Lectures on
Representations Of Complex Semi-Simple Lie Groups

By

Thomas J. Enright

Tata Institute of Fundamental Research

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Preface

These notes are the slightly revised lecture notes from lectures given at the Tata Institute during Winter 1980. The purpose of the lectures was to describe a factorial correspondence between the theory of admissible representations for a complex semisimple Lie group and the theory of highest weight modules for a semisimple Lie algebra. A detailed description of the main results of this correspondence is given in section one.

A first draft of these notes was prepared by Vyjayanthi Sunder. I am grateful to her for her careful work. It is also a pleasure to thank my colleagues at the Tata Institute for their hospitality to my wife and me during our stay at the institute. Finally, I wish to acknowledge the support of the National Science Foundation and the Alfred P. Sloan Foundation.

Contents

Chapter 1

Introduction and summary of results

Within the theory of representations of Lie groups, the case of compact 2
groups is distinguished by the simplicity and completeness of the theory. The fundamental results here are classical and go back to the work of H. Cartan and H. Weyl. For the noncompact case, the representation theory is far from complete although in this case an elaborate theory does exist. The foundations of this theory were developed by Harish-Chandra and the contributions to this field during the last twenty-five years have been extensive. The subcase of the noncompact case of complex Lie groups is of interest due to the special simplicity of the theory for these groups and the unusual parallels which exist between this theory and the theory for the compact case. The representation theory of complex semisimple Lie groups has a long history beginning in 1950 with the fundamental work of Gel'fand and Naimark [20]. Aspects of this history are described in the introduction to the expository article [9] by Duflo and also the survey article [37] by Zelobenko.

Let G be a connected complex semisimple Lie group. One of the fundamental questions of representation theory is the description of $\varepsilon(G)$, the infinitesimal equivalence classes of irreducible representations of G. Although the parametrization of $\varepsilon(G)$ is essentially the same as in the compact case the proof of this fact lies much deeper in this case. The first result on the parametrization of $\varepsilon(G)$ was established by Parthasarathy, Ranga Rao and Varadarajan [30] for the subset of $\varepsilon(G)$ of spherical classes. Zelobenko [36] in 1969 gave the first parametrization of $\varepsilon(G)$ and in 1973 Langlands [27] described a program for general G which in this case specialized to the Zelobenko classification.

3 There are four questions related to any complete description of $\varepsilon(G)$ which naturally arise.

Question A: For any element $\pi \in \varepsilon(G)$, what is the formula for the (distribution) character of the class π ?

Question B: For any element $\pi \in \varepsilon(G)$, the restriction of π to a maximal compact subgroup of G splits into the direct sum of irreducible finite dimensional representations. What are the components and multiplicities of this direct sum ?

Question C: For compact groups all indecomposable representations are irreducible; for complex groups this is not so. What is the category of all representations of G ?

Question D: What is the subset of $\varepsilon(G)$ of unitarizable classes ?

Although $\varepsilon(G)$ has been determined by Zelobenko, his description does not give a solution to any of these questions. In these notes we give an alternate description of $\varepsilon(G)$ which will at the same time answer the above questions in terms of data associated with highest weight modules. The method applied will be to use functor introduced in [15] to relate irreducible highest weight modules to elements of $\varepsilon(G)$.

We now describe the results of these notes in some detail. In section two the definition and basic properties of the completion functors are summarized. The application of completion functors is the fundamental technique used here to construct modules for a Lie algebra. In section three invariant pairings and forms are introduced. The main result in this section is a splitting theorem which we now describe. Let $\mathfrak{L}$ be
4 a Lie algebra with subalgebra $\mathfrak{t}$ and let C be a category of $\mathfrak{L}$-modules. The category C is called $\mathfrak{t}$-semisimple if every short exact sequence of $\mathfrak{L}$-modules splits as a $\mathfrak{t}$-module sequence. Let m be a semisimple Lie algebra over $\mathbb{C}$, the complex numbers, and let $\mathfrak{h}$ be a Cartan subalgebra (CSA) and $\mathfrak{b}$ a Borel subalgebra of m with $\mathfrak{h} \subset \mathfrak{b}$. For any Lie algebra $\mathfrak{g}$, let $U(\mathfrak{g})$ denote the universal enveloping algebra. By the category $\mathscr{O}$

for m we mean the full subcategory of m-modules which are (i) finitely generated, (ii) weight modules for $\mathfrak{h}$ and (iii) $U(\mathfrak{b})$-locally finite. This category was introduced by Bernstein, Gel'fand and Gel'fand [2] in their study of irreducible highest weight modules and Verma modules. Let $\mathfrak{h}^*$ denote the algebraic dual of $\mathfrak{h}$ and for $\lambda \in \mathfrak{h}^*$, let $\mathbb{C}_\lambda$ denote the one dimensional $\mathfrak{b}$-module where $\mathfrak{h}$ acts by λ. Let δ denote half the sum of the roots of $\mathfrak{b}$ and let $M(\lambda)$ denote the Verma module with highest weight $\lambda - \delta$; i.e., $M(\lambda) \simeq U(m) \bigotimes_{U(\mathfrak{b})} \mathbb{C}_{\lambda-\delta}$. Following the notation in Dixmier [8], let $L(\lambda)$ denote the unique irreducible quotient of $M(\lambda)$. Now fix an irreducible Verma m-module M, and let $\mathscr{O} \otimes M$ denote the category of $m \times m$-modules of the form $A \otimes M$, where A is an object in $\mathscr{O}$. Using the theory of invariant forms we obtain: Proposition 3.9: If $\mathfrak{t}$ denotes the diagonal subalgebra of $m \times m$, then $\mathscr{O} \otimes M$ is a $\mathfrak{t}$-semisimple category of $m \times m$-modules. This proposition will play a central role in the remaining sections of these notes.

In section four, the definitions of a lattice of modules and the functor τ are recalled from [15], and their basic properties are summarized. Section five includes a description of the Zuckerman functors which allow the translation from modules with one infinitesimal character to modules with another.

Section six begins the study of modules for a complex semisimple 5
Lie algebra. We now introduce the notation necessary to complete the description of the main results.

Let $\mathfrak{g}_0$ be a complex semisimple Lie algebra with CSA $\mathfrak{h}_0$, roots Δ_0 and positive system of roots P_0. Let $\mathfrak{w}_0$ denote the Weyl group of Δ_0 and let n_0 (resp. n_0^-) denote the nilpotent Lie subalgebra spanned by the positive (resp. negative) root spaces of $\mathfrak{g}_0$. The subalgebra $\mathfrak{b}_0 = \mathfrak{h}_0 \oplus n_0$ is a Borel subalgebra of $\mathfrak{g}_0$. Let $\langle \cdot, \cdot \rangle$ denote the Killing form on $\mathfrak{h}_0^*$ and for $\lambda \in \mathfrak{h}^*$, $\alpha \in \Delta_0$, let $\lambda_\alpha = 2\langle \lambda, \alpha \rangle / \langle \alpha, \alpha \rangle$. Let $\mathbb{Z}$ denote the integers and $\Delta_\lambda = \{\alpha \in \Delta_0 \mid \lambda_\alpha \in \mathbb{Z}\}$. Δ_λ is a root system and $P_\lambda = P_0 \cap \Delta_\lambda$ is a positive system or roots for Δ_λ. Let $\mathfrak{w}_\lambda$ be the Weyl group for Δ_λ. Let δ_0 equal half the sum of the roots P_0. If $\lambda_\alpha \geq 0$ for all $\alpha \in P_0$, we call λ P_0-dominant.

By deleting the subscript 0, we denote the product of the algebra with itself; i.e., $\mathfrak{g} = \mathfrak{g}_0 \times \mathfrak{g}_0$, $\mathfrak{h} = \mathfrak{h}_0 \times \mathfrak{h}_0$, etc. Now $\mathfrak{h}^*$ and $\mathfrak{h}_0^* \times \mathfrak{h}_0^*$ are identified by the formula: $(\lambda, \lambda')(H, H') = \lambda(H) + \lambda'(H')$, $\forall \lambda,\ \lambda' \in \mathfrak{h}_0^*$. $H, H' \in \mathfrak{h}_0$. Then $P = (P_0 \times 0) \cup (0 \times P_0)$ is a positive system of roots for $\Delta = (\Delta_0 \times 0) \cup (0 \times \Delta_0)$. The Weyl group $\mathfrak{w}$ of Δ is the product $\mathfrak{w}_0 \times \mathfrak{w}_0$. Let $\mathfrak{L}$ denote the subset of $\mathfrak{h}^*$ of elements (λ, λ') with $\lambda + \lambda'$ integral (i.e., $\lambda_\alpha + \lambda'_\alpha \in \mathbb{Z}$ for all $\alpha \in P_0$). Let $\delta = (\delta_0, \delta_0)$.

Let $\mathfrak{t}$ denote the diagonal subalgebra of $\mathfrak{g}$ and for any subalgebra of $\mathfrak{g}$, let a subscript $\mathfrak{t}$ denote the intersection with $\mathfrak{t}$; e.g. $\mathfrak{h}_\mathfrak{t} = \mathfrak{h} \cap \mathfrak{t}$, $n_\mathfrak{t} = n \cap \mathfrak{t}$. For convenience we also write $\mathfrak{t} = \mathfrak{h}_\mathfrak{t}$. Let $\Delta_\mathfrak{t}$ equal the roots of $(\mathfrak{t}, \mathfrak{t})$ and let $P_\mathfrak{t} = \{(\alpha, 0) |_\mathfrak{t} \mid \alpha \in P_0\}$ (here $|_\mathfrak{t}$ denotes the restriction to $\mathfrak{t}$). One
6 checks easily that $P_\mathfrak{t}$ is a positive system of roots for $\Delta_\mathfrak{t}$. Let $\mathfrak{w}_\mathfrak{t}$ denote the diagonal subgroup of $\mathfrak{w}$. $\mathfrak{w}_\mathfrak{t}$ acts on $\mathfrak{t}^*$ and can be identified with the Weyl group of $\Delta_\mathfrak{t}$. The subset $\mathfrak{L}$ of $\mathfrak{h}^*$ is stable under $\mathfrak{w}_\mathfrak{t}$; and so, for $(\lambda, \lambda') \in \mathfrak{h}^*$ we let $[\lambda, \lambda']$ denote the $\mathfrak{w}_\mathfrak{t}$-orbit of (λ, λ'). We write $\mathfrak{L}/\mathfrak{w}_\mathfrak{t}$ for the set of $\mathfrak{w}_\mathfrak{t}$-orbits in $\mathfrak{L}$.

