

**Lectures on
Nonlinear Waves And Shocks**

**By
Cathleen S. Morawetz**

**Tata Institute Of Fundamental Research
Bombay
1981**

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Nonlinear Waves And Shocks**

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Cathleen S. Morawetz**

Lectures delivered at the
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**Notes by
P.S. Datti**

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Preface

THESE LECTURE NOTES are the outcome of a six week course with the Bangalore applied mathematics group of the Tata Institute. It is a pleasure to recall the enthusiasm and energy of the participants. The lectures were on introduction to certain aspects of gas dynamics concentrating on some of the currently most important nonlinear problems, important not only from the engineering or computational point of view but also because they offer great mathematical challenges. The notes, I hope, touch on both these aspects.

I am indebted to Professor K.G. Ramanathan for inviting me in the first place and for making my visit so enjoyable and stimulating. To P.S. Datti, who cooperated in writing these notes and worked and reworked them, go my very deep thanks. I would also like to thank P.P. Gopalakrishnan who reproduced them so well.

Cathleen S. Morawetz

New York
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Contents

Preface	v
1 The Traffic problem and a first order nonlinear equation	1
1.1 Introduction	1
1.2 Weak Solutions	6
1.3 Initial value problem	7
1.4 Initial value problem with shock	14
1.5 Modifications by diffusion and dissipation	25
1.6 Propagation of singularities in derivatives	26
1.7 Computing methods	28
2 One Dimensional gas dynamics	41
2.1 Equations of motion	41
2.2 Thermodynamical relations. Entropy:	43
2.3 One dimensional flow	45
2.4 Shock conditions	49
2.5 Contact Discontinuities	53
2.6 Shock Reflection	53
2.7 Hugoniot Curve. Shock Determinacy	55
2.8 Riemann Problem	60
2.9 Solution of initial value problem	63
2.10 Combustion. Detonations and deflagrations	71
2.11 Riemann problem with detonations and deflagrations	76
2.12 Internal mechanism	78

3	Two dimensional steady flow	81
3.1	Equations of motion	81
3.2	Classifications of flow equations	84
3.3	Supersonic Flow	85
3.4	Shock polar	86
3.5	Equations in the hodograph plane	89
3.6	Small Disturbance equation	92
3.7	Transonic flow	93
3.8	General theory of boundary value...	94
3.9	The boundary value problems of transonic wing flow . . .	101
3.10	Perturbation boundary value problem	104
3.11	Design by the method of complex characteristics	106
3.12	Numerical solution with shocks: Off...	110
3.13	Nozzle flow	113
	Bibliography	115

Chapter 1

The Traffic problem and a first order nonlinear equation

1.1 Introduction

Many physical laws occur as conservation laws. The most general such law in its differentiated form is given by

$$Y_t + (F(Y))_x = 0$$

where $Y = (Y_1, \dots, Y_n)$ is a vector valued function of x, t with $x \in \mathbb{R}^n$ and $F(Y)$ is a matrix valued function of Y . The term *conservation* comes from the fact that if $F(Y) \rightarrow 0$ as $|x| \rightarrow \infty$ then $\int Y dx$ is constant for all time, that is, these integrals are conserved.

In these notes we shall consider conservation laws in only two independent variables, either time and one space variable or two space variables.

The simplest case to treat is that of the single conservation equation with the variables t, x . Then $Y, F(Y)$ are scalar. We replace Y by N and consider

$$N_t + (F(N))_x = 0, \quad -\infty < x < \infty, \quad t > 0 \quad (1.1)$$

with $N(x, t) \in \mathbb{R}$.

We shall now formulate the traffic problem, first proposed, by Lighthill and Whitham [26]. Let $N(x, t)$ denote the density, the number of vehicles passing through the position x at time t on a highway. Let $u(x, t)$ be the average (local) velocity of the vehicles. Then in any section $[x_1, x_2]$ the *conservation equation* states that the total number of cars is preserved, or

$$\int_{x_1}^{x_2} N(x, t_2) dx - \int_{x_1}^{x_2} N(x, t_1) dx - \int_{t_1}^{t_1} N(x_1, t) u(x_1, t) dt + \\ + \int_{t_1}^{t_2} N(x_2, t) u(x_2, t) dt = 0.$$

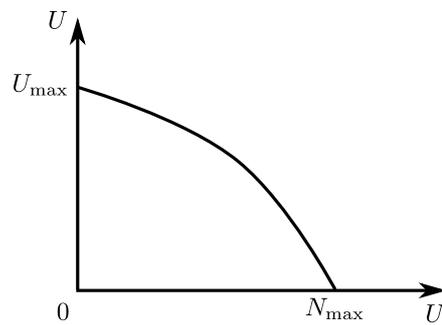
- 2 Assuming the quantities N, u to be smooth, we obtain in the limit $t_1 \rightarrow t_2$, that

$$\int_{x_1}^{x_2} N_t(x, t) dx + [Nu]_{x_1}^{x_2} = 0.$$

This is the integrated form of the conservation equation. As $x_1 \rightarrow x_2$, we obtain

$$N_t + (Nu)_x = 0. \quad (1.2)$$

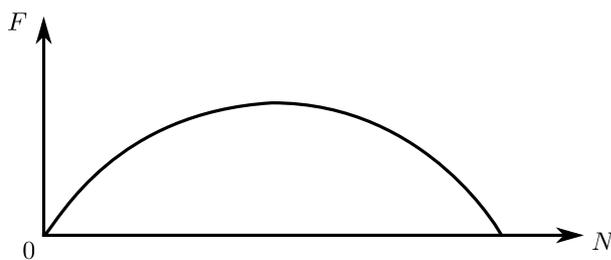
We can always put this equation in the form of (1.1) if we use $u = U(N)$. This assumption seems to be reasonable since drivers are supposed to increase or decrease their speed as the density N decreases or increases respectively. The maximum value of u occurs when $N = 0$ (the maximum is, say, the maximum allowed speed). And when N is maximum $u = 0$. Hence the graph of U plotted against N takes the form shown in figure 1.1.

**Fig. 1.1.**

Rewriting (1.2) as

$$N_t + (F(N))_x = 0,$$

where $F(N) = NU(N)$ is the flux of cars, we see that $F = 0$ when $N = 0$ and when N is maximum (in this case $U(N) = 0$). Hence the graph of the flux curve $F(N)$ will look as in figure 1.2. It could have the shapes as shown in figure 1.2 (a) and 1.2 (b), but these lead to certain difficulties in the theory. 3

**Fig. 1.2.**

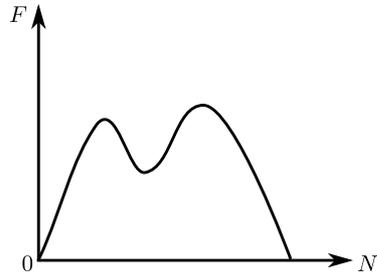


Fig. 1.2 (a).

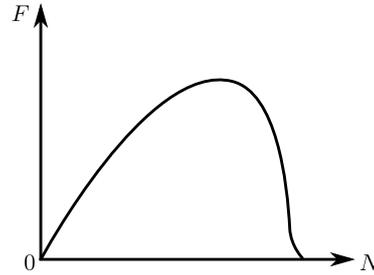


Fig. 1.2 (b).

Consider the simplest case in which the flow is steady, i.e., independent of time. Then from (1.3), we obtain

$$F(N) = \text{constant}.$$

In general, the line $F(N) = \text{constant}$, will cut the flux graph in two points as shown in figure (1.3).

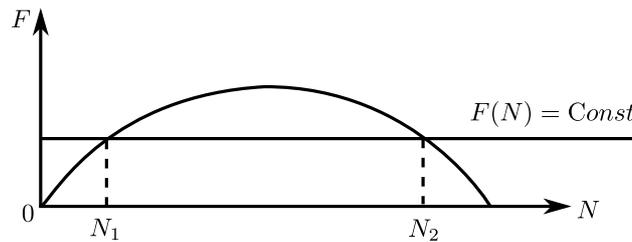


Fig. 1.3.

- 4 If we require continuity in N , then we have to take either $N = N_1$ or $N = N_2$. If we allow jump discontinuities in N , then uniqueness in the solution will be lost as shown in figure (1.4). Then thick line can be a candidate for the solution but the dotted line could also be a candidate for the solution.

So, for uniqueness, we need an additional condition which is called an *Entropy Condition*. The terminology will become clear when we study gas dynamics. Again this has to come from the physics of the problem at hand.

For the traffic problem at hand, we would like to add the *entropy condition* that infinite acceleration is impossible, i.e.

$$\frac{\partial U}{\partial x} < \infty \text{ (or } \frac{\partial N}{\partial x} > -\infty).$$

It turns out that there is then a unique solution to the initial value problem.

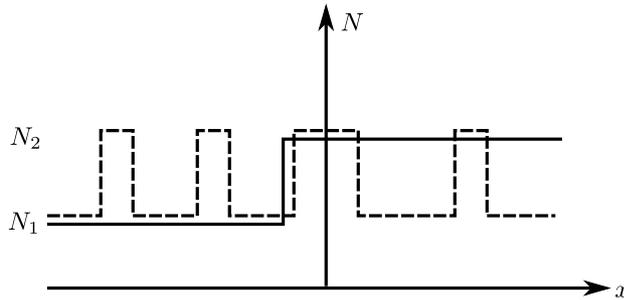


Fig. 1.4.

We also observe that for a given conservation law in differentiated form there are several equivalent conservation laws in differentiated form. For example, consider

$$N_t + (F(N))_x = 0.$$

Let $P(N)$ be any arbitrary integrable function of N . Put

$$Q(N) = \int_{N_*}^N P(M) dM.$$

Then

$$\begin{aligned} Q_t &= Q'(N) \cdot N_t \\ &= P(N) \cdot \left(-\frac{dF}{dN} \cdot N_x\right) \end{aligned}$$

$$= -(R(N))_x, \text{ where } R(N) = \int_{N_*}^N P(M) \cdot F'(M) dM.$$

Thus

$$Q_t + R_x = 0.$$

Hence to choose the *correct entropy condition*, and thus to get *uniqueness*, we should look at the *integrated form of the conservation law*.

If we allow discontinuities in the solution, it is a so-called *weak solution*. We shall now give a precise definition of a weak solution for a general first order nonlinear equation and then return to the traffic problem again.

1.2 Weak Solutions

Consider a first order nonlinear equation

$$N_t + (F(N))_x = 0, \quad -\infty < x < \infty, \quad t > 0. \quad (1.4)$$

- 6 Let N be a classical solution of (1.4). By this we mean that N is a C^1 function in x, t variables and satisfies (1.4) identically. Let now $\chi(x, t)$ be any C^∞ -function (even C^2 is enough) which vanishes for large $|x|$ and t and $t = 0$.

$$\int_{-\infty}^{\infty} \int_0^{\infty} \chi(x, t) \{N_t + (F(N))_x\} dt dx = 0.$$

Integrating by parts, we obtain

$$\int_{-\infty}^{\infty} \int_0^{\infty} (\chi_t N + \chi_x F(N)) dt dx = 0 \quad (1.5)$$

since the boundary terms vanish. Also, the entropy condition $\frac{\partial N}{\partial x} > -\infty$ becomes

$$\int_{-\infty}^{\infty} \int_0^{\infty} \Phi(x, t) \cdot \frac{\partial N}{\partial x} dt dx > -\infty$$

for all $\Phi \geq 0$, C^∞ function vanishing for large $|x|$ and t and $t = 0$. This again by integrating by parts, becomes

$$\int_{-\infty}^{\infty} \int_0^{\infty} \Phi_x N \, dt \, dx < \infty. \quad (1.6)$$

Motivated by this, we now define a weak solution.

Definition. A locally integrable function $N(x, t)$ is a weak solution of (1.4) if (1.5) and (1.6) hold under the conditions stated there.

Note that a classical solution (usually called strong solution) is necessarily a weak solution. Conversely, it can be seen that if N is a weak solution and N is of class C^1 then N is necessarily a strong solution.

1.3 Initial value problem

We now consider the problem of solving a general first order nonlinear equation

$$N_t + (F(N))_x = 0, \quad -\infty < x < \infty, \quad t > 0 \quad (1.7)$$

with initial data

$$N(x, 0) = \Phi(x), \quad -\infty < x < \infty. \quad (1.8)$$

If we put $G(N) = F'(N)$, (1.7) can be written as

$$N_t + G(N)N_x = 0. \quad (1.9)$$

This we solve by the method of characteristics. If we define a curve C in x, t plane by $dx/dt = G(N)$, we find that on C , (1.9) reduces to

$$\frac{dN}{dt} = 0,$$

or $N = \text{constant}$ on $dx/dt = G(N)$. Since this means $G(N)$ is also a constant we see that the characteristics are straight lines with slopes $(G(N))^{-1}$. If $x(0) = \xi$ we find that

$$N(x, t) = \Phi(\xi) \text{ on } dx/dt = G(N)$$

with $x(0) = \xi$.

Hence we have a solution in implicit form given by

$$\left. \begin{aligned} x &= \xi + tG(\Phi(\xi)) \\ N(x, t) &= \Phi(\xi) \end{aligned} \right\} \quad (1.10)$$

If we can determine $\xi = \xi(x, t)$ from the first equation, then we know N at (x, t) uniquely. But, however, this is not always the case; and trouble occurs if the characteristics intersect, as we shall see now. If $\xi = R$, $\xi = L$ are two points on the x -axis such that

$$G(\Phi(L)) > G(\Phi(R)) > 0$$

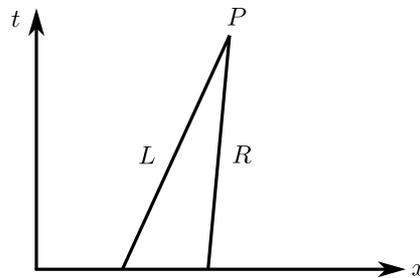


Fig. 1.5.

8

(See figure (1.5)) then the characteristic through R intersects the characteristic through L at P . Thus the value of ξ corresponding to the point P is *not* unique.

However, if the characteristics fan out, then clearly there will be a unique characteristic through every point and the solution will be determined uniquely. If

$$G(\Phi(L)) < G(\Phi(R)) \quad (1.11)$$

for $L < R$ then the characteristics in the x, t plane have decreasing slopes and they never intersect. In such a case, we obtain a unique continuous solution.

Since the solution is given implicitly by (1.10) another way of looking at the breakdown, when (1.11) does not hold is to try to solve the

first equation in (1.10) using implicit function theorem. the functional relation

$$f(x, t, \xi) = 0$$

can be represented in a single-valued way by

$$\xi = g(x, t)$$

if and only if $f_\xi \neq 0$. Put

$$f(x, t, \xi) = tG(\Phi(\xi)) - (x - \xi).$$

Then

$$f_\xi = t \cdot \frac{dG}{dN} \cdot \frac{d\Phi}{d\xi} + 1.$$

Hence, if

$$\frac{dG}{dN} \cdot \frac{d\Phi}{d\xi}$$

is always positive for all positive t we have $f_\xi > 0$. On the other hand, if this expression changes sign, there is a finite time, called breaking time, at which $f_\xi = 0$. We note from the relations

$$N_t = -\frac{\Phi' \cdot G(N)}{1 + t\Phi'G'}, \quad N_x = \frac{G(N)}{1 + t\Phi'G'}$$

that the derivatives in N will blow up at the time of breaking. We also note that the breakdown in the solution can happen even if the initial data are very smooth. Suppose the flux curve is convex, so that $G' = F'' < 0$. Now, if the initial data Φ is very smooth and tends to zero as $|x| \rightarrow \infty$, then Φ' will change its sign and there is always a time at which the solution will become singular.

This is true in higher dimensions also. John, F. [19] has investigated the question of how late with 3-space variables and a nonlinear equation, a breakdown can occur. Note in the above that as $|\Phi'| \rightarrow 0$ the breaking time $t \rightarrow \infty$. Klainerman [21] has shown for the nonlinear analogue of the wave equation.

$$N_{tt} - N + \sum_{i,j} a_{ij}(\nabla N, N_t) \frac{\partial^2 N}{\partial x_i \partial x_j} = 0$$

that if the number of space variables is greater than or equal to 6, then compact initial data can propagate smoothing for all time. This leaves open the analogous important question in 2 and 3 dimensions. 10

Exercise 1.1. Describe the solution of

$$N_t + (F(N))_x = 0$$

with

$$\begin{aligned} F(N) &= 16 - N & \text{if } N < 4 \\ &= 0 & \text{if } N > 4 \end{aligned}$$

for the initial data

- a) $N(x, 0) = 2 - \tanh x$
 b) $N(x, 0) = 2 - \tanh x$ if $x < 0$
 $= 2 - \frac{1}{2} \tanh x$ if $x > 0$.

Where does the discontinuity in N_x move ?

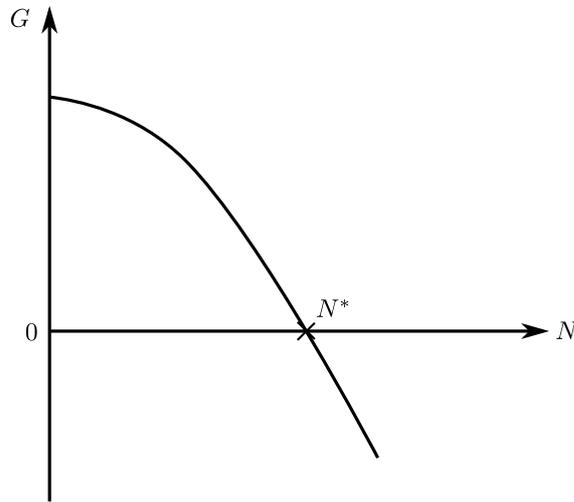
Exercise 1.2. Work out the corresponding theory for

$$U_t + a(U)U_x + b(U)U_y = 0$$

and determine conditions on a and b which lead to discontinuities for all compact initial data.

We now return to our traffic problem and ask how the condition (1.11) should be interpreted as a condition on the density at some given time.

- 11** We refer to the flux curve (see figure (1.2)). Suppose F attains a maximum value at density N_* . Then F is increasing in $[0, N_*]$ and decreasing in $[N_*, N_{\text{Max}}]$. N_{Max} is the maximum density at which $F = 0$. Hence $G(N) = dF/dN$ is decreasing in $[0, N_*]$, equal to zero at N_* and again decreasing (negatively) beyond N_* . Hence the graph of $G(N)$ will look as shown in figure (1.6).

**Fig. 1.6.**

If the traffic problem is accelerating at the given time, i.e. cars to the right going faster than those to the left then N is a decreasing function of x . Thus

$$N(L) > N(R)$$

for all pairs L, R such that $L < R$ and thus

$$G(N(L)) < G(N(R))$$

(unless the density has increased beyond N_*), so that (1.11) holds and we have a unique continuous solution. However, the tail of the $G(N)$ curve for $N > N_*$ was fixed up artificially to give us a smooth curve and we should properly speaking ignore this region.

If the traffic is decelerating then $N(L) < N(R)$ for all pairs L, R with $L < R$ and we will not be able to obtain a continuous solution.

We are then forced back to re-examine our model. The conservation of the number of vehicles still holds, but we can allow discontinuities in density. 12

We have the conservation law

$$\oint_{\epsilon} (Ndx - Fdt) = 0$$

for every closed curve ϵ in the x, t plane or equivalently

$$\frac{d}{dt} \int_{x_1}^{x_2} N(x, t) dx + [F(N)]_{x_1}^{x_2} = 0 \quad (1.12)$$

for every segment $[x_1, x_2]$ if N has jump discontinuities. We ask for piecewise smooth solutions N and investigate what happens across a discontinuity in N as in the steady case. Such a discontinuity is called a *shock* and the curve on which it lies is the *shock curve*.

A discontinuity in N implies, of course, a discontinuity in the speed U . These discontinuities represent instant or rather infinite decelerations. In many situations, the presence of a discontinuity represents a pile-up and to say the least, the model breaks down. However, densities have been defined in an average sense and if the traffic is this enough a sharp deceleration can occur and we can study what happens to it. Ofcourse, the relation of speed to density *inside* this narrow region represented by the discontinuity cannot be the old one.

13 On the other hand, one might also ask, why not allow discontinuities in an accelerating situation ? It is not quite clear that in a traffic situation one should not but again it involves violating the speed density relation in some narrow region and this time we would lose uniqueness.

Let $x = s(t)$ be a C^1 -function representing the shocks curve. We work with the conservation law given by (1.12). Let $x_1 < s(t) < x_2$ at some time. Then from (1.12) we obtain

$$\frac{d}{dt} \int_{x_1}^{s(t)-} N dx + \int_{s(t)+}^{x_2} N dx + F(N)(x_2) - F(N)(x_1) = 0.$$

Differentiating under the integral sign and taking the limits

$$\left. \begin{array}{l} x_1 \rightarrow s(t)- \\ x_2 \rightarrow s(t)+ \end{array} \right\}$$

we obtain

$$-S(N_R - N_L) + (F_R - F_L) = 0,$$

where we denote ds/dt by S and N_R, N_L and F_R, F_L are the corresponding limiting values from right and left respectively. We then have

$$S = \frac{F_R - F_L}{N_R - N_L} \quad (1.13)$$

S is called *shock speed* and $(N_R - N_L)$ *shock strength*. The equation (1.13) is called a *jump condition*.

Exercise 1.3. If N is a piecewise smooth weak solution of $N_t + (F(N))_x = 0$, show that the jump condition holds across the line of discontinuity. Further, show that as the shock strength goes to zero, the shock speed becomes the characteristic speed.

We want to show that by allowing shocks we can solve any initial value problem uniquely and that we can study in particular where the shock travels and how strong it gets. We first look at *constant states*. 14

The simplest problem to solve is the transition from one constant density, say, N_R to another constant density N_L . The deceleration requirement is that $N_R > N_L$. The shock speed, by (1.13), is the slope of the line segment connecting two points on the flux curve; see figure (1.7).

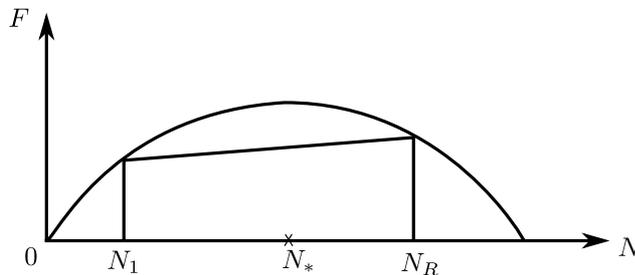


Fig. 1.7.

It is positive or negative depending on the relationship of N_L and N_R to N_* . A more important question is whether it is more or less than the speed of the traffic.

Generally speaking, relative to the traffic ahead at time t , the shock is always retreating, i.e., $S < U_R$. For, since $F = NU$.

$$\begin{aligned} S - U_R &= \frac{F_R - F_L}{N_R - N_L} - U_R \\ &= \frac{N_L(U_R - U_L)}{N_R - N_L} < 0, \text{ since } N_R > N_L. \end{aligned}$$

- 15 Similarly, $S < U_L$. Thus, all traffic ahead of the shock remains ahead but all traffic behind it eventually hits it and decelerates. The path history of cars is illustrated in figure (1.8).

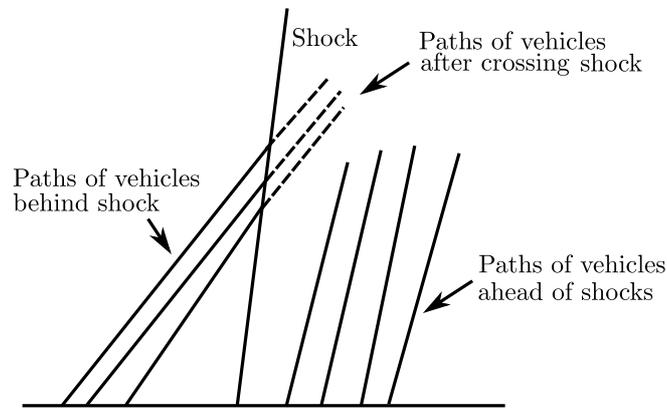


Fig. 1.8.

1.4 Initial value problem with shock

We already know from the constant configuration (and it can be proved generally) that the vehicles after crossing the shock move off at a slower speed. However, the behaviour of characteristics is quite different; they hit the shock from both sides. In the x, t plane the slope of the characteristic satisfies $dx/dt = G(N) = dF/dN$. From the $G - N$ curve (figure (1.7)), we see that not only $G(N_R) < 0$, but $G(N_R) < S$, for $N > N_*$ since $N_R > N_L$ and F is convex. Hence the characteristics ahead of a shock

run into it. Behind the shock the reverse is true, $G(N_L) > S$ and again the characteristics hit the shock (figure (1.9)).

