Lectures on Moduli of Curves

By

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Introduction

These notes are based on some lectures given at TIFR during January and February 1980. The object of the lectures was to construct a projective moduli space for stable curves of genus $g \geq 2$ using Mumford's geometric invariant theory.

The general plan of the notes is as follows: Chapter 0 consists of preliminaries. In particular, for $m >> 0$, we review how to attach to each space curve $C \subset \mathbb{P}^n$ a point in some projective space called the m^{th} Hilbert point of C. We then consider the question of the stability of the m^{th} Hilbert point in the sense of geometric invariant theory. Our first main result in Chapter 1 is that if C is smooth and $d \geq 20(g-1)$, then the m^{th} Hilbert point of C is stable. Our second main result in Chapter 1 is that if the m^{th} Hilbert point is semi-stable, then the curve is semi-stable as a curve. In Chapter 2, we use the results of Chapter 1 to give an indirect proof that the n-canonical embedding of a stable curve is stable if $n \geq 10$, and to construct the projective moduli space for stable curves. As corollaries, we obtain proofs of the stable reduction theorem for curves, and of the irreducibility of the moduli space for smooth curves.

Historically speaking, Mumford used his theory to construct a quasi-projective moduli space for smooth curves by studying the stability of the Chow points of spaces curves. Mumford and Deligne [1] introduced the concept of stable curve in their proof of the irreducibility of the moduli space of curves of genus $g \geq 2$, and later F.Knudsen established the existence of a projective moduli space for stable curves. In 1974, Mumford and I realized that the n-canonical model of a stable curve was stable in the invariant theory sense if $n >> 0$. Mumford then showed that the Chow point of the n-canonical model of a stable curve is stable if $n \geq 5$, [7]. Our treatment here parallels that of Mumford, except for technical points arising from the difference between Chow and Hilbert points. (I believe one could use Hilbert point methods in the case $n \geq 5$).

I wish to thank D.R. Gokhale, who filled in many gaps in the original lectures. I also wish to thank TIFR for inviting me for a most enjoyable visit and my audience, especially C.S. Seshadri, for their comments and patience.

Notation

The following notations will be used without further comment.

K	a fixed algebraically closed field
K^*	multiplicative group of non-zero elements in K
$\mathbb{A}^N$	affine N-space over K
$\mathbb{P}^N$	projective N-space over K
$GL(N+1)$	group of invertible $(N+1)\times(N+1)$ matrices over K.
$SL(N+1)$	group of elements in $GL(N+1)$ with determinant 1.
$PGL(N+1)$	$GL(N+1)$/scalar multiples of the Identity matrix.
$PGL(N+1)(R)$	group of invertible $(N+1)\times(N+1)$ matrices over a ring R/scalar multiples of the Identity matrix.
$1-ps\lambda$	One parameter subgroup of an algebraic group Let X be a projective scheme and let F be a coherent 0_X module.
$H^i(X,F)$	i^{th} cohomology of X with coefficients in F
$h^i(X,F)$	$\dim H^i(X,F)$
$\chi(F)$	$\sum(-1)^i h^i(X,F)$
$\#S$	cardinality of a set S.

Contents

Chapter 0

Preliminaries

In this introductory Chapter we recall, **1**

A) some basic definitions and standard results in Geometric Invariant Theory;

B) the definition of a Hilbert point of a curve;

C) the definition of a Hilbert scheme;

D) the definition and simple properties of a stable curve;

E) some basic definitions and standard results in Deformation theory.

A) Geometric Invariant Theory

Let G be a reductive algebraic group acting on an algebraic scheme X. It is natural to ask whether X has a quotient by G, which is reasonably good, say, in the sense of the following definition.

Definition 0.0.0. *In the above situation a* good quotient *of X by G is a morphism $f : X \to Y$ of algebraic schemes, satisfying,*

i) f is surjective, affine and G-invariant (i.e. $f(gx) = f(x)$ for all $g \in G$, $x \in X$);

ii) $f_*(O_X)^G = O_Y$, ($f_*(O_X)$ is the direct image of O_X and $f_*(O_X)^G$ is the sheaf of G-invariants in $f_*(O_X)$);

iii) *if F is a G-invariant closed subset of X then $f(F)$ is closed in Y* 2
and if F_1 and F_2 are G-invariant closed subsets of X such that $F_1 \cap F_2 = \phi$ then $f(F_1) \cap f(F_2) = \phi$.

Definition 0.0.1. *With the same notations as above, a* geometric quotient *of X by G is a morphism $f : X \to Y$ of algebraic schemes, satisfying,*

i) *f is a good quotient of X by G;*

ii) *for every y in Y the fibre $f^{-1}(y)$ is exactly one orbit. (In particular the orbits are closed).*

It is easy to see that a quotient (good or geometric) is unique up to isomorphism (if it exists).

Example 0.0.2. Consider the natural action of $GL(N)$ on affine N-space $\mathbb{A}^N$. Clearly $\mathbb{A}^N - \{0\}$ is a single orbit in $\mathbb{A}^N$ which is not closed. Hence a geometric quotient of $\mathbb{A}^N$ by $GL(N)$ does not exists.

Now suppose that $X \subset \mathbb{P}^N$ is a projective algebraic scheme and G is a reductive algebraic group acting on X via a representation $\varphi : G \to GL(N+1)$.

Definition 0.0.3. *In the above situation a point $x \in X$ is called* **semi-stable** *if there exists a non constant G-invariant homogeneous polynomial F such $F(x) \neq 0$.*

3 *Put $X^{ss} = \{x \in X | x$ is semistable $\}$. Clearly X^{ss} is open in X.*

Definition 0.0.4. *With the same notation as above, a point $x \in X$ is called* stable *, if,*

i) $\dim 0(x) = \dim G$, *($0(x)$ denotes the orbit of x);*

ii) *there exists a non constant G-invariant homogeneous polynomial F such that $F(x) \neq 0$ and for every y_0 in $X_F = \{y \in X | F(y) \neq 0\}$, $0(y_0)$ is closed in X_F.*

Put $X^s = \{x \in X | x$ is stable $\}$. Note that the set $\{x \in X | \dim(0(x)) = \dim G\}$ is open X because $\dim(0(x))$ is a lower semicontinuous function *of x. Now it is immediate that X^s is open in X. Both X^{ss} and X^s can be empty, however in the case when they are non empty we have the following theorem.*

Theorem 0.0.5. *There exists a projective algebraic scheme Y and a morphism $f_{ss} : X^{ss} \to Y$ such that f_{ss} is a good quotient of X^{ss} by G. Further there exists an open subset U of Y such that $f_{ss}^{-1}(U) = X^s$ and $f_s : X^s \to U$ is a geometric quotient of X^s by G.*

There is a test for semistability using one parameter subgroups.

Definition 0.0.6. *Let G be an algebraic group. A* one parameter subgroup λ (abbreviated as 1- $ps\lambda$) *of G is defined to be a nontrivial homomorphism $\lambda : G_m \to G$ of algebraic groups.* 4

Let G be a reductive algebraic group acting on a projective algebraic scheme $X \subset \mathbb{P}^N$ via a representation $\varphi : G \to GL(N+1)$. Given a $1 - ps\lambda$ of G, there is an induced action of λ on the affine $(N+1)$-space $\mathbb{A}^{N+1}$. This action can be diagonalized, i.e., there exists a basis $e_0, e_1, \ldots, e_N$ of (the vector spacer) $\mathbb{A}^{N+1}$ such that the action of λ on $\mathbb{A}^{N+1}$ is given by $\lambda(t)e_i = t^{r_i}e_i$, $t \in K^*$, $r_i \in \mathbb{Z}$, $(0 \leq i \leq N)$. Let $x = \sum_{i=0}^{N} x_i e_i$ be a point in $\mathbb{A}^{N+1} - \{0\}$, $(x_i \in K, 0 \leq i \leq N)$. Clearly $\lambda(t)x = \sum_{i=0}^{N} x_i e_i$. The point $x \in \mathbb{A}^{N+1} - \{0\}$ represents a point, say $\overline{x}$, in $\mathbb{P}^N$

Definition 0.0.7. *With λ and x as above we define $\mu(\overline{x}, \lambda) = -\max.$ $\{r_i | x_i \neq 0\}$.*

It can be shown that $\mu(\overline{x}, \lambda)$ is independent of the basis $e_0, e_1, \ldots, e_N$ and the point x, so that the above definition makes sense.

Definition 0.0.8. *With the same notations as above a point $\overline{x} \in X$ is called* λ-semistable (respectively λ -stable) *if $\mu(\overline{x}, \lambda) \leq 0$ (respectively $\mu(\overline{x}, \lambda) < 0$).*

Semistability (respectively stability) and λ-semistability (respectively λ-stability) of a point $\overline{x}$ are related by the following theorem.

Theorem 0.0.9. *With the same notations as above, $\overline{x}$ is semistable $\Longleftrightarrow$* 5

$\overline{x}$ is λ-semistable for every $1-ps\lambda$ of G, and $\overline{x}$ is stable $\Longleftrightarrow$ $\overline{x}$ is λ-stable for every $1-ps\lambda$ of G.

It follows from the above theorem that to show that a point $\overline{x} \in X$ is not semistable it suffices to find a single $1-ps\lambda$ of G such that $\overline{x}$ is not λ-semistable.

The proofs of the results in this section can be found in [5].

B) Hilbert point of a curve

Let $X \subset \mathbb{P}^N$ be a complete curve. Let L be the restriction of $O_{\mathbb{P}^N}(1)$ to X. Recall that $\chi(L^m) = h^o(X, L^m) - h^l(X, L^m)$ is a polynomial in m, say $P(m)$.

By Serre's vanishing theorem there exists an integer m' such that all integers $m > m'$, $H^l(H, L^m) = 0$ and the restriction

$$\varphi_m : H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m)) \to H^o(X, L^m) \text{ is surjective.}$$

Assume now that $m > m'$. Taking the $P(m)^{\text{th}}$ exterior powers, we get,

$$\overset{P(m)}{\Lambda}\varphi_m : \overset{P(m)}{\Lambda} H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m)) \to \overset{P(m)}{\Lambda} H^o(X, L^m) \simeq K,$$

a point in the projective space $\mathbb{P}(\overset{P(m)}{\Lambda} H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m)))$. (For a vector space V, $\mathbb{P}(V)$ denotes the projective space associated to V, in the sense of Grothendieck i.e. $\mathbb{P}(V)$ is the space consisting of equivalence classes of nonzero linear forms on V.)

6 **Definition 0.1.0.** *In the above situation the point $\overset{P(m)}{\Lambda}\varphi_m \in \mathbb{P}(\overset{P(m)}{\Lambda} H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m)))$ is defined to be the m^{th}* Hilbert point of a curve X.

Choose a basis $X_0, X_1, \ldots, X_N$ of $H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(1))$. Consider the action of $GL(N+1)$ (and hence of $SL(N+1)$) on $H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(1))$, defined by,

$$[a_{ij}].X_P = \sum_{j=0}^{N} a_{pj}X_j, [a_{ij}] \in GL(N+1), \quad (0 \leq p \leq N).$$

We have an induced action of $SL(N+1)$ on $\mathbb{P}(\overset{P(m)}{\Lambda} H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m))$ described as follows.

Recall that $H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m))$ *has the basis* $B_m = \{M_1, M_2, \ldots, M_{\alpha_m}\}$ *consisting of monomials of degree* m *in* $X_0, X_1, \ldots, X_N$, $(\alpha_m = h^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m))$. $SL(N+1)$ *acts on* $H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m))$, *with the action given by,*

$$g.X_o^{\gamma_o} X_l^{\gamma_1} \cdots X_N^{\gamma_N} = H_o^{\gamma_o} H_l^{\gamma_l} \cdots H_N^{\gamma_N},$$
$$(X_o^{\gamma_o} X_l^{\gamma_l} \cdots X_N^{\gamma_N} \in B_m, g \in SL(N+1), H_p = g.X_p,\ 0 \le p \le N).$$

Hence there is an action of $SL(N+l)$ *on* $\overset{P(m)}{\Lambda} H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m))$, *as follows. Recall that* $M_{i_1} \Lambda M_{i_2} \Lambda \cdots \Lambda M_{i_{P(m)}}$ $(1 \le i_1 < i_2 < \cdots < i_{P(m)} \le \alpha_m)$ *is a basis of* $\overset{P(m)}{\Lambda} H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m))$. *The action of* $SL(N+l)$ *on this space is given by,* **7**

$$g \cdot (M_{i_1} \Lambda M_{i_2} \Lambda \ldots \Lambda M_{i_{P(m)}}) = gM_{i_1} \Lambda gM_{i_2} \Lambda \ldots \Lambda gM_{i_{P(m)}}, (g \in SL(N+1)).$$

Take the dual action $\overset{P(m)}{\Lambda} H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m))^*$ *which naturally gives an action of* $SL(N+1)$ *on* $\mathbb{P}(\overset{P(m)}{\Lambda} H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m)))$.

Let λ *be a* $1-ps$ *of* $SL(N+1)$. *When is the point* $H_m(X) \in \mathbb{P}(\overset{P(m)}{\Lambda} H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m)))\lambda$ *- semistable? We try to answer this question.*

There exists a basis $w_0, w_1, \ldots, w_N$ *of* $H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(1))$ *and integers* $r_0, r_1, \ldots, r_N$ *such that the action of* λ *on* $H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(1))$ *is given by*

$$\lambda(t)w_i = t^{r_i} w_i, t \in K^*, \quad (0 \le i \le N).$$

Let $B'_m = \{M'_1, M'_2, \ldots, M'_{\alpha_m}\}$ *be a basis of* $H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m))$ *consisting of monomials of degree* m *in* $w_0, w_1, \ldots, w_N$. *In this situation we make the following definition.*

Definition 0.1.1. *For a monomial* $M = w_0^{\gamma_0}\ w_1^{\gamma_1}, \ldots, w_N^{\gamma_N}$, *define its* λ-weight $w_\lambda(M)$, *by* $w_\lambda(M) = \sum_{i=0}^{N} \gamma_i r_i$ *and define,* total λ-weight of monomials $M''_1, M''_2, \ldots, M''_t$ *to be* $\sum_{i=0}^{t} w_\lambda(M''_i)$.

The vector space $\overset{P(m)}{\Lambda} H^o(\mathbb{P}^N,\ O_{\mathbb{P}^N}(m))$ has the following basic, **8**

$$\left\{M'_{i_1} \wedge M'_{i_2} \wedge \ldots \wedge M'_{i_{P(m)}}\right\}_{(1 \le i_1 < i_2 < \ldots\, i_{P(m)} \le \alpha_m)}$$

Let $\{M'_{i_1} \wedge M'_{i_2} \wedge \ldots \wedge M'^{*}_{i_{P(m)}}\}_{(1 \le i_1 < i_2 < \ldots\, i_{P(m)} \le \alpha_m)}$ be the basis of $\overset{P(m)}{\wedge} H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m))^*$ dual to the above basis of $\overset{P(m)}{\wedge} H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m))$. The action of λ on $\overset{P(m)}{\wedge} H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m))^*$ is given by,

$$\lambda(t)\,(M'_{i_1} \wedge M'_{i_2} \wedge \ldots \wedge M'^{*}_{i_{P(m)}}) = t^{-\theta}\,(M'_{i_1} \wedge M'_{i_2} \wedge \ldots \wedge M'^{*}_{i_{P(m)}}),$$
$$t \in K^*,\ \theta = \sum_{j=1}^{P(m)} w_\lambda(M'_{i_j}).$$

Write $H_m(X)$ as a linear combination of the vectors in the above basis of $\overset{P(m)}{\wedge} H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m))^*$.

$$H_m(X) = \sum \overset{P(m)}{\wedge} \varphi(M'_{i_1} \wedge \ldots \wedge M'_{i_{P(m)}})\,(M'_{i_1} \wedge M'_{i_2} \wedge \ldots \wedge M'^{*}_{i_{P(m)}})$$

By definition
$H_m(X)$ is λ-semistable (respectively λ-stable)
$\iff \mu(H_m(X), \lambda) \le 0$ (respectively < 0)
$\iff -\max.\left\{-\sum_{j=1}^{P(m)} w_\lambda(M'_{i_j})\right\} \le 0$ (respectively < 0),
9 where the maximum is taken over all $(i_1, i_2, \ldots, i_{P(m)})$ with $1 \le i_1 < i_2 < \ldots < i_{P(m)} \le \alpha_m$, such that $\overset{P(m)}{\wedge} \varphi_m\,(M'_{i_1} \wedge M'_{i_2} \wedge \ldots \wedge M'_{i_{P(m)}}) \ne 0$. Clearly

$$-\max.\left\{-\sum_{j=1}^{P(m)} w_\lambda\,(M'_{i_j})\right\} = \min.\left\{\sum_{j=1}^{P(m)} w_\lambda(M'_{i_j})\right\}.$$

Thus we have the following criterion.

$(*)$ In the above situation $H_m(X)$ is λ-semistable (respectively λ-stable) $\iff$ There exist monomials $M'_{i_1}, M'_{i_2}, \ldots, M'_{i_{P(m)}}$, $(1 \le i_1 < i_2 < \ldots < i_{P(m)} \le \alpha_m)$, in B'_m such that $\varphi_m(M'_{i_1}), \varphi_m(M'_{i_2}), \ldots, \varphi_m(M'_{i_{P(m)}})$ is a basis of $H^o(X, L^m)$ and $\sum_{j=1}^{P(m)} w_\lambda\,(M'_{i_j}) \le 0$ (respectively < 0).

Let λ be a $1-ps$ of $GL(N+1)$. There exists a basis $\{w_0, w_1, \ldots, w_N\}$ of $H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(1))$ and integers $r_0, r_1, \ldots, r_N$ such that the induced action of λ on $H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(1))$ is given by,

$$\lambda(t)\ w_i = t^{r_i} w_i, \quad t \in K^*, (0 \le i \le N).$$

Put $\sum\limits_{i=0}^{N} r_i = r$. Define a $1-ps\lambda'$ of $SL(N+1)$ so that the action of λ' on $H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(1))$ is given by

$$\lambda'(t)w_i = t^{r'_i} w_i, t \in K^*, r'_i = (N+1)r_i - r, \quad (0 \le i \le N).$$

Definition 0.1.2. *In the above situation the* $1-ps\lambda'$ *of* $SL(N+1)$ *is said to be the* $1-ps$ *of* $SL(N+1)$ *associated to the* $1-ps\lambda$ *of* $GL(N+1)$*. We want to rewrite the condition* $(*)$ *for* λ'*−semistability (respectively* λ*-stability) of* $H_m(X)$ *in terms of* λ*−weights of the monomials.*

Note that for a monomial $M \in H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m))$, $w_{\lambda'}(M) = (N+1)w_\lambda(M) - rm$. It follows that,

$$\sum_{j=1}^{P(m)} w_{\lambda'}(M'_{i_j}) \le (\text{ respectively } < 0)$$

$$\Longleftrightarrow$$

$$(N+1)\sum_{j=1}^{P(m)} w_\lambda(M'_{i_j}) - P(m)\ \text{rm} \le 0 \ \ (\text{ respectively } < 0)$$

$$\Longleftrightarrow$$

$$\sum_{j=1}^{P(m)} \frac{w_{\lambda'}(M'_{i_j})}{mP(m)} \le \frac{r}{N+1}, \ \ (\text{ respectively } < \frac{r}{N+1})$$

Thus we have the following criterion

$(**)$ With the same notations as above,

$H_m(X) \in \mathbb{P}(\overset{P(m)}{\Lambda} H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m)))$ is λ' -semistable (respectively λ' -stable) $\Leftrightarrow$ there exist monomials M'_{i_1}, M'_{i_2} $, \ldots, M'_{i_{P(m)}}$, $(1 \le i_1 < i_2 <$

$\cdots < i_{P(m)} \leq \alpha_m$) in B'_m such that $\left\{\varphi_m(M'_{i_1}), \varphi_m(M'_{i_2}), \ldots, \varphi_m(M'_{i_{P(m)}})\right\}$ is a basis of $H^o(X, L^m)$ and $\dfrac{\sum_{j=1}^{P(m)} w_{\lambda'}(M'_{i_j})}{mP(m)} \leq \dfrac{r}{N+1}$, (respectively $< \dfrac{r}{N+1}$).

C) Hilbert Scheme

11 Consider the projective space $\mathbb{P}^N$ over $\operatorname{Spec}\mathbb{Z}$. Look at all closed subschemes of $\mathbb{P}^N$, flat over $\mathbb{Z}$, with a fixed Hilbert polynomial say $P(m)$. A fundamental existence theorem says that there exists a scheme, projective over $\operatorname{Spec}\mathbb{Z}$, parametrizing all these closed subschemes of $\mathbb{P}^N$. In fact we have the following stronger version of the theorem.

Let Sch denote the category of locally noetherian schemes. Define a functor $\mathrm{Hilb}^P_{\mathbb{P}^N}$ form Sch to the category of sets as follows.

For S in Sch, $\mathrm{Hilb}^P_{\mathbb{P}^N}(S)$ = The set of all closed subschemes W of $\mathbb{P}^N \times S$, flat over S such that for every $s \in S$ the induced closed subscheme W_s of $\mathbb{P}^N_{k(s)}$ has Hilbert polynomial $P(m)$.

Theorem 0.1.3. *The functor* $\mathrm{Hilb}^P_{\mathbb{P}^N}$ *is representable and is represented by a scheme projective over spec* $\mathbb{Z}$.

Let H denote the scheme representing the functor $\mathrm{Hilb}^P_{\mathbb{P}^N}$. Thus for all S in Sch, $\mathrm{Hilb}^P_{\mathbb{P}^N}(S) \simeq \mathrm{Hom}(S, H)$. In particular $\mathrm{Hilb}^P_{\mathbb{P}^N}(H) \simeq \mathrm{Hom}(H, H)$. Let Z be the closed subscheme of $\mathbb{P}^N \times H$ which corresponds to the identity morphism $i \in \mathrm{Hom}(H, H)$, under the above isomorphism. We call Z, the universal closed subscheme. It has the following universal property.

12 Given a scheme S in Sch and a scheme $Y \in \mathrm{Hilb}^P_{\mathbb{P}^N}(S)$, there exists a unique morphism $f : S \to H$ such that $(1 \times f)^* Z \simeq Y$.

A proof of the above theorem and other details can be fond in [2], [6].

D) Stable Curves (in the sense of Deligne - Mumford (1))

Definition 0.1.4. *Let S be any scheme.* A stable (respectively semistable) curve of genus $g \geq 2$ *over S s a proper flat morphism $\pi : X \to S$ such that for all $s \in S$ the fibre X_s of π over s, satisfies,* *i)* X_s *is a reduced, connected scheme of dim 1 with* $h^1(X_s, O_{X_s}) = g$*;*

ii) *each singular point of* X_s *is an ordinary double point;*

iii) *if* E *is an irreducible component of* X_s *such* $E \simeq \mathbb{P}^1$ *then* E *meets the other component of* X_s *in at least 3 points, (respectively 2 points).*

We now quote some results on stable curves which will be needed in the sequel.

Theorem 0.1.5. *If* $\pi : C \to \operatorname{Spec} K$ *is a stable curve then* $H^1(C, \omega^n_{C/K}) = 0$ *for* $n \geq 2$ *and* $\omega^n_{C/K}$ *is very ample for* $n \geq 3$, *(*$\omega_{C/K}$ *denotes the dualizing sheaf of* X*).*

We have the following consequences of the above theorem.

Let $\pi : X \to S$ be a stable curve of genus $g \geq 2$. It follows from **13**
the above theorem that for all $s \in S$ and for $n \geq 2$, $H^1(X_s, \omega^n_{X/S} \otimes O_{X_s}) = 0$. This implies that $\pi_*(\omega^n_{X/S})$ is locally free and there are natural isomorphisms

$$\pi_*(\omega^n_{X/S} \otimes k(s) \simeq H^o(X_s, \omega^n_{X/S} \otimes O_{X_s}), \quad \text{(cf. EGA, Chapter 3, §7).}$$

Hence for $n \geq 3$ the relatively very ample line bundle $\omega^n_{X/S}$ gives an embedding of X into the projective bundle $\mathbb{P}(\pi_*\omega^n_{X/S})$ over S, associated to the locally sheaf $\pi_*(\omega^n_{X/S})$ on S. Thus X can be realized as a family of curves C in $\mathbb{P}^{n(2g-2)-g}$ with the Hilbert polynomial of C given by $P(m) = n(2g-2)m - g + 1$.

Let $p : X \to S$ and $q : Y \to S$ be two stable curves. Define a functor $\mathrm{Isom}_S(X, Y)$ form the category of S- schemes, Sch S, to the category of sets, as follows.