A $\mathfrak{g}$-module M will be called an admissible $(\mathfrak{g}, \mathfrak{t})$-module of (i) M is finitely generated, (ii) M is $U(\mathfrak{t})$-locally finite, and (iii) for each $\mathfrak{t}$-module E, $\mathrm{Hom}_\mathfrak{t}(E, M)$ is finite dimensional. If G is the complex connected and simply connected Lie group with Lie algebra $\mathfrak{g}_0$, then the theory of representations for G is equivalent to the theory of admissible $(\mathfrak{g}, \mathfrak{t})$ modules. For irreducible representations this equivalence is implied by Harish-Chandra's subquotient theorem. Refinements of this theorem have established the equivalence for all (quasi-simple) representations of G (cf. [29]). With this equivalence in mind, we offer solutions to the questions above rephrased in the category of admissible $(\mathfrak{g}, \mathfrak{t})$-modules. We begin with a description of the irreducible admissible $(\mathfrak{g}, \mathfrak{t})$-modules.

Let τ denote the lattice functor for the algebra $\mathfrak{t}$ determined by the positive system $P_\mathfrak{t}$ (cf. § 4). For $\underline{\lambda} = (\lambda, \lambda') \in \mathfrak{L}$, we say $\underline{\lambda}$ satisfies (6.5) if (i) λ' is -P_λ-dominant and (ii) if $\alpha \in P_\lambda$ and $\lambda'_\alpha = 0$ then $\lambda_\alpha \in -\mathbb{N}$ ($\mathbb{N} = \{0, 1, 2, \ldots\}$). Each orbit $[\lambda, \lambda']$ contains one or more elements which satisfy (6.5). Now for $\underline{\lambda} \in \mathfrak{L}$ satisfying (6.5) define a $\mathfrak{g}$-module $Z(\underline{\lambda})$ to be the image under τ of the irreducible highest weight module $L(\underline{\lambda})$; i.e., $Z(\underline{\lambda}) = \tau L(\underline{\lambda})$. The basic result of these notes is:

7 **Theorem 10.8.** *The map* $\underline{\lambda} \mapsto Z(\underline{\lambda})$ *induces a bijection of* $\mathfrak{L}/\mathfrak{w}_\mathfrak{t}$ *onto the equivalence classes of irreducible admissible* $(\mathfrak{g}, \mathfrak{t})$*-modules.*

The starting point for both the proof of Theorem 10.8 as well as a detailed description of the modules $Z(\underline{\lambda})$ is the connection between Verma modules and principal series modules of $\mathfrak{g}$. Let $Q = (P_0 \times 0) \cup (0 \times -P_0)$. Then Q is a positive system for Δ. Let δ_Q equal half the sum of the roots in Q and let $\overline{M}(\lambda)$ denote the $\mathfrak{g}$-Verma module with Q-highest weight $\lambda - \delta_Q$, $\lambda \in \mathfrak{h}^*$. For $\lambda \in \mathfrak{h}^*$, let $X(\lambda)$ denote the submodule of $U(\mathfrak{t})$-locally finite vectors in the algebraic dual of $\overline{M}(-\lambda)$. The $\mathfrak{g}$-modules $X(\lambda)$, $\lambda \in \mathfrak{L}$, are called the principal series modules of $\mathfrak{g}$. These modules are isomorphic to the $\mathfrak{g}$-modules of K-finite vectors of the principal series of G (here K is the analytic subgroup of G with complexified Lie algebra $\mathfrak{t}$). Section eight includes the definition and basic properties of the principal series modules. The correspondence between Verma modules and principal series modules is given as:

Theorem 9.1. *Let $\underline{\lambda} = (\lambda, \lambda') \in \mathfrak{L}$ and assume $M(\lambda')$ is an irreducible Verma module. Then $\tau M(\underline{\lambda})$ and $X(\underline{\lambda})$ are isomorphic $\mathfrak{g}$-modules.*

By our remarks above regarding Proposition 3.9, $\mathcal{O} \otimes M(\lambda')$ is a $\mathfrak{t}$-semisimple category of $\mathfrak{g}$-modules; and so, τ is exact on this category. This exactness and Theorem 9.1 give character formulae for the modules $Z(\underline{\lambda})$. Let $ch\,A$ denote the formal character of the $\mathfrak{g}_0$-module A in $\mathcal{O}$ and let $E(\underline{\mu})$ denote the distribution character of the principal series module $X(\underline{\mu})$, $\underline{\mu} \in \mathfrak{L}$, and $\Theta(\underline{\mu})$ the distribution character of $Z(\underline{\mu})$.

Proposition 9.15. *Let $\underline{\lambda} = (\lambda, \lambda') \in \mathfrak{L}$ and assume $\underline{\lambda}$ satisfies (6.5). Fix integers $m(s\lambda)$, $s \in \mathfrak{w}_\lambda$, such that $chL(\lambda) = \sum_{s\lambda \in \mathfrak{w}_\lambda \cdot \lambda} m(s\lambda)\, chM(s\lambda)$. Then* **8**

$$\Theta(\underline{\lambda}) = \sum_{s\lambda \in \mathfrak{w}_\lambda \cdot \lambda} m(s\lambda) E(s\lambda, \lambda').$$

This is the solution to Question A given in terms of data associated with the category $\mathcal{O}$. Using this result and the known $\mathfrak{t}$-module structure of the principal series modules (cf. (8.3)), Proposition 9.15 also gives an answer to Question B. However, we will obtain an answer to Question B by specializing a somewhat stronger result.

In the classical case of compact semisimple Lie groups, the weight space structure of the irreducible representations can be obtained directly from the Bernstein, Gel'fand and Gel'fand resolution of finite dimensional modules by sums of Verma modules [1]. We approach the $\mathfrak{t}$-structure of $Z(\underline{\lambda})$ from a similar point of view. A resolution of $Z(\underline{\lambda})$ is given in terms of the modules in the lattice above $L(\underline{\lambda})$.

The simple reflections generate the Weyl group $\mathfrak{w}_{\mathfrak{t}}$. An expression of s as a product of simple reflections is called reduced if the number of simple reflections is a minimum. This minimum number is called the length of s and is denoted $\ell(s)$. If $d = \operatorname{card} P_{\mathfrak{t}}$ and $s \in \mathfrak{w}_{\mathfrak{t}}$, then $0 \le \ell(s) \le d$. Moreover, $\ell(s) = 0$ implies $s = 1$ and $\ell(s) = d$ implies s is the unique element with $sP_{\mathfrak{t}} = -P_{\mathfrak{t}}$. The resolution of $Z(\underline{\lambda})$ is included as a special case of the general result:

9 **Proposition 7.1.** Let $\lambda \in \mathfrak{h}_0^*$ and assume the $\mathfrak{g}_0$-Verma module $M(\lambda)$ is irreducible. Let B be a $\mathfrak{g}$-module in $\mathscr{O} \otimes M(\lambda)$ with integral $\mathfrak{t}$-weights. Let B_s, $s \in \mathfrak{w}_{\mathfrak{t}}$, be a lattice of modules above B and for $0 \le i \le d$, put $\mathscr{B}_i = \sum_{\ell(s)=i} B_s$. Then there is a resolution

$$0 \to \mathscr{B}_d \to \ldots \to \mathscr{B}_0 \to \tau B \to 0.$$

From this resolution and standard properties of lattices we obtain a $\mathfrak{t}$-multiplicity formula for the modules $Z(\underline{\lambda})$. Let $\mu \in \mathfrak{h}_0^*$ be P_0-dominant integral and put $\mu_1 = (\mu, 0)\,|_t$. Let F denote the irreducible finite dimensional $\mathfrak{t}$-module with extreme weight μ_1. For any $\nu \in \mathfrak{h}_0^*$, let a subscript ν denote the weight space of weight ν. By specializing Corollary 7.12 we have:

Corollary. *Let* $\underline{\lambda} = (\lambda, \lambda') \in \mathfrak{L}$ *and assume* $\underline{\lambda}$ *satisfies (6.5). Then*

$$\dim \operatorname{Hom}_{\mathfrak{t}}(F, Z(\underline{\lambda})) = \sum_{s \in \mathfrak{w}_0} (-1)^{d-\ell(s)} \dim L(\lambda)_{s(\mu+\delta_0)-\lambda'}.$$

So far our theme has been the reduction of questions for admissible $(\mathfrak{g}, \mathfrak{t})$-modules to related questions for highest weight modules. In section eleven we invert this theme and use the correspondence between highest weight and admissible $(\mathfrak{g}, \mathfrak{t})$-modules to obtain certain skew-symmetry properties for characters of irreducible highest weight modules. Fix $\lambda \in \mathfrak{h}_0^*$ and write $ch\ L(\lambda) = \sum m(s\lambda) chM(s\lambda)$ with the sum over the $\mathfrak{w}_\lambda$ orbit of λ ($\mathfrak{w}_\lambda$ is the Weyl group of Δ_λ). In Proposition 11.2

10 we assert the following skew-symmetry property: Let α a simple root of P_λ and assume $\lambda_\alpha \in \mathbb{N}^*$ (positive integers). Then $m(s_\alpha s\lambda) + m(s\lambda) = 0$ for all $s\lambda$. This skew-symmetry property follows easily from a determination of those $\underline{\lambda} \in \mathfrak{L}$ for which $\tau L(\underline{\lambda})$ equals zero (cf. Proposition 10.6).