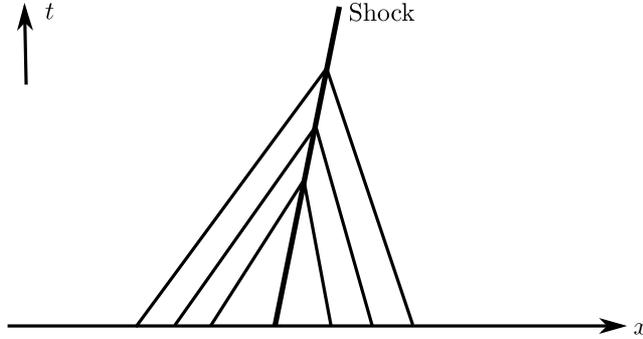


Fig. 1.9.

16

If we knew how to *lay down* the shock across overlapping characteristics, we could solve the *initial value problem*. The shock starts when the continuous method breaks down, i.e., when $\partial N/\partial x$ becomes infinite at some time and plane. Let this, for the sake of argument, be $(0,0)$. Then initial slope of the shock is the characteristic slope and we integrate from this point $(0,0)$. The shock velocity

$$\begin{aligned} \frac{dx}{dt} = S &= \frac{F_R - F_L}{N_R - N_L} \\ &= \frac{F(\Phi(\xi_R)) - F(\Phi(\xi_L))}{\Phi(\xi_R) - \Phi(\xi_L)} \end{aligned}$$

where Φ is the given initial data and $\xi_R(x, t), \xi_L(x, t)$ are the abscissae of the appropriate right and left characteristics. To find this curve, a useful approximation is often made. We use the subscripts R and L in an obvious manner and note two possibilities of getting Taylor expansions for the shock speed in terms of the shock strength $N_R - N_L$.

$$\begin{aligned} S = \frac{F_R - F_L}{N_R - N_L} &= \left(\frac{dF}{dN}\right)_R + \frac{1}{2}\left(\frac{d^2F}{dN^2}\right)_R(N_R - N_L) + O((N_R - N_L)^2) \\ &= \left(\frac{dF}{dN}\right)_L + \frac{1}{2}\left(\frac{d^2F}{dN^2}\right)_L(N_L - N_R) + O((N_L - N_R)^2) \end{aligned}$$

17 as $N_R \rightarrow N_L$. Adding the two expressions,

$$2S = \left(\left(\frac{dF}{dN} \right)_R + \left(\frac{dF}{dN} \right)_L \right) + \frac{1}{2} \left(\left(\frac{d^2F}{dN^2} \right)_R - \left(\frac{d^2F}{dN^2} \right)_L \right) \\ (N_R - N_L) + O((N_R - N_L)^2).$$

or, expanding $\left(\frac{d^2F}{dN^2} \right)_R$ and $\left(\frac{d^2F}{dN^2} \right)_L$ in terms of $(N_R - N_L)$, we see that

$$S = \frac{1}{2} \left(\left(\frac{dF}{dN} \right)_R + \left(\frac{dF}{dN} \right)_L \right) + O((N_R - N_L)^2).$$

This means the shock speed is the average of the slopes of the two characteristics with a second order error. Note, if F is quadratic the formula is exact.

To find the whole flow in the (x, t) -plane, we now have a relatively simple recipe:

1. Carry N along the characteristics coming from the initial line, i.e., lines of slope $(dF/dN)^{-1} = G^{-1}$.
2. Note the region where they are crossed.
3. From the points on each line $t = \text{constant}$ where such a region starts, start a shock.
4. Integrate the differential equation for the shocks,

$$\frac{dx}{dt} = g(x, t),$$

where $g(x, t) = \frac{1}{2}(G_R + G_L)$ is a good approximation.

18 This procedure is quite straight forward until shocks intersect. But there the initial value problem can be restarted provided we can solve the local problem with constant states on each side. Note if initial data have a discontinuity then the two sides can be connected either by a shock or by a degenerate solution of (1.10) with $x = tG(N)$ called a centered wave. However, interaction shows that the number of shocks decreases.

The real numerical difficulty of the foregoing procedure lies in inverting the variable and finding ξ as a function of x, t . For this reason, it is often preferable to use a difference scheme. Before going into this, we now give some existence and uniqueness theorems.

Remarks . We note the difference in shock speed if we use different conservation laws that are equivalent in the smooth case. From

$$N_t + F_x = 0,$$

we have seen that, Q satisfies

$$Q_t + T_x = 0$$

where

$$Q = \int^N P(M)dM$$

and

$$T = \int^N P(M)F'(M)dM.$$

Considering shock speeds, we see that

$$\begin{aligned} S &= \frac{T_R - T_L}{Q_R - Q_L} \\ &= \frac{\int_{N_L}^{N_R} P(N)F'(N)dN}{\int_{N_L}^{N_R} P(N)dN} \end{aligned}$$

In the limits as $N_L \rightarrow N_R$ we see that S tends to the characteristic speed, **19** the same for both the forms. But for $N_R \neq N_L$ we can obtain a wide range of shock speeds by the choice of $P(N)$.

To get uniqueness, as we shall see, we introduce the following *entropy condition* to go with a specific weak conservation law:

The characteristics starting on either side of the discontinuity curve (shock curve) when continued in the direction of increasing time t intersect the line of discontinuity.

This means the characteristics issuing from a shock go backwards in time (see figure (1.9)). This will be the case if

$$F'(N_L) > S > F'(N_R)$$

where N_L, N_R are the values of N on left and right of the shock curve respectively and S is the shock speed.

Hereafter, in this section, by a shock, we mean a discontinuity satisfying the jump condition and this entropy condition. For the following results, which we are going to derive, we refer to P.D. Lax [23]. The most important feature there is the role of the L^1 norm. A piecewise continuous function such as a solution with shocks will have its first derivatives in L^1 but not in L^2 . Consider

$$N_t + F_x = 0, \quad -\infty < x < \infty, \quad t > 0. \quad (1.14)$$

We assume F is convex¹ and a C^2 -function.

20 Exercise 1.4. Let $f : (a, b) \rightarrow \mathbb{R}$ be a C^2 -function. Then f is convex iff $f''(x) > 0$ for all $x \in (a, b)$. Further show that f satisfies the inequality

$$f(x) \geq f(y) + (x - y)f'(y) \quad (1.15)$$

for all $x, y \in (a, b)$.

Theorem . Let N, M be two piecewise continuous solutions of (1.14) whose discontinuities are only shocks. Then $\|N - M\|(t)$ is a decreasing function of t where

$$\|N - M\|(t) = \int_{-\infty}^{\infty} |N(x, t) - M(x, t)| dx.$$

(We assume the integral exists).

¹In the traffic problem F is concave but the problem can be transformed to this case.

Corollary (Uniqueness Theorem) *If $N = M$ at time $t = 0$, then $N \equiv M$.*

Proof. Let $x = y_n(t)$ be the points such that $(N - M)$ has the sign of $(-1)^n$ in $y_n(t) < x < y_{n+1}(t)$. Then

$$\|N - M\|(t) = \sum (-1)^n \int_{y_n}^{y_{n+1}} (N - M) dx.$$

There are two cases:

(i) Suppose $N = M$ on a curve which is not a shock curve of either solution. Then $G(N) = G(M)$. Thus the two sets of characteristics have the same slopes and hence coincide. Hence there is a segment or a point on every line $t = \text{constant}$ where $N = M$. If it is a point, the curve on which $N = M$ is a characteristic; if it is a line segment then the whole region swept out by the characteristics from the segment satisfies $N = M$.

(ii) The curve where $N = M$ is a shock. Consider

21

$$\begin{aligned} \frac{d}{dt} \|N - M\|(t) = \sum (-1)^n \left\{ \int_{y_n}^{y_{n+1}} (N_t - M_t) dx + \right. \\ \left. + (N - M) \Big|_{y_{n+1}} \frac{dy_{n+1}}{dt} - (N - M) \Big|_{y_n} \frac{dy_n}{dt} \right\}. \end{aligned}$$

Consider the term in the bracket; from equation (1.14) it is

$$\int_{y_n}^{y_{n+1}} (F(M)_x - F(N)_x) dx + (N - M) \frac{dy}{dt} \Big|_{y_n}^{y_{n+1}} = [F(m) - F(N) + (N - M) \frac{dy}{dt}]_{y_n}^{y_{n+1}}, \quad (1.16)$$

where y_n are now points of discontinuity of N or M . The contribution from other points y_n vanishes because $N = M$ and $F(M) = F(N)$. Next we calculate the contribution at the upper point y_{n+1} . Suppose N has a discontinuity at y_{n+1} and M does not; the other cases are analogous.

Now

$$\frac{dy}{dt} = S = \frac{F(N_L) - F(N_R)}{N_L - N_R}.$$

Since $(N - M)$ changes sign and M does not have a discontinuity at y_{n+1} , we must have

$$N_R < M < N_L.$$

The contribution of the term in the bracket in (1.16) at y_{n+1} is thus, with $N = N_L$, given by

$$\begin{aligned} & F(M) - F(N_L) + (N_L - M) \cdot \frac{F(N_L) - F(N_R)}{N_L - N_R} = \\ & = F(M) - \left\{ \frac{M - N_R}{N_L - N_R} F(N_L) + \frac{N_L - M}{N_L - N_R} F(N_R) \right\} \end{aligned}$$

22

≤ 0 , (by the convexity of F since $N_R < M < N_L$). Since $M < N_L$, $(N_L - M)$ is positive and therefore n is even and hence the contribution is negative. Arguing on the same lines, we find similarly that the contribution from the lower point y_n is also negative as well as the contribution when both N and M have shocks. This completes the proof. \square

Remark. A similar estimate which yields uniqueness can be made under alternative conditions on F , other than convexity.

Theorem (A minimum principle) Consider the initial value problem

$$\begin{aligned} N_t + F_x &= 0, \quad -\infty < x < \infty, \quad t > 0, \\ N(x, 0) &= \Phi(x). \end{aligned}$$

Let $N(x, t)$ be a continuous and differentiable solution. Let $\Phi \in L'$ (It suffices to assume that Φ vanishes for large negative x). Put

$$I(x, y, t) = \int_{-\infty}^y \Phi(s) ds + tH\left(\frac{x-y}{t}\right),$$

where

$$H(L) = MG(M) - F(M), \quad M = G^{-1}(L), \quad G = dF/dM. \quad (1.17)$$

Then $N(x, t) = G^{-1}\left(\frac{x-y}{t}\right)$ where y minimizes $I(x, y, t)$. (Note $G(N) = F'(N)$ is an increasing function of N by our assumption on F ; so that the inverse G^{-1} exists).

23 *Proof.* Let

$$U(x, t) = \int_{-\infty}^x N(y, t) dy, \quad (1.18)$$

then

$$U_x = N. \quad (1.19)$$

Since N satisfies the differential equation, we obtain

$$U_t + F(U_x) = 0; \quad (1.20)$$

here we have adjusted the integration constant by putting

$$F(0) = 0. \quad (1.21)$$

If (1.15) is applied with U_x and any number M , we obtain

$$F(U_x) \geq F(M) + (U_x - M)G(M).$$

Or using (1.20),

$$U_t + G(M)U_x \leq MG(M) - F(M). \quad (1.22)$$

Let y denote the intercept on the x -axis of the line given by $dx/dt = G(M)$ or

$$(x - y)/t = G(M), \quad t > 0. \quad (1.23)$$

Integrating (1.22) along this line w.r.t. t from $t = 0$, we obtain

$$U(x, t) - U(y, 0) \leq t[MG(M) - F(M)]. \quad (1.24)$$

From (1.23), we have

$$G^{-1}\left(\frac{x - y}{t}\right) = M. \quad (1.25)$$

If H is defined by (1.17), we see that

$$U(x, t) \leq U(y, 0) + tH\left(\frac{x - y}{t}\right) = I(x, y, t). \quad (1.26)$$

We also note that

$$dH/dL = G^{-1}(L).$$

Let $G(0) = c$; then $G^{-1}(c) = 0$. Since $F(0) = 0$, we have $H(c) = 0$ and this is its minimum value. The inequality in (1.26) holds for all choices of y . In particular for the value of y for which M , given by (1.25), equals $N(x, t)$, the sign of equality holds in (1.24) along the whole characteristic $dx/dt = G(N)$ issuing from (x, t) . Therefore, the sign of equality holds in (1.26). This completes the proof. \square

Remark 1. The above theorem holds also for generalised (weak) solutions whose discontinuities are shocks. For, relation (1.20) is the integral form of the conservation law and so relation (1.26) is also valid for generalised solutions. Since all discontinuities are shocks every point (x, t) can be connected to a point y on the initial line by a backward characteristic (entropy condition). For this choice of M equality holds in (1.26).

Remark 2. The converse of the result given in remark 1 is also true.

An estimate for large t . In the first theorem, we have found an explicit formula for the solution of an initial value problem in terms of its initial value. Recall

$$N(x, t) = G^{-1}\left(\frac{x-y}{t}\right) \quad (1.27)$$

25 where y minimizes

$$I(x, y, t) = \int_{-\infty}^y \Phi(s)ds + tH\left(\frac{x-y}{t}\right). \quad (1.28)$$

We have also seen that H takes its minimum values at $F(0) = c$ and $H(c) = 0$. Let

$$k = \frac{1}{2}(G^{-1})'(c) = \frac{1}{2}H''(c). \quad (1.29)$$

Assuming F is strictly convex, we have $k > 0$. Suppose there are two positive constants k_1, k_2 such that

$$2k_1 < H'' < 2k_2. \quad (1.30)$$

It follows then from $H(c) = 0$, $H'(0) = 0$ that

$$H(L) \geq k_1(L - c)^2.$$

Thus

$$tH\left(\frac{x-y}{t}\right) \geq \frac{k_1}{t}(x-y-ct)^2. \quad (1.31)$$

Now we assume $\Phi \in L^1$ and let $\|\Phi\|$ denotes its L^1 -norm. Then using (1.28) and (1.31), we conclude

$$-\|\Phi\| + \frac{k_1}{t}(x-y-ct)^2 \leq I(x, y, t). \quad (1.32)$$

But

$$I(x, x-ct, t) = \int_{-\infty}^{x-ct} \Phi(s) ds \leq \|\Phi\|$$

and hence by the minimum principle

$$I(x, y, t) \leq \|\Phi\|$$

at the minimizing function y . Combined with (1.31), we obtain

$$\left| \frac{x-y}{t} - c \right| \leq \left\{ \frac{2\|\Phi\|}{k_1 t} \right\}^{1/2} = \tilde{m}/\sqrt{t}, \quad \text{say.} \quad (1.33)$$

From $(G^{-1})' = H'' < 2k_2$ and $G^{-1}(c) = 0$, we have

26

$$|G^{-1}(L)| \leq 2k_2|L - c|,$$

or

$$\begin{aligned} |G^{-1}\left(\frac{x-y}{t}\right)| &\leq 2k_2 \left| \frac{x-y}{t} - c \right| \\ &\leq 2k_2 \cdot \tilde{m}/\sqrt{t} = m/\sqrt{t}, \quad \text{say.} \end{aligned}$$

Thus

$$|N(x, t)| \leq m/\sqrt{t}, \quad \text{from (1.27).}$$

Now suppose that Φ vanishes outside $(-A, A)$; then

$$\int_{-\infty}^x \Phi(s) ds = 0, \quad \text{if } x < -A$$

$$= \text{constant, if } x > A.$$

According to (1.33), the minimum value of y lies in the interval

$$x - ct - \tilde{m} \sqrt{t} \leq y \leq x - ct + \tilde{m} \sqrt{t}.$$

If $x < ct - \tilde{m} \sqrt{t} - A$, then $y < -A$, where the value of

$$\int_{-\infty}^y \Phi(s) ds$$

is independent of y and therefore the minimum of I is attained at the point which minimizes $tH(\frac{x-y}{t})$; this value is $y = x - ct$. Similarly, if $x > ct + \tilde{m} \sqrt{t} + A$, the minimizing value is $y = x - ct$. Since $G^{-1}(c) = 0$, we conclude from $N(x, t) = G^{-1}(\frac{x-y}{t})$ that $N(x, t) = 0$ for x outside the interval $(ct - \tilde{m} \sqrt{t} - A, ct + \tilde{m} \sqrt{t} + A)$. This can be restated as: Every solution N whose initial value vanishes outside a finite interval, at time t vanishes outside on terval whose length is $O(\sqrt{t})$ and inside this interval it is $O(1/\sqrt{t})$.

27

In fact, the result can be improved. We state the theorem without proof.

Theorem. Define the 2-parameter family of functions $M(p, q)$, $p, q \geq 0$ as

$$M(x, t; p, q) = \begin{cases} (x - ct)/tF''(0), & \text{for } ct - \sqrt{pt} < x < ct + \sqrt{qt}, \\ 0, & \text{otherwise} \end{cases}$$

Let $N(x, t)$ be any solution with shocks of

$$N_t + (F(N))_x = 0$$

where F is convex, $F'(0) = c$. Then

$$\|N - M(p, q)\|(t)$$

tends to zero as $t \rightarrow \infty$ where $\|\cdot\|$ is the L^1 -norm and

$$p = -2F''(0) \min_y \int_{-\infty}^y \Phi(s) ds$$

$$q = -2F''(0) \max_y \int_y^{\infty} \Phi(s) ds.$$

1.5 Modifications by diffusion and dissipation

Consider the first order nonlinear equation

$$N_t + NN_x = 0, \quad -\infty < x < \infty, \quad t > 0, \quad (1.34)$$

$$N(x, 0) = \Phi(x),$$

whose solution is implicitly given by

$$x = \xi + t\Phi(\xi)$$

$$N(x, t) = \Phi(\xi).$$

We have seen that however smooth Φ may be the solution becomes **28** discontinuous after a finite time, if Φ has compact support. We want to consider a different model in which we add an extra *dissipative term*:

$$N_t + NN_x = \alpha N_{xx}, \quad (1.35)$$

where α is a small positive number. This is the well-known Burgers' equation that has been utilized by Burgers as a mathematical model of turbulence. By a nonlinear transformation (Cole-Hopf transformation), it is known that (1.35) can be reduced to the linear heat equation

$$\Psi_t = \alpha \Psi_{xx}, \quad (1.36)$$

where

$$N = -2\alpha \frac{\Psi_x}{\Psi}.$$

Solving (1.36) a complete solution of (1.35) with $N(x, 0) = \Phi(x)$ is given by

$$N(x, t) = \frac{\int_{-\infty}^{\infty} \frac{x-\eta}{t} e^{-\chi/2\alpha} d\eta}{\int_{-\infty}^{\infty} e^{-\chi/2\alpha} d\eta},$$

where

$$\chi(\eta; x, t) = \int_0^\eta \Phi(\eta') d\eta' + (x - \eta)^2 / 2t$$

29 Hence in the case of Burgers' equation a solution exists for all time ever for discontinuous (bounded, measurable) functions Φ . It can be shown, by the method of steepest descent, that as $\alpha \rightarrow 0$ the solution of the Burgers' equation becomes, asymptotically, a solution of (1.34).

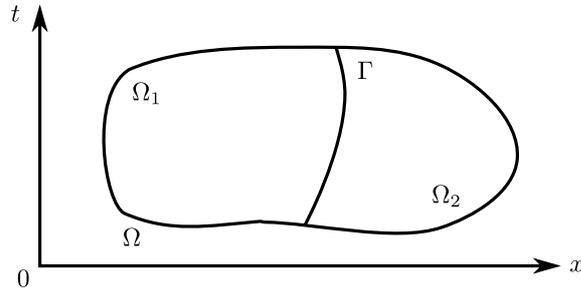
Now instead of putting an extra dissipative term, we put a dispersive term and consider

$$N_t + N_{xx} = \beta N_{xxx}. \quad (1.37)$$

This is the Kortaweg-deVries equation, originally formulated in the study of shallow water theory. Recently, P.D. Lax and D. Levermore have developed the theory of the connection between this equation and shock theory in the limit $\beta \rightarrow 0$. Many other mathematical properties of this equation have been studied in recent years.

1.6 Propagation of singularities in derivatives

Let $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$ be an open set in (x, t) plane (see figure 1.10). Let $f(x, t)$ be a continuous function in Ω . Suppose f_x, f_t are continuous in Ω_1 and Ω_2 and have finite limits as they approach Γ from either side. Let Γ be described by a smooth curve

**Fig. 1.10.**

$x = s(t)$. Let $[\cdot]$ denote the jump across Γ . Then a calculation shows **30**

$$\frac{d}{dt}[f] = [f_x]s' + [f_t]. \quad (1.38)$$

Consider now the equation

$$N_t + G(N)N_x = 0 \quad (1.39)$$

in Ω . Let N be a continuous function in Ω , having discontinuities in N_x , N_t across Γ , and satisfying (1.39) in Ω_1, Ω_2 . Applying (1.38) to N , we obtain

$$[N_x]s' + [N_t] = 0, \quad (1.40)$$

since $[N] = 0$.

Since N satisfies (1.39) in Ω_1 and Ω_2 , we obtain as we approach Γ that

$$[N_t] + G(N)[N_x] = 0. \quad (1.41)$$

Let $\lambda = [N_x]$; then from (1.40) $[N_t] = -\lambda s'$. Thus from (1.41), we obtain

$$-\lambda s' + G(N)\lambda = 0.$$

Assuming $\lambda \neq 0$, we obtain $s' = G(N)$. But then this curve is precisely a characteristic. Hence the discontinuities in N_x , N_t propagate along characteristics.

Note. This is true in higher dimensions also.

Let us look more closely at the singularities in N_x, N_t . To do this, we put $p = N_x, q = N_t$. Then in the regions where p, q are continuous they satisfy the equations

$$p_t + G(N)p_x + G'(N)p^2 = 0, \quad (1.42)$$

$$q_t + G(N)q_x + G'(N)pq = 0 \quad (1.43)$$

31 respectively. Solving (1.42) by the method of characteristics, we find

$$p = \frac{1}{\int_t G'(N)dt - c} \text{ on } dx/dt = G(N),$$

where c is a constant. Note that the characteristics for these equations and the original equation are the same. Note that since at the time of breaking p becomes infinite, the constant c must become the integral in the denominator at that time.

1.7 Computing methods

Although we have obtained a method of solving a nonlinear equation, it may be difficult to obtain the solution explicitly using the method described. In practice, it is preferable to use a difference scheme.

We first consider a linear equation

$$U_t + aU_x = 0$$

where ' a ' is a constant, with initial condition $U(x, 0) = \Phi(x)$. Let the domain be approximated by a rectangle. Divide this rectangle into small rectangles of length ' h ' and width ' k ' (see the figure 1.11). We want to find an approximate value of U on this mesh of points.

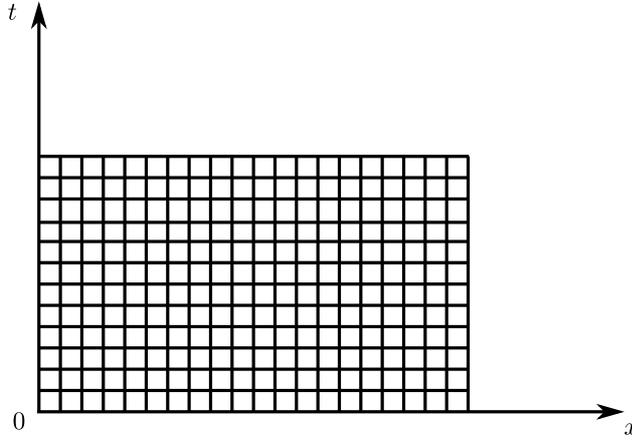


Fig. 1.11.

32

Let $U(x, t) = U(hi, kj) = U_i^j$. The simplest way to approximate $\partial U/\partial t$, at (hi, kj) , would be by

$$\frac{U_i^{j+1} - U_i^j}{k}$$

making an error of order k . We shall this is too inaccurate. If a similar approximation is made for $\partial U/\partial x$ we obtain the following difference equation

$$\frac{U_i^{j+1} - U_i^j}{k} + a \cdot \frac{U_{i+1}^j - U_i^j}{h} = 0$$

or

$$U_i^{j+1} = U_i^j - \frac{ak}{h}(U_{i+1}^j - U_i^j) \quad (1.44)$$

Its simplicity lies in the fact that it requires the values only on the first row, which will be given by the initial data. Each evaluation, however, gives an error of order h^2 or $k^2 = \alpha^2 h^2$, say, and if one substitutes consecutively, the value U_i^j is obtained and involves using the approximation $j(j-1)/2$ times, i.e., making an error of order $j(j-1)h^2/2$. Now, if j is a mesh corresponding to final time T , $j = T/k = T/h$ and, hence, the error behaves like T^2 . Therefore, we would have to keep T small in order to avoid an error that is of the same size as the solution. In spite of

33

this possibility, the scheme is sometimes useful because of its simplicity. Thus in moving from an ordinary differential equation, we see that we must make a scheme that has more consistent accuracy.

We look now instead for a scheme that is accurate in the equation or *consistent with the equation* to second order in the difference h taking k, h of the same order of magnitude.