$\mathrm{Isom}_S(X, Y)(S') =$ The set of S'- isomorphisms between $X \underset{S}{\times} S'$ and $Y \underset{S}{\times} S'$.

Theorem 0.1.6. *The functor* $\mathrm{Isom}_S(X, Y)$ *is represented by a scheme* $\overline{\mathrm{Isom}}_S(X, Y)$, *quasiprojective over* S. *(cf. [3]).*

Let $\pi : C \to \operatorname{Spec} K$ be a stable curve. Let $t : \operatorname{Spec} \dfrac{K[\varepsilon]}{(\varepsilon^2)} \to \overline{\mathrm{Isom}}_K(C, C)$ be a tangent vector at a point $P \in \overline{\mathrm{Isom}}_K(C, C)$. By definition t corresponds to an automorphism of $C \times \operatorname{Spec} \dfrac{K[\varepsilon]}{(\varepsilon^2)}$ which is canonically identified with a vector field D defined on the whole of X. Now note the following lemma.

14 **Lemma 0.1.7.** *If $\pi : C' \to \operatorname{Spec} K$ is a stale curve then a vector field defined on the whole of C' is zero.*

Before we go to the proof of the lemma we deduce the following result. The lemma says that the tangent space to $\overline{\mathrm{Isom}}_K(C, C)$ at the point P is zero. Since P was an arbitrary point of $\overline{\mathrm{Isom}}_K(C, C)$ we see that $\overline{\mathrm{Isom}}_K(C, C)$ is finite. Thus we have the following theorem.

Theorem 0.1.8. *If $\pi : C \to \operatorname{Spec} K$ is a stable curve then the group of automorphisms of C is finite.*

Proof of the Lemma 0.1.7. Let D be a vector field defined on the whole of C'. Let $\overline{C'}$ be the normalization of C'. Since the only singularities of C' are ordinary double points, D naturally corresponds to a vector field $\overline{D}$ on $\overline{C'}$, such that $\overline{D}$ vanishes at all points of $\overline{C'}$ which lie over the double points of C'. It follows that if E is an irreducible component of C' such that $\overline{E}$, the normalization of E, has genus ≥ 2 then $\overline{D}$ vanishes on $\overline{E}$ and hence D vanishes on E.

Now consider the components E of C' such that $\overline{E}$ has genus ≤ 1. We have the following possibilities for E.

i) E is a nonsingular curve of genus 0.

ii) E has one double point, $\overline{E}$ has genus 0.

iii) E has at least two double points, $\overline{E}$ has genus 0.

iv) E is a nonsingular curve of genus 1.

v) E has at least one double point, $\overline{E}$ has genus 1.

15 In the cases when E has genus 0, $\overline{D}$ has at least 3 zeroes and when $\overline{E}$ has genus 1, $\overline{D}$ has at least one zero. It follows that $\overline{D}$ must be zero on $\overline{E}$ in each of the above cases. This proves the lemma.

For the proofs of the results in this section we refer to [1].

E) Deformation Theory

In this section we consider complete curves X such that,

i) X is reduced, connected;

ii) if $P \in X$ is a singular point of X then P is necessarily ordinary double point, i.e., $\hat{O}_{X,P} \simeq \dfrac{K[[x,y]]}{(xy)}$, ($\hat{O}_{X,P}$ denotes the completion of the local ring $O_{X,P}$ of X at P).

It is clear that such a curve X is a local complete intersection.

Definition 0.1.9. *A (flat) deformation of X over a complete local K-algebra A is a flat morphism $\varphi : \overline{X} \to \operatorname{Spec} A$ such that the special fibre of φ (i.e., the fibre over the closed point of Spec A) is isomorphic to X.*

Recall that the set of first order deformation of X (i.e. deformations over Spec $\dfrac{K[\varphi]}{(\varepsilon^2)}$) is canonically identified with $\operatorname{Ext}^1(\Omega^1_X, O_X)$, ($\Omega^1_X$ is the sheaf of Kahler differentials on X).

Note the following lemma. 16

Lemma 0.2.0. $\operatorname{Ext}^1(\Omega^1_X, O_X) = 0$.

Proof. The result follows from the following observations.

i) We have the following spectral sequence.

$$H^p(X, \underline{\operatorname{Ext}}^q(\Omega^1_X, O_X)) \Rightarrow \operatorname{Ext}^{p+q}(\Omega^1_X, O_X).$$

Since $\dim X = 1$, $H^2(X, \underline{\operatorname{Ext}}^o(\Omega^1_X, O_X)) = 0$. Since Ω^1_X is locally free except at a finite number of points, ($\underline{\operatorname{Ext}}^1(\Omega^1_X, O_X)$ has support at only finitely many points and hence $H^1(X, \underline{\operatorname{Ext}}^1(\Omega^1_X, O_X)) = 0$.

ii) Locally X can be embedded in an affine N-space $\mathbb{A}^N$. Let I be be the ideal sheaf defining X in $\mathbb{A}^N$. Ω^1_X has the following free resolution.

$$0 \to \frac{I}{I^2} \to \Omega^1_{\mathbb{A}^N} \otimes O_X \to \Omega^1_X \to 0$$

It follows that $\underline{\mathrm{Ext}}^2(\Omega^1_X, O_X)) = 0$.

Now it is immediate that $\underline{\mathrm{Ext}}^2(\Omega^1_X, O_X)) = 0$. Thus there are no obstructions to lifting deformations over Spec $\frac{A}{J}$ to deformations of over Spec A (A denotes an Artin local ring with residue field K, J an ideal in A). Equivalently the functor of deformations of X over an Artin local $K-$algebra is formally smooth. We have the following theorem. □

17 **Theorem 0.2.1.** *There exists a formal scheme $\tilde{X}$ and a proper flat morphism $\eta : \tilde{X} \to \mathrm{Spec}\, K[[t_1, t_2, \ldots, t_r]] = T$, ($r = \dim \mathrm{Ext}^1(\Omega^1_X, O_X)$), such that the special fibre of η is isomorphic to X. Further the morphism η has the following properties.*

i) Given a deformation $\overline{X} \to \mathrm{Spec}\, A$ of X over an Artin local K-algebra A, there exists a morphism $\rho : \mathrm{Spec}\, A \to T$ such that $\overline{X} \to \mathrm{Spec}\, A$ is obtained form $\eta : \tilde{X} \to T$ by the base change $\rho : \mathrm{Spec}\, A \to T$.

ii) In the case when $A \simeq \frac{K[\omega]}{(\omega^2)}$ the above morphism ρ is unique so that the tangent space of T at the closed point is canonically isomorphic to $\mathrm{Ext}^1(\Omega^1_X, O_X)$.

$\eta : \tilde{X} \to T$ is called a versal deformation space for X.

In the case when $\mathrm{Ext}^o(\Omega^1_X, O_X) = 0, \eta : \tilde{X} \to T$ is universal i.e. the morphism ρ is always unique. Thus if X is a stable curve then a versal deformation is universal (cf. lemma 0.1.7 page 10). Further since the invertible sheaf $\omega_{\tilde{X}/T}$ is relatively ample, $\tilde{X}$ is the formal completion of a unique scheme, proper and flat over T. We have the following theorem.

Theorem 0.2.2. *If X is a stable curve then the versal deformation $\eta : \tilde{X} \to T$ is universal and algebraizable.*

Another fact about $\eta : \tilde{X} \to T$ (X a stable curve) is that generic fibre of η is nonsingular.

F) In this section we prove some results and make a few definitions which will be needed in the sequel. We first prove Clifford's theorem for a reduced curve with ordinary double points. The proof in this generality is due to Gieseker and Morrison. **18**

Theorem 0.2.3 (Clifford's theorem)**.** *Let X be a reduced curve with only nodes and let L be a a line bundle on X generated by global sections. If* $H^1(X, L) \neq 0$*, there is a curve* $C \subset X$ *so that*

$$h^o(C, L) \leq \frac{\deg_C L}{2} + 1$$

Proof. Since $H^1(X, L) \neq 0$, $H^o(X, L^{-1} \otimes \omega_X) \neq 0$, ($\omega_X$ is the dualizing sheaf of X). So there is a non-zero $\varphi : L \to \omega_D$. We can find a curve $C \subset X$ so that φ is not identically zero on each component of C, but φ vanishes at all points $C \cap \overline{X - C} = \{P_1, \ldots, P_k\}$. Since $\omega_C = \omega_X(-P_1 \ldots - P_k)$, we actually obtain

$$\varphi : L_C \to \omega_C.$$

Choose a basis $s_1, \ldots, s_r$ of $\operatorname{Hom}(L_C, \omega_C)$ so that $\varphi = s_1$. We can choose a basis $t_1 \ldots t_p$ of $H^o(L_C)$ so that t_1 does not vanish at the zeros of s_1 nor any singular point of C. Suppose

$$a_1[s_1, t_1] + a_2[s_1, t_2] + \cdots = b_2[s_2, t_1] + b_3[s_3, t_1] + \cdots$$

where $[s, t]$ is in $H^o(C, \omega_C)$. Then **19**

$$[s_1, t] = [s, t_1]$$

where $t \in H^o(C, L_C)$ and s is a linear combination of $s_2, \ldots, s_r$. Hence t is a multiple of t_1, since t vanishes where t_1 does. Hence s is a multiple of s_1, contradicting the independence of the s_i's. So

$$h^o(L_C) - h^o(L_C^{-1} \otimes \omega_C) \leq g + 1$$

and

$$h^o(L_C) + h^o(L_C^{-1} \otimes \omega_C) \leq \deg_C(L) + 1 - g.$$

Adding gives the desired result. □

Lemma 0.2.4. *Fix two integers $g \geq 2$, $d \geq 20(g-1)$ and put $N = d - g$. There exists a constant $\varepsilon > 0$ such that for all integers not all zero, $r_0 \leq r_1 \leq \cdots \leq r_N$, $\sum\limits_{i=0}^{N} r_i = 0$ and for all integers $0 = e_0 \leq e_1 \leq \cdots \leq e_N = d$, satisfying,*

i) if $e_i > 2g-2$ then $e_i \geq i + g$,

ii) if $e_i \leq 2g-2$ then $e_i \geq 2i$,

there exists a sequence of integers $0 = i_1 < i_2 < \cdots < i_k = N$, making the following inequality true.

$$\sum_{t=1}^{k-1} (r_{i_{t+1}} - r_{i_t}) \frac{(e_{i_{t+1}} + e_{i_t})}{2} > r_N e_N + \varepsilon (r_N - r_0) \tag{1}$$

Proof. We use the following combinatorial lemma proved by Morrison. □

Fix integers $0 = e_0 \leq e_1 \leq \cdots \leq e_N$. Define a function

$$T(r_0, r_1, \ldots, r_N) = \min_{0=i_1<\cdots<i_k=N} \left[\sum_{t=1}^{k-1} (r_{i_t} - r_{i_{t+1}}) \frac{(e_{i_t} + e_{i_{t+1}})}{2} \right],$$

where $r_0 \geq r_1 \geq \ldots \geq r_N = 0$ are numbers with $\sum\limits_{i=0}^{N} r_i = 1$. Then maximum value of T is $T_{\max} = \dfrac{1}{2} \max\limits_{i \in \{1, \ldots N\}} \dfrac{e_i^2}{ie_i - \sum\limits_{j=1}^{i-1} e_j}$

We modify inequality (1) as follows.

Let $r_i' = r_i + \mid r_0 \mid, R = \sum\limits_{i=0}^{N} r_i' = (N+1)|r_0|, r_i'' = \dfrac{r_i'}{R}, (0 \leq i \leq N)$. Inequality (1) can be easily seen to be equivalent to the following inequality

$$\sum_{t=1}^{k-1} (r''_{i_{t+1}} - r''_{i_t}) \frac{(e_{i_{t+1}} + e_{i_t})}{2} > e_N (r''_N - \frac{1}{N+1}) + \varepsilon r''_N.$$

Here $0 = r''_0 \leq r''_1 \leq \cdots \leq r''_N, \sum_{i=0}^{N} r''_i = 1$. Transferring we get,

$$e_N r''_N - \sum_{t=1}^{k-1}(r''_{i_{t+1}} - r''_{i_t})\frac{e_{i_{t+1}} + e_{i_t}}{2} < \frac{e_N}{N+1} - \varepsilon\, r''_N, \quad \text{i.e.}$$

$$\sum_{t=1}^{k-1}(r''_{i_{t+1}} - r''_{i_t})\frac{(e_N - e_{i_{t+1}} + e_N - e_{i_t})}{2} < \frac{e_N}{N+1} - \varepsilon\, r''_N.$$

For $0 < i < N$, let $e'_i = e_N - e_{N-i}$ and $r'''_i = r''_{N-i}$. Thus we have $0 = e'_0 \leq e'_1 \leq \cdots \leq e'_N = d,\ \ r'''_0 \geq r'''_1 \geq \cdots \geq r'''_N, \sum_{i=0}^{N} r'''_i = 1$.
Also it follows from conditions i) and ii) that, **21**

i) if $e'_i < d - (2g-2)$ then $e'_i \leq i$,

ii) if $e'_i \geq d - (2g-2)$ then $e_i \leq g + 2i - N$.

The last inequality can be written as,

$$\sum_{t=1}^{k-1}(r'''_{N-i_{t+1}} - r'''_{N-i_t})\frac{e'_{N-i_{t+1}} + e'_{N-i_t}}{2} < \frac{e_N}{N+1} - tr''_N$$

It follows from Morrison's combinatorial lemma that there exists $i \in \{1, 2, \ldots, N\}$ and a sequence of integers, $0 = N - i_k < N - i_{k-1} < \cdots < N - i_1 = N$ such that the following inequality is true.

$$\sum_{t=1}^{k-1}(r'''_{N-i_{t+1}} - r'''_{N-i_t})\frac{(e'_{N-i_{t+1}} + e'_{N-i_t})}{2} < \frac{1}{2}\frac{e'^2_i}{ie'_i - \sum_{j=1}^{i-1} e'_j}$$

Thus to prove the lemma it suffices to prove that there exists an $\varepsilon > 0$ such that for any sequence of integers $0 = e'_0 < e'_1 < \cdots < e'_N = d$ as above and for all $1 \leq i \leq N$,

$$\frac{1}{2}\ \frac{e'^2_i}{ie'_i - \sum_{j=1}^{i-1} e'_j} < \frac{d}{N+1} - \ \varepsilon.$$

This can be easily checked using the bounds on $e'_0, e'_1, \ldots, e'_N$.

Let $X = \bigcup\limits_{i=1}^{p} X_i$ be a curve, (X_i is an irreducible component of X $1 \leq$
22 $i \leq p$). Let $\pi_i : \overline{X}_{\text{red}} \to X_{\text{ired}}$ be the normalization of X_{ired} and let $\overline{X}$ be the disjoint union $\overline{X} = \bigcup\limits_{i=1}^{p} \overline{X}_{\text{ired}}$. We have closed immersions (inclusions) $\eta_i : X_{\text{ired}} \to X$. Let $\pi' : \overline{X} \to X_{\text{ired}}$ be the morphism such that the restriction of the morphism $\pi = \eta o \pi'$ to $\overline{X}_{\text{ired}}$ is the morphism $\eta_i o \pi_i$, $(1 \leq i \leq p)$.

Definition 0.2.5. *The morphism $\pi : \overline{X} \to X$ is defined to be the normalization of X.*

Let V be a vector space of dimension n and let

$$o \subset V_1 \subset V_2 \subset \cdots \subset V_r = V, \tag{F}$$

be a filtration of V. Put $n_i = \dim V_i$, $(1 \leq i \leq r)$.

Definition 0.2.6. *In the above situation a basis $v_1, v_2, \ldots, v_n$ of V is said to be a basis relative to the filtration (F) if $v_1, v_2, \ldots, v_{n_1}$ is a basis of V_1; $v_1, v_2, \ldots, v_{n_1}, v_{n_{1+1}}, \ldots, v_{n_2}$ is a basis of V_2, etc.*

Chapter 1

Stability of Curves

Fix a polynomial $P(m) = dm - g + 1$ where g and d are integers with 23
$g \geq 2$ and $d \geq 20(g-1)$. Put $N = d - g$. In this chapter we prove that there exists an integer m_o such that if X is a connected nonsingular (nondegenerate) curve in $\mathbb{P}^N$ with Hilbert polynomial $P(m)$ then the m_o^{th} Hilbert point of X, $H_{m_o}(X) \in \mathbb{P}(\overset{P(m_o)}{\Lambda})H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m_o)))$ (cf. definition 0.1.0 page 4) is stable for the natural action of $SL(N+1)$ on $\mathbb{P}(\overset{P(m_o)}{\Lambda} H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m_o)))$ (cf. definition 0.0.4 page 2). We prove further that if X is a connected curve in $\mathbb{P}^N$, with Hilbert polynomial $P(m)$ such that the m_o^{th} Hilbert point of X, $H_{m_o}(X) \in \mathbb{P}(\overset{P(m_o)}{\Lambda} H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m_o)))$ is semistable, then X is semistable in the sense of definition 0.1.4 (page 8).

Recall that all curve X in $\mathbb{P}^N$, such that the Hilbert polynomial of X is $P(m)$, are parametrized by a projective algebraic scheme, say H (cf. Hilbert scheme, page 8). Let $Z \xrightarrow{\text{inclusion}} \mathbb{P}^N \times H$ be the universal closed subscheme and let $Z \xrightarrow{p_H} H$ be the composite

$$Z \xrightarrow{\text{inclusion}} \mathbb{P}^N \times H \xrightarrow{\text{projection}} H.$$

$Z \xrightarrow{p_H} H$ can be viewed as a family of curves parametrized by H such that for all geometric points $h \in H$ the fibre X_h of $Z \xrightarrow{p_H} H$ over h is a curve in $\mathbb{P}^N_{k(h)}$ and $P(m)$ is the polynomial of X_h.

Notation:

24 i) By “a curve in the family $Z_H \xrightarrow{p_H} H$” we mean the fibre of $Z_H \xrightarrow{p_H} H$ over a closed point of H, which is connected.

ii) X denotes a curve in the family $Z_H \xrightarrow{p_H} H$.

iii) I_X denotes the ideal sheaf of nilpotents in O_X.

iv) $\pi : \overline{X} \to X$ denotes the normalization of X.

v) L denotes the restrictions of $O_{\mathbb{P}^N}(1)$ to X.

vi) L' denotes the line bundle $\pi^* L$ on $\overline{X}$.

vii) φ_m denotes the natural restriction,

$$\varphi_m : H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m)) \to H^o(X, L^m), \; m \in \mathbb{Z}.$$

viii) By a nondegenerate curve in the family $Z_H \xrightarrow{p_H} H$ we mean a curve in the family $Z_H \xrightarrow{p_H} H$, which is a nondegenerate curve in $\mathbb{P}^N$.

Note the following assertions. There exists positive integers m', m'', m''', q_1, q_2, q_3, μ_1, μ_2 with $m''' > m'$, $m'' > 2$, $q_3 > q_1$, $\mu_1 > \mu_2$ such that for every curve X in the family $Z_H \xrightarrow{p_H} H$, the following is true:

i) For all integers $m > m'$, $H^1(X, L^m) = 0 = H^1(\overline{X}, L'^m)$.

ii) $I_X^{q_1} = 0$.

25 iii) $h^o(X, I_X) \leq q_2$.

iv) For every complete subcurve C of X, $h^o(C, O_C) \leq q_3$.

v) For every point $P \in X$ and for all integers $r \geq 0$, $\dim \dfrac{O_{X,P}}{m^r_{X,P}} \leq \mu_1 r + \mu_2$, ($O_{X,p}$ is the local ring X at P and $m_{X,P}$ is the maximal ideal in $O_{X,P}$).

vi) For every subcurve C of X, for every point $P \in C$ and for all integers $m'', m > r \geq m''$, $H^1(C, I^{m-r} \otimes L^m_C) = 0$, ($I$ is the ideal subsheaf of O_C defining the point $P \in C$).

vii) For a geometric point $h \in H$ let X_h denotes the fibre of $Z \xrightarrow{p_H} H$ over $h \in H$. For $m > m'$ let $\psi_m : H \to \mathbb{P}(\overset{P(m)}{\Lambda})H^o(\mathbb{P}^N, O_{\mathbb{P}}{}^N(m)))$ be the morphism defined by $\psi_m(h) = H_m(X_h)$. For all integer $m \geq m'''$, ψ_m is a closed immersion.

We do not try prove these assertions as these can be proved by standard arguments.

Fix a basis $X_0, X_1, \ldots, X_N$ of $H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(1))$. Consider the action of $GL(N+1)$ (and hence $SL(N+1)$) on $H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(1))$, defined by

$$[a_{ij}].X_p = \sum_{j=o}^{N} a_{pj} X_j,\ [a_{ij}] \in GL(N+1),\quad (0 \leq p \leq N).$$

The above action of $SL(N+1)$ on $H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(1))$ induces an action of $SL(N+1)$ on $\mathbb{P}(\overset{P(m)}{\Lambda})H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m_o))$, (cf. page 4).

In the above situation we have the following theorem.

Theorem 1.0.0. *There exists an integer* $m_o >$ max. $\{m''', d\overline{q}(3d + m'' + 5)\}$ *such that for every nondegenerate nonsingular curve X in the family* $Z_H \xrightarrow{p_H} H$, *the* m_o^{th} *Hilbert point of X,* 26

$H_{m_o}(X) \in \mathbb{P}(\overset{P(m_o)}{\Lambda} H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m_o)))$ *is stable*

Remark. It will follow from the proof that there exist infinitely many integers $m >$ max .$\{m''', d\overline{q}(3d + m'' + 5)\}$ such that for every nondegenerate nonsingular curve X in the family $Z_H \xrightarrow{p_H} H$, the m^{th} Hilbert point of X,

$H_m(X) \in \mathbb{P}(\overset{P(m_o)}{\Lambda} H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m)))$ is stable.

Proof. It suffices tn prove that there exists an integer $m_o >$ max .$\{m''', d\overline{q}(3d + m'' + 5)\}$, such that, for every nondegenerate nonsingular curve X in the family $Z_H \xrightarrow{p_H} H$ and for every $1 - ps\lambda$ of $SL(N+1)$, the m_o^{th} Hilbert point of X, $H_{m_o}(X) \in \mathbb{P}(\overset{P(m_o)}{\Lambda} H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m_o)))$ is λ-stable, (cf. theorem 0.0.9 page 3). ◻

Let X be a nondegenerate nonsingular curve in the family $Z_H \xrightarrow{p_H} H$ and let λ be a $1-ps$ of $SL(N+1)$. There exists a basis of $H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(1))$, say, $w_0, w_1, \ldots, w_N$, and integers $r_0 \leq r_1 \leq \cdots \leq r_N$, $\sum_{i=0}^{N} r_i = 0$, such that the action of λ on $H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(1))$ is given by,

$$\lambda(t)w_i = t^{r_i}w_i, \quad t \in K^*, \quad (0 \leq i \leq N)$$

It is easily seen that the natural restriction map $\varphi_1 : H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(1)) \to$
27 $H^o(X, L)$ is an isomorphism. Let $\varphi_1(w_i) = w'_i$, $0 \leq i \leq N$. Let F_{j-1} be the invertible subsheaf of L generated by $w'_o, w'_1, \ldots, w'_{j-1}$, deg $F_{j-1} = e_{j-1}$, $1 \leq j \leq N+1$. Note that the integers $e_0, e_1, \ldots, e_N$ satisfy,

i) if $e_j > 2g-2$ then $e_j \geq j+g$,

ii) if $e_j \leq 2g-2$ then $e_j \geq 2j$.

This is immediate by the Riemann-Roch theorem and Clifford's theorem.

It follows from the combinatorial lemma 0.2.4 (page 14) that there exists a constant $\varepsilon > 0$ such that for all integers, $0 = e'_0 \leq e'_1 \leq \cdots \leq e'_N = d$, satisfying conditions i) and ii) and for all integers $r'_0 \leq r'_1 \leq \cdots \leq r'_N$, $\sum_{i=0}^{N} r'_i = 0$; there exist integers $0 = i_1 < i_2 < \cdots < i_k, = N$ such that the following inequality holds

$$\sum_{t=1}^{k'-1}(r'_{i_{t+l}} - r'_{i_t})\frac{(e'_{i_{t+1}} + e'_{i_t})}{2} > r'_N e'_N + \varepsilon(r'_N - r'_o).$$

In particular, there exist integer $0 = i_1 < i_2 < \cdots < i_k = N$, such that,

$$\sum_{t=1}^{k-1}(r_{i_{t+1}} - r_{i_t})\frac{e_{i_{t+l}} + e_{i_t}}{2} > r_N e_N + \varepsilon(r_N - r_o).$$

Recall that for all positive integers p and n, $H^o(\mathbb{P}^N, O_{\mathbb{P}^N}((p+1)n))$ has a basis $B_{(p+1)n} = \{M_1, M_2, \ldots, M_{\alpha_{(p+1)n}}\}$ consisting of monomials for
28 degree $(p+1)n$ in $w_0, w_1, \ldots, w_N$, $(\alpha_{(p+1)n}) = h^o(\mathbb{P}^N, O_{\mathbb{P}^N}(p+1)n)$.