Section twelve contains a review of some standard concepts from homological algebra, as well as, a somewhat special definition of certain derived functors. Let $\mathcal{O}_{\mathfrak{t}}$ denote the category $\mathcal{O}$ for the data $(\mathfrak{t}, \mathfrak{t}, P_{\mathfrak{t}})$ and let n denote the category of $\mathfrak{g}$-modules whose underlying $\mathfrak{t}$-modules lie in $\mathcal{O}_{\mathfrak{t}}$. Fix a set $\psi \subseteq t^*$ of dominant integral elements. For any $\mathfrak{t}$-module L, put L' equal to the span of the $n_{\mathfrak{t}}$-invariants of weight μ, $\mu \in \psi$. Then, a complex

$$\ldots A_i \to \ldots \to A_0 \to A \to 0$$

in the category n is called a ψ-resolution if (i) A_i is projective, $i \in \mathbb{N}$ (ii) A_i is generated as a $\mathfrak{g}$-module by A'_i, $i \in \mathbb{N}$, and (iii) $0 \to A'_i \to \ldots \to A'_0 \to A' \to 0$ is exact. For $\mu \in \mathfrak{t}^*$ and $P_{\mathfrak{t}}$-dominant integral, let $M(\mu)$ denote the $\mathfrak{t}$-Verma module. If $r \in \mathfrak{w}_{\mathfrak{t}}$ is the element of maximal length, then $M(r\mu) \subseteq M(\mu)$; and moreover, this subspace is unique. For $\nu \in \mathfrak{t}^*$, put $U(\nu) = U(\mathfrak{g}) \bigotimes_{U(\mathfrak{t})} M(\nu)$. The inclusion $M(r\mu) \subseteq M(\mu)$ induces the inclusion $U(r\mu) \subseteq U(\mu)$, $\mu \in \psi$. The task of section twelve is to give a definition of an additive covariant functor σ_0 on n having the property that, for $\mu \in \psi$, $\sigma_0 U(\mu) = U(r\mu)$. Next with projective resolutions replaced with ψ-projective resolutions, we definite the additive covariant functors σ_i, $i \in \mathbb{N}$, as the derived functors of σ_0. The setting of this 11
section is quite general with $\mathfrak{t}$ a reductive Lie algebra and $\mathfrak{g}$ an arbitrary Lie algebra.

In section thirteen we return to the setting of complex Lie algebras and attempt to compute the values of σ_i on the subcategory of n of admissible $(\mathfrak{g}, \mathfrak{t})$-modules. Fix $\underline{\lambda} = (\lambda, \lambda') \in \mathfrak{L}$ and assume $\mathrm{Re}\,\lambda'_\alpha << 0$, $\alpha \in P_0$. Let $\wedge\mathfrak{t}$ denote the exterior algebra of $\mathfrak{t}$ and let r be the element of $\mathfrak{w}_{\mathfrak{t}}$ of maximal length. Put $\psi = \{r(s\lambda, \lambda')\ |_t + \xi | s \in \mathfrak{w}_0$ and ξ is a weight of $\wedge\mathfrak{t}\}$. Let σ_i be the derived functors associated with ψ as in section twelve. Let χ denote the $Z(\mathfrak{g})$ character parametrized by the $\mathfrak{w}$-orbit of $\underline{\lambda}$ and let $\mathfrak{U}$ denote the category of admissible $(\mathfrak{g}, \mathfrak{t})$-modules having generalized infinitesimal character χ (i.e., $z - \chi(z)$ is locally nilpotent for all $z \in Z(\mathfrak{g})$). Let $\mathfrak{B}$ denote the full subcategory of the category of $\mathfrak{g}$-modules with objects which are finitely generated, have precisely $L(s\lambda, \lambda')$, $s \in \mathfrak{w}_0$, as irreducible objects, and are weight modules for $\mathfrak{t}$. The basic result in this section is:

Proposition 13.13. *Assume* $\operatorname{Re} \lambda'_\alpha << 0$, $\alpha \in P_0$. *Then the lattice functor* τ *gives a natural equivalence of categories;* $\tau : \mathfrak{B} \xrightarrow{\sim} \mathfrak{U}$. *When restricted to* $\mathfrak{U}$, σ_i *is exact for* $i \in \mathbb{N}$*; and* $\sigma_i \equiv 0$ *for* $i \in \mathbb{N}^*$. *Moreover,* $\sigma_0 : \mathfrak{U} \xrightarrow{\sim} \mathfrak{B}$ *and is a natural inverse to* τ.

Using translation functors we obtain:

Theorem 13.2. *Let* $\underline{\lambda} = (\lambda, \lambda') \in \mathfrak{L}$ *and assume* (λ') *is irreducible and* λ' *is regular. Then the lattice functor* τ *gives a natural equivalence;*

$$\tau : \mathfrak{B} \xrightarrow{\sim} \mathfrak{U}.$$

12 The map $B \mapsto B/(0, n_0^-) \cdot B$ induces a natural equivalence of $\mathfrak{B}$ onto the category of finitely generated $\mathfrak{g}_0$-modules which are $U(\mathfrak{b}_0)$-locally finite, have generalized $Z(\mathfrak{g}_0)$-character with orbit $\mathfrak{w}_0 \cdot \lambda$ and have generalized $\mathfrak{g}_0$-weights ν with $\nu + \lambda'$ integral. Therefore Theorem 13.2 gives an equivalence of the category $\mathfrak{U}$ and a category of highest weight $\mathfrak{g}_0$-modules. The reader may wish to compare this with a similar equivalence of categories established by Bernstein, Gel'fand and Gel'fand in [3]. The category studied in [3] is the subcategory of $\mathfrak{U}$ of objects which have $1 \otimes Z(\mathfrak{g}_0)$ characters instead of generalized characters and the results in [3] hold without the restriction that λ' is regular.

In section fourteen we begin a study of the question of unitarizability of the modules $Z(\underline{\lambda})$, $\underline{\lambda} \in \mathfrak{L}$. The main result of this section gives a necessary and sufficient condition for $Z(\underline{\lambda})$ to be unitarizable in terms of certain invariant Hermitian pairings of highest weight modules.

In the last two sections we summarize several additional results on complex groups as well as another point of view in analyzing the admissible $(\mathfrak{g}, \mathfrak{t})$-modules. Section fifteen includes a number of partial results on the question of unitarizability obtained by a variety of techniques. The main results here are the description of the unitarizable representations with regular integral infinitesimal character as unitarily induced

representations of the group and the resulting description of the representations with relative Lie algebra cohomology. In section sixteen, we describe a technique of constructing admissible $(\mathfrak{g}, \mathfrak{k})$-modules by derived functors introduced by G. Zuckerman. We then describe how the 13
first part of the program followed by these notes could equally well have been completed with the lattice functor τ replaced by the derived functor in the "middle" dimension. The section ends with an example which shows the lattice functor and the "middle" dimension functor are not equivalent.

As mentioned earlier, the first description of the irreducible admissible $(\mathfrak{g}, \mathfrak{k})$-modules was given by Zelobenko. We now describe the Zelobenko classification, compare it with ours and match the parameters. Let $\underline{\lambda} \in \mathfrak{L}$ and put $\mu = \underline{\lambda} \mid_{\mathfrak{t}}$. By Frobenius reciprocity the irreducible $\mathfrak{k}$-module with extreme weight μ occurs with multiplicity one in $X(\underline{\lambda})$ (cf. §8); and so, we let $X(\underline{\lambda})$ denote the ($\mathfrak{g}$-module) subquotient of $X(\underline{\lambda})$ which contains this $\mathfrak{k}$-module. These $\mathfrak{g}$-modules were first studied by Parthasarathy, Ranga Rao and Varadarajan [30] for the case $\mu = 0$ (the spherical case). The classification of Zelobenko [36] can be expressed as follows:

Theorem . *The map $\underline{\lambda} \mapsto \hat{X}(\underline{\lambda})$ induces a bijection of $\mathfrak{L}/\mathfrak{w}_{\mathfrak{t}}$ onto the equivalence classes of irreducible admissible $(\mathfrak{g}, \mathfrak{k})$-modules.*

The connection with our parameters is given as:

Proposition 10.5. *Let $\underline{\lambda} \in \mathfrak{L}$ and assume $\underline{\lambda}$ satisfies (6.5). Then $Z(\underline{\lambda})$ and $\hat{X}(\underline{\lambda})$ are isomorphic.*

There has been some recent work on the subject of these notes. The Bernstein Gel'fand article [3] offers a different and especially beautiful approach to Questions A, B and C. Their results for complex groups are 14
essentially the same as those described here. The question of the decomposition of the principal series modules can be interpreted as a question of primitive ideals in the enveloping algebra. From this point of view, A. Joseph [25] has obtained results for Questions A and B which are essentially the same as those described above. The conjectures of Kazhdan and Lustig [26] offer a precise formula for the integers $m(s\lambda)$ given in the character formulae above. A positive resolution of this conjec-

ture would offer a very satisfying description of the character theory of complex semisimple Lie groups.

Lastly we list a few conventions which will remain in force throughout the notes. We denote the integers (resp. nonnegative integers, positive integers) by $\mathbb{Z}$ (resp. $\mathbb{N}$, $\mathbb{N}^*$). The symbol $\Leftrightarrow$ will be used in place of the term if and only if. For a Lie algebra $\mathfrak{g}$, let $U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$ and let $Z(\mathfrak{g})$ denote the center of $U(\mathfrak{g})$.

Chapter 2

The completion functors

In this section we collect the definitions and the main results on com- 15
pletion functors. For omitted proofs consult §3 of [15] or [33].

Let $\mathfrak{a}$ denote the Lie algebra $sl(2, \mathbb{C})$. Choose a basis H, X, Y of $\mathfrak{a}$ with $[H, X] = 2X$, $[H, Y] = -2Y$ and $[X, Y] = H$. Such a basis of $\mathfrak{a}$ is called a standard basis. Put $\mathfrak{h} = \mathbb{C} \cdot H$ and $\mathfrak{b} = \mathfrak{h} \oplus \mathbb{C} \cdot X$. Then $\mathfrak{h}$ is a CSA of $\mathfrak{a}$ and $\mathfrak{b}$ is a Borel subalgebra.