We note that a derivative is much more accurately described by a difference ratio that straddles equally the point where the approximation is made. Thus using

$$\frac{U_i^{j+1} - U_i^{j-1}}{2k}$$

for $\partial U/\partial t$ the derivative is accurate to second order in k . This can be seen as follows. Let $W(t)$ possess a Taylor's series. Then

$$\begin{aligned} W(t + \Delta t) &= W(t) + W'(t) \cdot \Delta t + W''(t) \cdot (\Delta t)^2/2 + 0((\Delta t)^3), \\ W(t - \Delta t) &= W(t) - W'(t) \cdot \Delta t + W''(t) \cdot (\Delta t)^2/2 + 0((\Delta t)^3). \end{aligned}$$

On subtraction, we obtain

$$W'(t) = \frac{W(t + \Delta t) - W(t - \Delta t)}{2\Delta t} + 0((\Delta t)^2).$$

Thus, we are led to a scheme consistent to $O(h^2)$:

$$\frac{U_i^{j+1} - U_i^{j-1}}{2k} + a_i^j \frac{U_{i+1}^j - U_{i-1}^j}{2h} = 0$$

34 or

$$U_i^{j+1} = U_i^{j-1} - \frac{a_i^j k}{h} (U_{i+1}^j - U_{i-1}^j) \quad (1.45)$$

Here 'a' is considered as a function of x, t . This algorithm gives us a new row of values from neighbours in the two preceding rows.

Its apparent disadvantage is that it requires the values on two rows to start with, but we have only one. The simplest way to get the second row is to use the first scheme (1.44). This leads to an error of order h^2 in the second row. This error is just carried through, but not compounded. From the algorithm, we see that the value at the point (i, j) is obtained

successively from a pyramid of mesh points and that the algorithm is applied $j(j-1)/2$ times again; but the error is of order h^3 at each step, and hence the error is of order $j^2 h^3$. Since $j \simeq T/h$, T being the final time, we see that the net error is of order h .

This is then a consistent scheme but it is still possible for it to be unstable, i.e., to have the property that it amplifies the error.

Stability. (1.45) (and also (1.44)) is a difference scheme with constant coefficients. A constant raised to a power plays the role that an exponential plays in differential equations. So, we look for solutions of (1.45) in the form $\xi^i \zeta^j$, i.e., $U_i^j = \xi^i \zeta^j$. Substituting this in (1.45), we obtain a relation between ξ and ζ . 35

$$\zeta = \zeta^{-1} - \frac{ak}{h}(\xi - \xi^{-1}).$$

Thus for every ξ there are two possible value for ζ . We are, of course, looking for real solutions and we could generate the solutions with real ζ . But, clearly, if ξ and ζ are complex, then

$$(\xi^i \zeta^j + \bar{\xi}^i \bar{\zeta}^j)/2 \quad \text{and} \quad (\xi^i \zeta^j - \bar{\xi}^i \bar{\zeta}^j)/2\sqrt{-1},$$

will both be real solutions. For $j = 0$, these solutions are $|\xi|^i \cos(i \arg \xi)$, $|\xi|^i \sin(i \arg \xi)$. If we take $|\xi| = 1$ and put $\arg \xi/h = n$, $n = 0, \pm 1, \pm 2, \dots$ the corresponding values for $j = 0$ then become $\cos(nih)$ and $\sin(nih)$, which can be used to describe any initial data by a Fourier series approximation. Let us consider first $\cos(nih)$. At an arbitrary row j its value, since $|\xi| = 1$, is $\text{Re}[\exp(\sqrt{-1} inh)\zeta^j]$ where we can take ζ to be either root of

$$\begin{aligned} \zeta &= \zeta^{-1} - \frac{ak}{h}(e^{\sqrt{-1}.nh} - e^{-\sqrt{-1}.nh}) \\ &= \zeta^{-1} - \frac{2ak}{h}\sqrt{-1} \cdot \sin(nh). \end{aligned} \quad (1.46)$$

Note that the absolute value of the product of the roots is 1. So, if one of the roots has absolute value different from 1, then there is a root with absolute value greater than 1. Suppose $j = T/k$; then the solution is

$$|\zeta|^{T/k} \text{Re}[e^{\sqrt{-1}(inh)} \cdot e^{\sqrt{-1} \cdot T/k \cdot \arg \zeta}],$$

If the mesh size k shrinks, this solution behaves like $\exp((T \log \zeta)/k)$ and it also oscillates. The exponential factor goes to infinity as T tends to infinity. A small multiple of this solution even, say, of order k^3 , for the sake of argument, still goes to infinity. Such a small multiple can easily represent an error and hence, the error amplifies as the mesh size shrinks, i.e., the scheme is unstable unless $|\zeta| = 1$. In that case set $\zeta = \exp(\sqrt{-1}\Phi)$, with Φ a real. From (1.46), we then obtain

$$2\sqrt{-1}\sin\Phi = -2\sqrt{-1}\frac{ak}{h}\sin(nh)$$

or

$$\sin\Phi = -\frac{ak}{h}\sin(nh).$$

But this has a solution iff $|\frac{ak}{h}| \leq 1$, and hence if this is the case the difference scheme is stable. If $|\frac{ak}{h}| > 1$, then there is a root ζ with $|\zeta| > 1$ and the difference scheme is unstable.

What has been established here is the Courant-Friedrichs-Lewy condition for the particularly simple hyperbolic system of one equation.

Note. If we look at the characteristics, we see that h/k must be greater than characteristic speed for stability. Otherwise, we are trying to evaluate a solution at points whose domains of dependence include points from which we are drawing no data.

37 Exercise 1.5. Find the stability criterion for (1.44).

We now consider a difference scheme for the nonlinear equation²

$$U_t + F_x = 0, \quad -\infty < x < \infty, \quad t > 0. \quad (1.47)$$

Here U is a scalar and $F(U)$ is smooth. Initial data will be prescribed for (1.47):

$$U(x, 0) = \Phi(x). \quad (1.48)$$

We know that in general smooth solutions of (1.47) do not exist for all time however smooth Φ may be; we have to consider weak solutions. We recall the definition:

²The nonlinear system can be handled in the same way both formally and numerically provided the speeds of propagation are always distinct.

Definition. A locally integrable function $U(x, t)$ is a weak solution of (1.47) with initial data (1.48) if

$$\iint_{t>0} (W_t U + W_x F) dx dt + \int W(x, 0) \Phi(x) dx = 0 \quad (1.49)$$

is satisfied for all smooth functions W which vanish for large $|x|$, t and $t = 0$; we call such functions, test functions.

We now propose and discuss a difference scheme for getting an approximate solution to (1.47) with initial data (1.48).

Choose $G : \mathbb{R}^{2\ell} \rightarrow \mathbb{R}$, smooth enough, related to F by the requirement

$$\underbrace{G(u, \dots, u)}_{2\ell\text{-arguments}} = F(u). \quad (1.50)$$

For k an integer put $U_k = U(x + k\Delta x, t)$ where Δx is step size in x -direction; similarly, let Δt be step size in t -direction. Define

38

$$G(x + \frac{1}{2}\Delta x) = G(U_{-\ell+1}, U_{-\ell+2}, \dots, U_\ell)$$

and

$$G(x - \frac{1}{2}\Delta x) = G(U_{-\ell}, U_{-\ell+1}, \dots, U_{\ell-1}).$$

We now consider the following difference analog of (1.47)

$$\frac{\Delta U}{\Delta t} + \frac{\Delta G}{\Delta x} = 0 \quad (1.51)$$

where

$$\begin{aligned} \Delta U &= U(x, t + \Delta t) - U(x, t), \\ \Delta G &= G(x + \frac{1}{2}\Delta x) - G(x - \frac{1}{2}\Delta x). \end{aligned}$$

It follows from (1.51) that

$$U(x, t + \Delta t) = U(x, t) - \lambda \Delta G, \quad (1.51)'$$

where $\lambda = \Delta t / \Delta x$.

We claim that the difference scheme (1.51), as a consequence of (1.50), is ‘consistent’ with the differential equation (1.47) in the following sense: denote by $V(x, t)$ the solution of the difference scheme where, we have taken $V(x, 0) = \Phi(x)$. Here V is defined for non-integer multiples t of Δt , for the sake of convenience, as equal to $V(x, t')$, $t' = [t/\Delta t]\Delta t$. Ofcourse, V depends on $\Delta x, \Delta t$.

Then the following theorem holds.

Theorem. *Assume that as $\Delta x, \Delta t \rightarrow 0$, $V(x, t) \rightarrow U(x, t)$ boundedly a.e. Then $U(x, t)$ is a weak solution of (1.47) with initial data (1.48).*

39 *Proof.* Multiply (1.51) throughout by Δt and by any test function W and then integrate with respect to x to obtain

$$\int W(x, t) \frac{\Delta V}{\Delta t} dx \Delta t + \int W(x, t) \frac{\Delta G}{\Delta x} dx \Delta t = 0.$$

Now sum over all t which are integral multiples of Δt and carry out summation by parts in the first integral; we obtain

$$\begin{aligned} \sum_{t>0} \int \frac{W(x, t - \Delta t) - W(x, t)}{\Delta t} V(x, t) dx \Delta t - \int W(x, 0) \Phi(x) dx + \\ + \sum_t \int W(x, t) \frac{\Delta G}{\Delta x} dx \Delta t = 0. \end{aligned}$$

In the last integral replace x by $x - \frac{1}{2} \Delta x$ in first term and by $x + \frac{1}{2} \Delta x$ in second term; we finally obtain

$$\begin{aligned} \sum_{t>0} \int \frac{W(x, t - \Delta t) - W(x, t)}{\Delta t} V(x, t) dx \Delta t - \\ - \int W(x, 0) \Phi(x) dx \\ - \sum_t \int \frac{W(x + 1/2 \Delta x) - W(x - 1/2 \Delta x)}{\Delta x} G dx \Delta t, \end{aligned}$$

where G stands for $G(V_1, \dots, V_{2\ell})$, $V_1, \dots, V_{2\ell}$ denoting values of V at 2ℓ points which are distributed symmetrically around (x, t) and have distance Δx from each other. If $V \rightarrow U$ boundedly a.e. as $\Delta x, \Delta t \rightarrow 0$ so do $V_1, \dots, V_{2\ell}$ and therefore

$$G(V_1, \dots, V_{2\ell}) \rightarrow G(U, \dots, U) = F(U) \quad \text{by (1.50).}$$

The proof is complete. \square

The real difficulty is to find when $V(x, t) \rightarrow U(x, t)$ boundedly.

We turn to the problem of choosing G and minimizing the truncation error. Let $U(x, t)$ be an exact smooth (C^2 is enough) solution of (1.47). It will then satisfy difference equation (1.51)' only approximately; the deviation of right side from the left side of (1.51)' is called truncation error. It is easily seen that, in view of (1.50), then truncation error is $0(\Delta t^2)$. We shall now show, by taking $\ell = 1$, that G can be so chosen that the truncation error is $0(\Delta t^3)$. Let

$$U(x, t + \Delta t) = U(x, t) + \Delta t U_t + \frac{1}{2}(\Delta t)^2 U_{tt} + 0(\Delta t^3) \quad (1.52)$$

be a Taylor series up to terms of second order.

From (1.47), we obtain the following:

$$\begin{aligned} U_t + F_x &= 0 \\ U_{tt} + (A^2 U_x)_x &= 0. \end{aligned} \quad (1.53)$$

The second of these equations follows from the calculation with $dF/dU = A$,

$$U_{tt} = -F_{xt} = -F_{tx} = -(AU_t)_x = -(AF_x)_x = -(A^2 U_x)_x.$$

What is significant is that all t derivatives are exact x derivatives and therefore can be approximated by exact x differences. Substituting (1.53) in (1.52), we obtain

$$U(x, t + \Delta t) = U(x, t) + (\Delta t F + \frac{1}{2}(\Delta t)^2 A^2 U_x)_x + 0(\Delta t^3). \quad (1.52)'$$

Comparing (1.51)' and (1.52)', we see that the truncation error is $0(\Delta t^3)$, iff

$$\frac{\Delta G}{\Delta x} = (F + \frac{1}{2}\Delta t A^2 U_x)_x + 0(\Delta t^2).$$

41 From this, we can easily determine the form that G must take.

Theorem. *The truncation error in the difference scheme (1.51)' is $0(\Delta t^3)$ iff*

$$G(a, b) = \frac{F(a) + F(b)}{2} + \frac{1}{2}\lambda A^2 \cdot (b - a) \quad (1.54)$$

plus terms which are $0(|a - b|^2)$ for $(a - b)$ small.

The quantity A^2 in (1.54) shall be taken as $1/2\{A^2(a) + A^2(b)\}$ for the sake of symmetry more than anything else; any other choice would make a difference that is quadratic in $(a - b)$.

Denote the function in (1.54) by G_o ; any permissible G can then be written in the form

$$G = G_o + \frac{1}{2}Q(a, b) \cdot (b - a) \quad (1.55)$$

where $Q(a, b)$ vanishes for $a = b$. Substituting (1.55) in (1.51)', we see that

$$U(x, t + \Delta t) = U(x, t) + \lambda \Delta' F + \frac{1}{2}\lambda^2 \Delta A^2 \Delta U + \frac{1}{2}\lambda \Delta Q \Delta U, \quad (1.56)$$

where $\Delta' = 1/2[T(\Delta x) - T(-\Delta x)]$ and $\Delta = T(\frac{1}{2}\Delta x) - T(-\frac{1}{2}\Delta x)$, $T(s)$ being the shift operator of the independent variable by an amount s . We shall call Q the *artificial viscosity*.

The difference equation (1.56) expresses the value of U at time $t + \Delta t$ as a nonlinear function of U at time t ; we shall write this as

$$U(t + \Delta t) = NU(t). \quad (1.56)'$$

42 The value of the solution of the difference equation at some later time $k\Delta t$ is obtained from the initial values by the application k -times power of the operator N .

Our aim is to show that the difference scheme (1.56) is convergent if the size of λ is suitably restricted. In the case of linear equations, it is well-known and easy to show that convergence is equivalent to stability defined as the uniform boundedness of all powers N^k of N with some fixed range $k\Delta t \leq T$, a problem which we have studied previously. In the nonlinear case, following Von Neumann the convergence of the scheme would depend on the stability of the *first variation* of the operator N . The first variation of N , by definition, is a linear difference operator with variable coefficients; Von Neumann has conjectured that such an operator is stable iff all the *localized* operators associated with it, i.e., the operators obtained by replacing the variable coefficients by their value at some given point, are stable.

We content ourselves with:

Theorem. *If the Courant-Friedrichs-Lewy condition*

$$\frac{\Delta x}{\Delta t} \geq |A|_{\max} \quad (1.57)$$

is satisfied, the difference equation (1.56) satisfies Von Neumann's condition of stability that the linearized equation is stable.

Proof. The first variation of the operator N can be easily computed and is given by

$$I + \lambda A \Delta' + \frac{1}{2} \lambda^2 A^2 \Delta^2 + 0(\Delta x), \quad (1.58)$$

where Δ' , Δ are as before, and $0(\Delta x)$ denotes an operator bounded in norm by $|\Delta x|$, provided we are perturbing in the neighbourhood of a smoothly varying solution, i.e., one where neighbouring values differ by $0(\Delta x)$. In this case, the influence of the additional viscosity term is $0(\Delta x)$; see Remark 1 below. 43

To *localise* the operator (1.58), we replace A by its value at some point. After making a Fourier transform, the operator Δ' becomes multiplication by $i \sin \alpha$, and the operator Δ multiplication by $2i \sin(1/2)\alpha$, so that $(1/2)\Delta^2$ becomes multiplication by $\cos \alpha - 1$; here $\alpha = \xi \Delta x$, ξ being dual variable. Hence the amplification function of the operator (1.58) is

$$I + i\lambda A \sin \alpha + \lambda^2 A^2 (\cos \alpha - 1) + 0(\Delta x). \quad (1.59)$$

Since the eigenvalues k of A are real, the eigenvalues ν of the matrix (1.59) are given by

$$\begin{aligned} |\nu|^2 &= (1 - k^2(1 - \cos \alpha))^2 + k^2 \sin^2 \alpha + 0(\Delta x) \\ &= 1 - 2k^2(1 - \cos \alpha) + k^4(1 - \cos \alpha)^2 + k^2(1 - \cos^2 \alpha) + 0(\Delta x) \\ &= 1 - (k^2 - k^4)(1 - \cos \alpha)^2 + 0(\Delta x). \end{aligned}$$

By our assumption (1.57) $|k| \leq 1$. Thus $|\nu| \leq 1 + 0(\Delta x)$.

The proof of completed. \square

Remark 1. We have seen above that the quadratic terms, $Q(a, b)$, appearing in a representation of $G(a, b)$ influence neither the order of the truncation error nor the stability of the scheme at points where the solution varies smoothly. The terms do influence, however, at the points where solution varies rapidly, e.g., across a shock. For a detailed analysis, we refer to Lax and Wendroff [24].

As a second example, we consider the Lax-Friedrichs scheme for (1.47). We add a dissipative term ϵU_{xx} with $\epsilon = 2\Delta x^2/\Delta t$. This scheme is given by

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{2\Delta x} [F(U_{j+1}^n) - F(U_{j-1}^n)] + \frac{1}{2}(U_{j+1}^n + U_{j-1}^n - 2U_j^n), \quad (1.60)$$

where U_j^n abbreviates $U(j\Delta x, n\Delta t)$. We want to establish convergence via the contraction mapping principle. We write (1.60) as

$$U^{n+1} = T(U^n). \quad (1.60)'$$

Then T maps sequence $\{U_j\}_{j=-\infty}^{\infty}$ to a sequence $\{T(U)\}_{j=-\infty}^{\infty}$ according to

$$\{T(U)\}_j = U_j - \frac{\Delta t}{2\Delta x} [F(U_{j+1}) - F(U_{j-1})] + \frac{1}{2}(U_{j+1} + U_{j-1} - 2U_j).$$

Let ℓ^1 be the space of all summable sequences $\{U_j\}_{j=-\infty}^{\infty}$ with usual ordering: if $U, V \in \ell^1$ say $U \leq V$ if $U_j \leq V_j$ for all j . Let $a < 0 < b$ and put

$$C = \{U \in \ell^1 : a < U_j < b \text{ for all } j\}.$$

Assuming $F(U_j) - F(U_{-j}) \rightarrow 0$ as $j \rightarrow \infty$ it can be seen easily that

$$\sum_{j=-\infty}^{\infty} \{T(U)\}_j = \sum_{j=-\infty}^{\infty} U_j.$$

Thus T is *integral preserving*. It is also easy to see that if $\Delta t / \Delta x |F'(\Omega)| < 1$, $a \leq \Omega \leq b$, then T is order preserving, i.e., $U \leq V$.

$T(U) \leq T(V)$. Then it follows, by the following theorem, that T is also a contraction. Thus the scheme is convergent.

Theorem. Let Ω be a measurable space with a positive measure and $T : L^1(\Omega) \rightarrow L^1(\Omega)$ satisfy

$$\int_{\Omega} Tf = \int_{\Omega} f.$$

Let $C \subset L^1(\Omega)$ be such that whenever $f, g \in C$, $\max(f, g) \in C$. Then the following statements are equivalent.

- i) $f, g \in C$, $f \leq g$ a.e. $T(f) \leq T(g)$: order preserving.
- ii) $\int_{\Omega} (Tf - Tg)^+ \leq \int_{\Omega} (f - g)^+$, $f, g \in C$ where $r^+ = \max(r, 0)$.
- iii) $\int_{\Omega} |Tf - Tg| \leq \int_{\Omega} |f - g|$, $f, g \in C$: contraction.

For proof, we refer to Crandall and Tartar [7].

Chapter 2

One Dimensional gas dynamics

2.1 Equations of motion

These equations of motion completely characterise smooth movement of a fluid. They express the physical laws: 46

- i) Conservation of mass,
- ii) Conservation of momentum and
- iii) Conservation of energy.

We first consider conservation of mass. For this, the rate of change of mass in any volume element V of the fluid is balanced by the flow across ∂V , the boundary of V . If \underline{n} denotes the outward unit normal to ∂V , then the normal component of velocity across ∂V is $\underline{n} \cdot \underline{u}$, where $\underline{u} = (u_1, u_2, u_3)$ is the velocity vector of the fluid (we are mainly interested in 1, 2 or 3 dimensional flows). Then the net flow across the boundary between two times t_1, t_2 is

$$- \int_{t_1}^{t_2} \int_{\partial V} \rho(\underline{n} \cdot \underline{u}) d\sigma dt,$$

where ρ is the density of the fluid and $d\sigma$ denotes the surface measure. This must be balanced by the change in total mass between the times t_1 , t_2 . Hence

$$\left[\int_V \rho dV \right]_{t_1}^{t_2} = - \int_{t_1}^{t_2} \int_{\partial V} \rho(\underline{n} \cdot \underline{u}) d\sigma dt,$$

or taking the limit as $t_1 \rightarrow t_2$,

$$\frac{d}{dt} \int_V \rho dV + \int_{\partial V} \rho(\underline{n} \cdot \underline{u}) d\sigma = 0.$$

47 Using divergence theorem this can be written as

$$\frac{d}{dt} \int_V \rho dV + \int_V \operatorname{div}(\rho \underline{u}) dV = 0. \quad (2.1)$$

In a similar way, the equation for the net change in the i^{th} component of the momentum is

$$\frac{d}{dt} \int_V \rho u_i dV + \int_{\partial V} [\rho u_i (\underline{n} \cdot \underline{u}) + p n_i] d\sigma = 0. \quad (2.2)$$

The first term is the rate of change of total momentum inside V , the second term is the transport of the momentum across the boundary and the third is the rate of change of momentum produced by the pressure p . Here, we are neglecting other forces such as gravity, viscosity, etc.

The total energy density per unit volume consists of the kinetic energy $\rho |\underline{u}|^2 / 2$ of the motion of particles plus the internal energy ρe of the molecular motion. For energy balance, we then have after neglecting heat conduction, viscosity, etc.

$$\frac{d}{dt} \int_V (\rho |\underline{u}|^2 / 2 + \rho e) dV + \int_{\partial V} \{ (\rho |\underline{u}|^2 + \rho e) \underline{n} \cdot \underline{u} + p \underline{n} \cdot \underline{u} \} d\sigma = 0. \quad (2.3)$$

The first term in the surface integral is again the contribution from energy transport across the boundary and the second term is the rate of work by the pressure p at the boundary.

If discontinuities are allowed in the flow, we have to work with (2.1) - (2.3) (In fact they are not quite enough). But if all the quantities are smooth, we can differentiate under the integral sign. We then obtain the differential equations

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \underline{u}) = 0, \quad (2.1(a))$$

$$\frac{\partial}{\partial t}(\rho u_i) + \operatorname{div}(\rho u_i \underline{u}) + \frac{\partial p}{\partial x_i} = 0, \quad i = 1, 2, 3, \quad (2.2(a))$$

$$\frac{\partial}{\partial t}(\rho |\underline{u}|^2 / 2 + \rho e) + \operatorname{div}[\rho \underline{u}(\rho |\underline{u}|^2 / 2 + e + \frac{p}{\rho})] = 0. \quad (2.3(a))$$

48

2.2 Thermodynamical relations. Entropy:

Consider the quantity $de + pd(1/\rho)$. When a small energy is added to a mass of gas some of the energy is the work of changing the volume $1/\rho$ to $1/\rho + d(1/\rho)$. This energy is $pd(1/\rho)$; the rest de is heat put into the system. Since the quantity de is a perfect differential, there exist functions $S(p, \rho)$, $T(p, \rho)$ such that

$$de + pd(1/\rho) = TdS \quad (2.4)$$

T is the absolute temperature and S is the entropy. The relation (2.4) may be viewed as a way of defining temperature and entropy up to an arbitrary function.

Suppose we treat p , $\tau = \rho^{-1}$ as the independent variables, then we will have

$$de = -pd\tau + \alpha dp + \beta d\tau,$$

where α, β are given functions and we want to express $\alpha dp + \beta d\tau$ as TdS , where T, S are functions of p, τ . We want to show this representation is essentially unique. First from the compatibility relations for de , we have

$$(-p + \beta)_p = \alpha_\tau, \quad (*)$$

and from the one for S we have

49

$$(\alpha/T)_\tau = (\beta/T)_p$$

or

$$\alpha\left(\frac{1}{T}\right)_\tau - \beta\left(\frac{1}{T}\right)_p + (\alpha_\tau - \beta_p)\frac{1}{T} = 0$$

or

$$\alpha\left(\frac{1}{T}\right)_\tau - \beta\left(\frac{1}{T}\right)_p - \frac{1}{T} = 0, \quad \text{from (*).}$$

This is a first order equation for $1/T$ which we may solve provided $a^2 + \beta^2 \neq 0$, which is a reasonable thermodynamic assumption. Once T is determined S is found from $dS = (\alpha/T)dp + (\beta/T)d\tau$. With respect to uniqueness, suppose we have two such representations, T, S and T', S' . Then $TdS = T'dS'$ or $S_\tau/S_p = S'_\tau/S'_p$ and hence $F(S, S') = 0$ for some F or with an assumption of monotonicity $S' = s(S)$ and then $T' = \frac{T}{ds/dS}$. Thus the temperature and entropy are uniquely determined up to a scaling factor $s(S)$.