Let V_{i_t} be the subspace of $H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(1))$, generated by $S_{i_t} = \{w_0, w_1, \ldots, w_{i_t}\}$, $(1 \le t \le k)$. For all integers t_1, t_2, s with $1 \le t_1 < t_2 \le k$ and $0 \le s \le p$ let $(V_{i_{t_1}}^{p-s} \cdot V_{i_{t_2}}^{s} \cdot V_N)^n$ be the subspace of $H^o(\mathbb{P}^N, O_{\mathbb{P}^N}((p+1)n))$ generated by elements w of the type $w = v_1 v_2 \ldots v_n$, where $v_r (1 \le r \le n)$ is as follows.

For $s = 0$, $v_r = x_{r_1} x_{r_2} \ldots x_{r_p} z_r$, $(x_{r_j} \in S_{i_{t_1}}, 1 \le j \le p, z_r \in S_{i_k})$;
for $0 \le s \le p$, $v_r = x_{r_1} x_{r_2}, \ldots x_{r(p-s)} y_{r_1} y_{r_2} \ldots y_{r_s} z_r$
$(x_{r_j} \in S_{i_{t_l}}, 1 \le j < p - s, y_{r_j} \in S_{i_{t_2}}, 1 \le j \le S;\ z_r \in S_{i_k})$;
for $s = p$, $v_r = y_{r_1} y_{r_2} \ldots y_{r_p} z_r$ $(y_{r_j} \in S_{i_{t_2}}, 1 \le j \le p, z_r \in S_{i_k})$
These subspaces define the following filtration of $H^o(\mathbb{P}^N, O_{\mathbb{P}^N}((p+1)^n))$.

$$\begin{aligned} &0 \subset (V_{i_1}^p \cdot V_{i_2}^o \cdot V_N)^n \subset (V_{i_1}^{p-1} \cdot V_{i_2}^1 \cdot V_N)^n \subset \cdots \subset (V_{i_1}^1 \cdot V_{i_2}^{p-1} \cdot V_n)^n \\ &\subset (V_{i_2}^p \cdot V_{i_3}^o \cdot V_N)^n \subset (V_{i_2}^{p-1} \cdot V_{i_3}^1 \cdot V_N)^n \subset \ldots\ldots \quad \ldots \\ &\subset (V_{i_t}^p \cdot V_{i_{t+1}}^o \cdot V_N)^n \subset (V_{i_t}^{p-1} \cdot V_{i_{t+1}}^1 \cdot V_N)^n \subset \cdots \subset (V_{i_t}^{p-s} \cdot V_{i_{t+1}}^s \cdot V_N)^n \subset \ldots \\ &\subset (V_{i_{k-1}}^p \cdot V_{i_k}^o \cdot V_N)^n \subset (V_{i_{k-1}}^{p-1} \cdot V_{i_k}^1 \cdot V_N)^n \subset \ldots \subset (V_{i_{k-1}}^1 \cdot V_{i_k}^{p-1} \cdot V_N)^n \\ &\subset (V_{i_{k-1}}^o \cdot V_{i_k}^p \cdot V_N)^n = H^o(\mathbb{P}^N, O_{\mathbb{P}^N}((p+1)n)), \qquad \text{(F)} \end{aligned}$$

Assume now that $(p+1)n > m'$ so that the natural restriction map $\varphi_{(p+1)n} : H^o(\mathbb{P}^N, O_{\mathbb{P}^N}((p+1)n)) \to H^o(X, L^{(p+1)n})$ is surjective. For 29
integer $0 \le s \le p$ and $1 \le t \le k$, let $(\bar{V}_{i_t}^{p-s} \cdot \bar{V}_{i_{t+1}}^s \cdot \bar{V}_N)^n = \varphi_{(p+1)n}(V_{i_t}^{p-s} \cdot V_{i_{t+1}}^s \cdot V_N)^n \subset H^o(X, L^{(p+1)n})$. We have the following filtration of $H^o(X, L^{(p+1)n})$.

$$\begin{aligned} &0 \subset (\bar{V}_{i_1}^p \cdot \bar{V}_{i_2}^o \cdot \bar{V}_N)^n \subset (\bar{V}_{i_1}^{p-1} \cdot \bar{V}_{i_2} \cdot V_N)^n \subset \cdots \subset (\bar{V}_{i_t}^{p-s} \cdot \bar{V}_{i_{t+1}}^s \cdot \bar{V}_n)^n \subset \ldots \\ &(\bar{V}_{i_{k-1}}^o \cdot \bar{V}_{i_k}^p \cdot \bar{V}_N)^n = H^o(X, L^{(p+1)}n) \qquad (\bar{F}) \end{aligned}$$

Rewrite the basis $B_{(p+1)n}$ as
$B_{(p+1)n} = \{M'_1, M'_2, \ldots, M'_{P((p+1)n)}, M'_{P((p+1)n)+1}, \ldots M'_{\alpha_{(p+1)n}}\}$ so that $M'_1, M'_2, \ldots M'_{P((p+1)n)'}$ is a basis of $H^o(X, L^{(p+1)}n)$ relative to the filtration $(\bar{F})$ and $M'_{P((p+1)n)+1}, \ldots M'_{\alpha_{(p+1)n}}$ are the rest of the monomials in $B_{(p+1)n}$ in some order.

Let X' be a nondegenerate nonsingular curve in the family $Z_H \xrightarrow{p_H} H, L'$ be the restriction of $O_{\mathbb{P}^N}(1)$ to X', $X'_0, X'_1, \ldots, X'_N$ be a basis of $H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(1))$. Let F'_{j-1} be the invertible subsheaf of L' generated by

the images of $X'_0, X'_1, \ldots, X'_{j-1}$ under the natural restriction $\varphi'_1 : H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(1)) \to H^o(X', L'), (1 \leq j \leq N+1)$. We claim that there exists an integer n' such that for all integers $n > n'$, $0 \leq t_1 < t_2 \leq N$ for all nonsingular curves X' in the family $z_H \xrightarrow{p_H} H$ and for all invertible
30 sheaves F'_{t_1}, F'_{t_2}, L' as above,

$$(\overline{V}'^{p-s}_{t_1} \cdot \overline{V}'^{s}_{t_2} \cdot \overline{V}'_N)^n = H^o(X', (F'^{p-s}_{t_1} \otimes F'^{s}_{t_2} \otimes L')^n),$$

$$\left[(\overline{V}'^{p-s}_{t_1} \cdot \overline{V}'^{s}_{t_2} \cdot \overline{V}'_N)^n \text{ is defined in the same way as } (\overline{V}^{p-s}_{t_1} \cdot \overline{V}^{s}_{t_2} \cdot \overline{V}_N)^n\right].$$

Indeed, $F'^{p-s}_{t_1}$ and $F'^{s}_{t_2}$ are generated by the sections in $\overline{V}'^{p-s}_{t_1}$ and $\overline{V}'^{s}_{t_2}$, and the linear system V'_N is very ample. Thus the linear system $W = \overline{V}'^{p-s}_{t_1} \cdot \overline{V}'^{s}_{t_2} \cdot \overline{V}'_N$ is very ample and generates $M = \overline{F}'^{p-s}_{t_1} \otimes \overline{F}'^{s}_{t_2} \otimes L'$. Let $\psi : X' \to \mathbb{P}(W)$ be the projective embedding derived from W and let I be the ideal of $\psi(X')$. For $n >> 0$, $H^1(\mathbb{P}(W), I(n)) = 0$. For such n, the map from W^n to $H^o(X', M^n)$ is onto. Our claim follows provided we can pick n' independent of X' and integers t_1, t_2. This can be done using standard techniques. Thus for integers $0 < t_1 < t_2 < N$, $0 \leq s \leq p$, $n > n'$ we have

$$(\overline{V}^{p-s}_{t_1} \cdot \overline{V}^{s}_{t_2} \cdot \overline{V}_N)^n = H^o(X, (F^{p-s}_{t_1} \otimes F^{s}_{t_2} \otimes L)^n).$$

Choose integers p_o and n_o such that $p_o > \max.\{d+g, \dfrac{2d+1}{\varepsilon}\}$, $n_o > \max.\{p_o, n'\}$ and $m_o = (p_o+1)n_o > \max.\{m''', d\overline{g}(3d + m'' + 5)\}$. It then follows by the Riemann-Roch theorem that

$$\dim(\overline{V}^{p_o-s}_{i_t} \cdot \overline{V}^{s}_{i_{t+l}} \cdot \overline{V}_N)^{n_o} = n_o((p_o - s)e_{i_t} + se_{i_{t+l}} + e_N) - g + 1,$$
$$(0 \leq s \leq p, \quad 1 \leq t \leq k).$$

31 We now estimate,

$$\text{total } \lambda\text{-weight of } M'_1, M'_2, \ldots, M'_{P(m_o)} = \sum_{i=1}^{P(m_o)} w_\lambda(M'_i),$$

(cf. definition 0.1.1 page 5). Note that a monomial $M \in (V^{p_0-s}_{i_t} \cdot V^{s}_{i_{t+1}} \cdot V_N)^{n_o} - (V^{p_o-s+1} \cdot V^{s-1}_{i_{t+1}} \cdot V_N)^{n_o}$ has λ-weight $w_\lambda(M) \leq n_o((p_o - s)r_{i_t} + sr_{i_{t+1}} + r_N)$.

$$
\begin{aligned}
\sum_{i=1}^{P(m_o)} w_\lambda(M_i') &< n_o(p_o r_{i_1} + r_N)(\dim \overline{V}_{i_1}^{p_o} \cdot \overline{V}_{i_2}^{o} \cdot \overline{V}_N)^{n_o} \\
&+ n_o((p_o - 1)r_{i_1} + r_{i_2} + r_N)(\dim(\overline{V}_{i_1}^{p_o-1} \cdot \overline{V}_{i_2}^{1} \cdot \overline{V}_N)^{n_o} - (\dim \overline{V}_{i_1}^{p_o} \cdot \overline{V}_{i_2}^{0} \cdot \overline{V}_N)^{n_o} \\
&+ n_o((p_o - 2)r_{i_1} + 2r_{i_2} + r_N)(\dim(\overline{V}_{i_1}^{p_o-2} \cdot \overline{V}_{i_2}^{2} \cdot \overline{V}_N)^{n_o} - (\dim \overline{V}_{i_1}^{p_o-1} \cdot \overline{V}_{i_2}^{1} \cdot \overline{V}_N)^{n_o}) \\
&+ \cdots \qquad \cdots \qquad \cdots \qquad \cdots \qquad \cdots \\
&+ n_o((p_o - s)r_{i_t} + sr_{i_{t+1}} + r_N)(\dim(\overline{V}_{i_t}^{p_o-s} \cdot \overline{V}_{i_{t+l}}^{s} \cdot \overline{V}_N)^{n_o} - \dim(\overline{V}_{i_1}^{p_o-s+1} \cdot \overline{V}_{i_{t+1}}^{s-1} \cdot \overline{V}_N)^{n_o}) \\
&+ \cdots \qquad \cdots \qquad \cdots \qquad \cdots \qquad \cdots \\
&+ n_o(p_o r_{i_k} + r_N)(\dim(\overline{V}_{i_{k-1}}^{o} \cdot \overline{V}_{i_k}^{p_o} \cdot \overline{V}_N)^{n_o} - (\dim \overline{V}_{i_{k-1}}^{1} \cdot \overline{V}_{i_k}^{p_o-1} \cdot \overline{V}_N)^{n_o} \\
&= n_o(p_o r_{i_1} + r_N)(n_o(p_o e_{i_1} + e_N) - g + 1) \\
&+ n_o^2((p_o - 1)r_{i_1} + r_{i_2} + r_N)(e_{i_2} - e_{i_1}) + n_o^2((p_o - 2)r_{i_1} + r_{i_2} + r_N)(e_{i_2} - e_{i_1}) \\
&+ \cdots + n_o^2((p_o - s)r_{i_t} + sr_{i_{t+1}} + r_N)(e_{i_{t+1}} - e_{i_t}) + \cdots + n_o^2(p_o r_{i_k} + r_N)(e_{i_k} - e_{i_{k-1}})
\end{aligned}
$$

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$$
\begin{aligned}
&= n_o^2 p_o r_{i_1} e_N + n_o^2 r_N e_N + n_o(p_o r_{i_1} + r_N)(1-g)\\
&+ n_o^2 \sum_{s=1}^{p_o}((p_o - s) r_{i_1} + s r_{i_2} + r_N)(e_{i_2} - e_{i_1}) + \cdots \qquad \cdots\\
&+ n_o^2 \sum_{s=1}^{p_o}((p_o - s) r_{i_t} + s r_{i_{t+1}} + r_N)(e_{i_{t+1}} - e_{i_t}) + \cdots \qquad \cdots\\
&+ n_o^2 \sum_{s=1}^{p_o}((p_o - s) r_{i_{k-1}} + s r_{i_k} + r_N)(e_{i_k} - e_{i_{k-1}}),\\
&\qquad\qquad (\because e_{i_1} = e_o = 0) < n_o^2 r_N e_N + n_o p_o r_{i_1}(1-g)\\
&+ n_o^2\Big[\frac{(p_o-1)p_o r_{i_1}}{2} + \frac{p_o(p_o+1) r_{i_2}}{2} + p_o r_N\Big](e_{i_2} - e_{i_1}) + \cdots\\
&+ n_o^2\Big[\frac{(p_o-1)p_o r_{i_t}}{2} + \frac{p_o(p_o+1) r_{i_{t+1}}}{2} + p_o r_N\Big](e_{i_{t+1}} - e_{i_t}) + \cdots\\
&+ n_o^2\Big[\frac{(p_o-1)p_o r_{i_{k-1}}}{2} + \frac{p_o(p_o+1) r_{i_k}}{2} + p_o r_N\Big](e_{i_k} - e_{i_{k-1}})\\
&\qquad\qquad \because (r_{i_1} = r_o < 0,\ r_N(1-g) < 0)\\
&= n_o^2 r_N e_N + n_o p_o r_{i_l}(1-g)\\
&+ \frac{n_o^2 p_o^2}{2}\Big[(r_{i_2} + r_{i_1})(e_{i_2} - e_{i_1}) + \cdots + (r_{i_{t+1}} + r_{i_t})(e_{i_{t+1}} - e_{i_t}) + \cdots\\
&+ (r_{i_k} + r_{i_{k-1}})(e_{i_k} - e_{i_{k-1}})\Big] + \frac{n_o^2 p_o}{2}\Big[(r_{i_2} - r_{i_1})(e_{i_2} - e_{i_1}) + \cdots\\
&+ (r_{i_{t-1}} - r_{i_t})(e_{i_{t+l}}) + \cdots + (r_{i_k} - r_{i_{k-1}})(e_{i_k} - e_{i_{k-1}})\\
&+ 2 r_N \sum_{t=1}^{k-1}(e_{i_{t+1}} - e_{i_t})\Big]
\end{aligned}
$$

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$$
= n_o^2 r_N e_N + n_o p_o r_{i_1}(1-g) n_o^2 p_o^2 \sum_{t=1}^{k-1}(r_{i_{t+1}} + r_{i_t})\frac{(e_{i_{t+1}} - e_{i_t})}{2}
$$

$$+ n_o^2 p_o (\sum_{t=1}^{k-1} (r_{i_{t+1}} - r_{i_t})) \frac{(e_{i_{t+1}} - e_{i_t})}{2} + r_N (e_{i_k} - e_{i_1}))$$

$$< n_o^2 r_N e_N + n_o^2 p_o n_o^{-1} r_{i_1} (1 - g)$$

$$+ n_o^2 p_o^2 \Big[r_N e_N - \sum_{t=1}^{k-1} (r_{i_{t+1}} - r_{i_t}) \frac{(e_{i_{t+1}} + e_{i_t})}{2} \Big]$$

$$+ n_o^2 p_o \Big[\sum_{t=1}^{k-1} \frac{(r_{i_{t+1}} - r_{i_1}) e_{i_k}}{2} + r_N e_{i_k} \Big],$$

$$(\because -(r_{i_{t+1}} - r_{i_t}) e_{i_k} \leq 0, \quad e_{i_1} = e_o = 0)$$

$$< n_o^2 p_o^2 \Big[r_N e_N - \sum_{t=1}^{k-1} (r_{i_{t+1}} - r_{i_t}) \frac{(e_{i_{t+1}} + e_{i_t})}{2} \Big]$$

$$+ n_o^2 p_o \Big[\frac{e_{i_k}(r_{i_k} - r_{i_l})}{2} + r_N e_{i_k} + \Big] + n_o^2 (r_N e_N + r_{i_1}(1 - g))$$

$$(\because p_o n_o^{-1} < 1, \; r_{i_1}(1 - g) > 0)$$

$$< n_o^2 \Big[-\varepsilon (r_N - r_o) p_o^2 + p_o (\frac{d(r_N - r_o)}{2} + d r_N) + d r_N + r_o (1 - g) \Big]$$

(This follows from the lemma (page 27) and the facts that $r_{i_1} = r_o$, $e_{i_k} = e_N = d$, $r_{i_k} = r_N$).

$$= n_o^2 (r_N - r_o) \Big[-\varepsilon p_o^2 + p_o (\frac{d}{2} + \frac{d r_N}{r_N - r_o}) + \frac{d r_N}{r_N - r_o} + \frac{r_o(1 - g)}{r_N - r_o} \Big]$$

$$< n_o^2 (r_N - r_o) \Big[-\varepsilon p_o^2 + \frac{3 d p_o}{2} + d + g - 1 \Big]$$

$$(\because \frac{r_N}{r_N - r_o} < 1, \quad \frac{r_o(1 - g)}{r_N - r_o} < g - 1)$$

$$= n_o^2 p_o (r_N - r_o) \Big[-\varepsilon p_o + \frac{3d}{2} + \frac{d + g - 1}{p_o} \Big]$$

$$< 0, (\because p_o > \max.\{d + g, \frac{2d + 1}{\varepsilon}\})$$

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It is immediate from the above estimate and criterion (∗) (page 6) that the point $H_{m_o}(x) \in \mathbb{P}(\overset{P(m_o)}{\Lambda} H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m_o)))$ is λ-stable. Further, by our choice of the numbers ε, p_o and $(p_o + 1)n_o = m_o$ it is clear from the above calculation that for every nonsingular curve X' in the family $Z_H \xrightarrow{p_H} H$, for every $1-ps\lambda'$ of $SL(N+1)$, the point $H_{m_o}(X') \in \mathbb{P}(\overset{P(m_o)}{\Lambda} H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m_o)))$ is λ'-stable. This proves the result.

Now consider the closed immersion (cf. page 19), $\psi_{m_o} : H \to \mathbb{P}(\overset{P(m_o)}{\Lambda} H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m_o)))$ where m_o is the integer fixed in the above theorem 1.0.0. Let $\mathbb{P}(\overset{P(m_o)}{\Lambda} H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m_o)))^{ss}$ be the open subset of $\mathbb{P}(\overset{P(m_o)}{\Lambda} H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m_o)))$ consisting of semistable points and let V be the inverse image of this open set by the morphism ψ_{m_o}. Let $Z_V = p_H^{-1}(V)$. By restricting the morphism p_H to Z_V we obtain a family $Z_V \xrightarrow{p_V} V$ of curves X, such that the m_o^{th} Hilbert point of X, $H_{m_o}(X) \in \mathbb{P}(\overset{P(m_o)}{\Lambda} H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m_o)))$ is semistable. The above theorem 1.0.0. asserts that the family $Z_V \xrightarrow{p_V} V$ contains all the nondegenerate nonsingular curves in the family $Z_H \xrightarrow{p_H} H$.

We are now ready to state the main theorem of these is lecture notes.

35 **Theorem 1.0.1.** *Every curve X in the family $Z_V \xrightarrow{p_V} V$ is semistable in the sense of definition 0.1.4. (page 8). Further trace of the linear system $|D|$ on X is complete, ($|D|$ is the complete linear system on $\mathbb{P}^N$ corresponding to the line bundle $O_{\mathbb{P}^N}(1)$ on $\mathbb{P}^N$).*

Idea of the proof: The proof of the above theorem is divided in the following propositions 1.0.2, 1.0.3., ..., 1.0.9. The proofs of the propositions 1.0.2, 1.0.3., ..., 1.0.6. are on the same lines as follows. Assume that the proposition is not true, i.e., let X be a curve in the family $Z_V \xrightarrow{p_V} V$ which does not have the property stated in the proposition. Using this assumption we are able to produce a $1-ps\lambda'$ of $SL(N+1)$ such that the m_o^{th} Hilbert point of $X, H_{m_o}(X) \in \mathbb{P}(\overset{P(m_o)}{\Lambda} H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m_o)))$ is not λ'-semistable.

In particular it follows that $H_{m_o}(X)$ is not semistable, (cf. theorem 0.0.9 page 3). This contradiction then proves the proposition.

In proposition 1.0.7 we prove an important inequality (cf. inequality $(*')$, proposition 1.0.7, page 55) which follows from the inequality in criterion $(**)$ (page 7), used in the case of a particular $1-ps\lambda'$ of $SL(N+1)$ and the integer m_o. Propositions 1.0.8 and 1.0.9 are proved using the above inequality.

Proposition 1.0.2. *Every curve X in the family $Z_V \xrightarrow{p_V} V$ is a nondegenerate curve in $\mathbb{P}^N$ i.e. X is not contained in any hyperplane in $\mathbb{P}^N$.*

Proof. Suppose that the result is not true i.e. suppose that there exists a curve X in the family $Z_V \xrightarrow{p_V} V$ such that $X \subset \mathbb{P}^N$ is a degenerate curve. **36**
We will show that this leads to the contradiction that the m_o^{th} Hilbert point of X, $H_{m_o}(X) \in \mathbb{P}(\overset{P(m_o)}{\Lambda} H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m_o)))$ is not semistable. This contradiction will then prove the result. □

That X is a degenerate curve in $\mathbb{P}^N$ means that the restriction map $\bar{\varphi}_1 : H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(1)) \to H^o(X_{red'}\ L_{X_{\text{red}}})$ has nontrivial kernel, say W_o. Let $\dim W_o = N_o$. Choose a basis of $W_1 = H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(1))$ relative to the filtration $0 \subset W_o \subset W_1$, say $w_o, w_1, \ldots, w_{N_o-1}, \ldots, w_N$, (cf. definition 0.2.6 page 16).

Let λ be a $1-ps$ of $GL(N+1)$ such that the induced action of λ on W_1 is given by,

$$\lambda(t)w_i = w_i, \quad t \in K^*, \quad (0 \le i \le N_o - 1),$$
$$\lambda(t)w_i = w_i, \quad t \in K^*, \quad (N_o \le i \le N).$$

Let λ' be the $1-ps$ of $SL(N+1)$ associated to the $1-ps\lambda$ of $GL(N+1)$, (cf. definition 0.1.2, page 7). The rest of the proof consists of showing that $H_{m_o}(X)$ is not λ'-semistable.

Assume now that $m > m'$ so that $H^1(X, L^m) = 0$ and the restriction

$$\varphi_m : H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m)) \to H^o(X, L^m) \text{ surjective.}$$

Let $B_m = \{M_1, M_2, \ldots, M_{\alpha_m}\}$ be a basis of $H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m))$ consisting of monomials of degree m in $w_0, w_1, \ldots, w_N$, $(\alpha_m = h^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m)))$. **37**
Recall that we have chosen the integer q_1 such that $I_X^{q_1} = 0$ where I_X denotes the ideal sheaf of nilpotents in O_X, (cf. page 18).