We now define Verma modules for $\mathfrak{a}$. For $\lambda \in \mathfrak{h}^*$, let $\mathbb{C}_\lambda$ denote the one dimensional $\mathfrak{b}$-module where X acts by zero and H acts by multiplication by $\lambda(H)$. Put $V_\lambda = U(\mathfrak{a}) \bigotimes_{U(\mathfrak{b})} \mathbb{C}_\lambda \cdot V_\lambda$ is the Verma with highest weight λ. It will be convenient to identify $\mathfrak{h}^*$ and $\mathbb{C}$ by $\lambda \rightleftarrows \lambda(H)$.

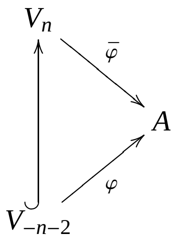
2.1 Proposition. *(i) For $\lambda \in \mathbb{C}$, V_λ is irreducible unless $\lambda \in \mathbb{N}$.*
(ii) For $n \in \mathbb{N}$, we have an inclusion $V_{-n-2} \hookrightarrow V_n \cdot F_n = V_n/V_{-n-2}$ is the irreducible finite dimensional module with highest weight n.

2.2 Definition. *Let $\mathscr{I} = \mathscr{I}(\mathfrak{a})$ denote the category of $\mathfrak{a}$-modules A which satisfy: (i) H acts semisimply on A with integral eigenvalues; i.e., $A = \bigoplus_{n \in \mathbb{Z}} A_n$ with $A_n = \{a \in A | H \cdot a = na\}$, (ii) X acts locally nilpotently on A and (iii) as a module over $U(\mathbb{C} \cdot Y)$, A is torsion free.*

Since $U(\mathbb{C} \cdot Y)$ is a *p.i.d.*, if $A \in \mathscr{I}$ is finitely generated then A is a 16
free module over $U(\mathbb{C} \cdot Y)$. For any $A \in \mathscr{I}$, let A^X equal the subspace of

A annihilated by X, A_n equal the subspace where H acts by eigenvalue n and $A_n^X = A^X \cap A_n$.

2.3 Definition. *For $A \in \mathscr{I}$, we say A is complete if for all $n \in \mathbb{N}$ and all $\mathfrak{a}$-module maps $\varphi : V_{-n-2} \to A$ there exists a unique $\mathfrak{a}$-module map $\bar{\varphi} : V_n \to A$ such that the following diagram is commutative:*



We leave as an exercise the verification that (2.3) is equivalent to the alternate definition:

2.4 Definition. *For $A \in \mathscr{I}$, A is complete if for all $n \in \mathbb{N}$, Y^{n+1}: $A_n^X \xrightarrow{\sim} A_{-n-2}^X$ is an isomorphism.*

2.5 Definition. *For $A, B \in \mathscr{I}$, B is called the completion of A if (i) B is complete, (ii) there is an injection $i : A \hookrightarrow B$ with B/iA locally $U(\mathfrak{a})$-finite.*

2.6 Proposition. *For $A \in \mathscr{I}$, A admits a unique completion which we denote by $C(A)$. Moreover, for any finite dimensional $\mathfrak{a}$-module F, $F \otimes C(A) \simeq C(F \otimes A)$.*

17 **2.7 Theorem.** *The assignment $A \mapsto C(A)$ is a covariant functor on $\mathscr{I}$.*

We let $C(\cdot)$ denote the functor given by (2.7) and we call this the completion functor associated with the standard basis H, X, Y of $\mathfrak{a}$.

Next we consider the case of an ambient Lie algebra. Let $\mathfrak{g}$ be a Lie algebra with $\mathfrak{a} \subseteq \mathfrak{g}$.

2.8 Definition. *Let $\mathscr{I}_\mathfrak{g} = \mathscr{I}_\mathfrak{g}(\mathfrak{a})$ denote the category of $\mathfrak{g}$-modules whose underlying $\mathfrak{a}$-modules lie in $\mathscr{I}$.*

For any $\mathfrak{g}$-module A, let $A_\mathfrak{a}$ denote the underlying $\mathfrak{a}$-module.

2.9 Proposition. *For $A \in \mathscr{I}_{\mathfrak{g}}$, $C(A_{\mathfrak{a}})$ admits unique $\mathfrak{g}$-module structure such that $A \hookrightarrow C(A_{\mathfrak{a}})$ is an inclusion of $\mathfrak{g}$-modules.*

For $A \in \mathscr{I}_{\mathfrak{g}}$, let $C(A)$ denote the unique $\mathfrak{g}$-module given by (2.9). Then (2.7) generalizes to:

2.10 Theorem. *The assignment $A \to C(A)$ is a covariant functor on $\mathscr{I}_{\mathfrak{g}}$.*

Chapter 3

Invariant pairings and forms

In this section invariant pairings and forms are introduced. The main 18
result derived is the result on splitting discussed in the introduction, Proposition 3.9. For omitted proofs consult [15, § 6].

Let m be a reductive Lie algebra with CSA $\mathfrak{h}$, set of roots Δ and positive system of roots Q. Fix an involutive antiautomorphism σ of m such that σ equals the identity on $\mathfrak{h}$. Let σ also denote the extension of σ to an antiautomorphism of $U(m)$, the universal enveloping algebra of m.

3.1 Definition. *For m-modules A and B, a bilinear map $\varphi : A \times B \to \mathbb{C}$ will be called an invariant pairing if $\varphi(xa, b) = \varphi(a, x^\sigma b)$, $x \in U(m)$, $a \in A$, $b \in B$. Let $Inv_m(A, B)$ denote the vector space of invariant pairings of A and B. When $A = B$, we write $Inv_m(A)$ in place of $Inv_m(A, A)$ and call elements of $Inv_m(A)$ invariant forms on A.*

Let $\mathscr{B}(m)$ denote the category of m-modules which are weight modules for $\mathfrak{h}$ with finite dimensional weight spaces. For $A \in \mathscr{B}(m)$, let A^σ denote the $U(\mathfrak{h})$-locally finite vectors in the algebraic dual to A. For $x \in U(m)$, $a \in A$ and $a' \in A^\sigma$, define $(xa')(a) = a'(x^\sigma a)$. With this action, A^σ becomes an m-module.

3.2 Lemma. *For $A \in \mathscr{B}(m)$, (i) $A^\sigma \in \mathscr{B}(m)$; in fact, A and A^σ are isomorphic as $\mathfrak{h}$-modules, (ii) $A \simeq (A^\sigma)^\sigma$ and (iii) A is irreducible $\Leftrightarrow A^\sigma$ is irreducible.*

19 **3.3 Lemma.** *For $A \in \mathscr{B}(m)$, A admits a nondegenerate invariant form if and only if $A \simeq A^{\sigma}$.*

3.4 Proposition. *For $A \in \mathscr{B}(m)$, if A has a Jordan-Hölder series of length r then* $\dim Inv_m(A) \leq r^2$.

Recall now our notation for Verma modules and their irreducible quotients, i.e., $M(\lambda)$ and $L(\lambda)$, $\lambda \in \mathfrak{h}^*$.

3.5 Proposition. *For $\lambda \in \mathfrak{h}^*$,* $\dim Inv_m(M(\lambda)) = \dim Inv_m(L(\lambda)) = 1$. *Moreover, the forms on $M(\lambda)$ are the pull backs of the nondegenerate forms on $L(\lambda)$.*

For $\alpha \in \Delta$, let a subscript α denote the root space. Now fix a Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \sum_{\alpha \in Q} m_\alpha$ of m and put $n^- = \sum_{-\alpha \in Q} m_\alpha$. Let $\mathcal{J} = \mathcal{J}(b)$ denote the category of finite dimensional $\mathfrak{b}$-modules which are weight modules for $\mathfrak{h}$. For any $L \in \mathcal{J}$, put $U(L) = U(m) \bigotimes_{U(\mathfrak{b})} L$. We now can state the very useful technical result:

3.6 Proposition. *For any $L \in \mathcal{J}$, let res denote the restrict map from $U(L)$ to $1 \otimes L$. Then, for $L, L' \in \mathcal{J}$, res induces a natural vector space isomorphism*

$$res : Inv_m(U(L), U(L')) \xrightarrow{\sim} Inv_{\mathfrak{h}}(L, L').$$

The critical role played by (3.6) involves the splitting of certain short exact sequences. Several of the following results are examples of this type result.

20 **3.7 Lemma.** *Let $0 \to A \to B \to C \to 0$ be a short exact sequence in $\mathcal{J}$. Assume $U(A)$ and $U(C)$ admit nondegenerate invariant forms. Then $U(B)$ also admits a nondegenerate invariant form and the induced sequence $0 \to U(A) \to U(B) \to U(C) \to 0$ is split exact.*

Proof. Let φ_A and φ_C denote nondegenerate invariant forms on $U(A)$ and $U(C)$ respectively. Every short exact sequence in $\mathcal{J}$ splits. So we write $B = A \oplus A'$, A' an $\mathfrak{h}$-submodule of B. Clearly $A' \simeq C$; and so, $Inv_{\mathfrak{h}}(A') \simeq Inv_{\mathfrak{h}}(C)$. Let φ_1 be the element in $Inv_{\mathfrak{h}}(A')$ corresponding to $res\varphi_C$. Put $\varphi \in Inv_{\mathfrak{h}}(B)$ equal to the orthogonal sum of res φ_A and φ_1; and let $\varphi_B \in Inv_m(U(B))$ be determined by res $\varphi_B = \varphi$. Now φ_B restricted to $U(A)$ equals φ_A; and so, the restriction is non-degenerate and we have the orthogonal decomposition $U(B) = U(A) \oplus U(A)^{\perp}$. Then $U(A)^{\perp} \simeq U(C)$ and so the induced short exact sequence splits. But then $U(B)$ admits a nondegenerate invariant form, the orthogonal sum of φ_A and φ_C. This completes the proof. □

3.8 Definition. *Let $\mathfrak{L}$ be a Lie algebra with subalgebra $\mathfrak{k}$ and let C be a category of $\mathcal{L}$-modules. The category C is called $\mathfrak{k}$-semisimple if every short exact sequence in C splits as a sequence of $\mathfrak{k}$-modules.*

Note. If $\mathfrak{k} \neq \mathfrak{L}$ then objects in a $\mathfrak{k}$-semisimple category of $\mathfrak{L}$-modules need not be isomorphic to direct sums of irreducible $\mathfrak{k}$-modules.