Suppose, we know for some medium that e is a function of τ and S , where $\tau = 1/\rho$, specific volume. Then we see that p is also a function of ρ, S . We assume

$$p = f(\rho, S) \quad \text{or} \quad p = g(\tau, S).$$

It is a fundamental property of almost all media that, entropy remaining constant, pressure is an increasing function of ρ or equivalently decreasing function of τ . Thus

$$f_\rho > 0 \quad \text{and} \quad g_\tau < 0.$$

- 50 For any value of S , the function $g(\tau, S)$ is generally convex w.r.t. τ . Henceforth, we shall assume this:

$$g_{\tau\tau}(\tau, S) > 0.$$

Alternative forms of equations of motion:

Introducing the operator

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_j \cdot \frac{\partial}{\partial x_j},$$

the equations (2.1(a)) - (2.3(a)) can be written as

$$\frac{D\rho}{Dt} + \frac{\partial u_j}{\partial x_j} = 0, \quad (2.5)$$

$$\rho \frac{Du_i}{Dt} + \frac{\partial p}{\partial x_i} = 0, \quad i = 1, 2, 3, \quad (2.6)$$

$$\rho \frac{De}{Dt} + p \cdot \frac{\partial u_j}{\partial x_j} = 0, \quad (2.7)$$

or using (2.5), (2.7) can be written as

$$\frac{De}{Dt} - \frac{p}{\rho^2} \frac{D\rho}{Dt} = 0.$$

Then from the thermodynamic relation (2.4), we obtain

$$T \frac{DS}{Dt} = 0. \quad (2.8)$$

This is to say, entropy remains constant following a particle. Flows satisfying (2.8) are called *adiabatic*. It follows from that, if the fluid initially has uniform entropy then entropy remains constant throughout as long as the flow is continuous. In such a case $p = f(\rho)$. Such flows are called *isentropic*.

2.3 One dimensional flow

The equations reduce to

$$\left. \begin{aligned} \rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + (\rho u^2)_x + p_x &= 0, \\ S_t + u S_x &= 0. \end{aligned} \right\} \quad (2.9)$$

The second equation in (2.9) can be written, using the first equation, as **51**

$$\rho u_t + \rho u u_x + p_x = 0.$$

The third equation follows from (2.8). It is convenient to use p, u . Then the system (2.9) may be written as

$$\begin{aligned} p_t + up_x + \rho c^2 u_x &= 0, \\ \rho u_t + p_x + \rho u u_x &= 0, \\ S_t + u S_x &= 0, \end{aligned} \quad (2.9(a))$$

where $c^2 = (\partial p / \partial \rho)_{S = \text{constant}}$; c is called the sound speed. System (2.9(a)) is a typical nonlinear hyperbolic system.

Another such a system, the Lundquist equations, occurs in magneto-hydrodynamics. It involves coupling Maxwell's equations with the classical equations of gas dynamics.

Let E, B stand for electric and magnetic field vectors and let J, u denote the electric current and the flow velocity. If p is pressure and $c^2 = dp/d\rho$ is the sound velocity, then the equations are

$$\begin{aligned} \frac{\partial B}{\partial t} + (\nabla \cdot u)B - (B \cdot \nabla)u &= 0, \\ \rho \frac{\partial u}{\partial t} + \nabla p + B \times (\nabla \times B) &= 0, \\ c^2/\rho \cdot \frac{\partial \rho}{\partial t} + c^2 \nabla \cdot u &= 0, \\ \frac{dS}{dt} &= 0. \end{aligned}$$

- 52 Another related hyperbolic system occurs in the theory of shallow water where (2.5) and (2.6) hold with the depth playing the role of density ρ and p/ρ^2 is constant.

We turn to the study of the system (2.9(a)). We consider isentropic flow so that the third equation drops out. Adding and subtracting the other two equations, we arrive at

$$\begin{aligned} p_t + (u + c)p_x + \rho c\{u_t + (u + c)u_x\} &= 0, \\ p_t + (u - c)p_x - \rho c\{u_t + (u - c)u_x\} &= 0. \end{aligned}$$

We choose the characteristics C_+, C_- so that the derivatives are along them. Thus for

$$C_+ : \frac{dx}{dt} = u + c,$$

$$C_- : \frac{dx}{dt} = u - c$$

and then

$$\begin{aligned} \frac{dp}{dt} + \rho c \frac{du}{dt} &= 0 \quad \text{on } C_+ \\ \frac{dp}{dt} - \rho c \frac{du}{dt} &= 0 \quad \text{on } C_-. \end{aligned}$$

Integrating these two equations, we obtain

$$\begin{aligned} \ell(\rho) + u &= \text{constant on } C_+, \\ \ell(\rho) - u &= \text{constant on } C_-, \end{aligned}$$

where

$$\ell(\rho) = \int \frac{c(\rho)d\rho}{\rho}$$

The quantities $\ell(\rho) \pm u$ are called *Riemann invariants*.

Exercise 2.1. Reduce the system (2.9(a)) (omitting the last equation) 53 for slow speeds nearly constant density to the wave equation (in u or in p). Show also that discontinuities in derivatives propagate with sound speed.

Exercise 2.2. Find the speeds of propagation of the one dimensional Lundquist equations.

Simple Waves. Piston Problem: In addition to being isentropic, if the flow has one Riemann invariant constant throughout, the solution is called a *simple wave*. Such a wave lies next to a constant state since all the characteristics of one kind issuing from the constant state carry a constant Riemann invariant.

To illustrate, how such waves can be produced, we consider the piston problem as a basic model. Consider the waves produced by the movement of a piston at the end of a tube and gas at rest in a constant state ahead of it. Provided shocks do not appear, the waves carry a constant Riemann invariant from the constant state ahead.

Initially, the gas has velocity $u = 0$, sound speed c_0 and $S = S_0$ in $s \geq 0, t = 0$. The piston path is itself a particle path and the flow is isentropic.

Since $u - c < u$ the C_- characteristics start on the x - axis in the uniform region (see figure 2.1). On each C_-

$$\ell(\rho) - u = \text{constant}; \quad (2.10)$$

- 54 Since $u = 0$ initially, we conclude that the constant is $\ell(\rho_0)$, where ρ_0 corresponds to the initial density. Since this constant is the same for C_- characteristics, we conclude that $\ell(\rho) - u$ is the Riemann invariant that is constant throughout. We now use the other characteristic to find the other quantities.

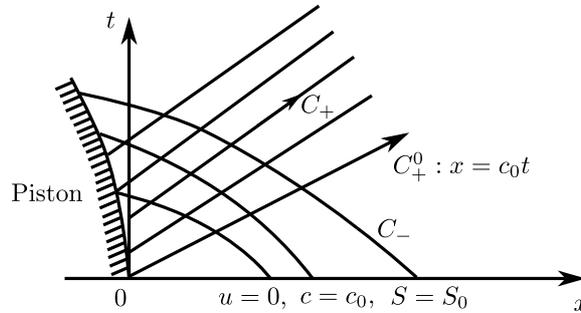


Fig. 2.1.

For those C_+ which originate from the x -axis, we see that

$$\ell(\rho) + u = \ell(\rho_0) \quad (2.11)$$

From (2.10), (2.11), we conclude that $u = 0, c = c_0$ in the region covered by these C_+ characteristics. Let C_+^0 separate the C_+ characteristics that meet the piston from those which meet the x - axis.

Since the image of the flow in (ρ, u) space lies entirely on the curve $\ell(\rho) - u = \text{constant}$ each point on the curve represents a curve in the flow. At that point the value of $\ell(\rho) + u$ determines u, ρ and hence the slope $dx/dt = u + c$, of C_+ which is constant along that characteristic. Hence

ρ, u are constant on each $C_+ : \frac{dx}{dt} = u + c$.

To complete the solution, we should use the boundary conditions, 55 given on the piston. Let the piston path be given by $x = X(t)$. Then the boundary condition is

$$u = \dot{X}(t) \quad \text{on} \quad x = X(t).$$

With this, we can obtain the complete solution:

$$\begin{aligned} x &= (\dot{X}(\tau) + c)(t - \tau) \\ u &= \dot{X}(\tau) \end{aligned}$$

Since the C_+ characteristics are straight lines with the slope dx/dt increasing with u , it is clear that the characteristics will overlap if u increases on the piston i.e., if $\ddot{X}(t) > 0$ for any t (this is the case when the piston accelerates into the gas). This is typical of nonlinear breaking, (see p8) and shocks will be formed. We need to reexamine the arguments leading to the constancy of entropy and of one of the Riemann invariants. But for motions with $\ddot{X}(t) \leq 0$, we have constructed a complete solution.

2.4 Shock conditions

We suppose the flow is one dimensional and allow jump discontinuities, denoted by $[\cdot]$, in the flow. We must work with the equations (2.1) - (2.3). Let the region V collapse on a slit around the discontinuity and we find the discontinuity conditions:

$$\begin{aligned} -U[\rho] + [\rho u] &= 0 \\ -U[\rho u] + [\rho u^2 + p] &= 0 \\ -U[\rho^2/2 + \rho e] + [(\rho u^2/2 + \rho e + p)u] &= 0, \end{aligned}$$

where U is the discontinuity velocity. We denote by subscripts (0), (1) 56 the two states. As a particle moves across the discontinuity, it moves from the *front* of the discontinuity across the discontinuity to *behind* the discontinuity. The entropy must *increase* in this direction.

For convenience, we introduce the relative velocities

$$v_i = U - u_i, \quad i = 0, 1.$$

In the velocity frame where the discontinuity is at rest, since the equations are independent of the coordinate system, we obtain from the above

$$\rho_0 v_0 = \rho_1 v_1 = m, \quad (2.12)$$

where m is the *mass flux* through the surface;

$$\rho_0 v_0^2 + p_0 = \rho_1 v_1^2 + p_1 = p, \quad (2.13)$$

where p is total *momentum flux* and finally the *energy flux* condition:

$$m(v_0^2/2 + e_0 + p_0\tau_0) = m(v_1^2/2 + e_1 + p_1\tau_1). \quad (2.14)$$

According to the second law of thermodynamics entropy can only increase. Hence across a discontinuity

$$S_0 \leq S_1$$

or

$$mS_0 \leq mS_1 \quad (2.15)$$

Two types of discontinuity surfaces are distinguished by the cases $m = 0$ and $m \neq 0$. The case $m \neq 0$ corresponds to mass flowing across and the discontinuity surfaces are called *check* fronts; the case $m = 0$ corresponds to a *contact* or *slip* surface.

We first consider the case of a shock, $m \neq 0$. Then from equation (2.14), we obtain

$$\frac{1}{2}v_0^2 + e_0 + p_0\tau_0 = \frac{1}{2}v_1^2 + e_1 + p_1\tau_1.$$

If we introduce *enthalpy*, $i = e + p\tau$, we then obtain

$$v_0^2/2 + i_0 = v_1^2/2 + i_1 \quad (2.16)$$

If we use (2.12) and (2.13), we obtain

$$\tau_0(p_0 - p_1) = v_0(v_1 - v_0),$$

$$\tau_1(p_0 - p_1) = v_1(v_1 - v_0)$$

and hence adding these two equations, we obtain

$$(\tau_0 + \tau_1)(p_1 - p_0) = v_0^2 - v_1^2 \quad (2.17)$$

Equation (2.16) becomes

$$(p_1 - p_0)(\tau_0 + \tau_1)/2 = i_1 - i_0, \quad (2.18)$$

or, since $i = e + p\tau$,

$$(\tau_0 - \tau_1)(p_1 + p_0)/2 = e_1 - e_0. \quad (2.19)$$

Since (2.18) and (2.19) involve only thermodynamical quantities, they are particularly useful. They were first studied by Hugoniot and (2.19) is called the *Hugoniot relation*. The relation (2.19) can be interpreted by stating that the increase in energy across a shock is due to the work done by the mean of the pressures in performing the *compression*.

The most common example considered is a polytropic gas, with $p = A(S)\rho^\gamma$. We note the following relations: (refer Serrin¹)

$$\log A = S/c_v, \quad e = p/(\gamma - 1)\rho, \quad c^2 = \gamma p/\rho,$$

where c_v is the specific heat at constant volume. It is often useful to use formulas for polytropic gases with the parameter

$$M = \frac{U - u_0}{c_0}$$

which is the *Mach number* of the shock relative to the flow ahead and is a useful measure of strength. Using (2.12) and (2.13) and the definition of M , we arrive at the following:

$$\left| \frac{u_1 - u_0}{c_0} \right| = \frac{2(M^2 - 1)}{(\gamma + 1)M},$$

$$\rho_1/\rho_0 = (\gamma + 1)M^2 / \{(\gamma - 1)M^2 + 2\},$$

¹See, Flügge, S. (Ed.), *Encyclopedia of Physics*, VIII/1, Springer-Verlag (1959).

$$(p_1 - p_0)/p_0 = 2\gamma(M^2 - 1)/(\gamma + 1),$$

$$c_1/c_0 = \frac{\{2\gamma M^2 - (\gamma - 1)\}^{1/2} \cdot \{(\gamma - 1)M^2 + 2\}^{1/2}}{(\gamma + 1)M}.$$

Introducing an alternative strength parameter

$$z = \frac{p_1 - p_0}{p_0},$$

the above set of equations can be written as

$$M = \left\{ 1 + \frac{(\gamma + 1)z}{2\gamma} \right\}^{1/2}, \quad (2.20)$$

$$\left| \frac{u_1 - u_0}{c_0} \right| = \frac{z}{\gamma \left\{ 1 + \frac{(\gamma + 1)z}{2\gamma} \right\}^{1/2}}, \quad (2.21)$$

$$\rho_1/\rho_0 = \left\{ 1 + \frac{(\gamma + 1)z}{2\gamma} \right\} / \left\{ 1 + \frac{(\gamma - 1)z}{2\gamma} \right\}, \quad (2.22)$$

$$c_1/c_0 = \left\{ \frac{(1 + z)(1 + \frac{(\gamma - 1)z}{2\gamma})}{1 + \frac{(\gamma + 1)z}{2\gamma}} \right\}^{1/2} \quad (2.23)$$

- 59 From the relation for S , obtained from A , and the relation (2.22), we obtain

$$(S_1 - S_0)/c_v = \log \left\{ \frac{(1 + z)(1 + \frac{(\gamma - 1)z}{2\gamma})z}{(1 + \frac{(\gamma + 1)z}{2\gamma})\gamma} \right\} \quad (2.24)$$

Since $S_1 > S_0$ across a shock, we obtain, as is easily seen by expanding that $z > 0$. Hence $p_1 > p_0$, and from the above relations, we then obtain

$$\rho_1 > \rho_0, \quad c_1 > c_0, \quad u_1 > u_0, \quad M > 1.$$

From (2.20), it is clear that $U > u_0 + c_0$. From the relations (2.20), (2.21) and (2.23), it then follows that,

$$u_1 + c_1 > U.$$

Thus a polytropic shock is always compressive with $p_1 > p_0$ and it is supersonic viewed from ahead and subsonic from behind.

These facts are true quite generally, but we will discuss this later, when we study the Riemann problem.

2.5 Contact Discontinuities

If the mass flux m through the surface of discontinuity is zero, then $v_1 = v_0 = 0$, so that $u_1 = u_0 = U$. Then we have $p_0 = p_1$ from (2.13) and (2.14) is automatically satisfied. Such a discontinuity surface as indicated before, is called a *contact* or *slip surface*. **60**

The flow velocity is continuous across the contact surfaces in one dimensional flow, but in higher dimensions the tangential component of the velocity vector may suffer a discontinuity across a contact surface, while the normal component relative to the surface is always zero. For details see, e.g., Courant–Friedrichs [6].

2.6 Shock Reflection

A simple example of determining a flow with shocks is provided by the reflection of a shock from a wall which can also be solved exactly. A shock with a state behind of prescribed velocity hits a wall and is reflected. We seek the pressure after reflection.

Let the subscripts 0, 1 refer to the states ahead of and behind the incident shock, and subscript 2 refer to the state behind the reflected shock, see figure 2.2.

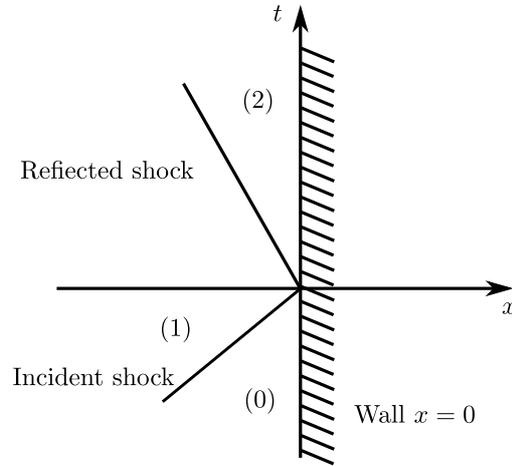


Fig. 2.2.

- 61 If the shock strength of the incident shock is $z_I = (p_1 - p_0)/p_0$, the state (1) can be determined by the relations (2.20) - (2.24). If

$$z_R = (p_2 - p_1)/p_1$$

is the strength of the reflected shock, we obtain, with suitable change in the sign of the velocities since the reflected shock travels in the opposite direction to the incoming shock, from (2.21) that

$$\left| \frac{u_1 - u_2}{c_1} \right| = \frac{z_R}{\gamma \left\{ 1 + \frac{(\gamma+1)z_R}{2\gamma} \right\}^{1/2}}.$$

Next to the wall, the gas must be at rest and hence $u_2 = u_0 = 0$. Now, u_1 and c_1 can be found in terms of z_I . After doing this, we obtain

$$\frac{z_I}{\gamma \left\{ (1 + z_I) \left(1 + \frac{(\gamma-1)z_I}{2\gamma} \right) \right\}^{1/2}} = \frac{z_R}{\gamma \left\{ 1 + \frac{(\gamma+1)z_R}{2\gamma} \right\}^{1/2}}$$

This is a quadratic in z_R and it is easily seen that the relevant solution is

$$z_R = \frac{z_I}{1 + \frac{(\gamma-1)z_I}{2\gamma}}$$

For weak shocks $z_I \rightarrow 0$ and hence $z_R \simeq z_I$ and

$$p_2 - p_0 \simeq 2(p_1 - p_0).$$

So, for the acoustic case, the pressure is doubled as is well known. Also 62
for strong shocks $z_I \rightarrow \infty$. This implies $z_R = \frac{2\gamma}{\gamma - 1}$ and therefore

$$\frac{p_2}{p_1} = \frac{3\gamma - 1}{\gamma - 1} = 8$$

for $\gamma = 1.4$. Hence there is a large gain in pressure after reflection. This phenomenon is even more striking in spherically symmetric flows when a shock is reflected at the center of symmetry.

2.7 Hugoniot Curve. Shock Determinacy

We recall the Hugoniot relation (2.19):

$$e_1 - e_0 = (\tau_0 - \tau_1) (p_1 + p_0)/2.$$

We regard all quantities such as energy e , entropy S etc. as functions of τ, p . We define the *Hugoniot function*

$$H(\tau, p) = e(\tau, p) - e(\tau_0, p_0) + (\tau - \tau_0) (p + p_0)/2.$$

If (τ_0, p_0) is fixed, the graph of the points (τ, p) which satisfy $H(\tau, p) = 0$ is the *Hugoniot curve*. As we shall see with $p > p_0$ it represents all possible states that can be reached with (τ_0, p_0) ahead of the *shock*. For $p < p_0$ the curve represents the states that can be ahead of (τ_0, p_0) . The big advantage in considering this curve is that it involves no velocities.

For polytropic gases, we have

$$e = \frac{p\tau}{\gamma - 1} = \frac{p\tau(1 - \mu^2)}{2\mu^2},$$

where

$$\mu^2 = \frac{(\gamma - 1)}{(\gamma + 1)}.$$

Hence

$$2\mu^2 H(\tau, p) = (\tau - \mu^2 \tau_0)p - (\tau_0 - \mu^2 \tau)p_0.$$

Hence the Hugoniot curve, specifically the Hugoniot curve with center (τ_0, p_0) , is a rectangular hyperbola with the left asymptote

$$\tau = \mu^2 \tau_0 = \tau_{\min} > 0.$$

(See figure 2.3)

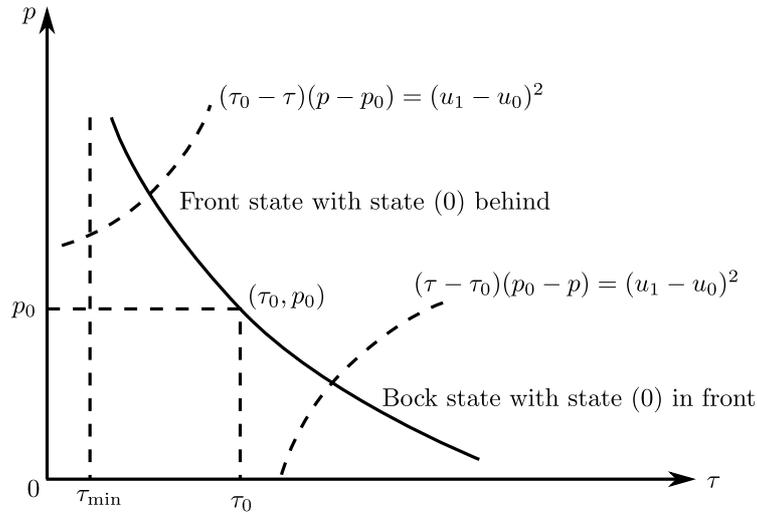


Fig. 2.3. The Hugoniot curve is the heavy curve.

We shall see that under wide conditions on the Hugoniot curve the state (τ, p) that can lie ahead or behind (τ_0, p_0) and connected by a shock, can be found exactly.

64

The precise statements are:

1. The state (0), i.e., in which (τ_0, p_0) is given, and the shock velocity U determine the complete state (1) on the other side of the shock front.
2. The state (0) and the velocity u_1 determine the speed of the shock front and the complete state (1) if it is specified whether the state (0) should be ahead of or behind the shock front.

3. The state (0) and the pressure p_1 determine the speed of the shock front and the complete state (1).

To prove these statements, we make the following assumptions on the Hugoniot curve $H(\tau, p) = 0$ with center (τ_0, p_0) :

- (H1) Along the Hugoniot curve, the pressure varies from zero to infinity and the value of τ exceeds τ_{\min} .
- (H2) The Hugoniot curve is strictly decreasing, i.e. $\frac{dp}{d\tau} < 0$ along the curve.
- (H3) Every ray through (τ_0, p_0) intersects the Hugoniot curve at exactly one point and at (τ_0, p_0) , $\frac{d^2p}{d\tau^2} > 0$.

(All three conditions are satisfied by the polytropic gases).

We are now in a position to prove statements (1) - (3) made above provided *we assume pressure increases across a shock*. It follows from the shock condition (2.12) and (2.13) that

$$-m^2 = p_0^2 v_0^2 = \frac{p_1 - p_0}{\tau_1 - \tau_0} \quad (2.25)$$

So, to find (τ_1, p_1) , we need to find the intersection of the curve with the line through (τ_0, p_0) and with slope $-m^2$. Then (H3) assumes there is just one such intersection. The velocity u_1 can then be found through (2.12), $u_1 = U - m\tau_1$. This proves statement (1). 65

As far as second statement is concerned, it can be easily derived from (2.12) and (2.13) that

$$-(\tau_0 - \tau_1)(p_0 - p_1) = (v_1 - v_0)^2. \quad (2.26)$$

From the data given, $(u_1 - u_0)^2 = (v_1 - v_0)^2$, is known. Hence to find (τ_1, p_1) it suffices to find the intersection of the hyperbola

$$(\tau_0 - \tau)(p_0 - p) = -(v_1 - v_0)^2$$

with the Hugoniot curve. The slope of the hyperbola is $m^2 > 0$. Hence from (H2), it follows that there are just two such intersections, corresponding to the two possibilities that the state (0) lies ahead of or behind

the shock front (see figure 2.3). The shock velocity and the state (1) can then be found easily using shock conditions.

To prove statement (3), assume the state (0) and p_1 are given. The assumptions (H1) and (H2) assert that there is exactly one τ_1 such that $H(\tau_1, p_1) = 0$. The other quantities to determine the state (1) completely are then found essentially as above.

66 We shall discuss a few more qualitative statements about the shock transition using the Hugoniot relation; in particular, some of the properties of the shock transition were already discussed in an earlier section for polytropic gases. The main thing we are going to prove here is the following:

The increase of entropy across a shock is of the third order in the shock strength and the shock is compressive.

