Lectures on
Partial Differential Equations

By

G.B. Folland

Tata Institute of Fundamental Research
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Lectures on Partial Differential Equations

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Notes by

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Preface

This book consists of the notes for a course I gave at the T.I.F.R. Center in Bangalore from September 20 to November 20, 1981. The purpose of the course was to introduce the students in the Programme in Application of Mathematics to the applications of Fourier analysis-by which I mean the study of convolution operators as well as the Fourier transform itself-to partial differential equations. Faced with the problem of covering a reasonably broad spectrum of material in such a short time, I had to be selective in the choice of topics. I could not develop any one subject in a really thorough manner; rather, my aim was to present the essential features of some techniques that are well worth knowing and to derive a few interesting results which are illustrative of these techniques. This does not mean that I have dealt only with general machinery; indeed, the emphasis in Chapter 2 is on very concrete calculation with distributions and Fourier transforms-because the methods of performing such calculations are also well worth knowing.

If these notes suffer from the defect of incompleteness, they posses the corresponding virtue of brevity. They may therefore be of value to the reader who wishes to be introduced to some useful ideas without having to plough through a systematic treatise. More detailed accounts of the subjects discussed here can be found in the books of Folland [1], Stein [2], Taylor [3], and Treves [4].

No specific knowledge of partial differential equations or Fourier Analysis is presupposed in these notes, although some prior acquittance with the former is desirable. The main prerequisite is a familiarity with the subjects usually gathered under the rubic "real analysis": measure and integration, and the elements of point set topology and functional analysis. In addition, the reader is expected to be acquainted with the basic facts about distributions as presented, for example, in Rudin [7].

I wish to express my gratitude to professor K.G. Ramanathan for inviting me to Bangalore, and to professor S.Raghavan and the staff of the T.I.F.R. Center for making my visit there a most enjoyable one. I also wish to thank Mr S. Thangavelu and Mr K.T. Joseph for their painstaking job of writing up the notes.

Contents

Chapter 1

Preliminaries

IN THIS CHAPTER, we will study some basic results about convolu- 1
tion and the Fourier transform.

1 General Theorems About Convolutions

We will begin with a theorem about integral operators.

Theorem 1.1. *Let K be a measurable function on $\mathbb{R}^n \times \mathbb{R}^n$ such that, for some $c > 0$,*
$\int |K(x,y)|dy \leq c$, $\int |K(x,y)|dx \leq c$, *for every x, y in $\mathbb{R}^n$.*

If $1 \leq p \leq \infty$ and $f \in L^p(\mathbb{R}^n)$, then the function Tf, defined by $Tf(x) = \int K(x,y)f(y)dy$ for almost every x in $\mathbb{R}^n$, belongs to $L^p(\mathbb{R}^n)$ and further,

$$\|Tf\|_p \leq c\|f\|_p.$$

Proof. If $p = \infty$, the hypothesis $\int |K(x,y)|dx \leq c$ is superfluous and the conclusion of the theorem is obvious. If $p < \infty$, let q denote the conjugate exponent. Then, by Hölder 's inequality,

$$|Tf(x)| \leq \left\{ \int |K(x,y)|dy \right\}^{1/q} \left\{ \int |K(x,y)|\,|f(y)|^p dy \right\}^{1/p}$$

$$\leq c^{1/q}\left\{\int |K(x,y)|\,|f(y)|^p dy\right\}^{1/p}.$$

□

From this we have,

$$\begin{aligned}\int |Tf(x)|^p dx &\leq c^{p/q}\iint |K(x,y)||f(y)|^p dydx\\ &\leq c^{1+p/q}\int |f(y)|^p dx = c^{1+p/q}\|f\|_p^p.\end{aligned}$$

2 Therefore $\|Tf\|_p \leq c\|f\|_p$.

Next, we define the convolution of two locally integrable functions.

Definition 1.2. *Let f and g be two locally integrable functions. The* convolution *of f and g, denoted by $f * g$, is define by*

$$(f*g)(x) = \int f(x-y)g(y)dy = \int f(y)g(x-y)dy = (g*f)(x),$$

provided that the integrals in question exist.

The basic theorem on convolution is the following theorem, called *Young's inequality*.

Theorem 1.3 (Young's Inequality)**.** *Let $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, for $1 \leq p \leq \infty$. Then $f * g \in L^p(\mathbb{R}^n)$ and*

$$\|f*g\|_p \leq \|g\|_p\|f\|_1.$$

Proof. Take $K(x,y) = f(x-y)$ in Theorem 1.1. Then $K(x,y)$ satisfies all the conditions of Theorem 1.1. and the conclusion follows immediately.

□

The next theorem underlies one of the most important uses of convolution. Before coming to the theorem, let us prove the following

Lemma 1.4. *For a function f defined on $\mathbb{R}^n$ and x in $\mathbb{R}^n$, we define a function f^x by $f^x(y) = f(y-x)$. If $f \in L^p, 1 \leq p < \infty$, then $\lim_{x\to O}\|f^x - f\|_p = 0$.*

3 *Proof.* If g is a compactly supported continuous function, then g is uniformly continuous, and so g^x converges to g uniformly as x tends to 0. □

Further, for $|x| \leq 1, g^x$ and g are supported in a common compact set. Therefore, $\lim\limits_{x\to 0} ||g^x - g||_p = 0$. Given $f \in L^p$, we can find a function g which is continuous and compactly supported such that $||f - g||_p < \in /3$ for $\in > 0$. But then $||g^x - f^x||_p < \in /3$ also holds. Therefore

$$\begin{aligned} ||f^x - f||_p &\leq ||f^x - g^x||_p + ||g^x - g||_p + ||g - f||_p \\ &\leq 2 \in /3 + ||g^x - g||_p. \end{aligned}$$

Since $\lim x \to 0 ||g^x - g||_p = 0$, we can choose x close to 0 so that $||g^x - g||_p < \in /3$. Then $||f^x - f|| < \in$ and this proves the lemme since $\in$ is arbitrary.

Remark 1.5. *The above lemma is false when $p = \infty$.* Indeed, "$f^x \to f$ in L^∞" means precisely that f is uniformly continuous.

Let us now make *two important observations about convolutions* which we shall use without comment later on.

i) $\operatorname{Supp}(f * g) \subset \operatorname{Supp} f + \operatorname{Supp} g$, where

$$A + B = \{x + 1 : x \in A, y \in B\}.$$

ii) If f is of class C^k and $\partial^\alpha(|\alpha| \leq k)$ and g satisfy appropriate conditions so that differentiation under the integral sign is justified, then $f * g$ is of class C^k and $\partial^\alpha(f * g) = (\partial^\alpha f) * g$.

Theorem 1.6. *Let $g \in L^1(\mathbb{R}^n)$ and $\int g(x)dx = a$. Let $g_\in(x) = \in^{-n} g(x/\in)$ for $\in > 0$. Then, we have the following:* 4

i) *If $f \in L^p(\mathbb{R}^n)$, $p < \infty$, $f * g_\in$ converges to af in L^p as $\in$ tends to 0.*

ii) *If f is bounded and continuous, then $f * g_\in$ converge to af uniformly on compact sets as $\in$ tends to 0.*

Proof. By the change of variable $x \to \in x$, we see that $\int g_\in(x)dx = a$ for all $\in > 0$. Now

$$\begin{aligned}(f * g_\in)(x) - af(x) &= \int f(x-y)g_\in(y)dy - \int f(x)g_\in(y)dy \\ &= \int [f(x-y) - f(x)]\, g_\in(y)dy \\ &= \int [f(x - \in y) - f(x)]g(y)dy \\ &= \int [f^{\in y}(x) - f(x)]g(y)dy.\end{aligned}$$

□

If $f \in L^p$ and $p < \infty$, we apply Minkoswski's inequality for integrals to obtain

$$\|f * g_\in - af\|_p \leq \int \|f^{\in y} - f\|_p |g(y)|dy.$$

The function $y \to \|f^{\in y} - f\|_p$ is bounded by $2\|f\|_p$ and tends to 0 as $\in$ tends to 0 for each y, by lemma 1.4. Therefore, we can apply Lebesgue Dominated Convergence theorem to get the desired result.

On the other hand, suppose f is bounded and continuous. Let K be any compact subset of $\mathbb{R}^n$. Given $\delta > 0$, choose a compact set G $\mathbb{R}^n$ such that

$$\int\limits_{\mathbb{R}^n - G} |g(y)|dy < \delta.$$

5 Then

$$\begin{aligned}\text{Sup}_{x \in k}(|f * g_\epsilon)(x) - af(x)| &\leq 2\delta\|f\|_\infty \\ &+ \text{Sup}_{(x,y)\epsilon K \times G} |f(x - \epsilon y) - f(x)| \int\limits_G |g|dy.\end{aligned}$$

Since f is uniformly continuous on the compact set K the second term tends to 0 as $\in$ to 0. Since δ is arbitrary, we see that

$$\sup x \in K|(f * g_\epsilon)(x) - af(x)| \to 0 \text{ as } \epsilon \to 0.$$

Hence the theorem is proved.

Corollary 1.7. *The space $C_o^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^{\ltimes})$ for $1 \le p < \infty$.*

Proof. Let

$$\begin{aligned} \phi(x) &= e^{-1/(1-|x|^2)} && \text{for } |x| < 1 \\ &= 0 && \text{for } |x| \le 1 \end{aligned}$$

Then $\phi \epsilon C_o^\infty(\mathbb{R}^n)$ and $\int \phi(x)dx = 1/a > 0$. If $f \epsilon L^p$ and has compact support, then $a(f * \phi_\epsilon)\epsilon C_o^\infty(\mathbb{R}^n)$ and by theorem 1.6, $a(f * \phi_\epsilon)$ converges to f in L^p as ϵ tends to 0. Since L^p functions with compact support are dense in L^p, this completes the proof. □

Proposition 1.8. *Suppose $K \subset \mathbb{R}^n$ is compact and $\Omega \supset K$ be an open subset of $\mathbb{R}^n$. Then there exists a C_o^∞ function ϕ such that $\phi(x) = 1$ for $x \epsilon K$ and* $\operatorname{Supp} \phi \subset \Omega$.

Proof. Let $V = \{x \in \Omega : d(x, K) \le \frac{1}{2}\delta\}$ where $\delta = d(K, \mathbb{R}^n \backslash \Omega)$. Choose 6
a $\phi_o \epsilon C_o^\infty$ such that $\operatorname{Supp} \phi_o \subset B\left(0, \frac{1}{2}\delta\right)$ and $\int \phi_o(x)dx = 1$. Define

$$\phi(x) = \int_V \phi_o(x-y)dy = (\phi_0 * X_V)(x).$$

Then $\phi(x)$ is a function with the required properties. □

2 The Fourier Transform

In this section, we will give a rapid introduction to the theory of the Fourier transform.

For a function $f \epsilon L^1(\mathbb{R}^n)$, the *Fourier transform* of the function f, denoted by $\hat{f}$, is defined by

$$\hat{f}(\xi) = \int e^{-2\pi i x.\xi} f(x)dx, \ \xi \epsilon \mathbb{R}^n.$$

Remark 1.9. Our definition of $\hat{f}$ differs from some other in the placement of the factor of 2π.

BASIC PROPERTIES OF THE FOURIER TRANSFORM

For

(1.10) $$f \epsilon L^1, \|\hat{f}\|_\infty \leq \|f\|_1.$$

The proof of this is trivial.

For

(1.11) $$f, g \in L^1, (f * g)\hat{}\,(\xi) = \hat{f}(\xi)\hat{g}(\xi).$$

Indeed,

$$\begin{aligned}(f * g)\hat{}(\xi) &= \iint e^{-2\pi i x.\xi} f(y) g(x-y)\,dy\,dx\\ &= \iint e^{-2\pi i(x-y)\cdot\xi} g(x-y) e^{-2\pi i Y.\xi} f(y)\,dy\,dx\\ &= \int e^{-2\pi i(x-y)\cdot\xi} g(x-y)\,dx \int f(y) e^{-2\pi i y.\xi}\,dy\\ &= \hat{f}(\xi)\hat{g}(\xi).\end{aligned}$$

7 Let us now consider the Fourier transform in the Schwartz class $S = S(\mathbb{R}^n)$.

Proposition 1.12. *For $f \in S$, we have the following:*

i) $\hat{f} \epsilon C^\infty(\mathbb{R}^n)$ *and* $\partial^\beta \hat{f} = \hat{g}$ *where* $g(x) = (-2\pi i x)^\beta f(x)$.

ii) $(\partial^\beta f)\hat{}(\xi) = (2\pi i \xi)^\beta \hat{f}(\xi)$.

Proof. i) Differentiation under the integral sign proves this.

ii) For this, we use integration by parts.

$$\begin{aligned}(\partial^\beta f)\hat{}(\xi) &= \int e^{-2\pi i x.\xi} (\partial^\beta f)(x)\,dx\\ &= (-1)^{|\beta|} \int \partial_\beta [e^{-2\pi i x.\xi}] f(x)\,dx\\ &= (-1)^{|\beta|} (-2\pi i \xi)^\beta \int e^{-2\pi i x\cdot\xi} f(x)\,dx\\ &= (2\pi i \xi)^\beta \hat{f}(\xi).\end{aligned}$$

□

Corollary 1.13. *If $f \epsilon S$, then $\hat{f} \in S$ also.*

Proof. For multi-indices α and β, using proposition 1.12, we have

$$\begin{aligned}
\xi^{\alpha}(\partial^{\beta}\hat{f})(\xi) &= \xi^{\alpha}((-2\pi ix)^{\beta}f(x))\hat{}\,(\xi)\\
&= (2\pi i)^{-|\alpha|}[\partial^{\alpha}((-2\pi ix)^{\beta}f(x))\hat{}(\xi)]\\
&= (-1)^{|\beta|}(2\pi i)^{|\beta|-|\alpha|}(\partial^{\alpha}(x^{\beta}f(x)))\hat{}\,(\xi)
\end{aligned}$$

Since $f \epsilon S, \partial^{\alpha}(x^{\beta}f(x))\epsilon L^1$ and hence $(\partial^{\alpha}(x^{\beta}f(x)))\hat{}\,\epsilon L^{\infty}$. Thus ξ^{α} $(\partial^{\beta}\hat{f})$ is bounded. Since α and β are arbitrary, this proves that $\hat{f}\epsilon S$. ◻

Corollary 1.14 (RIEMANN-LEBESGUE LEMMA)**.** *If $f\epsilon L^1$, then $\hat{f}$ is continuous and vanishes at ∞.*

Proof. By Corollary 1.13, this is true for $f\epsilon S$. Since S is dense in L^1 **8**
and $\|\hat{f}\|_{\infty} \leq \|f\|_1$, the same in true for all $f\epsilon L^1$. ◻

Let us now compute the Fourier transform of the Gaussian.

Theorem 1.15. *Let $f(x) = e^{-\pi a|x|^2}$, Re $a > 0$. Then, $\hat{f}(\xi) = a^{-n/2}$ $e^{-a^{-1}}\pi|\xi|^2$.*

Proof.

$$\hat{f}(\xi) = \int e^{-2\pi ix.\xi}e^{-a\pi|x|^2}dx$$

i.e.,
$$\hat{f}(\xi) = \prod_{j=1}^{n}\int_{-\infty}^{\infty} e^{-2\pi ix_j\xi_j}e^{-a\pi x_j^2}dx_j.$$

Thus it suffices to consider the case $n = 1$. Further, we can take $a = 1$ by making the change of variable $x \to a^{-1/2}x$. ◻

Thus we are assuming $f(x) = e^{-\pi x^2}, x\epsilon\mathbb{R}$. Observe that $f'(x) + 2\pi x f(x) = 0$. Taking the Fourier transform, we obtain

$$2\pi i\xi\hat{f}(\xi) + i\hat{f}'(\xi) = 0.$$

Hence

$$\hat{f}'(\xi)/f(\xi) = -2\pi\xi$$

which, on integration, gives $f(\xi) = ce^{-\pi\xi^2}$, c being a constant.

The constant c is given by

$$c = \hat{f}(0) = \int\limits_{-\infty}^{\infty} e^{-\pi x^2} dx = 1.$$

Therefore $\hat{f}(\xi) = e^{-\pi\xi^2}$, which completes the proof.

We now derive the *Fourier inversion formula for the Schwartz class* S.

9 Let us define $f^V(\xi) = \int e^{2\pi ix\cdot\xi} f(x)dx = \hat{f}(-\xi)$.

Theorem 1.16 (Fouries Inversion Theorem)**.** *For* $f\epsilon(\hat{f})^\vee = f$.

Proof. First, observe that for $f, g\epsilon L^1$, $\int f\hat{g} = \int \hat{f}g$. In fact,

$$\begin{aligned}
\int \hat{f}(x)g(x)dx &= \iint e^{-2\pi iy\cdot x} f(y)g(x)dydx \\
&= \int \left[\int e^{-2\pi iy\cdot x} g(x)dx\right] f(y)dy \\
&= \int \hat{g}(y)f(y)dy.
\end{aligned}$$

□

Given $\epsilon > 0$ and x in $\mathbb{R}^n$, take the function ϕ defined by

$$\phi(\xi) = e^{-2\pi ix\cdot\xi - \pi\epsilon^2|\xi|^2}.$$

Now

$$\begin{aligned}
\hat{\phi}(y) &= \int e^{-2\pi iy\cdot\xi} e^{-2\pi ix\cdot\xi - \pi\epsilon^2|\xi|^2} d\xi \\
&= \int e^{-2\pi i(y-x)\cdot\xi_e - \pi\epsilon^2|\xi|^2} d\xi \\
&= \in^{-n} e^{-\pi\in^{-2}|x-y|^2}.
\end{aligned}$$

If we take $g(x) = e^{-\pi|x|^2}$ and define $g_\epsilon(x) = \epsilon^{-n} g(x/\epsilon)$, then

$$\hat{\phi}(y) = g_\epsilon(x - y).$$

Therefore

$$\begin{aligned}\int e^{2\pi ix\cdot\xi}\hat{f}(\xi)e^{-\pi\epsilon^2|\xi|^2}d\xi &= \int \hat{f}(\epsilon)\phi(\xi)d\xi\\ &= \int f(y)\hat{\phi}(y)dy\\ &= \int f(y)g_\epsilon(x-y)dy\\ &= (f*g_\epsilon)(x)\end{aligned}$$

But as ϵ tends to 0, $(f * g_\epsilon)$ converges to f, by Theorem 1.6 and 10
clearly

$$\int e^{2\pi i\cdot x\xi}\hat{f}(\xi)e^{-\pi\epsilon^2|\xi|^2}d\xi \to \int \hat{f}(\xi)e^{2\pi i\cdot x.\xi_{d\xi}}$$

Therefore $(\hat{f})^\vee = f$.

Corollary 1.17. *The Fourier transform is an isomorphism of S onto S.*

Next, we prove the *Plancherel Theorem*.

Theorem 1.18. *The Fourier transform uniquely extends to a unitary map of $L^2(\mathbb{R}^n)$ onto itself.*

Proof. For $f\epsilon S$, define $\tilde{f}(x) = \overline{f(-x)}$. Then it is easily checked that $\hat{\tilde{f}} = \bar{\hat{f}}$, so that

$$\begin{aligned}\|f\|_2^2 &= \int |f(x)|^2dx\\ &= \int f(x)\tilde{f}(-x)dx\\ &= (f*\tilde{f})(0)\\ &= \int (f_*\tilde{f})\hat{}\,(\xi)d\xi\\ &= \int \hat{f}(\xi)\hat{\tilde{f}}(\xi)d\xi\\ &= \int \hat{f}(\xi)\overline{\hat{f}}(\xi)d\xi = \|\hat{f}\|_2^2.\end{aligned}$$

Therefore, the Fourier transform extends continuously to an isometry of L^2. It is a unitary transformation, since its image S is dense in L^2. □

Let us observe how the Fourier transform interacts with translations, rotations and dilations.

11 (1.19) *The Fourier transform and translation*: If $f^x(y) f(y-x)$ then

$$\begin{aligned}\hat{f}^x(\xi) &= \int e^{2\pi i y\cdot\xi} f(y-x)dy\\ &= \int e^{2\pi i(z+x)\cdot\xi} f(z)dz \text{ (by putting } y-x=z)\\ &= \int e^{2\pi i x\cdot\xi} \hat{f}(\xi).\end{aligned}$$

(1.20) *The Fourier transform and rotations (orthogonal transformations):*

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an orthogonal transformation. Then

$$\begin{aligned}(f\circ T)\hat{}(\xi) &= \int e^{-2\pi i x\cdot\xi}(foT)(x)dx\\ &= \int e^{-2\pi i T^{-1}y\cdot\xi} f(y)dy \text{ (by putting } y=Tx)\\ &= \int e^{-2\pi i y\cdot T\xi} f(y)dy\\ &= \hat{f}(T\xi) = (\hat{f}oT)(\xi).\end{aligned}$$

Thus, $(foT)\hat{} = \hat{f}oT$ i.e. $\hat{}$ commutes with rotations.

(1.21) *The Fourier transform and dilation*: Let $f_r(x) = r^{-n} f(x/r)$.

Then

$$\begin{aligned}\hat{f}_r(\xi) &= \int e^{-2\pi i x\cdot\xi} r^{-n} f(x/r)dx\\ &= \int e^{-2\pi i y\cdot\xi} f(y)dy = \hat{f}(r\xi).\end{aligned}$$

The last equation suggests, roughly: the more spread out f is, the more $\hat{f}$ will be concentrated at the origin and vice versa. This notion can be put in a precise form as follows.

(1.22) *HEISENBERG INEQUALITY* ($n = 1$*: For* $f \epsilon S(\mathbb{R})$*, we have*

$$\|xf(x)\|_2\|\xi\hat{f}(\xi)\|_2 \geq (1/4\pi)\|f\|_2^2.$$

Proof. Observe that **12**

$$\frac{d}{dx}(xf(x)) = x\frac{df}{dx}(x) + f(x).$$

Thus

$$\begin{aligned}
\|f\|_2^2 &= \int f(x)\overline{f(x)}dx \\
&= \int \overline{f(x)}\left[\frac{d}{dx}(xf(x)) - x\frac{df}{dx}(x)\right]dx \\
&= -\int xf(x)\frac{d\overline{f}}{dx}(x)dx - \int x\frac{df}{dx}(x)\overline{f}(x)dx \\
&= -2Re\int xf(x)\frac{d\overline{f}}{dx}(x)dx. \\
&\leq 2\|xf(x)\|_2\|\frac{d\overline{f}}{dx}\|_2 \text{ (by Cauchy-Schwarz)}
\end{aligned}$$

i.e., $$\|f\|_2^2 \leq 2\|xf(x)\|_2\|\frac{df}{dx}\|_2.$$

But

$$\widehat{\left(\frac{d\overline{f}}{dx}\right)}(\xi) = 2\pi i\xi\hat{f}(\xi).$$

Therefore,

$$\|f\|_2^2 \leq 2.2\pi\|xf(x)\|_2\|\xi\hat{f}(\xi)\|_2$$

or

$$\|xf(x)\|_2\|\xi f(\xi)\|_2 \geq (1/4\pi)\|f\|_2^2.$$

□

A GENERALISATION TO *n* VARIABLES

We replace x by x_j, $\dfrac{d}{dx}$ by $\dfrac{\partial}{\partial x_j}$. Also for any $a_j, b_j \epsilon \mathbb{R}$, we can replace x_j and $\dfrac{\partial}{\partial x_j}$ by $x_j - a_j$ and $\dfrac{\partial}{\partial x_j} - b_j$ respectively. The same proof then yields:

(1.23) $$\|(x_j - a_j) f(x)\|_2 \|(\xi_j - b_j)\hat{f}(\xi)\|_2 \geq (1/4\pi)\|f\|_2^2.$$

13 Let us now take $\|f\|_2 = 1$ and $A = (a_1, a_2, \ldots, a_n) \epsilon \mathbb{R}^n$. Let f be small outside a small nighbourhood of A.

In this case, $\|(x_j - a_j) f(x)\|_2$ will be small. Consequently, the other factor on the left in (1.23) has to be large. That is, if the mass of f is concentrated near one point, the mass of $\hat{f}$ cannot be concentrated near any point.

Remark 1.24. If we take

$$a_j = \int x_i |f(x)|^2 dx, b_j = \int \xi_j |\hat{f}(\xi)|^2 d\xi,$$

then inequality (1.23) is the mathematical formulation of the *position-momentum uncertainty relation* in Quantum Mechanics.

3 Some Results From the Theory of Distributions

In this section, let us recall briefly some results from the theory of distributions. (For a more detailed treatment, see, for example [7] or [8]).

In the sequel, $\mathcal{D}'(\Omega)$ will denote the *space of distributions* on the open set $\Omega \subset \mathbb{R}^n$ which is the dual space of $C_o^\infty(\Omega)$. When $\Omega = \mathbb{R}^n$, we will simply write $\mathcal{D}'$ instead of $\mathcal{D}'(\mathbb{R}^n)$. In the same way, $S' = S'(\mathbb{R}^n)$ will denote the *space of tempered distributions with and* $E' = E'(\mathbb{R}^n)$ will stand for the *space of distributions with compact support*.

The value of a distribution $u \epsilon \mathcal{D}'$ at a function $\phi \epsilon C_o^\infty$ will be denoted by $< u, \phi >$. If u is a locally integrable function, then u defines a distri-
14 bution by $< u, \phi > = \int u(x)\phi(x) dx$. It will sometimes be convenient to

write $\int u(x)\phi(x)dx$ for $< u, \phi >$, when u is an arbitrary distribution.

The convergence in $\mathcal{D}'$ is the *weak convergence* defined by the following:

$$u_n, u\epsilon\mathcal{D}', u_n \to \text{ in means } < u_n, \phi > \to < u, \phi > \text{ for every } \phi \text{ in } C_o^\infty.$$

Let us now recall briefly *certain operations on distributions.*

(1.25) We can multiply a distribution $u\epsilon\mathcal{D}'$ by a C^∞ function ϕ to get another distribution ϕu which is defined by

$$< \phi u, \psi > = < u, \phi\psi >$$

A C^∞ function ψ is said to be *tempered* if, for every multiindex $\alpha, \partial^\alpha\phi$ grows at most polynomially at ∞. We can multiply an $u, \epsilon S'$ by a tempered function to get another tempered distribution. The definition is same as in the previous case.

(1.26) If $u\epsilon\mathcal{D}'$ and $f\epsilon C_o^\infty$, we define the *convolution* $u * f$ by $(u * f)(x) = < u, f_x >$ where $f_x(x) = f(x - y)$.

The function $u * f$ is C^∞ and when $u\epsilon E, u * f$ is in C_o^∞. The convolution $* : \mathcal{D}' \times C_o^\infty \to C^\infty$ can be extended to a map from $\mathcal{D}' \times E'$ to $\mathcal{D}'$. Namely, if $u\epsilon\mathcal{D}', v\epsilon E'$ and $\phi\epsilon C_o^\infty, < u * v, \phi > = < u, \tilde{v} * \phi >$ where $\tilde{v}$ is defined by $< \tilde{v}, \psi > = \int v(x)\psi(-x)dx$. The associative law

$$u * (v * w) - (u * w) * w \text{ holds for } u, v, w\epsilon\mathcal{D}'$$

provided that at most one of them does not have compact support. For **15**
$u\epsilon S'$ and $f\epsilon S, u * f$ can also defined in the same way and $u * f$ is a tempered C^∞ function.

(1.27) Since the Fourier transform is an isomorphism of S onto S. and $\int f\hat{g} = \int \hat{f}g$, the Fourier transform extends by duality to an isomorphism of S' onto S'.

For $u\epsilon E' \subset S'$, we have $\hat{u}(\xi) = < u, e^{-2\pi i(\cdot)\cdot\xi} >$ which is an entire analytic function.

Chapter 2

Partial Differential Operators with Constant Coefficients

1 Local Solvability and Fundamental Solution

For the sake of convenience in taking the Fourier transforms, from now 16
onwards, we will use the differential monomials

$$D^\alpha = (2\pi i)^{|\alpha|}\partial^\alpha. \text{ Thus } (D^\alpha)\hat{}(\xi) = \xi^\alpha \hat{f}(\xi).$$

By a *partial differential operator with constant coefficients,*we mean a differential operator L of the form

$$L = \sum_{|\alpha|\leq k} a_\alpha D^\alpha, a_\alpha \epsilon \mathbb{C}$$

Further, we assume that $\sum\limits_{|\alpha|\leq k} |a_\alpha| \neq 0$. In this case, we say that the operator L is of order k. If we write $P(\xi) = \sum\limits_{|\alpha|\leq k} a_\alpha \xi^\alpha$, then we have $L = P(D)$.

For $u \epsilon S$, taking the Fourier transform, we see that

$$(P(D)u)\hat{}(\xi) = \sum_{|\alpha|\leq k} a_\alpha \xi^\alpha \hat{u}(\xi) = P(\xi)\hat{u}(\xi).$$

Let us now consider the following problem:

Given f in C^{∞}, we want to find a distribution u such that $P(D)u = f$.

We say that the differential operator L is *locally solvable at* $x_{\circ}\epsilon\mathbb{R}^{n}$, if there is a solution of the above problem in some neighbourhood of the point $x_{\circ}$ for any f in C^{∞}.

17 **Remark 2.1.** We can assume that f has compact support. To see this, we can take any ϕ in C_{o}^{∞} such that $\phi = 1$ in some nighbourhood of the point x_{o}. If solve the problem $P(D)u = f\phi$ near $x_{\circ}$, then u is a solution of our original problem since we have $f\phi = f$ in a neighbourhood of x_{o}.

In the following theorem we give an affirmative answer to the question of local solvability of $L = P(D)$. The simple proof exhibited here is due to L. Nirenberg.

Theorem 2.2. *Let* $L = \sum\limits_{|\alpha|\leq k} a_{\alpha}D^{\alpha}$ *be a differential operator with constant coefficients. If* $u\epsilon C_{o}^{\infty}$, *there is a* C^{∞} *function* u *satisfying* $Lu = f$ *on* $\mathbb{R}^{n}$.

Proof. Taking the Fourier transform of the equation $Lu = P(D)u = f$, we see that $P(\xi)\hat{u}(\xi) = \hat{f}(\xi)$. It is natural to try to define u by the formula

$$u(x) = \int e^{2\pi ix\cdot\xi}\hat{f}(\xi)/P(\xi)\xi.$$

In general, P will have many zeros: so there will be a problem in applying the inverse Fourier transform to $\hat{f}/P$. But things are not so bad. Since $f\epsilon C_{\circ}^{\infty}$, $\hat{f}$ is an entire function of $\xi\epsilon\mathbb{C}^{n}$ and P is obviously entire. Hence we can deform the contour of integration to avoid the zeros of $P(\xi)$. ◻

To make this precise, let us choose a unit vector η so that $\sum\limits_{|\alpha|\leq k} a_{\alpha}\eta^{\alpha} \neq 0$. By a rotation of coordinates, we can assume that $\eta = (0, 0, \ldots, 0, 1)$.
18 Multiplying by a constant, we can also assume that $a_{\alpha 0} = 1$ where $\alpha_{o} = (0, 0, \ldots, 0, k)$. Then we have $P(\xi) = \xi_{n}^{k}$+(lower order terms in ξ_{n}). Denote $\xi = (\xi', \xi_{n})$ with $\xi' = (\xi_{1}, \ldots, \xi_{n-1})$ in $\mathbb{R}^{n-1}$.

Consider $P(\xi) = P(\xi', \xi_n)$ as a polynomial in the last variable ξ_n in $\mathbb{C}$ for ξ' in $\mathbb{R}^{n-1}$. Let $\lambda_1(\xi'), \ldots, \lambda_k(\xi')$ be the roots of $P(\xi', \xi_n) = 0$ arranged so that if $i \leq j$.

$\Im\lambda_i(\xi') \leq \Im\lambda_j(\xi')$ and $\operatorname{Re}\lambda_i(\xi') \leq \operatorname{Re}\lambda_i(\xi')$ when $\Im\lambda_i(\xi') = \Im\lambda_j(\xi')$.

Since the roots of a polynomial depends continuously on the coefficients we see that $\lambda_j(\xi')$ are continuous in ξ'. To proceed further, we need the help of two Lemmas.

Lemma 2.3. *There is a measurable function $\phi : \mathbb{R}^{n-1} \to [-k-1, k+1]$ such that for all ξ' in $\mathbb{R}^{n-1}$ $\min\limits_{1\leq j\leq k}\{|\phi(\xi') - Im\lambda_j(\xi')|\} \geq 1$.*

Proof. Left as an exercise to the reader : (cf. G.B. Folland [1]).

(The idea is that at least one of the $k+1$ intervals $[-k-1, k+1], [-k-1, k+1] \ldots, [-k-1, k+1]$ must contain none of the k points $\operatorname{Im}\lambda_j(\xi')$, $j = 1, 2, \ldots, k$). □

Lemma 2.4. *Let $P(\xi) = \xi_n^k$+(lower order terms) and let $N(P)$ be the set $\{\zeta \epsilon \mathbb{C}^n : P(\zeta) = 0\}$. Let $d(\xi, N(P))$. Then, we have*

$$|P(\xi)| \geq (d(\xi)/2)^k.$$

Proof. Take ξ in $\mathbb{R}^n$ such that $P(\xi) \neq 0$. Let $\eta = (0, 0, \ldots 0, 1)$ and define $g(z) = P(\xi + z\eta)$ for z in $\mathbb{C}$. This g is a polynomial in one complex variable z. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the zeros of the polynomial g. Then 19

$$g(z) = c(z - \lambda_1)\cdots(z - \lambda_k)$$

so that

$$\left|\frac{g(z)}{g(0)}\right| = \prod_{j=1}^{k}\left|1 - \frac{z}{\lambda_j}\right|.$$

Since $\xi + \lambda_j\eta \,\epsilon\, N(P)$ $|\lambda_j| \geq d(\xi)$ so that when $|z| \leq d(\xi)$, $\left|\dfrac{g(z)}{g(0)}\right| \leq 2^k$. Also

$$|g^{(k)}(0)| = \left|\frac{k!}{2\pi i}\int_{|\zeta|=d(\xi)} q(\zeta)\zeta^{-k-1}d\zeta\right| \leq \frac{k!}{2}\frac{|g(0)|}{d(\xi)^{k+1}}2^k 2\pi d(\xi)$$

i.e. $|g^{(k)}(0)| \leq k!|g(0)|2^k d(\xi)^{-k}$. But $g(0) = P(\xi)$ and

$$|g^{(k)}(0)| = \frac{\partial^k}{\partial \xi_n^k} P(\xi) = k!$$

Therefore,

$$k! \leq k!|g(0)|2^k d(\xi)^{-k}$$

or

$$|P(\xi)| \geq (d(\xi)/2)^k$$

Hence the lemma is proved. □

Returning to the proof of the theorem, consider the function

$$u(x) = \int\limits_{\mathbb{R}^{n-1}} \int\limits_{IM\xi_n = \phi(\xi')} e^{2\phi i x \cdot \xi} (\hat{f}(\xi)/P(\xi)) d\xi_n d\xi.$$

By Lemmas 2.3 and 2.4 we have

$$|P(\xi)| \geq (d(\xi)/2)^k \geq 2^{-k} \text{ along } Im\xi_n = \phi(\xi').$$

20 Since $f \epsilon C_\circ^\infty$, $\hat{f}(\xi)$ is rapidly decreasing as $|Re\xi|$ tends to ∞ as along as $|Im\xi|$ stays bounded, so the integral converges absolutely and uniformly together with all derivatives defining a C^∞ function u.

Finally, by Cauchy's theorem,

$$\begin{aligned} P(D)u(x) &= \int\limits_{\mathbb{R}^{n-1}} \int\limits_{IM\xi_n = \phi(\xi')} e^{2\phi i x \cdot \xi} (\hat{f}(\xi)/P(\xi)) d\xi_n d\xi' \\ &= \int_{\mathbb{R}^n} e^{2\phi i x \cdot \xi} (\hat{f}(\xi) d\xi = f(x). \end{aligned}$$

This completes the proof of Theorem 2.2.

Let us now consider the local solvability of $L = P(D)$ in the case when f is a distribution.

As before, we remark that it suffices to take $f \epsilon E'$. Further, it is enough to consider the case where $f = \delta$. Indeed, if K satisfies $P(D)K = \delta$, then we have for any $f \epsilon E', P(D)(K * f) = P(D)K * f = \delta * f = f$.

Definition 2.5. *A distribution K satisfying $P(D)K = \delta$ is called a fundamental solution or elementary solution of the differential operator $L = P(D)$.*

A remarkable theorem due to MALGRANGE AND EHRENPREIS states that every differential operator $P(D)$ with constant coefficients has a fundamental solution. In fact, we can prove this result by a simple extension of the preceding argument.

Theorem 2.6. Every partial differential operator $P(D)$ with constant co- **21**
efficients has a fundamental solution.