Let $\mathcal{O}$ denote the BGG category $\mathcal{O}$ for m. Fix an m-module M and let $\mathfrak{k}$ (resp. $\mathfrak{b}_{\mathfrak{k}}$, $n_{\mathfrak{k}}^{-}$) denote the diagonal subalgebra in $m \times m$ (resp. $\mathfrak{b} \times \mathfrak{b}$, $n^{-} \times n^{-}$). Let $\mathcal{O} \otimes M$ denote the category of $m \times m$-modules with objects of the form $A \otimes M$, $A \in \mathcal{O}$.

3.9 Proposition. *For any irreducible Verma module M, $\mathcal{O} \otimes M$ is a $\mathfrak{k}$-semisimple category of $m \times m$-modules. Moreover, each object in $\mathcal{O} \otimes M$ admits a nondegenerate $\mathfrak{k}$-invariant form.* **21**

To prove (3.9) we need the following lemmas.

3.10 Lemma. *Let $\lambda \in \mathfrak{h}^*$ and assume $M(\lambda)$ is irreducible. Let $A \in \mathcal{O}$. Then there exists a natural isomorphism of $\mathfrak{k}$-modules*

$$A \otimes M(\lambda) \xrightarrow{\sim} U = U(\mathfrak{k}) \bigotimes_{U(\mathfrak{b}_{\mathfrak{k}})} (A \otimes \mathbb{C}_{\lambda-\delta}).$$

Proof. The injection $A \otimes \mathbb{C}_{\lambda-\delta} \hookrightarrow A \otimes M(\lambda)$ is a $\mathfrak{b}_{\mathfrak{k}}$-module map and so induces a unique $\mathfrak{k}$-module map $i : U \to A \otimes M(\lambda)$. Both U and $A \otimes M(\lambda)$ are free $U(n_{\mathfrak{k}}^{-})$-modules with basis $A \otimes \mathbb{C}_{\lambda-\delta}$; and thus, i is an isomorphism. □

3.11 Lemma. *Let A be an absolutely simple m-module. Then the map $B \mapsto B \otimes A$ gives an isomorphism of the category $\mathcal{O}$ of m-modules with the category $\mathcal{O} \times A$ of $m \times m$-modules.*

Proof. Let C be a submodule of $B \otimes A$. For $c \in C$, write $c = \sum b_i \otimes a_i$, a_i linearly independent. Since A is absolutely simple choose $u_i \in U(m)$ such that $u_i a_j = \delta_{ij} a_j$. Then $b_i \otimes a_i = (1 \otimes u_i)\, c \in C$. This proves C has the form $B' \otimes A$ for some submodule B' of B. The lemma follows easily from this fact. □

Proof of 3.9. Let $0 \to A \to B \to C \to 0$ be a short exact sequence in $\mathscr{O}$ and put $M = M(\lambda)$. Then

$$0 \to A \otimes M \to B \otimes M \to C \otimes M \to 0 \tag{3.12}$$

22 is a short exact sequence in $\mathscr{O} \otimes M$ and all such are of this form. We now proceed by induction on the length of a Jordan-Hölder series for B. If this length is one then either A or C is zero and the sequence splits. Also then $B \otimes M$ is an irreducible object in $\mathscr{O} \otimes M$; and so, by (3.5) $B \otimes M$ admits a nondegenerate $m \times m$ (hence $\mathfrak{t}$)-invariant form. Now assume the length is greater than one and both A and C are nonzero. By the induction hypothesis both $A \otimes M$ and $C \otimes M$ admit nondegenerate $\mathfrak{t}$-invariant forms. Using (3.10) and (3.7), we conclude that $B \otimes M$ admits a nondegenerate $\mathfrak{t}$-invariant form and that the sequence (3.12) splits. This completes the proof.

3.13 Corollary. *Let M be an irreducible Verma module. Then the map $B \to B \otimes M$ gives an isomorphism of categories $\mathscr{O}$ and $\mathscr{O} \otimes M$.*

We complete this section by describing the main results on the transfer of an invariant pairing from a pair of modules A, B to their completions $C(A)$, $C(B)$. Let $\mathfrak{a} \simeq sl(2, \mathbb{C})$ be a subalgebra of m and *assume* the standard basis H, X, Y is given with $\sigma(H) = H$, $\sigma X = Y$, $\sigma Y = X$.

3.14 Theorem. *Let $A, B \in \mathscr{I}_m(\mathfrak{a})$ and $\varphi \in Inv(A, B)$. Then there exists a unique m-invariant pairing $C(\varphi)$ of $C(A)$ and $C(B)$ such that*

(i) *$C(\varphi)$ is zero on $A \times C(B)$ and $C(A) \times B$.*

(ii) *for $n \in \mathbb{N}$, $a \in C(A)^X_n$, $b \in C(B)^X_n$ we have*

$$C(\varphi)(a, b) = \frac{1}{n!(n+1)!}\varphi(Y^{n+1}a, Y^{n+1}b).$$

23

Moreover, if F and F' are finite dimensional m-modules and φ_0 is an invariant pairing of F and F', then $C(\varphi_0 \otimes \varphi) = \varphi_0 \otimes C(\varphi)$, both being pairings on $C(F \otimes A) \times C(F' \otimes B)$ (cf. 2.6).

For a discussion of the choice of constants $\dfrac{1}{n!(n+1)!}$ in (3.14) and a result on their essential uniqueness the reader may wish to consult §8 of [15].

3.15 Remark. Let m_0 be a real form of m with $m_0 \cap \mathfrak{h}$ a real form of $\mathfrak{h}$. Let σ denote an involutive conjugate linear antiautomorphism of $U(m)$ which equals (-1)·identity on m_0. Using this σ, we may define invariant sesquilinear pairings by replacing in (3.1) bilinear φ by sesquilinear φ. With the obvious modifications, (3.6) and (3.12) remain true for sesquilinear pairings.

As usual a sesquilinear form φ on $A \times A$ is called a Hermitian form if $\varphi(a, a') = \overline{\varphi(a', a)}$.

Chapter 4

Lattices and the functor τ

Let m be a reductive Lie algebra with notation as in § 3. Let $\mathfrak{w}$ be the Weyl group of $(m, \mathfrak{h})$ and let t_0 be the unique element of $\mathfrak{w}$ with $t_0 Q = -Q$. If $\ell(\cdot)$ is the length function on $\mathfrak{w}$, then $\ell(t_0) = \operatorname{card} Q$. Put $n^{\pm} = \sum_{\alpha \in Q} m_{\pm\alpha}$. For $\alpha \in Q$, choose a subalgebra $\mathfrak{a}^{(\alpha)}$ of m with $\mathfrak{a}^{(\alpha)} \sim sl(2, \mathbb{C})$ and having a standard basis $\overline{H}_\alpha$, X_α, Y_α. Let C_α denote the completion functor defined on the category $\mathscr{I}_m(a^{(\alpha)})$. 24

4.1 Definition. *Let $\mathfrak{g}$ be a Lie algebra with $m \subseteq \mathfrak{g}$. Let $\mathscr{I}_{\mathfrak{g}}(m)$ denote the category of $\mathfrak{g}$-modules A which satisfy the following conditions:*

(i) A is a weight for $\mathfrak{h}$ with integral weights (ii) for $\alpha \in Q$ and simple, X_α acts locally nilpotently on A and (iii) as a $U(n^-)$-module, A is torsion free.

4.2 Lemma. *Let $\alpha \in Q$ be simple. Then C_α maps $\mathscr{I}_{\mathfrak{g}}(m)$ into itself.*

For a proof of (4.2) and other results given without proof consult [15, §4].

Remark. It is essential to assume α simple. Otherwise the assertion of the lemma is false.

4.3 Definition. *Let $A \in \mathscr{I}_{\mathfrak{g}}(m)$. By a lattice above A, we mean a set of $\mathfrak{g}$-modules A_s, $s \in \mathfrak{w}$, in $\mathscr{I}_{\mathfrak{g}}(m)$ such that: (i) $A = A_{t_0}$ and (ii) if $s \in \mathfrak{w}$ and $\alpha \in Q$ is simple with $\ell(s_\alpha s) = \ell(s) + l$, then $A_{s_\alpha s} \subset A_s$ and $A_s \simeq C_\alpha(A_{s_\alpha s})$.*

25 The fundamental result of this section has been proved independently by Bouaziz [5] and Deodhar [7]. This result is

4.4 Theorem. *Let $s \in \mathfrak{w}$ and let $s = s_{\beta_1} \cdots s_{\beta_r} = s_{\gamma_1} \cdots s_{\gamma_r}$ be two reduced expressions for s. Then the two composite functors $C_{\beta_1} \circ \cdots \circ C_{\beta_r}$ and $C_{\gamma_1} \circ \cdots \circ C_{\gamma_r}$ are naturally equivalent on $\mathscr{I}_{\mathfrak{g}}(m)$.*

This theorem implies the existence of a lattice.

4.5 Corollary. *Let $A \in \mathscr{I}_{\mathfrak{g}}(m)$. With notations as above, put $A_{st_0} = C_{\beta_1} \circ \cdots circ C_{\beta_r}(A)$. Then A_s, $s \in \mathfrak{w}$, is a lattice above A. Moreover if A'_s, $s \in \mathfrak{w}$, is another lattice above A then there exist a unique isomorphism $\varphi : A'_1 \xrightarrow{\sim} A_1$ which extends the identity map A_{t_0} to A'_{t_0}. For $s \in \mathfrak{w}$, φ gives an isomorphism $\varphi : A_s \xrightarrow{\sim} A'_s$.*

The proof of (4.4) reduces easily to the case where m has rank two. In this case both proofs then relay on an ingenious use of specific identities in the universal enveloping algebra $U(m)$.