By the shock strength, here, we mean one of the quantities

$$\rho_1 - \rho_0, \quad p_1 - p_0, \quad \text{or} \quad |v_1 - v_0|.$$

We can consider, because of (H2), the Hugoniot curve as $p = G(\tau)$; in particular, we can consider τ as independent variable. Now along the Hugoniot curve $dH = 0$. So,

$$2de + (\tau - \tau_0)dp + (p - p_0)d\tau = 0.$$

But $de + pd\tau = TdS$ and therefore we obtain

$$2TdS - (p - p_0)d\tau + (\tau - \tau_0)dp = 0 \quad (2.27)$$

and hence

$$dS = 0 \quad \text{at} \quad (\tau_0, p_0), \quad \text{the center.} \quad (2.28)$$

Differentiating (2.27), we obtain

$$2d(TdS) + (\tau - \tau_0)d^2p = 0, \quad (2.29)$$

and hence again at the center (τ_0, p_0)

$$d(TdS) = dTdS + Td^2S = 0$$

and therefore from (2.28)

$$d^2S = 0.$$

Thus the change in entropy is at least of third order. We show it is of third order. Differentiating (2.29), we obtain, at the center (τ_0, p_0) ,

$$2Td^3S + d\tau d^2p = 0.$$

Thus since $d^2p/d\tau^2 > 0$ at (τ_0, p_0) , we have

$$d^3S \geq 0 \quad \text{if} \quad d\tau \leq 0 \quad \text{at} \quad (\tau_0, p_0)$$

which proves entropy change is of third order exactly.

Furthermore, since the entropy must increase, we must have $d\tau < 0$ which means the shock is compressive and hence the upper branch of the Hugoniot curve represents states behind the state (τ_0, p_0) as was asserted earlier.

Exercise. Piston at uniform speed: It is a simple exercise to find the flow if a piston is moved with uniform speed *into* a gas at rest. The speed of the flow behind the shock is that of the piston and so we are in case 2.

Remark. We need the notion of vorticity in three dimensions. The vorticity is defined by

$$\underline{\omega} = \text{curl } \underline{u}.$$

Claim. *The change in vorticity across a shock is also of third order. In a steady flow the vorticity vector $\underline{\omega}$ and velocity vector \underline{u} satisfy*

$$\nabla(|\underline{u}|^2/2) + (\underline{\omega} \times \underline{u}) + \frac{1}{\rho} \nabla p = \text{constant}.$$

Note that if $\underline{\omega} \equiv 0$, i.e., if the flow is irrotational, we have Bernoulli's law. Taking a scalar product with a vector ξ , $|\xi| = 1$, lying in the shock surface and letting $[\cdot]$ to denote difference across the shock, we have

$$\left[\frac{d}{ds} (|\underline{u}|^2/2 + p/\rho) - p \frac{d}{ds} (1/\rho) \right] - [\xi \cdot (\underline{u} \times \underline{\omega})] = 0,$$

where the differentiation w.r.t. arc length along the shock is denoted by d/ds ; but since

$$[|u|^2/2 + p/\rho] = -[e]$$

and

$$de + pd(1/\rho) = TdS,$$

we obtain

$$[T \frac{dS}{ds}] = [\underline{\omega} \cdot (\underline{\xi} \times \underline{u})].$$

Thus the change in $\underline{\omega}$ is of third order in the shock strength if $\omega = 0$ on the side since we may choose co-ordinates so that u is locally normal.

2.8 Riemann Problem

We now turn to another important initial value problem, the Riemann problem. It is also referred as the shock tube problem. It is important both theoretically as we shall see and because of its practicability; it is the main device for producing fast chemical reaction fronts. The Riemann problem can be stated as follows:

Given two states they can always be connected by a “fan wave” consisting of a centered rarefaction wave, a shock and a discontinuity.

69 The two different states will be separated by a thin diaphragm up to time $t = 0$. Then the diaphragm is instantaneously removed and we have to find the flow. Without loss of generality we can always assume that $u_R = 0$ and $p_R < p_L$. The subscripts R, L refer to right and left states respectively.

We first define a *state behind* or *SB* curve associated with a state (τ_R, p_R) . It consists of two branches, the upper branch for $\tau < \tau_R$ is a Hugoniot curve and for $\tau > \tau_R$ it is a curve $p = p(\tau)$ at constant entropy and corresponds to a centered rarefaction wave leading from (τ_R, p_R) and on which $u \pm \ell(\rho)$, see section 2.3, is a constant.

Any point on the *SB* curve, thus defines a unique transition from (τ_R, p_R) to a new state by means of a shock or a rarefaction. For the

right side, these must be facing so that particle paths enter them, i.e., to the right.

Similarly, we define a *state behind* or *SB* curve for (τ_L, p_L) in the same way again remembering to face the appropriate waves. i.e., to the left.

We also define a map from the *SB* curve of (τ_R, p_R) to that of (τ_L, p_L) on the lines $p = \text{constant}$. Since the *SB* curves are easily seen to be monotonic, this map is invertible and is represented by $\tilde{\tau}(\tau)$ if τ lies on the *SB* curve of (τ_R, p_R) and by $\tilde{\tau}^{-1}(\tau)$ if τ lies on the *SB* curve of (τ_L, p_L) . See figure 2.4.

The solution of the Riemann problem is found in the (τ, p) plane by connecting (τ_R, p_R) to (τ_L, p_L) by at

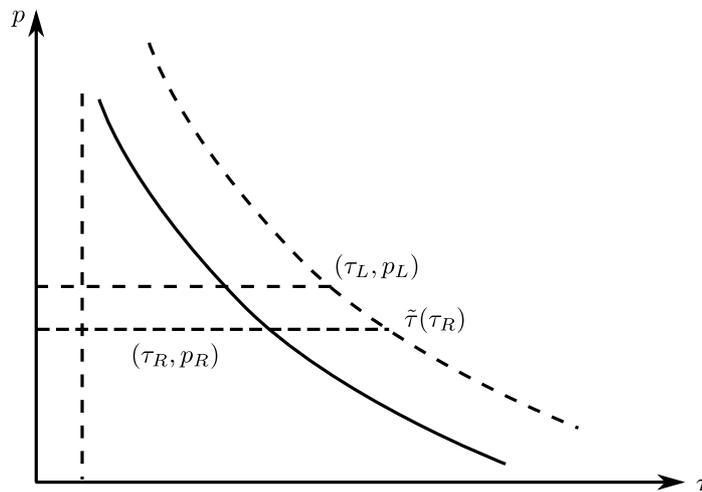


Fig. 2.4.

70

most two in between states, each representing states behind (τ_R, p_R) and (τ_L, p_L) respectively and connected by a slip or contact discontinuity which appears as the horizontal line $p = \text{constant}$. The velocity u in the two states must be the same. From the right hand side, it is determined either by the shock condition

$$u = [(p_R - p) (\tau - \tau_R)]^{1/2},$$

using $u_R = 0$ or by the rarefaction formula

$$u = -\ell_R(\tau_R) + \ell_R(\tau)$$

and on the left side by

$$u - u_L = -[(p_L - p)(\tau - \tau_L)]^{1/2}$$

71 or

$$u - u_L = \ell_L(\tau_L) - \ell_L(\tau).$$

We note that these formulas are continuous at $\tau = \tau_R$ and $\tau = \tau_L$ respectively. We further note that we cannot “add in” rarefaction waves or shock waves as part of the states to be connected because we cannot match the velocities.

To solve the Riemann problem we have only to show that we can always find a horizontal segment where the values of u on the intersections with the two SB curves are the same.

But for $p \rightarrow \infty$ these intersections correspond to the shock portions of the SB curves, the two values of τ are approaching their minima and the difference between u on the left SB curve (\tilde{u}_L) and on the right (\tilde{u}_R) satisfies

$$\tilde{u}_L - \tilde{u}_R \rightarrow -\infty.$$

On the other hand as $\tau \rightarrow \infty$ a line $p = \text{constant}$ intersects the two rarefaction sections of the SB curves. Both $\ell_R(\tau)$ and $\ell_L(\tau) \rightarrow 0$ since

$$\ell = \int_0^p c/\rho d\rho.$$

Hence

$$\tilde{u}_L - \tilde{u}_R \rightarrow u_L + \ell_L(\tau_L) + \ell_R(\tau_R).$$

Hence if

$$u_L + \ell_L(\tau_L) + \ell_R(\tau_R) > 0$$

72 there is always a solution. On the other hand if

$$u_L < -\ell_L(\tau_L) - \ell_R(\tau_R)$$

we have two completed rarefaction waves and there is a vacuum in between.

Thus every Riemann problem can be solved. The configurations are as follows:

If u_L lies in the interval

$$[-\ell_L(\tau_L) + \ell_L(\tilde{\tau}(\tau_R)), \{(p_R - p(\tilde{\tau}^{-1}(\tau_L)) (\tilde{\tau}^{-1}(\tau_L) - \tau_R))^{1/2}\}]$$

we have a rarefaction from the left and a shock from the right. This includes the case $u_L < 0$ since $\tilde{\tau}(\tau_R) < \tau_L$. If u_L lies in

$$[-\ell_L(\tau_L) - \ell_R(\tau_R), \ell_L(\tilde{\tau}(\tau_R)) - \ell_L(\tau_L)]$$

we have two rarefactions possibly with a vacuum and if

$$u_L > \{(p_R - p(\tilde{\tau}^{-1}(\tau_L))) (\tilde{\tau}^{-1}(\tau_L) - \tau_R)\}^{1/2}$$

there are two shocks.

It is also quite easy to show that $d(\tilde{u}_L - \tilde{u}_R) > 0$ provided the Hugoniot curves are star-shaped, which shows that within the class of fan waves our solution is unique.

However, full uniqueness follows only by using a contraction theorem, see, for example, Oleinik [34] or Keyfitz [20].

2.9 Solution of initial value problem

It was originally proposed by Godunov that the initial value problem (IVP) should be solved approximately by considering the initial data as approximately piecewise constant and then solving a set of Riemann problems, each by what we shall call a fan solution. 73

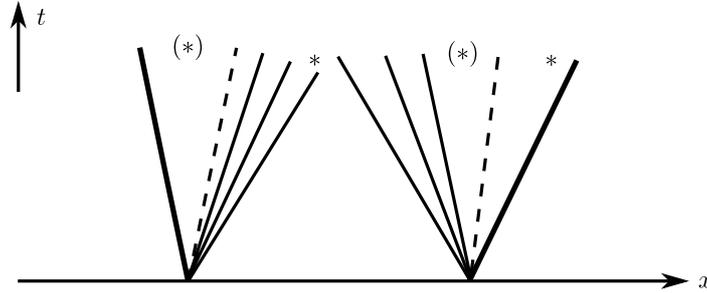


Fig. 2.5.

This solution is considered up to time Δt where $\Delta t/\Delta x < 1/2S$ where S is the maximum of shock speeds or characteristic speeds that occur. This cuts out intersections. At the next time t we have to replace the resulting data again by a piecewise constant solution. It turns out that we must choose this value and place it at $*$ and it must be chosen as the state values at a random point between the $(*)$'s (See Fig. 2.5). Alternatively (Liu) the values at a point that sweeps out the interval regularly will do.

We assume this set up and state the theorem, show what actually happens in some special cases but first a few summary remarks.

The approach is based on Lax [23]. We are looking for a weak solution of

$$u_t + (F(u))_x = 0 \quad (2.30)$$

74 satisfying an entropy condition (stated below). Here u is an n -vector and F is a vector valued function. We assume this system is hyperbolic, i.e., the matrix $F'(u)$ has n real and distinct eigenvalues for all u in some relevant domain. We arrange these eigenvalues, $\lambda_k(u)$, in increasing order

$$\lambda_1 < \lambda_2 < \dots < \lambda_n. \quad (2.31)$$

We also assume the system (2.30) is genuinely nonlinear in a sense to be chosen. A weak solution of (2.36) means a bounded measurable function u such that

$$\int (\chi_t u + \chi_x F(u)) dx = 0 \quad (2.32a)$$

for $\chi \in C_0^\infty$ and a weak solution with initial value $u(x, 0) = \phi(x)$ means a weak solution satisfying

$$\int_{t>0} (\chi_t u + \chi_x F(u)) dx + \int \chi(x, 0) \phi(x) dx = 0 \quad (2.32b)$$

for all smooth vectors χ vanishing for large $|x| + t$. A piecewise continuous solution is a weak piecewise continuous solution and hence satisfies the jump condition across a discontinuity:

$$S[u_K] = [F_k], \quad k = 1, 2, \dots, n, \quad (2.33)$$

where S is the speed of propagation of discontinuity and $[\cdot]$ denotes the jump across the discontinuity.

We now formulate an entropy condition by requiring the following to hold: For some k , $l \leq k \leq n$,

$$\left. \begin{array}{l} \text{while} \\ \lambda_k(u_\ell) > S > \lambda_k(u_r) \\ \lambda_{k-1}(u_\ell) < S < \lambda_{k+1}(u_r) \end{array} \right\} \quad (2.34)$$

Here u_ℓ and u_r are the states to the left and right of the discontinuity respectively. The eigenvalues λ'_k s are also called characteristic speeds. The condition (2.34) says the k^{th} characteristic meets the discontinuity from the left and the $n - (k - 1)$ characteristic from the right, the total being equal to $(n + 1)$ and thus $(n + 1)$ quantities will determine $(2n)$ unknowns and the 'shock speed'. This agrees with gas dynamics and combustion. 75

A discontinuity across which (2.33) and (2.34) hold is called a k -th shock and S will be called a shock speed.

Suppose for some k , $1 \leq k \leq n$, $\text{grad}_u \lambda_k \neq 0$ and is *not* orthogonal to r_k the corresponding eigenvector. If this is so, we say k^{th} fan is *genuinely nonlinear*. We normalise r_k so that

$$r_k \cdot \text{grad}_u \lambda_k = 1. \quad (2.35)$$

If on the otherhand $r_k \cdot \text{grad}_u \lambda_k = 0$ we say the k^{th} fan is *linearly degenerate*.

We consider an example from gas dynamics. The equations read (See (2.9(a)))

$$\begin{aligned} p_t + up_x + \rho c^2 u_x &= 0 \\ \rho u_t + p_x + \rho u u_x &= 0 \\ S_t + S_x u &= 0. \end{aligned}$$

76 Here the matrix $F'(u)$ is given by

$$\begin{pmatrix} u & \rho c^2 & 0 \\ \rho^{-1} & u & 0 \\ 0 & 0 & u \end{pmatrix}$$

and $\lambda = u$ is an eigenvalue and the corresponding eigenvector is given by

$$\begin{pmatrix} 0 \\ 0 \\ u_3 \end{pmatrix}$$

where $u_3 \neq 0$ is arbitrary. Thus the characteristic field corresponding to this eigenvalue is linearly degenerate. (This actually leads to special difficulties in computation, see, e.g., Harten [17]).

We now state the main result.

Theorem 1. *The set of states u_r which are connected to u_ℓ for $|u_r - u_\ell|$ sufficiently small through a k -shock from a smooth one parameter family $u_r = u(\epsilon)$, $-\epsilon_0 < \epsilon \leq 0$, $u(0) = u_\ell$; the shock speed is also a smooth function of ϵ .*

Remark. The entropy condition gives the one sided interval.

We now turn to an important class of solutions, centred rarefaction waves; these are the solutions which depend only on the ratio $(x-x_0)/(t-t_0)$, x_0, t_0 are the centre of the wave.

Let u be a rarefaction wave centred at the origin:

$$u(x, t) = h(x/t). \quad (2.36)$$

77 Substituting this in (2.36), we see that

$$-\frac{x}{t}h' + \frac{1}{t}F'(u)h' = 0 \quad (2.37)$$

where $'$ denotes differentiation with respect to $\xi = x/t$. Thus $\xi = \lambda_k$ is an eigenvalue of $F'(u)$ and h' is a corresponding eigenvector; h is called a k -rarefaction wave.

in view of (2.35), we can take

$$h' = r(h). \quad (2.38)$$

Put $\lambda = \lambda(u_\ell)$; (2.38) has a unique solution satisfying the initial condition

$$h(\lambda) = u_\ell; \quad (2.39)$$

h is defined for all ξ close enough to λ .

Let $\epsilon \geq 0$ be such that h is defined for $\lambda + \epsilon$; write $u_r = h(\lambda + \epsilon)$.

We now construct the following piecewise smooth function $u(x, t)$ for $t \geq 0$ (See Figure 2.6).

$$u(x, t) = \begin{cases} u_\ell & \text{for } x < \lambda t \\ h(x/t) & \text{for } \lambda t \leq x \leq (\lambda + \epsilon)t \\ u_r & \text{for } (\lambda + \epsilon)t < x \end{cases} \quad (2.40)$$

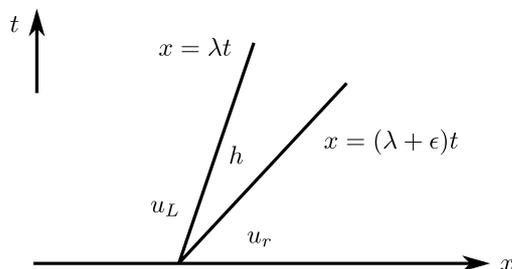


Fig. 2.6.

This function u satisfies the differential equation (2.36) in each of three regions, and is continuous across the lines separating the regions. We shall say that in u the states u_ℓ and u_r are connected by a centred k -rarefaction wave. 78

Theorem 2. *There exists to every state u_ℓ a one parameter family of states $u_r = u(\epsilon)$, $0 \leq \epsilon < \epsilon_0$, connected to u_ℓ by a k - rarefaction wave.*

We now turn to another important problem, Riemann problem, in which the initial value given are

$$\begin{aligned} u &= u_0 \quad \text{for } x < 0 \\ u &= u_1 \quad \text{for } x > 0. \end{aligned}$$

Theorem 3. *There always exists a solution (a fan wave) to the Riemann problem if $|u_0 - u_1|$ is sufficiently small.*

The proof uses only the implicit function theorem and we refer to Lax [23].

Norms: What can we expect: $u_x \in L^1$, not L^2 , at best since u has jump discontinuities. This suggests looking for a solution u of bounded variation.

We now state a theorem due to Glimm and for the proof we refer to Glimm [15].

Theorem 4 (Glimm). *Let (2.30) be hyperbolic, strictly (genuinely) non-linear and F be smooth in a neighbourhood of $u = \tilde{v}$, a constant vector. Then there is a $K < \infty$ and a $\delta > 0$ with the following property:*

79 *If the initial values $u(x, 0)$ are given so that²*

$$d_1 = \|u(\cdot, 0) - \tilde{v}\|_\infty + T.V.u(\cdot, 0) < \delta,$$

then there exists a weak solution of (2.36) for all $x, t \geq 0$ with initial values $u(x, 0)$ such that

$$\|u - \tilde{v}\|_\infty \leq K\|u(\cdot, 0) - \tilde{v}\|_\infty \quad (2.41a)$$

$$T.V. u(\cdot, t) \leq K(T.V.u(\cdot, 0)), \quad t \geq 0 \quad (2.41b)$$

$$\int_{-\infty}^{+\infty} |u(x, t_1) - u(x, t_2)| dx \leq K|t_1 - t_2|(T.V.u(\cdot, 0)) \quad (2.41c)$$

²T.V. = Total variation

For a restricted class including gas dynamics and if

$$\|u(\cdot, 0) - \tilde{v}\|_{\infty}(1 + T.V.u(\cdot, 0)) \leq \delta$$

then there exists a solution satisfying (2.41a) and (2.41b).

We now describe an approximate method developed by Glim to solve any initial value problem $u(x, 0) = u_o(x)$ when the oscillation of $u_o(x)$ is small. The solution u is obtained as the limit of approximate solutions u_h , as $h \rightarrow 0$, which are constructed as follows:

(I) $u_h(x, 0)$ is a piecewise constant approximation to $u_o(x)$

$$u_h(x, 0) = m_j \quad \text{for} \quad jh < x < (j+1)h, \quad j = 0, \pm 1, \dots, \quad (2.42)$$

where m_j is some kind of mean value of $u_o(x)$ in the interval $(jh, (j+1)h)$.

(II) For $0 \leq t < h/\lambda$, $u_h(x, t)$ is the exact solution of (2.30) with initial values $u(x, 0)$ given by (2.42); here λ is an upper bound for $2|\lambda_k(u)|$. This exact solution is constructed by solving the Riemann IVP's, **80**

$$u(x, 0) = \begin{cases} m_{j-1} & \text{for } x < jh, \\ m_j & \text{for } jh, x, \end{cases} \quad (2.43)$$

$j = 0, \pm 1, \dots$. Since the oscillation of u_o is small, m_{j-1} and m_j are close and so by Theorem 3, this IVP has a solution consisting of constant states separated by shocks or rarefaction waves issuing from the points $x = jh, t = 0$ (See Figure 2.7). As long as

$$t < h/\lambda \quad (2.44)$$

these waves do not intersect each other and so the solutions of the IVP (2.43) can be combined into a single exact solution u_h .

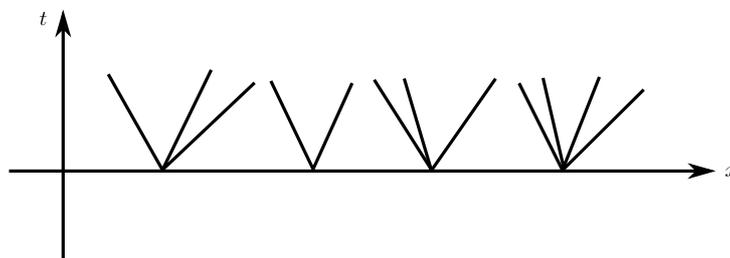


Fig. 2.7.

(III) We repeat the process, with $t = h/\lambda$ as new initial time in place of $t = 0$.

81

It is not at all obvious that this process yields an approximate solution u_h which is defined for all t ; to prove this one must show that the oscillation of $u(x, nh)$ remains small, uniformly for $n = 1, 2, \dots$ and so that one can solve Riemann IVPs (2.43). This estimate turns out to depend very sensitively on the kind of average used to compute the mean values m_j . Glimm has used the following method to compute m_j :

A sequence of random numbers $\alpha_1, \alpha_2, \dots$ uniformly distributed in $[0, 1]$ is chosen; m_j^n then mean value of $u(x, nh/\lambda)$ over the interval $(jh, (j+1)h)$ is taken to be

$$m_j^n = u(jh + \alpha_n h, nh/\lambda). \quad (2.45)$$

Glimm proves the following.

Theorem 5. *A subsequence of u_h converges in L^1 with respect to x , to a weak solution of (2.30), uniformly in t , and for almost all choices of $\{\alpha_n\}$.*

For the proof we refer to Glimm [15] and we illustrate here by an example; see Lax [23].

Consider a Riemann IVP

$$u(x, 0) = \begin{cases} u_\ell & \text{for } x < 0 \\ u_r & \text{for } x > 0 \end{cases}$$

where u_ℓ and u_r are so chosen that the exact solution u consists of the two states u_ℓ, u_r separated by a shock,

$$u(x, t) = \begin{cases} u_\ell & \text{if } x < st, \\ u_r & \text{if } x > st, \end{cases} \quad (2.46)$$

where s is the shock speed. We may take $\lambda > |s|$. Assume $s > 0$. **82**
Glimm's recipe gives

$$u_h(x, h/\lambda) = \begin{cases} u_\ell & \text{if } x < J_1 h \\ u_r & \text{if } J_1 h < x \end{cases}$$

where

$$J_1 = \begin{cases} 1 & \text{if } \alpha_1 < s/\lambda \\ 0 & \text{if } s/\lambda < \alpha_1 \end{cases}$$

Repeating this procedure n times, we obtain

$$u_h(x, nh/\lambda) = \begin{cases} u_\ell & \text{for } x < J_n h, \\ u_r & \text{for } J_n h < x, \end{cases}$$

where $J_n =$ number of $\alpha'_j s, j = 1, 2, \dots, n$, such that $\alpha_j, s/\lambda$. Since $\{\alpha_j\}$ is a uniformly distributed random sequence in $[0, 1]$

$$\frac{J_n}{n} \rightarrow \frac{s}{\lambda}$$

with probability 1; this tells the approximate solution tends almost surely to the exact solution.

One would like to prove more about the solution of the initial value problem. In particular the uniqueness of the solution. Some results are contained in DiPerna [8] but they are not applicable to the general initial value problem because they do not admit shock formations within the flow.

2.10 Combustion. Detonations and deflagrations

In this section, we present a brief account of the elementary theory of **83**

detonation and deflagration waves. These differ from shocks as the increased pressure releases energy and converts one gas into another in a chemical reaction. We denote this energy, per unit mass, the energy of formation, by g and the total energy

$$E = e + g$$

where e is internal energy. In this case the equations of the gas dynamic are

$$\begin{aligned}\rho_t + (\rho u)_x &= 0 \\ (\rho u)_t + (\rho u^2 + p)_x &= 0, \\ \tilde{E}_t + ((\tilde{E} + p)u)_x &= 0,\end{aligned}\tag{2.47}$$

where ρ is the density, u is the velocity, p is the gas pressure. But \tilde{E} depends on the precise nature of the gas and is given by

$$\tilde{E} = \rho E + \rho u^2/2.$$

Let the subscript 0 refer to unburnt gas and 1 to burnt gas. Let the unburnt gas be to the right of the reaction zone and let U be the velocity of the reaction zone. Then the two laws of conservation of mass and momentum are identical with the corresponding laws for shock fronts and we have

$$\rho_0 v_0 = \rho_1 v_1 = m,\tag{2.48}$$

$$p_0 + \rho_0 v_0^2 = p_1 + \rho_1 v_1^2\tag{2.49}$$

84 where $v_i = u_i - U$, $i = 0, 1$, are the relative velocities. The law of conservation of energy now takes the form:

$$E^{(o)}(\tau_o, p_o) + p_o \tau_o + v_o^2/2 = E^{(1)}(\tau_1, p_1) + p_1 \tau_1 + v_1^2/2\tag{2.50}$$

where $E^{(1)}$ and $E^{(o)}$ are two different energy functions. As in the case of shock fronts we consider the Hugoniot function

$$H(\tau_1, p_1; \tau_o, p_o) = E^{(1)}(\tau_1, p_1) - E^{(o)}(\tau_o, p_o) + (\tau_1 - \tau_o)(p_1 + p_o)/2.$$

It should be stressed that the conservation of energy (2.50) is entirely different from that in the case of shock fronts but that any law derived from the conservation of mass and momentum still holds.