Proof. Proceeding as in the previous theorem, we try to define

$$K(x) = \int\limits_{\mathbb{R}^{n-1}} \int\limits_{Im\xi_n=\phi(\xi')} e^{2\Pi i x\cdot\xi}(p(\xi))^{-1} d\xi_n d\xi'.$$

Here, however, the integral may diverge at infinity. So, we consider the polynomial

$$P_N(\xi) = P(\xi)(1 + 4\Pi^2 \sum_{j=1}^{n} \xi_j^2)N$$

where N is a large positive integer. Let

$$K_N(x) = \int\limits_{\mathbb{R}^{n-1}} \int\limits_{Im\xi_n=\phi(\xi')} e^{2\Pi i x\cdot\xi}(P_N(\xi))^{-1} d\xi_n d\xi'$$

where ϕ is chosen appropriately for the polynomial P_N.Then on the region of integration, we have

$$|P_N(\xi)| \geq c(1 + |\xi|^2)^N;$$

so the integral will converge when $N > n/2$. We claim that $P_N(D)K_N = \delta$. ☐

To see this, take $\phi\epsilon C_o^\infty$ and observe that the transpose of $P_N(D)$ is $P_N(-D)$. Thus

$$< P_N(D)K_N, \phi > =< K_N, P_N(-D)\phi >$$

$$= \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^{n-1}} \int\limits_{Im\xi_n=\phi(\xi')} e^{2\Pi ix\cdot\xi} \frac{(P_N(-D)\phi)(x)}{P_N(\xi)} d\xi_n d\xi' dx$$

$$= \int\limits_{\mathbb{R}^{n-1}} \int\limits_{Im\xi_n=\phi(\xi')} (P_N(\xi))^{-1} d\xi_n d\xi' \int\limits_{\mathbb{R}^n} e^{2\Pi ix\cdot\xi} P_N(-D)\phi(x)dx$$

$$= \int\limits_{\mathbb{R}^{n-1}} \int\limits_{Im\xi_n=\phi(\xi')} \hat{\phi}(-\xi)d\xi_n d\xi'$$

$$= \int\limits_{\mathbb{R}^n} \hat{\phi}(-\xi)d\xi = \phi(0) =< \delta, \phi > .$$

22

Thus,

$$\delta = P_N(D)K_N = P(D)(1-\Delta)^N K_N$$

so if we put $K = (1 - \cdot\Delta)^N K_N$ we have $P(D)K = \delta$. Thus K is a fundamental solution of $P(D)$.

2 Regularity Properties of Differential Operators

Definition 2.7. *The singular support of a distribution $f \epsilon D'$ is defined to be the complement of the largest open set on which f is a C^∞ function. The singular support of f will be denoted by sing* supp f.

Definition 2.8. *Let $L = \sum\limits_{|\alpha|\leq K} a_\alpha(x)D^\alpha$ where $a_\alpha \epsilon C^\infty$ be a differential operator. L is said to be hypoelliptic, if and only if, for any $u\epsilon\mathcal{D}'$, sing* supp u $\subset$ *sing supp Lu. In other words, L is hypoelliptic if and only if for any open set $\Omega \subset \mathbb{R}^n$ and any $u\epsilon\mathcal{D}'(\Omega)$, "$Lu\epsilon C^\infty(\Omega)$ implies $u\epsilon C^\infty(\Omega)$".*

Remark 2.9. The operator L is said to be *elliptic* at $x\epsilon\mathbb{R}^n$ if

$$\sum_{|\alpha|=k} a_\alpha(x)\xi^\alpha \neq 0 \text{ for every } \xi \text{ in } \epsilon\mathbb{R}^n\backslash\{0\}.$$

L is said to be *elliptic* if L is elliptic at every $x\epsilon\mathbb{R}^n$. Elliptic operators
23 are hypoelliptic, as we shall prove later. This accounts for the name hypoelliptic.

We know that an ordinary differential operator with C^∞ coefficients is hypoelliptic as long as the top order coefficient is non-zero. But this is not the case with partial differential operators as seen from the following

Example 2.10. Take the operator $L = \dfrac{\partial^2}{\partial x \partial y}$ in $\mathbb{R}^2$. The general solution of the equation $Lu = 0$ is given by $u(x, y) = f(x) + g(y)$ where f and g are arbitrary C^1 functions. This shows that L is not hypoelliptic.

We observe that if L is hypoelliptic, then every fundamental solution of L is a C^∞ function in $\mathbb{R}^n \backslash \{0\}$.

In the case of partial differential operators with constant coefficients, this is also sufficient for hypoellipticity. Indeed, we have the following

Theorem 2.11. Let L be a partial differential operator with constant coefficients. Then the following are equivalent:

a) *L is hypoelliptic.*

b) *Every fundamental solution of L is C^∞ in$\mathbb{R}^2\backslash\{0\}$.*

c) *At least one fundamental solution of L is C^∞ in $\mathbb{R}^2\backslash\{0\}$.*

Proof. That (a) simple (b) follows from the above observation and that (b) implies (c) is completely trivial. The only nontrivial part we need to prove is that (c) implies (a). To prove this implication, we need □

Lemma 2.12. *Suppose $f \epsilon \mathscr{D}'$ is such that f is C^∞ in $\mathbb{R}^n\backslash\{0\}$ and $g \epsilon E'$.* **24**
Then sing $\operatorname{supp}(f * g) \subset \operatorname{supp} g$.

Proof. Suppose $x \notin \operatorname{supp} g$. We will show that $f * g$ is C^∞ in a neighbourhood of x. □

Since $x \notin \operatorname{supp} g$, there exists an $\epsilon > 0$ such that $B(x, \epsilon) \cap \operatorname{supp} g = \phi$. Choose $\phi \epsilon C_0^\infty(B(0, \epsilon/2))$ such that $\phi = 1$ on $B(0, \epsilon/4)$. Now, $f * g = (\phi f) * g + (1 - \phi) f * g$. Since $(1 - \phi) f$ is a C^∞ function, $(1 - \phi) f * g$ is C^∞. Also

$$\operatorname{Supp}(\phi f * g) \subset \operatorname{Supp} \phi f + \operatorname{Supp} g$$

$$\subset \{y : d(y, \operatorname{supp} g) \leq \epsilon/2\}$$

which does not intersect $B(x, \epsilon/2)$. Therefore, in $B(x, \epsilon/2)$, $f * g = (1 - \phi) f * G$ which is C^∞.

Proof of theorem 2.11. Let K be a fundamental solution of L such that K is C^∞ in $\mathbb{R}^n \backslash \{0\}$. Suppose $u \epsilon \mathcal{D}'$ and $Lu \epsilon C^\infty(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is open.

For $x \epsilon \Omega$, pick $\epsilon > 0$ small enough so that $\overline{B}(x, \epsilon) \subset \Omega$. Choose $\phi \epsilon C_o^\infty(B(x, \epsilon))$ so that $\phi = 1$ on $B(x, \epsilon/2)$. Then $L(\phi u) = \phi Lu + v$ where $v = 0$ on $B(x, \epsilon/2)$ and also outside $B(x, \epsilon)$. We write

$$K * L(\phi u) = K * \phi Lu + K * v.$$

ϕLu is a C_o^∞ function so that $K * Lu$ is a C^∞ function. Also $K * v$ is a C^∞ function on the ball $B(x, \epsilon/2)$ by Lemma 2.12.

25 Therefore $K * L(\phi u)$ is a C^∞ function on $B(x, \epsilon/2)$. But $K * L(\phi u) = LK * \phi u = \delta * \phi u = \phi u$. Thus, ϕu is a C^∞ function on $B(x, \epsilon/2)$. Since $\phi = 1$ on $B(x, \epsilon/2), u$ is a C^∞ function on $B(x, \epsilon/2)$. Since x is arbitrary, this completes the proof.

3 Basic Operators in Mathematical Physics

In this section, we introduce the three basic operators in Mathematical Physics. In the following sections, we shall compute fundamental solutions for these operators and show how they can be applied to solve boundary value problems and to yield other information.

(i) *LAPLACE OPERATOR*: $\Delta = \sum\limits_{j=1}^{n} \dfrac{\partial^2}{\partial x_j^2}$ in $\mathbb{R}^n$. If $u(x)$ represents electromagnetic potential (or gravitational potential) and ρ denotes the charge (resp. mass) density, then they are connected by the equation $\Delta u = -4\pi\rho$. If the region contains no charge, i.e., if $\rho = 0, \Delta u = 0$. This equation is called the *homogeneous Laplace equation*, and its solutions are called *harmonic functions*.

(ii) *HEAT OPERATOR*: $L = \dfrac{\partial}{\partial t} - \Delta$ in $\mathbb{R}^{n+1}$. If $u(x,t)$ represents the temperature of a homogeneous body at the position x and time t, then u satisfies the heat equation $\dfrac{\partial u}{\partial t} - \Delta u = 0$ in $\mathbb{R}^{n+1}$.

(iii) *WAVE OPERATOR*: $\Box = \dfrac{\partial^2}{\partial t^2} - \Delta$ in $\mathbb{R}^{n+1}$. If $u(x,t)$ represents the amplitude of an electromagnetic wave in vacuum at position x and time t, then u satisfies the *wave equation* $\Box u = 0$ in $\mathbb{R}^{n+1}$. **26**

The equation $\Box u = 0$ can also be used to describe other types of wave phenomena although in most cases, it is only an approximation valid for small amplitudes. More generally, the equation $\Box u = f$ describes waves subject to a driving force f.

The Laplace operator Δ is an ingredient in all the above examples. The reason for this is that the basic laws of Physics are invariant under translation and rotation of coordinates which severely restricts the differential operators which can occur in them. Indeed, we have

Theorem 2.13. *Let L be a differential operator which is invariant under rotations and translations. Then there exists a polynomial Q in one variable with constant coefficients such that $L = Q(\Delta)$.*

Proof. Let $L = P(D) = \sum\limits_{|\alpha| \leq K} a_\alpha D^\alpha$. Since $P(D)$ is invariant under translations, a_α are constants. Let T be any rotation. We have

$$\begin{aligned} P(T\xi)\hat{\phi}(T\xi) &= (P(D)\phi)\hat{}(T\xi) \\ &= (P(D)\phi \circ T)\hat{}(\xi) \\ &= (P(D)(\phi \circ T))\hat{}(\xi) \\ &= P(\xi)\hat{\phi}(T\xi), \text{ for every } \phi. \end{aligned}$$

Thus $P(T\xi) = P(\xi)$ so that P is rotation - invariant. $\square$

Write $P(\xi) = \sum\limits_{j=0}^{k} P_j(\xi)$ where $P_j(\xi)$ is the part of P which is homogeneous of degree j. We claim that each P_j is rotation - invariant. Indeed, if t is any real number, **27**

$P(t\xi) = \sum\limits_{j=0}^{k} t^j P_j(T\xi)$. For a rotation T, since $P(\xi) = P(T\xi)$, we have

$$P(t\xi) = \sum_{j=0}^{k} t^j P_j(T\xi)$$

so that
$$\sum_{j=0}^{k} t^j (P_j(T\xi) - P_j(\xi)) = 0.$$

This being true for all t, it follows that $P_j \circ T = P_j$. But the only rotation-invariant functions which are homogeneous of degree j are of the form $P_j(\xi) = c_j|\xi|^j$ where c_j are constants.

Indeed, since P_j is rotation-invariant, $P_j(\xi)$ depends only on $|\xi|$ and thus, for $|\xi| \neq 0$,

$$P_j(\xi) = P_j\left(|\xi|\frac{\xi}{|\xi|}\right) = |\xi|^j P_j\left(\frac{\xi}{|\xi|}\right) = c_j|\xi|^j.$$

Since $|\xi|^j$ is not a polynomial when j is odd, $c_j = 0$ in that case. Therefore,
$$P(\xi) = \sum c_{2j}|\xi|^{2j}.$$

Taking $Q(x) = \sum (c_{2j}/(-4\pi^2)^j) x^j$, we get $P(D) = Q(\Delta)$.

Remark 2.14. This theorem applies to scalar differential operators. If one considers operators on vector or tensor functions, there are first or-
28 der operators which are translation - and rotation - invariant, namely, the familiar operators grad, curl, div of 3-dimensional vector analysis and their n-dimensional generalisations.

Definition 2.15. *A function $F(x)$ is said to be* ***radial*** *, if there is a function f of one variable such that $F(x) = f(|x|) = f(r), r = |x|$.*

When F is radial and $F \epsilon L^1(\mathbb{R}^n)$, we have

$$\int\limits_{\mathbb{R}^n} F(x)dx = \int\limits_0^\infty \int\limits_{|x|=1} f(r) r^{n-1} d\sigma(x) dr$$

where $d\sigma(x)$ is the surface measure on S^{n-1}, the unit sphere in $\mathbb{R}^n$.

Thus

$$\int_{\mathbb{R}^n} F(x)dx = \omega_n \int_0^\infty f(r)r^{n-1}dr$$

where ω_n is the area of S^{n-1}.

Let us now calculate ω_n. We have

$$\int e^{-\pi|x|^2}dx = 1.$$

Now

$$\begin{aligned}\int e^{-\pi|x|^2}dx &= \omega_n \int_0^\infty e^{-\pi r^2} r^{n-1}dr\\ &= \frac{\omega_n}{2\pi}\int_0^\infty e^{-s}(s/\pi)^{n/2-1}ds,\ s=\pi r^2\\ &= \frac{\omega_n}{2\pi^{n/2}}\int_0^\infty e^{-s}s^{n/2-1}ds\\ &= \omega_n\Gamma(n/2)/(2\pi^{n/2}).\end{aligned}$$

Therefore

$$\omega_n = 2\pi^{n/2}/\Gamma(n/2).$$

4 Laplace Operator

First, let us find a fundamental solution of Δ, i.e., we want to find a distribution K such that $\Delta K = \delta$. **29**

Since Δ commute with rotations and δ has the same property, we observe that if u is a fundamental solution of Δ and T is a rotation, then $u \circ T$ is also a fundamental solution. We therefore expect to find a fundamental solution K which is radial.

Let us tarry a little to compute the Laplacian of a radial function F. Let$F(x) = f(r), r = |x|$. Then,

$$\begin{aligned}\Delta F(x) &= \sum_{j=1}^{n} \frac{\partial}{\partial x_j}[f'(r)x_j/r] \\ &= \sum_{j=1}^{n}\left(f''(r)\frac{x_j^2}{r^2} + \frac{f'(r)}{r} - \frac{f'(r)}{r^3}x_j^2\right) \\ &= f''(r) + ((n-1)/r)f'(r).\end{aligned}$$

Now set $K(x) = f(r)$. If K is to be a fundamental solution of Δ, we must have

$$f''(r) + ((n-1)/r)f'(r) = 0 \text{ on } (0, \infty).$$

From this equation

$$f''(r)/f''(r) = -(n-1)/r.$$

Integrating, we get

$$f'(r) = c_\circ r^{1-n}$$

with a constant $c_\circ$. One more integration yields

$$\begin{aligned}f(r) &= c_1 r^{2-n} + c_2 \text{ when } n \neq 2 \\ &= c_1 \log r + c_2 \text{ when } n = 2.\end{aligned}$$

30

Since constants are solutions of the homogeneous Laplace equation, we may assume that $c_2 = 0$. Thus if we set $F(x) = |x|^{2-n}(n \neq 2)$ or $F(x) = \log|x|(n = 2)$, we expect to find that $\Delta F = c\delta$ for some $c \neq 0$, and then $K = c^{-1}F$ will be our fundamental solution.

In fact, we have

Theorem 2.16. *If $F(x) = |x|^{2-n}$ on $\mathbb{R}^n(n \neq 2)$, then $\Delta F = (2-n)\omega_n\delta$, where ω_n is the area of the unit sphere in $\mathbb{R}^n$,* *Proof.* For $\epsilon > 0$, we define $F_\epsilon(x) = (\epsilon^2 + |x|^2)^{(2-n)/2}$. Then F_ϵ is a C^∞ function on $\mathbb{R}^n$, and an application of Lebesgue's Dominated Convergence Theorem reveals that F_ϵ converges to F as ϵ tends to 0, in the sense of distributions. Therefore ΔF_ϵ converges to ΔF, in the same sense. Let us now compute ΔF_ϵ.

$$\begin{aligned}\Delta F_\epsilon(x) &= \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left\{(2-n)(|x|^2+\epsilon^2)^{-n/2}x_j\right\}\\ &= \sum_{j=1}^{n}\left\{(2-n)(|x|^2+\epsilon^2)^{-n/2} + (2-n)(-n)(|x|^2+\epsilon^2)^{(-n/2)-1}x_j^2\right\}\\ &= (2-n)(|x|^2+\epsilon^2)^{(-n/2)-1}n\epsilon^2.\end{aligned}$$

Thus we see that $\Delta F_\epsilon \epsilon L^1(\mathbb{R}^n)$. A simple computation shows that $\Delta F_\epsilon(x) = \epsilon^{-n}\Delta F_1(x/\epsilon)$; so, by Theorem 1.6, ΔF_ϵ tends to $(\int \Delta F_1)\delta$ as ϵ tends to 0. Therefore $\Delta F = (\int \Delta F_1)\delta$, and we need only to compute $\int \Delta F_1$.

$$\begin{aligned}\int \Delta F_1 &= n(2-n)\int (1+|x|^2)^{(-n/2)-1}dx\\ &= n(2-n)\omega_n \int_0^\infty (1+r^2)^{(-n/2)-1}r^{n-1}dr.\end{aligned}$$

Putting $r^2+1 = s$, we see that **31**

$$\begin{aligned}\int \Delta F_1 &= \frac{1}{2}n(2-n)\omega_n \int_1^\infty s^{(-n/2)-1}(s-1)^{(n-1)/2}ds\\ &= \frac{1}{2}n(2-n)\omega_n \int_1^\infty s^{-2}(1-\frac{1}{s})^{(n/2)-1}ds\\ &= \frac{1}{2}n(2-n)\omega_n \int_0^\infty (1-\sigma)^{(n/2)-1}d\sigma, \sigma = \frac{1}{s}\\ &= \frac{1}{2}n(2-n)\omega_n(2/n) = (2-n)\omega_n\end{aligned}$$

which completes the proof. □

Exercise. *Show that if* $F(x) = \log|x|$ *on* $\mathbb{R}^2$*, then* $\Delta F = 2\pi\delta$.

Corollary 2.17. *Let* $K(x) = \begin{cases} \dfrac{|x|^{2-n}}{(2-n)\omega_n} (n \neq 2) \\ \frac{1}{2\pi} \log|x| (n = 2) \end{cases}$

Then K *is a fundamental solution of the Laplacian.*

Corollary 2.18. Δ *is hypoelliptic.*

Proof. Follows from Theorem 2.11. □

It is also instructive to compute the fundamental solution of Δ by the Fourier transform method.

If K is a fundamental solution of Δ, we have $\Delta(k * f) = \Delta K * f = f$. Since $(\Delta g)\hat{}(\xi) = -4\pi^2|\xi|^2\hat{g}(\xi)$, we get

$$-4\pi^2|\xi|^2\hat{K}(\xi)\hat{f}(\xi) = \hat{f}(\xi),$$

32 so that, at least formally,

$$\hat{K}(\xi) = -1/(4\pi^2|\xi|^2).$$

We observe that when $n > 2$, the function $-1/(4\pi^2|\xi|^2)$ is locally integrable and so defines a tempered distribution. We want to show that its inverse Fourier transform is our fundamental solution K. For that purpose, we will prove a more general theorem.

Theorem 2.19. *For* $0 < \alpha < n$*, let* F_α *be the locally integrable function* $F_\alpha(\xi) = |\xi|^{-\alpha}$*. Then*

$$F_\alpha^\vee(x) = \left(\Gamma\left(\frac{1}{2}(n-\alpha)\right)/\Gamma\left(\frac{1}{2}\alpha\right)\right)\pi^{\alpha-n/2}|x|^{\alpha-n}.$$

Before proving this theorem, let us pause a minute to observe that it implies

$$\begin{aligned} F_2^\vee(x) &= \Gamma((n/2)-1)\pi^{2-n/2}|x|^{2-n} = (\Gamma(n/2)/(n/2-1))\pi^{2-n/2}|x|^{2-n} \\ &= (-4\pi^2/((2-n)\omega_n))|x|^{2-n}, \text{ so that } (-F_2/4\pi^2))^\vee = K \end{aligned}$$

as desired.

Proof of theorem 2.19. The idea of the proof is to express F_α as a weighted average of Gaussian functions, whose Fourier transforms we can compute.

To begin with,

$$\int_0^\infty e^{-rt} t^{\alpha-1} dt = \int_0^\infty e^{-s} s^{\alpha-1} r^{-\alpha} ds = r^{-\alpha}\Gamma(\alpha), (r > 0, \alpha > 0).$$

In other words, for any $r > 0$ and $\alpha > 0$, **33**

$$r^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-rt} t^{\alpha-1} dt.$$

Taking $r = \pi \mid \xi \mid^2$ and replacing α by $\alpha/2$, then

$$\mid \xi \mid^{-\alpha} = \frac{\pi^{\alpha/2}}{\Gamma(\alpha/2)} \int_0^\infty e^{-\pi|\xi|^2 t} t^{(\alpha/2)-1} dt$$

which is the promised formula for F_α as a weighted average of Gaussians. Formally, we can write

$$\begin{aligned}
\int_{\mathbb{R}^n} e^{2\pi i x.\xi} \mid \xi \mid^{-\alpha} d\xi &= \frac{\pi^{\alpha/2}}{\Gamma(\alpha/2)} \int_{\mathbb{R}^n} \int_0^\infty e^{2\pi i x.\xi_{e}-\pi|\xi|^2 t} t^{\alpha/2-1} dt d\xi \\
&= \frac{\pi^{\alpha/2}}{\Gamma(\alpha/2)} \int_0^\infty (e^{\pi|\xi|^2 t})^{v}(x) t^{(\alpha/2)-1} dt \\
&= \frac{\pi^{\alpha/2}}{\Gamma(\alpha/2)} \int_0^\infty e^{-\pi(|x|^2)/t}\ t^{-n/2}\ t^{\alpha/2-1} dt \\
&= \frac{\pi^{\alpha/2}}{\Gamma(\alpha/2)} \int_0^\infty e^{-\pi|x|^2 S} S^{(n-\alpha)/2-1} ds \\
&= \frac{\pi^{\alpha/2}}{\Gamma(\alpha/2)} \frac{\Gamma((n-\alpha)/2)}{n^{(n-\alpha)/2}} \mid x \mid^{\alpha-n}.
\end{aligned}$$

In this computation, the change of order of the integration is unfortunately not justified, because the double integral is not absolutely convergent. This is not surprising, since F_α is not an L^1 function; instead, we must use the definition of the Fourier transform for distributions.

For every $\phi \epsilon S$,

$$< e^{-\pi|\xi|^{2t}}, \hat{\phi} > = < e^{-\pi|\xi|^{2t}})\hat{}, \phi >,$$

i.e.,
$$\int\limits_{\mathbb{R}^n} e^{-\pi|\xi|^{2t}} \hat{\phi}(\xi) d\xi = \int\limits_{\mathbb{R}^n} t^{-n/2} e^{-(\pi/t)|x|^2} \phi(x) dx.$$

34 Now multiply both sides by $t^{(\alpha/2)-1}$ dt and integrate from 0 to ∞.

$$\int\limits_0^\infty \int\limits_{\mathbb{R}^n} e^{-\pi|\xi|^{2t}} t^{(\alpha/2)-1} \hat{\phi}(\xi) d\xi \, dt = \int\limits_0^\infty \int\limits_{\mathbb{R}^n} e^{-\pi/t|x|^2} t^{(\alpha-n)/2-1} \phi(x) dx \, dt.$$

The change of order in the integral is permitted now and integrating, we obtain,

$$\frac{\Gamma(\alpha/2)}{\pi^{\alpha/2}} \int\limits_{\mathbb{R}^n} \hat{\phi}(\xi) \mid \xi \mid^{-\alpha} d\xi = \frac{\Gamma((n-\alpha)/2)}{\pi^{(n-\alpha)/2}} \int\limits_{\mathbb{R}^n} \mid x \mid^{\alpha-n} \phi(x) \, dx$$

i.e.,
$$<\mid \xi \mid^{-\alpha}, \hat{\phi} > = < \frac{\Gamma((n-\alpha)/2)}{(\Gamma(\alpha/2)} \pi^{\alpha-n/2} \mid x \mid^{\alpha-n}, \phi > .$$

This shows that

$$F_\alpha^\vee(x) = \frac{\Gamma((n-\alpha)/2)}{\Gamma(\alpha/2)} \pi^{\alpha-n/2} \mid x \mid^{\alpha-n}$$

as desired.

This analysis does not suffice to explain $-(4\pi^2 \mid \xi \mid^2)^{-1}$ as (in some sense) the Fourier transform of $(2\pi)^{-1} \log \mid x \mid$ in the case $n = 2$. One way to proceed is to define $G_\alpha(\xi) = (2\pi \mid \xi \mid)^{-\alpha}$ and to study the behaviour of G_α and $G_\alpha^\vee$ as α tends to 2. This analysis can be carried out just as easily in n dimensions where the problem is to study G_α and G_α^v as α tends to n.

Let $R_\alpha = G_\alpha^\vee$ for $0 < \alpha < n$. (R_α called the *Riesz potential of order* α). By the preceding theorem, we have

$$R_\alpha(x) = \frac{(\Gamma((n-\alpha)/2)}{2^\alpha \pi^{n/2}\Gamma(\alpha/2)} \mid x \mid^{\alpha-n} .$$

35

Moreover, if $f \epsilon S$,

$$(-\Delta)^{\alpha/2}(f * R_\alpha) = f,$$

in the sense that $(2\pi \mid \xi \mid)^\alpha (f * R_\alpha)\hat{}(\xi) = \hat{f}(\xi)$.

As α tends to n, the Γ-function in R_α blows up. However, since $(2\pi \mid \xi \mid)^{\alpha_\delta} = 0$, we see that $(-\Delta)^{\alpha/2} 1 = 0$; so we can replace R_α by $R_\alpha - c, c$ being a constant, still having

$$(-\Delta)^{\alpha/2}(f * (R_\alpha - c)) = f.$$

If we choose $c = c_\alpha$ appropriately, we can arrange that $R_\alpha - c_\alpha$ will have a limit as α tends to n. In fact, let us take

$$c_\alpha = \frac{\Gamma((n-\alpha)/2)}{2^\alpha \pi^{n/2}\Gamma(\alpha/2)} \text{ and define } R'_\alpha = R_\alpha - c_\alpha.$$

Then

$$\begin{aligned} R'_\alpha(x) &= \frac{\Gamma((n-\alpha)/2)}{2^\alpha n^{n/2}\Gamma(a/2)} (\mid x \mid^{\alpha-n} -1) \\ &= \frac{2\Gamma((n-\alpha)/2)+1}{2^\alpha \pi^{n/2}\Gamma(\alpha/2)} \; \frac{(\mid x \mid)^{\alpha-n} - 1}{n-\alpha}. \end{aligned}$$

Letting α tend to n, we get in the limiting case

$$R'_n(x) = \frac{-2^{1-n}}{\pi^{n/2}\Gamma(n/2)} \log \mid x \mid .$$

when $n = 2, -R'_2(x)\dfrac{1}{2\pi} \log \mid x \mid$ and $\Delta(f * (-R'_2)) = f$, so we recover our fundamental solution for the case $n = 2$.

It remains to relate the function $G_n(\xi) = (2\pi \mid \xi \mid)^{-n}$ to the Fourier **36** transform of the tempered distribution F'_n. One way to make G_n into a distribution is the following.

Define a functional F on S by

$$< F, \phi >= \int\limits_{|\xi|\leq 1} \frac{\phi(\xi)-\phi(0)}{(2\pi \mid \xi \mid)^n} d\xi + \int\limits_{|\xi|>1} \frac{\phi(\xi)}{(2\pi \mid \xi \mid)^n} d\xi.$$

Note that the first integral converges, since, in view of the mean value theorem, $\mid \phi(\xi) - \phi(0) \mid \leq c \mid \xi \mid$. It is easy to see that F is indeed a tempered distribution. Further, we observe that when $\phi(0) = 0$, $< F, \phi >= \int \phi(\xi) G_n(\xi) d\xi$, i.e., F agrees with G_n on $\mathbb{R}^n \backslash \{0\}$.

Just as R'_n was obtained from R_n by subtracting off an infinite constant, F is obtained from G_n by subtracting off an infinite multiple of the δ function. This suggests that F is essentially the Fourier transform of R'_n. In fact, one has the following

Exercise. *Show that* $F - \widehat{(R'_n)}$ *is a multiple of* δ.

We now examine a few of the basic properties of harmonic functions, i.e., functions satisfying $\Delta u = 0$. *We shall need the following results from advanced calculus.*

Theorem 2.20 (DIVERGENCE THEOREM)**.** Let Ω be a bounded open set in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let v be the unit outward normal vector on $\partial\Omega$. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 function on $\bar{\Omega}$, *the closure of* Ω. Then we have

$$\int\limits_{\partial\Omega} F.\, v \, d\sigma = \int\limits_{\Omega} div\ F\ dx = \int\limits_{\Omega} \sum_{j=1}^{n} \frac{\partial F_j}{\partial x_j} dx.$$

37

CONSEQUENCES OF THEOREM 2.20

(2.21) When we take $F = \text{grad}\ u, F.v = \text{grad}\ u.v = \dfrac{\partial u}{\partial v}$, the normal derivative of u, and div F = div grad $u = \Delta u$. Therefore

$$\int\limits_{\partial\Omega} \frac{\partial u}{\partial v} d\sigma = \int_{\Omega} \Delta u\ dx.$$

(2.22) When we take $F = u \text{ grad } v - v \text{ grad } u$, div $F = u\Delta v - v\Delta u$. Therefore

$$\int\limits_{\partial\Omega} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) d\sigma = \int\limits_{\Omega} (u\Delta v - v\Delta u) dx.$$

This formula is known as *Green's formula.*

For harmonic functions, we have the following *mean value theorem*.

Theorem 2.23. Let u be harmonic in $B(x_0, r)$. Then,for any $\rho < r$, we have

$$u(x_0) = \frac{1}{\omega_n \rho^{n-1}} \int\limits_{|x-x_0|=\rho} u(x) d\sigma(x).$$

That is, $u(x_0)$ is the mean value of u on any sphere centred at x_0

Proof. Without loss of generality, assume that $x_0 = 0$. Since $\Delta u = 0$ and Δ is hypoelliptic, u is *a* C^∞ function. Now formally,

$$\begin{aligned} u(0) &=< \delta, u > \\ &= \int\limits_{|x|<\rho} \delta(x) u(x) dx \\ &= \int\limits_{|x|<\rho} \Delta K(x) u(x) dx \\ &= \int\limits_{|x|<\rho} (u\Delta K - K\Delta u) dx \\ &= \int\limits_{|x|=\rho} \left(u\frac{\partial K}{\partial \nu} - K\frac{\partial u}{\partial \nu} \right) d\sigma, \text{ by (2.22).} \end{aligned}$$

□ **38**

Of course, we are cheating here by applying Green's formula to the non-smooth function K. Nonetheless the result is correct, and we leave it as an exercise for the reader to justify it rigorously (either approximate K by C^∞ functions as in the proof of Theorem 2.16 or apply Green's formula to the region $\epsilon <| x |< \rho$ and let ϵ tend to 0.)

On the circle $\mid x \mid = \rho, K$ is a constant and so $\int\limits_{|x|=\rho} K\dfrac{\partial u}{\partial v}\, d\sigma =$ Const $\int\limits_{|x|=\rho} \dfrac{\partial u}{\partial v} d\sigma = \text{Const}\,. \int\limits_{|x|\leq\rho} \Delta\, u\; d\sigma = 0.$

Further on $\mid x \mid = \rho. \dfrac{\partial}{\partial v} = \dfrac{\partial}{\partial r}$, the radial derivative. Therefore

$$\frac{\partial K}{\partial v} = 1/(\omega_n \rho^{n-1}) \text{ on } \mid x \mid = \rho.$$

Thus we have

$$u(0) = \frac{1}{\omega_n \rho^{n-1}} \int\limits_{|x|=\rho} u(x) d\sigma \quad \text{as desired.}$$

As a corollary to the mean value theorem, we can derive the *maximum principle for harmonic functions.*

39 **Theorem 2.24.** *Suppose Ω is a connected open set in $\mathbb{R}^n$. Let u be a real-valued function harmonic in Ω. If $A = \sup\limits_{\Omega} u(x)$, then either $u(x) < A$ for all $x \epsilon \Omega$ or $u(x) \equiv A$ on Ω.*

Proof. Suppose that $u(x_0) = A$ for some $x_0 \epsilon \Omega$. From the mean value theorem

$$u(x_0) = \frac{1}{\omega_n \rho^{n-1}} \int\limits_{|x-x_0|=\rho} u(x) d\sigma(x)$$

where ρ is small enough so that $\{x : \mid x - x_o \mid \leq \rho\} \subset \Omega$. By our assumption, the integral does not exceed A. If $u(x_1) < A$ for some point of $B(x_0, \rho)$ then $u(x) < A$ in some neighbourhood of x_1 by continuity. Taking $r = \mid x_1 - x_0 \mid < \rho$, we have

$$u(x_0) = \frac{1}{\omega_n r^{n-1}} \int\limits_{|x-x_o|=r} u(x) d\sigma(x) < A,$$

a contradiction. Therefore, if we set $\Omega_1 = \{x \epsilon \Omega : u(x) = A\}$, then Ω_1 is an open subset of Ω and $\Omega_1 \neq \phi$. Further $\Omega_2 = \{x \epsilon \Omega : u(x) < A\}$ is also on open subset of Ω and $\Omega_1 \cup \Omega_2 = \Omega$. The connectedness of Ω and the non-emptiness of Ω_1 force Ω_2 to be empty. Thus $u(x) \equiv A$ on Ω. $\square$

Corollary 2.25. *If Ω is a bounded open set in $\mathbb{R}^n, u\epsilon C(\overline{\Omega})$ and $\Delta u = 0$ in Ω, then*

$$\sup_{\Omega} \mid u(x) \mid = \sup_{\partial\Omega} \mid u(x) \mid .$$

Proof. Since the function u is continuous on the compact set $\overline{\Omega}, \mid u(x) \mid$ attains its supremum on $\overline{\Omega}$ at some point x_0. By multiplying u by a constant, we may assume that $u(x_0) = \mid u(x_0) \mid$. □

If $x_0 \epsilon \partial\Omega$, the corollary is proved. Otherwise $x_0 \epsilon \Omega$ and by the previ- 40
ous theorem, on the connected component of containing x_0, Re $u(x) = u(x_0)$ and hence Im $u(x) = 0$. Since u is continuous on $\overline{\Omega}, u(x) = u(x_0)$ for all $x \epsilon \partial\Omega$. Thus $\sup\limits_{\Omega} \mid u(x) \mid = \sup\limits_{\partial\Omega} \mid u(x) \mid$.

Corollary 2.26. *If u and v are in $C(\overline{\Omega}), \Omega$ is bounded $\Delta u = \Delta v = 0$ on Ω and $u = v$ on $\partial\Omega$, then $u = v$ everywhere.*

Proof. Apply the previous corollary to $u - v$. □

The following boundary value problem for Laplace's equation, known as the *Dirichlet problem*, is of fundamental importance: given a function f on $\partial\Omega$, find a function u such that $\Delta u = 0$ on Ω and $u = f$ on $\partial\Omega$.

When Ω is bounded and $f \epsilon C(\partial\Omega)$ the uniqueness of the solution, if it exists at all, is assured by corollary 2.26. The problem of proving the existence of solution of the Dirichlet problem is highly nontrivial. We shall solve a special case, namely, when Ω is a half-space and then indicate how similar ideas can be applied for other regions.

First of all a word about notation: we will replace $\mathbb{R}^n$ by $\mathbb{R}^{n+1}$ with coordinates $(x_1, x_2, \ldots, x_n, t)$. Our Laplacian in $\mathbb{R}^{n+1}$ will be denoted by $\partial_t^2 + \Delta$ where $\partial_t = \dfrac{\partial}{\partial t}$ and $\Delta = \sum\limits_{j=1}^{n} \dfrac{\partial^2}{\partial x_j}$. Now our *Dirichlet Problem* is the following :

given a function f on $\mathbb{R}^n$, find a function u such that

$$(\partial_t^2 + \Delta)\, u(x,t) = 0, x \epsilon \mathbb{R}^n, t > 0 \tag{2.27}$$
$$u(x, o) = f(x), x \epsilon \mathbb{R}^n.$$

Since the half space is unbounded, the uniqueness argument given above does not apply. Indeed, without some assumption on the behaviour of u at ∞, uniqueness does not hold. For example, if $u(x, t)$ is a solution, so is $u(x, t) + t$. However, we have **41**

Theorem 2.28. *Let u be a continuous function on* $\mathbb{R}^n \times [0, \infty)$ *satisfying*

i) $(\Delta + \partial_t^2)u = 0$ *on* $\mathbb{R}^n \times (0, \infty)$,

ii) $u(x, o) = 0$ *for* $x \epsilon \mathbb{R}^n$, *and*

iii) *u vanishes at* ∞.

Then $U \equiv 0$ *on* $\mathbb{R}^n \times [0, \infty)$.

Proof. Applying the maximum principle for u on $B(0, R) \times (0, T)$, we see that maximum of $| u |$ on $B(0, R) \times (0, T)$ tends to zero as R, T tend ∞. Hence $u \equiv 0$. □

Remark 2.29. Hypothesis (iii) in Theorem 2.28 can be replaced by (iii): u is bounded on $\mathbb{R}^n \times [0, \infty)$. But this requires a deeper argument (See Folland [1]).