From the corresponding result for completions we have:

4.6 Proposition. *Let $A \in \mathscr{I}_{\mathfrak{g}}(m)$ and let F be a finite dimensional $\mathfrak{g}$-module. If A_s, $s \in \mathfrak{w}$, is a lattice above A, then $F \otimes A_s$, $s \in \mathfrak{w}$, is a lattice above $F \otimes A$.*

4.7 Definition. *For $A \in \mathscr{I}_{\mathfrak{g}}(m)$, let A_s, $s \in \mathfrak{w}$, be a lattice above A. Define $\tau(A)$ to be the quotient $A_1 / \sum\limits_{s \neq 1} A_s$.*

26 Note that $\tau(A)$ may be the zero module. Let $\mathfrak{U}(\mathfrak{g}, m)$ be the category of $(\mathfrak{g}, m)$-modules; i.e., $\mathfrak{g}$-modules which are $U(m)$-locally finite.

4.8 Theorem. *The map $A \mapsto \tau(A)$ is a covariant additive functor from the category $\mathscr{I}_{\mathfrak{g}}(m)$ into the category $\mathfrak{U}(\mathfrak{g}, m)$. Also τ commutes with the functor of tensoring by a finite dimensional $\mathfrak{g}$-module.*

This theorem is an immediate consequence of (2.7), (4.5) and (4.6).

4.9 Warning: In general, completion functors are left exact but nor right exact. However, in general, the functor τ is neither right nor left exact.

We complete this section by describing the transfer of invariant pairings by the functor τ. Let σ be an antiautomorphism of $\mathfrak{g}$ with $\sigma m = m$ and σ equal the identity on the CSA $\mathfrak{h}$ of m. Invariant pairings and forms on $\mathfrak{g}$-modules and m-modules are defined using this σ (cf. (3.1)). Let $A, B \in \mathscr{I}_{\mathfrak{g}}(m)$ and $\varphi \in Inv_{\mathfrak{g}}(A, B)$. Let $t_0 = s_{\gamma_1} \cdots s_{\gamma_d}$ be a reduced expression for t_0 and, using (3.14), let $\varphi_1 = C_{\gamma_1} \circ \cdots \circ C_{\gamma_d}(\varphi)$. Then, if A_s, B_s, $s \in \mathfrak{w}$, are lattices above A and B respectively, φ_1 is a $\mathfrak{g}$-invariant pairing of A_1 and B_1. In [17] it is shown that φ_1 is independent of the choice of reduced expression; and so, φ_1 is zero on $A_1 \times \sum\limits_{s \neq 1} B_s$ and $\sum\limits_{s \neq 1} A_s \times B_1$. Therefore we have:

4.10 Theorem. *φ_1 induces a pairing of $\tau(A)$ and $\tau(B)$.*

This theorem is proved in [17].

For any $\mathfrak{b}$-module L, put $U(L) = U(m) \bigotimes\limits_{U(\mathfrak{b})} L$. The main result of **27**
[17] is:

4.11 Proposition. *Let L and M be locally finite $\mathfrak{b}$-modules which are weight modules for $\mathfrak{h}$ with integral weights and finite dimensional weight spaces. Assume that $U(L)$ and $U(M)$ admit nondegenerate M-invariant forms. Then the map $\tau : Inv_m(U(L), U(M)) \to Inv_m(\tau U(L), \tau U(M))$ is surjective. Further-more, τ maps nondegenerate pairings to nondegenerate pairings.*

Let notation be as in (3.9) and let τ be the functor defined on the category $\mathscr{I}_{m \times m}(\mathfrak{t})$.

4.12 Corollary. *Let M be an irreducible Verma m-module. For A, B objects in the category $\mathscr{O} \otimes M$, $\tau : Inv_{\mathfrak{t}}(A, B) \to Inv_{\mathfrak{t}}(\tau A, \tau B)$ is surjective. Also, if $\varphi \in Inv_{\mathfrak{t}}(A, B)$ is nondegenerate then $\tau\varphi$ is nondegenerate.*

The corollary follows from (3.10), (3.9) and (4.11).

Chapter 5

Translation functors

In this section we described the Zuckerman functors which allow the translation from modules with one infinitesimal character to modules with another. Let $\mathfrak{g}_0$ be a real semisimple Lie algebra with complexification $\mathfrak{g}$. With our standard notation $\mathfrak{g}$ has a CSA $\mathfrak{h}$, roots Δ, positive system P and Weyl group $\mathfrak{w}$. For $\nu \in \mathfrak{h}^*$, let χ_ν denote the $Z(\mathfrak{g})$ character which is the infinitesimal character of the Verma module $M(\nu)$. By the Chevalley restriction theorem we know that every character of $Z(\mathfrak{g})$ has the form χ_ν and, for $\nu, \nu' \in \mathfrak{h}^*$, $\chi_\nu = \chi_{\nu'}$ precisely when ν and ν' lie in the same $\mathfrak{w}$-orbit. For $\nu \in \mathfrak{h}^*$ and any $\mathfrak{g}$-module A, let $P_\nu A$ be the maximal $\mathfrak{g}$-submodule of A where elements $z - \chi_\nu(z)$, $z \in Z(\mathfrak{g})$, are locally nilpotent. $P_\nu A$ is called the generalized eigen subspace of A for χ_ν. 28

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the complexification of a Cartan decomposition of $\mathfrak{g}_0$ and let $\mathfrak{U}$ denote the category of admissible $(\mathfrak{g}, \mathfrak{k})$-modules. Let $\mathscr{O}$ denote the BGG category $\mathscr{O}$ for $\mathfrak{g}$. For $A \in \mathfrak{U}$ (resp. $\mathscr{O}$), A is the direct sum of its subspaces $P_\nu A$, $\nu \in \mathfrak{h}^*$. For $\nu \in \mathfrak{h}^*$, let $\mathfrak{U}_\nu$ (resp. $\mathscr{O}_\nu$) denote the full subcategory of $\mathfrak{U}$ with all objects A in $\mathfrak{U}$ (resp. $\mathscr{O}$), such that $P_\nu A = A$.

Fix $\lambda \in \mathfrak{h}^*$ and let Δ_λ, P_λ and $\mathfrak{w}_\lambda$ be as in § 1. Let F_μ be the irreducible finite dimensional $\mathfrak{g}$-module with extreme weight μ. Assume, for $\alpha \in \Delta_\lambda$, $\lambda_\alpha \mu_\alpha \in \mathbb{N}$. Then define functors $\varphi^\lambda_{\lambda+\mu}$ and $\psi^{\lambda+\mu}_\lambda$ by:

$$\varphi^\lambda_{\lambda+\mu} A = P_{\lambda+\mu}(F_\mu \otimes P_\lambda(A)),\ \psi^{\lambda+\mu}_\lambda(A) = P_\lambda(F^*_\mu \otimes P_{\lambda+\mu}(A)). \tag{5.1}$$

29 We say that λ and $\lambda+\mu$ are equisingular if they have the same stabilizer in $\mathfrak{w}$. For convenience we write $\varphi = \varphi^{\lambda}_{\lambda+\mu}$ and $\psi = \psi^{\lambda+\mu}_{\lambda}$.

5.2 Theorem. (i) *φ is injective on both $\mathfrak{U}_\lambda$ and $\mathscr{O}_\lambda$.*

(ii) *If λ and $\lambda+\mu$ are equisingular, then we have natural equivalences: $\varphi : \mathfrak{U}_\lambda \xrightarrow{\sim} \mathfrak{U}_{\lambda+\mu}$, $\varphi : \mathscr{O}_\lambda \xrightarrow{\sim} \mathscr{O}_{\lambda+\mu}$. Moreover, ψ is the natural inverse to φ in both cases.*

(iii) *If $A \in \mathfrak{U}$ or $\mathscr{O}$ and is irreducible, then ψA is either zero or irreducible.*

(iv) *If $A \in \mathfrak{U}_\lambda$ (resp. $\mathscr{O}_\lambda$) and is irreducible, then there is an irreducible $B \in \mathfrak{U}_{\lambda+\mu}$ (resp. $\mathscr{O}_{\lambda+\mu}$) with $\psi B \simeq A$.*

Assertions (i) and (ii) are due to Zuckerman [38]; while (iii) and (iv) have been established for the category $\mathscr{O}$ by Jantzen [24] and for the category $\mathfrak{U}$ by Vogan [34]. For the category $\mathscr{O}$, assertions (iii) and (iv) are implied by the following proposition. For the category of admissible modules for a complex Lie group, we give a proof of (iii) and (iv) in (10.9).

Recall the notation for Verma modules and their irreducible quotients.

5.3 Proposition. *With notation as in (5.2), we have:*

(i) $\psi M(\lambda+\mu) = M(\lambda)$

(ii) $\psi L(\lambda+\mu) = L(\lambda)$ *or zero*

(iii) *Let $Q = P \cap \{\alpha \mid \lambda_\alpha = 0\}$. Then $\psi L(\lambda+\mu) = L(\lambda)$ if and only if $\lambda+\mu$ is $-Q$-dominant.*

30 (iv) *If λ and $\lambda+\mu$ are equisingular, then*

$$\varphi M(\lambda) = M(\lambda+\mu), \quad \varphi L(\lambda) = L(\lambda+\mu).$$

Proof. $F^*_\mu \otimes M(\lambda+\mu) \simeq U(\mathfrak{g}) \bigotimes_{U(\mathfrak{b})} (F^*_\mu \otimes \mathbb{C}_{\lambda+\mu-\delta})$. Let $0 = L_0 \subset \ldots \subset L_r = F^*_\mu \otimes \mathbb{C}_{\lambda+\mu-\delta}$ be a flag of $\mathfrak{b}$-modules for L_r with $L_i/L_{i-1} \simeq \mathbb{C}_{\nu_i-\delta}$,

$1 \le i \le r$. Then by inducing form $U(\mathfrak{b})$ to $U(\mathfrak{g})$, $F_\mu^* \otimes M(\lambda+\mu)$ has a flag of submodules $0 = N_0 \subset \ldots \subset N_r$ with $N_i/N_{i-1} \simeq M(\nu_i)$. By assumption, λ and μ lie in the same Weyl chamber for Δ_λ; and so, the only element ν_i in the $\mathfrak{w}$-orbit of λ is λ itself. This proves (i). □

The functor ψ is exact. So by (i), $\psi L(\lambda+\mu)$ is a quotient of $M(\lambda)$. However, $L(\lambda+\mu)$ and F_μ^* both admit nondegenerate invariant forms (cf. (3.5)); and so, $\psi L(\lambda+\mu)$ admits a nondegenerate invariant form. Therefore by (3.5), $\psi L(\lambda+\mu)$ is either zero or $L(\lambda)$.