Using (2.48) and (2.49), (2.50) can be written as

$$H(\tau_1, p_1; \tau_o, p_o) = 0. \quad (2.51)$$

As in the previous case, the graph of (τ_1, p_1) in the (τ, p) plane, which satisfies (2.51) for fixed (τ_o, p_o) will be called the *Hugoniot curve with center* (τ_o, p_o) . For polytropic gases, we have

$$e = \frac{p\tau}{\gamma - 1}, \quad \gamma \geq 1.$$

If we set $\Delta = g_o - g_1$ ($\Delta \leq 0$ for an exothermic process) and $\mu^2 = \frac{\gamma - 1}{\gamma + 1}$, we find

$$0 = 2\mu^2 H = -p_o(\tau_o - \mu^2\tau_1) + p_1(\tau_1 - \mu^2\tau_o) - 2\mu^2\Delta. \quad (2.52)$$

As in the previous case, this is a rectangular hyperbola. This time the point (τ_o, p_o) does not lie on the hyperbola because of the extra term due to Δ . In fact, if $\Delta \leq 0$, (τ_o, p_o) lies below the Hugoniot curve; see figure 2.8. We assume this in general. 85

The lines through (τ_o, p_o) tangent to $H = 0$ are called *Rayleigh lines*. Their points of tangency, S_1 and S_2 , are called *Chapman-Jouguet (CH) points*. The portion $p > p_o$, $\tau > \tau_o$, of the Hugoniot curve is omitted because it corresponds to the impossible case in which $m^2 < 0$. Here we have used the relation

$$-m^2 = \frac{p_o - p_1}{\tau_o - \tau_1},$$

which follows from (2.48) and (2.49). The upper portion of the Hugoniot curve corresponds to *detonations* (increase in pressure); the portion above S_1 corresponds to strong detonations and below S_1 to weak detonations. The lower portion of the Hugoniot curve corresponds to *deflagrations* (decreases in pressure). These two pieces might appear to be playing the role of a “state behind” curve and we might anticipate that shocks and detonations are linked while deflagrations and simple waves

are related. However, considerations of the internal mechanism appears to eliminate deflagrations and weak detonations except in special cases.

The relative speeds of fronts governed by the Hugoniot curve can be determined by differentiation.

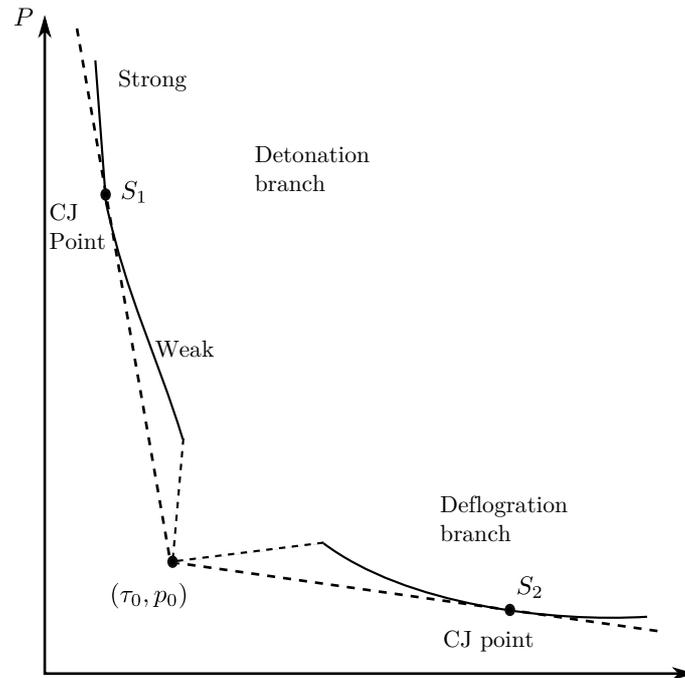


Fig. 2.8. The Hugoniot curve for exothermic gas flow

86

First of all,

$$0 = dH(\tau, p) = TdS + \frac{1}{2}\{(\tau - \tau_o)dp + (p - p_o)d\tau\}.$$

Thus for a Chapman-Jouguet process where $(\tau - \tau_o)dp + (p - p_o)d\tau = 0$ we have $dS = 0$. So at this point

$$\frac{p - p_o}{\tau - \tau_o} = \frac{dp}{d\tau} = \frac{\partial p}{\partial \tau} \Big|_1 = -\frac{1}{\tau_1^2} c_1^2.$$

Thus by the mechanical conditions

$$v_1^2 = -\tau_1^2(p - p_o)/(\tau - \tau_o) = c_1^2.$$

That is, the state behind is sonic relative to the front. On the other hand

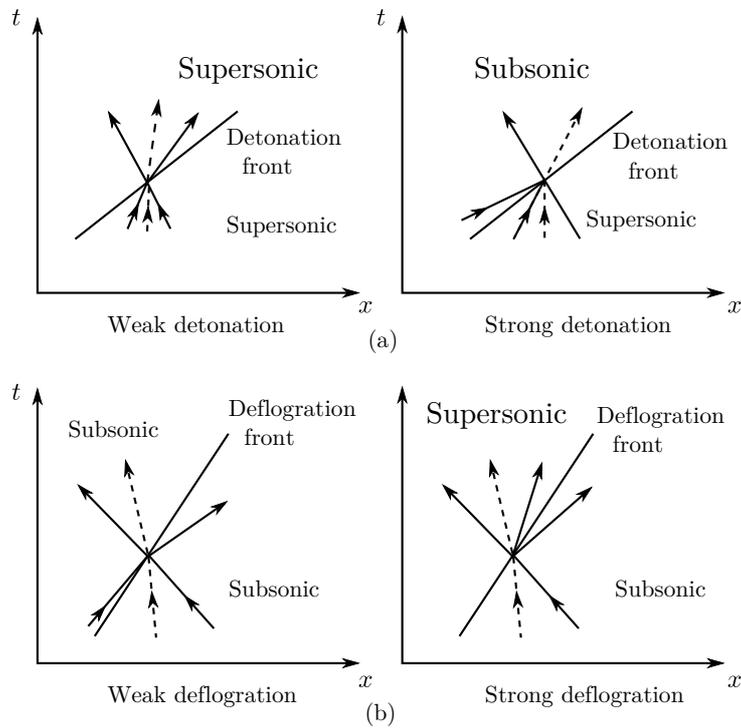
$$v_o^2 = -\tau_o^2(p - p_o)/(\tau - \tau_o)$$

and along the Hugoniot curve

$$dv_1^2 = \tau_o^2((p - p_o)d\tau - (\tau - \tau_o)dp)$$

so that v_o has stationary value for a Chapman-Jouguet process and furthermore it is a minimum. Hence the flow ahead of a strong or weak detonation is supersonic and of a strong or weak deflagration is subsonic provided the Hugoniot curve has the usual convexity. 87

By looking at the Hugoniot curve for the state behind in a similar way we find that v_1^2 is monotonic on each branch. Hence the flow is supersonic behind a weak detonation, subsonic behind a strong detonation, supersonic behind a strong deflagration and subsonic behind a weak deflagration. See Fig. 2.9.

**Fig. 2.9.**

88 This leads to Figure 2.9 which shows the way the three characteristics can leave and enter a front. Each characteristic is carrying one datum. Thus for a strong detonation, one piece of data must be given along with the state in front and the two remaining quantities are then determined. See the corresponding shock wave problem.

2.11 Riemann problem with detonations and deflagrations

We assume there are only two gases, burnt and unburnt, and that the unburnt gas is on the right with $u_R = 0$. Then there are the following possibilities:

To the right of the contact discontinuity or slip line either

- a) a strong detonation or
- b) a CJ detonation followed by a rarefaction wave or
- c) a rarefaction wave

and on the left either a shock or a rarefaction.

This includes the unlikely case (c) when the pressure in the unburnt gas p_R exceeds that in the burnt p_L . Weak detonations and all deflagrations have been eliminated for other reasons to be discussed later.

The SB curve for the left state is the same in section 2.8. The SB curve for the right state row consists of the strong detonation branch connected at the Chapman-Jouguet point to an adiabatic curve $p = p(\tau)$ and there is also the rarefaction curve through p_R for use in case (c).

Exactly the same kind of argument then shows the Riemann problem for the case $p_R < p_L$ can be used again, i.e., connecting the two continuous curves by a line segment and matching up the values of u which are given by the same formulas unless there is a $C - J$ detonation when

$$\tilde{u}_R = c(\tau^*, S^*) - \ell_{CJ}(\tau, S^*) + \ell_{CJ}(\tau^*, S^*),$$

and \tilde{u}_L as before.

Here, τ^*, S^* are the specific volume and entropy at the Chapman-Jouguet detonation and ℓ_{CJ} is the ' ℓ ' corresponding to that state.

The original shock wave argument continues to work for $p_R > p_L$ if a shock or rarefaction on the left and a rarefaction on the right are generated. The excluded case is the two-shock case because the unburnt gas is detonated. For

$$u_L < [(p_L - p(\tilde{\tau}(T_R)))(\tilde{\tau}(\tau_R) - \tau_L)]^{1/2}$$

there will be a solution without detonations. If u_L does not satisfy this inequality, then there is a detonation. However, uniqueness is lacking.

2.12 Internal mechanism

We now investigate why only certain processes take place by looking into the internal mechanism of the front. We look for steady state solutions in one dimension where we now assume the flow has viscosity and heat conductivity. Let the temperature be θ . The conservation of mass remains the same as before:

$$\rho v = \rho_o v_o = m, \text{ a constant.}$$

- 90 We seek a continuous flow that moves from a constant state at $x = -\infty$ to a constant state at $x = +\infty$. We assume the flow is moving from right (unburnt) to left (burnt) and the reaction front has fixed velocity, so $m > 0$, and $x = +\infty$ is ahead, $x = -\infty$ is behind. The conservation of momentum is given by

$$\rho v^2 + p - \mu \frac{dv}{dx} = p, \text{ a constant,}$$

where μ is the coefficient of viscosity.

Next, at any stage the gas has internal energy $E^{(\epsilon)}(\theta)$ and the energy balance is influenced by heat conduction. Denoting by λ , the coefficient of heat conduction, the energy balance is written as:

$$-\lambda \frac{d\theta}{dx} + m \left\{ E^{(\epsilon)}(\theta) + \frac{1}{2} v^2 \right\} + v \left\{ p - \mu \frac{dv}{dx} \right\} = mQ = \text{constant.}$$

Following Friedrichs (see [11]), we assume the balance between burnt and unburnt gases is

$$-v \frac{d\epsilon}{dx} + (1 - \epsilon) S(\theta) = 0,$$

where we try to avoid specifying too much about $S(\theta)$. So, we have three autonomous equations and they have *singular points* exactly at possible end states.

We write $E^{(0)}(\theta_o) = E_o$ and $E^{(1)}(\theta_1) = E_1$ and investigate the singular points and find that at $x = -\infty$ (state '0') there are in general solutions behaving like

$$e^{\alpha_1 x}, e^{\alpha_2 x}, e^{\alpha_3 x},$$

91 where $\alpha_1 > 0$, $\alpha_2 = 0$ and

$$\begin{aligned} \alpha_3 > 0 & \quad \text{if } v_o \geq c_o \\ \alpha_3 < 0 & \quad \text{if } v_o < c_o. \end{aligned}$$

Here the suffix 0 corresponds to the unburnt gas and c_o is the sound speed in the unburnt gas.

The manifold of regular solutions (i.e., tending to constant as $x \rightarrow -\infty$) has two free parameters if $v_o \geq c_o$ and only one if $v_o < c_o$. Call the number of parameters γ_o . Analogously, at the other end, we find the manifold of regular solutions has one free parameter if $v_1 > c_1$ and two if $v_1 \leq c_1$; call the number of free parameters γ_1 .

When can we even hope to find a solution going from state (0) to state (1)? We have three quantities to solve for. We must impose $(3 - \gamma_o)$ initial conditions to leave the state (0) and γ_1 parameters characterise regular solution at $+\infty$. But one is used up by an arbitrary shift in x . Hence $(\gamma_1 - 1)$ parameters are to be chosen subject to $(3 - \gamma_o)$ conditions. So we need $\gamma_1 - 1 \geq 3 - \gamma_o$, i.e., $4 - \gamma_o - \gamma_1 \leq 0$ or else *some other quantity must be specially chosen*. Thus:

$$\begin{aligned} \text{Strong detonation} \quad v_o \geq c_o, \quad \gamma_o = 2 \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad 4 - \gamma_o - \gamma_1 = 0 \\ v_1 \leq c_1, \quad \gamma_1 = 2 \end{aligned}$$

$$\begin{aligned} \text{Weak detonation} \quad v_o \geq c_o, \quad \gamma_o = 2 \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad 4 - \gamma_o - \gamma_1 = 1 \\ v_1 \geq c_1, \quad \gamma_1 = 1 \end{aligned}$$

92

$$\begin{aligned} \text{Strong deflagration} \quad v_o \leq c_o, \quad \gamma_o = 1 \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad 4 - \gamma_o - \gamma_1 = 2 \\ v_1 \geq c_1, \quad \gamma_1 = 1 \end{aligned}$$

$$\begin{aligned} \text{Weak deflagration} \quad v_o \leq c_o, \quad \gamma_o = 1 \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad 4 - \gamma_o - \gamma_1 = 1 \\ v_1 \leq c_1, \quad \gamma_1 = 2 \end{aligned}$$

So over determinacy the internal mechanism matches the *under* determinacy of the characteristic problem as given in § 2.11.

When the topology of the solution curves is worked out, it turns out that

- (i) All strong detonations are possible,
- (ii) *CJ* detonations are only possible if one of the parameters satisfies a special condition,
- (iii) Weak detonations are not possible except under very special conditions,
- (iv) Weak deflagrations are as in (iii) and
- (v) Strong deflagrations do not exist.

Chapter 3

Two dimensional steady flow

3.1 Equations of motion

The next in simplicity to one dimensional flow is a two dimensional steady flow which is also irrotational. We use the following notations throughout this chapter. 93

Let ρ be the density of the gas and p be the pressure. They are considered as functions of the cartesian co-ordinates x, y . Let u, v denote the velocity components along the x -axis and y -axis respectively. The equations of motion reduce to

Conservation of mass:

$$(\rho u)_x + (\rho v)_y = 0. \quad (3.1)$$

Conservation of momentum:

$$(\rho u^2)_x + (\rho uv)_y + p_x = 0 \quad (3.2a)$$

$$(\rho uv)_x + (\rho v^2)_y + p_y = 0. \quad (3.2b)$$

We restrict ourselves to situations with weak shocks and assume the flow is isentropic, i.e., $p = p(\rho)$ with $p'(\rho) > 0$. The irrotational condition implies

$$u_y = v_x. \quad (3.3)$$

With these assumptions the equations for momentum reduce to

$$\begin{aligned} uu_x + vu_y + \frac{1}{\rho}c^2\rho_x &= 0, \\ uv_x + vv_y + \frac{1}{\rho}c^2\rho_y &= 0, \end{aligned}$$

94 where $c = (dp/d\rho)^{1/2}$ is the sound speed. Equivalently,

$$\nabla\left(\frac{1}{2}(u^2 + v^2) + \int \frac{c^2}{\rho}d\rho\right) = 0,$$

which in turn implies Bernoulli's law

$$\frac{1}{2}q^{*2} = \frac{1}{2}(u^2 + v^2) + i(\rho)$$

is constant, where $i(\rho) = \int \frac{c^2}{\rho}d\rho$ and q^* is the speed at zero density.

We recall that across a steady shock the following relations hold:

$$[\rho(\vec{q} - \vec{U}) \cdot \vec{n}] = 0, \quad (3.4)$$

$$[(\vec{q} - \vec{U}) \times \vec{n}] = 0, \quad (3.5)$$

$$[\rho((\vec{q} - \vec{U}) \cdot \vec{n})^2 + p] = 0, \quad (3.6)$$

$$\left[\frac{1}{2}|\vec{q} - \vec{U}|^2 + e(p, \rho) + \frac{p}{\rho}\right] = 0, \quad (3.7)$$

where $\vec{q} = (u, v)$ is the velocity vector, \vec{U} is the shock speed, \vec{n} is normal to the shock and $[\cdot]$ denotes the jump across the shock. Using the fact that the entropy and vorticity changes are to be neglected, see Chapter 2, we see that (3.7) may be written as

$$\left[\frac{1}{2}|\vec{q} - \vec{U}|^2 + i(\rho)\right] = 0$$

or equivalently,

$$[q^{*2}] - [\vec{q}] \cdot \vec{U} = 0.$$

95 In a frame of reference where the normal component of the shock velocity is zero, we have the tangential component of \vec{q} is continuous and so $[\vec{q}] \cdot \vec{U} = 0$. Hence q^* is continuous. This, along with the assumption that the entropy is constant, replaces the conservation of energy and the normal momentum equation. In fact energy and normal momentum equations are not conserved. We are left with the two shock conditions given by the conservation of mass and the continuity of the tangential component of velocity.

Thus we need to consider the flow which satisfies the conservation laws:

$$(\rho u)_x + (\rho v)_y = 0, \quad (3.1a)$$

$$u_y - v_x = 0, \quad (3.3a)$$

in their integrated form where ρ as a functions of the speed \vec{q} is given by Bernoulli's law

$$\frac{1}{2}q^2 + i(\rho) = \frac{1}{2}q^{*2}, \quad \text{where } q^2 = u^2 + v^2.$$

Equation (3.3a) implies there is a functions $\phi(x, y)$, called the *potential function*, such that

$$\phi_x = u, \quad \phi_y = v.$$

Similarly, (3.1a) implies, there exists a function $\psi(x, y)$, called the *stream function*, such that

$$\psi_y = \rho u, \quad -\psi_x = \rho v.$$

The shock conditions reduce to the conditions:

$$\phi, \psi \text{ are continuous.}$$

From the definitions of ϕ, ψ we see that if we introduce $w = u - \sqrt{-1}v$ 96 the equations of motion reduce to

$$d\phi + \frac{\sqrt{-1}d\psi}{\rho(|w|)} = wdz, \quad (3.8)$$

where $z = x + \sqrt{-1}y$, and $\rho(|w|)$ is given by Bernoulli's law. From (3.8) any number of alternative equations are easily written down on the basis of the perfect differential properties of (3.8) as we shall see in the section on hodograph transformations.

3.2 Classifications of flow equations

From Bernoulli's law,

$$(u du + v dv) + \frac{c^2}{\rho} d\rho = 0.$$

Using the mass equation (3.1) and the above we obtain the equation

$$(c^2 - u^2)u_x - uv(u_y + v_x) + (c^2 - v^2)v_x = 0 \quad (3.9)$$

or for the potential ϕ the well-known governing nonlinear equation:

$$(c^2 - u^2)\phi_{xx} - 2uv\phi_{xy} + (c^2 - v^2)\phi_{yy} = 0, \quad (3.10)$$

where c, u, v are functions of $\nabla\phi$. It is convenient to introduce the *Mach number* M :

$$M = q/c.$$

The characteristics of the partial differential equation (3.10) are given by

$$\frac{dy}{dx} = \frac{uv \pm c^2 \sqrt{M^2 - 1}}{c^2 - u^2} \quad (3.11)$$

- 97 The equation (3.10) is hyperbolic, elliptic or parabolic for $M > 1$, $M < 1$ or $M = 1$ respectively. In the first case, the flow is said to be *supersonic*, in the second subsonic and in the third *sonic*. Among these, the first two types of flows are more or less thoroughly studied and the theory is understood if not complete. But when both $M > 1$ and $M < 1$ occur in a single flow, we call it *transonic*. This mixed case has many open problems.

Remark. By Bernoulli's law,

$$q dq + \frac{c^2}{\rho} d\rho = 0.$$

Therefore,

$$\frac{d}{dq}(\rho q) = \rho \left(1 - \frac{q^2}{c^2}\right)$$

At the sonic line $q^2 = c^2$, therefore, as in the case of a scalar conservation equation (Chapter 1), ρq has a maximum.

3.3 Supersonic Flow

The supersonic case is equivalent to the hyperbolic systems we have already studied where we may treat x or y as a “time” variable. Thus Cauchy problems may be solved. The characteristics are given by (3.11):

$$\frac{dy}{dx} = \frac{uv \pm c^2 \sqrt{M^2 - 1}}{c^2 - u^2}.$$

By looking at the case $v = 0$, the path of a particle is given by $dy/dx = 0$, we see that the path bisects the characteristics. Note that as $M \rightarrow 1$ the characteristics become perpendicular to the direction of the flow and tangent to each other. 98

There are simple waves, Riemann invariants and a solution to the analogue of the Riemann problem and the piston problem. However, the two elementary flows of greatest interest correspond to the flow past a bend in the wall (see the figures below).

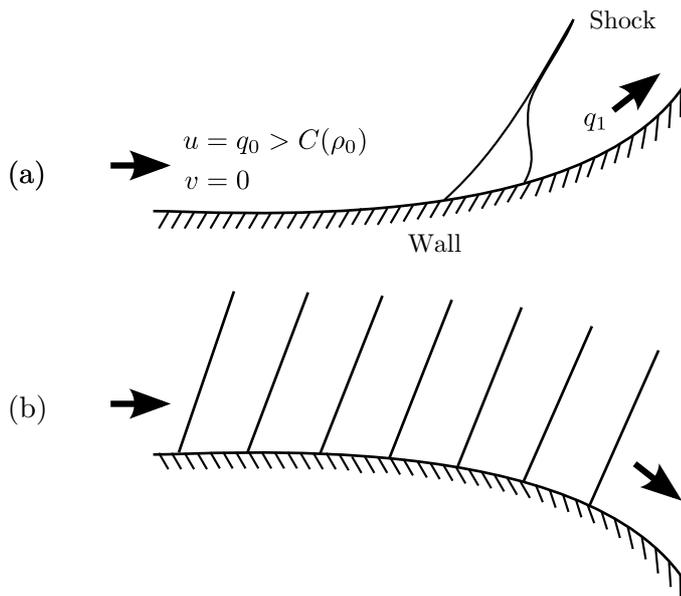


Fig. 3.1.

99 A continuous flow is described by a simple wave whenever it is adjacent to constant state. An inward bend (Fig. 3.1(a)) causes the characteristics to form a cusped envelope and hence a shock. An outward bend (Fig. 3.1(b)) has a continuous rarefaction wave possibly ending at zero density and the escape velocity which, by Bernoulli's law, is equal to q^* . A sharp straight bend yields a rarefaction wave or a straight shock (Figures 3.2 (a) and 3.2(b)).

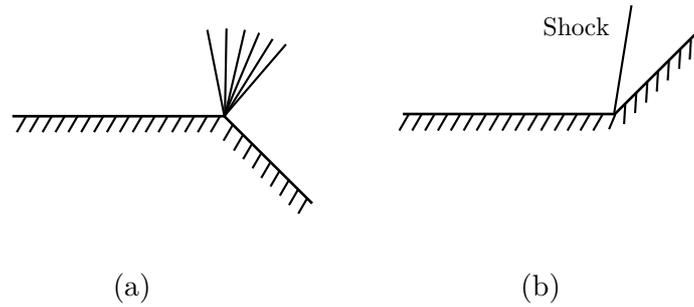


Fig. 3.2.

These problems are easily solved by noting that the image of the rarefaction wave is a characteristic in the (u, v) -plane and the image of the possible states behind the shock is given by a so called *shock polar*. It is convenient to look at these in (θ, q) - plane, where θ is the flow angle $\tan^{-1}(v/u)$.