For $0 \leq s \leq q_1 - 1 \leq m$ let $W_o^{q_1-s} \cdot W_1^{m-q_1+s}$ be the subspace of $H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m))$ generated by elements w of the type

$$w = x_1 x_2 \cdots x_{q_1-s} y_1 y_2 \cdots y_{m-q_1+s} \begin{bmatrix} x_i \in W_o, & 1 \leq i \leq q_1 - s \\ y_i \in W_1, & 1 \leq i \leq m - q_1 + s \end{bmatrix}$$

Put $W_o^o.W_1^m = H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m))$. We have the following filtration of $H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m))$,

$$\begin{aligned} 0 \subset W_o^{q_1}.W_1^{m-q_1} \subset W_o^{q_1-1}.W_1^{m-q_1+1} \subset W_o^{q_1-2}.W_1^{m-q_1+2} \subset \\ \ldots W_o^1.W_1^{m-1} \subset W_o^o.W_1^m = H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m)), \end{aligned} \tag{F}$$

For $\leq s \leq q_1 < m$ let,

$$\begin{aligned} \bar{W}_o^{q_1-s}.\bar{W}_1^{m-q_1+s} &= \varphi_m(W_o^{q_1-s}.W_1^{m-q_1+s}) \\ &\subset H^o(X, L^m), \dim \bar{W}_o^{q_1-s}.\bar{W}_1^{m-q_1+s} = \beta_s. \end{aligned}$$

These subspaces define the following filtration of $H^o(X, L^m)$.

$$\begin{aligned} 0 = \bar{W}_o^{q_1}.\bar{W}_1^{m-q_1} \subset \bar{W}_o^{q_1-1}.\bar{W}_1^{m-q_1+1} \subset \bar{W}_o^{q_1-2}.\bar{W}_1^{m-q_1+2} \subset \\ \ldots \bar{W}_o^1.\bar{W}_1^{m-1} \subset \bar{W}_o^o.\bar{W}_1^m = H^o(X, L^m), \end{aligned} \tag{$\overline{F}$}$$

Rewrite the basis B_m as $B_m = \{M'_1, M'_2, \ldots, M'_{P(m)}, M'_{P(m)+1}, \ldots, M'_{\alpha_m}\}$ such that $\{\varphi_m(M'_1), \varphi_m(M'_2) \ldots, \varphi_m(M'_{P(m)})\}$ is a basis of $H^o(X, L^m)$ relative to the filtration $(\overline{F})$ and $M'_{P(m)+1}, M'_{P(m)+2}, \ldots, M'_{\alpha_m}$ are the rest of
38 the monomials in B_m in some order. We now estimate, total λ-weight of $M'_1, M'_2, \ldots, M'_{P(m)} = \sum_{i=1}^{P(m)} w_\lambda(M'_i)$, (cf. definition 0.1.1 page 5). It follows from the definition of λ, that a monomial $M \in W_o^{q_1-s}.W_1^{m-q_1+s} - W_o^{q_1-s+1}.W_1^{m-q_1+s-1}$ has λ-weight $W_\lambda(m) = m - q_1 + s$, $(1 \leq s \leq q_1)$.

$$\begin{aligned} \sum_{i=1}^{P(m)} w_\lambda(M'_i) = m(\beta_{q_1} - \beta_{q_1-1}) + (m-1)(\beta_{q_1-1} - \beta_{q_1-2}) + \\ \ldots + (m - q_1 + 1)\beta_1 \\ = m\beta_{q_1} - \sum_{s=1}^{q_1-1} \beta_s \geq m(dm - g + 1) - (q_1 - 1)(dm - g + 1) \end{aligned}$$

$$(\because \beta_s \leq h^o(X, L^m) = dm - g + 1, 1 \leq s \leq q_1 - 1)$$

Thus $\sum_{i=1}^{P(m)} w_\lambda(M'_i) \geq (m - q_1 + 1)(dm - g + 1), \quad (E_1)$

total λ-weight of $w_o, w_1, \ldots, w_N = \sum_{i=0}^{N} w_\lambda(w_i)$

$$\begin{aligned} &= \dim W_1 - \dim W_o, \text{ (Follows from the definition of } \lambda \text{)} \\ &= d - g + 1 - \dim W_o \leq d - g, \qquad (\because \dim W_o \geq 1) \end{aligned}$$

Thus $\sum_{i=0}^{N} w_\lambda(w_i) \leq d - g, \quad (E_2)$

We are now ready to get the contradiction that the m_o^{th} Hilbert point of $X, H_{m_o}(X)$ is not semistable.

If $H_m(X)$ is λ' semistable then there exists monomials $M'_{i_1} M'_{i_2}, \ldots, M'_{i_{P(m)}}$ in $B'_m(1 \leq i_1 < i_2 < \cdots < i_{P(m)} \leq \alpha_m)$, such that $\{\varphi_m(M'_{i_1}), \varphi_m(M'_{i_2}), \ldots, \varphi_m(M'_{i_{P(m)}})\}$ is a basis of $H^o(X, L^m)$ and

$$\frac{\sum_{i=1}^{P(m)} w_\lambda(M'_{i_j})}{mP(m)} \leq \frac{\sum_{i=0}^{N} w_\lambda(w_i)}{d - g + 1}$$

(cf. criterion $(**)$ page 7).

Observe that $\sum_{i=1}^{P(m)} w_\lambda(M'_i) \leq \sum_{j=1}^{P(m)} w_\lambda(M'_{i_j})$. Note the following. **39**

$$H_m(X) \text{ is } \lambda' - \text{ semistable } \Rightarrow \frac{\sum_{i=1}^{P(m)} w_\lambda(M'_{i_j})}{m(dm - g + 1)} \leq \frac{\sum_{i=0}^{N} w_\lambda(w_i)}{d - g + 1}$$

$$\begin{aligned} &\Rightarrow \frac{(m - q_1 + 1)(dm - g + 1)}{m(dm - g + 1)} \leq \frac{d - g}{d - g + 1} \text{ (Follows from } (E_1), (E_2)) \\ &\Rightarrow 1 - \frac{q_1 - 1}{m} \leq \frac{d - g}{d - g + 1} \Rightarrow \frac{1}{d - g + 1} \leq \frac{q_1 - 1}{m} \\ &\Rightarrow m \leq (d - g + 1)(q_1 - 1) \Rightarrow m < m_o \end{aligned}$$

$$(\because (d-g+1)(q_1-1) < m_o)$$

Thus $H_{m_o}(X)$ is not λ'-semistable. In particular it follows that $H_{m_o}(X)$ is not semistable. (cf. theorem 0.0.9 page 3. This contradiction proves the result.

The above proof can be considered as the prototype of the proofs of the next propositions 1.0.3., 1.0.4., 1.0.5., 1.0.6.

Proposition 1.0.3. *Every curve X in the family $Z_V \xrightarrow{p_V} V$ is generically reduced i.e. the local ring of X, at each generic point of X, is reduced.*

Proof. Assume the contrary. Let X be a curve in the family $Z_V \xrightarrow{p_V} V$ such that X is not generically reduced. Write $X = \bigcup_{i=1}^{p} X_i$, ($X_i$, an irreducible component of X, $1 \leq i \leq p$) so that the local ring of X at the generic point of X_1 is not reduced. We will show that this leads
40 to the contradiction that the m_o^{th} Hilbert point of X, $H_{m_o}(X) \in \mathbb{P}(\overset{P(m_o)}{\Lambda}(\mathbb{P}^N, O_{\mathbb{P}^N}(m_o)))$ is not semistable. This contradiction will then prove the result. ☐

Let $\deg_{X_{\text{ired}}} L = e_i$, $1 \leq i \leq p$. It is easy to see that $\deg L = d = \sum_{i=1}^{p} k_i e_i$ for some positive integers $k_1, k_2, \ldots, k_p$, with $k_1 \geq 2$. Let W_o be the kernel of the natural restriction map

$$\bar{\varphi_1} : H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(1)) \to H^o(X_{\text{lred}}, L_{X_{\text{lred}}}).$$

Claim : $W_o \neq 0$.

Proof of the Claim: Look at the exact sequence,

$$0 \longrightarrow 0_{X_{\text{lred}}} \longrightarrow L_{X_{\text{lred}}} \longrightarrow O_{D_1} \to 0,$$

where D_1 is a divisor on X_{lred}, corresponding to the line bundle $L_{X_{\text{lred}}}$ on X_{lred}, such that D_1 has support in the smooth locus of X_1. It follows from the corresponding long exact cohomology sequence that

$$h^o(X_{\text{lred}}, L_{X_{\text{lred}}}) \leq h^o(X_{\text{lred}}, O_{D_1}) + h^o(X_{\text{lred}}, O_{X_{\text{lred}}}) = e_1 + 1.$$

41 Now note the following.

$$W_o = 0 \Rightarrow d - g + 1 = h^o(\mathbb{P}^N, O_{\mathbb{P}^N}(1)) \leq h^o(X_{\text{1red}}, L_{X_{\text{1red}}}) \leq e_1 + 1$$

$$\Rightarrow d - g \leq e_1 \Rightarrow k_1(d-g) + \sum_{i=2}^{p} k_i e_i \leq k_1 e_1 + \sum_{i=2}^{p} k_i e_i = d$$
$$\Rightarrow (k_1 - 1)d \leq k_1 g - \sum_{i=2}^{p} k_i e_i \leq k_1 g \Rightarrow \frac{(k_1-1)d}{k_1} \leq g$$
$$\Rightarrow \frac{d}{2} \leq g, (\because k_1 \geq 2 \therefore \frac{k_1 - 1}{k_1} \geq \frac{1}{2})$$
$$d \leq 2g.$$

It is immediate from the above contradiction that $W_o \neq 0$. Also in view of the Proposition 1.0.2 (page 27) it follows that X cannot be irreducible.

Let $\dim W_o = N_o$. Choose a basis of $W_1 = H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(1))$ relative to the filtration $0 \subset W_o \subset W_1$, say $\{w_o, w_1, \ldots, w_{N_o - 1}, w_{N_o}, \ldots, w_N\}$, (cf. Definition 0.2.6 page 16).
Let λ be a $1-ps$ of $GL(N+1)$ such that the action of λ on W_1 is given by,

$$\lambda(t)w_i = w_i, \quad t \in K^*, (o \leq i \leq N_o - 1),$$
$$\lambda(t)w_i = tw_i, \quad t \in K^*, (N_o \leq i \leq N).$$

Let λ' be the $1-ps$ of $SL(N+1)$ associated to the $1-ps$ λ of $GL(N+1)$, (cf. definition 0.1.2 page 7). The rest of the proof consists of showing that $H_{m_o}(X)$ is not λ' -semistable.

Assume now that $m > m'$ so that $H^1(X, L^m) = 0$ and the restriction 42
$\varphi_m : (H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m)) \rightarrow H^o(X, L^m)$ is surjective. Recall that H^o $(\mathbb{P}, O_{\mathbb{P}^N}(m))$ has a basis $B_m = \{M_1, M_2, \ldots, M_{\alpha_m}\}$ consisting of monomials lof degree m in $w_o, w_1, \ldots, w_N$, $(\alpha_m = h^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m)))$.

For $0 \leq r \leq m$, let $W_o^{m-r}.W_1^r$ be the subspace of $H^o(\mathbb{P}, O_{\mathbb{P}^N}(m))$ generated by elements w of the following type.

For $r = 0$,

$$w = x_1x_2, \dots, x_m, \quad (x_j \in W_o, \quad 1 \leq j \leq m),$$

for $0 < r < m$,

$$w = x_1x_2, \dots, x_{m-r}y_1y_2, \dots, y_r,$$
$$(x_j \in W_o, 1 \leq j \leq m-r; y_j \in W_1, 1 \leq j \leq r),$$

for $r = m$,

$$w = y_1y_2, \dots, y_m, \quad (y_j \in W_1, \quad 1 \leq j \leq m).$$

We have the following filtration of $H^o(\mathbb{P}, O_{\mathbb{P}^N}(m))$.

$$\begin{aligned} 0 &\subset W_o^m.W_1^o \subset W_o^{m-1}\cdot W_1^1 \subset W_o^{m-2}.W_1^2 \subset \cdots \\ &\subset W_o^{q_1}.W_1^{m-q_1} \subset W_o^{q_1-1}.W_1^{m-q_1+1} \subset \cdots \subset W_o^o.W_1^m = H^o(\mathbb{P}, O_{\mathbb{P}^N}(m)), \end{aligned} \tag{F}$$

Let $\bar{W}_o^{m-r}.\bar{W}_1^r = \varphi_m(W_o^{m-r}.W_1^r) \subset H^o(X, L^m), \dim \bar{W}_o^{m-r}.\bar{W}_1^r = \beta_r, 0 \leq r \leq m$.

43 These subspaces define the following filtration of $H^o(X, L^m)$.

$$\begin{aligned} 0 &\subset \bar{W}_o^m.\bar{W}_1^o.\bar{W}_0^{m-1}\cdot \bar{W}_1^1 \subset \bar{W}_0^{m-2}.\bar{W}_1^2 \subset \cdots \\ &\subset \bar{W}_0^{q_1}.\bar{W}_1^{m-q_1} \subset \bar{W}_0^{q_1-1}.\bar{W}_1^{m-q_1} \subset \cdots \subset \bar{W}_0^o.\bar{W}_1^m = H^o(X, L^m), \end{aligned} \tag{$\bar{F}$}$$

Rewrite the basis $B_m = \{M_1', M_2', \dots, M_{P(m)}', \dots, M_{\alpha_m}'\}$ such that $\{\varphi_m(M_1'), \varphi(M_2'), \dots, \varphi_m(M_{P(m)}')\}$ is a basis of $\bar{H}^o(X, L^m)$ relative to the filtration $(\bar{F})$ and $M_{P(m)}', M_{P(m)+1}', \dots, M_{\alpha_m}'$ are the rest of the monomials in B_m in some order.

Let C be the closure of $X - X_1$ in X. Since X is connected, there exists a (closed) point, say $P \epsilon X_1 \cap C$.

Claim. *C can be given the structure of a closed subscheme of X such that the kernal of the restriction map $\varphi_m' : H^o(X, L^m) \to H^o(C, L_C^m)$ intersects $\bar{W}_o^{m-r}.\bar{W}_1^r$ in the null space i.e.* $\bar{W}_o^{m-r}.\bar{W}_1^r \cap \operatorname{kernel} \varphi_m' = 0$, $0 \leq r \leq m - q_1$.

The proof of the above claim is somewhat technical. Hence, assuming the claim we will prove the proposition and then we will go to the proof of the claim.

Let I denote the ideal subsheaf of O_C defining the point $P \in C$.

The exact sequence $0 \to I^{m-r} \otimes L^m_C \to L^m_C \to \dfrac{O_C}{I^{m-r}} \otimes L^m_C \to 0$, given the following long exact sequence.

$$0 \to H^o(C, I^{m-r} \otimes L^m_C) \to H^o(C, L^m_C) \to H^o(C, \frac{O_C}{I^{m-r}} \otimes L^m_C) \\ \to H^1(C, I^{m-r} \otimes L^m_C) \to 0$$

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Now make the following observations.

i) $h^o(C, L^m_C) = \chi(L^m_C) = \deg_C L^m + \chi(O_C) \leq (d - 2e_1)m + h^o(C, O_C) < (d - 2e_1)m + q_3$ (cf. assertion iv, page 18).

ii) Since $\dfrac{O_C}{I^{m-r}} \otimes L^m_C$ has support only at the point $P \in C$,

$h^o(C, \dfrac{O_C}{I^{m-r}} \otimes L^m_C) = \dim \left[\dfrac{O_{C,P}}{m^{m-r}_{C,P}}\right] \geq m - r,$

($O_{C,P}$ is the local ring of C at P and $m_{C,P}$ is the maximal ideal in $O_{C,P}$).

iii) Note that $h^o(C, \dfrac{O_C}{I^{m-r}} \otimes L^m_C) = \dim \left[\dfrac{O_{C,P}}{m^{m-r}_{C,P}}\right] \leq \mu_1(m - r) + \mu_2$ (cf. assertion v, page 18). Hence it follows from the above long exact cohomology sequence that $h^1(C, I^{m-r} \otimes L^m_C) \leq \mu_1(m - r) + \mu_2$, $0 \leq r \leq m'' - 1$.

iv) For $m'' \leq r \leq m - q_1$, $H^1(C, I^{m-r} \otimes L^m_C) = 0$ (cf. assertion vi page 18).

v) By definition the image of $\bar{W}^{m-r}_o . \bar{W}^r_1 \subset H^o(X, L^m)$ under the restriction φ'_m is contained in the subspace $H^o(C, I^{m-r} \otimes L^m_C)$ of $H^o(C, L^m_C)$.

It follows that, **45**

$$\beta_r = \dim(\bar{W}^{m-r}_o . \bar{W}^r_1) \leq h^o(C, I^{m-r} \otimes L^m_C)$$

$$= h^o(C, L_C^m) - h^o(C, \frac{O_C}{I^{m-r}} \otimes L_C^m) + h^1(C, I^{m-r} \otimes L_C^m)$$
$$\leq (d-2e_1)m + q_3 + r - m + \mu_1(m-r) + \mu_2, \quad (0 \leq r \leq m''-1)$$
$$\beta_r \leq (d-2e_1)m + q_3 + r - m, \quad (m'' \leq r \leq m - q_1)$$
$$\beta_r \leq dm - g + 1, \quad (m - q_1 + 1 \leq r \leq m-1)$$

(For the last inequality note that, $\beta_r \leq h^o(X, L^m) = dm - g + 1$).

We now estimate total λ-weight of $M'_1, M'_2, \ldots, M'_{P(m)} = \sum_{i=1}^{P(m)} w_\lambda(M'_i)$.

Note that a monomial $M \in W_o^{m-r}W_1^r - W_o^{m-r+1}W_1^{r-1}$ has λ-weight $w_\lambda(M) = r$.

$$\sum_{i=1}^{P(m)} w_\lambda(M'_i) = \sum_{r=1}^{m} r(\beta_r - \beta_{r-1}) = m\beta_m - \sum_{r=0}^{m-1}\beta_r$$
$$= m\beta_m - \sum_{r=0}^{m''-1}\beta_r - \sum_{r=m''}^{m-q_1}\beta_r - \sum_{r=m-q_1+1}^{m-1}\beta_r$$
$$\geq m(dm-g+1) - \sum_{r=0}^{m-q_1}\Big[(d-2e_1)m + q_3 + r - m\Big]$$
$$- \sum_{r=0}^{m''-1}\Big[\mu(m-r) + \mu_2\Big] - \sum_{r=m-q_1+1}^{m-1}\Big[dm - g + 1\Big]$$
$$= dm^2 + m(1-g) - \Big[(m-q_1+1)(d-2e_1)m + (m-q_1+1)q_3$$
$$+ \frac{(m-q_1)(m-q_1+1)}{2} - (m-q_1+1)m\Big]$$
$$- \Big[\mu m m'' - \mu_1\frac{m''-1}{2} + \mu_2 m''\Big] - (q_1-1)(dm-g+1)$$
$$= (2e_1 + \frac{1}{2})m^2 - m\Big[(1-q_1)(d-2e_1) + q_3 + \frac{1}{2} - q_1 - (1-q_1) + \mu_1 m''$$
$$+ s(q_1-1) + g - 1\Big] + (q_1-1)q_3 - \frac{q_1(q_1-1)}{2} + \frac{\mu_1 m''(m''-1)}{2}$$
$$- \mu_2 m'' + (q_1-1)(g-1)$$
$$\geq (2e_1 + \frac{1}{2})m^2 - m\Big[(q_1-1)(2e_1+1) + (q_3-q_1) + \mu_1 m'' + g - \frac{1}{2}\Big]$$
$$= \left(2e_1 + \frac{1}{2}\right)m^2 - mS,$$

$$\left(S = (q_1 - 1)(2e_1 + 1) + (q_3 - q_1) + \mu_1 m'' + g - \frac{1}{2}\right)$$

Thus, 46

$$\sum_{i=1}^{P(m)} w_\lambda(M'_i) \geq (2e_1 + \frac{1}{2})m^2 - mS, \qquad (E_l)$$

Clearly,

$\sum\limits_{i=0}^{N} w_\lambda(w_i) = \dim W_1 - \dim W_o,$ (Follows from the definition of λ)

$$\leq h^o(X_{\mathrm{lred}}, L_{X_{\mathrm{lred}}}) = e_1 + 1, \qquad (E_2)$$

We are now ready to get the contradiction that m_o^{th} Hilbert point of X, $H_{m_o}(X)$ is not $\lambda'-$ semistable.

If $H_m(X)$ is $\lambda'-$ semistable ($m > m'$) then there exists monomials $M'_{i_1}, M'_{i_2}, \ldots, M'_{i_{P(m)}}(1 \leq i_1 < i_2 < \cdots < i_{P(m)} \leq \alpha_m)$ such that $\{\varphi_m(M'_{i_1}), \varphi_m(M'_{i_2}), \ldots, \varphi_m(M'_{i_{P(m)}})\}$ is a basis of $H_o(X, L^m)$ and

$$\frac{\sum\limits_{j=1}^{P(m)} w_\lambda(M'_{i_j})}{m(dm - g + 1)} < \frac{\sum\limits_{i=0}^{N} w_\lambda(w_i)}{d - g + 1},$$

(cf. criterion (∗∗) page 7). It is easy to see that

$$\sum_{i=1}^{P(m)} w_\lambda(M'_i) \leq \sum_{j=1} P(m) w_\lambda(M'_{i_j}).$$

Thus, 47

$H_m(X)$ is λ'−semistable ($m > m'$)

$$\Rightarrow \frac{\sum\limits_{i=1}^{P(m)} w_\lambda(M'_i)}{m(dm - g + 1)} \leq \frac{\sum\limits_{i=0}^{N} w_\lambda(w_i)}{d - g + 1}$$

$$\Rightarrow \frac{(2e_1 + \frac{1}{2})m^2 - ms}{m(dm - g + 1)} \leq \frac{e_1 + 1}{d - g + 1}, \qquad \text{(Follows from } (E_1) \text{ and } (E_2))$$

$$\Rightarrow \frac{(2e_1+\frac{1}{2})-\frac{S}{m}}{d} \le \frac{e_1+1}{d-g+1}$$
$$\Rightarrow (d-g+1)(2e_1+\frac{1}{2}) - \frac{S(d-g+1)}{m} < d(e_1+1)$$
$$\Rightarrow d(e_1-\frac{1}{2})-(g-1)(2e_1+\frac{1}{2}) \le \frac{S(d-g+1)}{m} \Rightarrow 1 \le \frac{S(d-g+1)}{m},$$
$$(\because d(e_1-\frac{1}{2})-(g-1)(2e_1+\frac{1}{2}) \ge 1)$$
$$\Rightarrow m \le S(d-g+1) \Rightarrow m < m_o, \qquad (\because m_o > S(d-g+1))$$

This proves that the m_o^{th} Hilbert point of $X, H_{m_o}(X)$ is not $\lambda'-$semistable. In particular, it follows that $H_{m_o}(X)$ is not semistable, (cf. theorem 0.0.9 page 3). This contradiction proves the result.

It remains to prove the claim. Let $P_1, P_2, \ldots, P_t$ be all the associated (closed) points of X, (a point $Q \in X$ is called an associated point of X if the maximal ideal in the local ring $0_{X,P}$ of X at P is associated to the zero ideal). Choose a finite affine open cover $\{U_i\}$ of X such that any of
48 the points $P_1, P_2, \ldots, P_t$ belongs to exactly one of the $U_i's$ in $\{U_i\}$ and further L_{U_i} is trivial for every U_i in $\{U_i\}$.

Let $U_i \simeq \operatorname{Spec} A_i$ and let $U_i \bigcap U_k \simeq \operatorname{Spec} A_{ik}$. In the ring A_i let $(0) = \sum_{j=1}^{n_i} q_{ij}$ be a primary decomposition of the zero ideal with q_{ij}, $p_{ij}-$primary for some prime ideal p_{ij} in A_i, $(1 \le j \le n_i)$. We can assume without loss of generality that in those U_i such that $U_i \bigcap X_1 \ne \emptyset$, $X_{1\,\text{red}}$ is defined by the prime ideal q_{i1}.

Define an ideal subsheaf J of O_X as follows. If $U_i \bigcap X_1 \ne \emptyset$, then in U_i, J is defined by $\bigcap_{j=2}^{n_i} q_{ij}$. If $U_i \bigcap X_1 = \emptyset$ then J is defined by $\sum_{j=1}^{n_i} q_{ij} = (0)$. In $\operatorname{Spec} A_{ik} = U_i \bigcap U_k (i \ne k)$ there is no associated (closed) point of X hence all the primary ideals in a primary decomposition of the zero ideal in A_{ik} are minimal and hence are uniquely determined. Thus the above construction indeed defines an ideal sheaf. Let C be the closed subscheme of X defined by the ideal J. Let, $\varphi'_m : H^o(X, L^m) \underset{-}{\rightarrow} H_o(\underline{C}, L_C^m)$ be the natural restriction. We now proceed to prove that $W_0^{m-r} \cdot W_1^r \bigcap \text{Kernel } \varphi'_m = 0$, $(0 \le r \le m - q_1)$.