The functor ψ is exact; and so, by (i) and (ii), for some $L(\xi)$ occurring in $M(\lambda+\mu)$, $\psi L(\xi) = L(\lambda)$. Choose $s \in \mathfrak{w}$ with $\xi = s(\lambda+\mu)$. Then, again by exactness of ψ, $L(\lambda)$ must occur as a quotient of $\psi M(\xi) = M(s\lambda)$. Thus $s\lambda = \lambda$ and then if $\lambda+\mu$ is -Q-dominant, $\xi = \lambda+\mu$. This proves half of (iii). Now assume $\lambda+\mu$ is not -Q-dominant and choose $\alpha \in Q$ with $(\lambda+\mu)_\alpha \in \mathbb{N}^*$. Put $\xi = s_\alpha(\lambda+\mu)$. Now $M(\xi) \subset M(\lambda+\mu)$ and $\psi M(\xi) = M(\lambda) = \psi M(\lambda) = M(\lambda) = \psi M(\lambda+\mu)$. But ψ is exact; and so, $\psi A = 0$ for any component of $M(\lambda+\mu)/M(\xi)$. Thus $\psi L(\lambda+\mu) = 0$, completing the proof of (iii). Statement (iv) follows from (i), (ii) and (iii) using (5.2) (ii).

5.4 Proposition. *Let* $\mathfrak{t} \subset \mathfrak{g}$ *be a subalgebra and let* τ *be the functor defined on* $\mathscr{I}_\mathfrak{g}(\mathfrak{t})$ *as in § 4. Then* τ *commutes with both* φ *and* ψ.

Proof. By (4.8), τ commutes with tensoring by finite dimensional $\mathfrak{g}$-modules. Also, completions preserve generalized infinitesimal character; so, P_ν commutes with completions, $\nu \in \mathfrak{h}^*$. These two facts give (5.4). □

Chapter 6

Construction of irreducible admissible $(\mathfrak{g}, \mathfrak{k})$-modules

In this section we begin the study of the representations of a connected complex semisimple Lie group G. The main result is the construction of irreducible $(\mathfrak{g}, \mathfrak{k})$-modules. If $\mathfrak{g}_0$ is the Lie algebra of G then $\mathfrak{g}_0$ is a complex Lie algebra. We fix a compact real form $\mathfrak{k}_0$ of $\mathfrak{g}_0$ and let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be a Cartan decomposition. At this point, it will be more useful to think of $\mathfrak{g}_0$ as a real Lie algebra admitting a map J defined by multiplication by $\sqrt{-1}$. We have $J(x) = \sqrt{-1}x$, $x \in \mathfrak{g}_0$, and $J^2 = -1$. Now we fix a CSA $\mathfrak{t}_0$ of $\mathfrak{k}_0$ and put $\mathfrak{a}_0 = J\mathfrak{t}_0$. Then $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ is a CSA of $\mathfrak{g}_0$. **31**

Considering $\mathfrak{g}_0$ as a real Lie algebra, it is especially useful to have the following complexification. Let c denote the conjugation of $\mathfrak{g}_0$ with respect to the real form $\mathfrak{k}_0$. Put $\mathfrak{g} = \mathfrak{g}_0 \times \mathfrak{g}_0$ and define $i : \mathfrak{g}_0 \to \mathfrak{g}$ by $i(x) = (x, x^c)$ for $x \in \mathfrak{g}_0$. Clearly i is an injection and $(\mathfrak{g}, i)$ is a complexification of $\mathfrak{g}_0$. Letting $\mathfrak{k}$ equal the diagonal in $\mathfrak{g}$, $\mathfrak{k}$ is naturally the complexification of $\mathfrak{k}_0$ and, abstractly, $\mathfrak{k}$ is isomorphic to the complex Lie algebra $\mathfrak{g}_0$. Similarly, putting $\mathfrak{p} = \{(x, -x) \mid x \in \mathfrak{g}_0\}$, $\mathfrak{p}$ is the complexification of $\mathfrak{p}_0$, $\mathfrak{p}_0 = J\mathfrak{k}_0$ and $\mathfrak{g} \oplus \mathfrak{p}$. Let θ denote the Cartan involution giving this decomposition of $\mathfrak{g}$. Next we put $\mathfrak{h} = \mathfrak{h}_0 \times \mathfrak{h}_0$, $\mathfrak{t} = \mathfrak{t}_0 \times \mathfrak{t}_0$, and $\mathfrak{a} = \mathfrak{a}_0 \times \mathfrak{a}_0$. Then $\mathfrak{h}$ and $\mathfrak{t}$ are CSAs of $\mathfrak{g}$ and $\mathfrak{k}$ respectively. The algebra $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}$ and $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$. Note also that for any real Lie algebra denoted by a lower case German script letter, deletion of a subscripted zero gives the complexification of the algebra in $\mathfrak{g}$.

6.1 Conventions for roots and Weyl groups

32 We identify $\mathfrak{h}^*$ with $\mathfrak{h}_0^* \times \mathfrak{h}_0^*$ by the formula:

$$(\lambda, \lambda')(H, H') = \lambda(H) + \lambda'(H'), \quad \forall (H, H') \in \mathfrak{h}.$$

Let Δ, Δ_0 and $\Delta_\mathfrak{t}$ denote the set of roots of $(\mathfrak{g}, \mathfrak{h})$, $(\mathfrak{g}_0, \mathfrak{h}_0)$ and $(\mathfrak{t}, \mathfrak{t})$ respectively. One checks that $\Delta = \Delta_0 \times \{0\} \cup \{0\} \times \Delta_0$. Let P_0 be a positive system for Δ_0. Then $P = P_0 \times \{0\} \cup \{0\} \times P_0$ is a θ-stable positive system for Δ. Also if we put $P_\mathfrak{t} = \{(\alpha, 0) |_\mathfrak{t} | \ \alpha \in P_0\}$, then $P_\mathfrak{t}$ is a positive system for $\Delta_\mathfrak{t}$. For $\nu \in \mathfrak{h}_0^*$ let $\Delta_\nu = \{\alpha \in \Delta \mid \nu_\alpha \in \mathbb{Z}\}$. Then Δ_ν is a root system and we let $P_\nu = P_0 \cap \Delta_\nu$.

Let $\mathfrak{w}$, $\mathfrak{w}_0$ and $\mathfrak{w}_\mathfrak{t}$ be the Weyl groups of Δ, Δ_0 and $\Delta_\mathfrak{t}$ respectively. Then $\mathfrak{w} = \mathfrak{w}_0 \times \mathfrak{w}_0$ and $\mathfrak{w}_\mathfrak{t}$ is the diagonal in $\mathfrak{w}$.

6.2 Conventions for highest weight modules

Let $n^\pm$, $n_0^\pm$, $\mathfrak{b}$ and $\mathfrak{b}_0$ denote the nilpotent and Borel subalgebras of $\mathfrak{g}$ and $\mathfrak{g}_0$ respectively associated with the positive systems P and P_0. We have: $n^\pm = n_0^\pm \times n_0^\pm$, $\mathfrak{b} = \mathfrak{b}_0 \times \mathfrak{b}_0$. Moreover, if we put $n_\mathfrak{t}^\pm = n^\pm \cap \mathfrak{t}$ and $\mathfrak{b}_\mathfrak{t} = \mathfrak{b} \cap \mathfrak{t}$, then $n_\mathfrak{t}^\pm$ and $\mathfrak{b}_\mathfrak{t}$ are the nilpotent and Borel subalgebras of $\mathfrak{t}$ associated with $P_\mathfrak{t}$. Let δ, δ_0 and $\delta_\mathfrak{t}$ denote half the sum of the elements of P, P_0 and $P_\mathfrak{t}$ respectively. If $\mu \in \mathfrak{h}^*$ (resp. $\mathfrak{h}_0^*$, $\mathfrak{t}^*$) then $M(\mu)$ denotes the Verma module with P (resp. P_0, $P_\mathfrak{t}$)-highest weight $\mu - \delta$ (resp. $\mu - \delta_0$, $\mu - \delta_\mathfrak{t}$) and $L(\mu)$ denotes the unique irreducible quotient of $M(\mu)$.

6.3 Convention on product actions

33 Let $\mathfrak{L}_0$ be a Lie algebra, A and B two $\mathfrak{L}_0$-modules. Put $\mathfrak{L} = \mathfrak{L}_0 \times \mathfrak{L}_0$. Then $A \otimes B$ becomes an $\mathfrak{L}$-module under the action: $(X, Y)(a \otimes b) = Xa \otimes b + a \otimes Y \cdot b$, for $X, Y \in \mathfrak{L}_0$, $a \in A$, $b \in B$. Furthermore, A and B are both irreducible if and only if $A \otimes B$ is an irreducible $\mathfrak{L}$-module.