3.4 Shock polar

From the discontinuity conditions in the form

$$[\rho u]dy/dx - [\rho v] = 0,$$

$$[u] + [v]dy/dx = 0,$$

100 where dy/dx is the slope of the shock, we find that with an initial state $(q_o, 0)$ with $\rho_o = \rho(q_o)$, if the state behind the shock is $(q \cos \theta, q \sin \theta)$, then

$$q \cos \theta = \frac{\rho q^2 + \rho_o q_o^2}{q_o(\rho + \rho_o)}. \quad (3.12)$$

In the limit of a weak shock, we note that a shock can become weak in two ways: either the shock becomes characteristic or the flow on both sides becomes sonic. To see this, we set

$$F = \rho q, \quad F_o = \rho_o q_o, \quad F/F_o = 1 + \delta G \quad \text{and} \quad q/q_o = 1 + \delta p.$$

Then we obtain from (3.12)

$$\cos \theta = 1 + \frac{\delta p \cdot \delta G}{2 + \delta p + \delta G}$$

or

$$\sin^2 \frac{\theta}{2} = -\frac{1}{2} \frac{\delta p \delta G}{2 + \delta p + \delta G}.$$

To a first order approximation, we have

$$\delta G = \delta p \cdot \rho_o \cdot dF/dq.$$

Hence, if the shock is weak we have $\theta \rightarrow 0$ and $\delta p \rightarrow 0$, and we obtain

$$\theta^2 = (\delta p)^2 (-dF/dq),$$

and the shock is characteristic with

$$\theta = \pm \delta p (-dF/dq)^{1/2},$$

provided $dF/dq \neq 0$. However, if the velocities on both sides of the shock are close to sonic, if we approach the limit appropriately: **101**

$$\theta^2 = (\delta p)^3.$$

Note that in the normal shock case $\theta = 0$ and $\delta G \equiv 0$. The shock polar for the polytropic case is as illustrated in Fig. 3.3 (See Bers [2]).

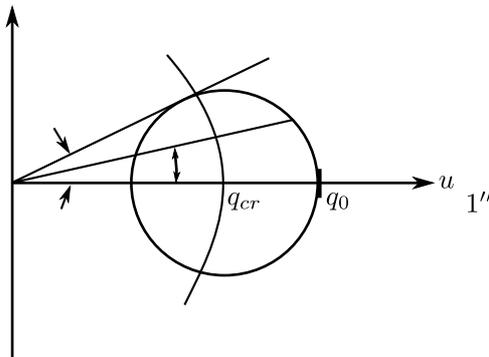


Fig. 3.3. SHOCK POLAR

An important problem is the detached problem. Suppose a projectile or wing is moving in a fluid with supersonic speed. If the projectile (or wing) is round the speed vanishes at the tip (stagnation point). There is a shock in front, *not* an attached shock. The shock is curved and the flow is constant in front because of the curvature behind the shock, the flow is not strictly isentropic or irrotational. If the shock strength is moderate a situation that occurs if the speed at infinity is not too high we may still use the conditions of irrotationality and isentropy. Then there is a subsonic region around the nose.

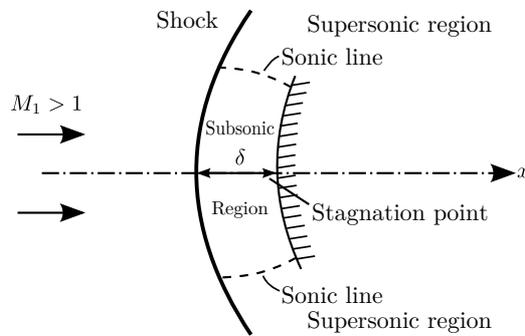


Fig. 3.4. DETACHED SHOCK OVER A BLUFF BODY

The problem is to solve (3.1) and (3.3), a mixed system, with boundary condition $u dy - v dx = 0$ on the object. The shock is a free boundary in the flow represented in the u, v plane by the shock polar. From (3.12) one sees that the flow behind the shock is subsonic on the axis, so there is a region of subsonic flow. However, like the nozzle flow discussed in §12 there are no transonic difficulties such as those described in §7-11. See Bauer et al. [1].

3.5 Equations in the hodograph plane

By the hodograph plane, we shall mean either (u, v) or $(q, 0)$ or even $(\sigma(q), \theta)$ plane, whichever is convenient since they correspond to simple mappings except near $q = 0$. However near $q = 0$, the equation for the potential ϕ behaves like $\Delta\phi = 0$ and we have a very complete understanding of the flow. The special function $\sigma(q)$ is chosen to simplify the equations. From (3.8), we obtain

$$q^{-1} e^{\sqrt{-1}\theta} (d\phi + \frac{\sqrt{-1}d\psi}{\rho}) = dx + \sqrt{-1}dy.$$

Hence the left hand side is a perfect differential. Considering ϕ, ψ as functions of σ, θ , where $\sigma = \sigma(q)$, we find

$$q^{-1} e^{\sqrt{-1}\theta} (\phi_\theta + \frac{\sqrt{-1}\psi_\theta}{\rho}) d\theta + q^{-1} e^{\sqrt{-1}\theta} (\phi_\sigma + \frac{\sqrt{-1}\psi_\sigma}{\rho}) d\sigma$$

is a perfect differential. This requires

$$\{q^{-1} e^{\sqrt{-1}\theta} (\phi_\theta + \frac{\sqrt{-1}\psi_\theta}{\rho})\}_\sigma = \{q^{-1} e^{\sqrt{-1}\theta} (\phi_\sigma + \frac{\sqrt{-1}\psi_\sigma}{\rho})\}_\theta.$$

Thus,

$$(q^{-1})_\sigma \phi_\theta + \rho^{-1} q^{-1} \psi_\sigma = 0, \quad (3.13)$$

$$(\rho^{-1} q^{-1})_\sigma \psi_\theta - q^{-1} \phi_\sigma = 0. \quad (3.14)$$

For the transonic range, it is convenient to introduce

$$\sigma = \int^q q^{-1} \rho \, dq.$$

Then equations (3.13) and (3.14) reduce to

$$\phi_\theta = \psi_\sigma,$$

and

$$\phi_\sigma = -K(\sigma)\psi_\theta,$$

where

$$K(\sigma) = \frac{1}{\rho^3 q^2} \frac{d(\rho q)}{dq}$$

104 is a function of σ only. Thus

$$K(\sigma) \psi_{\theta\theta} + \psi_{\sigma\sigma} = 0. \quad (3.15)$$

Note that the equation is elliptic or hyperbolic according as $K(\sigma)$ is positive or negative, i.e., as σ is positive or negative. Note also that the characteristics of (3.15) are given by

$$\theta = \pm \int_0^\sigma \sqrt{-K(\sigma)} d\sigma + \text{constant}. \quad (3.16)$$

The Legendre Transformation: In solving perturbation problems, it will be convenient to use the Legendre transformation which we introduce now. Recall equations (3.1a) and (3.9). If we regard x, y as functions of u, v the equations reduce to

$$x_v - y_u = 0 \quad (3.17)$$

$$(c^2 - u^2)y_v + uv(x_v + y_u) + (c^2 - v^2)x_u = 0.$$

By (3.17), there is a function $\chi(u, v)$, *Legendre transformation*, satisfying

$$x = \chi_u, \quad y = \chi_v.$$

The relation between the potential ϕ and χ is then given by

$$\phi = xu + yv - \chi$$

as is easily seen. Similarly, we can introduce a *Legendre transformation* $\tilde{\chi}$ such that the stream function ψ can be written as

$$\psi = -\rho xv + \rho yu - \tilde{\chi}.$$

It is clear that we can consider both $\chi, \tilde{\chi}$ as functions of q, θ .

105

Combining the definitions of $\chi, \tilde{\chi}$ with (3.8), we find that

$$\begin{aligned} d\chi &= xdu + ydv = x(\cos \theta dq - q \sin \theta d\theta) + y(-\sin \theta dq - q \cos \theta d\theta), \\ d\tilde{\chi} &= yd(\rho u) - xd(\rho v) = y(\cos \theta \frac{d(\rho q)}{dq} dq - \sin \theta \cdot \rho q d\theta) \\ &\quad - x(-\sin \theta \frac{d(\rho q)}{dq} dq - \cos \theta \rho q d\theta). \end{aligned}$$

From which it follows that

$$\chi_\theta = -\left(\frac{d(\rho q)}{dq}\right)^{-1} \tilde{\chi}_q, \quad \chi_q = (\rho q)^{-1} \tilde{\chi}_\theta.$$

We set

$$d\hat{\sigma} = dq/\rho q \quad \text{and} \quad \hat{K} = \rho q \frac{d(\rho q)}{dq}$$

and the equations become

$$\hat{K}\chi_\theta = -\tilde{\chi}_{\hat{\sigma}}, \quad \chi_{\hat{\sigma}} = \tilde{\chi}_\theta,$$

where $\hat{K}, \hat{\sigma}$ are not the same as K and σ but have the same basic properties.

Furthermore,

$$\hat{K}\chi_{\theta\theta} + \chi_{\hat{\sigma}\hat{\sigma}} = 0. \quad (3.18)$$

We shall use this equation later in connection with perturbation problems.

See Bers [2] for more details on these equations and for early work on subsonic and transonic theory.

3.6 Small Disturbance equation

Von Karman's model nonlinear mixed equation

106

$$\phi_x \phi_{xx} + \phi_{yy} = 0, \quad (3.19)$$

elliptic for $\phi_x > 0$, hyperbolic for $\phi_x < 0$, can be derived from (3.8) by expanding θ about '0' and q about the sonic value c_o . From

$$\rho d\phi + \sqrt{-1} d\psi = \rho q e^{-\sqrt{-1}\theta} dz$$

we have with $\phi = c_o x - \phi$, $\psi = \rho_o c_o y + \psi$,

$$\begin{aligned} & (\rho_o + \rho - \rho_o) (c_o dx - d\phi) + i(\rho_o c_o dy + d\psi) \\ &= (\rho_o q_o + \frac{1}{2} \frac{d^2(\rho q)}{dq^2} \Big|_o (q - c_o)^2 + \dots) (1 - i\theta + \dots) dz \end{aligned}$$

so that from the highest order terms

$$-\rho_o d\phi + id\psi = [A(q - c_o)^2 - \rho_o q_o i\theta] dz - B(q - c_o) dx,$$

where

$$A = \frac{1}{2} d^2(\rho q)/dq^2 \Big|_o \quad \text{and} \quad B = d\rho/dq \Big|_o \cdot c_o.$$

Here, we have used

$$d(\rho q)/dq \Big|_o = 0 \quad \text{and} \quad q_o = c_o.$$

Thus

$$\begin{aligned} \rho_o \phi_x &= B(q - c_o) - A(q - c_o)^2 \\ \psi_x &= -\rho_o q_o \theta \\ -\rho_o \phi_y &= \rho_o q_o \theta \\ \psi_y &= A(q - c_o)^2. \end{aligned}$$

107 The small disturbance equations are obtained by eliminating θ , q and thus

$$\phi_x^2 = \rho_o^{-2} B^2 A^{-1} \psi_y, \quad \text{to first order in } \psi_y \propto (q - c_o)^2,$$

$$\rho_o \phi_y = \psi_x.$$

But $A < 0$ so that after rescaling ($x \rightarrow x$ and $y \rightarrow (-\frac{1}{2}\rho_o^{-1}B^2A^{-1})^{-1/2}y$) we obtain

$$\begin{aligned}\phi_x^2 &= (-\frac{1}{2}\rho_o^{-1}B^2A^{-1})^{1/2}(-2\rho_o^{-1})\psi_y, \\ \rho_o(-\frac{1}{2}\rho_o^{-1}B^2A^{-1})^{-1/2}\phi_y &= \psi_x.\end{aligned}$$

These equations in turn yield (3.19).

On the other hand, by expanding the hodograph equations we obtain

$$\begin{aligned}\sigma\phi_\sigma &= -\psi_\theta \\ \phi_\theta &= \psi_\sigma.\end{aligned}$$

3.7 Transonic flow

We limit our discussion to a few problems centered around transonic wing flow.

The first question is whether we can find a wing shape which at a prescribed subsonic speed at infinity has a smooth flow. Experimental observation suggested in the forties that perhaps such a steady flow did not exist as the speed came close to sonic at infinity and locally supersonic in a region next to wing. However, Lighthill [25] showed how to construct such a wing shape. But this wing was not constructed possibly because the prevailing sense was that in any case it would be unstable. Frankl' and Guderley, [10], [16], proposed that the explanation for the instability lay in the fact that the boundary value problem was ill-posed in the sense of Hadamard if the flow was required to be smooth. This we proved by the author [29].

The implication was that in general flows with transonic regions would have shocks, not detached as in supersonic flow but arising in the supersonic region or cutting it off. This is in fact the case and the shocks have a strong effect on the drag. However, this effect is much less than the drag produced at supersonic speeds by the detached shocks. So that it has proved very useful to use such wings. In [33], Nieuwland has

designed an algorithm for finding, not a transonic wing, but a smooth transonic cross-sections.

In the following sections, we shall discuss the relevant boundary value problems and the related general theory of mixed equations. Then we shall describe the design of aerofoil shapes developed by Garabedian et al., and the method introduced by Murman and Cole for finding the flows at off design Mach numbers.

3.8 General theory of boundary value problems for mixed equations

The mixed equations were first investigated by Tricomi, see [36], for the equation that bears his name and is the hodograph equation for the small disturbance equation:

$$\sigma\psi_{\theta\theta} + \psi_{\sigma\sigma} = 0, \quad (3.20)$$

109 see the end of § 7.

A sample theorem on a boundary value problem which illustrates that the standard Dirichlet problem would be over determined is:

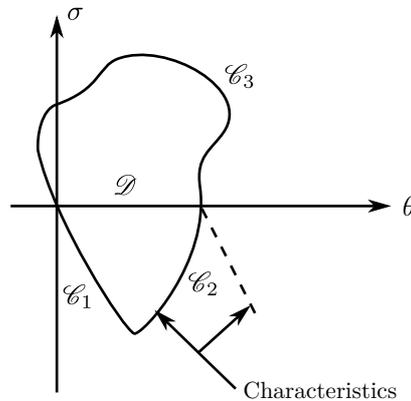


Fig. 3.5. TRICOMI PROBLEM

Theorem. Suppose ψ satisfies (3.20), where ψ is prescribed on the curve C_3 and the characteristic C_2 . Let D be the region enclosed by the curves

C_3, C_1 and C_2 (C_1 is also a characteristic). Suppose the arc C_3 is star-like¹ with respect to the origin:

$$\theta d\sigma - \sigma d\theta > 0.$$

Under these conditions, the solution is unique for ψ_σ, ψ_θ continuous throughout the closure of D (See Garabedian [13]).

It is reasonable to expect such a theorem. Consider the still simpler mixed equation (Lavrentev, Bitsadze [22]).

$$(\operatorname{sgn} \sigma)\psi_{\theta\theta} + \psi_{\sigma\sigma} = 0 \quad (3.21)$$

and the same boundary data. Here C_1 and C_2 are the straight characteristics. Let F be the value of ψ on $\sigma = 0$. Then $G = \partial\psi/\partial\sigma|_{\sigma=0}$ found by solving an elliptic problem is a linear functional of F , in $\sigma \geq 0$. On the other hand solving the wave equation, for $\sigma < 0$, one finds that the data on C_2 alone determine another linear relation between F and G . It is reasonable to expect that one can eliminate G and solve for F uniquely. Then ψ is easily seen to be unique. 110

There are a variety of methods for proving the theorem and also establishing the existence of the solution. The method we shall use is by an estimate. Without loss of generality, we can consider the nonhomogeneous equation with homogeneous boundary data. We rewrite the equation as a system by setting

$$\omega = (\omega_1, \omega_2)^t \quad \text{with} \quad \omega_1 = \psi_\theta, \quad \omega_2 = \psi_\sigma$$

and thus

$$\begin{aligned} A\omega_\theta + B\omega_\sigma &= f \\ \omega_1 d\theta + \omega_2 d\sigma &= 0 \quad \text{on} \quad C_2 + C_3 \end{aligned} \quad (3.22)$$

where

$$A = \begin{pmatrix} \sigma & 0 \\ 0 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (3.23)$$

¹This condition means, as a point moves along C_3 in the counterclockwise direction the angle which it makes the θ -axis is increasing.

which is a symmetric system.

111 We seek a matrix C with a certain property: Take the ‘scalar product’ of (3.22) with C , where AC and BC are symmetric. Integrate by parts over the domain D . Apply the boundary condition. Choose C so that the boundary and area integrals are positive definite. Denote the area integral by $Q(\omega, \omega)$. Then

$$Q(\omega, \omega) = (f, C\omega),$$

where (\cdot, \cdot) is defined for any two column vector functions

$$f = (f_1, f_2)^t, \quad g = (g_1, g_2)^t$$

by

$$(f, g) = \iint_D (f_1 g_1 + f_2 g_2) dx dy.$$

Thus if $f \equiv 0$ then $\omega \equiv 0$ which proves uniqueness.

We now proceed to find C . If

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

we find

$$(C\omega, A\omega_\theta) = (AC\omega, \omega_\theta) = \frac{1}{2}(AC\omega, \omega) - \frac{1}{2}((AC)_\theta\omega, \omega),$$

provided AC is symmetric, i.e., $\sigma c_{12} = -c_{21}$, and

$$(C\omega, B\omega_\sigma) = \frac{1}{2}(BC\omega, \omega)_\sigma - \frac{1}{2}((BC))\sigma\omega, \omega),$$

provided BC is symmetric, i.e. $c_{11} = c_{22}$. Hence the boundary term is, by Green’s theorem,

$$\frac{1}{2} \int \omega_1^2 \sigma (bd\sigma + cd\theta) + 2\omega_1\omega_2(\sigma cd\sigma - bd\theta) - \omega_2^2 (bd\sigma - cd\theta).$$

Here $c_{11} = c_{22} = b$ and $c_{12} = c$.

The contribution to this term from $C_2 + C_3$ where $\psi = 0$ and we may write

$$\omega_1 = \alpha d\sigma, \quad \omega_2 = -\alpha d\theta \quad (3.25)$$

112 is

$$\frac{1}{2} \int_{C_2+C_3} \alpha^2 (bd\sigma - cd\theta) (\sigma d\sigma^2 + d\theta^2).$$

And the integral on the characteristic C_1 where $d\theta + \sqrt{-\sigma}d\sigma = 0$ is

$$-\frac{1}{2} \int_{C_1} \{(\sqrt{-\sigma}\omega_1 - \omega_2)^2 (b + c\sqrt{-\sigma})\} d\sigma.$$

Hence to fulfil the required positivity C must satisfy:

$(AC)_\theta + (BC)_\sigma$ is a negative definite matrix

$$\begin{aligned} bd\sigma - cd\theta &\geq 0 \quad \text{on } C_2 + C_3 \\ (b + c\sqrt{-\sigma})d\sigma &\leq 0 \quad \text{on } C_1. \end{aligned}$$

We then find the explicit requirements,

$$\begin{aligned} \sigma b_\theta - (\sigma c)_\sigma &\leq 0 \\ -b_\theta + c_\sigma &\leq 0 \\ (\sigma c_\theta + b_\sigma)^2 &\leq (\sigma b_\theta - (\sigma c)_\sigma)(-b_\theta + c_\sigma) \end{aligned} \quad (3.26)$$

in D and the boundary conditions

$$\begin{aligned} (bd\sigma - cd\theta) (\sigma d\sigma^2 + d\theta^2) &\leq 0 \quad \text{on } C_2 + C_3 \\ (b - \sqrt{-\sigma}c) &\geq 0 \quad \text{on } C_1 \end{aligned}$$

The choice $b = \theta$, $c = \sigma$ for $\sigma \geq 0$, $c = 0$ for $\sigma \leq 0$ satisfies these requirements. This completes the uniqueness theorem. Note that the requirements would also be fulfilled if C_2 was not characteristic but satisfied only $\sigma d\sigma^2 + d\theta^2 \geq 0$ and that the star-shaped condition on C_2 could be freed up by changing b and c . 113

A weak solution $\omega \in H_*$ of (3.22) and (3.25) satisfies

$$-(Lv, \omega) = (v, f)$$

for all smooth vectors v such that

$$v_1 = 0 \quad \text{on} \quad C_2 + C_3$$

and, since the matrix $A + \sqrt{-\sigma}B$ is singular, such that

$$v_2 - v_1 \sqrt{-\sigma} = 0 \quad \text{on} \quad C_1,$$

where $L = A \partial/\partial\theta + B\partial/\partial\sigma$ and H_* is defined below.

This is the adjoint problem. To use the projection theorem it would suffice to find a Hilbert space for which (v, f) would be a bounded linear functional of Lv for v satisfying boundary conditions. But (v, f) is a bounded linear functional in v in some weighted L^2 -space and f in its corresponding space. So, we have to find an appropriate space in which v is bounded in terms of Lv if

$$v_1 = 0 \quad \text{on} \quad C_1 + C_2 + C_3. \quad (3.27)$$

We proceed to the details.

Let H_* be the Hilbert space of all pairs of measurable functions $u = (u_1, u_2)$ for which the norm

$$\|u\|_*^2 = \iint_D (ru_1^2 + u_2^2) d\theta d\sigma$$

is finite; the inner product is given by

$$(u, v)_* = \iint_D (ru_1v_1 + u_2v_2) d\theta d\sigma,$$

114 where $r^2 = \theta^2 + \sigma^2$.

Let V be the set of all function $\omega = (\omega_1, \omega_2)$ with continuous derivatives and such that

$$\begin{aligned} \omega &= (0, 0) \quad \text{at} \quad r = 0, \\ \omega_1 &= 0 \quad \text{on} \quad C_1 + C_2 + C_3, \end{aligned}$$

and

$$\iint_D \left(\frac{1}{r} (L\omega)_1^2 + (L\omega)_2^2 \right) d\theta d\sigma < \infty$$

Let H^* denote the Hilbert space of all measurable functions $u = (u_1, u_2)$ for which

$$\|u\|^* = \left\{ \iint_D \left(\frac{1}{r} u_1^2 + u_2^2 \right) d\theta d\sigma \right\}^{1/2}$$

is finite; the inner product is

$$(u, v)^* = \iint_D \left(\frac{1}{r} u_1 v_1 + u_2 v_2 \right) d\theta d\sigma.$$

Note that $LV \subset H^*$. We now state the following.

Theorem. *There exists a weak solution of (3.22) and (3.25) for every $f \in H^*$.*

To prove the theorem we require the following lemma which will be proved later.

Lemma. *For all $v \in V$, $f \in H^*$*

$$|(v, f)| \leq B \|Lv\| * \|f\|^*,$$

where B is a constant.

Proof of the Theorem; For $v \in V$ define

115

$$G(Lv) = (v, f).$$

By the lemma G is bounded on $LV \subset H^*$. Thus by Hahn-Banach theorem G can be extended to H^* as a bounded linear functional. Thus by classical Riesz's representation theorem there is a $t \in H^*$ such that

$$(Lv, t) = (v, f) \quad \text{for all } v \in V.$$

The function ω defined by $\omega_1 = -t_1/\Omega$, $\omega_2 = -t_2$ will belong to H_* and satisfies $-(Lv, \omega) = (v, f)$. Thus ω is the required weak solution and this completes the proof.

Proof of the Lemma: By Schwarz's inequality, we obtain

$$|(v, f)| \leq \|v\|_* \|f\|^*.$$

Thus the proof will be complete if we prove the a-priori estimate

$$\|v\|_* \leq B \|Lv\|^*,$$

where B is a constant.

We proceed to do this, Set

$$v = C\tilde{v}. \quad (3.28)$$

Again consider

$$(Lv, \tilde{v}) = (Av_\theta + Bv_\sigma, \tilde{v}) = (A(C\tilde{v})_\theta + B(C\tilde{v})_\sigma, \tilde{v}).$$

Again by rearranging terms properly, we can integrate by parts. The boundary condition (3.7) becomes $b\tilde{v}_1 + c\tilde{v}_2 = 0$. Set

$$(Lv, \tilde{v}) = I_1 + I_2,$$

116 where I_1 is area integral and I_2 is surface integral. It is easy to see that if (3.26) is satisfied then I_1 is positive definite; also I_2 is non-negative. Thus

$$|(Lv, \tilde{v})| \geq I_1.$$

On the other hand, for any $\lambda > 0$,

$$|(Lv, \tilde{v})| \leq \lambda \|Lv\|^{*2} + \frac{1}{\lambda} \|\tilde{v}\|_*^2$$

and hence

$$I_1 \leq \lambda \|Lv\|^{*2} + \frac{1}{\lambda} \|\tilde{v}\|_*^2, \quad \lambda > 0.$$

Thus if $\|\tilde{v}\|_*$ can be estimated in terms of I_1 then by choosing λ sufficiently large, we can estimate $\|\tilde{v}\|_*$ in terms of $\|Lv\|^*$. For the same choice of b and c one can estimate $\|\tilde{v}\|_*$ in terms of I_1 . This estimate is obtained by the same method as was used in the uniqueness theorem.

Once the existence of the weak solution has been established, one proceeds to determine whether it has in fact some better properties but we shall not do that here except to say that since the smoothness properties are local, elliptic methods suffice in elliptic regions, hyperbolic in hyperbolic regions, and that the solution is a strong solution everywhere.

For general description, see Morawetz [30]. See also Osher [35] for a different approach.

3.9 The boundary value problems of transonic wing flow

The two most important boundary value problems for transonic wing flow are for the nonlinear solution and for the linearized flow about it.