Let $s \in \overline{W}_o^{m-r} \cdot \overline{W}_1^{-r} \bigcap$ Kernel φ'_m. It suffices to prove that for every open set U_i in the cover U_i, the restriction s_i of s to U_i is zero. Let γ_i be the isomorphism $L_{U_i} \simeq \tilde{A}_i$. Let $\gamma_i(s_i) = q_i$. If $U_i \bigcap X_1 = \emptyset$ **49**
since $s_i \in$ Kernel φ'_m means that $s_i = 0$. If $U_i \bigcap X_1 \neq \phi$ write $a_i = b_1 b_2 \cdots b_{m-r} c_1 c_2 \cdots c_r$ where $b_1, b_2, \ldots, b_{m-r}$ are the images of the sections in W_o and $c_1, c_2, \ldots, c_r$ are the images of the sections in W_1, under the isomorphism $I_{U_i} \simeq \tilde{A}_i$. Since $m - r \geq q_1$, $a_i \in p_{i1}^{q_1}$. It is easy to see that since $(\bigcap_{j=1}^{n_i} p_{ij})^{q_1} = 0$ and p_{i1} is a minimal prime, $p_{i1}^{q_1} \subset q_{i1}$. Thus $a_i \in q_{i1}$. Now $s_i \in$ Kernel φ'_m, hence $a_i \in \bigcap_{n=2}^{n_i} q_{ij}$. It follows that $a_i \in \bigcap_{j=1}^{n_i} q_{ij} = 0$ i.e. $s_i = 0$. This completes the proof of the claim.

Now we want to prove that for every curve X in the family $Z_V \xrightarrow{p_V} V$, only singularities of X_{red} are ordinary double points. This will follow from the next three propositions 1.0.4., 1.0.5., 1.0.6.

Proposition 1.0.4. *Let X be a curve in the family $Z_V \xrightarrow{p_V} V$. Every singular point of X_{red} is a double point.*

Proof. Assume the contrary i.e. assume that there exists a point P of multiplicity ≥ 3 on X_{red}. We will show that this leads to the contradiction that the m_o^{th} Hilbert point of X, $H_{m_o}(X) \in \mathbb{P}(\overset{P(m_o)}{\Lambda} H_o(\mathbb{P}^N, O_{\mathbb{P}^N}(m_o)))$ is not semistable and then the result will follow by the contradiction. □

Let $\varphi : H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(1)) \to k(P)$ be the evaluation map, where $k(P)$
is the residue field at the point $P \in X$. It is clear that $W_o =$ kernel φ has **50**
dimension N. Choose a basis of W_o, say $w_0, w_1, \ldots, w_{N-1}$ and extend it to a basis of $W_1 = H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(1))$ by adding a vector, say w_N.

Let λ be a $1 - ps$ of $SL(N+1)$ such that the induced action of λ on W_1 is given by,

$$\lambda(t) w_i = w_i, t \in K^*, (0 \leq i \leq N-1)$$
$$\lambda(t) w_N = t w_N, t \in K^*.$$

The rest of the proof consists of showing that $H_{m_o}(X)$ is not λ- semistable.

Let $\pi : \bar{X} \to X$ be the normalization of X (cf. definition 0.2.5 page 16) and let $L' = \pi^* L$. Assume now that $m > m'$, so that $H^1(x, L^m) = 0 = H^1(\bar{X}, L'^m)$ and the restriction

$\varphi_m : H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m)) \to H^o(X, L^m)$ is surjective. Recall that $H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m))$ has a basis $B_m = \{M_1, M_2, \ldots, M_{\alpha_m}\}$ consisting of monomials of degree m in $w_0, w_1, \ldots, w_N$.

For $0 \leq r \leq m$ let $W_o^{m-r}.W_1^r$ be the subspace of $H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m))$ generated by elements w of the following type.

For $r = 0$,

$$w = x_1 x_2 \cdots x_m, (x_j \in W_o, i \leq j \leq m);$$

for $0 < r < m$,

$$w = x_1 x_2 \cdots x_{m-r} y_1 y_2 \cdots y_r, (x_j \in W_o, 1 \leq j \leq m-r; y_j \in W_1, 1 \leq j \leq r);$$

51 for $r = m$,

$$w = y_1 y_2 \cdots y_m, \quad (y_j \in W_1, 1 \leq j \leq m).$$

We have the following filtration of $H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m))$.

$$0 \subset W_o^m.W_1^o \subset W_o^{m-1}.W_1^1 \subset \cdots \subset W_o^o.W_1^m = H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m)), \qquad \text{(F)}$$

For $0 \leq r \leq m$ let,

$$\bar{W}_o^{m-r}.\bar{W}_1^r = \varphi_m(W_o^{m-r}.W_1^r) \subset H^o(X, L^m), \beta_r = \dim \bar{W}_0^{m-r}.\bar{W}^r.$$

These subspaces define the following filtration of $H^o(X, L^m)$.

$$0 \subset \bar{W}_o^m \cdot \bar{W}_1^o \subset \bar{W}_o^m.\bar{W}_1^1 \subset \cdots \subset \bar{W}_0^o \cdot \bar{W}_1^m = H^o(X, L^m), \qquad (\bar{F})$$

Rewrite the basis B_m as $B_m = \{M'_1, M'_2, \ldots, M'_{P(m)}, M'_{P(m)+1}, \ldots, M'_{\alpha_m}\}$ such that $\{\varphi_m(M'_1), \varphi_m(M'_2), \ldots, \varphi_m(M'_{P(m)})\}$ is a basis of $H^o(X, L^m)$ relative to the filtration $\bar{F}$ (cf. definition 0.2.6 page 16) and $M'_{P(m)+1}$, $M'_{P(m)+2}, \ldots, M'_{\alpha_m}$ are the rest of the monomials in B_m in some order.

Since P is a point of multiplicity ≥ 3 on X_{red}, we have the following cases.

i) There exists exactly one component of X_{red}, say X_1, passing through P;

ii) There exist exactly two components of X_{red}, say X_1 and X_2, passing through P;

iii) There exist at least three components of X_{red}, say X_1, X_2, X_3, passing through P.

In the first case choose three points, say P_1, P_2, P_3 (not necessarily dis- 52
tinct), from the fibre $\pi^{-1}(P)$ of π over P. In the second case note that at least one of the components X_1 and X_2, say X_1, has degree ≥ 3 in $\mathbb{P}^N$ and $P \in X_1$ is a singular point of X_1. Choose three points, say P_1, P_2, P_3 from the fibre $\pi^{-1}(P)$ with $P_1, P_2 \in \overline{X}_1$ (not necessarily distinct), $P_3 \in \overline{X}_2$, ($\overline{X}^1$ denotes the normalization of X_1 etc.). In the third case choose 3 points, say, P_1, P_2, P_3 from the fibre $\pi^{-1}(P)$ with $P_1 \in \overline{X}_1$, $P_2 \in \overline{X}_2$, $P_3 \in \overline{X}_3$. In each of the above cases let D denote the divisor $P_1 + P_2 P_3$ on $\overline{X}$.

We have homomorphisms, $\pi_{m_*} : H^o(X, L^m) \to H^o(\overline{X}, L^m)$. By definition the image of $\overline{W}_0^{m-r} \cdot \overline{W}_1^r \subset H^o(\underline{X}, L^m)$ under the homomorphism π_{m_*} is contained in the subspace $H^o(\overline{X}, L^m((r-m)D))$ of $H^o(\overline{X}, L'^m)$, $(0 \leq r \leq m-1)$. It follows that for $0 \leq r \leq m-1$,

$$\begin{aligned}\beta_r &= \dim \overline{W}_0^{m-r}.\overline{W}_1^r \leq h^o(\overline{X}, L'^m((r-m)D)) + \dim(\text{kernel}\, \pi_{m*}) \\ &= dm + 3(r-m) - g_{\overline{X}} + 1 + h^1(\overline{X}, L'^m((r-m)D)) + \dim(\text{kernel}\, \pi_{m*})\end{aligned}$$

(The last equality follows from the Riemann-Roch theorem).

Claim. *i)* $\dim(\text{kernel}\, \pi_{m*}) < q_2$,

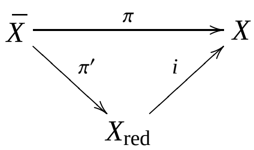
(q_2 is the integer given in assertion iii) page 18).

ii) $h^1(\overline{X}, L'^m((r-m)D)) \leq 3(m-r)$, $\quad (0 \leq r \leq q = 2g-2)$.

iii) $h^1(\overline{X}, L'^m((r-m)D)) = 0$, $\quad (q+1 \leq r \leq m-1)$.

Proof of the Claim: 53

i) Recall that the morphism $\pi : \overline{X} \to X$ has the following factorization.



This gives the following commutative diagram.

$$\begin{array}{ccc} H^o(\overline{X}, L'^m) & \xleftarrow{\pi_{m*}} & H^o(X, L^m) \\ & {\nwarrow}^{\pi'_{m*}} \quad {\swarrow}^{i_{m*}} & \\ & H^o(X_{\text{red}}, L^m_{X_{\text{red}}}) & \end{array}$$

Since the homomorphism π'_{m*} is injective, kernel π_{m*} = kernel i_{m*}. Let I_X be the ideal sheaf of nilpotents in O_X. I_X has finite support, X being generically reduced (cf. Proposition 1.0.3. page 30). Recall that $h^o(X, I_X) < q_2$. Consider the cohomology exact sequence, given by the following exact sequence.

$$0 \to I_X \otimes L^m \to L^m \to L^m_{X_{\text{red}}} \to 0$$

It follows that kernel π_{m*} = kernel $i_{m*} = H^o(X, I_X \otimes L^m)$ and hence dim(kernel $\pi_{m*}) = h^o(X, I_X \otimes L^m) = h^o(X, I_X) < q_2$.

ii) In view of the fact that $H^1(\overline{X}, L'^m) = 0$, this is immediate from the long exact cohomology sequence associated to the exact sequence

$$0 \to L'^m((r-m)D)) \to L'^m \to O_{(m-r)D} \to 0$$

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iii) This follows from the following general fact. Let C be an integral nonsingular curve of genus g_C and let M be a line bundle on C of degree $\geq 2g_C - 1$. Then $H^1(C, M) = 0$.

It follows from the claim and the last inequality that

$$\beta_r = \dim(\overline{W}_0^{m-r}.\overline{W}_1^r) \leq dm + 3(r-m) - g_{\overline{X}} + 1 + 3(m-r) + q_2, (0 \leq r \leq q),$$
$$\beta_r = \dim(\overline{W}_0^{m-r}.\overline{W}_1^r) \leq dm + 3(r-m) - g_{\overline{X}} + 1 + q_2, \quad (q+1 \leq r \leq m-1).$$

We now estimate total λ-weight of $M'_1, M'_2, \ldots, M'_{P(m)} = \sum_{i=1}^{P(m)} w_\lambda(M'_i)$.

Note that a monomial $M \in W_o^{m-r}.W_1^r - W_o^{m-r+1}.W_1^{r-1}$ has λ-weight r.

$$\begin{aligned}
&\sum_{i=1}^{P(m)} w_\lambda(M_i') = \sum_{r=1}^{m} r(\beta_r - \beta_{r-1}) = m\beta_m - \sum_{r=0}^{m-1}\beta_r \\
&\geq m(dm - g + 1) - \sum_{r=0}^{m-1}(dm + 3(r-m) - g_{\bar X} + 1 + q_2) - \sum_{r=0}^{q} 3(m-r) \\
&= \frac{3m^2}{2} - m(g - g_{\bar X} + 3q + \frac{3}{2} + q_2) + \frac{q(q+1)}{2} \\
&> \frac{3m^2}{2} - mS, \quad (S = (g - q_{\bar X} + 3q + \frac{3}{2} + q_2)).
\end{aligned}$$

$$\text{Thus } \sum_{i=1}^{P(m)} w_\lambda(M_i') > \frac{3m^2}{2} - mS, \qquad (E_1)$$

Clearly, total λ-weight of $w_0, w_1, \ldots, w_N = \sum\limits_{i=0}^{N} w_\lambda(w_i) = 1, \qquad (E_2)$.

We are now ready to get the contradiction that the m_o^{th} Hilbert point of X, $H_m(X)$ is not semistable.

If $H_m(X)$ is λ-semistable, $(m > m')$, then there exist monomials **55**
$M_{i_1}', M_{i_2}', \ldots, M_{i_{P(m)}}'$ in B_m, $(1 \leq i_1 < i_2 < \cdots < i_{P(m)} \leq \alpha_m)$ such that $\{\varphi_m(M_{i_1}'), \varphi_m(M_{i_2}'), \ldots, \varphi_m(M_{i_{P(m)}}')\}$ is a basis of $H^o(X, L^m)$ and

$$\frac{\sum\limits_{j=1}^{P(m)} w_\lambda(M_{i_j}')}{mP(m)} \leq \frac{\sum\limits_{i=0}^{N} w_\lambda(w_i)}{d - g + 1},$$

(cf. criterion $(**)$ page 7). It is easily seen that

$$\sum_{i=1}^{P(m)} w_\lambda(M_i') \leq \sum_{j=1}^{P(m)} w_\lambda(M_{i_j}').$$

Now note the following.

$H_m(X)$ is λ-semistable, $(m > m')$,

$$\Rightarrow \frac{\sum_{i=1}^{P(m)} w_\lambda(M_i')}{m(dm - g + 1)} \leq \frac{\sum_{i=0}^{N} w_\lambda(w_i)}{d - g + 1}$$

$$\Rightarrow \frac{\frac{3m^2}{2} - mS}{m(dm-g+1)} \leq \frac{1}{d-g+1} \quad \text{(Follows from } (E_1) \text{ and } (E_2))$$
$$\Rightarrow \frac{\frac{3}{2} - \frac{S}{m}}{d} \leq \frac{1}{d-g+1} \Rightarrow \frac{3}{2}(d-g+1) - \frac{S(d-g+1)}{m} \leq d$$
$$\Rightarrow \frac{1}{2}(d-3g+3) \leq \frac{S(d-g+1)}{m} \Rightarrow m \leq \frac{2S(d-g+1)}{d-3g+3} \leq 4S,$$
$$(\because d \geq 20(g-1) \therefore \frac{d-g+1}{d-3g+3} \leq 2)$$
$$\Rightarrow m < m_o, \qquad (\because m_o > 4S).$$

56 It follows that $H_{m_o}(X)$ is not λ-semistable and hence $H_{m_o}(X)$ is not semistable. (cf. theorem 0.0.9. page 3). This contradiction proves that the only singularities of X_{red} are double points.

Thus we have proved that if X is a curve in the family $Z_V \xrightarrow{p_V} V$ and $P \in X_{\text{red}}$ is a singular point, then P is necessarily a double point. The singular point P is either a cusp or a tacnode or an ordinary double point. The next two propositions will exclude the first two possibilities and this will prove that if X is a curve in the family $Z_V \xrightarrow{p_V} V$ then only singularities of X_{red} are ordinary double points.

Proposition 1.0.5. *If X is a curve in the family $Z_V \xrightarrow{p_V} V$ then X_{red} can not have a cusp singularity.*

Proof. If the result were not true then there exists a curve, say X, in the family $Z_V \xrightarrow{p_V} V$ and a point $Q \in X_{\text{red}}$ such that Q is a cusp. Let Y be the unique irreducible component of X passing through the point $Q \in X$. Let C be the closure of $X - Y$ in X. Let $\pi : \overline{X} \to X$ be the normalization of X. By definition, $\overline{X}$ is a disjoint union of $\overline{Y}_{\text{red}}$ and $\overline{C}_{\text{red}}$. Choose a point $P \in \overline{Y}_{\text{red}}$ such that $\pi(P) = Q$. Since the point $Q \in X$ is a cusp. the morphism π is ramified at the point $P \in \overline{X}$. We will show that this leads to the contradiction that the m_o^{th} Hilbert point of X, $H_{m_o}(X) \in \mathbb{P}(\overset{P(m_o)}{\Lambda} H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m_o)))$ is not semistable. The result will then follow by the contradiction. □

Since the morphism π is ramified at the point $P \in \overline{Y}_{\text{red}} \subset \overline{X}$, Y_{red} is singular and hence $\deg_{Y_{\text{red}}} L \geq 3$, (an integral curve of degree ≤ 2 in $\mathbb{P}^N$

is either a line or a conic and hence nonsingular). Since the curve X is 57
not contained in any hyperplane in $\mathbb{P}^N$ (cf. proposition 1.0.2. page 27), we think of $W_3 = H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(1))$ as a subspace of $H^o(X, L)$. Let,

$$W_o = \{s \in W_3 | \pi_* s \text{ vanishes to order } \geq 3 \text{ at } P\}, \dim W_o = N_o,$$
$$W_1 = \{s \in W_3 | \pi_* s \text{ vanishes to order } \geq 2 \text{ at } P\}, \dim W_1 = N_1.$$

Choose a basis of W_3, relative to the filtration $0 \subset W_o \subset W_1 \subset W_3$, say, $\{w_1, w_2, \ldots, w_{N_o}, w_{N_o+1}, \ldots, w_{N_1}, w_{N_1+1}, \ldots, w_{N+1}\}$ (cf. definition 0.2.6 page 16). Let λ be a $1-ps$ of $GL(N+1)$ such that the action of λ on W_3 is given by,

$$\lambda(t)w_i = w_i, t \in K^*, (1 \leq i \leq N_o),$$
$$\lambda(t)w_i = tw_i, t \in K^*, (N_o + 1 \leq i \leq N_1)$$
$$\lambda(t)w_i = t^3 w_i, t \in K^*, (N_1 + 1 \leq i \leq N + 1).$$

There exists a $1 - ps\lambda'$ of $SL(N+1)$, associated to the $1 - ps\lambda$ of $GL(N+1)$ (cf. definition 0.1.2 page 7). The rest of the proof consists of showing that the m_o^{th} Hilbert point of X, X,

$$H_{m_o}(X) \in \mathbb{P}(\overset{P(m_o)}{\Lambda} H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m_o)))$$

is not λ'-semistable.

Assume now that $m > m'$ so that $H^1(X, L^m) = 0 = H^1(\bar{X}, L'^m)$ and the restriction $\varphi_m : H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m)) \to H^o(X, L^m)$ is surjective. Recall that $H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m))$ has a basis consisting of monomials of degree m in $w_1, w_2, \ldots, w_{N+1}$, say $B_m = \{M_1, M_2, \ldots, M_{\alpha_m}\}$, $(\alpha_m = h^o(P^N, O_{\mathbb{P}^N}(m)))$. Let Ω_i^m be the subspace of $H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m))$ spanned by

$$\{M \in B_m | w_\lambda(M) \leq i\}, \quad (0 \leq i \leq 3m).$$

We have the following filtration of $H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m))$. 58

$$0 \subset \Omega_o^m \subset \Omega_1^m \subset \cdots \subset \Omega_{3m}^m = H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m)), \qquad \text{(F)}$$

Let $\bar{\Omega}_i^m = \varphi_m(\Omega_i^m) \subset H^o(X, L^m), \beta_i = \dim \bar{\Omega}_i^m, (0 \leq i \leq 3m)$.

The above subspaces give the following filtration of $H^o(X, L^m)$.

$$0 \subset \bar{\Omega}_0^m \subset \bar{\Omega}_1^m \subset \ldots \subset \bar{\Omega}_{3m}^m = H^o(X, L^m), \qquad (\bar{F})$$

Rewrite the basis B_m as $B_m = \{M'_1, M'_2, \ldots, M'_{P(m)}, M'_{P(m)+1}, \ldots, M'_{\alpha_m}\}$ so that $\{\varphi_m(M'_1)\,\varphi_m(M'_2), \ldots, \varphi_m(M'_{P(m)})\}$ is a basis of $H^o(X, L^m)$ relative to the filtration $(\bar{F})$ and $M'_{P(m)+1}, M'_{P(m)+2}, \ldots, M'_{\alpha_m}$, are the rest of the monomials in B_m in some order.

The morphism π gives homomorphisms

$$\pi_{m^*} : H^o(X, L^m) \longrightarrow H^o(\bar{X}, L'^m).$$

Claim. *The image of $\bar{\Omega}_i^m$ under the homomorphism π_{m*} is contained in the subspace $H^o(\bar{X}, L'^m((-3+i)P))$ of $H^o(\bar{X}, L'^m)$, $(0 \leq i \leq 3m)$.*

Proof of the claim: First observe that for $i = 0$ the claim follows from definition. Now it suffices to prove that if M is a monomial in B_m such that $M \in \Omega_i^m - \Omega_{i-1}^m$ then $\pi_{m*}(M) \in H^o(\bar{X}, L'^m((-3m+i)P)$, $(1 \leq i \leq 3m)$.

Let $M \in \Omega_i^m - \Omega_{i-1}^m$. Suppose that M has i_0 factors from $\{w_1, w_2, \ldots, w_{N_0}\}$, i_1 factors from $\{w_{N_0+1}, w_{N_0+2}, \ldots, w_{N_1}\}$ and i_3 factors from $\{w_{N_1+1}, w_{N_1+2}, \ldots, w_{N_1}\}$. It follows that,

$$i_0 + i_1 + i_3 = m \text{ and } i_1 + 3i_3 = i.$$

59 By definition $\pi_{m*}(M) \in H^o(\bar{X}, L'^m((-3i_0 - 2i_1)P))$. Now note that $3m - i = 3(i_0 + i_1 + i_3) - (i_1 + 3i_3) = 3i_0 + 2i_1$. This proves the claim.

It follows from the above claim and the Riemann-Roch theorem that for $0 \leq i \leq 3m - 1$.

$$\begin{aligned} \beta_i &= \dim \bar{\Omega}_i^m \leq h^o(\bar{X}, L'^m((-3m+i)P)) + \dim(\text{kernel } \pi_{m*}) \\ &= \dim -3m + i - g_{\bar{X}} + 1 + h^1(\bar{X}, L'^m((-3m+i)P)) + \dim(\text{kernel } \pi_{m*}) \end{aligned}$$

Claim.

i) $\dim(\text{kernel}\, \pi_{m*}) < q_2$.

ii) $h^1(\bar{X}, L'^m((-3m+i)D)) \leq 3m - i, \quad (0 \leq i \leq q = 2g - 2).$

iii) $h^1(\overline{X}, L'^m((-3m+i)D)) = 0, \quad (q+1 \le i \le 3m-1).$

(D denotes the divisor on $\overline{X}$, supported at $P \in \overline{X}$, with multiplicity one).

Proof of the claim:

i) Since the curve X is generically reduced,

$\dim(\text{kernel}\, \pi_{m*}) = h^o(X, I_X) < q_2$, (cf. page 39).

ii) In view of the fact that $H^1(\overline{X}, L'^m) = 0$, this follows from the long exact cohomology sequence associated to the following exact sequence

$$0 \longrightarrow L'^m((-3m+i)D) \longrightarrow L'^m \longrightarrow\!\!\!\!\to O_{(3m-i)D} \to 0$$

iii) Use the following general fact. If C is an integral smooth curve of genus g_C and if M is a line bundle on C with $\deg M \ge 2g_C - 1$ then $H^1(C, M) = 0$. **60**

Hence,

$$\beta_i = \dim \overline{\Omega}_i^m \le dm - 3m + i - g_{\overline{X}} + 1 + 3m - i + q_2, (0 \le i \le q),$$
$$\beta_i = \dim \overline{\Omega}_i^m \le dm - 3m + i - g_{\overline{X}} + 1 + q_2, \quad (q+1 \le i \le 3m-1).$$

We now estimate, total λ-weight of

$$M'_1, M'_2, \ldots, M'_{P(m)} = \sum_{i=1}^{P(m)} w_\lambda(M'_i).$$

Note that a monomial $M \in \Omega_i^m - \Omega_{i-1}^m$, has λ–weight $w_\lambda(M) = i$.

$$\sum_{i=1}^{P(m)} w_\lambda(M'_i) = \sum_{i=1}^{3m} i(\beta_i - \beta_{i-1}) = 3m\beta_{3m} - \sum_{i=0}^{3m-1} \beta_i$$
$$\ge 3m(dm - g + 1) - \sum_{i=0}^{3m-1} (dm - 3m + i - g_{\overline{X}} + 1 + q_2) - \sum_{i=0}^{q} (3m - i)$$

$$\geq \frac{9m^2}{2} - 3m(g - g_{\bar{X}} + q_2 + q + \frac{1}{2})$$
$$= \frac{9m^2}{2} - mS, (S = 3(g - g_{\bar{X}} + q_2 + q + \frac{1}{2}))$$
Thus, $\sum_{i=0}^{P(m)} w_\lambda(M'_i) \geq \frac{9m^2}{2} - mS,$ (E_1)

Next we estimate, total λ-weight of $w_1, w_2, \ldots, w_{N+1} = \sum_{i=1}^{N+1} w_\lambda(w_i)$. Recall that we have agreed to view $W_3 = H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(1))$ as a subspace of $H^o(X, L)$. Now observe that dim $\frac{W_3}{W_1} \leq 1$ and dim $\frac{W_1}{W_0} \leq 1$. To see this note that the image of W_0 (respectively W_1) under the homomorphism $\pi_* : H^o(X, L) \to H^o(\bar{X}, L')$ is contained in the subspace $H^o(\bar{X}, L'(-3p))$
61 (respectively $H^o(\bar{X}, L'(-2P))$) of $H^o(\bar{X}, L')$. Now use the assumption that π is ramified at P and use the following exact sequences

$$0 \longrightarrow L'(-P) \longrightarrow L' \longrightarrow k(P) \longrightarrow 0$$
$$0 \longrightarrow L'(-3P) \longrightarrow L'(-2P) \longrightarrow k(P) \longrightarrow 0$$

$(k(P))$ = the residue field at the point $P \in \bar{X}$.