To solve the Dirichelt problem, we apply the Fourier transform in the variable x. We denote by $\tilde{u}(\xi, t)$ the Fourier transform

$$\tilde{u}(\xi, t) = \int e^{-2\pi i x.\xi} u(x, t) dx.$$

42 If we take Fourier transform of (2.27), we obtain

$$(\partial_t^2 - 4\pi^2 \mid \xi \mid^2)\tilde{u}(\xi, t) = o \text{ on } \mathbb{R}^n \times (0, \infty)$$
$$\tilde{u}(\xi, o) = \hat{f}(\xi) \text{ on } \mathbb{R}^n.$$

The general solution of the ordinary differential equation (ODE) $(\partial_t^2 - 4\pi^2 \mid \xi \mid^2)\, \tilde{u}(\xi, t) = 0$ is given by $\tilde{u}(\xi, t) = a(\xi)e^{-2\pi|\xi|t} + b(\xi)e^{2\pi|\xi|t}$ and the initial condition is $a(\xi) + b(\xi) = \hat{f}(\xi)$.

This formula for $\tilde{u}$ will define a tempered distribution in ξ, provided that as $| \xi |$ tends to ∞, $| a(\xi) |$ grows at most polynomially and $| b(\xi) |$

decreases faster than exponentially. We remove the non- uniqueness by requiring that u should satisfy good estimates in terms of f which are uniform in t-for example, that u should be bounded if f is bounded. This clearly forces $b(\xi) = 0$, since $e^{2\pi|\xi|t}$ blows up as t tends to ∞. Thus we take $\tilde{u}(\xi, t) = \hat{f}(\xi)e^{-2\pi|\xi|}t$. Taking the inverse Fourier transform in the variable ξ, $u(x, t) = (f * P_t)(x)$ where $P_t(x) = [e^{2\pi t(|.|)}]^{\vee}(x)$ is the *Poisson Kernel* for $\mathbb{R}^n \times [0, \infty)$.

We now compute P_t explicitly. When $n = 1$, this is easy:

$$\begin{aligned} P_t(x) &= \int\limits_{-\infty}^{0} e^{2\pi ix\xi} e^{2\pi\xi t} d\xi + \int\limits_{0}^{\infty} e^{2\pi ix\xi} e^{-2\pi\xi t} d\xi \\ &= \frac{1}{2\pi}[(t + ix)^{-1} + (t - ix)^{-1}] = \frac{t}{\pi}(x^2 + t^2)^{-1}. \end{aligned}$$

To compute $P_t(x)$ for the case $n > 1$, as in the proof of Theorem 2.19, the idea is to express $e^{-2\pi|\xi|t}$ as a weighted average of Gaussians. Here's how! 43

Lemma 2.30. *When $\beta > 0$, $e^{-\beta} = \int\limits_0^\infty \frac{e^{-s}e^{-\beta^2/4s}}{\sqrt{(\pi s)}} ds$.*

Proof. First, observe that

$$e^{-\beta} = \frac{1}{\pi} \int\limits_{-\infty}^{\infty} \frac{e^{i\beta\tau}}{\tau^2 + 1} d\tau.$$

□

This can be proved by using the residue theorem or by applying the Fourier inversion theorem to $P_1(x) = \frac{1}{\pi}(1 + x^2)^{-1}$ on $\mathbb{R}^1$. We also have

$$\frac{1}{\tau^2 + 1} = \int\limits_0^\infty e^{-(\tau^2+1)s} ds.$$

Therefore,

$$e^{-\beta'} = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty} e^{i2\pi\tau} e^{-(\tau^2+1)s} ds\, d\tau.$$

Putting $\tau = 2\pi\sigma$ and changing the order of integration,

$$e^{=\beta} = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} e^{i2\pi\sigma\beta} e^{-s} e^{-4\pi^2\sigma^2 s} d\sigma\, ds$$

i.e.,
$$e^{-\beta} = 2\int_{0}^{\infty} e^{-2}(4\pi s)^{-1/2} e^{\beta^2/4s} ds$$
$$= \int_{0}^{\infty} \frac{e^{-s}e^{\beta^2/4s}}{\sqrt{(\pi s)}} ds$$

which is the required expression.

Returning to the computation of $P_t(x)$, we have

$$P_t(x) = \int e^{2\pi i x.\xi} e^{-2\pi|\xi|t} d\xi.$$

44 Taking $\beta = 2\pi \mid \xi \mid t$ in the lemma, we get

$$P_t(x) = \int\int_{0}^{\infty} e^{2\pi i x.\xi} \frac{e^{-s}}{\sqrt{\pi s}} e^{(4\pi^2|\xi|^2 t^2)/4s} ds d\xi$$
$$= \int_{0}^{\infty} \frac{e^{-s}}{\sqrt{\pi s}} \int e^{2\pi i x\xi} e^{-(\pi^2|\xi|^2 t^2)/S} d\xi . ds$$
$$= \int_{0}^{\infty} \frac{e^{-s}}{\sqrt{\pi s}} \left(\frac{\pi t^2}{s}\right) - n/2 e^{-(s|x|^2)/t^2} ds$$
$$= \pi^{-(n+1)/2} t^{-n} \int_{0}^{\infty} e^{-s(1+(|x|^2/t^2))} s^{(n-1)/2}\ ds$$

$$= \frac{\pi^{-(n+1)/2} t^{-n} \Gamma(n+1)/2}{\left(1+\frac{|x|^2}{t^2}\right)(n+1/2)}.$$

Thus we have

$$P_t(x) = \Gamma((n+1)/2)\pi^{(n+1)/2} t/(t^2 + |\,x\,|^2)^{(n+1)/2} \tag{2.31}$$

In particular, we see that $P_t \epsilon L^1 \cap L^\infty$, so that $f * P_t$ is well defined if, for example, $f \epsilon L^p, 1 \leq p \leq \infty$. That $(\Delta + \partial_t^2)P_t = 0$ follows by taking the Fourier transform, and hence

$$(\Delta + \partial_t^2)(f * P_t) = f * (\Delta + \partial_t^2)P_t = 0 \text{ for any } f.$$

Further note that

$$P_t(x) = t^{-n} P_1(x/t) \text{ and } \int P_1(x)dx = \hat{P}_1(0) = 1.$$

Therefore, by Theorem 1.6, when $f \epsilon L^p, 1 \leq p < \infty$, $f * P_t$ tends to f in the L^p norm and when f is bounded and continuous $f * P_t$ tends to f uniformly on compact sets as t tends to 0.

If we take f continuous and bounded, then for $u(x,t) = P_t * f$, we have $\lim\limits_{t \to 0} u(x,t) = u(x,0) = f(x)$. Thus the function $u(x,t) = (f * P_t)(x)$ 45
is a solution of the Dirichlet Problem for the half space.

Remark 2.32. The Poisson Kernel P_t is closely related to the fundamental solution K_{n+1} of the Laplacian in $\mathbb{R}^{n+1}$. Indeed

$$K_{n+1}(x,t) = \frac{1}{(1-n)\omega_{n+1}}(|x|^2 + t^2)^{-((n-1)/2)}$$

and hence

$$P_t(x) = 2\partial_t K_{n+1}(x,t).$$

Exercise. *Check the above equation using the Fourier transform. (Start with $(\partial_t^2 + \Delta)K_{n+1} = \delta(x)\delta(t)$ and take the Fourier transform in the variable x to obtain*

$$(\partial_t^2 - 4\pi^2|\xi|^2)\tilde{K}_{n+1} = \delta(t).$$

Solve this equation to obtain

$$\tilde{K}_{n+1}(\xi, t) = \frac{e^{2\pi|\xi||t|}}{4\pi|\xi|}.$$

Then

$$\partial_t \tilde{K}_{n+1} = \frac{1}{2} e^{-2\pi|\xi|t}, t > 0, \textit{so that } \partial_t \tilde{K}_{n+1} = \frac{1}{2}\hat{P}_t).$$

Our formula (2.31) for P_t makes sense even when $t < 0$ and we have $P_{-t}(x) = -P_t(x)$, so that $\lim\limits_{t \to o\pm} f * P_t = \pm f$. We further observe that

$$f *_{(x)} P_t = f(x)\delta(t) *_{(x,t)^2} \partial_t K_{n+1}(x, t)$$

where $*_{(}x)$ and $*_{(x,t)}$ mean convolution on $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$ respectively.

46 Therefore,

$$f *_{(x)} P_t = f(x)\delta'(t) *_{(x,t)} 2K_{n+1}(x, t)$$

i.e.,
$$u(x, t) = 2 f(x)\delta'(t) *_{(} x, t) K_{n+1}(x, t).$$

Form this, we

$$(\delta_t^2 + \Delta)u(x, t) = 2 f(x)\delta'(t) * \delta(x)\delta(t) = 2 f(x)\delta'(t).$$

Exercise. *Show directly that if*

i) $(\Delta + \partial_t^2)u = 0$ *on* $\mathbb{R}^{n+1} \backslash \mathbb{R}^n \times \{0\}$

ii) $u(x, -t) = -u(x, t)$

iii) $\lim\limits_{t \to 0\pm} u(x, t) = \pm f(x)$, *then u is a distribution solution of* $(\partial_t^2 + \Delta)u = 2 f(x)\delta'(t)$.

[To avoid technicalities, assume f is sufficiently smooth so that $\lim\limits_{t \to 0\pm} \dfrac{\partial u}{\partial t}$ exists; e.g. $f \epsilon C^2$ is sufficient].

We now indicate, without giving any proof, how these ideas can be used to solve the Dirichlet Problem in a bounded open set Ω in $\mathbb{R}^2$. We assume that the boundary $\partial\Omega$ of Ω is of class C^2.

We know that the solution of the Dirichlet Problem in the case when $\Omega = \mathbb{R}^{n-1} \times (0, \infty)$ is given by $u = f * 2\dfrac{\partial K}{\partial x_n}$ (the convolution is in $\mathbb{R}^{n-1}$). we note that $\dfrac{\partial K}{\partial x_n}$ is the inward normal derivative of K. For bounded Ω with C^2 boundary let us consider

$$u(x) = 2 \int\limits_{\partial\Omega} f(y) \frac{\partial}{\partial \nu_y} K(x-y) d\sigma(y).$$

Here $\sigma(y)$ is the surface measure on the boundary, ν is the unit outward normal on $\partial\Omega$, and $\dfrac{\partial\phi}{\partial\nu_y}(x,y) = \Sigma \nu_j \dfrac{\partial\phi}{\partial y_j}$. The minus sign in $K(x-y)$ compensates the switch from inward to outward normal. 47

Since $\Delta K(x-y) = \delta(x-y)$ we see that $\Delta u = 0$ in Ω. What about the behaviour of u on $\partial\Omega$? It turns out that if u is defined as above, then

$$u \epsilon C(\overline{\Omega}) \text{ and } u|_{\partial\Omega} = f + Tf$$

where T is a compact integral operator on $L^2(\partial\Omega)$ or $C(\partial\Omega)$. Hence if we can find ϕ in $C(\partial\Omega)$ such that $\phi + T\phi = f$, and we set

$$u(x) = 2 \int\limits_{\partial\Omega} \phi(y) \frac{\partial}{\partial \nu_y} K(x-y) d\sigma(y)$$

then u satisfies $\Delta u = 0$ in Ω and further $u|_{\partial\Omega} = \phi + T\phi = f$. Hence the Dirichlet problem is reduced to solving the equation $\phi + T\phi = f$, and for this purpose, the classical *Fredholm - Riesz theory* is available. The upshot is that a solution to the Dirichlet problem always exists (provided, as we have assumed, that $\partial\Omega$ is of class C^2). See Folland [1] for a detailed treatment.

5 The Heat Operator

The *Heat operator* is given by $\partial_t - \Delta$. We want to find a distribution K such that $(\partial_t - \Delta)K = \delta(x)\delta(t)$. Taking the Fourier transform in both variables we have 48

(2.33) $$\hat{K}(\xi, \tau) = (2\pi i + 4\pi^2 |\xi|^2)^{-1}.$$

Exercise. *Show that $\hat{K}$ is locally integrable near the origin, and hence defines a tempered distribution.*

$\hat{K}$ is not globally integrable, however; so computing its inverse Fourier transform directly is a bit tricky. Instead, if we take the Fourier transform in the variable x only,

$$(\partial_t + 4\pi^2|\xi|^2)\,\tilde{K}(\xi, t) = \delta(t).$$

we can solve this *ODE* get

$$\tilde{K}(\xi, t) = \begin{cases} a(\xi)e^{-4\pi^2|\xi|^2 t}, & t > 0 \\ b(\xi)e^{-4\pi^2|\xi|^2 t}, & t < 0 \end{cases}$$

with $a(\xi) - b(\xi) = 1$.

As in the previous section, there is some latitude in the choice of a and b, but the most natural choice is the one which makes $\tilde{K}$ tempered in t as well as ξ, namely $a = 1$, $b = 0$. So,

$$\tilde{K}(\xi, t) = \begin{cases} e^{-4\pi^2|\xi|^2 t}, t > 0 \\ 0 \text{ otherwise }. \end{cases}$$

49 Taking the inverse Fourier transform,

$$K(x, t) = \begin{cases} (4\pi t)^{-n/2} e^{(|x|^2/4t)}, & t > 0 \\ 0, & t > 0. \end{cases}$$

This is a fundamental solution of the heat operator.

Remark 2.34. If we take the Fourier transform of $\tilde{K}(\xi, t)$ in the t variable, we obtain

$$\begin{aligned} \tilde{K}(\xi, \tau) &= \int_{-\infty}^{\infty} \tilde{K}(\xi, t) e^{2\pi\,\mathrm{i}\,t\,\tau} dt \\ &= \int_0^{\infty} e^{-t(4\pi^2|\xi|^2 + \pi\, i\, \tau)} dt \\ &= (4\pi^2|\xi|^2 + 2\pi i\tau)^{-1}, \end{aligned}$$

thus recovering formula (2.33).

Exercise . *Show that this really works, i.e., the iterated Fourier transform of K first in x, then in t, is the distribution Fourier transform of K in all variables.*

OBSERVATIONS: From the formula for K, we get

(1) $K(x,t)$ vanishes to infinite order as t tends to 0 when $x \neq 0$ and hence is C^∞ on $\mathbb{R}^{n+1} \backslash \{(0,0)\}$. Therefore by Theorem 2.11, $\partial_t - \Delta$ is hypoelliptic.

(2) $K(x,t) = t^{-n/2} K\left(\dfrac{x}{t}, 1\right)$. Thus if we set $K(x,1) = \phi(x)$, then $K(x,\epsilon^2)$ $= \epsilon^{-n}\phi(x/\epsilon) = \phi_\epsilon(x)$; so, as t tends to $0, K(x,t)$ tends to δ, Theorem 1.6. From this, we infer 50

(3) If $f \epsilon L^p$ and if we set $u(x,t) = f *_{(x)} K(x,t)$, then

$$\begin{aligned}(\partial - \Delta)u &= 0 \text{ for } t > 0\\ u(x,0) &= f(x).\end{aligned}$$

i.e., as t tends to $0, \|u(.,t) - f\|_P$ converges to 0. Thus we have solved the initial value problem for the homogeneous heat equation, when $f \epsilon L^p$. Actually, since $K(x,t)$ decreases so rapidly as $|x|$ tends to ∞, this works for much wider classes of $f's$. The convolution $f *_{(x)} K(.,t)$ make sense, for example, if $|f(x)| < Ce^{|x|^{2-\epsilon}}$. If f is also continuous, it is not hard to see that $f *_{(x)} K(.,t)$ converges to f uniformly on compact sets as t tends to 0.

It is now a simple matter to solve the inhomogeneous initial value problem:

$$\begin{aligned}(\partial_t - \Delta)u &= F \quad \text{on } \mathbb{R}^n \times (0,\infty),\\ u(x,0) &= f(x) \text{ on } \mathbb{R}^n.\end{aligned}$$

If we take $u_1 = F *_{(x,t)} K$ and

$$u_2 = (f - u_1(.,0)) *_{(x)} K,$$

then we see that

$$(\partial_t - \Delta)u_1 = F \text{ on } \mathbb{R}^n \times (0,\infty), (\partial_t - \Delta)u_2 = 0$$

on $\mathbb{R}^n \times (0,\infty)$, and $(u_1 + u_2)(x,0) = f(x)$. Thus $u = u_1 + u_2$ solves the **51**
problem.

As another application of the fundamental solution K, we can derive the *Weierstrass approximation theorem*.

Theorem 2.35. (WEIERSTRASS) If f is continuous with compact support on $\mathbb{R}^n$, then, for any compact $\Omega \subset \mathbb{R}^n$, there exists a sequence (P_m) of polynomials such that P_m converges to f uniformly on Ω

Proof. Let $u(x,t) = (f *_{(x)} K(.,t))(x)$. Then $u(x,t)$ converges to $f(x)$ uniformly as t tends to 0. Moreover, for any $t > 0$,

$$u(x,t) = \int f(y)(4\pi t)^{-n/2} e^{-\sum_{j=1}^{n}(x_j - y_j)^2/4t} dy$$

is an entire holomorphic function of $x \in \mathbb{C}^n$. So $u(.,t)$ can be uniformly approximated on any compact set by partial sums of its Taylor series. ◻

6 The Wave Operator

If we take the Fourier transform of the equation

$$(\partial_t^2 - \Delta)K = \delta(x)\delta(t)$$

in both variables x and t, we have, formally

$$\hat{K}(\xi,\tau) = (4\pi^2|\xi|^2 - 4\pi^2\tau^2)^{-1}. \tag{2.36}$$

52 This function $\hat{K}$ is not locally integrable, so it is not clear how to interpret it as a distribution. Again, it is better to take the Fourier transform in the x variable obtaining

$$(\partial_t^2 + 4\pi^2|\xi|^2)\tilde{K}(\xi,t) = \delta(t).$$

Solving this *ODE*,

$$\tilde{K}(\xi, t) = a_{\pm}(\xi)e^{2\pi i|\xi|t} + b_{\pm}(\xi)e^{-2\pi i|\xi|t} \text{ for } t/|t| = \pm 1$$

and the coefficients $a_{\pm}$, $b_{\pm}$ must be determined so that

$$\tilde{K}(\xi, 0+) = \tilde{K}(\xi, 0-), \partial_t\tilde{K}(\xi, 0+) - \partial_t K(\xi, 0 =) = 1.$$

This gives two equations in four unknowns. In contrast to the situation with the Dirichlet problem and the heat operator, there is no way to narrow down the choices further by imposing growth restrictions on K as $|t|$ tends to ∞. Rather, it is a characteristic feature of the wave operator that one can adapt the choice of fundamental solution to the problem at hand. The two which we shall use, called K_+ and $K_,$are the ones supported in the half - space $t \geq 0$ and $t \leq 0$. K_+ and K_- are thus determined by the requirements $a_- = b_- = 0$ and $a_+ = b_+ = 0$ respectively, from which one easily sees that

$$\tilde{K}_+(\xi, t) = H(t)\frac{\sin 2\pi|\xi|t}{2\pi|\xi|}$$
$$\tilde{K}_-(\xi, t) = -H(-t)\frac{\sin 2\pi|\xi|t}{2\pi|\xi|} = \tilde{K}_+(\xi, -t)$$

where H is the *Heaviside function*, i.e., the characteristic function of **53**
$[0, \infty)$. Let us compute the Fourier transforms of $\tilde{K}_+$ and $\tilde{K}_-$ in t to see how to make sense out of (2.36). $\tilde{K}_+$ and $\tilde{K}_-$ are not integrable on t, but it is easy to approximate them in the distribution topology by integrable functions whose Fourier transforms we can calculate.Indeed, if we set

$$\tilde{K}_+^{\epsilon}(\xi, t) = e^{-2\pi\epsilon t}H(t)\frac{\sin 2\pi|\xi|t}{2\pi|\xi|}, \epsilon > 0,$$

then $\tilde{K}_+^{\epsilon}$ is an integrable function and $\tilde{K}_+^{\epsilon}$ converges to $\tilde{K}_+$ is S' as ϵ tends to 0. Therefore $\tilde{K}_+ = \lim\limits_{\epsilon\to 0}\tilde{K}_+^{\epsilon}$ where

$$\hat{K}_+(\xi, \tau) = \int\limits_0^{\infty} e^{2\pi\epsilon t - 2\pi \text{ it } \tau}\frac{\sin 2\pi|\xi|t}{2\pi|\xi|}dt$$

$$= \int\limits_0^\infty e^{2\pi\epsilon t - 2\pi \text{ it } \tau} \frac{(e^{\pi i|\xi|t} - e^{\pi i|\xi|t})}{4\pi i|\xi|} dt$$
$$= (4\pi^2)^{-1}(|\xi|^2 - (\tau - i\epsilon)^2)^{-1}.$$

Exercise. *Prove that* $\hat{K}_- = \lim\limits_{\epsilon \to 0} \hat{K}^\epsilon_-$ *in* S' *where*

$$\hat{K}^\epsilon_-(\xi, \tau) = (4\pi^2)^{-1}(|\xi|^2 - (\tau + i\epsilon)^2)^{-1}.$$

Thus we have two distinct ways of making the function $[4\pi^2(|\xi|^2 - \tau^2)]^{-1}$ into a tempered distribution. The difference $\tilde{K}_+ - \tilde{K}_-$ is of course a distribution supported on the cone $|\xi| = |\tau|$.

54 We now propose to use the fundamental solutions K_+ and K_- to solve the *Initial Value Problem or Cauchy Problem*, for the operator:

$$(2.37) \qquad \begin{aligned} (\partial_t^2 - \Delta)u &= f \text{ on } \mathbb{R}^{n+1} \\ u(x, o) &= u_o(x), \\ \partial_t u(x, 0) &= u_1(x), \end{aligned}$$

where u_0, u_1, f are given functions.

For the moment, we assume that $u_0, u_1 \epsilon S$ and $f \epsilon C^\infty(\mathbb{R}^n_x))$ (That is, $t \to f(., t)$ is a C^∞ function with values in $S(\mathbb{R}^n)$).

Taking the Fourier transform in the variable x in (2.37),

$$\begin{aligned} (\partial_t^2 + 4\pi^2|\xi|^2)\,\tilde{u}(\xi, t) &= \tilde{f}(\xi, t) \\ \tilde{u}(\xi, 0) &= \tilde{u}_0(\xi) \\ \partial_t \tilde{u}(\xi, 0) &= \tilde{u}_0(\xi). \end{aligned}$$

When $f = 0$, the general solution of the *ODE* is

$$\tilde{u}(\xi, t) = A(\xi)\sin 2\pi|\xi|t + B(\xi)\cos 2\pi|\xi|t,$$
$$A(\xi) = \frac{\hat{u}_1(\xi)}{2\pi|\xi|} \text{ and } B(\xi) = \tilde{u}_0(\xi).$$

when $u_0 = u_1 = 0$, the solution is

$$\tilde{u}(\xi, t) = \int\limits_0^t \tilde{f}(\xi, s) \frac{\sin(2\pi|\xi|(t - s))}{2\pi|\xi|} ds$$

55 (This may be derived by variation of parameters; in any case it is easy to check that this u is in fact the solution). Therefore, the solution for the general case is given by

$$\tilde{u}(\xi, t) = \hat{u}_0(\xi)\cos 2\pi|\xi|t + \frac{\hat{u}_1(\xi)}{2\pi|\xi|}\sin 2\pi|\xi|t + \\ + \int_0^t \tilde{f}(\xi, s)\frac{\sin(2\pi|\xi|(t-s))}{2\pi|\xi|}ds.$$

For $t > 0$, we can rewrite this as

$$\tilde{u}(\xi, t) = \hat{u}_1(\xi)\tilde{K}_+(\xi, t) + \hat{u}_0(\xi)\partial_t\tilde{K}_+(\xi, t) + \int_0^t \tilde{f}(\xi, s)\tilde{f}(\xi, s)K_+(\xi, t-s)ds.$$

Since $K_+(\xi, t-s) = 0$ for $t < s$,

$$\tilde{u}(\xi, t) = \hat{u}_1(\xi)\tilde{K}_+(\xi, t) + \hat{u}_0(\xi)\partial_t\tilde{K}_+(\xi, t) + \int_0^t \tilde{f}(\xi, s)\tilde{f}(\xi, s)K_+(\xi, t-s)ds.$$

Take the inverse Fourier transform:

$$u(x, t) = (u_1 *_{(x)} K_+) + u_0 *_{(x)} \partial_t K_+) + (H(t)f *_{(x,t)} K_+).$$

Likewise, for $t < 0$

$$u(x, t) = -(u_1 *_{(x)} K_-) - u_0 *_{(x)} \partial_t K_-) + ((H(-t)f) *_{(x,t)} K_-).$$

So, for arbitrary t, our solution u can be expressed as

$$\begin{aligned} u(x, t) = (u_1 *_{(x)} (K_+ - K_)) + u_0 *_{(x)} \partial_t(K_+ - K_-) \\ + (H(t)f *_{(x,t)} K_+) + (H(-t)f *_{(x,t)} K_)). \end{aligned} \tag{2.38}$$

So far, we have avoided the question of computing K_+ and K_- explicitly. Indeed, since $\tilde{K}_+$ and $\tilde{K}_-$ are not L^1 functions, it is not an easy matter to calculate their inverse Fourier transform, 56

However, for the case $n = 1$, we can find K_+ and K_- by solving the wave wave equation directly. We have

$$\partial_t^2 - \Delta = \partial_t^2 - \partial_x^2.$$

If we make the change of variables $\xi = x+t, n = x-t$, then the wave operator becomes $\partial_t^2 - \partial_x^2 = -4\dfrac{\partial^2}{\partial\xi\partial\eta}$. The general solution of $\dfrac{\partial^2 u}{\partial\xi\partial\eta} = 0$ is given by

$$u(\xi, n) = f(\xi) + g(\eta)$$

where f and g are arbitrary functions. Therefore

$$u(x,t) = f(x+t) + g(x-t).$$

To solve $(\partial_t^2 - \partial_x^2)u = 0, u(x,0) = u_0(x), \partial_t u(x,0) = u_1(x)$ we must have

$$\begin{aligned} u_0(x) &= f(x) + g(x) \\ u_1(x) &= f'(x) - g'(x). \end{aligned}$$

From these equations, we have

$$\begin{aligned} f'(x) &= \frac{1}{2}(u_0'(x) + u_1(x)) \\ g'(x) &= \frac{1}{2}(u_0'(x) - u_1(x)). \end{aligned}$$

57 Thus $u(x,t)$ is given by

$$u(x,t) = \frac{1}{2}(u_0(x+t) + u_0(x-t)) + \frac{1}{2}\int\limits_{x-t}^{x+t} u_1(s)ds.$$

Comparing with the previous formula (2.38), we find that

$$K_+(x,t) = \frac{1}{2}H(t-|x|), K_-(x,t) = \frac{1}{2}H(-t-|x|).$$

Exercise. *Compute $\tilde{K}_\pm$ directly from these formulas.*

It turns out that, for $n = 2$,

$$K_{\pm}(x,t) = \frac{1}{2\pi\sqrt{t^2 - x^2}} H(\pm t - |x|),$$

and, for $n = 3$,

$$K_{\pm}(x,t) = \frac{\pm 1}{4\pi t}\delta(\pm t - |x|).$$

For $n = 1, 2, K_+, K_-$ are functions; for $n = 3, K_+, K_-$ are not functions but they are measures. For $n > 3, K_+, K_-$ are neither functions nor measures; they are more singular distributions. The exact formula for $K_\pm$ is rather messy and we shall not write it out; it may be read off from Theorem 5.13 and 5.14 of Folland [1] in view of our formula (2.38). *The most important qualitative feature of* $K_\pm$, however, *is that it is always supported in the light cone* $\{(x,t) : \pm t > |x|\}$ and we shall now prove this as a consequence of the following result.

Theorem 2.39. *Suppose u is a C^2 function on $\{(x,t) : t \geq o\}$ such that $(\partial_t^2 - \triangle)u = 0$ for $t > 0$ and $u = \partial_t u = 0$ on the set $B_0 = \{(x,0) : |x - x_0| \leq t_0\}$. Them u vanishes on $\Omega = \{(x,t) : 0 \leq t \leq t_0, |x - x_0| \leq t_0 - t\}$.* **58**

Proof. Assume u is real valued; otherwise, we can consider the real and imaginary parts separately. Let

$$B_t = \{x : |x - x_0| \leq t_0 - t\} \text{ and } E(t) = \frac{1}{2}\int\limits_{B_t} |\operatorname{grad}_{x,t} u|^2 dx$$

i.e.,
$$E(t) = \frac{1}{2}\int_{B_t}\left[(\partial_t u)^2 + \sum_1^n \left(\frac{\partial u}{\partial x_j}\right)^2\right] dx. \qquad \square$$

Then

$$\frac{dE}{dt} = \frac{1}{2}\int\limits_{B_t} 2\left[\frac{\partial u}{\partial t}.\frac{\partial^2 u}{\partial_t^2} + \sum_1^n \frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_j \partial t}\right] dx - $$
$$- \frac{1}{2}\int\limits_{\partial B_t}\left[\left(\frac{\partial u}{\partial t}\right)^2 + \sum_1^n \left(\frac{\partial u}{\partial x_j}\right)^2\right] d\sigma(x).$$

(The second term comes from the change in the region B_t. If this is not clear, write the derivative as a limit of difference quotients and work it our).

Since

$$\sum \frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_j \partial t} + \sum \frac{\partial^2 u}{\partial_{x_j^2}}\frac{\partial u}{\partial t} = div\left(\frac{\partial u}{\partial t} grad_x u\right),$$

applying the divergence theorem and using $(\partial_t^2 - \triangle)u = 0$, we obtain

$$\frac{dE}{dt} = \int\limits_{\partial B_t} \left[\frac{\partial u}{\partial t}\frac{\partial u}{\partial \nu} - \frac{1}{2}|\operatorname{grad}_{x,t} u|^2\right] d\sigma$$

59 where ν is the unit normal to B_t in $\mathbb{R}^n$. Now

$$\begin{aligned} |\frac{\partial u}{\partial t}\frac{\partial u}{\partial \nu}| &< \frac{1}{2}\left[|\frac{\partial u}{\partial t}|^2 + |\frac{\partial u}{\partial \nu}|^2\right] \\ &< \frac{1}{2}\left[|\frac{\partial u}{\partial t}|^2 + |\operatorname{grad}_x u|^2 = \frac{1}{2}|\operatorname{grad}_{x,t} u|^2\right]. \end{aligned}$$

Thus we see that the integrand is non-positive, and hence $\frac{dE}{dt} \leq 0$. Also $E(0) = 0$ since $u = \partial_t u = 0$ on B_0, so $E(t) \leq 0$. But $E(t) \geq 0$ by definition, so $E(t) = 0$. This implies that $\operatorname{grad}_{x,t} u = 0$ on $\Omega = \bigcup\limits_{t \leq t_0} B_t$ and since $u = 0$ on B_0, we conclude that $u = 0$ on Ω.

Corollary 2.40. *Suppose* $u \epsilon C^2$ *on* $\mathbb{R}^n \times [0, \infty), (\partial_t^2 - \triangle)u = 0$ *for* $t > 0, u(x, 0) = u_0(x), \partial_t u(x, 0) = u_1(x)$. *If* $\Omega_0 = (\operatorname{supp} u_0) \cup (\operatorname{supp} u_1)$, *then* $\operatorname{supp} u \subset \Omega = \{(x, t) : d(x, \Omega_0) \leq t\}$.

(The set Ω is the union of the forward light cones with vertices in Ω_0).

Proof. Suppose $(x, t_0) \notin \Omega$. Then for some $\epsilon > 0$, the set

$$B_0 = \{x : d(x, x_0) \leq t_0 + \epsilon\} \text{ is disjoint from } \Omega_0.$$

Therefore, by Theorem 2.39, $u = 0$ on the cone

$$\{(x,t) : |x - x_0| \leq t_0 + \epsilon - t, 0 \leq t \leq t_0 + \epsilon\}.$$

In particular, $u = 0$ on a neighbourhood of (x_0, t_0), i.e., (x_0, t_0) is not in the support of u. ◻

Corollary 2.41.

$$\begin{aligned} &\mathrm{Supp} K_+ \subset \{(x,t) : t \geq |x|\} \\ &\mathrm{Supp} K_+ \subset \{(x,t) : -t \geq |x|\}. \end{aligned}$$

60

Proof. Pick $a\phi\epsilon C_0^\alpha(B(0,1))$ such that $\int \phi = 1$. Put

$$\phi_\epsilon(x) = \epsilon^{-n}\phi(\epsilon^{-1}x). \text{ Let } u_\epsilon(x,t) = \phi_\epsilon *_{(x)} K_+, t > 0.$$

◻

Then $(\partial_t^2 - \triangle)u_\epsilon = 0, u(x,0) = 0, \partial_t u(x,0) = \phi_\epsilon(x)$ and u_ϵ is C^α. By the previous corollary

$$\mathrm{supp}\, u_\epsilon \subset \{(x,t) : |x| \leq t + \epsilon\}.$$

Now u_ϵ converges to K_+ in S', as ϵ tends to 0. Therefore,

$$\mathrm{supp}\, K_+ \subset \{(x,t) : |x| \leq t\}.$$

The result for K_- follows then since $K_-(x,t) = K_+(x,-t)$.

Remarks 2.42. (i) One could also deduce the above result from our formulas for $\tilde{K}_\pm$ by using the *Paley-Wiener theorem*.

(ii) Actually for $n = 3, 5, 7, \ldots$, it turns out that $\mathrm{supp}\, K_\pm = \{(x,t) : |x| = \pm t\}$. This is known as the *Huygens principle*. See Folland [1].

(iii) The distributions $K_{\pm}$ are smooth functions of t (except at $t = 0$) with values in $E'(\mathbb{R}^n)$. Therefore, we now see that our formula (2.38) for the solution of the Cauchy problem makes sense even when $u_0, u_1, \epsilon D'(\mathbb{R}^n)$ and $f \epsilon C(\mathbb{R}_t, D'(\mathbb{R}^n_x))$, and it is easily checked **61**
that u thus defined still solves the Cauchy problem in the sense of distributions. Corollary 2.40 remains valid in this more general setting, as can seen by an approximation argument as in the proof of Corollary 2.41.

EXERCISE If $(\partial_t^2 - \triangle)u = f$ on $\mathbb{R}^{n+1}$, $u(x,0) = u_0(x)$, and $\partial_t u(x,0) = u_1(x)$, figure out how $\operatorname{supp} u$ is related to $\operatorname{supp} f$, $\operatorname{supp} u_0$ and $\operatorname{supp} u_1$.

Chapter 3

L^2 Sobolev Spaces

THERE ARE MANY ways of measuring smoothness properties of functions in terms of various norms. Often it is convenient to use L^2 norms, since L^2 interacts nicely with the Fourier transforms. In this chapter, we set up a precise theory of L^2 differentiability and use it to prove Hormander's theorem on the hypoellipticity of constant coefficient differential operators. **62**

1 General Theory of L^2 Sobolev Spaces

Definition 3.1. *For a non-negative integer k, the* Sobolev space H_k *is defined to be the space of all tempered distributions all of whose derivatives of order less than or equal to k are in L^2.*

Thus
$$H_k = \{f \epsilon S' : D^\alpha f \epsilon L^2(\mathbb{R}^n) \text{ for } 0 \leq |\alpha| \leq k\}.$$

From the definition, we note that $f \epsilon H_k$ if and only if $\xi^\alpha \hat{f}(\xi) \epsilon L^2$ for $0 \leq |\alpha| \leq k$.

Proposition 3.2. $f \epsilon H_k$ if and only if $(1 + |\xi|^2)^{k/2} \hat{f} \epsilon L^2$.

Proof. First assume that $(1 + |\xi|^2)^{k/2} \hat{f} \epsilon L^2$. Since, for $|\xi| \geq 1$,
$$|\xi^\alpha| \leq |\xi|^k \leq (1 + |\xi|^2)^{k/2} \text{ for all } |\alpha| \leq k$$

and for $|\xi| < 1$, **63**

$$|\xi^\alpha| \leq 1 \leq (1+|\xi|^2)^{k/2} \text{ for all } |\alpha| \leq k,$$

we have
$$|\xi^\alpha \hat{f}(\xi)| \leq (1+|\xi|^2)^{k/2} \hat{f}(\xi)$$

and hence
$$\xi^\alpha \hat{f} \epsilon L^2 \text{ for } |\alpha| \leq k \text{ which implies } f \epsilon H_k. \qquad \square$$

$\square$

Conversely, assume that $f \epsilon H_k$. Since $|\xi|^k$ and $\sum\limits_{j=1}^{n} |\xi_j|^k$ are homogeneous of degree k and nonvanishing for $\xi \neq 0$, we have

$$(1+|\xi|^2)^{k/2} \leq c_0(1+|\xi|^k) \leq c_0 \left(1 + c\sum_{j=1}^{n} |\xi_j|^k\right)$$

so that
$$\|(1+|\xi|^2)^{k/2} \hat{f}\|_2 \leq c_0 \|f\|_2 + c_0 c \sum_{j=1}^{n} \|\partial_j^k f\|_2$$

which shows that $(1+|\xi|^2)^{k/2} \hat{f} \epsilon L^2$.

The characterisation of H_k given in the above proposition immediately suggests a generalisation to non-integral values of k which turns out to be very useful.

