in \mathbb{S}^3_>$: left Cauchy-Green strain tensor

$E = E(u) = \frac{1}{2}(C - I) = \frac{1}{2}(\nabla u^T + \nabla u + \nabla u^T \nabla u)$: Green-St Venant strain tensor

$\epsilon(u) = \frac{1}{2}(\nabla u^T + \nabla u)$: linearized strain tensor

$\rho_R(X_R) \in \mathbb{R}$: density per unit mass at $X_R \in \mathscr{B}_R$

$b_R(X_R) \in \mathbb{R}^3$: body force density per unit mass at $X_R \in \mathscr{B}_R$

$f = \rho_R b_R : \Omega \to \mathbb{R}^3$

$t_{1R}(X_R) \in \mathbb{R}^3$: applied surface force density per unit area of $\partial\mathscr{B}_R$ at $X_R \in \partial\mathscr{B}_R$

$g = t_{1R} : \Gamma_1 \to \mathbb{R}^3$

$t_R(X_R, n_R), X_R \in \mathscr{B}_R, |n_R| = 1$: first Piola-Kirchhoff stress vector in $\mathscr{B}_R$

$T_R(X_R)$: first Piola-Kirchhoff stress tensor at $X_R \in \mathscr{B}_R$

$(t_{ij}) = T_R$: $\Omega \to \mathbb{M}^3$

$\hat{T}_R$: response functions for $T_R = \hat{T}_R(F)$, $F \in \mathbb{M}^3_+$

$\sum_R(X_R) = \nabla_\phi(X_R)^{-1} T_R(X_R)$: second Piola-Kirchhoff stress tensor at $X_R \in \mathscr{B}_R$

$(\sigma_{ij}) = \sum_R$: $\Omega \to \mathbb{S}^3$

$\hat{\sum}_R, \overline{\sum}_R, \sum_R^*$: response functions for $\sum_R = \hat{\sum}_R(F) = \overline{\sum}_R(C) = \sum_R^*(E)$, with $F \in \mathbb{M}^3_+$, $C = F^T F = I + 2E$

$\sigma^*(E)$: $\sum_R^*(E) = \lambda(\operatorname{tr} E)I + 2\mu E + 0(E)$

λ, μ : Lamé's constants

$\nu = \dfrac{\lambda}{2(\lambda + \mu)}$: Poisson's ratio

$E = \dfrac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$	:	Young's modulus
$a_{ijk\ell} = \lambda\delta_{ij}\delta_{k\ell} + 2\mu\delta_{ik}\delta_{j\ell}$	:	elasticity coefficients for isotropic materials
$\mathscr{W}$	:	stored energy function $\left(\dfrac{\partial\mathscr{W}}{\partial F}(R) = \hat{T}_R(F), F \in \mathbb{M}^3_+\right)$
$\overline{\mathscr{W}}$	:	stored energy function in terms of $C = F^T F(\mathscr{W}(F) = \overline{\mathscr{W}}(C))$
$\mathscr{W}^*$	:	stored energy function in terms of $E(\mathscr{W}(F) = \overline{\mathscr{W}}(I + 2E) = \mathscr{W}^*(E))$
ϕ	:	stored energy function in terms of $l_c : \mathscr{W}(F) = \phi(l_c), C = F^T F$
$W(\psi) = \int_{\mathscr{B}_R} \mathscr{W}(\nabla\psi)dX_R = \int_\Omega \mathscr{W}(\nabla\psi)dx$	:	strain energy
$I(\psi) = W(\psi) - \{B(\psi) + T_1(\psi)\}$	:	total energy

$$B(\psi) = \int_{\mathscr{B}_R} \rho_R b_R \cdot \psi dX_R = \int_\Omega f \cdot \psi dx \text{ (for dead loads)}$$

$$T_l(\psi) = \int_{\partial\mathscr{B}_{lR}} t_{lR} \cdot \psi dA_R = \int_{\Gamma_1} g \cdot \psi da \text{ (for dead loads)}$$

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