It follows that dim $\frac{W_3}{W_1} \leq 1$, dim $\frac{W_1}{W_0} \leq 1$. The above considerations imply that total λ-weight of $w_1, w_2, \ldots, w_{N+1} = \sum_{i=1}^{N+1} w_\lambda(w_i) \leq 4,$ (E_2).

We are now ready to get the contradiction that the m_0^{th} Hilbert point of X, $H_{m_o}(X) \in \mathbb{P}(\overset{P(m_o)}{\Lambda} H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m_o)))$ is not $\lambda'-$ semistable.

If $H_m(X)$ is λ'-semistable $(m > m')$, then there exist monomials $M'_{i_1}, M'_{i_2}, \ldots, M'_{i_{P(m)}}$, $(1 \leq i_1 < i_2 < \ldots < i_{P(m)} \leq \alpha_m)$, such that $\{\varphi_m(M'_{i_1}), \varphi_m, (M'_{i_2}), \ldots, \varphi_m(M'_{i_{P(m)}})\}$ is a basis of $H^o(X, L^m)$ relative to the filtration $(\bar{F})$ and

$$\frac{\sum_{j=1}^{P(m)} w_\lambda(M'_{i_j})}{mP(m)} \leq \frac{\sum_{i=1}^{N+1} w_\lambda(w'_i)}{d - g + 1}, \qquad \text{(cf. criterion (**) page 7).}$$

It is easily seen that $\sum_{i=1}^{P(m)} w_\lambda(M'_i) \leq \sum_{j=1}^{P(m)} w_\lambda(M'_{i_j})$. Thus,

$$H_m(X) \text{ is } \lambda'\text{-semistable} \Rightarrow \frac{\sum_{j=1}^{P(m)} w_\lambda(M'_i)}{m(dm-g+1)} \leq \frac{\sum_{i=1}^{N+1} w_\lambda(w'_i)}{d-g+1}$$

$$\Rightarrow \frac{\frac{9m^2}{2} - mS}{m(dm-g+1)} \leq \frac{4}{d-g+1}, \quad \text{(Follows from the estimates}(E_1),(E_2))$$

$$\Rightarrow \frac{\frac{9}{2} - \frac{S}{m}}{d} \leq \frac{4}{d-g+1} \Rightarrow \frac{9(d-g+1)}{2} - \frac{(d-g+1)S}{m} \leq 4d$$

$$\Rightarrow \frac{1}{2}(d-9g+9) \leq \frac{(d-g+1)S}{m} \Rightarrow m \leq \frac{2(d-g+1)S}{d-9g+9}$$

$$\Rightarrow m \leq 4S \quad (\because d \geq 20(g-1) \therefore \frac{d-g+1}{d-9g+9} \leq 2)$$

$$\Rightarrow m < m_o \quad (\because m_o > 4S).$$

It follows that $H_{m_o}(X)$ is not λ'-semistable and hence $H_{m_o}(X)$ is not semistable (cf. theorem 0.0.9. page 3). The result now follows by the contradiction. **62**

Proposition 1.0.6. *Let X be a curve in the family* $Z_V \xrightarrow{p_V} V$. X_{red} *cannot have a tacnode singularity.*

Proof. Assume the contrary i.e. let X be a curve in the family $Z_V \xrightarrow{p_V} V$ such that X_{red} has a tacnode singularity at a point P say. □

Let $X = \bigcup_{i=1}^{p} X_i$ be the decomposition of X in to irreducible components. Let $\pi : \overline{X} \to X$ be the normalization of X so that $\overline{X} = \bigcup_{i=1}^{p} \overline{X}_{i\,\text{red}}$ (cf. definition 0.2.5 page 16) and let $L' = \pi^* L$. That P is a tacnode means,

i) $p > 1$;

ii) there exist components $\overline{X}_i$ and $\overline{X}_j (i \neq j)$ of X and points $Q_1 \in \overline{X}_{i\,\text{red}}, Q_2 \in \overline{X}_{j\,\text{red}}$ such that $\pi(Q_1) = P = \pi(Q_2)$;

iii) $X_{i\,\text{red}}$ and $X_{j\,\text{red}}$ have a common tangent at P. **63**

Since X is not contained in any hyperplane in $\mathbb{P}^N$ (cf. Proposition 1.0.2. page 27), we can think of $W_2 = H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(1))$ as a subspace $H^o(X, L)$. Define two subspaces of W_2 as follows.

$$W_0 = \{s \in W_2 | \pi_* s \text{ vanishes to order } \geq 2 \text{ at } Q_1 \text{ and } Q_2\}$$
$$W_1 = \{s \in W_2 | \pi_* s \text{ vanishes to order } \geq 1 \text{ at } Q_1 \text{ and } Q_2\}$$

Note that since P is a double point of X_{red} and $X_{i\,\text{red}}$ and $X_{j\,\text{red}}$ have a common tangent at P,

$$\pi_* s (s \in W_2) \text{ vanishes to order } \geq 2 \text{ at } Q_1$$
$$\Leftrightarrow$$
$$\pi_* s (s \in W_2) \text{ vanishes to order } \geq 2 \text{ at } Q_2.$$

Let $\dim W_0 = N_0$, $\dim W_1 = N_1$. Choose a basis of W_2 relative to the filtration $0 \subset W_0 \subset W_1 \subset W_2$, say $\{w_1, w_2, \ldots, w_{N_0}, w_{N_0+1}, \ldots, w_{N_1}, w_{N_1+1}, \ldots, w_{N+1}\}$, (cf. definition 0.2.6 page 16).

Let λ be a $1 - ps$ of $GL(N+1)$ such that the action of λ on W_2 is given by,

$$\lambda(t) w_i = w_i, t \in K^*, (1 \leq i \leq N_0),$$
$$\lambda(t) w_i = t w_i, t \in K^*, (N_0 + 1 \leq i \leq N_1),$$
$$\lambda(t) w_i = t^2 w_i, t \in K^*, (N_1 + 1 \leq i \leq N + 1).$$

64 There exists a $1 - ps\lambda'$ of $SL(N+1)$ associated to the $1 - ps\lambda$ of $GL(N+1)$, (cf. definition 0.1.2 page 7). We will show that the m_0^{th}. Hilbert point of X, $H_{m_o}(X)$ is not λ'-semistable. In particular it will follow that $H_{m_o}(X)$ is not semistable, (cf. theorem 0.0.9. page 3), and the result will then follow by the contradiction.

Assume now that $m > m'$ so that $H^1(X, L^m) = 0 = H^1(\bar{X}, L'^m)$ and the restriction $\varphi_m : H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m)) \to H^o(X, L^m)$ is surjective. Recall that $H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m))$ has a basis $B_m = \{M_1, M_2, \ldots, M_{\alpha_m}\}$ consisting of monomials of degree m in $w_1, w_2, \ldots, w_{N+1}$. For $0 \leq r \leq m$ let $W_0^{m-r}.W_1^r$ be the subspace of $H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m))$ generated by elements w of the following type.

For $r = 0$,
$$w = x_1 x_2 \dots x_m, (x_s \in W_0, 1 \leq s \leq m);$$
for $0 < r < m$,
$$w = x_1 x_2 \dots x_{m-r} y_1 y_2 \dots y_r, (x_s \in W_0, 1 \leq s \leq m-r, y_s \in W_1, 1 \leq s \leq r);$$
for $r = m$
$$w = y_1 y_2 \dots y_m, \quad (y_s \in W_1, 1 \leq s \leq m).$$

Similarly, for $0 \leq r \leq m$, let $W_1^{m-r}.W_2^r$ be the subspace of $H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m))$ generated by elements w' of the following type.
For $r = 0$,
$$w' = x'_1 x'_2 \cdots x'_m, (x'_s \in W_1, 1 \leq s \leq m);$$
for $0 < r < m$, 65
$$w' = x'_1 x'_2 \dots x'_{m-r} y'_1 y'_2 \dots y'_r, \\ (x'_s \in W_1, 1 \leq s \leq m-r, y'_s \in W_2, 1 \leq s \leq r);$$
for $r = m$,
$$w' = y'_1 y_2 \dots y'_m, \quad (y'_s \in W_2, 1 \leq s \leq m).$$

We have the following filtration of $H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m))$;
$$0 \subset W_0^m.W_1^o \subset W_0^{m-1}.W_1^1 \subset \cdots \subset W_0^o.W_1^m = W_1^m.W_2^o \subset W_1^{m-1}.W_2^1 \subset \cdots \\ W_1^o.W_2^m = H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m)) \tag{F}$$

Let
$$\bar{W}_0^{m-r}.\bar{W}_1^r = \varphi(W_0^{m-r}.W_1^r) \subset H^o(X, L^m), \dim \bar{W}_0^{m-r}.\bar{W}_1^r = \gamma_r, \\ \bar{W}_1^{m-r}.\bar{W}_2^r = \varphi_m(W_1^{m-r}.W_2^r) \subset H^o(X, L^m), \dim \bar{W}_1^{m-r}.\bar{W}_2^r = \beta_r, (0 \leq r \leq m).$$

These subspace define the following filtration of $H^o(X, L^m)$
$$0 \subset \bar{W}_0^m.\bar{W}_1^o \subset \bar{W}_0^{m-1}.\bar{W}_1^1 \subset \cdots \subset \bar{W}_0^o.\bar{W}_1^m = \bar{W}_1^m.\bar{W}_2^o \subset \bar{W}_1^{m-1}.\bar{W}_2^1 \subset \dots \\ \bar{W}_1^o.\bar{W}_2^m = H^o(X, L^m) \tag{$\bar{F}$}$$

Rewrite the basis B_m as

$B_m = \{M'_1, M'_2, \ldots, M'_{P(m)}, M'_{P(m)+1}, \ldots, M'_{\alpha_m}\}$ so that $\{\varphi_m(M'_1), \varphi_m(M'_2), \ldots, \varphi_m(M'_{P(m)})\}$ is a basis of $H^o(x, L^m)$ relative to the filtration (F) (cf. definition 0.2.6 page 16) and $M'_{P(m)+1}, M'_{P(m)+2}, \ldots, M'_{\alpha_m}$ are the rest of the monomials in B_m in some order.

66 For the rest of the proof we consider the following cases.

Case 1. $\deg_{X_{i\,\mathrm{red}}} L \geq 2$, $\deg_{X_{j\,\mathrm{red}}} L \geq 2$.

Case 2. $\deg_{X_{i\,\mathrm{red}}} L = 1$, $\deg_{X_{j\,\mathrm{red}}} L \geq 2$.

Case 3. $\deg_{X_{i\,\mathrm{red}}} L \geq 2$, $\deg_{X_{j\,\mathrm{red}}} L = 1$.

(Since the point $P \in X$ is a tacnode, these are the only possibilities). We will give proofs in cases 1) and 2) and then case 3) will follow from case 2), by interchanging the roles of i and j.

Case 1. The morphism $\pi : \bar{X} \to X$ gives homomorphisms $\pi_{m*} : H^o(X, L^m) \to H^o(\bar{X}, L'^m)$. Let D denote the divisor $Q_1 + Q_2$ on $\bar{X}$. By definition, the image of $\bar{W}_0^{m-r}.\bar{W}_1^r$ (respectively $\bar{W}_1^{m-r}.\bar{W}_2^r) \subset H^o(X, L^m)$ under the homomorphism π_{m*} is contained in the subspace $H^o(\bar{X}, L'^m((r-2m)D))$ (respectively $H^o(\bar{X}, L'^m((r-m)D))$ of $H^o(\bar{X}, L'^m)$, $(0 \leq r \leq m-1)$.

It follows from the Riemann-Roch theorem that for $0 \leq r \leq m-1$

$$\begin{aligned}\gamma_r &= \dim(\bar{W}_0^{m-r}.\bar{W}_1^r) \leq h^o(\bar{X}, L'^m((r-2m)D)) + \dim(\text{kernel}\,\pi_{m*})\\ &= dm + 2r - 4m - g_{\bar{X}} + 1 + h^1(\bar{X}, L'^m((r-2m)D) + \dim(\text{kernel}\,\pi_{m*})\end{aligned}$$

and

$$\begin{aligned}\beta_r &= \dim(\bar{W}_1^{m-r}.\bar{W}_2^r) \leq h^o(\bar{X}, L'^m((r-m)D) + \dim(\text{kernel}\,\pi_{m*})\\ &= dm + 2r - 2m - g_{\bar{X}} + 1 + h^1(\bar{X}, L'^m((r-m)D) + \dim(\text{kernel}\,\pi_{m*})\end{aligned}$$

Claim:

67 i) $\dim(\text{kernel}\,\pi_{m*}) < q_2$,

ii) $h^1(\bar{X}, L'^m((r-2m)D) \leq 4m - 2r$, $(0 \leq r \leq q = 2g-2)$,

iii) $h^1(\overline{X}, L'^m((r-2m)D) = 0, (q+1 \leq r \leq m-1)$,

iv) $h^1(\overline{X}, L'^m((r-m)D) = 0, (0 \leq r \leq m-1)$.

Proof of the Claim:

i) Since the curve X is generically reduced, (cf. proposition 1.0.3. page 30), for all integers m, $\dim(\mathrm{kernel}\,\pi_{m*}) = h^o(X, I_X) < q_2$, (cf. page 39).

ii) In view of the fact that $H^1(\overline{X}, L'^m) = 0$, this is immediate from the long exact cohomology sequence associated to the following exact sequence,

$$0 \to L'^m((r-2m)D) \to L'^m \to O_{(2m-r)D} \to 0.$$

iii) Recall that $\deg_{X_{i\,\mathrm{red}}} L \geq 2 \leq \deg_{X_{j\,\mathrm{red}}} L$ and use the following general fact. If C is an integral smooth curve of genus g_C and if M is a line bundle of C with $\deg M \geq 2g_C - 1$ then $H^1(C, M) = 0$.

iv) This follows from the same reasoning as above.

It follows from these considerations that,

$$\gamma_r \leq dm + 2r - 4m - g_{\overline{X}} + 1 + 4m - 2r + q_2, \quad (0 \leq r \leq q = 2g-2),$$
$$\gamma_r \leq dm + 2r - 4m - g_{\overline{X}} + 1 + q_2, \quad (q+1 \leq r \leq m-1),$$
$$\beta_r \leq dm + 2r - 2m - g_{\overline{X}} + 1 + q_2, \quad (0 \leq r \leq m-1).$$

We now estimate total λ-weight of $M'_1, M'_2, \ldots, M'_{P(m)} = \sum_{i=1}^{P(m)} w_\lambda(M'_i)$. **68**
Note that a monomial $M \in W_0^{m-r}.W_1^r - W_0^{m-r+1}.W_1^{r-1}$ has λ-weight $w_\lambda(M) = r$ and a monomial

$$M' \in W_1^{m-r} W_2^r - W_1^{m-r+1} W_2^{r-1} \text{ has } \lambda - \text{weight } w_\lambda(M') = m + r.$$

$$\sum_{i=1}^{P(m)} w_\lambda(M'_i) = \sum_{r=1}^{m} (m+r)(\beta_r - \beta_{r-1}) + \sum_{r=1}^{m} r(\gamma_r - \gamma_{r-1})$$

$$= 2m\beta_m - \sum_{r=0}^{m-1}\beta_r - \sum_{r=0}^{m-1}\gamma_r$$

$$\geq 2m(dm - g + 1) - \sum_{r=0}^{m-1}(dm + 2r - 2m - g_{\bar{X}} + 1 + q_2)$$

$$- \sum_{r=0}^{m-1}(dm + 2r - 4m - g_{\bar{X}} + 1 + q_2) - \sum_{r=0}^{q}(4m - 2r)$$

$$= 2m(dm - g + 1) - \sum_{r=0}^{m-1}(2dm + 4r - 6m - 2g_{\bar{X}} + 2 + 2q_2)$$

$$- 4m(q + 1) + q(q + 1)$$

$$> 4m^2 - mS, (S = 2(g - g_{\bar{X}} + q_2 + 2q + 1)).$$

Thus, $$\sum_{i=1}^{P(m)} w_\lambda(M_i') > 4m^2 - mS, \qquad (E_1)$$

We now estimate total λ-weight of $w_1, w_2, \ldots, w_{N+1} = \sum_{i=1}^{N+1} w_\lambda(M_i')$. The morphism $\pi : \bar{X} \to X$ gives a homomorphism $\pi* : H^o(X, L) \to H^o(\bar{X}, L')$. Recall that we have agreed to view $W_2 = H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(1))$ as a subspace of $H^o(X, L)$. Clearly $W_2 \bigcap \text{kernel}\, \pi_* = 0$. By definition the image of W_0 (respectively W_1) under the homomorphism π_* is contained in the subspace $H^o(\bar{X}, L'(-2Q_1))$ (respectively $H^o(\bar{X}, L'(-Q_1))$) of $H^o(\bar{X}, L')$.

69 Now it is immediate from the following exact sequences that $\dim \dfrac{W_1}{W_0} \leq 1$, $\dim \dfrac{W_2}{W_1} \leq 1$.

$$0 \to L'(-2Q_1) \to L'(-Q_1) \to k(Q_1) \to 0$$
$$0 \to L'(-Q_1) \to L' \to k(Q_1) \to 0$$

($k(Q_1)$ = the residue field the point $Q_1 \in \bar{X}$).

It follows that $\sum_{i=1}^{N+1} w_\lambda(w_i) \leq 3, \qquad (E_2)$

We are now ready to get the contradiction that the m_0^{th} Hilbert point of X, $H_{m_o}(X)$ is not λ'-semistable.

If $H_m(X)$ is semistable $(m > m')$ then there exist monomials M'_{i_1}, $M'_{i_2}, \ldots, M'_{i_{P(m)}}$ in B_m such that
$\{\varphi_m(M'_{i_1}), \varphi_m(M'_{i_2}), \ldots, \varphi_m(M'_{i_{P(m)}})\}$ is a basis of $H^o(X, L^m)$
and $\dfrac{\sum_{j=1}^{P(m)} w_\lambda(M'_{i_j})}{mP(m)} \leq \dfrac{\sum_{i=1}^{N+1} w_\lambda(w_i)}{d-g+1}$ (cf. criterion $(**)$ page 7).
Observe that $\sum\limits_{i=1}^{P(m)} w_\lambda(M'_i) \leq \sum\limits_{j=1}^{P(m)} w_\lambda(M'_{i_j})$.
Hence,

$$H_m(X) \text{ is } \lambda' - \text{ semistable } (m > m') \Rightarrow \frac{\sum_{j=1}^{P(m)} w_\lambda(M'_i)}{m(dm-g+1)} \leq \frac{\sum_{i=1}^{N+1} w_\lambda(w_i)}{d-g+1}$$

$$\Rightarrow \frac{4m^2 - mS}{m(dm-g+1)} \leq \frac{3}{d-g+1} \qquad \text{(Follows from } (E_1), (E_2))$$

$$\Rightarrow \frac{4 - \frac{S}{m}}{d} \leq \frac{3}{d-g+1} \Rightarrow d - 4g + 4 \leq \frac{(d-g+1)S}{m}$$

$$\Longrightarrow m \leq \frac{(d-g+1)S}{(d-4g+4)} \leq 2S \quad (\because d \geq 20(g-1) \therefore \frac{d-g+1}{d-4g+4} \leq 2)$$

$$\Longrightarrow m < m_o \quad (\because m_o > 2S)$$

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Thus $H_{m_o}(X)$ is not λ'−semistable and hence $H_{m_o}(X)$ is not semistable (cf. theorem 0.0.9. page 3). This contradiction proves the result in case 1).

Case 2. The proof in this case is on the same lines. Recall that we have homomorphisms $\pi_m : H^o(X, L^m) \to H^o(\bar{X}, L'^m), (m > m')$, $\pi_* : H^o(X, L) \to H^o(\bar{X}, L')$ and that we have agreed to view $H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(1)) = W_2$ as a subspace of $H^o(X, L)$. Let Y be the closure of $X - X_i$ in X. Clearly a section $\pi_* s (s \in W_o)$ vanishes on $\bar{X}_{i\,\text{red}}$. Hence the image of $\bar{W}_o^{m-r}.\bar{W}_1^r$, $(0 \leq r \leq m-1)$, under the homomorphism π_{m*} is contained in the subspace
$H^o(\bar{Y}_{\text{red}}, L'^m_{\bar{Y}_{\text{red}}}(r-2m)Q_2)) \subset H^o(\bar{X}, L'^m)$, ($\bar{Y}_{\text{red}}$ is the normalization of Y_{red}).
It follows that, for $0 \leq r \leq m-1$

$$\gamma_r = \dim(\bar{W}_o^{m-r} \cdot \bar{W}_1^r) \leq h^o(\bar{Y}_{\text{red}}, L'_{\bar{Y}_{\text{red}}})((r-2m)Q_2)) + (\text{kernel}\, \pi_{m*})$$

$$= (d-1)m + r - 2m - g_{\overline{Y}} + 1 + h^1(\overline{Y}_{\text{red}}, L'^m_{\overline{Y}_{\text{red}}})((r-2m)Q_2))$$
$$+ \dim(\text{kernel}\,\pi_{m*})$$

Recall that,

$$\beta_r = \dim(\overline{W}_1^{m-r} \cdot \overline{W}_2^r) \leq h^o(\overline{X}, L'^m((r-m)D)) + \dim(\text{kernel}\,\pi_{m*}).$$
$$= dm + 2r - 2m - g_{\overline{X}} + 1 + h^1(\overline{X}, L'^m((r-m)d)) + \dim(\text{kernel}\,\pi_{m*}).$$

As before we have,

i) $\dim(\text{kernel}\,\pi_{m*}) < q_2$

71 ii) $h^1(\overline{Y}, L'^m_{\overline{Y}_{\text{red}}}((r-2m)Q_2)) = 0, \quad (0 \leq r \leq q = 2q-2).$

iii) $h^1(\overline{Y}, L'^m_{\overline{Y}_{\text{red}}}((r-2m)Q_2)) = 0, \quad (q+1 \leq r \leq m-1).$

iv) $h^1(\overline{X}, L'^m((r-m)D)) \leq 2m - 2r, \quad (0 \leq r \leq q = 2g-2).$

v) $h^1(\overline{X}, L'^m((r-m)D)) = 0, \quad (q+1 \leq r \leq m-1).$

Thus,

$$\gamma_r \leq (d-1)m + r - 2m - g_{\overline{Y}} + 1 + 2m - r + q_2, \quad (0 \leq r \leq q),$$
$$\gamma_r \leq (d-1)m + r - 2m - g_{\overline{Y}} + 1 + q_2, \quad (q+1 \leq r \leq m-1),$$
$$\beta_r \leq dm + 2r - 2m - g_{\overline{X}} + 1 + 2m - 2r + q_2, \quad (0 \leq r \leq q),$$
$$\beta_r \leq dm + 2r - 2m - g_{\overline{X}} + 1 + q_2, \quad (q+1 \leq r \leq m-1)$$

We want to estimate $\sum_{i=1}^{P(m)} w_\lambda(M'_i)$. Recall that, a monomial
$M \in W_0^{m-r} \cdot W_1^r - W_0^{m-r+1}.W_1^{r-1}$ has $\lambda-$weight r, a monomial
$M \in W_1^{m-r} \cdot W_2^r - W_1^{m-r+1}.W_2^{r-1}$ has $\lambda-$weight $m+r$

$$\sum_{i=1}^{P(m)} w_\lambda(M'_i) = \sum_{r=1}^{m}(m+r)(\beta_r - \beta_{r-1}) + \sum_{r=1}^{m} r(\gamma_r - \gamma_{r-1})$$
$$= 2m\beta_m - \sum_{r=0}^{m-1}\beta_r - \sum_{r=0}^{m-1}\gamma_r.$$

$$\geq 2m(dm-g+1)-\sum_{r=0}^{m-1}(dm+2r-2m-g_{\bar{X}}+1+q_2)$$
$$-\sum_{r=0}^{q}(2m-2r)-\sum_{r=0}^{m-1}((d-1)m+r-2m-g_{\bar{Y}}+1+q_2)$$
$$-\sum_{r=0}^{q}(2m-r)>\frac{7m^2}{2}-m(2g-g_{\bar{X}}-g_{\bar{Y}}+2q_2+4q-\frac{3}{2}).$$

Put $(2g-g_{\bar{X}}-g_{\bar{Y}}+2q_2+4q-\frac{3}{2})=S$. Thus we have the following 72
estimate.

$$\sum_{i=1}^{P(m)} w_\lambda(M_i^{'}) > \frac{7m^2}{2}-mS, \qquad (E_1')$$

We have already seen that total $\lambda-$weight of

$$w_1,w_2,\ldots,w_{N+1}=\sum_{i=1}^{N+1} w_\lambda(w_i)\leq 3, \qquad (E_2)$$

As in case 1) we have, $H_m(X)$ is $\lambda'-$semistable $\Longrightarrow \dfrac{\frac{7}{2}m^2-mS}{m(dm-g+1)} \leq \dfrac{3}{d-g+1}$ (Follows as in the previous case from criterion $(**)$ (page 7) and the estimates (E_1') and (E_2))

$$\Longrightarrow \frac{\frac{7}{2}-\frac{S}{m}}{d}\leq\frac{3}{d-g+1}\Longrightarrow \frac{1}{2}(d-7g+7)\leq\frac{(d-g+1)S}{m}$$
$$\Longrightarrow m\leq\frac{2(d-g+1)S}{d-7g+7}\leq 4S\,(\because\ d\geq 20(g-1)\ \therefore\ \frac{d-g+1}{d-7g-7}\leq 2).$$

Since $m_o > 4S$, the above inequality implies that $H_{m_o}(X)$ is not semistable. This contradiction concludes the proof as in case 1.