Definition 3.3. *For $s \epsilon \mathbb{R}$, we define the operator $\Lambda^s : S \to S$ by*

$$(\Lambda^s f)\hat{}(\xi) = (1+|\xi|^2)^{s/2} \hat{f}(\xi).$$

64 *In other words, $\Lambda^s = \left(I - \dfrac{\triangle}{4\pi^2}\right)^{s/2}$. Clearly Λ^s maps S continuously onto itself. We can therefore extend Λ^s continuously from S' onto itself.*

The *Sobolev space of order s* is defined by

$$H_s = \{f \epsilon S' : \Lambda^s f \epsilon L^2\}.$$

We equip H_s with the norm $\|f\|_{(s)} = \|\Lambda^s f\|_2$. If s is a positive integer, the proof of Proposition 3.2 shows that $\| \quad \|_{(s)}$ is equivalent to the norm

$$\|f\| = \Sigma_{0 \leq |\alpha| \leq s} \|D^\alpha f\|_2.$$

PROPERTIES OF H_s

(i) H_s is a Hilbert space with the scalar product defined by $(u, v)_{(s)} = (\Lambda^s u, \Lambda^s v)$. Here the scalar product on the right is that in L^2. The Fourier transform is a unitary isomorphism between H_s and the space of functions which are square integrable with respect to the measure $(1 + |\xi|^2)^s d\xi$.

(ii) For every $s \epsilon \mathbb{R}, S$ is dense in H_s.

(iii) If $s > t, H_s \subset H_t$ with continuous imbedding. In fact, for $u \epsilon H_s$, $||u||_{(t)} \leq ||u||_{(s)}$. In particular, $H_s \subset L^2$ for $s > 0$.

(iv) D^α is a bounded operator from H_s into $H_{s-|\alpha|} s \epsilon \mathbb{R}$. **65**

(v) If $f \epsilon H_{-s}$, then f as a linear functional on S extends continuously to H_s and $||f||_{(-s)}$ is the norm of f in $(H_s)^*$. So we can identify $(H_s)^*$ with $H_{(-s)}$. For $f \epsilon H_{-s}, g \epsilon H_s$ the pairing is given by

$$< f, g >=< \Lambda^{-s} f, \Lambda^s g >= \int \hat{f} \hat{g}.$$

(If $s = 0$, this identification of $H_0 = L^2$ with its dual is the complex conjugate of the usual one).

(vi) The norm $||.||_{(s)}$ is translation invariant. Indeed, if $g(x) = f(x - x_0)$, then $\hat{g}(\xi) = e^{2\pi i x_0 . \xi} \hat{f}(\xi)$ and hence $||f||_{(s)} = ||g||_{(s)}$.

Proposition 3.4. *For $s > n/2$, we have $\delta \epsilon H_{-s}$.*

Proof. $(1 + |\xi|^2)^{-s/2} \hat{\delta} = (1 + |\xi|^2)^{-s/2} \epsilon L^2$, whenever $s > n/2$. This is a consequence of the following observation : □

$$\int (1 + |\xi|^2)^{-s} d\xi \sim 1 + \int_1^\infty r^{-2s} r^{n-1} dr < \infty \text{ if and only if } s > n/2.$$

As an immediate consequence of this proposition, we have

Corollary 3.5. *For $s > n/2 + |\alpha|, D^\alpha \delta \epsilon H_{-s}$.*

Theorem 3.6 (SOBOLEV IMBEDDING THEOREM). *For $s > \frac{n}{2} +$ 66
$k, H_s \subset C^k$. Further, we have*

$$\sum_{|\alpha| \leq k} \sup_{\mathbb{R}^N} |D^\alpha f| \leq C_{sk} ||f||_{(s)}. \tag{3.7}$$

Proof. Since S is dense in H_s, it suffices to prove (3.7) for $f \epsilon S$. Let δ_x denote the Dirac measure at x. For $|\alpha| \leq k$, since $s > \frac{n}{2} + |\alpha|, D^\alpha \delta_x \epsilon H_{-s}$. □

Since

$$< D^\alpha \delta_x, f >= (-1)^{|\alpha|} < \delta_x, D^\alpha f >= (-1)^{|\alpha|}(D^\alpha f)(x)$$

and $||D^\alpha \delta_x||_{(-s)}$ is independent of x,

$$\begin{aligned}\sum_{|\alpha| \leq k} \sup_{\mathbb{R}^n} |D^\alpha f(x)| &= \sum_{|\alpha| \leq k} \sup_{\mathbb{R}^n} |D^\alpha \delta_x, f > | \\ &\leq \sum_{|\alpha| \leq k} \sup_{\mathbb{R}^n} |D^\alpha \delta_x||_{(-s)} ||f||_{(s)} = C_{sk} ||f||_{(s)}\end{aligned}$$

Now, given $u \epsilon H_s$, choose a sequence (u_j) in S such that $||u - u_j||_{(s)}$ converges to 0, as j tends to ∞. The above inequality with $f = u_i - u_j$ shows that $(D^\alpha u_j)$ is a Cauchy sequence in the uniform norm for $|\alpha| \leq k$; so its limit $D^\alpha u$ is continuous.

Corollary 3.8. *If $u \epsilon H_s$ for all $s \epsilon \mathbb{R}$, then $u \epsilon C^\infty$ i.e., $\bigcap_s H_s \subset C^\infty$.*

This argument can be extended to show that if $s > \frac{n}{2} + k$, then
67 elements of H_s and their derivatives of order less than or equal to k are not merely continuous but actually Hölder continuous.

Proposition 3.9. *If $0 < \alpha < 1$ and $s = \frac{n}{2} + \alpha$, then $||\delta_x - \delta_y||_{(-s)} \leq C_\alpha |x - y|^\alpha$*

Proof. We have

$$||\delta_x - \delta_y||^2_{(-s)} = \int |e^{-2\pi i x.\xi} - e^{-2\pi i y.\xi}|^2 (1 + |\xi|^2)^{-s} d\xi.$$

Let R be a positive number, to be fixed later. When $|\xi| \leq R$ we use the estimate $|e^{-2\pi ix.\xi} - e^{-2\pi iy.\xi}| \leq 2\pi|\xi||x-y|$ (by the mean value theorem) and when $|\xi| > R$, we use $|e^{-2\pi ix.\xi} - e{-}2\pi iy.\xi| \leq 2$. Then we have

$$\begin{aligned}
||\delta_x - \delta_y||^2_{(-s)} &\leq 4\pi^2|x-y|^2 \int\limits_{|\xi|\leq R} |\xi|^2(1+|\xi|^2)^{-s} d\xi + 4 \int\limits_{|\xi|>R} (1+|\xi|^2)^{-s} d\xi \\
&\leq c\left[|x-y|^2 \int\limits_0^R (1+r^2)^{-s} r^{n+1} dr + \int\limits_0^R r^{-2s+n-1} dr\right] \\
&\leq c'\left[|x-y|^2 R^{-2s+n+2} + R^{-2s+n}\right] \quad \text{as} \quad \frac{n}{2} < s < \frac{n}{2}+1 \\
&= c'\left[|x-y|^2 R^{2-2\alpha} + R^{-2\alpha}\right]
\end{aligned}$$

When we take $R = |x-y|^{-1}$ we get our result. $\square$

Exercise . *Show that the above argument does not work when $\alpha = 1$. Instead, we get $||\delta_x - \delta_y||_{(-s)} \leq c|x-y||\log|x-y||^{1/2}$ when x is near y. What happens when $\alpha > 1$?*

Corollary 3.10. *Let $0 < \alpha < 1$ and Λ_α = bounded functions g : $\sup\limits_{x,y}$* **68**
$\dfrac{|g(x)-g(y)|}{|x-y|^\alpha} < \infty$. If $s = \dfrac{n}{2} + \alpha + l$ and $f \epsilon H_s$, then $D^\beta f \epsilon \Lambda_\alpha$ for $|\beta| \leq k$.

Remark 3.11. We shall obtain an analogue of this result for L^p norms in Chapter 5.

The following lemma will be used in several arguments hereafter.

Lemma 3.12. *For all $\xi, n \epsilon \mathbb{R}^n$ and $s \epsilon \mathbb{R}$, we have*

$$\left[\frac{1+|\xi|^2}{1+|\eta|^2}\right]^s < 2^{|s|}(1+|\xi-\eta|^2)^{|s|}.$$

Proof. $|\xi| \leq |\eta| + |\xi-\eta|$ gives

$$|\xi|^2 < 2(|\eta|^2 + |\xi-\eta|^2) \text{ so that}$$
$$(1+|\xi|^2) < 2(1+|\eta|^2)(1+|\xi-\eta|^2).$$

$\square$

If $s > 0$, then raise both sides to the s^{th} power. If $s < 0$, interchange ξ and η and raise to the $-s^{th}$ power.

Proposition 3.13. If $\phi\epsilon S$, then the operator $f \to \phi f$ is bounded for all s.

Proof. The operator $f \to \phi f$ is bounded on H_s if and only if the operator of $g \to \Lambda^s \phi \Lambda^{-s} g$ is bounded on L^2, as one sees by □

69 setting $g = \wedge^s f$. But

$$\begin{aligned}(\wedge^s \phi^{-s} g)^\wedge(\xi) &= (1 + |\xi|^2)^{s/2} (\phi \wedge^{-s} g)\hat{}(\xi) \\ &= (1 + |\xi|^2)^{s/2} \left[\hat{\phi}(\xi) * (\wedge^{-s} g)\hat{}(\xi)\right] \\ &= (1 + |\xi|^2)^{s/2} \int \hat{\phi}(\xi - \eta)(1 + |\eta|^2)\hat{g}(\eta) d\eta \\ &= \int \hat{g}(\eta) K(\xi, \eta) d\eta\end{aligned}$$

where

$$K(\xi, \eta) = (1 + \eta|^2)^{-s/2} (1 + |\xi|^2)^{s/2} \hat{\phi}(\xi - \eta).$$

By lemma 3.12,

$$|K(\xi, \eta)| < 2^{|s|/2} (1 + |\xi - \eta|^2)^{|s|/2} |\hat{\phi}(\xi - \eta)|.$$

Therefore, since $\hat{\phi}$ is rapidly decreasing at ∞,

$$\int |K(\xi, \eta)| d\xi \leq c \text{ for every } \eta \epsilon \mathbb{R}^n,$$
$$\int |K(\xi, \eta)| d\eta \leq c \text{ for every } \eta \epsilon \mathbb{R}^n.$$

Thus from Theorem 1.1, the operator with kernel K is bounded on L^2. Hence our proposition is proved.

The spaces H_s are defined on $\mathbb{R}^n$ globally by means of the Fourier transform. Frequently, it is more appropriate to consider the following versions of these spaces.

Definition 3.14. *If $\Omega \subset \mathbb{R}^n$ is open and $s\epsilon\mathbb{R}$, we define $H_s^{\text{loc}}\Omega = \{f\epsilon\mathcal{D}'(\Omega) : \forall\Omega' \Subset \Omega, \exists\, g_{\Omega}, \epsilon H_s$ such that*

$$g_{\Omega'} = f \text{ on } \Omega'\}.$$

Proposition 3.15. *$f\epsilon H_s^{\text{loc}}(\Omega)$ if and only if $\phi f\epsilon H_s$ for $\phi\epsilon C_0^\infty(\Omega)$.* 70

Proof. If $f\epsilon H_s^{\text{loc}}(\Omega)$ and $\phi\epsilon C_0^\infty(\Omega)$, then there exists $g\epsilon H_s$ such that $f = g$ on supp ϕ. Therefore $\phi f = \phi g\epsilon H_s$ by proposition 3.13. □

Conversely, if $\phi f\epsilon H_s$ for all $\phi\epsilon C_0^\infty(\Omega')$, and $\Omega' \Subset \Omega$, choose $\phi\epsilon C_0^\infty(\infty)$ with $\phi \equiv 1$ on Ω'. Then $\phi f\epsilon H_s$ and $f = \phi f$ on Ω'.

Corollary 3.16. *If $L = \sum\limits_{|\propto|\leq k} a_\propto(x)D^\propto$ with $a_\propto\epsilon C^\infty(\Omega)$, then L maps $H_s^{\text{loc}}(\Omega)$ into $H_{s-k}^{\text{loc}}(\Omega)$ for all $s\epsilon\mathbb{R}$.*

It is a consequence of the Arzela-Ascoli theorem that if (u_j) is a sequence of C^k functions such that $|u_j|$ and $|\partial^\alpha u_j|(|\alpha| \leq k)$ are bounded on compact set uniformly in j, there exists a subsequence (v_j) of (u_j) such that $(\partial^\alpha v_j)$ converges uniformly on compact set for $|\alpha| \leq k-1$. In particular, if the u_j's are supported in a common compact set, then $(\partial^\alpha v_j)$ converges uniformly.

There is an analogue of this result for H_s spaces.

Lemma 3.17. *Suppose (u_k) is a sequence of C^∞functions supported in a fixed compact set Ω such that $\sup\limits_k ||u_k||_{(s)} < \infty$. Then there exists a subsequence which converges in the H_t norm for all $t < s$.*

Proof. Pick a $\phi\epsilon C_0^\infty$ such that $\phi = 1$ on Ω so that $u_k = \phi u_k$ and hence $\hat{u}_k = \hat{\phi}_* \hat{u}_k$. Then 71

$$\begin{aligned}(1+|\xi|^2)^{s/2}|\hat{u}_k(\xi)| &= (1+|\xi|^2)^{s/2}\left|\int \hat{\phi}(\xi-\eta)\hat{u}_k(\eta)d\eta\right| \\ &\leq \int |\hat{\phi}(\xi-\eta)||\hat{u}_k(\eta)|2^{|s|/2}(1+|\eta|^2)^{s/2}(1+|\xi-\eta|^2)^{|s|/2}d\eta \\ &\leq 2^{|s|/2}||\phi||_{(|s|)}||u_k||_{(s)} \leq c_{-1} \text{ independent of } k.\end{aligned}$$

Likewise, we have

$$(1+|\xi|^2)^{s/2}|\partial_j\hat{u}_k(\xi)| \leq 2^{|s|/2}||2\pi x_j\phi(x)||_{|s|}||u_k||_{(s)} \leq c_2$$

independently of k. Therefore, by the Arzela-Ascoli theorem there exists a subsequence $(\hat{v}_k)$ of $(\hat{u}_k)$ which converges uniformly on compact sets. For $t \leq s$,

$$\begin{aligned}
\|v_j - v_k\|^2_{(t)} &= \int (1+|\xi|^2)^t |\hat{v}_j - \hat{v}_k|^2 d\xi \\
&= \int\limits_{|\xi|\leq R} (1+|\xi|^2)^t |\hat{v}_j - \hat{v}_k|^2 d\xi + \int\limits_{|\xi|>R} (1+|\xi|^2)^t |\hat{v}_j - \hat{v}_k|^2 d\xi \\
&< (1+R^2)^{\max(t,0)} \sup_{|\xi|\leq R} |\hat{v}_j(\xi) - \hat{v}_k(\xi)|^2 \int\limits_{|\xi|\leq R} d\xi + \\
&\quad + (1+R^2)^{t-s} \int_{|\xi|>R} [(1+|\xi|^2)^s |\hat{v}_j(\xi) - \hat{v}_k(\xi)|^2 d\xi] \\
&\leq c(l+R^2)^{n+|t|} \sup_{|\xi|\leq R} |\hat{v}_j(\xi) - \hat{v}_j(\xi)|^2 + (1+R^2)^{t-s} \|v_j - v_k\|^2_{(s)}.
\end{aligned}$$

□

Given $\epsilon > 0$, choose R large enough so that the second term is less
72 than $\epsilon/2$ for all j and k. This is possible since $\|v_j - v_k\|_{(s)} \leq c$ and $t - s < 0$. Then for j and k large enough the first term is less than $\epsilon/2$, since $(\hat{v}_k)$ converges uniformly on compact sets. Thus we see that (v_k) is a Cauchy sequence in $H_{t'}$ and since H_t is complete we are done.

Remark 3.12. Lemma 3.17 is false, if we do not assume that all the $u'_k s$ have support in a fixed compact set. For example, for $u \epsilon C_0^\infty$ and $x_k \epsilon \mathbb{R}^n$ with $|x_k|$ tending to ∞, define $u_k(x) = u(x - x_k)$. Then the invariance of H_s norms shows that $\|u_k\|_{(s)} = \|u\|_{(s)}$. But no subsequence of (u_k) converges, in any H_t. For, if a subsequence (v_k) of (u_k) converges, it must converge to 0, since u_k converges to 0 in S'. But then $\lim \|v_k\|_{(t)} = 0$ which is not the case.

Theorem 3.20 (RELLICH THEOREM)**.** *Let $H_s^0(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in H_s. If Ω is bounded and $t < s$, the inclusion $H_s^0(\Omega) \hookrightarrow H_t$ is compact, i.e., bounded sets in $H_s^0(\Omega)$ are relatively compact in H_t.*

Proof. Let (u_k) be a sequence in $H_s^0(\Omega)$. To each k, find a $v_k \epsilon C_0^\infty(\Omega)$

such that $||u_k - v_k||_{(s)} \leq \dfrac{1}{k}$. Then we have $||v_k||_{(s)} \leq ||u_k||_{(s)} + | \leq c$ (independent of k.) Therefore, by lemma 3.17, a subsequence (w_k) of (v_k) exists such that (w_k) converges in H_t. If (u'_k) is the subsequence of (u_k) corresponding to the sequence (w_k), we have **73**

$$\begin{aligned} ||u'_i - u'_j||_{(t)} &< ||u'_i - w_i||_{(t)} + ||w_i - w_j||_{(t)} + ||w_j - u'_j||_{(t)} \\ &< \frac{1}{i} + \frac{1}{j} + ||w_i - w_j||_{(t)} \to 0 \text{ as } ||i, j \to \infty. \end{aligned}$$

Hence (u'_k) converges in H_t. ◻

I the proof of the next theorem, we will use the technique of complex interpolation, which is based on the following result from elementary complex analysis called *'Three lines lemma'*.

Lemma 3.20. *Suppose $F(z)$ is analytic in $o < ReZ < 1$, continuous and bounded on $0 \leq ReZ \leq 1$. If $|F(1 + it)| \leq c_0$ and $|F(l + it)| \leq c_1$, then $F(s + it) \leq c_0^{1-s} c_1^s$, for $0 < s < 1$.*

Proof. If $\epsilon > 0$, the function

$$g_\epsilon(z) = c_0^{z-1} c_1^{z-1} e^{\epsilon(z^2 - z)} f(z)$$

satisfies the hypotheses with c_0 and c_1 replaced by 1, and also $|g_\epsilon(z)|$ converges to 0 as $|Imz| \to \infty$ for $0 \leq ReZ \leq 1$. From the maximum modulus principle, it follows that $|g(z)| \leq 1$ for $0 \leq Rez \leq 1$ and letting ϵ tend to 0, we obtain the desired result. ◻

Theorem 3.21. *Suppose that $-\infty < s_0 < s_1 < \infty$ and T is a bounded linear operator H_{s_0} such that $T|H_{s_1}$ is bounded on H_{s_1}. Then $T|H_s$ is bounded on H_s for all s with $s_0 \leq s \leq s_1$.*

Proof. Our hypothesis means that **74**

$$\wedge^{s_0} T \wedge^{-s_0} \text{ and } \wedge^{s_1} T \wedge^{-s_1}$$

are bounded operators on L^2. For $0 \leq Rez \leq 1$ we define

$$s_z = (l - z)s_0 + zs_1 \text{ and } T_z = \wedge^{s_z} T \wedge^{-s_z}.$$

◻

Then what we wish to prove is that T_z is bounded on L^2 for $0 \leq z \leq 1$. Observe that when $w = x + iy$, $\wedge^w$, $\wedge^x \wedge^{iy}$ and

$$(\wedge^{iy} f)\hat{}(\xi) = (1 + |\xi|^2)^{iy/2} \hat{f}^2(\xi) \text{ so that}$$
$$|(\wedge^{iy} f)\hat{}(\xi) = |\hat{f}(\xi)|.$$

Thus $\wedge^{iy}$ is unitary on H^s for all s.

For $\phi, \psi \epsilon S$, we define

$$F(z) = \int (T_z \phi)\psi =< \wedge^{s_z} T \wedge^{-s_z} \phi, \psi > .$$

Then

$$\begin{aligned}
|F(z)| &= | < \wedge^{s_z} T \wedge^{-s_z} \phi, \psi > | \\
&= | < T \wedge^{s_z} \phi, \wedge^{-s_z} \psi > | \\
&\leq ||T \wedge^{-s_z} \phi||_(s_0)|| \wedge^{-s_z} \psi||_{(-s_0)} \\
&\leq c|| \wedge^{-s_z} \phi||_{s_0} || \wedge^{s_z} ||_{(-s_0)} \\
&\leq c||\phi||_{(s_0 - s_1)Re\ z} ||\psi||_{(s_1 - s_0)Rez} \\
&\leq c||\phi||_{(0)} ||\psi||_{s_1 - s_0}
\end{aligned}$$

75 $F(z)$ is clearly an analytic function of z for $0 < Re\ z < 1$. Further, by our hypothesis on T, when $Rez = 0$, we have

$$|F(z)| \leq c_0 ||\phi||_{(0)} ||\psi||_{(0)}$$

and when $Rez = 1$, we have

$$|F(z)| \leq c_1 ||\phi||_{(0)} ||\psi||_{(0)}.$$

Therefore, by the Three lines lemma

$$|F(z)| \leq c_0^{1-z} c_0^z ||\phi||_{(0)} ||\psi||_{(0)} \text{ for } 0 < z < 1.$$

Finally, by the self duality of $H_0 = L^2$, this gives

$$||T_z \phi|| \leq c_0^{1-z} c_0^z ||\phi||_{(0)}$$

which completes the proof.

Remark 3.22. The same proof also yields the following *more general result:*

Suppose $\infty < s_0 < s_1 < \infty < t_0 < t_1 < \infty$. If T is a bounded linear operator from H_{s_0} to H_{t_0} whose restriction to H_{s_1} to H_{t_1}, then the restriction of T to H_{t_θ} is bounded from H_{t_θ} to H_{t_θ} for $0 < \theta < 1$ where $s_\theta = (1-\theta)s_0 + \theta s_1$ and $t_\theta = (1-\theta)t_0 + \theta t_1$.

As a consequence of this result, we obtain an easy proof that H_s^{loc} is invariant under smooth coordinate changes.

Theorem 3.23. *Suppose Ω and Ω' are open subsets of $\mathbb{R}^n$ and $\phi : \Omega \to \Omega'$ is a C^∞ diffeomorphism. Then the mapping $f \to f o\phi$ maps $H_s^{\text{loc}}(\Omega')$.* **76**
continuously onto $H_s^{\text{loc}}(\Omega)$.

Proof. The statement of the theorem is equivalent to the assertion that for any $\phi \epsilon C_0^\infty(\Omega')$, the map $Tf = (\phi f) \circ \phi$ is bounded on H_s for $s\epsilon\mathbb{R}$. If $s = 0, 1, 2, \ldots$, this follows from the chain rule and the fact that $H_s = \{f : D^\propto f \epsilon L^2 \text{ for } |\propto| \leq s\}$. By Theorem 3.21, it is true for all $s \geq 0$. But the adjoint of T is another map of the same form : $\square$

$T^* g = (\psi g)_\circ \psi$ where $\psi = \theta^{-1}$ and $\psi = \phi|J| \circ \Phi$, J being the Jacobian determinant of Φ^{-1}. Hence T^* is bounded on H_s for all $s \geq 0$ and by duality of H_s and H_{-s}, this yields the boundedness of T on H_s for $s < 0$.

Finally, we ask to what extend the H_s spaces include all distributions. Globally they do not, since, if $f \epsilon H_s$, then f is tempered and $\hat{f}$ is a function. But locally they do, as we see from the following result.

Proposition 3.24. *Every distribution with compact support lies in some H_s: i.e., $E' \subset \bigcup\limits_{s\in\mathbb{R}} H_s$.*

Proof. If $f \epsilon E'$, then it is a continuous linear functional on C^∞. Therefore, there exists a constant $c > 0$, a compact set K, and a nonnegative integer k such that

$$|< f, \phi >| \leq c \sum_{|\propto| \leq k} \sup_k |D^\propto \phi| \text{ for all } \phi \epsilon C^\infty,$$

i.e., $|< f, \phi >| \leq c \sum\limits_{|\alpha| \leq |k} \sup\limits_{\mathbb{R}^n} |D^\alpha \phi|$ for all $\phi \epsilon C^\infty$. **77**

By the Sobolev imbedding theorem,

$$\sum_{|\propto|\leq k} \sup_{\mathbb{R}^n} |D^\alpha \phi| \leq c' \|\phi\|_{(k+\frac{n}{2}+\epsilon)} \text{ for } \epsilon > 0.$$

Therefore, $| < f, \phi > | \leq c'' \|_{\left(k+\frac{n}{2}+\epsilon\right)}$ for all $\phi \epsilon S$ since S is dense in $H_{(k+\frac{n}{2}+\epsilon)}$, this shows that f is a continuous linear functional on $H_{\left(k+\frac{n}{2}+\epsilon\right)}$. Hence $f \epsilon H_{-\frac{n}{2}-k-\epsilon}$. □

Corollary 3.25. *If $f \epsilon D'(\Omega)$ and Ω' has compact closure in Ω then there exists s in $\mathbb{R}$ such that $f \epsilon H^{\text{loc}}_s(\Omega')$.*

2 Hypoelliptic Operators With Constant Coefficients

We now apply the machinery of Sobolev spaces to derive a criterion for the hypoellipticity of constant coefficient differential operators. First, we have a few preliminaries.

Definition 3.26. *Let P be a polynomial in n variables. For a multi-index $\propto$, $P^{(\propto)}$ will ne defined by $P^{(\propto)}(\xi) = (\dfrac{\partial}{\partial \xi})^\propto P(\xi)$.*

Proposition 3.27. ***LEIBNIZ RULE*** *When $f \epsilon C^\infty, g \epsilon D'$ and $P(D)$ is a constant-coefficient partial differential operator of order k, we have*

$$P(D)(fg) = \sum_{|\propto|\leq k} \frac{1}{\propto!} (P^{(\propto)}(D)g) D^\propto f).$$

The proof of this proposition is left as an exercise to the reader.

78 **Definition 3.28.** *We say that a polynomial P satisfies condition (H) if there exists a $\delta > 0$ such that*

$$\frac{|P^{(\propto)}(\xi)|}{|P(\xi)|} = 0 \ (|\xi|^{-\delta|\alpha|}) \ as \ |\delta| \to \infty, \forall \propto .$$

Theorem 3.29 (HÖRMANDER)**.** *If P satisfies condition (H), then $P(D)$ is hypoelliptic. More precisely, if f is in $D'\Omega)$and $P(D)$ $f H^{\text{loc}}_{s+k\delta}(\Omega)$, where δ is as in condition (H) and k is the degree of P.*

Proof. We first observe that the second assertion implies the first since $C^\infty(\Omega) = \bigcap\limits_{s\epsilon\mathbb{R}} H_s^{\text{loc}}(\Omega)$ by Corollary 3.8. □

Suppose therefore that $P(D)f\epsilon H_s^{\text{loc}}(\Omega)$, $f\epsilon D'(\Omega)$. Given $\phi\epsilon C_\circ^\infty(\Omega)$ we have to prove that $\phi f\epsilon H_{s+k\delta}$. Let Ω', be an open set such that $\Omega' \Subset \Omega$ and $\operatorname{supp}\phi \subset \Omega'$. By Corollary 3.25, therefore exists t in $\mathbb{R}$ such that $f\epsilon H_t^{\text{loc}}(\Omega')$. By decreasing t, we can some that $t = s + k - 1 - m\delta$ for assume positive integer m. Set $\phi_m = \phi$ and then choose $\phi_{m-1}, \phi_{m-2}, \ldots, \phi_0, \phi{-1}$ in $C_0^\infty(\Omega')$ such that $\phi_j = 1$ on $\operatorname{supp}\phi_{j+1}$.

Then $\phi_j P(D)f\epsilon H_S \subset H_{t-k+l+j\delta}$ for $0 \leq j \leq m$ and $\phi_{-1}f\epsilon H_t$. Now

$$P(D)(\phi_0 f) = \phi_0 P(D)f + \sum_{\propto\neq 0} \frac{1}{\propto!} P^{\propto}(D)(\phi_{-1}f)D^{\propto_{\phi_0}}$$

since $\theta_{-1} = 1$ on the support of ϕ_0. So, $P(D)(\phi_0 f)\epsilon H_{r-k+1}$. This means that

$$\int (1 + |\xi|^2)^{t-k+1} |P(\xi)(\phi_0 f)\hat{}(\xi)|^2 d\xi < \infty.$$

By condition (H) **79**

$$\int (1 + |\xi|^2)^{t-k+l+\delta|\delta|} |P^{\propto|}(\xi)(\phi_0 f)^(\xi)|^2 d\xi < \infty$$

This implies that $P^{(\alpha)}(D)(\phi_\circ f)\epsilon H_t - k + l + \delta|\delta|$.

Next,

$$P(D)(\phi_1 f) = \phi_1 P(D)f + \sum_{\propto\neq 0} \frac{1}{\propto!} p^{(\propto)}(D)(\phi_o f)D^{\propto}(\phi_1 f)$$

since $\phi_0 = 1$ on the support of ϕ_1, so $P(D)(\phi_1 f)\epsilon H_{t-k+1+\delta}$. By the same argument same argument as above,

$$P^{(D)}(\phi_1 f)\epsilon H_{t+k+l+\delta(1+|\propto|)}.$$

Continuing inductively, we obtain $P(D)(\phi_j f)\epsilon H_{t-k+l+j\delta}$, which implies that

$$P^{(\propto)}(D)(\phi_j f)\epsilon H_{t-k+l+\delta(j+|\propto|)}.$$

For $j = m$, we above

$$P^{(\propto)}(D)(\phi_m f)\epsilon H_{t-k+l+\delta(m+(\delta))} = H_{s+\delta|\propto|}.$$

If

$$P(\xi) = \sum_{|\propto|\leq k} a_{\propto}\xi^{\propto},$$

choose $\propto$ with $|\propto| = k$ such that $a_{\propto} \neq 0$. Then $P^{\propto}(\xi) = \propto! a_{\propto} \neq 0$. whence $\phi f = \phi_m f \epsilon H_{s+k\delta}$ and we are done.

80 **Remark 3.30.** The condition (H) is equivalent to the following apparently weaker condition

$$(H') : \frac{|p^{(\propto)(\xi)}|}{|P(\xi)|} \to 0 \text{ as } |\xi| \to \infty \text{ for } \propto \neq 0.$$

Condition (H') is in turn equivalent to

$$(H'') : |\mathrm{Im}\zeta| \to \infty, \text{ in the set } \{\zeta\epsilon\mathbb{C}^n : P(\zeta) = 0\}.$$

The converse of Hörmander's theorem is also true, i.e., hypoellipticity implies condition (H).

The proofs of these assertions can be found in Hörmander [6]. The logical order of the proofs is

$$(H) \Rightarrow hypoellipticity \Rightarrow (H'') \Rightarrow (H') \Rightarrow (H).$$

The implication $(H') \Rightarrow (H)$ requires the use of some results from (semi) algebraic geometry.

Definition 3.31. *$P(\xi) = \sum_{|\propto|\leq k} a_{\propto}\xi^{\propto}$ is called elliptic if $\sum_{|\propto|=k} a_{\propto}\xi^{\propto} \neq 0$ for every $\xi \neq 0$.*

EXERCISES

1. Prove that P is elliptic if and only if $|P(\xi)| \geq c_{\circ}|\xi|^k$ for large $|\xi|$.
2. Prove that P is elliptic if and only if P satisfies condition (H) with $\delta = 1$.

3. Prove that no P satisfies condition (H) with $\delta > 1$. (Hint: If $|\propto| = k, P^{(\propto)}$ is a constant).

81 4. Let P be elliptic and real valued. Define Q on $\mathbb{R}^{n+1}$ by $Q(\xi, \tau) = 2\pi i\tau + P(\xi)$, so that $Q(D_x, D_t) = \partial_t + P(D_x)$. Show that Q satisfies condition (H) with $\delta = 1/k$ where k is the degree of P and that $1/k$ is the best possible value of δ.

   (Hint : Consider the regions $|\xi|^k \leq |\tau|$ and $|\tau| \leq |\xi|^k$ separately).

Chapter 4

Basic Theory of Pseudo Differential Operators

1 Representation of Pseudo differential Operators

Let $L = \sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha$ be a partial differential operator with C^∞ coeffi- 82
cients on Ω. Using the Fourier transform, we can write

$$\begin{aligned}
(Lu)(x) &= \sum_{|\alpha| \leq k} a_\alpha(x) \int e^{2\pi i x.\xi} \hat{u}(\xi) \xi^\alpha d\xi \\
&= \int e^{2\pi i x.\xi} p(x, \xi) \hat{u}(\xi) d\xi
\end{aligned}$$

where $p(x, \xi) = \sum_{|\alpha| \leq k} a_\alpha(x) \xi^\alpha$. This representation suggests that $p(x, \xi)$ can be replaced by more general functions. So, we make the following definition.

Definition 4.1. *For an open set $\Omega \subset \mathbb{R}^n$ and a real number m we define $S^m(\Omega)$, the class of symbols of order m on Ω, by*

$$S^m(\Omega) = \{p \epsilon C^\infty(\Omega \times \mathbb{R}^n) : \forall \alpha, \beta, V\Omega' \subset \Omega,\ c = c_{\alpha\beta\Omega'}$$

such that $\sup_{x \epsilon \Omega'} |D_x^\beta D_\xi^\alpha p(x, \xi)| \leq c(1 + |\xi|)^{m - |\alpha|}\}$.

We note that $S^m(\Omega) \subset S^m$ whenever $m < m'$, and we set $S^{-\infty}(\Omega) = \bigcap\limits_{m \epsilon \mathbb{R}} S^m(\Omega)$.

Examples

(i) Let $p(x,\xi) = \sum\limits_{|\alpha| \leq k} a_\alpha(x)\xi^\alpha$ with $a_\alpha \epsilon C^\infty(\Omega)$. Then $p \epsilon S^k(\Omega)$.

83 (ii) Let $p(x,\xi) = \sum\limits_{j=1}^{N} a_j(x) f_j(\xi)$ where $a_f \epsilon C^\infty(\Omega)$ and $f_j \epsilon C^\infty(\mathbb{R}^n)$ is homogeneous of degree m_j large ξ : that is,

$$f_i(r\xi) = r^{m_j} f_j(\xi) \text{ for } |\xi| \geq c, r \geq 1.$$

In this case, $p \epsilon S^m(\Omega)$, where $m = \max\limits_{1 \leq j \leq N}\{m_j\}$.

(iii) Let $p(x,\xi) = (1+|\xi|^2)^{s/2}$. This p belongs to $S^s(\mathbb{R}^n)$.

(iv) Let $p(x,\xi) = \phi(\xi)\sin\log|\xi|$ with $\phi \epsilon C^\infty, \phi = 0$ near the origin and $\phi = 1$ when $|\xi| \geq 1$. Then $p \epsilon S^o(\mathbb{R}^n)$.

Remark 4.2. We observe that when $p \epsilon S^m(\Omega), D_x^\beta D_\xi^\alpha p \epsilon S^{m-|\alpha|}(\Omega)$.

Further, if $p \epsilon S^{m_1}(\Omega)$ and $q \epsilon S^{m_2}(\Omega)$,then $p + q \epsilon S^m(\Omega)$, where $m = \max\{m_1, m_2\}$ and $pq \epsilon S^{m_1+m_2}(\Omega)$.

Remark 4.3. Our symbol classes $S^m(\Omega)$ are special cases of Hormander's classes $S^m_{\rho,\delta}(\Omega)$. Namely, for $0 \leq \delta \leq \rho \leq 1$, and $m \epsilon \mathbb{R}$,

$$S^m_{\rho,\delta}(\Omega) = \{p \epsilon C^\infty(\Omega \times \mathbb{R}^n) : \forall \alpha, \beta, V\Omega' \Subset \Omega, c = c_{\alpha\beta\Omega'}$$

such that $$\sup_{x \epsilon \Omega'} |D_x^\beta D_\xi^\alpha p(x,\xi)| \leq c(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|}\}.$$

In this terminology, $S^m(\Omega) = S^m_{1,0}(\Omega)$.

Definition 4.4. *For $p \epsilon S^m(\Omega)$, we define the operator $p(x.D)$ on the domain $C_o^\infty(\Omega)$ by*

$$p(x,D)\,u(x) = \int e^{2\pi i x.\xi} p(x,\xi)\,\hat{u}(\xi) d\xi, u \epsilon C_o^\infty(\Omega).$$

(Sometimes, we shall denote $p(x, D)$ by p). Operators of the form
84 $p(x, D)$ with $p \epsilon S^m(\Omega)$ are called *pseudo differential operators of order* m on Ω.

The set of all pseudo differential operators of order m on Ω will denoted by $\Psi^m(\Omega)$. For brevity, we will sometimes write "$\Psi DO''$ instead of "pseudo differential operators".