In the physical plane, the flow potential ϕ satisfies (3.10), the boundary condition $\partial\phi/\partial n = 0$ on the wing and at infinity, $\nabla\phi$ is prescribed. We know from incompressible flow that we need, for a well-posed problem, the additional condition (the Kutta-Joukowski condition) that the circulation at infinity adjusts itself so that the flow past a wing with a cusp at the trailing edge has finite velocity. For compressible flows by Bernoulli's law, no infinite velocity is possible any way and the circulation adjustment is chosen to prevent. 117

It is not unreasonable to anticipate that the problem is overdetermined on the basis of the boundary value problems discussed earlier. There are two possibilities: Shocks in general and special smooth solutions.

The problem with shocks for a general aerofoil has only been tackled numerically. There is some indication that its perturbation problem is well-posed, see Morawetz [27].]

There exist numerical codes ("analysis" codes) for solving the boundary value problem using artificial viscosity or penalty methods. The original viscosity method of Murman and Cole [32] is described in § 12. It's basic ideas have been incorporated and considerably modified by Jameson [18] and can now be used on three dimensional problems. Bristeu et al. [4] has used finite elements and a penalty method.

The lag of the theory behind numerical experiment is not surprising especially when one realizes how limited the theory is with respect to one dimensional flow.

118 In the case of no shocks, the problem can be looked at in the hodograph plane. under the hodograph transformation, see § 5, the problem transforms into a free boundary value problem². Consider the symmetric wing section which has a singly curved image in the hodograph plane. The equation, say for ψ , (3.15), is linear. There is a prescribed singularity at the image point of the point at infinity. The boundary condition $\psi = 0$ is imposed on the axis $\theta = 0$ and the unknown image of the wing, see Fig. 3.6. But there is a second boundary condition because the flow angle θ is a prescribed function of the arc length on the wing. Using (3.8), this yields, on the free boundary,

$$d\phi = q(\sigma)e^{\sqrt{-1}\theta}(dX(\theta) + \sqrt{-1}dY(\theta)),$$

where $x = X(\theta)$, $y = Y(\theta)$ describe the wing. This is only one condition since $\tan \theta = dY/dX$. Then using

$$d\phi = \psi_{\sigma}d\theta - K\psi_{\theta}d\sigma$$

we have the extra condition on ψ :

$$-\psi_{\sigma}d\theta + K\psi_{\theta}d\sigma = q(\sigma).$$

Solutions for special shapes known as *super-critical* airfoils corresponding to smooth flow pasta wing can be found. One solves the boundary value problem without the last condition (3.31) using some smooth boundary and a well posed problem. But we know from § 8 that we cannot expect to solve the problem with full Dirichlet conditions. Instead use the following procedure:

²This approach has been explored by Brezis and Stampacchia [3] for subsonic flows

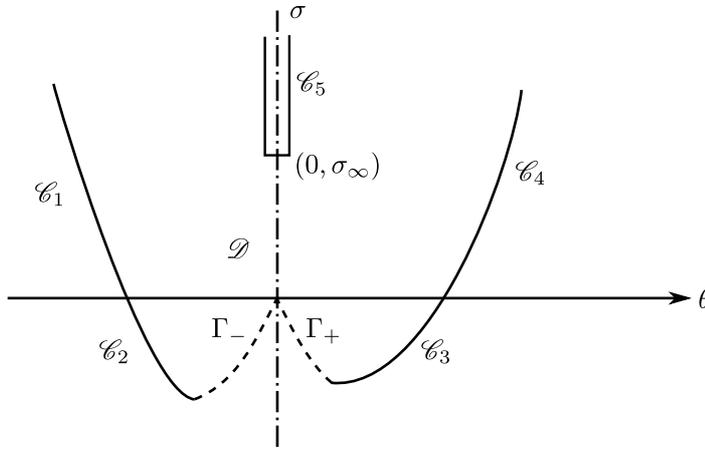


Fig. 3.6.

119

Solve the boundary value problem:

$$\begin{aligned} K\psi_{\theta\theta} + \psi_{\sigma\sigma} &= 0 \quad \text{in } D \\ \psi &= 0 \quad \text{on } C_1 + C_2 + C_3 + C_4 + C_5, \end{aligned}$$

with prescribed singularity at $(0, \sigma_\infty)$. Here $C_1 + C_2$ and $C_3 + C_4$ are smooth arcs satisfying

$$\sigma d\sigma^2 + d\theta^2 \geq 0,$$

and C_5 is a slit on σ -axis s.t. $\sigma_\infty \leq \sigma < \infty$ on C_5 .

The rest of the boundary of D consists of two characteristics Γ_- and Γ_+ issuing from the origin until C_2, C_3 are intersected. It can be shown by the methods of the preceding section that this is a well-posed problem. The singularity at $(0, \sigma_\infty)$ is a bit messy (see Gilbarg [14]; but for our purposes it suffices to treat it like the incompressible flow singularity which would require taking $K(\sigma_\infty) = 1$ and

$$\psi_\theta - \sqrt{-1}\psi_\sigma - A(\phi + \sqrt{-1}\sigma)^{3/2} = 0(|\phi + \sqrt{-1}\sigma|^{1/2}) \quad (3.32)$$

where A is related to the prescribed speed of the wing).

120

We now have part of a flow and part of a supercritical wing image ($C_1 + C_2 + C_3 + C_4$) in the hodograph plane. To choose the wing, we continue the solution ψ across the characteristic gap, Γ_+ and Γ_- , by solving the appropriate Goursat problem. It is not unreasonable to expect that there will be curve joining C_2 to C_3 on which $\psi = 0$ especially if the hyperbolic region is small. At this stage we have solved the hodograph problem. The next stage is to find the physical image using (3.8). A short calculation shows that this will fail if $K(\sigma)\psi_\theta^2 + \psi_\sigma^2$ changes sign (a limiting line occurs where $K(\sigma)\psi_\theta^2 + \psi_\sigma^2 = 0$). But again for sufficiently small hyperbolic regions this does not happen and we do in fact find a physical flow.

In the next section, we describe the method used by Garabedian to generate smooth flows. Fung et al. [12] has used the idea of finding first a purely subsonic flow with a sonic line separating two subsonic regions adjusting the equation of state and then continuing the flow from the sonic line into the smaller region using the right equation of state and finding a new profile.

There are other possibilities for generating supercritical wing sections all involving some form of unique continuation.

121 Having constructed a smooth flow and a wing section, one way or another, one asks what happens to this flow when it is a disturbed say by changing its tilt (angle of attack) or by changing its speed at infinity (Mach number). The evidence both numerical and experimental is that a shock develops at the rear sonic point on the wing profile. It increases the drag (and hence the fuel consumption). Theoretically there are no results on the nature of this shock flow.

3.10 Perturbation boundary value problem

It would be useful to know what flows close to supercritical (smooth transonic) flows look like. First one looks at the general perturbation problem assuming that the nearby flow is smooth. This leads to a contradiction. Then one asks what actually happens.

First, the perturbation equation has to be determined. The direct method from equations (3.9) is tedious. Instead, for the Legendre po-

tential

$$\chi(u, v) = xu + yv - \phi(x, y)$$

we see that the perturbation Legendre potential $\delta\chi$ satisfies

$$\delta\chi = \delta x \cdot u + \delta y \cdot v - \phi_x \delta x - \phi_y \delta y - \delta\phi$$

in terms of a perturbation potential $\delta\phi$ and the perturbations δx , δy . But $\phi_x = u$ and $\phi_y = v$ so that $\delta\chi = -\delta\phi$ to first order. Thus, in the hodograph variables θ , $\hat{\sigma}$ of the *undisturbed flow* since $\delta\chi$ satisfies (3.18) so does $\delta\phi$. Thus

$$\hat{K}(\delta\phi)_{\theta\theta} + (\delta\phi)_{\hat{\sigma}\hat{\sigma}} = 0. \quad (3.33)$$

On the perturbed boundary given by

122

$$y = Y(x) + \delta Y(x)$$

we find

$$\frac{\partial\phi}{\partial n}(x, Y + \delta Y) + \frac{\partial}{\partial n}(\delta\phi) = 0,$$

or to first order

$$\frac{\partial}{\partial n}\delta\phi - \left(\frac{\partial}{\partial n}\phi_y\right)\delta Y = 0. \quad (3.34)$$

At infinity, if there is a change in Mach number at infinity, there is a prescribed singularity of order 3/2 as in the unperturbed case; see (3.32).

If we restrict ourselves to solutions with continuous derivatives (no shocks) then one finds by using the methods of the first section of this chapter, that the problem is ill-posed if we fix the Mach number and change the profile, i.e., $\delta Y \neq 0$. For the symmetric profile see Morawetz [28], for the non-symmetric case Cook [5]. Very partial results exist if we change Mach number, Morawetz [29, III].

The simplest proof amounts to showing that the solution is uniquely determined up to a one parameter family by the prescribed data outside a ‘‘characteristic gap’’, i.e., by data on $C_1 + C_2 + C_3 + C_4$, see Fig. 3.6, since $\hat{\sigma} = \hat{\sigma}(\sigma)$. Therefore the function $\delta Y(x)$ cannot be arbitrary in the gap since $\delta\phi$ is determined by unique continuation from Γ_+ , Γ_- . A more elaborate proof shows that $\delta Y(x) = 0$ in the gap if $\delta Y \equiv 0$ on $C_1 + C_2 + C_3 + C_4 + C_5$.

The next question is how to find a well-posed perturbation problem that represents a disturbance with shocks. The answer probably lies in finding a suitable singular perturbation for the case where on the whole boundary the value of δY is given arbitrarily. This might be accomplished by admitting singularities into the perturbation velocities at the places where shocks are expected, i.e., the points where the sonic line hits the profile. Thus the perturbation flow velocities would be a small variation on the unperturbed flow velocities but not right at the sonic points on the boundary. See [31] where such a singular Dirichlet problem is solved. 123

3.11 Design by the method of complex characteristics

The method of complex characteristics has been introduced and used successfully by Bauer et al [1], in the computation of flows and profiles. It began with the computation of flows with Mach number greater than one at infinity where the object was to determine the subsonic flow behind an analytical shock, see § 4. Its full strength came in the computation of supercritical airfoils, i.e., transonic but shock free with some Mach number less than one at infinity. We sketch here the principles involved in one of the early computations.

We are given a velocity at infinity for a flow. The object is to find an airfoil and a smooth flow past it with this velocity at infinity and with somewhat indefinitely specified characteristics:

- 124
- (i) a large supersonic region,
 - (ii) a large decrease in the pressure (a large increase in the velocity) to control boundary layer separation on the forward end, and
 - (iii) a subsonic cusped trailing edge with the streamline from the upper surface meeting the one from the lower surface smoothly.

In the most recent work the object has been to specify the speed as a function of arc length along the airfoil but we will not describe this.

We write the equations of motion (3.3) and (3.9) as

$$SU_x + TU_y = 0,$$

where

$$U = \begin{pmatrix} u \\ v \end{pmatrix} \quad S = \begin{pmatrix} c^2 - u^2 & -uv \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} -uv & c^2 - v^2 \\ -1 & 0 \end{pmatrix}$$

and put this system in characteristic form formally. This yields the characteristic equations

$$\begin{aligned} y_\xi + \lambda_+ x_\xi &= 0, & u_\xi - \lambda_- v_\xi &= 0, \\ y_\eta + \lambda_- x_\eta &= 0, & u_\eta - \lambda_+ v_\eta &= 0, \end{aligned} \quad (3.35)$$

where

$$\lambda_\pm = \frac{uv \pm c \sqrt{q^2 - c^2}}{c^2 - u^2}.$$

Thus in the subsonic region where $q^2 < c^2$ the characteristics are complex. Note that ξ, η are functions of u, v .

Claim. ξ, η may be chosen so that in the real plane the solution is real. 125

Proof. Let $\omega = u - \sqrt{-1}v$, $\omega^* = u + \sqrt{-1}v$. Suppose $\xi = \xi(\omega, \omega^*)$ and write

$$\begin{aligned} \lambda_+(\omega, \omega^*) &= \lambda_1(\omega, \omega^*) + \sqrt{-1}\lambda_2(\omega, \omega^*) \\ \lambda_-(\omega, \omega^*) &= \lambda_1(\omega, \omega^*) - \sqrt{-1}\lambda_2(\omega, \omega^*), \end{aligned}$$

with $\lambda_1 = uv/(c^2 - u^2)$, $\lambda_2 = c \sqrt{q^2 - c^2}$, $\text{Re}(q^2 - c^2) > 0$. If u, v are real, then λ_1, λ_2 are real and

$$\lambda_+(\omega, \omega^*) = \overline{\lambda_-(\bar{\omega}^*, \bar{\omega})}$$

where a bar above denotes the complex conjugate. Hence

$$u_\xi(\omega, \omega^*) = \lambda_-(\omega, \omega^*)v_\xi(\omega, \omega^*)$$

and changing values of the variable $\omega \rightarrow \bar{\omega}^*$, $\omega^* \rightarrow \bar{\omega}$ and because u and v are real, we obtain

$$\overline{u_{\xi}(\omega, \omega^*)} = \overline{\lambda_+(\omega, \omega^*)} \overline{v_{\xi}(\omega, \omega^*)}$$

or

$$u_{\bar{\xi}}(\omega, \omega^*) = \lambda_+(\omega, \omega^*) v_{\bar{\xi}}(\omega, \omega^*).$$

This shows $\eta = \bar{\xi}$ is a possible characteristic variable in the subsonic region. \square

A simple example: Consider the Cauchy Riemann equations

$$\begin{aligned} u_x + v_y &= 0, \\ v_x - u_y &= 0. \end{aligned}$$

126 Here $\lambda_{\pm} = \pm \sqrt{-1}$ and therefore

$$\begin{aligned} y_{\xi} + \sqrt{-1}x_{\xi} &= 0, & u_{\xi} - \sqrt{-1}v_{\xi} &= 0, \\ y_{\eta} - \sqrt{-1}x_{\eta} &= 0, & u_{\eta} + \sqrt{-1}v_{\eta} &= 0. \end{aligned}$$

So, $\omega = u - \sqrt{-1}v$, $\bar{\omega} = u + \sqrt{-1}v$ are characteristic co-ordinates.

Note. Various problems could be solved in some limited region, for example, a Cauchy or Goursat problem. Now consider the elliptic region. There λ_{\pm} are complex but the system remains valid. We consider x, y, u, v as complex quantities. Note that $q^2 = u^2 + v^2$ and $c^2(q)$ is an analytic function of q . Of course, we look for solutions which are real for real x, y .

Remarks. (1) Since the solution is analytic and independent of the path the number of actually used real variables can be reduced to *three*.

(2) λ_{\pm} are analytic in some slit domain because of the singularities at $q \pm c = 0$. This surface $q \pm c = 0$ forms a two dimensional manifold in four dimensional space (ω, ω^*) .

We examine a difference scheme for a Goursat problem and first consider the real case. The difference scheme is given by

$$\begin{aligned} p(P) - p(Q) + \lambda_+(Q)(q(P) - q(Q)) &= 0, \\ p(P) - p(R) + \lambda_-(R)(q(P) - q(R)) &= 0, \end{aligned}$$

where we use p, q as variables. We want to solve for $p(P)$ and $q(P)$, see Fig. 3.7. 127

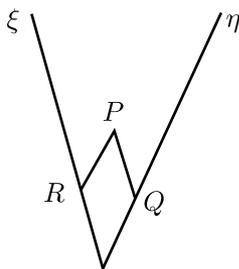


Fig. 3.7.

Here the data is prescribed on $\xi = 0, \eta = 0$ (For the Goursat problem see Garabedian [13] pp 118-119). We can solve for $p(P), q(Q)$ provided $\lambda_+ \neq \lambda_-$ or equivalently $q^2 \neq c^2$.

In the complex case, the same argument holds, but note that we should take $\overline{\Delta\xi} = \Delta\eta$ in the elliptic region. How do we guarantee that we have a real solution x, y for real u, v ? Note that the equations for x, y are linear and they have real coefficients if u, v are real. Hence $\text{Re}(x)$ and $\text{Re}(y)$ are solutions and these are real.

In practice, it has proved better to prescribe

$$\begin{aligned} x(\xi, \eta_0) &= f(\xi) + g(\eta_0) \\ x(\xi_0, \eta) &= g(\eta) + f(\xi_0), \end{aligned}$$

and to get real solutions in the real plane by choosing $f(\xi) = \overline{g(\overline{\xi})}$.

The following problems arise:

- (i) How to choose f so that a stagnation point appears before a singularity?

- 128 (ii) How to choose f so that the streamlines of the profile join smoothly at a trailing edge?

One usually chooses as a trial f, g the corresponding data for the Cauchy-Riemann equations and then adjust.

Remark. Existence up to a singular point follows by a Cauchy-Kowalewski type of argument for a Goursat problem.

3.12 Numerical solution with shocks: Off design computations

We now consider the problem of computing flows for supercritical airfoils at off design, e.g., with different velocities at infinity than specified. This problem was open for a number of years and the first numerical solution of a nonlinear mixed equation with shock was given by Murman and Cole [32]. We treat a similar case. Consider the small disturbance equation, (3.19),

$$\phi_x \phi_{xx} + \phi_{yy} = 0$$

which is elliptic for $\phi_x > 0$, hyperbolic for $\phi_x < 0$. Suppose we have the following data on the boundary of the region (see Figure 3.8):

$$\begin{aligned} \text{(i)} & \quad \phi_y(x, 0) \in C_o^\infty \\ \text{(ii)} & \quad \phi_y(x, b) = 0 \\ \text{(iii)} & \quad \phi_x(0, y) \quad \text{and} \quad \phi_x(a, y) \quad \text{are given.} \end{aligned} \tag{3.36}$$

The values of ϕ_y on the shaded segment corresponds to a given shape of airfoil $y = Y(x)$ in the small disturbance approximation.

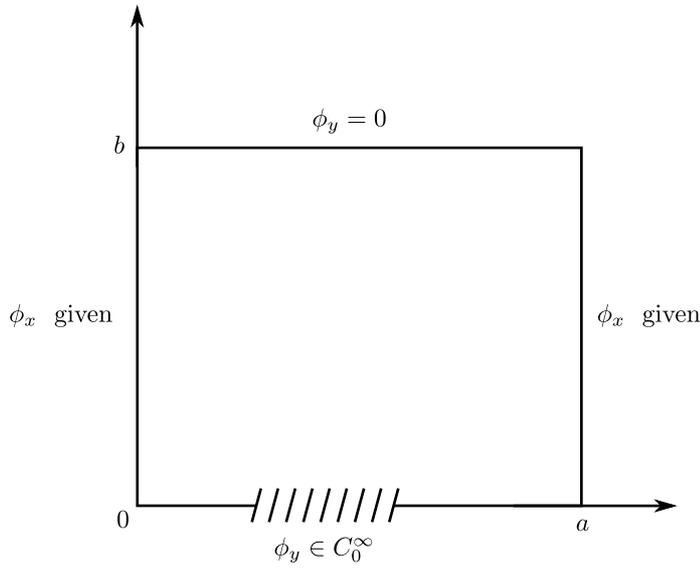


Fig. 3.8.

129

Remark. Even for this boundary value problem, there is still no existence theorem establishing a weak solution.

Let U_i^j represent an approximation for ϕ at the mesh point $x = i\Delta x$, $y = j\Delta y$. The form of difference scheme proposed for the elliptic region is:

$$\frac{1}{2\Delta x} \left[\left(\frac{U_{i+1}^j - U_i^j}{\Delta x} \right)^2 - \left(\frac{U_i^j - U_{i-1}^j}{\Delta x} \right)^2 \right] + \frac{1}{(\Delta y)^2} [U_i^{j+1} - 2U_i^j + U_i^{j-1}] = 0, \tag{3.37}$$

a second order accurate scheme. Note that in this form the difference analogue of $\oint \phi_x^2 dy - \phi_y dx = 0$. This is in so called conservation form. In the hyperbolic region we *retard* the x -differences and use

$$\frac{1}{2\Delta x} \left[\left(\frac{U_i^j - U_{i-1}^j}{\Delta x} \right)^2 - \left(\frac{U_{i-1}^j - U_{i-2}^j}{\Delta x} \right)^2 \right] + \frac{1}{(\Delta y)^2} [U_{i-1}^{j+1} - 2U_{i-1}^j + U_{i-1}^{j-1}] = 0 \tag{3.38}$$

This is accurate to second order for some equation of the form

$$\frac{1}{2}(\phi_x^2)_x + \phi_{yy} = (\epsilon(\phi_x^2)_x)_x \quad (3.39)$$

- 130 where $\epsilon = 0$ for $\phi_x > 0$. The scheme thus introduces *artificial dissipation* but only in the supersonic region, cf. Chapter I §§ 5 and 7.

Exercise. Investigate the one dimensional form of (3.39) and see how the solution depends on the parameter.

The object is to prescribe appropriate boundary conditions using (3.36) and then to solve (3.37) and (3.38) for U_i^j . For this a particular relaxation method is used which we will interpret as a time dependent problem on the original equation. Consider

$$\phi_{xt} = \phi_x \phi_{xx} + \phi_{yy},$$

with the same boundary conditions. Then solve numerically by differencing the following:

$$\tilde{\phi}_{xt} = \tilde{\phi}_x \tilde{\phi}_{xx} \quad \text{in} \quad n\Delta t \leq t \leq (n+1)\Delta t \quad (3.40)$$

$$\tilde{\phi}_{xt} = \tilde{\phi}_{yy} \quad \text{in} \quad (n+1)\Delta t \leq t \leq (n+2)\Delta t \quad (3.41)$$

Here Δt is small. Define

$$\phi^* = \begin{cases} \tilde{\phi} & \text{in} \quad n\Delta t \leq t \leq (n+1)\Delta t \\ \tilde{\tilde{\phi}} & \text{in} \quad (n+1)\Delta t < t \leq (n+2)\Delta t \end{cases} \quad (3.42)$$

This is an alternating direction scheme and it turns out that *if the quantities are smooth*, this alternate direction scheme implies

$$\phi^* \rightarrow \phi \quad \text{as} \quad \Delta t \rightarrow 0.$$

- 131 However, we do not expect a smooth solution; a shock occurs and across this shock ϕ_x increases in accordance with the entropy condition if the difference have been retarded in the x - direction and if the shock can be described by $x = X(y)$. Furthermore one believes that ϕ approaches a steady state as $t \rightarrow \infty$ and this steady state is the desired solution.

If we now difference the alternating direction time dependent scheme in time and space we get a particular relaxation scheme for finding U_i^j .

Alternative schemes have been suggested by Engquist, Osher [9], who show in fact that there exists a solution for their difference scheme. The main ideas of this method have been used by Jameson [18] and others to solve the full equations and in recent work the method has been applied to three dimensional flows where the computing difficulties are very great.

3.13 Nozzle flow

Another transition flow from subsonic to supersonic occurs in a nozzle but with much more stable behaviour.

The simplest example is a Meyer flow (see Bers [2]) where there is an elegant exact solution. From (3.8) one finds the equations for q, θ as functions of ϕ, ψ and one finds near $q = c$ that

$$S_\psi = \theta_\phi, \quad SS_\phi = \theta_\psi,$$

where S is related to σ . For every A ,

$$S = A\phi + \frac{A^2}{2}\psi^2$$

$$\theta = A^2\phi\psi + \frac{A^3}{6}\psi^3$$

is a solution. The flow in the ϕ, ψ plane has, of course, as streamlines the horizontals $\psi = \text{constant}$. Thus the wall of the corresponding flow in ϕ, ψ plane consists of two straight lines. 132

The sonic line is a parabola and the characteristics are $(\theta - \theta_0)^2 = \frac{4}{9}\sigma^3$ and there are four of them passing through $\phi = 0, \psi = 0$ given by $\phi = \pm \frac{A}{4}\psi^2$. Hence the mapping into the hodograph plane is not one-to-one but there is a fold and the region $\theta^2 < \frac{4}{9}\sigma^3$ is covered *three* times.

To design a general nozzle with prescribed speed at infinity, we therefore solve

$$K\psi_{\theta\theta} + \psi_{\sigma\sigma} = 0$$

in the hodograph plane σ, θ . Specify the singularity at σ_∞ by analogue with incompressible flow (see the Figure 3.9). Leave out the gap made by the characteristics through the origin.

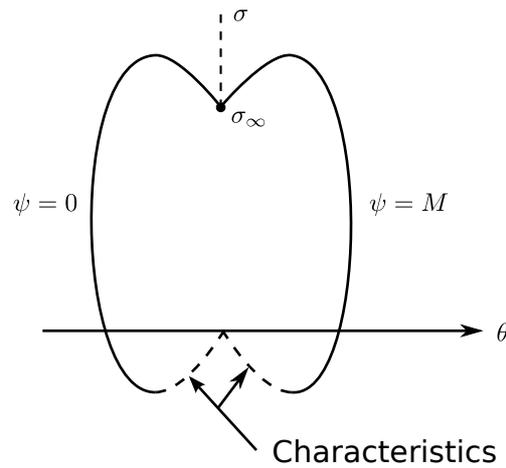


Fig. 3.9.

- 133 Next continue the flow across the characteristics to get two layers ending on the characteristics. Continue the flow on the third sheet into the whole quadrant bounded by four characteristics.

To continue the flow, one uses hyperbolic methods. The flow can be terminated by a shock and the outgoing flow from the nozzle will be subsonic.

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