The next step in the proof of Theorem 1.0.1. is to prove an important inequality which will be needed for the proof of the theorem.

Proposition 1.0.7. *Let X be a curve in the family $Z_V \xrightarrow{p_V} V$ such that X has at least two irreducible components. Let C be a reduced, connected,*

complete subcurve of $X (C \neq X)$ and let Y be the closure of $X - C$ in X, with the reduced structure. Recall that $\pi : \overline{X} \to X$ denotes the normalization of X. It follows from definition 0.2.5 (page 16) that $\overline{X}$ is a disjoint union of $\overline{C}$ (normalization of C) and $\overline{Y}$ (normalization of Y). Let $\varphi'_1 : H^o(\mathbb{P}^{\mathbb{N}}, O_{\mathbb{P}^{\mathbb{N}}}(1)) \to H^o(C, I_C)$ be the natural restriction map and let 73
$W_o = \operatorname{kernel} \varphi'_1$.

If there exist points $P_1, P_2, \ldots, P_k$ on $\overline{Y}$, satisfying

i) $\pi(P_i) \in Y \cap C$, $(1 \leq i \leq k)$,

ii) for every irreducible component $\overline{Y}_j$ of $\overline{Y}$

$\deg_{\overline{Y}_j}(L'_{\overline{Y}}(-D)) \geq 0$,

(D denotes the divisor $\sum\limits_{i=1}^{k} P_i$ on $\overline{Y}$, $L' = \pi^* L$), then the following inequality holds,

$$\frac{h^o(C, L_C)}{d - g + 1} \geq \frac{e + \frac{k}{2}}{d}, \qquad (e = \deg_C L) \qquad (*')$$

Proof. Let $\dim W_o = N_o$. Choose a basis of $W_1 = H^o(\mathbb{P}^{\mathbb{N}}, O_{\mathbb{P}^{\mathbb{N}}}(1))$ relative to the filtration $0 \subset W_o \subset W_1$, say $w_0, w_1, \ldots, w_{N_o - 1}, w_{N_o}, \ldots, w_N$. Let λ be a $1 - ps$ of $GL(N+1)$ such that the action of λ on W_1 is given by,

$$\lambda(t) w_i = w_i, t \in K^*, (0 \leq i \leq N_o - 1),$$
$$\lambda(t) w_i = t w_i, t \in K^*, (N_o \leq i \leq N).$$

74 Let λ' be the $1 - ps$ of $SL(N+1)$ associated to the $1 - ps \lambda$ of $GL(N+1)$, (cf. definition 0.1.2 page 7). Since the m_o^{th} Hilbert point of X, $H_{m_o}(X) \in \mathbb{P}(\overset{P(m_o)}{\wedge} H^o(\mathbb{P}^{\mathbb{N}}, O_{\mathbb{P}^{\mathbb{N}}}(m_o)))$ is λ'−semistable, (cf. theorem 0.0.9. page 3), criterion (∗∗) (page 7) is satisfied. The required inequality will follow from the inequality in the above mentioned criterion. □

Let $B_{m_o} = \{M_1, M_2, \ldots M_{\alpha_{m_o}}\}$ be a basis of $H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m_o))$ consisting of monomials of degree m_o in $w_0, w_1, \ldots, w_N$, ($\alpha_{m_o} = h^o(\mathbb{P}^N,$

$O_{\mathbb{P}^N}(m_o))$). For $0 \le r \le m$ let $w_o^{m_o-r} \cdot w_1^r$ be the subspace of $H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m_o))$ generated by elements w of the following type.

$$w = x_1 x_2 \cdots x_{m_o-r} y_1 y_2 \cdots y_r, (x_j \in W_o, 1 \le j \le m_o - r; y_j \in W_1, 1 \le j \le r)$$

As before, for $r = 0$, w is a product of x'^s_j only and for $r = m_o$, w is a product of y'^s_j only. We have the following filtration (F) of $H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m_o))$,

$$0 \subset W_o^{m_o} \cdot W_1^o \subset W_o^{m_o-1} \cdot W_1^1 \subset \cdots \subset W_o^o \cdot W_1^{m_o} = H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m_o)), \quad \text{(F)}$$

Note that if $W_o = 0$, then $W_0^{m-r} \cdot W_1^r = 0 (0 \le r \le m-1)$. Let
$\overline{W}_0^{m_o} \cdot \overline{W}_1^r = \varphi_{m_o}(W_0^{m_o-r} \cdot W_1^r) \subset H^o(X, L^{m_o}), \dim \overline{W}_0^{m_o-r} \cdot \overline{W}_1^r = \beta_r, (0 \le r \le m_o)$
These subspaces define the following filtration $(\overline{F})$ of $H^o(X, L^{m_o})$,

$$0 \subset \overline{W}_0^{m_o} \cdot \overline{W}_1^o \subset \overline{W}_0^{m_o-1} \cdot \overline{W}_1^1 \subset \cdots \subset \overline{W}_0^o \cdot \overline{W}_1^{m_o} = H^o(X, L^{m_o}), \quad (\overline{F})$$

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Rewrite the basis B_{m_o} as $B_{m_o} = \{M'_1, M'_2, \ldots, M'_{P(m_o)}, M'_{P(m_o)+1}, \ldots, M'_{\alpha_{m_o}}\}$ such that $\{\varphi_{m_o}(M'_1), \varphi_{m_o}(M'_2), \ldots, \varphi_{m_o}(M'_{P(m_o)})\}$ is a basis of $H^o(X, L^{m_o})$ relative to the filtration $(\overline{F})$ and $M'_{P(m_o)+1}$, $M'_{P(m_o)+2}, \ldots, M'_{\alpha_{m_o}}$ are the rest of the monomials in B_{m_o} in some order.

The morphism $\pi : \overline{X} \to X$ gives a homomorphism $\pi_{m_o *} : H^o(X, L^{m_o}) \to H^o(\overline{X}, L^{m_o})$. Since $\overline{X}$ is a disjoint union of $\overline{Y}$ (normalization of Y) and $\overline{C}$ (normalization of C), $H^o(\overline{X}, L^{m_o}) = H^o(\overline{Y}, L'^{m_o}_{\overline{Y}}) \oplus H^o(\overline{C}, L'^{m_o}_{\overline{C}})$. By definition the section in $\pi_{m_o *}(\overline{W}_0^{m-r} \cdot \overline{W}_1^r)(0 \le r \le m_o - 1)$ vanish on $\overline{C}$ and also vanish to order $\ge m_o - r$ at the points $P_1, P_2, \ldots, P_k$, therefore

$$\pi_{m_o *}(\overline{W}_0^{m-r} \cdot \overline{W}_1^r) \subset H^o(\overline{Y}, L'^{m_o}_{\overline{Y}}((r - m_o)D)) \subset H^o(\overline{X}, L'^{m_o})$$

(D denotes the divisor $\sum_{i=1}^{k} P_i$ on $\overline{Y}$). It follows that

$$\begin{aligned}
\beta_r = \dim \overline{W}_0^{m_o-r} \cdot \overline{W}_1^r &\le h^o(\overline{Y}, L'^{m_o}_{\overline{Y}}((r - m_o)D)) + \dim(\text{kernel}\, \pi_{m_o *}) \\
&= (d - e)m_o + k(r - m_o) - g_{\overline{Y}} + 1 + h^1(\overline{Y}, L'^{m_o}_{\overline{Y}}((r - m_o)D)) \\
&\quad + \dim(\text{kernel}\, \pi_{m_o *})
\end{aligned}$$

Claim:

i) $\dim(\text{kernel } \pi_{m_o *}) < q_2$,

ii) $h^1(\overline{Y}, L'^{m_o}_{\overline{Y}}((r - m_o)D)) \leq k(m_o - r), \quad (0 \leq r \leq q = 2g - 2)$,

iii) $h^1(\overline{Y}, L'^{m_o}_{\overline{Y}}((r - m_o)D)) = 0, \quad (q + 1 \leq r \leq m_o - 1)$

76 i) We have seen this (cf. page 39).

ii) The exact sequence

$$0 \to L'^{m_o}_{\overline{Y}}((r - m_o)D) \to L'^{m_o}_{\overline{Y}} \to O_{(m_o - r)D} \to 0$$

gives the long exact cohomology sequence

$$\cdots \to H^o(\overline{Y}, O_{(m_o - r)D}) \to H^1(\overline{Y}, L'^{m_o}_{\overline{Y}}((r - m_o)D)) \to H^1(\overline{Y}, L^{m_o}_{\overline{Y}}) \to \cdots$$

since $m_o > m', H^1(\overline{Y}, L'^{m_o}_{\overline{Y}}) = 0$. Hence

$$h^1(\overline{Y}, L'^{m_o}_{\overline{Y}}((r - m_o)D)) \leq h^o(\overline{Y}, O_{(m_o - r)D}) = k(m_o - r).$$

iii) Recall the condition ii) (page 56) and use the following general fact. If C' is an integral, smooth curve of genus $g_{C'}$ and if M is a line bundle on C' with $\deg M \geq 2g_{C'} - 1$ then $H^1(C', M) = 0$.

Hence,

$$\beta_r \leq (d - e)m_o + k(r - m_o) - g_{\overline{Y}} + 1 + k(m_o - r) + q_2, \qquad (0 \leq r \leq q)$$
$$\beta_r \leq (d - e)m_o + k(r - m_o) - g_{\overline{Y}} + 1 + q_2, \qquad (q + 1 \leq r \leq m_o - 1).$$

We make of following estimate.
total $\lambda-$ weight of

$$M'_1, M'_2, \ldots, M'_{(P(m_o))} = \sum_{i=1}^{P(m_o)} w_\lambda(M'_i)$$
$$= \sum_{r=1}^{m_o} r(\beta_r - \beta_{r-1}), (\because \text{ a monomial}$$

$$
\begin{aligned}
&M \in (W_0^{m-r}W_1^{r} - W_0^{m-r+1}W_1^{r-1})\\
&\text{has } \lambda - \text{ weight } r)\\
&= m_o\beta_{m_o} - \sum_{r=0}^{m_o-1}\beta_r\\
&\geq m_o(dm_o - g + 1) - \sum_{r=0}^{m_o-1}(m_o(d-e) + k(r - m_o)\\
&\quad - g_{\overline{Y}} + 1 + q_2) - \sum_{r=0}^{q} k(m_o - r)\\
&= \left(e + \frac{k}{2}\right)m_o^2 - m_o\left(g - g_{\overline{Y}} + q_2 + \frac{k}{2} + kq\right)\\
&\quad + \frac{q(q+1)}{2}
\end{aligned}
$$

Put $S = (g - g_{\overline{Y}} + q_2 + \frac{k}{2} + kq)$. Thus we get, 77

$$\sum_{i=1}^{P(m)} w_\lambda(M_i') > (e + \frac{k}{2})m_o^2 - m_o S, \qquad (E_1)$$

Note that the above inequality is true even if $W_o = 0$. Clearly, total $\lambda-$weight of $w_0, w_1, \ldots, w_N = \sum_{i=0}^{N} w_\lambda(w_i)$
$= \dim W_1 - \dim W_o < h^o(C, L_C)$, (Follows from the definition of λ)

$$\text{Thus we get } \sum_{i=0}^{N} w_\lambda(w_i) \leq h^o(C, L_C), \qquad (E_2)$$

If $W_o \neq 0$ then $\lambda' : G_m \to SL(N+1)$ is a nontrivial homomorphism i.e. a $1 - ps$ of $SL(N+1)$.

Since the m_o^{th} Hilbert point of X.

$H_{m_o}(X) \in P(\overset{P(m_o)}{\wedge} H^o\mathbb{P}^N, O_{\mathbb{P}^N}(m_o))$ is $\lambda'-$semistable, there exist monomials $M_{i_1}', M_{i_2}', \ldots, M_{i_{P(m_o)}}'(1 \leq i_1 < i_2 < \ldots < i_{P(m_o)} \leq \alpha_{m_o})$ in B_{m_o} such that $\{\varphi_{m_o}(M_{i_1}'), \varphi_{m_o}(M_{i_2}'), \ldots, \varphi_{m_o}(M_{i_{P(m_o)}}')\}$ is a basis of $H^o(X, L^{m_o})$ and

$$\frac{\sum_{j=1}^{P(m_o)} w_\lambda(M'_{i_j})}{m_o P(m_o)} \leq \frac{\sum_{i=0}^{N} w_\lambda(w_i)}{d-g+1},$$ (cf. criterion (∗∗), page 7).

It is easy to see that $\sum_{i=1}^{P(m_o)} w_\lambda(M'_i) \leq \sum_{j=1}^{P(m_o)} w_\lambda(M'_{i_j})$.

78 It follows that, $\dfrac{\sum_{i=1}^{P(m_o)} w_\lambda(M'_{i_j})}{m_o P(m_o)} \leq \dfrac{\sum_{i=0}^{N} w_\lambda(w_i)}{d-g+1}$,

$$\implies \frac{(e+\frac{k}{2})m_o^2 - m_o S}{m_o(dm_o - g + 1)} < \frac{h^o(C, L_C)}{d-g+1},$$ (Follows from (E_1) and (E_2)),

$$\implies \frac{e + \frac{k}{2} - \frac{S}{m_o}}{d} < \frac{h^o(C, L_C)}{d-g+1}.$$

Even if $W_o = 0$, we have, using (E_1) and (E_2)

$$\frac{(e+\frac{k}{2})m_o^2 - m_o S}{m_o(dm_o - g + 1)} < \frac{\sum_{j=1}^{P(m_o)} w_\lambda(M'_i)}{m_o(P(m_o))} = 1 < \frac{h^o(C, L_C)}{d-g+1}$$

$$\implies \frac{e + \frac{k}{2} - \dfrac{S}{m_o}}{d} < \frac{h^o(C, L_C)}{d-g+1}.$$

We claim that the above inequality implies that $\dfrac{e + \dfrac{k}{2}}{d} \leq \dfrac{h^o(C, L_C)}{d-g+1}$. In fact if the claim were not true i.e. if $\dfrac{e + \dfrac{k}{2}}{d} \leq \dfrac{h^o(C, L_C)}{d-g+1}$ then we get a contradiction as follows.

First note that

$$\frac{e + \frac{k}{2}}{d} > \frac{h^o(C, L_C)}{d-g+1} \implies (d-g+1)(e + \frac{k}{2}) - d(h^o(C, L_C)) \geq \frac{1}{2}.$$

Now,

$$\frac{e + \dfrac{k}{2} - \dfrac{S}{m_o}}{d} \leq \frac{h^o(C, L_C)}{d-g+1}, \quad \text{(proved)}$$

$$\implies (d-g+1)(e+\frac{k}{2}) - d(h^o(C, L_C)) \leq \frac{S(d-g+1)}{m_o}$$
$$\implies \frac{1}{2} \leq \frac{S(d-g+1)}{m_o} \implies m_o \leq 2S(d-g+1).$$

By our choice of the integer m_o, this is a contradiction. This proves the **79**
required inequality, $\dfrac{h^o(C, L_C)}{d-g+1} \geq \dfrac{e+\frac{k}{2}}{d}$.

Proposition 1.0.8. *Every curve X in the family $Z_V \xrightarrow{p_V} V$ is reduced.*

Proof. Let X be a curve in the family $Z_V \xrightarrow{p_V} V$ and let I_X be the ideal sheaf of nilpotents in O_X. We want to show that $I_X = 0$. For the moment consider the closed subscheme X_{red} of X defined by I_X. □

Claim: $H^1(X_{\text{red}}, L_{\text{red}}) = 0$, $(L_{\text{red}} = L_{X_{\text{red}}})$. First note that since the only singularities of X_{red} are ordinary double points (cf. proposition 1.0.4., 1.0.5., 1.0.6), X_{red} has a dualizing sheaf, say ω. If the claim were not true then we have $H^o(X_{\text{red}}, \omega \otimes L_{\text{red}}^{-1}) \simeq H^1(X_{\text{red}}, L_{\text{red}}) \neq 0$. So there exists a nonzero section $s \in H^o(X, \omega \otimes L_{\text{red}}^{-1})$ and a complete connected subcurve C of X_{red} such that s is not identically zero on any of the components of C but s vanishes at all points in $C \cap \overline{X - C}$. Observe that $C \neq \mathbb{P}^1$, hence $\deg_C L = e > 1$.

It follows from the proof of theorem 0.2.3. (page 13) that $h^o(C, L_C) - 1 \leq \frac{e}{2}$. Using the inequality $(*')$ (cf. proposition 1.0.7 page 55), we get,

$$\frac{e+\frac{1}{2}}{d} \leq \frac{h^o(C, L_C)}{d-g+1} \leq \frac{\frac{e}{2}+1}{d-g+1}$$

$$\implies de + \frac{d}{2} + (1-g)(e+\frac{1}{2}) \leq \frac{de}{2} + d \implies \frac{d(e-1)}{2} \leq (g-1)(e+\frac{1}{2})$$
$$\implies \frac{20(g-1)(e-1)}{2} \leq (g-1)(e+\frac{1}{2}) \quad (\because d \geq 20(g-1))$$
$$\implies 10e - 10 < e + \frac{1}{2} \quad (\because g \geq 2)$$

$$\Longrightarrow 9e < 10 + \frac{1}{2} \Longrightarrow e \leq 1.$$

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But we have already observed that $e > 1$. This contradiction proves the claim.

Now consider the following exact sequence.

$$0 \to I_X \otimes L \to L \to L_{\text{red}} \to 0, \tag{1}$$

We have the long exact cohomology sequence

$$\cdots \to H^1(X, I_X \otimes L) \to H^1(X, L) \to H^1(X, L_{\text{red}} \to 0).$$

Since I_X has finite support (cf. proposition 1.0.3., page 30), $H^1(X, I_X \otimes L) = 0$. We have seen that $H^1(X_{\text{red}}, L_{\text{red}}) = 0$. Hence we conclude from the above cohomology sequence that $H^1(X, L) = 0$. Recall that the restriction

$\overline{\varphi} : H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(1)) \to H^o(X_{\text{red}}, L_{\text{red}})$ is injective, (cf. proposition 1.0.2., page 27).

Thus

$$\begin{aligned} d - g + 1 = h^o(\mathbb{P}^N, O_{\mathbb{P}^N}(1)) &\leq h^o(X, L_{\text{red}}) \\ &= h^o(X, L) - h^o(X, I_X \otimes L) \quad \text{(Follows from (1))} \\ &= d - g + 1 - h^o(X, I_X \otimes L) \quad (\because h^o(X, L) = \chi(L) = d - g + 1) \\ &\Longrightarrow h^o(X, I_X \otimes L) = 0. \end{aligned}$$

81 Since $I_X \otimes L$ has finite support, it follows that $I_X = 0$ i.e. X is reduced.

It follows from the above proof that if X is a curve in the family $Z_V \xrightarrow{p_V} V$ then $H^1(X, L) = 0$. It is now immediate that trace of the linear system $|D|$ on X is complete. ($|D|$ is the complete linear system of $\mathbb{P}^N$ corresponding to the line bundle $O_{\mathbb{P}^N}(1)$ on $\mathbb{P}^N$).

Proposition 1.0.9. *Let X be a curve in the family $Z_V \xrightarrow{p_V} V$ and let Y be a nonsingular rational component of X i.e. $Y \simeq \mathbb{P}^1$, then Y meets the other components of X in at least two points.*

Proof. Let C be the closure of $X - Y$ in X with the reduced structure and let g_C be the genus of C (i.e. $g_C = h^1(C, O_C)$). □

Assume that the result is not true, i.e. assume that Y meets C in exactly one point, P say. Since X is connected, the above assumption implies that C is connected.

Claim: $g_C = g$.

Proof of the Claim: Since the curve X is reduced and the only singularities of X are ordinary double points (cf. propositions 1.0.8., 1.0.4., 1.0.5., 1.0.6.), we have the following exact sequence,

$$0 \to O_X \to O_Y \oplus O_C \to K \to 0.$$

Take the Euler characteristics.

$$\begin{aligned}
&\chi(O_Y) + \chi(O_C) = \chi(O_X) + 1\\
&\Longrightarrow 1 + 1 - h^1(C, O_C) = 1 - h^1(X, O_X) + 1, (\because Y \simeq P^1 \therefore \chi(O_Y) = 1)\\
&\Rightarrow g_C = h^1(C, O_C) = h^1(X, O_X) = g.
\end{aligned}$$

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Now apply the inequality $(*')$ of proposition 1.0.7 (page 55) to C

$$\frac{h^o(C, L_C)}{d - g + 1} \geq \frac{e + \frac{1}{2}}{d} > \frac{e}{d}$$

We have seen in the proof of the last proposition that $H^1(X, L) = 0$. Hence $H^1(C, L_C) = 0$. Since $\chi(L_C^m) = em - g_C + 1$, it follows from the above inequality that

$$\begin{aligned}
\frac{e - g_C + 1}{d - g + 1} = \frac{h^o(C, L_C)}{d - g + 1} > \frac{e}{d} &\Rightarrow de + d(1 - g_C) > de + e(1 - g)\\
&\Rightarrow e(g - 1) > d(g_C - 1) \Rightarrow e > d, (\because g_C = g \geq 2)
\end{aligned}$$

This contradiction concludes the proof.

Theorem 1.0.1. is now completely proved. Since every connected curve in the family $Z_V \xrightarrow{p_V} V$ is reduced, it can be easily seen that there exists an open (and closed) subscheme U of V parametrizing all the connected curves in the family $Z_V \xrightarrow{p_V} V$ i.e. if $h \in V$ such that the fibre X_h of p_V over h is connected then $h \in U \subset V$. By restricting the morphism p_V to $p_V^{-1}(U)$ we get a family $Z_U \xrightarrow{p_U} U$ of connected curves.

Let C be a complete, connected subcurve of a curve X in the family $Z_U \xrightarrow{p_U} U$. Let C' be the closure of $X - C$ in X. Let $\pi : \overline{X} \to X$ be the normalization of X and let $P_1, P_2, \ldots, P_k$ be all the points on $\overline{C'}$ (normalization of C') such that $\pi(P_i) \in C \cap C'$.

83 Assume that the following condition is satisfied.

i) For every irreducible component C'_j of C',

$$\deg_{\overline{C}_j} L' \geq \#(\overline{C}_{j'} \cap \{P_1, P_2, \ldots, P_k\}).$$

In this situation we proved the following inequality (cf. page 57).

$$\frac{h^o(C, L_C)}{d - g + 1} \geq \frac{e_C + \frac{k}{2}}{d} \qquad (e_C = \deg_C L) \qquad (*')$$

Note that if $C = X$ then the above inequality is trivially satisfied.