The next theorem states that $p(x, D)$ is a continuous linear map of $C_o^\infty(\Omega)$ into $C^\infty(\Omega$ which extends to $E'(\Omega)$. For the proof, we need a result which depends on

Lemma 4.5. *Let $p \epsilon S^m(\Omega)$ and $\phi \epsilon C_o^\infty(\Omega)$. Then, for each positive integer N, there exists $c_N > 0$ such that for all ξ, η in $\mathbb{R}^n$,*

$$|\int e^{2\pi i x.\eta} p(x, \xi)\phi(x)dx| \leq c_N(1 + |\xi|)^m(1 + |\eta|)^{-N}.$$

Proof. For any ξ and η in $\mathbb{R}^n$,
$|\eta^\alpha \int e^{2\pi i x.\eta} p(x, \xi)\phi(x)dx|$

$$= |\int D_x^\alpha e^{2\pi i x.\eta} p(x, \xi)\phi(x)dx|.$$

□

Integrating by parts, we have

$$|\eta^\alpha \int e^{2\pi i x.\xi} p(x, \xi)\phi(x)dx| = |\int e^{2\pi i x.\eta} D_x^\alpha(p(x, \xi)\phi(x))dx|$$
$$\leq C_\alpha(1 + |\xi|)^m \text{ for all } \alpha.$$

Therefore

$$\sum_{|\alpha| \leq N} |\eta^\alpha \int e^{2\pi i x.\eta} p(x, \xi)\phi(x)dx| \leq c'_N(1 + |\xi|)^m \text{ for all } N.$$

Since $(1 + |\eta|)^N \leq c \sum_{|\alpha| \leq N} |\eta^\alpha|$, the required result follows. 85

Corollary 4.6. *If $p\epsilon S^m(\Omega)$ and $\phi\epsilon C_o^\infty(\Omega)$, then the function*

$$g(\xi) = \int e^{2\pi ix.\xi} p(x,\xi)\phi(x)dx$$

is rapidly decreasing as ξ tends to ∞.

Proof. Set $\xi = \eta$ in the lemma. □

Theorem 4.7. *If $p\epsilon S^m(\Omega)$, then $p(x,D)$ is a continuous linear map of $C_o^\infty(\Omega)$ into $C^\infty(\Omega)$ which can be extended as a linear map from $E'(\Omega)$ into $D'(\Omega)$.*

Proof. For $u\epsilon C_o^\infty(\Omega)$,

$$p(x,D)u(x) = \int e^{2\pi ix.\xi} p(x,\xi)\hat{u}(\xi)d\xi.$$

The integral converges absolutely and uniformly on compact sets, as do the integrals

$$\int D_x^\alpha(e^{2\pi ix.\xi} p(x,\xi)\,\hat{u}(\xi)d\xi \text{ for all } \alpha,$$

since $p\epsilon S^m$ and $\hat{u}\epsilon\mathcal{S}$. □

This proves that $p(x,D)u\epsilon C^\infty(\Omega)$, and continuity of $p(x,D)$ from $C_o^\infty(\Omega)$ to $C^\infty(\Omega)$ is an easy exercise.

To prove the rest of the theorem, we will make use of Corollary 4.6.

For $u\epsilon E'(\Omega)$, we define $p(x,D)u$ as a functional on $C_o^\infty(\Omega)$ as follows:

$$\begin{aligned} < p(x,D)u,\phi > &= \iint p(x,\xi)\hat{u}(\xi)e^{2\pi ix.\xi}\phi(x)dxd\xi \\ &= \int g(\xi)\hat{u}(\xi)d\xi \end{aligned}$$

86 where

$$g(\xi) = \int e^{2\pi ix.\xi} p(x,\xi)\phi(x)dx, \text{ for } \phi\epsilon C_o^\infty(\Omega).$$

By Corollary 4.6, $g(\xi)$ is rapidly decreasing, while $\hat{u}$ is of polynomial growth; so the last integral is absolutely convergent and the functional on C_o^∞ thus defined is easily seen to be continuous. Moreover, if $u \epsilon C_o^\infty$, the double integral is absolutely convergent, and by interchanging the order of integration, we see that this definition of $p(x, D)u$ coincides with the original one. Hence we have extended $p(x, D)$ to a map from $E'(\Omega)$ to $\mathcal{D}(\Omega)$.

Remark 4.8. It follows easily from the above argument that $p(x, D)$ is sequentially from $E'(\Omega)$ to $\mathcal{D}(\Omega)$, i.e., if u_k converges to u in $E'(\Omega)$, then $p(x, D)u_k$ converges to $p(x, D)u$ in $\mathcal{D}'(\Omega)$. Actually, $p(x, D)$ is continuous from $E'(\Omega)$ to $\mathcal{D}'(\Omega)$: this follows from the fact (which we shall prove later) that the transpose of a pseudo differential operator is again a pseudo differential operator. Thus $p(x, D)^t : C_o^\infty(\Omega) \to C^\infty(\Omega)$ is continuous, and so, by duality, $p(x, D) = (p(x, D)^t)^t : E'(\Omega) \to \mathcal{D}'(\Omega)$ is continuous.

2 Distribution Kernels and the Pseudo Local Property

Definition 4.9. *Let T be an operator from $C_o^\infty(\Omega)$ to $C^\infty(\Omega)$. If there exists a distribution K on $\Omega \times \Omega$ such that*

$$< Tu, v >=< k, v \otimes u > \text{ for } u, v \epsilon C_o^\infty(\Omega),$$

we say that K is the distribution kernel *of the operator T.*

In this definition, $v \otimes u$ is defined by $(v \otimes u)(x, y) = v(x)u(y)$. Formally, this definition says that.

$$Tu(x) = \int K(x.y)u(y)dy.$$

K is uniquely determined since linear combinations of functions of the form $v \otimes u$ are dense in $C_o^\infty(\Omega \times \Omega)$. 87

If $p \epsilon S^m(\Omega)$, it is easy to compute the distribution kernel of $p(x, D)$. In fact,

$$< p(x, D)u, v > = \iint p(x, \xi)\hat{u}(\xi)e^{2\pi i x.\xi} v(x) d\xi dx$$

$$= \iint p(x,\xi) e^{2\pi i x.\xi} (v \otimes u)\hat{}_2(x,\xi) d\xi dx$$

where $(v \otimes u)\hat{}_2(x,\xi)$ means the Fourier transform in the second variable.

It follows immediately from the definition of K that

$$< K, w > = \iint e^{2\pi i x.\xi} p(x,\xi) \hat{w_2}(x,\xi) d\xi dx, \forall w \epsilon C_o^\infty(\Omega \times \Omega).$$

$$\text{or } < K, w > = \iiint e^{2\pi i (x-y).\xi} p(x,\xi) w(x,y) dy\, d\xi\, dx.$$

From this it is easy to see that $K(x,y) = p_2^v(x, x-y)$ where $p_2^v(x,.)$ is the inverse Fourier transform of the tempered distribution $p(x,.)$. In particular, this shows that p is uniquely determined by the operator $p(x,D)$. If $P \epsilon \Psi^m(\Omega)$, we shall sometimes denote the unique $p \epsilon S^m(\Omega)$ such that $p = p(x,D)$ by σ_p.

The following theorem gives precise results on the kernel K of $p(x,D)$.

Theorem 4.10. *The distribution kernel K of $p(x,D)$ with $p \epsilon S^m(\Omega)$ is in C^∞ on $(\Omega \times \Omega)/\Delta$ where $\Delta\{(x,x) : x \epsilon \Omega\}$ is the diagonal. More precisely, if $|\alpha| > M+n+j$ for a positive integer j, then $(x-y)^\alpha K(x,y) \epsilon C^j(\Omega \times \Omega)$.*

88 *Proof.* For $w \epsilon C_o^\infty(\Omega \times \Omega)$, let us compute $< (x-y)^\alpha K, w >$.

$$\begin{aligned} < (x-y)^\alpha K, w > &= < K, (x-y)^\alpha w > \\ &= \iint e^{2\pi i x.\xi} p(x,\xi)(x + D_\xi)^\alpha \hat{w_2}(x,\xi) d\xi\, dx \\ &= \iint \hat{w_2}(x,\xi)(x - D_\xi)^\alpha \{e^{2\pi i x.\xi} p(x,\xi)\} d\xi\, dx. \end{aligned}$$

□

Using the Leibniz formula, it is easily seen that

$$(x - D_\xi)^\alpha \{e^{2\pi i x.\xi} p(x,\xi)\} = (-D_\xi)^\alpha p(x,\xi).e^{2\pi i x.\xi}$$

Therefore

$$\begin{aligned} < (x-y)^\alpha K, w > &= \iint \hat{w_2}(x,\xi) e^{2\pi i x.\xi} (-D_\xi)^\alpha p(x,\xi) d\xi\, dx \\ &= \iiint w(x,y) e^{2\pi i (x-y).\xi} (-D_\xi)^\alpha p(x,\xi) dy\, d\xi\, dx. \end{aligned}$$

From the above expression, we infer that

$$(x-y)^\alpha K(x,y) = \int e^{2\pi i(x-y).\xi}(-D_\xi)^\alpha p(x,\xi)d\xi.$$

Since $|(-D_\xi)^\alpha p(x,\xi)| \leq c(1+|\xi|)^{m-|\alpha|}$, the integral converges absolutely and uniformly on compact sets whenever $m - |\alpha| < -n$. Also we can differentiate with respect to x and y j times under the integral sign provided $m - |\alpha| + j < -n$ or $|\alpha| > m + n + j$. Thus, we see that $(x-y)^\alpha K \epsilon C^j(\Omega \times \Omega)$.

Corollary 4.11 (PSEUDO LOCAL PROPERTY OF ΨDO)**.** *If* $P\epsilon\Psi^m(\Omega)$, *then, for all* $u\epsilon E'(\Omega)$, *sing* supp Pu *is contained in sing* supp u.

Proof. Let $u\epsilon E'$ and let V be an arbitrary neighbourhood of sing supp u. Take a $\phi\epsilon C_o^\infty(V)$ such that $\phi = 1$ on sing supp u. Then $u = \phi u + (1-\phi)u = u_1 + u_2$; u_2 is a C_o^∞ function and supp $u_1 \epsilon V$. ◻

Therefore, $Pu = Pu_1 + Pu_2$ and Pu_2 is a C^∞ function. Moreover, when $x_o \notin V$, **89**

$$Pu_1(x) = \int_V K(x,y)u_1(y)dy$$

is also a C^∞ function in a neighbourhood of x_o since $u_1(y) = 0$ for y near x_o. This implies that sing supp $Pu \subset V$. Since V is any neighbourhood of sing supp u, the corollary is proved.

Corollary 4.12. *If* $p\epsilon S^{-\infty}(\Omega)$, *then the distribution kernel K of* $p(x,D)$ *is in* $C^\infty(\Omega \times \Omega)$.

Proof. Follows from the theorem if we take $|\alpha| = 0$. ◻

Corollary 4.13. *If* $p\epsilon S^{-\infty}(\Omega)$, *then* $p(x,D)$, *maps* $E'(\Omega)$ *continuously into* $C^\infty(\Omega)$.

Proof. If $u\epsilon E'(\Omega)$, in view of Corollary 4.12, it is easily seen that $p(x,D)u$ is a smooth function defined by

$$P(x,D)u(x) = \int K(x,y)u(y)dy = < u, K(x,.) >,$$

whence the result follows. ◻

Definition 4.14. *A smoothing operator is a linear operator T which maps $E'(\Omega)$ continuously into $C^\infty(\Omega)$.*

If $K \epsilon C^\infty(\Omega \times \Omega)$, then the operator T defined by

$$(Tf)(x) = < K(x, .), f > = \int K(x, y) f(y) dy$$

is a smoothing operator. Conversely, every smoothing operator T can be given in the above form with $K(x, y) = (T\delta_y)(x)$.

As we have already remarked if $p \epsilon S^{-\infty}$, then the corresponding ΨDO is smoothing. However, not every smoothing operator is a ΨDO. For example, we have

90 **Proposition 4.15.** *Suppose $p(x, .)$ is a C^∞ function of x with values in E'. Then $p(x, D)$ (defined in the same way as in the case of $p \epsilon S^m(\Omega)$) is a smoothing operator.*

Proof. The distribution kernel K of $p(x, D)$ is given by

$$\begin{aligned} K(x, y) &= \int e^{2\pi i (x-y).\xi} p(x, \xi) d\xi \\ &= < p(x, .), e^{2\pi i (x-y).(.)} > \end{aligned}$$

and hence $K(x, y) \epsilon C^\infty$. Thus K defines a smoothing operator. ◻

Remark 4.16. Sometimes, it is convenient to enlarge the class of pseudo-differential operators of order m by including operators of the form $P + S$ where $P \epsilon \Psi^m(\Omega)$ and S is smoothing. However, the general philosophy is the following:

1. On the level of operators, smoothing operators are negligible.

2. On the level of symbols, what counts is the asymptotic behaviour at ∞, so that symbols of order $-\infty$ are negligible.

3 Asymptotic Expansions of Symbols

Definition 4.17. *Suppose $m_0 > m_1 > m_2 > \cdots m_j \epsilon \mathbb{R}, m_j$ tends to $-\infty$, and $p_j \epsilon S^{m_j}(\Omega), p \epsilon S^{m_o}(\Omega)$. We say that $p \sim \sum\limits_{j=0}^{\infty} p_j$ if $p - \sum\limits_{j<k} p_j \epsilon S^{m_k}(\Omega)$ for all k.*

Proposition 4.18. *Suppose m_j tends to $-\infty$ and $p_j \epsilon S^{mj}(\Omega)$. Then there exists a p in $S^{m_o}(\Omega)$ such that $p \sim \sum\limits^{\infty} j = 0 p_j$. This p is unique modulo $S^{-\infty}(\Omega)$.*

Proof. Let (Ω_n) be an increasing sequence of compact subsets of Ω whose union is Ω. Fix $\phi \epsilon C^\infty$ with $\phi = 1$ for $|\xi| \geq 1 >$ and $\phi = 0$ for $|\xi| \leq \frac{1}{2}$. ◻

Claim There exists a sequence $(t_j), t_j \geq 0$ and t_j tending to ∞ so rapidly that we have 91
(4.19)
$|D_x^\beta D_\xi^\alpha (\phi(\xi/t_j) p_j(x,\xi))| \leq 2^{-j}(1+|\xi|)^{m_{j-1}-|\alpha|}$ for $x \epsilon \Omega_i$, and $|\alpha|+|\beta|+i \leq j$.

Granted this, we define

$$p(x,\xi) = \sum_{j=0}^{\infty} \phi(\xi/t_j) p_j(x,\xi).$$

Note that for each x and ξ, the sum is finite. Using (4.19), it is straightforward to show that $p \epsilon S^{m_o}(\Omega)$ and $p \sim \sum\limits_{j=0}^{\infty} p_j$. Moreover, suppose that $q \epsilon S^{m_o}(\Omega)$ and $q \sim \sum\limits_{j=0}^{\infty} p_j$. Then

$$p - q = (p - \sum_{j<k} p_j) - (q - \sum_{j<k} p_j) \epsilon S^{m_k}(\Omega) \text{ for all } k.$$

Hence $p - q \epsilon S^{-\infty}$.

we now briefly indicate the steps involved in proving the claim :

i) First observe that $|D_\xi^\gamma \phi(\xi/t_j)| \leq c|\xi|^{-|\gamma|}$ uniformly in j.

ii) Hence we have $|D_x^\beta D_\xi^\alpha(\phi(\xi/t_j)p_j(x,\xi))| \leq c_j(1+|\xi|)^{m_j-|\alpha|}$ for $x\epsilon\Omega_i$, $|\alpha|+|\beta|+i \leq j$.

iii) Finally, pick t_j so large that "$|\xi| \geq t_j/2 \geq c_j(1+|\xi|)^{m_j-m_{j-1}} \leq 2^{-j}$" Details are left to the reader.

The following theorem provides a useful criterion for the asymptotic relation $p \sim \sum\limits_{j=0}^{\infty} p_j$ to hold.

92 **Theorem 4.20.** *Suppose $p_j\epsilon S^{m_j}, m_j$ tends to $-\infty$, and $p\epsilon C^\infty(\Omega\times\mathbb{R}^n)$ satisfies following conditions:*

i) for all α and β and all $\Omega' \Subset \Omega$, there exists $c>0$, $\mu\epsilon\mathbb{R}$such that

$$|D_x^\beta D_\xi^\alpha p(x,\xi)| \leq c(1+|\xi|)^\mu, x\epsilon\Omega, \quad \textit{and}$$

ii) there exists a sequence $(\mu_k), \mu_k$ tending to $-\infty$ so that $|p(x,\xi) - \sum\limits_{j<k} p_j(x,\xi)| \leq C_{\Omega'}(1+|\xi|)^{\mu_k}$ for $x\epsilon\Omega'$. Then $p\epsilon S^{m_o}(\Omega)$ and $p \sim \sum\limits_{j=0}^{\infty} p_j$.

To prove this theorem, we need the following

Lemma 4.21. *Let Ω_1 and Ω_2 be two compact subsets of $\mathbb{R}^n$such that Ω_1 is contained in the interior of Ω_2. Then there exists constants $c_1>0$ and $c_2>0$ such that for all $f\epsilon C^2(\Omega_2)$,*

$$\mathrm{Sup}_{\Omega_1}|\partial_j f|^2 \leq c_1(\mathrm{Sup}_{\Omega_2}|f|^2) + c_2(\mathrm{Sup}_{\Omega_2}|f|)(\mathrm{Sup}_{\Omega_2}|\partial_j^2 f|).$$

Proof. It suffices to assume that $n=1$ and f is real valued. With this reduction, the proof becomes an exercise in elementary calculus. This idea is roughly as follows: We wish to show that if $\| f \|_\infty$ and $\| f'' \|_\infty$ are both small, then so is $\| f' \|_\infty$ is small and $|f'(x_o)|$ is large, then $|f'|$ will be large in some sizable interval $[a,b]$ containing x_o. But then $|f(b)-f(a)|$ and hence $\| f \|_\infty$ is large. We leave it to the reader to work out the quantitative details of this argument. □

93 **Proof of the theorem.** *By Proposition 4.18, there exists q in* S^{m_o} *such that* $q \sim \sum_{j=0}^{\infty} p_j$; *so, it will suffice to show that* $p - q \epsilon S^{-\infty}$. *First,*

$$|p(x,\xi) - q(x,\xi)| = |(p(x,\xi) - \sum_{j<k} p_j(x,\xi)) - (q(x,\xi) - \sum_{j<k} p_j(x,\xi))|.$$

For

$$\Omega' \subset\subset \Omega, |p(x,\xi) - \sum_{j<k} p_j(x,\xi)| \leq c_\Omega, (1+|\xi|)^{\mu_k}, x\epsilon\Omega'.$$

Also $q \sim \sum_{j=0}^{\infty} p_j$. *These show that, for any N,*

$$|p(x,\xi)| - q(x,\xi)| \leq C_{N\Omega'}(1+|\xi|)^{-N}, x\epsilon\Omega'.$$

We want to prove such an estimate for $D_x^\beta D_\xi^\alpha(p-q)$ also. To this end, we will apply Lemma 4.21 to the function $(x,\eta) \to (p-q)(x,\xi+\eta)$ considering ξ as a parameter, and taking $\Omega_1 = \Omega' \times 0$ and Ω_2 a small neighbourhood of Ω_1. If $|\alpha| + |\beta| = 1$, we use the estimate just established for $p-q$, together with the hypothesis (i) on the second order derivatives of p; the lemma implies that $D_x^\beta D_\xi^\alpha(p-q)$ is rapidly decreasing for $|\alpha| + |\beta| = 1$. Combining this with the hypothesis (i) on the third order derivatives of p, we see that $D_x^\beta D_\xi^\alpha$ is rapidly decreasing for $|\alpha| + |\beta| = 2$. Proceeding by induction on $|\alpha| + |\beta|$ we get the required result.

4 Properly Supported Operators

Since pseudo differential operators map C_o^∞ to C^∞ rather then C_o^∞, it is generally not possible to compose two of them. The problem may be remedied by considering a more restricted class of operators, the so called "properly supported" ones.

Definition 4.22. *A subset K of* $\Omega \times \Omega$ *is said to be* proper *if, for any compact set* $\Omega' \subset \Omega$, *both* $\pi_1^{-1}(\Omega') \cap K$ *and* $\pi_2^{-1}(\Omega') \cap K$ *are compact.* 94

Here π_1 and π_2 are projections of $\Omega \times \Omega$ onto the first and second factors.

For example the diagonal $\Delta = \{(x, x) : x \epsilon \Omega\}$ is proper. Most of the proper subsets we will be considering are neighbourhoods of subsets of the diagonal.

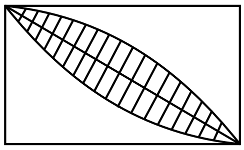


Figure 4.1: A proper set

Definition 4.23. *An operator $T : C^\infty_o(\Omega) \to C^\infty(\Omega)$ is said to be properly supported, if its distribution kernel K has proper support.*

Exercise. *Let $T = \sum a_\alpha(x) D^\alpha$ be a differential operator on Ω. Computer the distribution kernel K of T and show that supp K is a subset of the diagonal. Thus T is properly supported.*

If T is properly supported then it maps C^∞_o into itself, since supp $Tu \subset \pi_1(\pi_2^{-1}(\operatorname{supp} u) \cap \operatorname{supp} K)$, as is easily seen from the formula $Tu(x) = \int K(x, y)u(y)dy$. More generally, for any $\Omega' \subset\subset \Omega$, there exists $\Omega'' \subset\subset \Omega$ such that the values of Tu on Ω' depend only on the values of u on Ω'' namely, $\Omega'' = \pi_2(\pi^{-1}(\Omega') \cap \operatorname{supp} K)$. From this it follows that T can be extended to a map from $C^\infty(\Omega)$ to itself. In fact, if $u \epsilon C^\infty(\Omega)$ and Ω', Ω'' are as above, we define Tu on Ω' by

$$Tu|_{\Omega'} = T(\phi u)|_{\Omega},$$

95 where $\phi \epsilon C^\infty(\Omega)$ and $\phi = 1$ on Ω''. This definition is independent of the choice of ϕ and gives the same result on the intersection of two $\Omega' s$; so Tu is well defined on all of Ω.

If T is a properly supported pseudo differential operator, so that T extends to a map from $E'(\Omega)$ to $\mathcal{D}'(\Omega)$, the same arguments show that T maps $E'(\Omega)$ into itself and extends further to a map of $\mathcal{D}'(\Omega)$ to itself.

Suppose T and S are properly supported operators on $C^\infty(\Omega)$ with distribution kernels K and L which are C^∞ off the diagonal. Then TS

is an operator on $C^\infty(\Omega)$ with distribution kernel M formally given by $M(x, y) = \int K(x, z)L(z, y)dz$. In fact, if $x \neq y$, since $K(x, .)$ and $L(., Y)$ are smooth except at x and y, the product $K(x, .)L(., Y)$ is a well defined element of E', and the formula $M(x, y) =< K(x, .)L(., y), 1 >$ displays M as a C^∞ function off the diagonal.

Proposition 4.24. Supp *M is proper. Thus, TS is properly supported.*

Proof. Clearly,

$$\operatorname{Supp} M \subset \{(x, y) : \pi_2(\pi_1^{-1}(x) \cap \operatorname{supp} K) \cap \pi_1(\pi_2^{-1}(y) \cap \operatorname{supp} L) \neq \phi\}$$

Suppose $A \subset \Omega$ is compact and set $B = \pi_2(\pi_1^{-1}(A) \cap \operatorname{supp} K)$, then B is compact and

$$\begin{aligned}\pi_1^{-1}(A) \cap \operatorname{supp} M \subset A \times &\left\{y : B \cap \pi_1(\pi_2^{-1}(y) \cap \operatorname{supp} L) \neq \phi\right\}\\ = A \times &\left\{y : \pi_1^{-1}(B) \cap \pi_2^{-1}(y) \cap \operatorname{supp} L \neq \phi\right\}\\ = A \times &\,\pi_2(\pi_1^{-1}(B) \cap \operatorname{supp} L)\end{aligned}$$

which is compact. Likewise $\pi_2^{-1}(A) \cap \operatorname{supp} M$ is also compact. □

Exercise . *Suppose $A \subset (\Omega \times \Omega)$ is proper. Show that there exists a* **96**
properly supported $\phi \epsilon C^\infty(\Omega\times\Omega)$ such that $\phi = 1$ on A. (Hint: Let $\{\phi_j\} \subset C_o^\infty(\Omega \times \Omega)$ be a partition of unity on $\Omega \times \Omega$ and let $\phi = \sum\limits_{A \cap \operatorname{supp} \phi_j} = \phi^{\phi_j}$).

5 $\psi do'$s Defined by Multiple Symbols

We have arranged our definition of pseudo differential operators to agree with the usual convention for differential operators, according to which differentiations are performed first, followed by multiplication by the coefficients. However, in some situations (for example, computing adjoints), it is convenient to have a more flexible setup which allows multiplication operators both before and after differentiations. We therefore, introduce the following apparently more general class of operators (which, however, turns out to coincide with the class of ψDO, modulo smoothing operators).

Definition 4.25. *For Ω open in $\mathbb{R}^n$ and m real, we define the* class of multiple symbols of order m on Ω,

$$S^m(\Omega \times \Omega) = \{a \epsilon C^\infty(\Omega \times \mathbb{R}^n \times \Omega) : \forall \alpha, \beta, \forall \Omega' \subset\subset \Omega, \exists c = c_{\alpha\beta\Omega'}$$

such that $\sup\limits_{x,y\epsilon\Omega'} |D^\beta_{x,y} D^\alpha_\xi a(x, \xi, y)| \leq c(1 + |\xi|)^{m-|\alpha|}\}$

When $a \epsilon S^m(\Omega \times \Omega)$ we define the operator $A = a(x, D, y)$ by

$$Au(x) = \iint e^{2\pi i(x-y).\xi} a(x, \xi, y) u(y) dy d\xi$$

Here the integral must be interpreted as an iterated integral with integration performed first in y then in ξ as it is not absolutely convergent as a double integral.

97 We observe that if $a(x, \xi, y) = a(x, \xi)$ is independent of y, then $A = a(x, D)$; thus this class of operators include the $\psi DO's$. We also observe that different $a's$ may give rise to same operator. For example, if $a(x, \xi, y) = \phi(x)\psi(y)$ with $\psi, \phi \epsilon C^\infty_o(\Omega)$ and $\operatorname{supp} \psi \cap \operatorname{supp} \phi = \phi$, then $a \epsilon s^o(\Omega \times \Omega)$ and $a(x, D, y) = 0$.

Definition 4.26. *Given $a \epsilon S^m(\Omega \times \Omega)$ we define*

$$\sum_a = \{(x, y) \epsilon \Omega \times \Omega : (x, \xi, y) \epsilon \operatorname{supp}\ a \textit{ for some } \xi \epsilon \mathbb{R}^n\}.$$

Proposition 4.27. *Let $a \epsilon S^m(\Omega \times \Omega)$ and let K be the distribution kernel of $A = a(x, D, y)$. Then*

i) Supp $K \subset \sum_a$, *and*

ii) if the support of K is proper, then there exists $a' \epsilon S^m(\Omega \times \Omega)$ such that $a'(x, \xi, y) = a(x, \xi, y)$ when (x, y) is near the diagonal in $\Omega \times \Omega$, $\sum_{a,}$ is proper and $a(x, D, y) = a'(x, D, y)$.

Proof. The kernel K is given by

$$< K, w >= \iiint e^{2\pi i(x-y).\xi} a(x, \xi, y) w(x, y) dy d\xi dx.$$

From this, we see that $< K, w > . = 0$ whenever $\operatorname{supp} w \cap \sum_a = \phi$. Therefore, $\operatorname{supp} K \subset \sum_a$. This proves (i). □

To prove (ii), choose a properly supported $\phi\epsilon C^{\infty}(\Omega\times\Omega)$ with $\phi = 1$ on $\Delta\cup \operatorname{supp} K$, Δ being the diagonal. Set $a(x,\xi,y) = \phi(x,y)a(x,\xi,y)$. Then $\sum_{a'}\subset \operatorname{supp}\phi$ and hence $\sum_{a'}$ is proper. Also $a'(x,\xi,y) = a(x,\xi,y)$ when (x,y) is near the diagonal. Now,

$$\begin{aligned} < a(x,D,y)u, v > &=< K, v\otimes u > \\ &=< \phi K, v\otimes u > \\ &=< K, \phi(v\otimes u) > \\ &= \iiint e^{2\pi i(x-y)\cdot\xi} a(x,\xi,y)\phi(x,y)\times v(x)u(y)dyd\xi dx \\ &= \iiint e^{2\pi i(x-y)\cdot\xi} a'(x,\xi,y)u(y)dyd\xi dx \\ &=< a'(x,D,y)u, v > . \end{aligned}$$

98

Since v is arbitrary, we have $a(x,D,y) = a'(x,D,y)$.

Theorem 4.28. *Suppose $a\epsilon S^m(\Omega\times\Omega)$ and $A = a(x,D,y)$ is properly supported. Let $p(x,\xi) = e^{-2\pi i x.\xi}A(e^{2\pi i(.).\xi})(x)$. Then $p\epsilon S^m(\Omega)$ and $p(x,D) = A$. Further,*

$$p(x,\xi)\sim\sum_{a}\frac{1}{\alpha!}\partial_{\xi}^{\alpha}D_y^{\alpha}a(x,\xi,y)|_{y=x}.$$

Proof. For u in $C_o^{\infty}(\Omega)$, we have

$$u(x) = \int e^{2\pi i x.\xi}\hat{u}(\xi)d\xi.$$

Therefore, by linearity and continuity of A,

$$\begin{aligned} Au(x) &= \int A(e^{2\pi i(.).\xi})(x)\hat{u}(\xi)d\xi \\ &= \int e^{2\pi i x.\xi}p(x,\xi)\hat{u}(\xi)d\xi = p(x,D)u(x). \end{aligned}$$

□

By the proposition above, we can modify a so that $\sum_a$ is proper without affecting A or the behaviour of a along the diagonal $x = y$ (which is what enters into the asymptotic expansion of p). Set $b(x, \xi, y) = a(x, \xi, x + y)$. Then as a function of y, b has compact support. Indeed, 99
for fixed x and ξ, if $y \epsilon \{y' : b(x, \xi, y') \neq 0\}$ then $x + y \epsilon \pi_2(\pi_1^{-1}(x) \cap \sum_a)$ which is compact. Now,

$$\begin{aligned} p(x, \eta) &= e^{-2\pi i x.\eta} \iint e^{2\pi i(x-y).\xi} b(x, \xi, y - x) e^{2\pi i y.\eta} dy d\xi \\ &= e^{-2\pi i x.\eta} \iint e^{2\pi i z.\xi} b(x, \xi z) e^{2\pi i (z+x).\eta} dz d\xi \\ &= \int \hat{b}_3(x, \xi, \xi - \eta) d\xi = \int \hat{b}_3(x, \xi + \eta, \xi) d\xi, \end{aligned}$$

where $\hat{b}_3$ is the Fourier transform of b in the third variable. Since $b(x, \xi, .)$ is in C_o^∞, this calculation is justified and $\hat{b}_3(x, \eta, \xi)$ is rapidly decreasing in the variable ξ.

More precisely, since $a \epsilon S^m(\Omega \times \Omega)$, we have

$$|D_x^\beta D_\eta^\alpha \hat{b}_3(x, \eta, \xi)| \leq c_{\alpha\beta N}(1 + |\eta|)^{m-|\alpha|}(1 + |\xi|)^{-N}. \tag{4.29}$$

Since

$$\left(\frac{1 + |\eta|}{1 + |\xi|}\right)^s \leq (1 + |\xi - \eta|)^{|s|}, \text{ taking } \eta + \xi \text{ instead of } \eta$$

we have $(1 + |\xi + \eta|)^s \leq (1 + |\xi|)^s (1 + |\eta|)^{|s|}$. If we use this in

$$|D_x^\beta D_\eta^\alpha \hat{b}_3(x, \xi + \eta, \xi)| \leq c_{\alpha\beta N}(1 + |\xi + \eta|)^{m-|\alpha|}(1 + |\xi|)^{-N}$$

we get

$$|D_x^\beta D_\eta^\alpha \hat{b}_3(x, \xi + \eta, \xi)| \leq c'_{\alpha\beta N}(1 + |\xi|)^{-N+m-|\alpha|}(1 + |\eta|)^{|m|+|\alpha|}.$$

If we take N so that $-N + m - |\alpha| < -n$ we have

$$\begin{aligned} |D_x^\beta D_\eta^\alpha p(x, \eta)| &= \left| \int D_x^\beta D_\eta^\alpha \hat{b}_3(x, \xi + \eta, \xi) d\xi \right| \\ &\leq c_{\alpha\beta}(1 + |\eta|)^{|m|+|\alpha|} \end{aligned} \tag{4.30}$$

100 On the other hand, taking the Taylor expansion of $\hat{b}_3(x, \xi + \eta, \xi)$ in the middle argument about the point η, (4.29) gives

$$
\begin{aligned}
|\hat{b}_3(x, \xi + \eta, \xi) - \sum_{|\alpha|<k} \partial_\eta^\alpha \hat{b}_3(x, \eta, \xi) \frac{\xi^\alpha}{\alpha!}| \\
&\leq C|\xi|^k \sup_{\substack{|\alpha|=k \\ 0\leq t\leq 1}} |\partial_\eta^\alpha \hat{b}_3(x, \eta + t\xi, \xi)| \\
&\leq C_N|\xi|^k \sup_{0\leq t\leq 1} (1 + |\eta + t\xi|)^{m-k}(1 + |\xi|)^{-N}.
\end{aligned}
$$

When $|\xi| < \frac{1}{2}|\eta|$, taking $N = k$ we get

$$
|\hat{b}_3(x, \xi + \eta, \xi) - \sum_{|\alpha|<k} \partial_\eta^\alpha \hat{b}_3(x, \eta, \xi) \frac{\xi^\alpha}{\alpha!}| \leq c_k(1 + |\eta|)^{m-k}.
$$

When $|\xi| \geq \dfrac{1}{2}|\eta|$, we have

$$
|\hat{b}_3(x, \xi + \eta, \xi) - \sum_{|\alpha|<k} \partial_\eta^\alpha \hat{b}_3(x, \eta, \xi) \frac{\xi^\alpha}{\alpha!}| \leq C_N(1 + |\xi|)^{m+k-N}.
$$

(Actually, the exponent of $(1 + |\xi|)$) can be taken as $m - N$ when $m - k \geq 0$ and when $m - k < 0$). Also we have

$$
\begin{aligned}
\int \partial_\eta^\alpha \hat{b}_3(x, \eta, \xi)\xi^\alpha d\xi &= D_Y^\alpha \int e^{2\pi i Y\cdot\xi} \partial_\eta^\alpha \hat{b}_3(x, \eta, \xi) d\xi|_{Y=0} \\
&= D_Y^\alpha \partial_\eta^\alpha b(x, \eta, y)|_{Y=0} \\
&= D_Y^\alpha \partial_\eta^\alpha a(x, \eta, y)|_{y=x}.
\end{aligned}
$$

So finally

$$
\begin{aligned}
|p(x, \eta) - \sum_{|\alpha|<k} D_y^\alpha \partial_\eta^\alpha a(x, \eta, y) \frac{1}{\alpha!}|_{y=x}| \\
= |\int \hat{b}_3(x, \xi + \eta, \xi) d\xi - \sum_{|\alpha|<k} \frac{1}{\alpha!} \int \partial_\eta^\alpha \hat{b}_3(x, \eta, \xi)\xi^\alpha d\xi|
\end{aligned}
$$

$$\leq C_k \int\limits_{|\xi|<\frac{|\eta|}{2}} (1+|\eta|)^{m-k} d\xi + C_N \int\limits_{|\xi|\geq\frac{|\eta|}{2}} (1+|\xi|)^{m+k-N} d\xi.$$

Now 101

$$\int\limits_{|\xi|<\frac{|\eta|}{2}} C_k(1+|\eta|)^{m-k} d\xi = C'_k(1+|\eta|)^{m-k}|\eta|^n \leq C'_k(1+|\eta|)^{m-k+n}$$

In the second integral, we take $N > n+m+k$ so that

$$\int\limits_{|\xi|\geq\frac{|\eta|}{2}} C_N(1+|\xi|)^{m-N} d\xi \leq C'_N(1+|\eta|)^{m-N+n},$$

by the usual integration in polar coordinates. Therefore, since $N \geq k$ we obtain

$$|p(x,\eta) - \sum_{|\alpha|<k} \frac{1}{\alpha!} D^\alpha_y \sigma^\alpha_\eta a(x,\eta,y)|_{x=y}| \leq C_k(1+|\eta|)|^{\mu_k}$$

with $\mu_k = m-k+n$.

Combining this with (4.30), we see that all the conditions of Theorem 4.20 are satisfied, so we are done.

Corollary 4.31. *If $P\epsilon\psi^m$, there exists a $Q\epsilon\psi^m$ such that Q is properly supported and $P-Q$ is smoothing.*

Proof. If $P = p(x,D)$ choose $\phi\epsilon C^\infty(\Omega\times\Omega)$ which is properly supported and $\phi = 1$ near the diagonal. Set $a(x,\xi,y) = \phi(x,y)p(x,\xi)$. Then $a\epsilon S^m(\Omega\times\Omega)$ and by construction $\sum_a$ is proper. So $a(x,D,y) = Q\epsilon\psi^m$. If K_p and K_Q denote the distribution kernels of P and Q then it is easily seen that $K_Q = \phi K_p$. This shows that $K_Q - K_p$ vanishes near the diagonal: so, by Theorem 4.10, $K_Q - K_p$ is C^∞. Therefore, $Q-p$ is smoothing. ◻

Remark 4.32. The idea of multiple symbols can clearly be generalised. For example, if $a\epsilon C^\infty(\Omega\times\mathbb{R}^n\times\Omega\times\mathbb{R}^n)$ satisfies estimates of the form
102 $|D^\nu_{x,y}D^\alpha_\xi D^\beta_\eta a(x,\xi,y,\eta)| \leq C(1+\eta|)^{m_2-|\beta|}(1+|\xi|)^{m_1-|\alpha|}$, one can define an operator $A = a(x,D,y,D)$ by

$$Au(x) = \iiint e^{2\pi i(x.\xi - y.\xi + y.\eta)} a(x, \xi, y, \eta)\hat{u}(\eta) d\eta dy d\xi$$

which agree with our previous definitions, if a is independent of the last one or two variables. As one would expect, under reasonable restriction on supp a, it turns out that $A \epsilon \psi^{m_1+m_2}(\Omega)$. The reader may wish to amuse himself by working this out and computing the symbol of A.

6 Products and Adjoint of $\psi DO'S$

We are now in a position to compute products and adjoints of $\psi DO'S$. First we clarify our terminology. Let T S be linear operators from $C_o^\infty(\Omega)$ into $C^\infty(\Omega)$. We say that $S = T^t$ if $< Tu, v > = < u, Su >$ for $u, v \epsilon C_o^\infty(\Omega)$ and we say that $S = T^*$ if $< Tu, \bar{v} > = < u, \overline{Sv} >$ for $u, v \epsilon C_o^\infty(\Omega)$.

Remark 4.33. If T is properly supported, then T^t and T^* are also properly supported. Indeed, if K is the distribution kernel of T, then the distribution kernel of T^t is K^t and that of T^* is K^* where $K^t(x, y) = K(y, x)$ and $K^*(x, y) = K(\overline{y, x})$.

We recall that if $P \epsilon \psi^m(\Omega)$, we denote the corresponding symbol in $S^m(\Omega)$ by σ_p.

Theorem 4.34. *If $P \epsilon \psi^m(\Omega)$ is properly supported, then $P^t, P^* \epsilon \psi^m(\Omega)$ and $\sigma_p t(x, \xi) \sim \sum_\alpha \dfrac{(-1)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha D_x^\alpha \sigma_p(x, -\xi)$,*

$$\sigma_{p^*}(x, \xi) \sim \sum_a \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha \bar{\sigma}_p(x, \xi).$$

Proof. For $u, v \epsilon C_o^\infty(\Omega)$, we have **103**

$$\begin{aligned} < Pu, v > &= \iint e^{2\pi i x, \xi} p(x, \xi)\hat{u}(\xi) v(x) d\xi dx \\ &= \int \left(\int e^{2\pi i x, \xi} p(x, \xi) v(x) dx \right) \hat{u}(\xi) d\xi \end{aligned}$$

$$= \int g(\xi)\hat{u}(\xi)d\xi$$

where

$$g(\xi) = \int e^{2\pi i x,\xi} p(x,\xi)v(x)dx.$$

□

By Corollary 4.6, g us a rapidly decreasing function. Thus

$$< Pu, v >= \int g\hat{u} = \int u\hat{g} =< u, P^t_v >$$

so that

$$P^t v(y) = \hat{g}(y) = \iint e^{2\pi i(x-y).\xi} p(x,\xi)v(x)dxd\xi \tag{4.35}$$
$$= \iint e^{2\pi i(y-x).\xi} p(x,-\xi)v(x)dxd\xi$$
$$= a(x, D, y)v(y)$$

where $a(x,\xi y) = p(y. - \xi)$. Therefore, by Theorem 4.28,

$$P^t \epsilon \psi^m(\Omega) \text{ and } \sigma_{P^t}(x,\xi) \sim \sum_a \frac{(-1)^{|\alpha|}}{\alpha!} \partial^\alpha_\xi D^\alpha_x \sigma_P(x,-\xi).$$

The assertions about P^* follows along similar lines

Theorem 4.36. *If $P\epsilon\psi^{m_1}(\Omega)$ and $Q\epsilon\psi^{m_2}(\Omega)$ are properly supported, then $QP\epsilon\psi^{m_1+m_2}(\Omega)$ and*

$$\sigma_{QP}(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!} \partial^\alpha_\xi \sigma_Q(x,\xi) D^\alpha_x \sigma_p(x,\xi).$$

104 *Proof.* Since $P = (P^t)^t$, we have

$$(Pu)(x) = \iint e^{2\pi i(x-y)\cdot\xi} \sigma^t_P(y,-\xi)u(y)dyd\xi$$

by (4.35). □

In other words

$$(Pu)\hat{}(\xi) = \int e^{-2\pi iy\cdot\xi}\sigma_{P^t}(y,-xi)u(y)dy.$$

Therefore,

$$\begin{aligned}(QP)u(x) &= \iint e^{2\pi i(x-y)\cdot\xi}\sigma_Q(x,\xi)\sigma_{P^t}(y,-xi)u(y)dyd\xi\\ &= a(x,D,y)u(x)\end{aligned}$$

where $a(x,\xi,y) = \sigma_Q(x,\xi)\sigma_{P^t}(y,-\xi)$. Clearly $a\epsilon S^{m_1+m_2}(\Omega\times\Omega)$ which implies $QP\epsilon\psi^{m_1+m_2}(\Omega)$. Moreover

$$\begin{aligned}\sigma_{QP}(x,\xi) &\sim \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha D_y^\alpha(\sigma_Q(x,\xi)\sigma_{p^t}(y,-\xi))|_{y=x}\\ &\sim \sum_\alpha \frac{1}{\alpha!}\sum_{\beta+\nu=\alpha}\frac{\alpha}{\beta!\nu!}\partial_\xi^\beta\sigma_Q(x,\xi)\partial_\xi^\nu D_Y^\alpha\sigma_{P^t}(x,-\xi)\\ &\sim \sum_{\alpha,\nu}\sum_\delta \frac{(-1)^{|\delta|}}{\nu!\beta!\delta!}\partial_\xi^\beta\sigma_Q(x,\xi)D^{\beta+\nu+\delta}\partial_x^{\nu+\delta}\partial_\xi^{\nu+\delta}\sigma_P(x,-\xi)\\ &\sim \sum_{\beta,\lambda}\left(\sum_{\nu+\delta=\lambda}\frac{(-1)^{|\delta|}}{\nu!\beta!\delta!}\right)\frac{1}{\beta!}\partial_\xi^\beta\sigma_Q(x,\xi)D_x^{\beta+\lambda}\partial_\xi^\lambda\sigma_p(x,\xi)\end{aligned}$$

But $\sum\limits_{\nu+\delta=\lambda}\frac{(-1)^{|\delta|}}{\nu!\delta!} = (x_0-x_0)^\lambda$ with $x_0 = (1,1,\ldots,1)$

$$= \begin{cases}1 & \text{if } \lambda = 0\\ 0 & \text{if } \lambda \neq 0\end{cases}$$

Therefore $\sigma_{QP}(x,\xi) \sim \sum\limits_\beta \dfrac{1}{\beta!}\partial_\xi^\beta\sigma_Q(x,\xi)D_x^\beta\sigma_p(x,\xi)$.

Corollary 4.37 (PRODUCT RULES FOR $\psi DO'S$)**.**

If $Q = q(x,D)\epsilon\psi^m(\lambda)$ and $f\epsilon C^\infty(\Omega)$, then for any positive integer N, **105**
$q(x,D)(fu) = \sum\limits_{|\alpha|<N}\dfrac{1}{\alpha!}D^\alpha f[(\partial_\xi^\alpha q)(x,D)u] + T_Nu$ *where $T_N\epsilon\psi^{m-N}(\Omega)$.*

Corollary 4.38. *The correspondence* $p \to p(x, D)$ *and* $P \to \sigma_p$ *are * homomorphisms modulo lower order term i.e., if* $p\epsilon S^{m_1}(\Omega)$ *and* $q\epsilon S^{m_2}(\Omega)$*, then* $(pq)(x, D) - p(x, D)q(x, D)\epsilon\psi^{m_1+m_2-1}(\Omega)$ *and* $p(x, D)^* - \overline{p}(x, D)$ $\epsilon\psi^{m_1-1}$ *and also, if* $P\epsilon\psi^{m_1}(\Omega)$ *and* $Q\epsilon\psi^{m_2}(\Omega)$*, then* $\sigma_{PQ} - \sigma_P\sigma_Q$ $\epsilon S^{m_1+m_2-1}(\Omega)$*,* $\sigma_{P*} - \overline{\sigma} + p\epsilon S^{m_1-1}(\Omega)$*.*

Corollary 4.39. *If* $P\epsilon\psi^{m_1}(\Omega)$ *and* $Q\epsilon\psi^{m_2}(\Omega)$*, then* $[P, Q] = PQ - QP$ $\epsilon\ \psi^{m_1 m_2-1}(\Omega)$ *and* $\sigma_{[P,Q]} - \dfrac{1}{2\pi i}\{\sigma_P, \sigma_Q\}\epsilon S^{m_1 m_2-1}(\Omega)$ *where* $\{f, q\}$ *stands for the Poisson bracket defined by*

$$\{f, q\} = \sum \left(\frac{\partial f}{\partial \xi_j}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial \xi_j}\right).$$

Following the philosophy that smoothing operators are negligible it is a trivial matter to extend these results to non-properly supported operators.

Suppose $P\epsilon\psi^m(\Omega)$ is not properly supported. By corollary 4.31, we can write $P = P_1 + s$ where P_1 is properly supported and S is smoothing. Then $P^t = P_1^t + S$ where P_1^t is given by Theorem 4.34 and S^t is again smoothing (c.f. Remark 4.33). Likewise for P^*. If Q is a properly supported ψDO, the products PQ and QP are well defined as operators from $C_o^\infty(\Omega)$ to $C^\infty(\Omega)$. Again, we have $PQ = P_1Q + SQ$ and $QP = QP_1 + Qs$; P_1Q and QP_1 are described by Theorem 4.36, while SQ and QS are smoothing.

7 A Continuity Theorem for ψ Do on Sobolev Spaces

106 We now state and prove a continuity theorem for pseudo defferential operators acting on Sobolev spaces. We are indebted to Dr P.N. Srikanth for simplifying our original argument.

Theorem 4.40. *Suppose* $P = p(x, D)\epsilon\psi^m(\Omega)$*. Then*

i) $P : H_s \to H^{\text{loc}}_{s-m}(\Omega)$ *continuous for all* $s\epsilon\mathbb{R}$*.*

ii) *If P is properly supported, $P : H_s^{\text{loc}}(\Omega) \to H_{s-m}^{\text{loc}}(\Omega)$ continuously for all $s \epsilon \mathbb{R}$*

Proof. To established (i), we must show that the map $u \to \phi P u$ is bounded from $H_s \to H_{s-m}$ for any $\phi \epsilon C_o^\infty(\Omega)$. Replacing $p(x,\xi)$ by $\phi(x)p(x,\xi)$ we must show: □

(4.41) if $p \epsilon s^m(\mathbb{R})$ and $p(x,\xi) = 0$ for x outside a compact set, then $p = p(x, D)$ is bounded from H_s and H_{s-m} for every s in $\mathbb{R}$.

Suppose then that $u \epsilon H_s$. Then $pu \epsilon E'$, and from the definition of P on distributions, we see that

$$\begin{aligned}
(Pu)\hat{}(\eta) &=< Pu, e^{-2\pi i \eta.(.)} > \\
&= \iint e^{2\pi i(\xi-\eta).x} p(x,\xi)\hat{u}(\xi) dx d\xi \\
&= \int \hat{p}_1(\eta-\xi,\xi)\hat{u}(\xi) d\xi
\end{aligned}$$

and hence, if $V \epsilon S$,

$$< Pu, \bar{v} > = \int (Pu)\hat{}\hat{v} = \iint K(\eta,\xi) f(\xi)\bar{g}(\eta) d\xi d\eta$$

where
$$\begin{aligned}
f(\xi) &= (1+|\xi|^2)^{s/2}\hat{u}(\xi) \\
g(\eta) &= (1+|\eta|^2)^{(m-s)/2}\hat{v}(\eta)
\end{aligned}$$

and $K(\eta,\xi) = \hat{p}_1(\eta-\xi,\xi)(1+|\xi|^2)^{-s/2}(1+|\eta|^2)^{(s-m)/2}$. 107

We wish to estimate $K(\eta,\xi)$. For any multi-index α, since $p(.,\xi)\epsilon C_o^\infty$, we have

$$\begin{aligned}
|\zeta^\alpha \hat{p}_1(\zeta,\xi)| &= |\int (D_x^\alpha e^{-2\pi i x.\zeta)} p(x,\xi) dx| \\
&= |\int e^{-2\pi i x.\zeta)} (D_x^\alpha p(x,\xi) dx| \\
&\leq C_\alpha (1+|\xi|)^m,
\end{aligned}$$

so that for any positive integer N,

$$|\hat{p}_1(\zeta,\xi)| \leq C_N (1+|\xi|)^m (1+|\zeta|)^{-N}$$

and hence

$$\begin{aligned}|K(\eta,\xi)| &\le C_N(1+|\xi|)^{m-s}(1+|\eta|)^{s-m}(1+|\xi-\eta|)^N\\ &\le C_N(1+|\xi|)^{|m-s|-N}\end{aligned}$$

If we take $N > n + |m - s|$, we see that

$$\int |K(\eta,\xi)|d\eta \le C, \int |K(\eta,\xi)|d\xi \le C.$$

Therefore by Theorem 1.1 and the Schwarz inequality,

$$| < Pu, \overline{v} > | \le C||f||_2||g||_2 = C||u||_{(s)}||v||_{(m-s)}.$$

From this, it follows that $||Pu||_{(s-m)} \le C||u||_{(s)}$ for $u\epsilon H_s$ which establishes (4.41) and hence (i).

Now suppose P is properly supported and $u\epsilon H_s^{\text{loc}}(\Omega)$. If $\phi\epsilon C_o^\infty(\Omega)$ there exists $\Omega' \Subset \Omega$ such that the values of Pu on supp ϕ depend only the values of u on Ω. Thus if we pick $\phi'\epsilon C_o^\infty(\Omega)$ with $\phi' = 1$ on Ω', we
108 have $\phi Pu = \phi P(\phi' u)$. But $\phi' u\epsilon H_s$ and (4.41) ϕP is bounded from H_s to H_{s-m}. This establishes (ii) and completes the proof.

8 Elliptic Pseudo Differential Operators

Definition 4.42. *$P\epsilon\psi^m(\Omega)$ is said to be elliptic of order m if $|\sigma_P(x,\xi)| \ge C_\Omega, |\xi|^m$ for large $|\xi|$, for all $x\epsilon\Omega', \Omega' \Subset \Omega$.*

Definition 4.43. *If $P\epsilon\psi^m(\Omega)$, a* left (resp. right) parametrix of *P is a ψDOQ such that QP − I(resp. PQ -I) is smoothing.*

Theorem 4.44. *If $P\epsilon\psi^m$* is elliptic of order m and properly supported, then there exists a properly supported $Q\epsilon\psi^{-m}$ which is a two-sided parametrix for P.

Proof. Let $P = p(x, D)$. We will obtain $Q = q(x, D)$ with $q \sim \sum\limits_{j=0}^{\infty} q_j$ where the q_j's are defined recursively. Let $\zeta(x,\xi)$ be a C^∞ function such that $\zeta(x,\xi) = 1$ for large ξ and $\zeta(x,\xi) = 0$ in a neighbourhood of the zeros of p. Define $q_o(x,\xi) = \dfrac{\zeta(x,\xi)}{p(x,\xi)}$. Then $q_o \epsilon s^{-m}$. Let $Q_o = q_o(x,D)$. We have

$$\sigma_{Q_O^p} = q_0^p \mod S^{-1} = 1 \mod S^{-1} = 1 + r_1 \text{ with } r_1 \epsilon S^{-1}.$$

□

Let $q_1 = \dfrac{-r_1\zeta}{p}$, $Q_1 = q_1(x,D)$. Then

$$\begin{aligned}\sigma_{(Q_0+Q_1)P} &= \sigma_{Q_O^p} + \sigma_{Q_1^p} = 1 + r_1 - r_1\zeta \mod S^{-2}\\ &= 1 + r_2 \text{ with } r_2 \epsilon S^{-2}.\end{aligned}$$

Let $q_1 = \dfrac{-r_2\zeta}{p}$ etc. Having determined $q_1, q_2, \ldots q_j$ so that

$$\sigma_{(Q_0+Q_1+\cdots Q_j)P} = 1 + r_{j+1}, r_{j+1} \epsilon S^{-(j+1)}$$

set
$$q_{j+1} = \frac{-r_{j+1}\zeta}{p}.$$

109

Let Q be a properly supported operator with symbol $q \sim \sum\limits_{j=0}^{\infty} q_j$. Then $\sigma_{QP} = 1 \mod S^{-\infty}$, i.e., $QP - I$ is smoothing. Thus Q is a left parametrix. In the same way, we can construct a right parametrix Q'. Then if $S = QP - I$ and $S' = PQ' - I$, we have

$$QPQ' = SQ' + Q' = QS' + Q.$$

So $Q - Q' = SQ' - QS'$ is smoothing. Hence Q is a two sided parametrix.

Exercise. *What happens if P is not properly supported? (cf. the remarks at the end of §6).*

The left parametrices are used to prove regularity theorems and the right parametrices are used to prove existence theorems.

Indeed, we have:

Corollary 4.45 (ELLIPTIC REGUIARITY THEOREM)**.** *If P is elliptic of order m, $u\epsilon\mathcal{D}'(\Omega)$, $Pu\epsilon H_s^{\text{loc}}(\Omega)$ implies $u\epsilon H_{s+m}^{\text{loc}}(\Omega)$. In particular, P is hypoelliptic.*

Proof. Let Q be a left parametrix and set $S = QP - I$. Then $u = QPu - Su$. Since$Pu\epsilon H_s^{\text{loc}}(\Omega)$ and Q is properly supported, $Q\epsilon\psi^{-m}$ we have $QPu\epsilon H_{s+m}^{\text{loc}}(\Omega)$ by Theorem 4.40. Also $Su\epsilon C^\infty$ since S is smoothing. Hence $u\epsilon H_{s+m}^{\text{loc}}(\Omega)$. ◻

Theorem 4.46. *Every elliptic differential operator is locally solvable.*
110 *In fact, if P is elliptic on Ω, $f\epsilon\mathcal{D}'(\Omega)$ and $x_o\epsilon\Omega$, there exists $u\epsilon\mathcal{D}'(\Omega)$ such that $Pu = f$ in a neighbourhood of x_o. (Of course, if $f\epsilon C^\infty$, then $u\epsilon C^\infty$ near x_o, by the previous corollary).*

Proof. BY cutting f off away from x_o, we can assume that $f\epsilon E'$ and hence $f\epsilon H_s$ for some s. Let Q be a properly supported parametrix for P and let $S = PQ - I$. Then S is also properly supported. If $\Omega_1 \subset\subset \Omega$ is a neighbourhood of x_o then there exists $\Omega_2 \subset\subset \Omega$ such that the values of Su on Ω_1 depend only on the values of u on Ω_2. Pick $\phi_1, \phi_2\epsilon C_o^\infty(\Omega)$ such that $\phi_j \equiv 1$ on Ω_j, $j = 1, 2$ and set $Tu = \phi_1 S(\phi_2 u)$. ◻

Now observe the following :

i) $Tu = su$ on Ω_1.

ii) $T : H^s \to C_o^\infty(\text{suup } \phi_1)$ continuously.

From (ii) and the Arzela -Ascoli theorem, it follows that if (u_k) is a bounded sequence in $H_{s'}(Tu_k)$ has a convergent subsequence in $C_o^\infty(\text{suup } \phi_1)$ and hence in H_s. Therefore, T is compact on H_s. So the equation $(T + I)u = f$ can be solved if f is orthogonal (with respect to the pairing of H_s and H_{-s}) to the space $N = \{g : (T^* + I)g = 0\}$. This space N is a finite dimensional space of C^∞ functions so we can always make this happen by modifying f outside a small neighbourhood of x_o.

Indeed, pick a basis $g_1, g_2, \ldots, g_\nu$ for N and pick a neighbourhood U of x_o so small that $g_1, \ldots, g_\nu$ are linearly independent as functionals on $C_o^\infty(\Omega \backslash U)$. (Such a U exists; otherwise, by a limiting argument using the local compactness of N, we could find a nontrivial linear combination of

$g_1, \dots, g_\nu$ supported at $\{x_o\}$, which is absurd). Then we can make $f \perp N$ by adding to f a function in $C_o^\infty(\Omega \backslash U)$. But then

$$PQu = (I + S)u = (I + T)u \text{ on } \Omega_1 \text{ and } (I + T)u = f$$

in a neighbourhood of x_o. So Qu solves the problem. **111**

9 Wavefront Sets

We now introduce the notion of wavefront sets, which provides a precise way of describing the singularities of distributions: it specifies not only the points at which a distribution is not smooth but the directions in which it is not smooth.

All pseudo differential operators encountered in this section will be presumed to be properly supported, and by "ψDO" we shall always mean "properly supported ψDO".

Definition 4.47. (i) *Let Ω be an open set in $\mathbb{R}^n$. Then we define $T^o\Omega = \Omega \times (\mathbb{R}^n \backslash \{0\})$. (In coordinate - invariant terms, $T^o\Omega$ is the cotangent bundle of Ω with the zero section removed).*

(ii) *A set $S \subset T^o\Omega$ is called* conic *if $(x, \xi) \epsilon S \Rightarrow (x, r\xi) \epsilon S, \forall r > 0$*

(iii) *Suppose $P = p(x, D) \epsilon \psi^m(\Omega)$. Then $(x_o, \xi_o) \epsilon T^o\Omega$ is said to be non-characteristic for P if $|p(x, \xi)| \geq c|\xi|^m$ for $|\xi|$ large and (x, ξ) in some conic neighbourhood of (x_o, ξ_o).*

(iv) *The characteristic variety of P, denoted by char P, is defined by char $P = \{(x, \xi) \epsilon T^o\Omega : (x, \xi)$ is characteristic for $p\}$.*

(v) *Let $u \epsilon \mathcal{D}'(\Omega)$. The wavefront set of u, denoted by $WF(u)$ is defined by*

$$WF(U) = \cap \{\, charP : P \epsilon \psi(\Omega), Pu \epsilon C^\infty(\Omega)\}.$$

The restriction $P \epsilon \psi^0(\Omega)$ in the definition of $WF(u)$ is merely a convenient normalisation. We could allow ψDO of arbitrary order without changing anything, for if $P \epsilon \psi^m(\Omega)$ and $Q \epsilon \psi^{-m}(\Omega)$ is elliptic, then $QP \epsilon \psi^0(\Omega)$, char (QP) = char (P)(by Theorem 4.36), and $QPu \epsilon C^\infty$ if and only if $Pu \epsilon C^\infty$ (by Corollaries 4.11 and 4.45). **112**

Exercise. *When* $p = \sum_{|\alpha|=\leq m} a_\alpha(x)\xi^\alpha$ *show that the characteristic variety of the differential operator* $p = P(x, D)$ *is*

$$char\ P = \{(x, \xi) : \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha = 0\}.$$

The motivation for $WF(u)$ is as follows. If $u\epsilon E'$, to say that u is not smooth in the direction ξ_o should mean that $\hat{u}$ is not rapidly decreasing on the through ξ_o. On the other hand, if $Pu\epsilon C_o^\infty$ then $(Pu)\hat{}$ should be rapidly decreasing everywhere.

If these conditions both hold, then ξ_o must be characteristic for P. Localising these ideas, we arrive at $WF(u)$.

Thus $(x_o\xi_o)\epsilon WF(u)$ means roughly that u fails to be C^∞ at x_o in the direction ξ_o. For another interpretation of this statement, see Theorem 4.56 below. For the present, we show that $WF(u)$ is related to sing supp u as it should be. We denote the projection $T^o\Omega$ onto Ω by π.

Theorem 4.48. *For* $u\epsilon\mathcal{D}'(\Omega)$, *sing* supp $u = \pi(WF(u))$.

Proof. Suppose $x_0 \notin \sin g \operatorname{supp} u$. Then there exists $\phi\epsilon C_o^\infty$ with $\phi = 1$
113 near x_o and $\phi u\epsilon C_o^\infty$. But multiplication by ϕ is a ψDO of order 0. Call it P_ϕ. Then char $P_\phi = \pi^{-1}(\phi^{-1}(0))$. Since $\phi = 1$ near $x_o, \pi^{-1}(\phi^{-1}(0))$ is disjoint from $\pi^{-1}(x_o)$. Therefore, $\pi^{-1}(x_o) \cap WF(u) = \phi$, i.e., $x_o \notin \pi(WF(u))$. □

Conversely, suppose $x_o \notin \pi(WF(u))$. Then for each ξ with $|\xi| = 1$, there exists $P\epsilon\psi^0(\Omega)$ with $Pu\epsilon C^\infty(\Omega)$ and $(x_o, \xi) \notin$ char P.

Each char P is a closed conic set; so, by the compactness of the unit sphere, there exists a finite number of $\psi DO'$s say, $P_1, \ldots P_N\epsilon\psi^0(\Omega)$ with $P_j u\epsilon C^\infty(\Omega)$ and

$$\left(\bigcap_{j=1}^{N} \text{char } P_j\right) \cap \pi^{-1}(x_o) = \phi. \text{ Set } P = \sum_{j=1}^{N} P_j^* P_j.$$

Then P is elliptic near x_o and $Pu\epsilon C^\infty(\Omega)$. Therefore by Corollary 4.45, u is C^∞ near x_o, i.e., $x_o \notin \sin g \operatorname{supp} u$.

Definition 4.49. *Let $P = p(x, D)\epsilon\psi^m(\Omega)$ and U be an open conic set in $T^o\Omega$. We say that P has order $-\infty$ on U, if, for all closed conic sets $K \subset U$ with $\pi(K)$ compact for every positive integer N and multi-indices α and β, there exist constants $C_{\alpha\beta KN}$ such that*

$$|D_x^\beta D_\xi^\alpha p(x, \xi)| \leq C_{\alpha\beta KN}(1 + |\xi|)^{-N}, (x, \xi)\epsilon K.$$

The *essential support* of P is defined to be the smallest closed conic set outside of which P has order $-\infty$.

Exercise . *If P is a differential operator whose coefficients do not all vanish at any point of Ω, show that the essential support of P is $T\Omega$.*

Proposition 4.50. *Ess.* supp $PQ \subset$ *Ess.* supp $P\cap$ *Ess.* supp Q. **114**

Proof. This follows immediately from the expansion

$$\sigma_{PQ} \sim \sum \frac{1}{\alpha!}\partial_\xi^\alpha \sigma_P D_x^\alpha \sigma_C.$$

□

Lemma 4.51. *Let $(x_o, \xi_o)\epsilon T^o\Omega$ and U be any conic neighbourhood of (x_o, ξ_o). Then there exists a $P\epsilon\psi^0(\Omega)$ such that $(x_o, \xi_o) \notin$ char P but Ess.* supp $P \subset U$.

Proof. Choose $p_o(\xi)\epsilon C^\infty$ with the following properties:

i) $p_o(\xi) = 1$ for ξ near ξ_o, $p_o(\xi) = 0$ outside $\{\xi : (x_o\xi)\epsilon U\}$ and

ii) p_o is homogeneous of degree 0 for large $|\xi|$.

□

Then take $\phi\epsilon C^\infty(\pi(U))$ with $\phi = 1$ near x_o and put $p(x, \xi) = p_o(\xi)$ $\phi(x)$. This will do the job.

The following theorem and its corollary are refinements of Corollary 4.11 (pseudo local property of ψDO) and Corollary 4.45 (elliptic regularity theorem).

Theorem 4.52. *If $P\epsilon\psi^m(\Omega)$ and $u\epsilon D'(\Omega)$, then $WF(Pu) \subset WF(u)\cap$ Ess.* supp P.

Proof. If $(x_o, \xi_o) \notin$ Ess. supp P, by Lemma 4.51, we can find a $Q\epsilon\psi^0$ such that $(x_o, \xi_o) \notin$ char Q and Ess. supp $P\cap$ Ess. supp $Q = \phi$ Then $QP\epsilon\psi^{-\infty}$ by Proposition 4.50, so that $QP\epsilon C^\infty$. But this implies that $(x_o, \xi_o) \notin WF(Pu)$. □

Suppose now that $(x_o, \xi_o) \notin WF(u)$. Then there exists $A\epsilon\psi^0$ with AuϵC^∞ and $(x_o, \xi_o) \notin$ char A.

115 **Claim.** *There exist operators $B, C\epsilon\psi^0$ with $(x_o, \xi_o) \notin$ char B and $BP = CA$ mod $\psi^{-\infty}$.*

Granted this, $BPu = CAu + (C^\infty$ function) ϵC^∞ which implies that $(x_o, \xi_o) \notin WF(Pu)$.

To prove the claim, let A_o be elliptic with $\sigma_A = \sigma_A$ on a conic neighbourhood U of (x_o, ξ_o). Then Ess. supp$(A - A_o) \cap U = \phi$. By Lemma 4.51, choose B with $(x_o, \xi_o) \notin$ char B and Ess. supp $B \subset U$. Let E_o be a parametrix for A_o and set $C = BPE_o$.Then $CA = BPE_oA = BPE_o(A - A_o) + BPE_oA_o$. Since $E_oA_o = I$ mod $\psi^{-\infty}$ and $BPE_o(A - A_o)$ also belongs to $\psi^{-\infty}$ (by Proposition 4.50 again), $CA = BP$ mod $\psi^{-\infty}$. This completes the proof.

Corollary 4.53. *If P is elliptic, then $WF(u) = WF(Pu)$).*

Proof. By the theorem, $WF(Pu) \subset WF(u)$. If E is a parametrix for

$$
\begin{aligned}
P, \text{ then } WF(u) &= WF(EPu + C^\infty \text{ function}) \\
&= WF(EPu) \subset WF(Pu), \text{ by the theorem again} \\
\text{Hence} \quad WF(u) &= WF(Pu).
\end{aligned}
$$
□

Theorem 4.56. *Let $u\epsilon\mathcal{D}'(D)$. Then $(x_o, \xi_o) \notin WF(u)$ if and only if there exists $\phi\epsilon C_o^\infty(\mathbb{R}^n)$ such that $\phi(x_o) = 1$ and $(\phi u)\hat{}$ is rapidly decreasing on a conic neighbourhood of ξ_o.*

Proof. (Sufficiency) If such a ϕ exists, choose $p\epsilon S^0, p(x, \xi) = p(\xi)$ with $p(\xi) = 1$ near ξ_o and $p(\xi) = 0$ outside the region where $(\phi u)\hat{}$ is rapidly decreasing. Then $p(\phi u)\hat{}\epsilon S$ and hence $p(D)(\phi u)\epsilon S$ where $p(D) =$
116 $p(x, D)$. The operator $Pu = \phi p(D)(\phi u)$ is a pseudo differential operator of order 0 with symbol $\phi(x)^2 p(\xi)$ modulo S^{-1}, so $(x_o, \xi_o) \notin$ char P. This implies that $(x_o, \xi_o) \notin WF(u)$. □

(*Necessity*) Suppose $(x_o, \xi_o) \notin WF(u)$. Then there exists a neighbourhood U of x_o in Ω such that $(x_o, \xi_o) \notin WF(u)$ for all $x \epsilon U$. Choose $\phi \epsilon C_o^\infty(U)$ such that $\phi(x_o) = 1$. Then $(x_o, \xi_o) \notin WF(\phi u)$ for all $x \epsilon \mathbb{R}^n$. Let

$$\sum = \{\xi : (x, \xi) \epsilon WF(\phi u) \text{ for some} x\}.$$

This $\sum$ does not contain ξ_o and is a closed conic set. There exists $p(\xi) \epsilon S^0$, $p = 1$ on a conic neighbourhood of ξ_o and $p = 0$ on a neighbourhood of $\sum$, say $\sum_o$. Since $p(\xi) = 0$ on $\sum_o$, Ess. supp $p(D) \cap (\mathbb{R}^n \times \sum_o)^c$ and hence Ess. supp $p(D) \cap WF(\phi u) = \phi$. But this gives Ess. supp$(p(D)\phi u) = \phi$. So $p(D)\phi u \epsilon C^\infty$. We now *claim* that $p(D)\phi u \epsilon S$.

Accepting this, we see that $p(\phi u)\hat{} \epsilon S$ and in particular, $(\phi u)\hat{}$ is rapidly decreasing near ξ_o. To prove the *claim*, observe that

$$D^\beta(\xi^\alpha p(\xi)) = 0(1 + |\xi|)^{|\alpha| - |\beta|} \epsilon L^2 \text{ if } |\alpha| - |\beta| < -\frac{n}{2}.$$

So if we put $K(x) = \tilde{p}(x)$, $x^\beta D^\alpha K(x) \epsilon L^2$ if $|\alpha| - |\beta| < -\frac{n}{2}$. An application of Leibniz's rule and the the Sobolev imbedding theorem shows that $x^\beta D^\alpha K(x) \epsilon L^\infty$ if $|\beta| - |\alpha| > -\frac{3n}{2} + 2$. Hence K and all its derivatives are rapidly decreasing at ∞ and since $\phi u \epsilon E'$, the same is true of $p(D)\phi u = \phi u * K$.

10 Some Further Applications of Pseudo Defferential Operators

We conclude our discussion of pseudo differential operators by giving **117**
brief and informal descriptions of some further applications. We recall that an m^{th} order ordinary differential equation $u^{(m)} = F(x, u, u^{(1)}, \ldots u^{(m-1)})$ can be reduced to the first order system

$$\frac{d}{dx}\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} = \begin{pmatrix} u_2 \\ u_3 \\ \vdots \\ F(x, u_1, \ldots, u_m) \end{pmatrix}$$

simply by introducing the derivatives of u of order $< m$ as new variables, and this reduction is frequently a useful technical device. To do something similar for m^{th} order partial differential equations, however, is more problematical. If one simply introduces all partial derivatives of u of order $< m$ as new variables and writes down the first order differential relations they satisfy, one usually obtains more equations than there are unknowns, because of the equality of the mixed partials. Moreover, such a reduction usually does not preserve the character of the original equation. For example take the Laplace equation in two variables :

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f.$$

putting $u_1 = u, u_2 = \dfrac{\partial u}{\partial x}, u_3 = \dfrac{\partial}{\partial y}$ we have a 3×3 system given by

$$\begin{aligned} \frac{\partial u_1}{\partial x} - u_2 &= 0 \\ \frac{\partial u_1}{\partial y} - u_3 &= 0 \\ \frac{\partial u_2}{\partial x} + \frac{\partial u_3}{\partial y} &= f. \end{aligned}$$

118 The original equation is elliptic. But consider the matrix of the top order symbols of the 3×3 system obtained above. The matrix is given by

$$2\pi i \begin{pmatrix} \xi & 0 & 0 \\ \eta & 0 & 0 \\ 0 & \xi & \eta \end{pmatrix}$$

which is not invertible. This means that the first order system is not elliptic.

There is, however, a method, due to A.P. CALDERON, of reducing an m^{th} order linear partial differential equation to an $m \times m$ system of first order pseudo differential equations which preserves the characteristic variety of the equation in a sense which we shall make precise below. In this method, one of the variables is singled out to play a special role, so we shall suppose that we are working on $\mathbb{R}^{n+1}$ with coordinates $(x_1, x_2, \ldots, x_n, t)$.

Let L be a partial differential operator of order m on $\mathbb{R}^{n+1}$ such that the coefficient of ∂_t^m is nowhere vanishing. Dividing throughout by this coefficient, we can assume that L is of the following form:

$$L = \partial_t^m - \sum_{j=0}^{m-1} A_{m-j}(x, t, D_x)\partial_t^j.$$

119

Here A_{m-j} is a differential operator of order $\leq m - j$ in the x variable with coefficients depending on t.

We want to reduce the equation $Lv = f$ to a first order system. To this end we proceed as follows:

Using the familiar operators Λ^s with symbol $(1 + |\xi|)^{s/2}$ (acting in the x variables) we put

$$\begin{aligned}
u_1 &= \Lambda^{m-1} v \\
u_2 &= \Lambda^{m-2} \partial_{t^v} \\
u_3 &= \Lambda^{m-3} \partial_t^2 v \\
&\vdots \\
u_m &= \partial_t^{m-1} v.
\end{aligned}$$

For $j < m$, we observe that $\partial_t u_j = \Lambda u_{j+1}$. Therefore,

$$\begin{aligned}
\partial_t u_m &= f + \sum_{j=0}^{m-1} A_{m-j}(x, t, D_x)\partial_t^j v \\
&= f + \sum_{j=0}^{m-1} A_{m-j}(x, t, D_x)\Lambda^{j-m+1} u_{j+1} \\
&= f + \sum_{j=1}^{m} A_{m-j}(x, t, D_x)\Lambda^{j-m} u_j
\end{aligned}$$

Set $B_j(x, t, D_x) = A_{m-j+1}(x, t, D_x)\Lambda^{j-m}$. This B_j is a ΨDO of order 1 in the variable x. Then the equation $Lv = f$ is equivalent to

$$\partial_t u = Ku + \tilde{f}$$

where $u = (u_1, \ldots, u_m)^t$, $\tilde{f} = (0, 0, \ldots, f)^t$. (Here $(\cdots)^t$ denotes the 120
transpose of the row vector $(\cdots)$) and K is the matrix given by

$$K = \begin{pmatrix} 0 & \Lambda & 0 \cdots 0 \\ 0 & 0 & \Lambda \cdots 0 \\ . & . & \cdots \\ . & . & \cdots \\ 0 & 0 & 0 \cdots \Lambda \\ B_1 & B_2 & B_3 \cdots B_m \end{pmatrix}$$

This K is a matrix of first order pseudo differential operators in x, with coefficients depending on t.