We want to prove that the above inequality $(*')$ holds for every complete, connected subcurve C of every curve X in the family $Z_U \xrightarrow{p_U} U$, even if condition i) above is not satisfied. We prove the result by contradiction. So let X be a curve in the family $Z_U \xrightarrow{p_U} U$ and let C be a complete connected subcurve of X for which the inequality $(*')$ is not satisfied. Hence,

$$\frac{e_C - g_C + 1}{d - g + 1} = \frac{h^o(C, L_C)}{d - g + 1} < \frac{e_C + \frac{k}{2}}{d} \qquad (1)$$

($e_C = \deg_C L$, $k = \#(C \cap C')$, C' is the closure of $X - C$ in X)
We may assume that C is maximal in the sense that for every complete, connected subcurve C' of X, with $C \subsetneq C' \subset X$ the inequality $(*')$ holds.

Since the inequality $(*')$ does not hold for C, condition i) above is not satisfied i.e., for some irreducible component $\overline{C'_j}$ of $\overline{C'}$, $\deg_{\overline{C}_j} L <$
84 $\#(C \cap C_j) = \ell'$. Then $Y = C \cup C'_j$ is connected and the inequality $(*')$ holds for Y.

$$\frac{h^o(Y, L_Y)}{d - g + 1} \geq \frac{e_C + e'_{C_j} + \frac{k'}{2}}{d},$$

$(e_{C'_j} = \deg_{C'_j} L, k' = \#(Y \cap Y')$, Y' is the closure of $X - Y$ in X)

$$\Rightarrow \frac{e_C + e_{C'_j} - g_Y + 1}{d - g + 1} \geq \frac{e_C + e_{C'_j} + \frac{k'}{2}}{d}$$

$(\because \quad \chi(L_Y^m) = (e_C + e_{C'_j})m - g_Y + 1$ and $H^1(Y, L_Y) = 0)$

$$\Rightarrow \frac{e_C + e_{C'_j} - g_C - g_{C'_j} - \ell' + 2}{d - g + 1} \geq \frac{e_C + e_{C'_j} + \frac{k}{2} + \frac{k'' - \ell'}{2}}{d} \qquad (2)$$

$$(k'' = \#(C'_j \cap Y'), \ell' = \#(C \cap C'_j))$$

The last inequality follows from the following formula

$$g_Y = g_C + g_{C'_j} + \#(C \cap C'_j) - 1$$

Multiply the inequality (1) by -1 and add it to the inequality (2).

$$\frac{e_{C'_j} - g_{C'_j} - \ell' + 1}{d - g + 1} > \frac{e_{C'_j} + \frac{k'' - \ell'}{2}}{d} \qquad (3)$$

$$\begin{aligned}
&\Rightarrow e_{C'_j} - g_{C'_j} - \ell' + 1 > (\frac{d - g + 1}{d})(e_{C'_j} + \frac{k'' - \ell'}{2}) \\
&= (1 - \frac{g - 1}{d})(e_{C'_j} + \frac{k'' - \ell'}{2}) > \frac{19}{20}(e_{C'_j} + \frac{k'' - \ell'}{2}) \\
&\qquad (\because d \geq 20(g - 1)) \\
&\Rightarrow 20(e_{C'_j} - g_{C'_j} - \ell' + 1) > 19(e_{C'_j} + \frac{k'' - \ell'}{2}) \\
&\Rightarrow 20 > \frac{21\ell'}{2} - e_{C'_j} + 20g_{C'_j} + \frac{19k''}{2} > \frac{19}{2}\ell' + 20g_{C'_j} + \frac{19}{2}k'' \\
&\qquad (\because e_{C'_j} < \ell') \\
&\Rightarrow \ell' = 2, g_{C'_j} = 0, k'' = 0, \qquad (\because \ell' \geq 2)
\end{aligned}$$

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Since $e_{C'_j} < \ell'$ it follows that $e_{C'_j} = 1$.
Then the inequality (3) reads as $0 > 0$, a contradiction! This proves that the inequality $(*')$ holds for C. It is easy to see that the inequality $(*')$ holds for C even if C is not connected.
Thus we have the following proposition.

Proposition 1.0.10. *Let X be a curve in the family $Z_U \xrightarrow{p_U} U$, C be a complete, subcurve of X. Let $k = \#(C \cap C')$ (C' is the closure of $X - C$ in X). Then the following inequality holds.*

$$\frac{h^o(C, L_C)}{d - g + 1} \geq \frac{e_C + \frac{k}{2}}{d}, \qquad (e_C = \deg_C L)$$

The above result can also be stated in the following form.

Proposition 1.0.11. *Let X be a curve in the family $Z_U \xrightarrow{p_U} U$. Let ω_X be the dualizing sheaf of X. Put $\beta = \dfrac{d}{\deg \omega_X}$. For every complete subcurve C of X, we have,*

$$|\deg_C L - \beta \deg_C \omega_X| \leq \frac{k}{2} \qquad (*'')$$

($k = \#(C \cap C')$, C' is the closure of $X - C$ in X).

86 *Proof.* Since the inequality $(*')$ holds for C, simplifying we get,

$$\begin{aligned} \deg_C L = e_C &\geq \frac{d}{g-1}\left(g_C - 1 + \frac{k}{2}\right) - \frac{k}{2} = \frac{d}{2g-1}(2g_C - 2 + k) - \frac{k}{2} \\ &= \beta \deg_C \omega_X - \frac{k}{2} \qquad (\because \deg_C \omega_X = 2g_C - 2 + k) \\ &\Rightarrow \deg_C L - \beta \deg_C \omega_X \geq -\frac{k}{2}. \qquad (1) \end{aligned}$$

Since the inequality $(*')$ holds for C', we get, as above

$$\begin{aligned} \deg'_C L - \beta \deg'_C \omega_X &\geq -\frac{k}{2} \\ \Rightarrow \deg_C L - \beta \deg_C \omega_X &\leq \frac{k}{2} \qquad (2) \end{aligned}$$

$(\because \deg'_C L - \beta \deg'_C \omega_X = \beta \deg_C \omega_X - \deg_C L)$

The result follows from (1) and (2). □

Next proposition is an easy consequence of proposition 1.0.10.

Proposition 1.0.12. *Let X be a curve in the family $Z_U \xrightarrow{p_U} U$ and let $C' \subsetneqq X$ be a connected chain of smooth rational curves meeting C (closure of $X - C'$ in X) in two points. Then, i) C' is irreducible, ii)* $\deg'_C L = 1$.

Proof. In view of proposition 1.0.10 (page 66) we have the following inequality.

$$\frac{e_C - g_C + 1}{d - g + 1} = \frac{h^o(C, L_C)}{d - g + 1} \geq \frac{e_C + 1}{d}, \quad (e_C = \deg_C L)$$
$$\Rightarrow de_C - dg_C + d \geq de_C + d + (1 - g)\chi(e_C + 1)$$
$$\Rightarrow dg_C \leq (g - 1)(e_C + 1) \Rightarrow d \leq e_C + 1 \quad (\because g - 1 = g_C \geq 1)$$
$$\Rightarrow e_C = d - 1 \Rightarrow \deg_{C'} L = 1 \Rightarrow C' \text{ is irreducible}$$

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Now we make the following observation. Let X be a curve in the family $Z_U \xrightarrow{p_U} U$ and let ω_X be the dualizing sheaf of X. ω_X is a line bundle (X being Gorenstein) and ω_X^3 gives a morphism, $\psi_o : X \to \mathbb{P}^r$. The above proposition implies that the image X' of X under ψ_o is a stable curve and the fibre $X_{x'}$ of ψ_o over a point $x' \in X$ is either a point or $X_{x'} \simeq \mathbb{P}^1$ and $X_{x'}$ meets the rest of the curve in two points. X' is thus obtained by replacing each smooth rational component of X, meeting the rest of the curve in two points, by a node. □

Chapter 2

The Moduli Space of Curves

In this chapter we construct the Deligne-Mumford Moduli space of stable curves. We prove that it is a reduced and irreducible scheme projective over Spec K. **88**

We keep the same notations as in Chapter 1. Thus $Z_U \xrightarrow{p_U} U$ is a family of connected curves of genus $g \geq 2$ and degree $d \geq 20(g-1)$ in $\mathbb{P}^N$, $(N = d - g)$, such that the m_o^{th} Hilbert point of X, $H_{m_o}(X) \in \mathbb{P}(\overset{P(m_o)}{\wedge} H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m_o)))$ is semistable for the natural action of $SL(N+1)$ on $\mathbb{P}(\overset{P(m_o)}{\wedge} H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m_o)))$. Assume now onwards that $d = n(2g-2)$ where n is an integer, $n \geq 10$.

For a geometric point $h \in U$ let X_h be the fibre of $Z_U \xrightarrow{p_U} U$ over h, L_h be the restriction of $O_{\mathbb{P}^N}(1)$ to X_h, ω_{X_h} be the dualizing sheaf of X_h. It is easily seen that the set $U_C = \{h \in U | L_h \simeq \omega_{X_h}^n\}$ is constructable. We want to prove that U_C is a closed subscheme of U parametrizing all curves X_h in the family $Z_U \xrightarrow{p_U} U$ such that $L_h \simeq \omega_{X_h}^n$. The next proposition proves that U_C is a closed subset of U.

Proposition 2.0.0. *U_C is a closed subset of U.*

Proof. It suffices to prove the following. For every morphism Spec $R \xrightarrow{\alpha} U$, (R a discrete valuation ring), if the image of the generic point of Spec R is in U_C then the image of the special point of Spec R is also in U_C. □

89 Make the base change of $Z_U \xrightarrow{p_U} U$ by Spec $R \xrightarrow{\alpha} U$,

$$\begin{array}{ccc} Z_R & \longrightarrow & Z_U \\ \downarrow{\scriptstyle p_R} & & \downarrow{\scriptstyle p_U} \\ \operatorname{Spec} R & \xrightarrow{\alpha} & U \end{array}$$

The relatively very ample line bundle $O_{Z_U}(1)$ on Z_U induces a line bundle $O_{Z_R}(1)$ on Z_R. Let $\omega_{Z_R/R}$ be the relative dualizing sheaf on Z_R. Let h_o and h_1 be respectively, the special point and the generic point special point of spec R, and let X_{h_o} and X_{h_1} be the special and generic fibres. Write $X_{h_o} = \bigcup_{i=1}^{q'} C_i$ where C_i is an irreducible component of X_{h_o}. The restrictions of $O_{Z_R}(1)$ and $\omega^n_{Z_{R/R}}$ to X_{h_1} are isomorphic. This follows from the definition of U_C. To prove that $\alpha(h_o) \in U_C$ is equivalent to showing that the restrictions of the line bundles $\omega^n_{Z_{R/R}}$ and $O_{Z_R}(1)$ to X_{h_o} are isomorphic.

Write $O_{Z_R}(1) \simeq \omega^n_{Z_{R/R}} \otimes M$, where M is a line bundle on Z_R which on Z_R − {nodes of X_{h_o}} is of the form $O_{Z_R}(-\sum_{i=1}^{q'} r_i C_i)$. Let t be the uniformizing parameter of R. Tensoring $O_{Z_R}(1)$ with the trivial line bundle associated to the principal divisor $p_R^*((t^{\min(r_i)})$ we can assume that $r_i \geq 0$, $\min(r_i) = 0$ i.e. we can assume that M is an ideal sheaf.

Let $J = \bigcup_{r_i>0} C_i$, $J' = \bigcup_{r_i=0} C_i$. If g is the local equation of M then $g \not\equiv 0$ in any component of J' and $g(x) = 0$ for all $x \in J \cap J'$. Hence $\#(J \cap J') \leq \deg'_J M$. However, we have, $|\deg'_J M| = |\deg'_J L - n \deg'_J \omega_{X_{h_o}}| \leq$
90 $\# \dfrac{(J \cap J')}{2}$,

(cf. proposition 1.0.11, page 66). This forces $J' = X_{h_o}$ i.e. M is trivial. This proves the result.

Recall that U_C is precisely the set of points $h \in U$ such that the restriction of the line bundle $\omega^n_{Z_U/U} \otimes O_{Z_U}(-1)$ to the fibre of $Z_U \xrightarrow{p_U} U$ over h is trivial. Now using standard arguments (cf. [4], page 89) U_C is given the structure of a closed subscheme of U, having the following properties,

i) There exists a line bundle M' on U_C such that the restriction of $\omega^n_{Z_U/U} \otimes O_{Z_U}(-1)$ to $Z_{U_C} \times_U Z_U$ is isomorphic to $p^*_{U_C}(M')$, (p_{U_C} denotes the projection $Z_{U_C} \to U_C$);

ii) If $f : W \to U$ is any morphism such that for some line bundle M'' on W the line bundles $(1 \times f)^*(\omega^n_{Z_U/U} \times O_{Z_U}(-1))$ and $p^*_W(M'')$ on $W \underset{U}{\times} Z_U$ are isomorphic then f can be factored as $W \to U_C \to U$.

Theorem 2.0.1. *U_C is nonsingular.*

Proof. Let $h \in U_C$ be a closed point and let X_h be the fibre of $Z_{U_C} \xrightarrow{p_{U_C}} U_C$ over h. Let $\tilde{X}$ be the universal formal deformation of $X_h$over $T = \text{Spec}[[t_1, t_2, \ldots, t_3]], (s = \dim Ext^1(\Omega_{X_h/K}, O_{X_h}))$, (cf. theorem 0.2.2, page 12). ◻

Let $\eta : S = \text{Spec}\hat{O}_{U_{C,h}} \to U_C$ be the natural map. We have the following commutative diagram.

$$\begin{array}{ccc} Z_{U_{C,h}} & \longrightarrow & Z_{U_C} \\ \downarrow & & \downarrow \\ S & \xrightarrow{\ \eta\ } & U_C \end{array}$$

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It follows from theorem 0.2.2. (page 12) that there exists a unique morphism $f : S \to T$ such that $Z_{U_{C,h}} \simeq S \underset{T}{\times} \tilde{X}$ and the isomorphism restricted to the closed fibres is the identity morphism.

Claim: $f : S \to T$ is formally smooth , i.e.

$\hat{O}_{U_{C,h}} \simeq [[t_1, t_2, \ldots, t_s, t_{s+1}, \ldots, t_{s+s}]]$ for some nonnegative integer s'.

Note that if we prove the claim, the result will follow. Choose an isomorphism $\mathbb{P}(\pi_*(\omega^n_{X/T})) \simeq \mathbb{P}^{n(2g-2)-g} \times T$, (cf. stable curves, page 10). By the universal property of U_C we have a unique morphism $\gamma : T \to$

U_C such that the following diagram is commutative.

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & Z_{U_C} \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\gamma} & U_C \end{array}$$

Clearly γ has a factorization
92 $T \xrightarrow{\gamma} S \xrightarrow{\eta} C$ and γ' is a section of f.

Recall that $G = PGL(N+1)$ acts on V_C. Let S_h be the stabilizer of $h \in U_C$. Then S_h is finite and reduced. In fact if it were not then S_h would have a nonzero tangent vector i.e. there would be a $\frac{K[\varepsilon]}{(\varepsilon^2)}$ valued point of G centered at the identity which gives an automorphism of $X_h \times \frac{K[\varepsilon]}{\varepsilon^2} \subset \mathbb{P}^{n(2g-2)-g} \times \frac{K[\varepsilon]}{\varepsilon^2}$, hence a vector field defined on the whole of X_h. We have already seen that such a vector field is necessarily zero (cf. lemma 0.1.7. page 10). Thus the automorphism of $X_h \times \frac{K[\varepsilon]}{(\varepsilon^2)}$ must be identity and further since X_h is connected and nondegenerate in $\mathbb{P}^N$ (cf. proposition 1.0.2. page 27) the automorphism $\mathbb{P} \times \frac{K[\varepsilon]}{(\varepsilon^2)}$ must be identity. It follows that S_h is finite and reduced and hence the action of G on S is formally free which amounts to saying that S is formally a principal fibre bundle over T with group G. Therefore S is formally smooth over T.

Further U_C is contained in the open subset of U, parametrizing stable curves. To see this let $h \in U_C$ such that the fibre X_h of $Z_U \xrightarrow{p_U} U$ over h is semistable but not stable i.e. Y is a smooth rational component of X_h which meets the other components of X_h in exactly two points. But then the restriction of $\omega^n_{X_h}$ to Y is very ample and $\deg_Y \omega^n_{X_h} = n \deg_Y \omega_{X_h} = n(\deg \omega_Y + \#(Y \bigcap Y') = 0$,
(Y' is the closure of $X_h - Y$ in X_h).
93 This contradiction proves that X_h is necessarily a stable curve.

Recall that the morphism $U \to \mathbb{P}(\overset{P(m_o)}{\Lambda} H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m_o)))^{ss}$ defined by $\psi_{m_o}(h) = H_{m_o}(X_h)(h \in U)$ is a closed immersion and also it is a G-morphism. By Theorem 0.0.5 (page 3) $\mathbb{P}(\overset{P(m_o)}{\Lambda} H^o(\mathbb{P}^N, O_{\mathbb{P}^N}(m_o)))^{ss}$ has a good quotient by G which is projective. Then it is easy to see that U_C has a good quotient by G (denoted by U_C/G) which is projective. We are now ready to state the main theorem of this chapter.

Theorem 2.0.2. *U_C/G is a coarse moduli space of isomorphism classes of stable curves of genus g. Further U_C/G is reduced and irreducible.*

Proof. We first prove that every stable curve (in the sense of Definition 0.1.4. page 8) is represented in U_C/G. □

Let X be a stable curve of genus g. Let $\gamma : X' \to \operatorname{Spec} R$ be a deformation of X to a connected, smooth curve where R is a discrete valuation such that, K_1, the residue field of R is algebraically closed. Let K_2 be the quotient field of R. Note $X' \to \operatorname{Spec} R$ is a stable curve and hence can be realized as a family of curves in $\mathbb{P}^{n(2g-2)-g}$ (cf. Stable curves, page 8). Then by the universal property of the Hilbert scheme H we get a morphism $\rho : \operatorname{Spec} R \to H$. Since the generic fibre X'_{K_2} of γ is smooth, the image of $\operatorname{Spec} K_2 \subset \operatorname{Spec} R$ lies in the locally closed subscheme U_C of H (cf. Theorem 1.0.0 page 19). Since U_C/G is complete, (cf. Definition 4.1. page 526 [10]) there exists a morphism $\rho' : \operatorname{Spec} R \to U_C$ **94**
such that if $X'' \to \operatorname{Spec} R$ is the base change of $Z_{U_C} \to U_C$ by the morphism $\rho' : \operatorname{Spec} R \to U_C$ then the generic fibre X''_{K_2} of $X'' \to \operatorname{Spec} R$ is isomorphic X'_{K_2}. Now note the following lemma.

Lemma 2.0.3. *Let Y' and Y'' be two stable curves over a discrete valuation ring R with algebraically closed residue field. Let Y be the generic point of* Spec R *and assume that the generic fibres Y'_Y and Y''_Y are smooth. Then any isomorphism between Y'_Y and Y''_Y extends to an isomorphism between Y' and Y''. (cf. Lemma 1.12 [1])*

In view of the above lemma, it follows that the isomorphism between X'_{K_2} and X''_{K_2} can be extended to an isomorphism of X' and X'' over $\operatorname{Spec} R$. Observe that the isomorphism between X'_{K_1} and X''_{K_2} is induced by an automorphism of $\mathbb{P}^{n(2g-2)-g}$. Thus in the Hilbert scheme H, the two points representing the curves X'_{K_1} and X''_{K_1} lie in the same G-orbit. Now it is immediate that the morphism $\rho : \operatorname{Spec} R \to H$ factors as $\operatorname{Spec} R \to U_C \to H$.

Thus every stable curve is represented in U_C. Since G acts on U_C with finite isotropy (cf. Theorem 0.1.8. page 10) and with closed orbits (cf. Lemma 2.0.3) the good quotient U_C/G of U_C by G is a coarse moduli space for isomorphism classes of stable curves.

It remains to prove that U_C/G is reduced and irreducible. We know this to be true when characteristic of the ground field K is zero, [11]. So assume now that characteristic of K is positive.

95 Let R be a discrete valuation ring such that the quotient field of R has characteristic zero and K is the residue field of R.

Construct U_C over $\operatorname{Spec} R$ and call it $U_{C,R'}$ (Note that the method of our proof works over the base $\operatorname{Spec} R$, cf. [9]). Let G_R be the group $PGL(N+1)(R)$. Since $U_{C,R}/G_R$ is projective and the generic fibre of $U_{C,R}/G_R \to \operatorname{Spec} R$ is connected, Zariski's connectedness theorem shows that $U_{C,R}/G_R \otimes K$ is connected. Note that $U_{C,R}/G_R \otimes K = U_{C,R} \otimes K/G$ is just the orbit space, hence $U_{C,R} \otimes K = U_C$ is connected. We have already seen that U_C is smooth. Thus U_C is reduced and irreducible. Recall that the structure sheaf of U_C/G is the sheaf of invariants in the structure sheaf of U_C. Hence U_C/G is reduced and irreducible.

Appendix

Let X' be a reduced, complete, connected curve which has at most ordinary double points. Write $X' = \bigcup_{i=1}^{n} X'_i$ (X'_i an irreducible component of **96**
X'). Let L' be a line bundle on X' and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be positive rational numbers with $\sum \lambda_i = 1$. Following Oda-Seshadri [8] we say that the line bundle L' is (λ_i)-semistable, if for every complete, connected subcurve C of X', $\dfrac{\chi(L'_C)}{\chi(L')} \geq \sum_{X'_i \subset C} \lambda_i$, where the summation is taken over all i such that $X'_i \subset C$.

Now let X be a stable curve in the family $Z_U \xrightarrow{p_U} U$, (Notation as in Chapter 1, cf. page 64). Write $X = \bigcup_{i=1}^{n} X_i$, ($X_i$ an irreducible component of X). Let ω_X be the dualizing sheaf of X. Let $\lambda_i = \dfrac{\deg_{X_i} \omega_X}{\deg \omega_X} (1 \leq i \leq n)$. Let L be the very ample line bundle on X.

Proposition. Al. *The line bundle L is (λ_i)-semistable.*

Proof. For every complete, connected subcurve C of X, we have

$$\frac{e_C - g_C + 1}{d - g + 1} = \frac{X(L_C)}{X(L)} \geq \frac{e_C + \frac{k}{2}}{d}$$

($e_C = \deg_C L$, $k = \#(C \cap C')$, C' is the closure of $X - C$ in X), (cf. Proposition 1.0.10., page 66).

$$\Rightarrow \qquad de_C + d(1 - g_C) \geq de_C + \frac{dk}{2} + (1 - g)(e_C + \frac{k}{2})$$

$$\Rightarrow \quad (e_C + \frac{k}{2})(g-1) \geq d(g_C - 1 + \frac{k}{2}) = \frac{d}{2}(2g_C - 2 + k) = \frac{d(deg_C \omega_X)}{2}$$

$$\Rightarrow \quad \frac{e_C + \frac{k}{2}}{d} \geq \frac{deg_C \omega_X}{2g-2} = \sum_{X_i \subset C} \lambda_i$$

$$\Rightarrow \quad \frac{X(L_C)}{X(L)} \geq \sum_{X_i \subset C} \lambda_i$$

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For a complete subcurve Y of X, we apply the above inequality to each connected component C of X and by adding we get the result. □

Bibliography

[1] P. Deligne, D. Mumford The Irreducibility of the Space of Curves of Given Genus Publications Mathematiques, IHES 36 (1969), pages 75-110. **98**

A. Grothendieck

[2] Fondements de la geometrie algebrique

[Extraits du Seminaire Bourbaki 1957-1962]

[3] Techniques de descente et theoremes d'existence en geometrie algebrique, IV, Sem. Bourbaki, 221, 1960-1961.

David Mumford

[4] Abelian Varieties, Oxford University Press. (1974)

[5] Geometric Invariant Theory, Springer-Verlag (1965).

[6] Lectures on Curves on an Algebraic Surface.

Annals of Mathematics Studies, Number 59, Princeton University Press, (1966).

[7] Stability of Projective Varieties
L'Enseiqnement Mathematique
II^e Serie

Tome XXIII-Fascicule 1-2, 1977

pages 39-110.

Oda-Seshadri

[8] Compactifications of the Generalized Jacobian Variety, Transactions of the American Mathematical Society, Vol.253, 1979. pages 1-90.

99 C.S. Seshadri

[9] Geometric Reductivity over Arbitrary Base.

Advances in Mathematics, Vol.26, No.3, pages 225-274.

[10] Quotient Spaces Modulo Reductive Algebraic Groups.

Annals of Mathematics, Vol.95, No.3, pages 511-556.

André Weil

[11] Modules des surfaces de Riemann, Seminaire Bourbaki, exposé 168, 1958.