Let us now make a little digression to fix on some notations which will be used in the further development. For a $p \epsilon S^m$, a function $p_m(x, \xi)$ homogeneous of degree m in ξ is said to be the *Principal symbol* of $p(x, D)$ if $p - p_m$ agrees with an element of S^{m-1} for large $|\xi|$. We remark that not all $\Psi DO's$ have a principal symbol; but most of them that arise in practice do. Moreover, the principal symbol, if it exists, is clearly unique.

Examples

(i) If p is a polynomial,

$$p(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha,$$

then

$$p_m(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha.$$

121 (ii) If $p(x, \xi) = (1 + |\xi|^2)^{s/2}$, $p_s(x, \xi) = |\xi|^s$.

Returning to our discussion, let $a_k(x, t, \xi)$ be the principal symbol of $A_k(x, t, d_x)$ (Here we are considering A_k as an operator of order k, so if it happens to be of lower order, its principal symbol is zero). Then the principal symbol of B_j is $b_j(x, t\xi) = a_{m-j+1}(x, t, \xi)|\xi|^{j-m}$ and the principal symbol of k is the matrix K_1

$$K_1(x,t,\xi) = \begin{pmatrix} 0 & |\xi| & 0 & 0 & \cdots & 0 \\ 0 & 0 & |\xi| & 0 & \cdots & 0 \\ 0 & 0 & 0 & |\xi| & \cdots & 0 \\ . & . & . & & \cdots & \\ . & . & . & & \cdots & \\ . & . & . & & \cdots & \\ 0 & 0 & 0 & 0 & \cdots & |\xi| \\ b_1 & b_2 & b_3 & b_4 & \cdots & b_m \end{pmatrix}$$

The principal symbol of L, on the other hand, is

$$L_m(x,t,\xi,\tau) = (2\pi i\tau)^m - \sum_{j=0}^{m-1} a_{m-j}(x,t,\xi)(2\pi i\tau)^j$$

The characteristic variety of L, i.e., the set of zeros of L_m, can easily be read off from the matrix K as follows:

Proposition 4.57. *The eigenvalues of* $K_1(x,t,\xi)$ *are precisely* $2\pi i\tau_1, \ldots,$ $2\pi i\tau_m$ *where* $\tau_1, \tau_2, \ldots, \tau_m$ *are the roots of the polynomial* $L_m(x,t,\xi,.)$.

Proof. The characteristic polynomial $p(\lambda)$ of K_1 is the determinant of

$$\begin{pmatrix} \lambda & -|\xi| & 0 & \cdots & 0 \\ 0 & \lambda & -|\xi| & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & -|\xi| \\ -b_1 & -b_2 & -b_3 & \vdots & \lambda - b_m \end{pmatrix}$$

122

Expanding in minors along the last row, we get

$$\begin{aligned} p(\lambda) &= \sum_{j=1}^{m-1}(-1)^{m+j}(-b_j)\lambda^{j-1}(-|\xi|)^{m-j} + \lambda^{m-1}(\lambda - b_m) \\ &= \lambda^m - \lambda^{m-1}b_m - \sum_{j=1}^{m-1} b_j|\xi|^{m-1}\lambda^{j-1} \end{aligned}$$

$$= \lambda^m - \sum_{j=1}^{m} a_{m-j+1}\lambda^{j-1} = \lambda^m - \sum_{j=0}^{m-1} a_{m-j}\lambda^j$$
$$= L_m(x, t, \xi, (2\pi i)^{-1}\lambda\lambda.$$

This proves the proposition. □

It is also easy to incorporate boundary conditions into this scheme. Suppose we want to solve the equation $Lv = f$ with the boundary conditions $B_j v|_{t=0} = g_j, 1, 2, \ldots, \nu$ where

$$B_j = \sum_{k=0}^{m_j} b_j^k(x, t, D_x)\partial_t^k, b_j^k \text{ is of order } m_j - k \text{ and } m_j \leq m - 1.$$

If we set

$$B_j^k = \Lambda^{m-m_j-1} b_j^{k-1}(x, 0, D_x)\Lambda^{k-m},$$
$$\phi_j = \Lambda^{m-m_j-1} g_j$$

then with $u_j = \Lambda^{m-j}\partial_t^{j-1} v$ as above and $u_j^0(x) = u_j(x, 0)$, the boundary conditions will become

$$\sum_{k=1}^{m_j+1} B_j^k u_k^0 = \phi_j, j = 1, 2, \ldots \nu.$$

123

This is a system of zeroth order pseudo differential equations.

The above method of reduction is useful, for example, in the following *two important problems*.

(1) Cauchy Problem for Hyperbolic Equations

$$Lv = f, \partial_t^j v|_{t=0} = g_j, j = 0, 1, \ldots, m - 1.$$

Here the equation is said to be *hyperbolic* when the eigenvalues of the matrix K_1 (i.e., the principal symbol of K occurring in the linear system corresponding to $Lv = f$) are purely imaginary.Equivalently, the roots of $L_m(x, t, \xi, .)$ are real.

(2) Elliptic Boundary Value Prob Lems

$$Lu = f \text{ on } \Omega, B_j u = g_j \text{ on } \partial\Omega$$

where L is elliptic of order $2m$ and $j = 1, 2, \ldots m$. Here one works locally near a point $x_0 \epsilon \partial\Omega$ and makes a change of coordinates so that $\partial\Omega$ becomes a hyperplane near x_o. One then can apply Calderon's reduction technique, taking t as the variable normal to $\partial\Omega$. (See Michael Taylor [3]).

We shall now sketch an example of a somewhat different technique for applying $\psi DO'S$ to elliptic boundary value problems.

Let Ω be a bounded open set of $\mathbb{R}^n$ with C^∞ boundary $\partial\Omega$ Consider the problem

$$\Delta u = 0 \text{ in } \Omega, \partial_X u + au = g \text{ on } \partial\Omega, a \epsilon C^\infty(\partial\Omega).$$

Here X is a real a vector field on the boundary which is nowhere tangent to the boundary and $\partial_X u = \text{grad } u.\chi$. 124

If ν is the unit outward normal to $\partial\Omega$, by normalising we can assume that $\chi = \nu + \tau$ where τ is tangent to $\partial\Omega$. We want to use ΨDO to reduce this to the Dirichlet problem. Pretend for the moment that $\partial\Omega = \mathbb{R}^{n-1} x\{0\}$ and $\Omega = \{x : x_n < 0\}$ so that $\partial_\nu = \dfrac{\partial}{\partial x_n}$.

If $\Delta u = 0$ then $\partial_\nu^2 u = -\sum\limits_{j=1}^{n-1} \dfrac{\partial^2 u}{\partial x_j^2} = -\Delta_b u, \Delta_b$ being the Laplacian on the boundary. So formally $\partial_\nu u = \pm\sqrt{(\Delta_b u)}$. Indeed, if we compare this with our discussion of the Poisson kernel in Chapter 2(taking account of the fact that Ω is now the lower rather than the upper half space), we see that the equation $\partial_\nu u = \sqrt{(-\Delta_b)} u$ is correct if interpreted in terms of the Fourier transform in the variables $x' = (x_1, \ldots x_{n-1})$;

$$\partial_\nu \tilde{u}(\xi', x_n) = 2\pi|\xi'|\tilde{u}(\xi', x_n).$$

It turns out that something similar works for our original domain Ω. Namely, if u is smooth on $\bar{\Omega}$ and $\Delta u = 0$ on Ω then $\partial_\nu u|_{\partial\Omega} = P(u|_{\partial\Omega})$ where P is a pseudo differential operator of order 1 on $\partial\Omega$ which equals $\sqrt{(-\Delta_b)}$ modulo terms of order ≤ 0 where Δ_b is the Laplace -Beltrami operator on $\partial\Omega$. In particular, P is elliptic and has real principal symbol.

The boundary conditions become $Pv + |\partial_t v + av = g, v = u|_{\partial\Omega}$. This is a first order pseudo differential equation for v. It is elliptic (because ∂_τ has imaginary symbol). So it can be solved modulo smoothing oper-
125 ators. Since $\partial\Omega$ is compact. smoothing operators on $\partial\Omega$ are compact, so our boundary equation can be solved provided g is orthogonal to some finite dimensional space of smooth functions.

Having done this, we have reduced our original problem to the familiar Dirichlet problem $\Delta u = 0$ in Ω and $u|_{\partial\Omega} = v$ which has a unique solution.

Chapter 5

L^P and Lipschitz Estimates

OUR AIM IN this chapter is to study how to measure the smoothing **126**
properties of pseudo differential operators of non positive order in terms of various important function spaces. Most of the interesting results are obtained by considering operators of order $-\lambda$ with $0 \leq \lambda \leq n$. Indeed, if $P \epsilon \Psi^{-\lambda}$ with $\lambda > n$ and $K(x, y)$ is the distribution kernel of P then, by Theorem 4.10 we know that $K \epsilon C^j \Omega \times \Omega)$ when $j < \lambda - n$. So these operators can be studied by elementary methods. What is more, when $P \epsilon \Psi^{-\lambda}, D^\alpha P \epsilon \Psi^{-\lambda + |\alpha|}$. So, by a proper choice of α, we can make $0 \leq \lambda - |\alpha| \leq n$ and then study $D^\alpha P$ rather than P itself. Actually, we shall restrict attention to operators of order $-\lambda$ where $0 \leq \lambda < n$. The transitional case $\lambda = n$ requires a separate treatment to obtain sharp results; however, for many purpose, it suffices to make the trite observation that an operator of order $-n$ can be regarded as an operator of order $-n + \epsilon$. Further, we restrict our attention to $\Psi DO's$ whose symbols have asymptotic expansions

$$p(x, \xi) \sim \sum_{j=0}^{\infty} p_j(x, \xi)$$

where P_j is homogeneous of degree $-\lambda - j$. (These $p'_j s$ no longer belong to our symbol classes, being singular at $\xi = 0$, but we can still consider the operators $p_j(x, D)$.) By the preceding remarks, it will suffice to consider the operators corresponding to the individual terms in the series **127**

$\sum p_j$ whose degrees of homogeneity are between 0 and $-n$. Thus we are looking at $p(x, D)$ where $p(x, \xi)$ is C^∞ on $\mathbb{R}^n \times \mathbb{R}^n \backslash \{0\}$ and homogeneous of degree $-\lambda$ in ξ where $0 \leq \lambda < n$.

Since $\lambda < n$, for each $x, p(x, .)$ is locally integrable at the origin and hence defines a tempered distribution. Denoting the inverse Fourier transform of this distribution $p_2^\vee(x, .)$ we then have

$$\begin{aligned} p(x, D)u(x) &= \int e^{2\pi i x.\xi} p(x, \xi)\hat{u}(\xi)d\xi \\ &= (p_2^\vee(x, .) * u)(x) \\ &= \int p_2^\vee(x, x-y)u(y)dy. \end{aligned}$$

Thus we see that the distribution kernel of $p(x, D)$ is given by $K(x, y) = p_2^\vee(x, x-y)$.

Let us digress a little to make some remarks on homogeneous distributions.

Definition 5.1. *A distribution $f \epsilon S'$ is said to be* homogeneous of degree μ, *if $< f, \phi_r >= r^\mu < f, \phi >$ for all $\phi \epsilon S$ where ϕ_r is the function defined by $\phi_r(x) = r^{-n}\phi(x/r)$.*

Exercise.

1. Show that the above definition agrees with the usual definition of homogeneity when f is a locally integrable function.

2. Show that if $f \epsilon S'$ is homogeneous of degree μ, then $D^\alpha f$ is homogeneous of degree $\mu - |\alpha|$.

128 3. Show that $D^\alpha \delta$ is homogeneous of degree $-n - |\alpha|$ (where δ is the Dirac measure at 0).

Let us now prove a proposition concerning the Fourier transform of homogeneous distributions. It turns out that the Fourier transform of such a distribution is also a homogeneous one. Precisely, we have the following

Proposition 5.2. *If f is a tempered distribution, homogeneous of degree μ, then $\hat{f}$ is homogeneous of degree $-\mu - n$. If f is also C^∞ away from the origin, then the same is true of $\hat{f}$.*

Proof If f is homogeneous of degree μ, then for $\phi \epsilon S$

$$\begin{aligned} < \hat{f}, \phi_r > &=< f, (\phi_r)\hat{} >=< f, r^{-n}\hat{\phi}_{1/r} >= r^{-n-\mu} < f, \hat{\phi} > \\ &= r^{-\mu-n} < \hat{f}, \phi > . \end{aligned}$$

□

This proves the first assertion. To prove the second assertion, choose $\phi \epsilon C_0^\infty$ with $\phi = 1$ in a neighbourhood of the origin and write $f = \phi f + (1 - \phi)f$. Since $\phi f \epsilon E', (\phi f)\hat{}$ is C^∞ everywhere. On the other hand, $(1 - \phi)f$ is C^∞ and homogeneous of degree μ for large ξ, and hence lies in $S^\mu(\mathbb{R}^n)$. Let $p(x, \xi) = (1 - \phi)(\xi) f(\xi)$. The distribution kernel of $p(x, D)$ is given by

$$\begin{aligned} K(x, y) &= \int e^{-2\pi i(y-x),\xi}(1 - \phi)(\xi) f(\xi) d\xi \\ &= ((1 - \phi) f)\hat{}(y - x). \end{aligned}$$

Since K is C^∞ away from the diagonal, we see that $((1 - \phi) f)\hat{}$ is C^∞ away from the origin. Hence $\hat{f}$ is C^∞ on $\mathbb{R}^n \backslash \{0\}$.

Returning to our operators $p(x, D)$ with $p(x, \xi)$ homogeneous of de- **129**
gree $-\lambda$ in $\xi, 0 \leq \lambda < n$, by the proposition above, we have

$$p(x, D)u = \int K(x, x - y)u(y)dy$$

where $K = P_2^\vee$ is C^∞ on $\mathbb{R}^n \times \mathbb{R}^n \backslash \{0\}$ and homogeneous of degree $\lambda - n$ in the second variable.

Finally, we restrict attention to constant coefficient case, i.e., $K(x, x - y) = K(x - y)$. The essential ideas are already present in this case and the results we shall obtain can be generalised to the variable coefficient case in a rather routine fashion. Let us give a name to the objects we are finally going to study.

Definition 5.3. *A tempered distribution K which is homogeneous of degree $\lambda - n$ and C^∞ away from the origin is called a kernel of type λ. If K is a kernel of type λ, the operator $Tf = K * f$ is called an operator of type λ.*

We now classify the kernels of type $\lambda \geq 0$.

Proposition 5.4. *Suppose $\lambda > 0$ and $f \epsilon C^{\infty}(\mathbb{R}^n \backslash \{0\})$ is homogeneous of degree $\lambda - n$ in the sense of functions. Then f is locally integrable and defines a distribution F which is a kernel of type λ.*

Conversely, if F is a kernel of type λ and f is the function with which F agrees on $\mathbb{R}^n \backslash \{0\}$,then $< F, \phi >= \int f\phi$ for every ϕ in S.

Proof. Since $\lambda > 0$, f is locally integrable and of (at most) polynomial growth at ∞, so f defines an F in S'. It is easy to check that F is
130 homogeneous in the sense defined above, i.e.$< F, \phi_r >= r^{\lambda-n} < F, \phi >$, and so F is a kernel of type λ. □

For the converse, define G by $< G, \phi >=< F, \phi > - \int f\phi, \phi \in S$. Then G is a distribution supported at 0 and hence we have $G = \sum c_\alpha D^\alpha \delta$. Therefore,

$$< G, \phi_r >= \sum c_\alpha r^{-n-|\alpha|} (D^\alpha \phi)(0) = O(r^{-n}), \text{ as } r \to \infty.$$

On the other hand,

$$< G, \phi_r >= r^{-n-\lambda} < G, \phi >$$

and this is not $O(r^{-n})$ as r tends to ∞ unless $< G, \phi >= 0$. Hence $G = 0$ and this completes the proof.

Suppose that $f \epsilon C^{\infty}(\mathbb{R}^n \backslash \{0\})$ is homogeneous of degree $-n$. Then f is not locally integrable near 0 and so does not define a distribution in a trivial way. However, let us define

$$\mu_f = \int\limits_{|x|=1} f(x) d\sigma(x).$$

If $\mu_f = 0$ there is a canonical distribution associated with f which is called *principal value* of $f, PV(f)$, defined by

$$< PV(f), \phi >= \lim_{\epsilon \to 0} \int\limits_{|x|>\epsilon} f(X) dx.$$

To see that this limit exists, we observe that

$$\int\limits_{\epsilon<|x|<1} f(x)dx = \mu_f \int\limits_{\epsilon}^{1} r^{-1}dr = 0.$$

Hence

$$< PV(f), \phi >= \lim_{\epsilon\to 0} \int\limits_{\epsilon<|x|<1} f(x)(\phi(x) - \phi(0))dx + \int\limits_{|x|\geq 1} f(x)\phi(x)dx,$$

where the last integrals are absolutely convergent, since **131**

$$|\phi(x) - \phi(0)| \leq c|x|, \qquad \text{so that}$$

$$\int\limits_{|x|<1} |f(x)||\phi(x) - \phi(0)|dx \leq c \int\limits_{|x|<1} |x|^{-n+1}dx < \infty.$$

Further, the estimate on $\phi(x) - \phi(0)$ depends only on the first derivatives of ϕ via the mean value theorem, so it is easily verified that the functional $PV(f)$ is continuous on S. Finally, we observe that

$$\begin{aligned} < PV(f), \phi_r > &= \lim_{\epsilon\to 0} \int\limits_{|x|>\epsilon} f(x)r^{-n}\phi(x/r)dx \\ &= \lim_{\epsilon\to 0} \int\limits_{|y|>(\epsilon/r)} f(y)r^{-n}\phi(y)dy \\ &= r^{-n} < PV(f), \phi > . \end{aligned}$$

Thus $PV(f)$ is homogeneous of degree -n and so is a kernel of type 0.

The following theorem gives a sort of converse to the above results.

Theorem 5.5. *Suppose F is a kernel of type* 0 *and f is the function with which F agrees on $\mathbb{R}^n\backslash\{0\}$. Then $\mu_f = 0$ and $F = PV(f) + c\delta$, for some constant c.*

Proof. Define the functional G on S by

$$\square \qquad < G, \phi >= \int\limits_{|x|\leq 1} f(x)(\phi(x) - (\phi(0))dx + \int\limits_{|x|>1} f(x)\phi(x)dx.$$

By the same argument as above, G is a tempered distribution and $G = F$ on $\mathbb{R}^n/\{0\}$. Therefore, $G - F = \sum c_\alpha D^\alpha \delta$. Now, $< F, \phi_r >= r^{-n} < P, \phi >$ so that

$$< G, \phi_r > -r^{-n} < G, \phi >=< G - F\phi_r > -r^{-n} < G - F, \phi >= 0(r^{-n})$$

132 as r tends to ∞. On the other hand, for $r > 1$,

$$\begin{aligned} < G, \phi_r > - r^{-n} < G, \phi >= & \int\limits_{|x|\leq 1/r} r^{-n} f(x)(\phi(x) - \phi(0))dx \\ & + \int\limits_{|x|>1/r} r^{-n} f(x)\phi(x)dx - r^{-n} \int\limits_{|x|\leq 1} f(x)(\phi(x) - \phi(0))dx - r^{-n} \\ & \int\limits_{|x|>1} f(x)\phi(x)dx \\ & = r^{-n}\phi(0) \int\limits_{1r\leq|x|\leq 1} f(x)dx = r^{-n}\phi(0)\log r\mu_{f'} \quad \text{for every } \phi. \end{aligned}$$

This is not $0(r^{-n})$ as r tends to ∞ unless $\mu_f = 0$. Finally, $F - PV(f)$ is a kernel of type 0 which is supported at the origin and hence is a multiple of δ.

To study the boundedness of operators of type λ on L^p spaces, we need some concepts from measure theory.

Definition 5.6. *Let F be a measurable function defined on $\mathbb{R}^n$. Then the distribution function of f is the function $\delta_f : (0, \infty) \to [0, \infty]$ given by $\delta_f(t) = |E_t|$, where $E_t = \{x : |f(x)| > t\}$.*

From the distribution function of f, we can get a large amount of information regarding f. For example, we have

$$\int |f(x)|^p dx = -\int\limits_0^\infty t^p d\delta_f(t)$$

(To see this, observe that the Riemann sums for the Stieltjes integral on the right are approximating sums for the Lebesque integral on the left).

If $t^p \delta_f(t)$ converges to 0 as t tends to 0 and t tends to ∞, we can integrate by parts to obtain $\int |f(x)|^p dx = p \int\limits_0^\infty t^{p-1} \delta_f(t) dt$.

Using the concept of distribution functions, we will now define weak L^p spaces. 133

Definition 5.7. *For* $1 \leq p < \infty$ *we define weak* L^p *as* $\{f : \delta_f(t) \leq (c/t)^p$ *for some constant* $c\}$. *For* $f \epsilon$ *weak* L^p*, the smallest such constant* c *will be denoted by* $[f]_p$*. Thus, if* $f \epsilon$*weak* L^p*, we have*

$$\delta_f(t) \leq ([f]_p/t)^p.$$

For $p = \infty$, we set weak $L^\infty = L^\infty$.

Proposition 5.8. *CHEBYSHEV 'S INEQUALITY)* $L^p \subset$ *weak* L^p *and* $[f]_p \leq ||f||_p$.

Proof. For $f \epsilon L^p$, if $E_t = \{x : |f(x)| > t\}$,

$$||f||_p^p = \int\limits_{\mathbb{R}^n} |f(x)|^p dx \geq \int\limits_{E_t} |f(x)|^p dx \geq t^p |E_t|$$

i.e., $|E_t| \leq (||f||^p/t)p$.

From this, it follows that $f \epsilon$ weak L^p and $[f]_p \leq ||f||_p$. □

Remark 5.9. It is not true that L^p = weak L^p for $1 \leq p < \infty$. For example, $f(x) = |x|^{-n/p}$ belongs to weak L^p but not L^p.

Remark 5.10. The function $f \to [f]_p$ satisfies $[cf]_p = |c|[f]_p$.

But it fails to satisfy triangle inequality and hence is not a norm. However, since

$$\{|f + g| > t\} \subset \{|f|t/2\} \cup \{|g| > t/2\},$$

we have

$$\delta_{f+g}(t) \leq \delta_f(t/2) + \delta_g(t/2)$$

which gives, when f and g are in weak L^p, **134**

$$\delta_{f+g}(t) \leq (2^p[f]_p^p + 2^p[g]_p^p)/t^p \text{ so that}$$
$$[f+g]_p \leq 2([f]_p^p + [g]_p^p)^{1/p} \leq 2([f]^p + [g]_p).$$

The functional $[.]_p$ thus defines a topology on weak L^p which will turn it into a (non-locally convex) topological vector space.

Definition 5.11. *A linear operator T defined on a space of functions is said to be of* weak type $(p,q), 1 \leq p,q \leq \infty$, *if T is a bounded linear operator from L^p into weak L^q, i.e. for every $f \epsilon L^p, [Tf]_q \leq c\|f\|_p$ for some constant c independent of f.*

We will now state the Marcinkiewicz interpolation theorem and use it to prove generalisations of Young's inequality (Theorem 1.3).

Theorem 5.12 (J. Marcinkiewicz)**.** *Suppose T is of weak types (p_o,q_o) and (p_1,q_1) with $1 \leq p_i \leq q_i \leq \infty$, $p_0 < p_1, q_0 \neq q_1$, i.e., $['\Gamma f]^{q_i} \leq c_i\|f\|_{p_i}$ for $i = 0,1$. Then, if*

$$1/p_\theta = (1-\theta)/p_0 + (\theta/p_1), 1/q_\theta = (1-\theta)/q_0 + (\theta/q_1), 0 < \theta < 1,$$

T is bounded from L^{p_θ} to L^{q_θ} i.e., $\|Tf\|_{q_\theta} \leq c_\theta\|f\|_{p_\theta}$ where the constant c_θ depends only on $p_0, q_0, p_1, q_1, c_0, c_1$ *and θ.*

For the proof of this theorem, see *A.* Zygmund [5] or E.M.Stein [2].

Theorem 5.13 (GENERAL FROM OF YOUNG'S INEQUALITY)**.** *If $(1/p) + (1/q) - 1 = 1/r, 1 \leq p,q,r < \infty$, Then, for $f \epsilon L^p, g \epsilon L^q$ $f * g \epsilon L^r$ and we have $\|f * g\|_r \leq c_{pq}\|f\|_p\|g\|_q$.*

135 (In fact, $c_{pq} \leq 1$ for all p,q, although our proof does not yield this estimate).

Proof. Fixing $f \epsilon L^p$, consider the convolution operator $g \to T_q = f * g$. We know that for $g \epsilon L^1, Tg \epsilon L^p$ and $\|Tg\|_p \leq \|f\|_p\|g\|_1$. Further if p' is the conjugate of p, then, by Hölder's inequality $\|Tg\|_\infty \leq \|g\|_p, \|f\|_p$ for $g \epsilon L^{p'}$. Thus we see that the operator T is of weak types $(1,p)$ and

(p', ∞). Therefore, by the Marcinkiewicz interpolation theorem, T maps L^{p_θ} boundedly into L^{q_θ} for every $0 < \theta < 1$ with

$$(1/p_\theta) = 1 - \theta + (\theta/p') = 1 - (\theta/p) \text{ and } 1/q_\theta = (1/p) - \theta/p.$$

□

Given r with $(1/p) + (1/q) - 1 = (1/r)$, set $q_{\theta=r}$. Then $(1/p_\theta) = 1 + (1/r) - (1/p) = (1/q)$. Hence we get the required result.

Theorem 5.14 (WEAK TYPE YOUNG'S INEQUALITY)**.** *Let* $1 \le p < q < \infty$, $(1/p) + (1/q) > 1$ *and* $(1/p) + (1/q) - 1 = (1/r)$. *Suppose* $f \epsilon L^p$ *and* $g\epsilon$ *weak* L^q. *Then, we have*

a) $f * g$ *exists a.e. and is in weak* L^r, *and*

$$[f * g]_r \le c_{pq} \|f\|_p [g]_q$$

b) If $p > 1$, *then* $f * g \epsilon L^r$ *and* $\|f * g\|_r \le c'_{pq} \|f\|_p [g]_q$.

Proof. a) It suffices to assume that $\|f\|_p = [g]_p = 1$ and to show that $[f * g]_r \le c_{pq}$. Given $\alpha > 0$, let

$$M = (\alpha/2)^{p'/(p'-q)} (p'/(p'-q))^{-1/(p'-q)}$$

where p' is as usual the conjugate of p. Define

$$\begin{aligned} g_1(x) &= g(x), \text{ if } |g(x)| > M \\ &= 0 \text{ otherwise} \\ \text{and} \qquad g_2(x) &= g(x) - g_1(x). \qquad \text{Then} \end{aligned}$$

136

$$\delta_{f*g}(\alpha) \le \delta_{f*g_1}(\alpha/2) + \delta_{f*g_2}(\alpha/2)$$

and we shall estimate the quantities on the right separately. By Holder's inequality,

$$\|f * g_2\|_\infty \le \|f\|_p \|g_2\|_{p'} = \|g_2\|_{p'}.$$

Since $(1/q) - (1/p') = 1/r > 0$, we see that $p' - q > 0$, and hence

$$\begin{aligned}
||g_2||_{p'}^{p'} &= p' \int\limits_0^\infty t^{p'-1} \delta_{g_2}(t) dt \\
&= p' \int\limits_0^M t^{p'-1} (\delta_g(t) - \delta_g(M)) dt \\
&\leq p' \int\limits_0^M t^{p'-1} t^{-q_{dt}} = (p'/(p'-q)) M^{p'-q} = (\alpha/2)^{p'}.
\end{aligned}$$

Thus $(f * g_2)(x)$ exists at all points and $||f * g_2||_\infty \leq \alpha/2$. Consequently $\delta_{f*g_2}(\alpha/2) = 0$. Next consider

$$\begin{aligned}
||g_1||_1 &= \int\limits_0^\infty \delta_{g_1}(t) dt = \int\limits_0^M \delta_g(M) dt + \int\limits_M^\infty \delta_g(t) dt \\
&\leq \int\limits_0^M M^{-q} dt + \int\limits_M^\infty t^{-q} dt.
\end{aligned}$$

The integral $\int\limits_M^\infty t^{-q} dt$ converges since $q > 1$, and we obtain

$$||g_1||_1 \leq M^{1-q} + M^{1-q}/(q-1) = (q/(q-1)) M^{1-q}.$$

Also, by Chebyshev inequality,

$$\begin{aligned}
\delta_{f*g_1}\left(\frac{\alpha}{1}\right) &\leq \left(\frac{||f * g_1||}{\alpha/2} p\right) p \\
&\leq (2^p/\alpha^p)(q/(q-1))^p (\alpha/2)^{\frac{p'}{p'-q} p(1-q)} (p'/(p'-q))^{-\frac{p(1-q)}{p'-q}} \\
&= c_{pq} \alpha^{-r}
\end{aligned}$$

137

Hence (a) is proved.

b) The operator $f \to f * g$ is of weak type $(1, q)$ by (a). Also, if we choose $\overline{p} > p$ with $(1/\overline{p}) + (1/q) > 1$ and put $(1\overline{r}) = (1/\overline{p}) + (1/q) - 1$ then by (a), $f \to f * g$ is of weak type $(\overline{p}, \overline{r})$. Therefore, by Marcinkiewicz, $f \to f * g$ is bounded from L^p into L^{q_θ} with

$$1/p_\theta = 1 - \theta + (\theta/\overline{p}),\ (1/q_\theta) = ((1-\theta)/q) + \ (\theta/\overline{r}).$$

Putting $q_\theta = r$ we get $p_\theta = p$. Hence T maps L^p continuously to L^r and $\|f * g\|_r \leq c'_{pq}\|f\|_p[g]_q$. □

Corollary 5.15. *If T is an operator of type $\lambda, 0 < \lambda < n$, then T is bounded from L^p into L^q, whenever $1 < p < q < \infty$ and $1/q = 1/p - \lambda/n$. Also, T is of weak type $(1, n/(n-\lambda))$.*

Proof. If K is a kernel of type λ, we have

$$|K(x)| = |x|^{\lambda-n}K(x/|x|)| \leq c|x|^{\lambda-n},$$

which implies that $K\epsilon$ weak $L^{n/(n-\lambda)}$. Therefore, by Theorem 5.14(b), if $f \epsilon L^p$ and $Tf = K * f$ then $Tf \epsilon L^q$ where

$$(1/q) = (1/p) + ((n-\lambda(/n) - 1 = (1/p) - (\lambda/n) \text{ and } \|Tf\|_q \leq c[f]_p \leq c\|f\|_p.$$

□

Also we see that T is of weak type $(1, n/(n-\lambda))$ by Theorem 5.14 (a).

The limiting case of this result with $\lambda = 0$ is also true, but this is a **138**
much deeper theorem:

Theorem 5.16. *(Calderon - Zygmund) Operators of type 0 are bounded on $L^p, 1 < p < \infty$.*

Proof. Let T be an operator of type 0 with kernel K i.e., $Tf = K * f$. To begin with, we can regard T as a map from c_o^∞ to c^∞ and we shall show that $\|Tf\|_p \leq c_p\|f\|_p$ for $f \epsilon c_o^\infty$, $1 < p < \infty$, so that T extends uniquely to a bounded operator on L^p. □

We have $K = PV(k) + c\delta$ so that $Tf = PV(k) * f + cf$. Since the identity operator is continuous, we shall assume that $c = 0$ and also we shall identify K with k. The proof now proceed in several steps.

Step 1. T is bounded on L^2. Indeed, we have $(Tf)\hat{t} = \hat{k}\hat{f}.\hat{k}$ is smooth away from 0 and homogeneous of degree 0, hence is bounded on $\mathbb{R}^n$ Therefore,

$$\|Tf\|_2 = \|(Tf)\hat{}\|_2 \leq \|\hat{k}\|_\infty \|\hat{f}\|_2 = \|\hat{k}\|_\infty \|f\|_2.$$

Step 2. Fix a radial function $\phi \epsilon c_o^\infty$ with $\phi(x) = 1$ for $|x| \leq \dfrac{1}{2}$ and $\phi(x) = 0$ for $|x| \geq 1$. For $\epsilon > 0$, we define $k_\epsilon(x) = k(x)(1 - \phi(x/\epsilon))$ and $T_\epsilon f = k_\epsilon * f$. Then we claim that T_ϵ is bounded on L^2 uniformly in ϵ. To see this, we observe that

$$\begin{aligned}(T_\epsilon f)\hat{} = \hat{f}\hat{k}_\epsilon &= \hat{f}(k - k\phi(x/\epsilon))\hat{} \\ &= \hat{f}\hat{k} - \hat{f}(\hat{k} * (\phi(x/\epsilon))\hat{})\end{aligned}$$

which gives

$$\begin{aligned}\|T_\epsilon f\|_2 = \|(T_\epsilon f)\hat{}\|_2 &\leq \|\hat{f}\|_2 \|\hat{k}\|_\infty \{1 + \epsilon^n \int |\hat{\phi}(\epsilon\xi)| d\xi\} \\ &= \|f\|_2 \|\hat{k}\|_\infty \{1 + \|\hat{\phi}\|_1\}\end{aligned}$$

139 **Step 3.** T_ϵ is of weak type (1,1) uniformly in ϵ. The proof of this is more involved and will be given later.

Step 4. By steps 2 and 3, using Marcinkiewicz, we get that T_ϵ is bounded on L^p for $1 < p < 2$ uniformly in ϵ.

Step 5. T_ϵ is bounded on L^p, for $2 < p < \infty$, uniformly in ϵ. Indeed, for $f, g \epsilon C_o^\infty$,

$$\int (T_\epsilon f)(x)\, g(x)\, dx = \int f(x)\, (\tilde{T}_\epsilon g)(x)\, dx$$

with $\tilde{T}_\epsilon g = \tilde{k}_\epsilon * g$, $\tilde{k}_\epsilon = k_\epsilon(-x) = k(-x)(1 - \phi(x/\epsilon))$. Since $\tilde{k}_\epsilon$ satisfies the same conditions as k_ϵ, we see that $\tilde{T}_\epsilon$ is bounded on L^q for $1 < q < 2$ uniformly in ϵ. So if $(1/p) + (1/q) = 1$, $1 < q < 2$,

$$\|T_\epsilon f\|_p = \sup_{g \,\epsilon\, C_o^\infty} \frac{|\int (T_\epsilon f) g|}{\|g\|_q} \leq \sup_{g \epsilon C_o^\infty} \frac{\|\tilde{T}_\epsilon g\|_q}{\|g\|_q} \|f\|_p = c\|f\|_p.$$

Step 6. If $f\epsilon C_o^\infty$, $T_\epsilon f$ converges to Tf in the L^p norm, $1 \leq p \leq \infty$ as ϵ tends to 0. Since ϕ is radial and $\mu_k = 0$,

$$\int\limits_{|x|=r} \phi(x/\epsilon)k(x)d\sigma(x) = c \int\limits_{|x|=r} k(x)d\sigma(x) = 0;$$

so, for $\epsilon \leq 1$, we have

$$(T_\epsilon f)(x) \;=\; \int\limits_{|y|\leq 1} (f(x-y)-f(x))\,k(y)(1-\phi(y/\epsilon))dy + \int\limits_{|y|>1} f(x-y)\,k(y)dy$$

and hence $\quad (T_\epsilon f)(x) - (Tf)(x) \;=\; \int\limits_{|y|\leq\epsilon} f(x-y) - f(x)k(y)\phi(y/\epsilon)dy.$

140

Now $\operatorname{supp}(T_\epsilon f - Tf) \subset \{x : d(x, \operatorname{supp} f) \leq \epsilon\} \subset A$, a fixed compact set. Since, on compact sets, the uniform norm dominates all L^p norms, it suffices to show that $(T_\epsilon f - Tf)$ converges to 0 uniformly on A. But

$$||T_\epsilon f - f||_\infty \leq ||\operatorname{grad} f||_\infty \int\limits_{|y|\leq\epsilon} c|y||y|^{-n}dy \leq c'||\operatorname{grad} f||_\infty$$

$$\int\limits_0^\epsilon r^{1-n}r^{n-1}dr = c''\epsilon \to 0, \text{ as } \epsilon \to 0.$$

Step 7. If $f\epsilon L^p$ and $\eta > 0$, choose $g\epsilon C_o^\infty$ with $||g - f||_p < \eta$.

Then $||T_\epsilon f - T_\delta f||_p \leq ||T_\epsilon(f-g)||_p + ||T_\epsilon g - T_\delta g||_p + ||T_\delta(g-f)||_p \leq 2c\eta + ||T_\epsilon g - T_\delta g||_p$.

Since η is arbitrary and $||T_\epsilon g - T_\delta g||_p$ converges to 0 as ϵ, δ tend to 0 by step 6, we see that $(T_\epsilon f)$ is Cauchy in the L^p norm. Setting

$$Tf = \lim_{\epsilon\to 0} T_\epsilon f, ||Tf||_p \leq c||f||_p.$$

Thus the theorem is proved modulo Step 3.

Let us now proceed to the proof of Step 3. First, we need a lemma which will be used in the proof.

Lemma 5.17. *Suppose $F \epsilon L^1, F \geq 0$ and $\alpha > 0$. Then there exists a sequence (Q_k) of closed cubes with sides parallel to the coordinate axes and disjoint interiors such that* 141

a) $\alpha < \frac{1}{|Q_k|} \int\limits_{Q_k} F \leq 2^n \alpha$ *for all k,*

b) If $\Omega = \bigcup\limits_{1}^{\infty} Q_k$, *then* $|\Omega| \leq (1/\alpha) \|F\|_1$,

c) $F(x) \leq \alpha$ *for a.e.* $x \notin \Omega$.

Proof. Let $r = (\|F\|_1/\alpha)^{1/n}$ and for $j = 1, 2, \ldots$ let $\mathscr{Q}_j$ be the collection of closed cubes of side length $r/2^j$ and vertices in $(r/2^j)\mathbb{Z}^n$. Our sequence will be constructed in the following way. Put those cubes $Q \epsilon \mathscr{Q}_1$ in the sequence which satisfy $\alpha < \frac{1}{|Q|} \int\limits_Q F$. Then

$$\frac{1}{|Q|} \int\limits_Q F \leq \frac{1}{|Q|} \|F\|_1 = (2^n \alpha / \|F\|_1) \|F\|_1 = 2^n \alpha$$

so that the first condition is satisfied. □

Put those cubes $Q \epsilon \mathscr{Q}_2$ into the sequence which are not contained in one of the previously accepted cubes and satisfy

$$\alpha < \frac{1}{|Q|} \int\limits_Q F.$$

Inductively, put those $Q \epsilon \mathscr{Q}_j$ which are not contained in one of the previously accepted cubes and satisfy $\alpha < \frac{1}{|Q|} \int\limits_Q F$. If $Q \epsilon \mathscr{Q}_j$ is in the sequence and Q' is the cube in $\mathscr{Q}_{j-1}$ containing Q then

$$\frac{1}{|Q|} \int\limits_Q F \leq \frac{1}{|Q|} \int\limits_Q , F = \frac{2^n}{|Q'|} \int\limits_{Q'} F.$$

142 Since $Q \subset Q'$, Q' cannot be in the sequence and hence

$$\frac{1}{|Q'|}\int\limits_{Q'} F \le \alpha \text{ which then yields } \frac{1}{|Q|}\int\limits_{Q} F \le 2^n\alpha.$$

Thus the condition (a) is satisfied. Also,

$$|\Omega| = \sum_{1}^{\infty} |Q_k| \le \frac{1}{\alpha}\int\limits_{Q_k} F \le (1/\alpha)\|F\|_1; \text{ so } (b) \text{ follows .}$$

Finally, by the Lebesgue differentiation theorem,

$$\lim_{x\epsilon Q\epsilon \mathcal{Q}_j\to\infty} \frac{1}{|Q|}\int\limits_{Q} F = F(x) \text{ for a.e. } x.$$

So, if $x \notin \Omega$, $\frac{1}{|Q|}\int\limits_{Q} F \le \alpha$ for all those Q' s and hence $F(x) \le \alpha$ a.e. on $\mathbb{R}^n \backslash \Omega$.

Coming back to the proof of Step 3, given $f \epsilon L^1$ and $\alpha > 0$, let (Q_k) be the sequence of cubes as in the lemma with $F = |f|$. We write

$$f = g + \sum_{k=1}^{\infty} b_k \text{ with}$$

$$b_k(x) = \begin{cases} f(x) - \frac{1}{|Q_k|}\int\limits_{Q_k} f(y)dy, & \text{for } x\epsilon Q_k \\ 0, \text{otherwise} \end{cases}$$

and
$$g(x) = \begin{cases} \frac{1}{|Q_k|}\int\limits_{Q_k} f(y)dy, & \text{for } x\epsilon Q_k \\ f(x), & \text{for } x \notin \Omega. \end{cases}$$

Now me make the following observations :

a) Supp $b_k \subset Q_k$, $\int b_k = 0$ and

(5.18) $$\sum_{1}^{\infty} \|b_k\|_1 \le \sum_{1}^{\infty} 2\int\limits_{Q_k} |f| \le 2\sum_{1}^{\infty} 2^n\alpha|Q_k| \le 2^{n+1}\|f\|_1$$

b) $|g(x)| \leq 2^n\alpha$ for $x\epsilon\Omega$ and $|g(x)| \leq \alpha$ for a.e. $x \notin \Omega$.

Therefore, $|g(x)| \leq 2^n\alpha$ a.e. and

$$\|g\|_2^2 = \int\limits_{\Omega} |g|^2 + \int\limits_{\mathbb{R}^n-\Omega} |g|^2 \tag{5.19}$$
$$\leq (2^n\alpha)^2|\Omega| + \alpha \int\limits_{\mathbb{R}^n-\Omega} \|f\|$$
$$\leq (2^{2n}+1)\alpha\|f\|_1.$$

We put $\sum\limits_1^\infty b_k = b$ so that $T_\epsilon f = T_\epsilon g + T_\epsilon b$ and

$$|\{|T_\epsilon f| > \alpha\}| \leq |\{|T_\epsilon g| > \alpha/2\}| + |\{|T_\epsilon b| > \alpha/2\}|.$$

We shall show that both terms on the right are dominated by $\|f\|_{1/\alpha}$ uniformly in ϵ.

To estimate the first term on the right, we use Chebyshev inequality, Step 2 and the estimate (5.19) obtaining

$$|\{|T_\epsilon g| > \alpha/2\}| \leq (2/\alpha)\|T_\epsilon g\|_2)^2 \leq c(\|g\|_2/\alpha)^2 \leq c_1(\|f\|_1\alpha).$$

To estimate the second term, let y_k be the center of the cube Q_k and $\tilde{Q}_k$ be the cube centred at y_k but with length side $2\sqrt{n}$ times that of Q_k. We put $\bigcup\limits_1^\infty \tilde{Q}_k = \tilde{\Omega}$. Then

$$|\tilde{\Omega}| \leq \sum_1^\infty |\tilde{Q}_k| = \left(2\sqrt{n}\right)^n \sum_1^\infty |Q_k| \leq \left(2\sqrt{n}\right)^n \|f\|_1/\alpha = c(\|f\|_1/\alpha).$$

So
$$|\{|T_\epsilon b| > \alpha/2\} \leq |\tilde{\Omega}| + |\{|T_\epsilon b| > \alpha/2\}\backslash\tilde{\Omega}|$$
$$\leq c(\|f\|_1/\alpha) + |\{|T_\epsilon b| > \alpha/2\}\backslash\tilde{\Omega}|,$$

144 and it suffices to estimate $|\{|T_\epsilon| > \alpha/2\}/\tilde{\Omega}$. Since $\int\limits_{Q_k} b = 0$,

$$T_\epsilon b(x) = \int k_\epsilon(x-y)b(y)dy$$

$$= \sum_{1}^{\infty} \int_{Q_k} k_\epsilon(x-y)b(y)dy$$
$$= \sum_{1}^{\infty} \int_{Q_k} (k_\epsilon(x-y) - k_\epsilon(x-y_k))b(y)dy$$

Therefore, we have

$$|\{|T_\epsilon b| > \alpha/2\} \backslash \tilde{\Omega}|$$
$$\leq (2/\alpha) \int_{\mathbb{R}^n \backslash \tilde{\Omega}} |T_\epsilon b| dx$$
$$\leq (2/\alpha) \sum_{1}^{\infty} \int_{\mathbb{R}^n \backslash \tilde{\Omega}} \int_{Q_k} |(k_\epsilon(x-y) - k_\epsilon(x-y_k))||b(y)|dy\, dx$$
$$\leq (2/\alpha) \sum_{1}^{\infty} \int_{Q_k} \int_{\mathbb{R}^n \backslash \tilde{Q}_k} |(k_\epsilon(x-y) - k_\epsilon(x-y_k))||b(y)|dx\, dy$$

We now *claim* that $\int_{\mathbb{R}^n \backslash \tilde{Q}_k} |(k_\epsilon(x-y) - k_\epsilon(x-y_k))| dx \leq c$ independent of k and ϵ for $y \epsilon Q_k$.
Accepting this claim, by the estimate (5.18), we have

$$|\{|T_\epsilon| > \alpha/2\} \backslash \tilde{\Omega}| \leq (2/\alpha) c \sum_{1}^{\infty} \int_{Q_k} |b(y)| dy \leq 2^{n+2} c(\|f\|_1/\alpha).$$

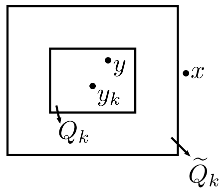
Thus $|\{|T_\epsilon b| > \alpha\}| \leq c_o(\|f\|_1/\alpha)$ which completes the proof of Step 3.

Returning to the *claim*, we observe that if $x \epsilon \mathbb{R}^n \backslash \tilde{Q}_k$ and $y \epsilon Q_k$, then $|x - y_k| \geq 2|y - y_k|$.

So, if we set $z = x - y_k, w = y - y_k$ we must show that **145**

$$\int_{|z|>2|w|} |(k_\epsilon(z-w) - k_\epsilon(z))| dz \leq c$$

independent of w and ϵ.



First consider the case $\epsilon = 1$. Now $k_1(z)$ is a C^∞ function which is homogeneous of degree $-n$ for large z. Therefore, $|gradk_1(z)| \leq c''|z|^{-n-1}$. By the mean value theorem,

$$\begin{aligned}|k_1(z-w) - k_1(z)| &\leq c'|w| \sup_{0<t<1} |z-tw|^{-n-1}\\ &\leq c''|w||z|^{-n-1} \text{for} |z| > 2|w|.\end{aligned}$$

So

$$\begin{aligned}&\int_{|z|>2|w|} |(k_1(z-w) - k_1(z))|dz\\ &\qquad \leq c'' \int_{|z|>2|w|} |w||z|^{-n-1} dz\\ &\qquad \leq c'''|w| \int_{2|w|}^{\infty} r^{-2} dr = c.\end{aligned}$$

Now for general ϵ,

$$k_\epsilon(z) = (1 - \phi(z/\epsilon))k(z/\epsilon)\epsilon^{-n} \text{ by the homogeneity of } k.$$

Therefore, if we set $z' = \epsilon^{-1}z$ and $w' = \epsilon^{-1}w$, we see that

$$\int_{|z|>2|w|} |(k_\epsilon(z-w) - k_\epsilon(z))|dz = \int_{|z'|>2|w'|} |(k_1(z'-w') - k_1(z'))|dz'$$

which is bounded by a constant, by the result for $\epsilon = 1$. Hence the *claim* above is proved.

146 To complete the picture, we should observe that operators of type

0 are not bounded on L^1 (and hence, by duality, not bounded on L^∞). Indeed, if T is an operator of type 0 with kernel k, $(Tf)\hat{} = \hat{k}\hat{f}$. Since $\hat{f}$ is homogeneous of degree 0, it has a discontinuity at 0 (unless $k = c\delta$). Thus if $f \epsilon L^1$, $(Tf)\hat{}$ is not continuous at 0 whenever $\hat{f}(0) \neq 0$ and this implies that Tf is not in L^1.

This reflects the fact that if $\int f = \hat{f}(0) \neq 0$, then Tf will not be integrable near ∞, because k is itself not integrable at ∞. However, there are also problems with the local integrability of Tf caused by the singularity of k at the origin. In fact, let $\phi \epsilon C_o^\infty$ be a radial function such that $\phi = 1$ near 0, and set $Sf = f * (\phi k)$. Then the argument used to prove Theorem 5.16 shows that S is bounded on L^p for $1 < p < \infty$. In this case $(\phi k)\hat{} = \hat{\phi} * \hat{k}$ is in C^∞, but still S is not bounded on L^1.

This follows from the following general fact.

Proposition 5.20. *If $k \epsilon S'$ and the operator $f \to k * f$ is bounded on L^1, then k is necessarily a finite Borel measure.*

Proof. Choose $\phi \epsilon C_o^\infty$ with $\int \phi = 1$ and put $\phi_\epsilon(x) = \epsilon^{-n}\phi(x/\epsilon)$. Then $\|\phi_\epsilon\|_1$ is independent of ϵ, so $\|\phi_\epsilon * k\|_1 \leq c$. Therefore there exists a sequence ϵ_k tending to 0 such that $\phi_{\epsilon_k} * k$ converges to a finite Borel measure μ in the weak* topology of measures and hence $\phi_{\epsilon_k} * k$ converges to μ in S' also. On the other hand, since (ϕ_{ϵ_k}) is an approximate identity, $\phi_{\epsilon_k} * k$ converges to k in S'. □

Hence $\mu = k$ and the proposition is proved. **147**

It is easy to see that kernels of type 0 are not measures even when truncated away from the origin, as their total variation in any neighbourhood of 0 is infinite. One can also see directly that they do not define bounded functionals on C_o.

Exercise . *Let $k(x) = 1/x$ on $\mathbb{R}$ and let f be a continuous compactly supported function such that $f(x) = (\log x)^{-1}$ for $0 < x < \frac{1}{2}$ and $f(x) = 0$ for $x \leq 0$. Show that*

$$\lim_{x \to 0-} (f * PV(k))(x) = \infty.$$

Generalise this to kernels of type 0 on $\mathbb{R}^n$, $n > 1$.

This example also shows that operators of type 0 do not map continuous functions into continuous functions. However, they do preserve Lipschitz or Hölder continuity, as we shall now see.

Definition 5.21. *For* $0 < \alpha < 1$, *we define* $|f|_\alpha = \sup_{x,y} \dfrac{|f(x+y) - f(x)|}{|y|^\alpha}$

$$\Lambda_\alpha = \{f : \|f\|_{\Lambda_\alpha} =^{def} \|f\|_\infty + |f|_\alpha < \infty\}.$$

Λ_α *is called* Lipschitz class *of order* α.

Remark 5.22. The definition makes perfectly good sense for $\alpha = 1$ (When $\alpha > 1$ it is an easy exercise to show that if $|f|_\alpha < \infty$ then f is constant). However, we shall not use this definition for $\alpha = 1$, because the theorems we wish to prove are false in this case.

We are going to prove, essentially, that operators of type 0 are boun-
148 ded on Λ_α. However, if k is a kernel of type 0 and $f\epsilon\Lambda_\alpha$, the integral defining $k * f$ will usually diverge f need not decay at ∞. Consequently, we shall work instead with $\Lambda^\alpha \cap L^p (1 < p < \infty)$, concerning which we have the following useful result.

Proposition 5.23. *If* $f\epsilon L^p, 1 \leq p < \infty$ *and* $|f|_\alpha < \infty$, *then* $f\epsilon\Lambda_\alpha$ *and* $\|f\|_\infty \leq c(\|f\|_p + |f|_\alpha)$. *Consequently,* $L^p \cap \Lambda_\alpha$ *is a Banach space with norm* $\|f\|_p + |f|_\alpha$.

Proof. Let
$$A_x = \left(\frac{|f(x)|}{2|f|_\alpha}\right)^{1/\alpha} \text{ for } x\epsilon\mathbb{R}^n.$$

Then $|f(y)| \geq |f(x)|/2$ for all y such that $|x - y| \leq A_x$. So

$$\int |f(y)|^p dy \geq \int\limits_{|x-y|\leq A_x} |f(y)|^p dy \geq |f(x)|^p 2^{-p} c' A_x^n$$
$$= c'' |f(x)|^{p+(n/\alpha)} |f|_\alpha^{-n/\alpha}$$

or
$$|f(x)|^{1+(n/\alpha p)} \leq c''' \|f\|_p |f|_\alpha^{n/alphap}.$$
□

Since this is true for all x, setting $\theta = n/p\alpha$ we have

$$\begin{aligned} \|f\|_\infty &\leq c'''\|f\|_p^{1/(1+\theta)}|f|_\alpha^{\theta/(1+\theta)} \\ &\leq c(\|f\|_p + |f|_\alpha). \end{aligned}$$

Theorem 5.24. *Operators of type* 0 *are bounded on* $\Lambda_\alpha \cap L^p(0 < \alpha < 1, < 1 < p < \infty)$.

Proof. Let $T : f \to K * f$ be an operator of type 0. Since we know that $\|Tf\|_p \leq c_p\|f\|_p$, by proposition 5.23, it will suffice to show $|Tf|_\alpha \leq c_\alpha|f|_\alpha$ for $0 < \alpha < 1$, $f \epsilon L^p \cap \Lambda_{alpha}$. As in the proof of Theorem 5.16, **149**
we may assume that $K = PV(k)$ and identify K with k. ◻

Given $y \epsilon \mathbb{R}^n \backslash \{0\}$ and $f \epsilon L^p \cap \Lambda_\alpha$ define

$$\begin{aligned} g(x) &= \int\limits_{|z| \leq 3|y|} k(z) f(x-z) dz \\ h(x) &= \int\limits_{|z| \leq 3|y|} k(z) f(x-z) dz \end{aligned}$$

so that $Tf = g + h$. Since $\mu_k = 0$, we have

$$\begin{aligned} |g(x)| &= |\int\limits_{|z| \leq 3|y|} k(z)(f(x-z) - f(x)) dz| \\ &\leq \int\limits_{|z| \leq 3|y|} c|z|^{-n}|f|_\alpha |z|^\alpha dz \leq c_1|f|_\alpha |y|^\alpha. \end{aligned}$$

Since this is true for all x,

$$|g(x+y) - g(x)| \leq 2c_1|f|_\alpha |y|^\alpha.$$

Next

$$h(x+y) = \lim_{\eta \to \infty} \int\limits_{3|y| < |z| < \eta} k(z)(f(x+y-z) - f(x)) dz$$

$$= \lim_{\eta\to\infty} \int\limits_{3|y|<|z+y|<\eta} k(z+y)(f(x-z) - f(x))dz.$$

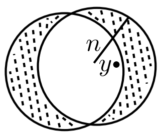
Therefore,

$$h(x+y) - h(x) = \lim_{\eta\to\infty} \int\limits_{3|y|<|z|<\eta} (k(z+y) - k(z))(f(x-z) - f(x))dz + \epsilon_1 + \epsilon_2$$

where ϵ_1 and ϵ_2 are errors coming from difference between the regions of integration.

ϵ_1 is the error coming from the difference between the regions $|z| < \eta$ and $|z+y| < \eta$.

150 The symmetric difference between these two regions is contained in the annulus $\eta - |y| < |z| < \eta + |y|$.



If z is in this region and $\eta \gg |y|$ then $|z| \approx |z+y| \approx \eta$ so that

$$|\epsilon_1| \leq c' \int\limits_{\eta-|y|<|z|<\eta+|y|} 2\eta^{-n} ||f||_\infty dz \leq c''\eta^{-n}||f||_\infty((\eta+|y|)^n - (\eta-|y|)^n)$$

$$= 0(\eta^{-1}) \to 0 \text{ as } \eta \to \infty.$$

The term ϵ_2 comes from the symmetric difference of the regions $|z| > 3|y|$ and $|z+y| > 3|y|$ which is contained in the annulus $2|y| < |z| < 4|y|$.

In this region $|z+y| \approx |z| \approx |y|$.

Therefore

$$|\epsilon_2| \leq c''' \int\limits_{2|y|<|z|<4|y|} |y|^{-n+\alpha}|f|_\alpha dz =$$

$$= c_2'|f|_\alpha |y|^{-n+\alpha}((4|y|)^n - (2|y|)^n)$$

$$= c_2|f|_\alpha |y|^\alpha.$$

Finally coming to the main term, we have

$$\begin{aligned}
|k(z+y)-k(z)| &\leq |y| \sup_{0<t<1} |\operatorname{grad} k(z+ty)| \\
&\leq |y| \sup_{0<t<1} |z+ty|^{-n-1} \\
&\leq c_o |y||z|^{-n-1} \text{ for } |z| \geq 3|y|.
\end{aligned}$$

Hence, since $\alpha < 1$ so that $-n-1+\alpha < -n$, **151**

$$\begin{aligned}
&\Big|\int\limits_{3|y|<|z|<\eta} (k(z+y)-k(z))(f(x-z)-f(x))dz\Big| \\
&\qquad \leq c_o \int\limits_{3|y|<|z|<\eta} |y||z|^{-n-1}|f|_\alpha |z|^\alpha dz \\
&\qquad \leq c_o \int\limits_{3|y|<|z|} |y||z|^{-n-1+\alpha}|f|_\alpha dz \\
&\qquad \leq c_3 |y||f|_\alpha |y|^{\alpha-1} = c_3 |y|^\alpha |f|_\alpha .
\end{aligned}$$

Therefore,

$$\frac{|Tf(x+y)-Tf(x)|}{|y|^\alpha} \leq c|f|_\alpha \text{and consequently } |Tf|_\alpha \leq c|f|_\alpha .$$

For kernels of positive type, we have the following result.

Theorem 5.25. *Suppose* $0 < \lambda < n, 1 < p < n/\lambda < q < Q$ *where* $Q = \infty$ *if* $\lambda \leq 1, Q = n/(\lambda-1)$, *if* $\lambda < 1$. *Let* $1/r = (1/p) - (\lambda/n)$ *and* $\alpha = \lambda - (n/q)$.*(Thus* $r < \infty$ *and* $0 < \alpha < 1$*). Thus operators of type* λ *are bounded from* $L^p \cap L^q$ *into* $L^r \cap \Lambda_\alpha$.

Proof. Let $Tf = k * f$ be an operator of type λ. By Corollary 5.15, T is bounded from L^p to L^r, so by proposition 5.23, it is enough to show that $|Tf|_\alpha \leq c||f||_q$.

$$\text{We have } (Tf)(x) = \int f(x-z)k(z)dz$$

$$\begin{aligned}(Tf)(x+y) &= \int f(x+y-z)k(z)dz \\ &= \int f(x-z)k(z+y)dz,\end{aligned}$$

so that

$$\begin{aligned}(Tf)(x+y) - Tf(x) &= \int f(x-z)(k(z+y)-k(z))dz \\ &= \int_{|z|\le 2|y|} f(x-z)(k(z+y)-k(z))dz \\ &\quad + \int_{|z|>2|y|} f(x-z)(k(z+y)-k(z))dz.\end{aligned}$$

152 □

If q' is the conjugate exponent of q,

$$\begin{aligned}&|\int_{|z|\le 2|y|} f(x-z)(k(z+y)-k(z))dz| \\ &\quad\le \|f\|_q \left\{\left(\int_{|w|\le 3|y|} |k(w)^{q'}dw\right)^{1/q'} + \left(\int_{|z|\le 2|y|} |k(z)^{q'}dz\right)^{1/q'}\right\} \\ &\quad\le 2\|f\|_q \left(\int_{|z|\le 3|y|} |k(z)|^{q'}dz\right)^{1/q'} \\ &\quad\le c_1\|f\|_q \left(\int_{|z|\le 3|y|} |z|^{(\lambda-n)q'}dz\right)^{1/q'} \\ &\quad\le c_1'\|f\|_q|y|^{-n+\lambda+(n/q')} = c_1'\|f\|_q|y|^{\alpha},\end{aligned}$$

by the definition of α

For the second integral, we estimate $k(z+y)-k(z)$ by the mean value theorem:

$$|k(z+y)-k(z)| \le |y| \sup_{0<t<1} |\mathrm{grad}k(z+ty)|$$

$$\leq c|y||z|^{\lambda-n-1} \text{for } |z| \geq 2|y|.$$

Thus

$$\Big|\int\limits_{|z|>2|y|} f(x-z)(k(z+y)-k(z))dz\Big|$$
$$\leq c||f||_q \left\{ \int\limits_{|z|>2|y|} \left(|y||z|^{\lambda-n-1}\right)^{q'} dz \right\}^{1/q'}$$
$$\leq c_2'||f||_q|y||y|^{\lambda-n-1+(n/q')} \text{ since } (\lambda-n-1)q' < -n$$
$$= c_2'||f||_q|y|^{\alpha}.$$

Hence 153

$$\frac{|Tf(x+y)-Tf(x)|}{|y|^{\alpha}} \leq c||f||_q \text{ which gives } |Tf|_{\alpha} \leq c||f||_q.$$

Remark 5.26. As in the preceding theorem, the reason for taking the domain of T to be $L^p \cap L^q$ instead of just L^q is that the integral defining Tf will usually diverge when f is merely in L^q.

Nonetheless, the point of these theorems is that operators of type 0 are in essence, bounded from L^q to Λ_{α} (for appropriate q and α). To make this precise without losing simplicity, one can observe that operators of type 0 map $\Lambda_{\alpha} \cap E'$ into Λ_{α} while operators of type $\lambda > 0$ map $L^q \cap E'$ into Λ_{α}.

We now introduce spaces of functions whose derivatives upto a certain order are in L^p or Λ_{α}.

Definition 5.27. *Suppose* $1 \leq p \leq \infty$ *and* k *is a positive integer. We define*

$$L_k^p = \{f : D^{\beta} f \epsilon L^p for 0 \leq |\beta| \leq k\}.$$

We equip L_k^p with the norm $||f||_{k,p} = \sum\limits_{|\beta|\leq k} ||D^{\beta} f||_p$. (Thus $L_k^2 = H_k$ in the notation of Chapter 3).

Definition 5.28. *Suppose k is a positive integer and $k < \alpha < k+1$. We define*

$$\Lambda_\alpha = \{f : D^\beta f \epsilon \Lambda_{\alpha-k} for 0 \leq |\beta| \leq k\}.$$

We equip Λ_α with the norm $||f||_{\Lambda_\alpha} = \sum_{|\beta|\leq k} ||D^\beta f||_{\Lambda_{\alpha-k}}$.

Remarks 5.29.

154 (i) $f \epsilon \Lambda_\alpha$ if and only if $D^\beta f$ is bounded and continuous for $0 \leq |\beta| \leq k$ and $D^\beta f \epsilon \Lambda_{\alpha-k}$ for $|\beta| = k$. Indeed, if $|\beta| < k$, for $|y| \leq 1$,

$$\frac{|D^\beta f(x+y) - D^\beta f(x)|}{|y|^{\alpha-k}} \leq \frac{|D^\beta f(x+y) - D^\beta f(x)|}{|y|} \leq c \sum_{|\nu|=|\beta|+1} ||D^\nu f||_\infty$$

(by the mean value theorem) and for

$$|y| > 1, \frac{|D^\beta f(x+y) - D^\beta f(x)|}{|y|^{\alpha-k}} \leq 2||D^\beta f||_\infty.$$

This shows that $D^\beta f \epsilon \Lambda_{\alpha-k}$ for $0 \leq |\beta| \leq k$ i.e., $f \epsilon \Lambda_\alpha$.

(ii) If $k < \alpha < k+1$ then $L^p_k \cap \Lambda_\alpha$ is a Banach space with the norm

$$\sum_{|\beta|\leq k} (||D^\beta f||_p + |D^\beta f|_{\alpha-k}).$$

This follows from the corresponding fact that $L^p \cap \Lambda_\alpha$ is a Banach space (Proposition 5.23).

Theorem 5.30. *Suppose $0 \leq \lambda < n, 1 < p < n/\lambda, 1/r = (1/p) - (\lambda/n)$ and $k = 0, 1, 2, \ldots$. Then we have:*

a) Operators of type λ are bounded from L^p_k into L^r_k.

b) If λ is an integer, $\lambda = 0, 1, 2, \ldots$ and $k < \alpha < k+1$, then operators of type λ are bounded from $L^p_k \cap \Lambda_\alpha$ to $L^r_{k+\lambda} \cap \Lambda_{\alpha+\lambda}$.

Proof. a) This is an easy consequence of Corollary 5.15 and Theorem 5.16, since convolution commutes with differentiation. In the same way (b) follows Theorem 5.24 when $\lambda = 0$. □

We now proceed by induction on λ. Therefore, assume that $\lambda \geq 1$.
155 Suppose $f \epsilon L^p_k \cap \Lambda_\alpha$. Then $\partial_j(f * k) = f * \partial_j k$ and $\partial_j k$ is a kernel of type $\lambda - 1$. So, if $1/s = (1/p) - (\lambda - 1)/n$, by our induction hypothesis $\partial_j(f * k) \epsilon L^s_{k+\lambda-1} \cap \Lambda_{\alpha+\lambda-1}$. Since $s < r < \infty, L^s \cap L^\infty \subset L^r$, so $\partial_j(f * k) \epsilon L^r_{k+\lambda-1} \cap \Lambda_{\alpha+\lambda-1}$. Also $f * k \epsilon L^r \cap C^1$, hence in $\Lambda_{\alpha+\lambda}$ provided it is bounded. But $\partial_j(f * k) \in \Lambda_{\alpha+\lambda-1}$ implies that $\partial_j(f * k)$ is bounded. By the mean value theorem, we then have

$$\frac{|(f * k)(x + y) - (f * k)(x)|}{|y|} \leq c.$$

This together with $f * k \epsilon L^r$ implies that f is bounded (by definition 5.21). Hence $f * k \epsilon L^r_{k+\lambda} \cap \Lambda_{\alpha+\lambda}$.

The above theorem can be generalised. For example, if $0 \leq \lambda < n$ one can show that operators of type λ map $\Lambda_\alpha \cap E'$ in to $\Lambda_{\alpha+\lambda}$.

Also generalisations of the L^p Sobolev spaces L^p_k can be given for non-integral values of k. In fact, a theorem due to CALDERON says that for $1 < p < \infty$, $f \epsilon L^p_k$ if and only if $\Lambda^k f \epsilon L^p$. (Here $\Lambda = (1 - \Delta)^{1/2}$).

Therefore, we can define L^p_s for any real s by

$L^p_s = \{f : \Lambda^s f \epsilon L^p\}$ with the norm $||f||_{s,p} = ||\Lambda^s f||_p$.

Then part (b) of the above theorem is still true for $0 \leq \lambda < n, \lambda$ not necessarily an integer in this case.

Refer to E.M. Stein [2].

We will now prove the *Sobolev imbedding theorem for L^p Sobolev spaces L^p_k with positive integral k*. This theorem can also be generalised 156
to L^p_s for s in $\mathbb{R}$.

Theorem 5.31 (SOBOLEV IMBEDDING THEOREM). *Suppose $1 < p < \infty$ and k a positive integer. If $k < n/p$, then $L^p_k \subset L^r$ for $1/r = (1/p) - (k/n)$ (Hence also $L^p_{k+j} \subset L^r_j$ for any j). If $k > n/p$, and $\alpha = k - n/p$ is not integer, then $L^p_k \subset \Lambda_\alpha$.*

Proof. Let N be the fundamental solution of Δ given by

$$N(x =) \begin{cases} (2 - n)^{-1} \omega_n^{-1} |x|^{2-n} \text{ for } n \neq 2 \\ (2\pi)^{-1} \log |x| \text{ for } n = 2. \end{cases}$$

Then $K_j(x) = \partial_j N(x) = \omega_n x_j |x|^{-n}$ (true for all n) is a kernel of type 1. □

Now, if $f \in L^p_k \cap E'$

$$f = f * \delta = f * \Delta N = f * \sum_{j=1}^{n} \partial_j k_j = \sum_{j=1}^{n} (\partial_j f * k_j).$$

Suppose $k = 1$. If $1 < n/p$, $\partial_j f \epsilon L^p \Rightarrow \partial_j f * k_j \epsilon L^r$ where $1/r = (1/p) - (1/n)$, by Theorem 5.25 and hence $f \epsilon L^r$. If $1 > n/p, \partial_j f * k_j \epsilon \Lambda_{1-(n/p)}$ by Theorem 5.25 which implies that $f \epsilon \Lambda_{1(n/p)}$. Thus the theorem is true for $k = 1$.

For $k > 1$, we proceed by induction. Now
$f \epsilon L^p_k \cap E' \Rightarrow f$ and $\partial_j f$ are in $L^p_{k-1} \cap E'$.

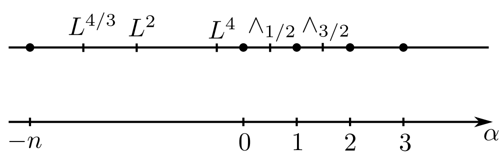
Therefore, if $p < n/(k-1)$ and $1/q = (1/p) - (k-1)/n$, we have f, $\partial_j f \epsilon L^q$, i.e., $f \in L^q_1$, while if $p > n/(k-1)$, we have f, $\partial_j f \epsilon \Lambda_{k-1-(n/p)}$, i.e., $f \epsilon \Lambda_{k-(n/p)}$. In the second case, we are done, and in the first case, we
157 apply the result for $k = 1$ to see that f is in the required space. Finally, it is easy to check that if we keep track of the norm inequalities that are implicit in the above arguments, we obtain

$$||f||_r \leq c||f||_{k,p} \text{ or } ||f||_{\Lambda_\alpha} \leq c||f||_{k,p},$$

as appropriate, for $f \epsilon L^p_k \cap E'$. Since $L^p_k \cap E'$ is clearly dense in L^p_k, the desired result follows immediately.

We can summarise these theorems in an elegant way using the following picture.

For $-n < \alpha < 0$ we define $x_\alpha = L^{n/|\alpha|}$ and when $\alpha > 0$, α not an integer, we define $x_\alpha = \Lambda_\alpha$.



(The small circle represent missing spaces)

In this terminology, we have

Theorem 5.32. *Operators of type λ map $x_\alpha \cap E'$ into $x_{\alpha+\lambda} o \leq \lambda < n$.*

Theorem 5.33 (SOBOLEV IMBEDDING THEOREM). *If $D^\beta f \epsilon x_\alpha$ for $0 \leq |\beta| \leq k$, then $f \in x_{\alpha+k}$.*

We now indicate how to fill the gaps in this picture at $\alpha = 0, 1, 2, \ldots$

For $\alpha = 1$, we define

$\Lambda_1 = \{f : f$ is continuous, bounded and

$$\sup_{x,y} \frac{|f(x+y) + f(x-y) - 2f(x)|}{|y|} < \infty\}.$$

For $k = 2, 3, 4, \ldots$ we define $\Lambda_k = \{f : D^\beta f \epsilon \Lambda_1$ for $|\beta| \leq k - 1\}$. The 158
sudden jump from first differences in the definition of Λ_α for $\alpha < 1$ to the second differences at $\alpha = 1$ is less mysterious than it seems at first, because it can be shown that if $0 < \alpha < 2$ then $f \epsilon \Lambda_\alpha$ if and only if f is bounded, continuous and satisfies

$$\sup_{x,y} \frac{|f(x+y) + f(x-y) - 2f(x)|}{|y|^\alpha} < \infty.$$

To fill the gap at $\alpha = 0$ we use the space BMO (" bounded mean oscillation") first introduced by F. JOHN and L. NIRENBERG in 1961, which is defined as follows.

For $f \epsilon L^1_{\text{loc}}(\mathbb{R}^n)$, we denote by $m_E f$ mean value of f over a measurable set $E \subset \mathbb{R}^n$, that is,

$$m_E f = \frac{1}{|E|} \int\limits_E f.$$

Let $\mathcal{Q}$ denote the collection of all cubes in $\mathbb{R}^n$ with sides parallel to the axes.

Definition 5.34. *$BMO = \{f \epsilon L^1_{\text{loc}}(\mathbb{R}^n) : \sup\limits_{Q \epsilon \mathcal{Q}} m_Q(|f - m_Q f|) < \infty\}$.*

Clearly $L^\infty \subset BMO$, for, $f \epsilon L^\infty \Rightarrow |m_Q f| \leq ||f||_\infty$ for every $Q \epsilon \mathcal{Q}$ and consequently

$$m_Q(|f - m_Q f|) \leq 2||f||_\infty.$$

It can be shown that $BMO \subset L^q_{\text{loc}}$ for every $q < \infty$.

If we define $x_\alpha = \Lambda_\alpha$ for $\alpha = 1, 2, \ldots$ and $X_0 = BMO$ then Theorem 5.32 remains valid for all $\alpha_\epsilon(-n, \infty)$, except that the Sobolev imbedding theorem for $\alpha = 0$ must be slightly modified as follows:

If $D^\beta f$ is in the closure of BMO $\cap E'$ in BMO for $|\beta| \leq k$ then $f \epsilon \Lambda_k$. 159

WARNING. *BMO is not an interpolation space between L^p and Λ_α i.e., it is not true that if T is a linear operator which is bounded on L^p for some $p < \infty$ and on Λ_α some $\alpha > 0$, then T is bounded on BMO.*

For proofs of the foregoing assertions, see E.M. Stein [2] and also the following papers :

(i) E.M Stein and A. Zygmund: Boundedness of translation invariant operators on Holder spaces and L^p spaces, Ann. Math 85 (11967)337-349, and

(ii) C. Fefferman and E.M. Stein : H^p spaces of several variables Acta Math. 129, (1972), 137-193.

As we indicated at the beginning of this chapter, the arguments we have developed can be extended in a fairly straightforward way to give estimates for ψDO with variable coefficients. We conclude by summarising the result in

Theorem 5.35. *[Let] $P = p(x, D)$ ba a properly supported ψDO of order $-\lambda$ on Ω, where $\lambda \geq 0$ and $p \sim \sum\limits_{j=0}^{\infty} p_j$ with $p_j(x, \xi)$ homogeneous of degree $\lambda - j$ for large $|\xi|$. Then P maps $L^p_k(\Omega, \mathrm{loc})$ into $L^p_{k+\lambda}(\Omega, \mathrm{loc})$ for $1 < p < \infty$, and in the terminology of Theorem 5.32 P maps $x_\alpha(\Omega, \mathrm{loc})$ into $x_{\alpha+\lambda}(\Omega, \mathrm{loc})$ for $-n < \alpha < \infty$.*

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