6.4 Lemma. *For* $\lambda, \lambda' \in \mathfrak{h}_0^*$, *there exist natural isomorphisms*

$$M(\lambda, \lambda') \xrightarrow{\sim} M(\lambda) \otimes M(\lambda'), \quad L(\lambda, \lambda') \xrightarrow{\sim} L(\lambda) \otimes L(\lambda').$$

Proof. The first isomorphism is induced by the natural isomorphism $U(\mathfrak{g}) \xrightarrow{\sim} U(\mathfrak{g}_0) \otimes U(\mathfrak{g}_0)$ while the second follows from the first and (6.3). □

We next define a set of representations of $\mathfrak{g}$ which will be the center of our study. Put $\mathfrak{L} = \{(\lambda, \lambda') \in \mathfrak{h}^* \mid \lambda + \lambda'$ is Δ_0 − integral $\}$. Let $\mathfrak{L}'$ denote the regular elements of $\mathfrak{L}$ is stable under the action of $\mathfrak{w}_\mathfrak{t}$ and we let $[\lambda, \lambda']$ denote the $\mathfrak{w}_\mathfrak{t}$-orbit of $(\lambda, \lambda') \in \mathfrak{L}$. Let $\mathfrak{L}/\mathfrak{w}_\mathfrak{t}$ denote the set of $\mathfrak{w}_\mathfrak{t}$-orbits.

6.5 Convention for $[\lambda, \lambda']$

Each orbit $[\lambda, \lambda']$ in $\mathfrak{L}$ contains an element (possibly several) (μ, μ') satisfying the following two conditions. Let $\alpha \in P_0$.

(i) μ' is -P_μ-dominant

(ii) if $\mu'_0 = 0$ then $\mu_\alpha \in -\mathbb{N}$.

We assume for convenience that for any orbit $[\lambda, \lambda']$, (λ, λ') itself satisfies (6.5). Note that (i) is equivalent to saying $M(\lambda')$ is irreducible.

6.6 Definition. *For* $\underline{\lambda} \in \mathfrak{L}$ *satisfying (6.5) and* τ *the lattice functor de-* 34
fined on the category $\mathscr{I}_\mathfrak{g}(\mathfrak{t})$*, define* $Z(\underline{\lambda}) = \tau(L(\underline{\lambda}))$.

Lemma 3.10 shows, among other things, that $L(\underline{\lambda})$ is an object of $\mathscr{I}_\mathfrak{g}(\mathfrak{t})$; and so, $Z(\underline{\lambda})$ is well defined.

6.7 Definition. *Let A be a module and* $A = A_d \supset A_{d-1} \supset \ldots \supset A_1 \supset A_0 = 0$ *be a flag of submodules. The flag of submodules* $\{A_i\}$ *is called a Jordan-Hölder series for A if* A_i/A_{i-1} *is irreducible,* $1 \leq i \leq d$. *The flag is said to be reducible to a Jordan-Hölder series for A if* A_i/A_{i-1} *is either zero or irreducible,* $1 \leq i \leq n$.

6.8 Proposition. *Let* $\underline{\lambda} = (\lambda, \lambda') \in \mathfrak{L}$ *and assume* $M(\lambda')$ *is irreducible.*

(i) *If* λ' *is regular then* τ *maps a Jordan-Hölder series for* $M(\underline{\lambda})$ *to a Jordan-Hölder series for* $M(\underline{\lambda})$.

(ii) *In general,* τ *maps a Jordan-Hölder series for* $M(\underline{\lambda})$ *to a flag of submodules reducible to a Jordan-Hölder series for* $\tau M(\underline{\lambda})$.

Moreover, if $M_r \supset \ldots \supset M_0 = 0$ *is a Jordan-Hölder series for* $M(\lambda)$*, then*

$$\tau(M_i/M_{i-1}) \simeq \tau M_i/\tau M_{i-1}. \qquad (*)$$

Proof. The category $\mathscr{O} \otimes M(\lambda')$ is $\mathfrak{t}$-semisimple by Proposition 3.9; and so, τ is exact on this subcategory of $\mathscr{I}_{\mathfrak{t}}(\mathfrak{t})$. This proves $(*)$. Using the translation functors (cf. Proposition 5.3) it is sufficient to prove (i) assuming

$$\lambda'_\alpha << 0 \quad \text{for all} \quad \alpha \in P_0. \qquad (\dagger)$$

35 □

For $1 \leq i \leq r$, put $N_i = M_i/M_{i-1}$. We need only show that $\tau(N_i)$ is irreducible $1 \leq i \leq r$. We begin with a result from [15], regarding the image of a Verma module under the functor τ.

6.9 Proposition. *Let* $\mu = (\lambda, \lambda') \mid_{\mathfrak{t}}$ *and assume* ($\dagger$) *holds. Then the irreducible* $\mathfrak{t}$*-module with extreme weight* μ *occurs in* $\tau M(\lambda, \lambda')$ *with multiplicity one and is* $\mathfrak{g}$ *cyclic.*

This result is a formulation of Proposition 10.3 [15] in our notation. Note that ($\dagger$) a decomposition $N_r = L \oplus B$ with B an irreducible $\mathfrak{t}$-Verma module with highest weight $\mu - 2\delta_{\mathfrak{t}}$ (μ as in (6.9)). τB is an irreducible $\mathfrak{t}$-module with extreme weight μ; and so, from (6.9) and the exactness of τ on $\mathscr{O} \otimes M(\lambda')$,

6.10

The irreducible $\mathfrak{t}$-module with extreme weight μ occurs in τN_r with multiplicity one and is $\mathfrak{g}$-cyclic.

We now show that τN_r is irreducible. By (3.5) and (4.12), τN_r admits a nondegenerate $\mathfrak{g}$-invariant form. If $L \subset \tau N_r$ is a proper submodule then by (6.10), $L^\perp$ contains the irreducible $\mathfrak{t}$-module with extreme

weight μ. This subspace is $\mathfrak{g}$-cyclic. So $L^{\perp} = \tau N_r$. But the form is nondegenerate; and thus, $L = 0$. This proves τN_r is irreducible. Now we proceed by induction on $r - i$. Assume $l \leq i < r$. There exists
$s \in \mathfrak{w}$ with $N_i \simeq L(s\lambda, \lambda')$ and such that $M(s^{\lambda}, \lambda')$ has a Jordan-Hölder 36
series of length strictly less than r. By induction (6.8) is true for any Jordan-Hölder series for $M(s\lambda, \lambda')$. This implies that τN_i is irreducible and completes the proof of (6.8).

6.11 Corollary. to (6.8). *Let* $\underline{\lambda} \in \mathfrak{L}$ *satisfy (6.5). Then* $Z(\underline{\lambda})$ *is an irreducible* $(\mathfrak{g}, \mathfrak{k})$*-module if* λ' *is regular and is irreducible or zero in general.*

Chapter 7

Resolutions of irreducible admissible $(\mathfrak{g}, \mathfrak{k})$-modules and $\mathfrak{k}$-multiplicity formulae

In [1] Bernstein, Gel'fand and Gel'fand have given a resolution of finite dimensional modules in terms of sums of Verma modules. These resolutions yield at once the character formulae as well as the weight space structure of the finite dimensional modules. This section will give the corresponding results for the irreducible $(\mathfrak{g}, \mathfrak{k})$-modules $Z(\underline{\lambda})$ defined in § 6. **37**

Recall the category $\mathscr{O} \otimes M(\lambda)$ from §3. Let $\ell(\cdot)$ denote the length function on the Weyl group $\mathfrak{w}_{\mathfrak{k}}$.

7.1 Proposition. *Let B be an object in $\mathscr{O} \otimes M(\lambda)$ with integral $\mathfrak{k}$-weights and assume $M(\lambda)$ is irreducible. Let B_s, $s \in \mathfrak{w}_{\mathfrak{k}}$, be a lattice above B and, for $0 \leq i \leq d = \mid P_{\mathfrak{k}} \mid$, put $\beta_i = \sum_{\ell(s)=i} B_s$. Then there is a resolution:*

$$0 \to \beta_d \to \ldots \to \beta_0 \to \tau B \to 0.$$

To proof will use the lemma:

7.2 Lemma. *Let C be any finite dimensional $\mathfrak{b}$-module which is a weight module for $\mathfrak{h}$. Then there exists $\nu \in \mathfrak{h}^*$ with $\nu_\alpha << 0$, $\forall \alpha \in P$, a finite dimensional $\mathfrak{g}$-module F and an injective $\mathfrak{b}$-module map $C \hookrightarrow F \otimes \mathbb{C}_\nu$.*

This lemma can be found as Lemma 4.7 in [15].

38 **7.3 Theorem.** *Let $\mu \in \mathfrak{t}^*$ be regular and -$P_{\mathfrak{t}}$-dominant integral. Let V_s, $s \in \mathfrak{w}_{\mathfrak{t}}$, be the lattice above the* $\mathfrak{t}$*-Verma module $M(\mu)$ and put $C_i = \sum_{\ell(s)=i} V_s$. Then there is a resolution*

$$0 \to C_d \to \ldots \to C_0 \tau M(\mu) \to 0.$$

Moreover, $\tau M(\mu)$ is the irreducible finite dimensional $\mathfrak{t}$*-module with extreme weight $\mu+\delta_{\mathfrak{t}}$. Each V_s is a Verma module and the maps $C_i \to C_{i-1}$, $1 \leq i \leq d$, are linear combinations of the inclusion maps among Verma modules.*

This theorem is a reformulation of the Bernstein Gel'fand Gel'fand result [1].

For a character χ of $Z(\mathfrak{t})$, let a superscript χ denote the generalized eigensubspace for χ. Since B is the direct sum of subspaces B^χ it is sufficient to give a resolution:

$$0 \to \beta_d^\chi \to \ldots \to \beta_0^\chi \tag{7.4}$$

By assumption we may write $B = A \otimes M(\lambda)$. Then by (3.10), we have a $\mathfrak{t}$-module isomorphism $B \simeq U(\mathfrak{t}) \bigotimes_{U(\mathfrak{b}_{\mathfrak{t}})} (A \otimes \mathbb{C}_{\lambda-\delta_0})$. Since $A \in \mathscr{O}$, there exists a finite dimensional $\mathfrak{b}$-module $C \subseteq A$ with B^χ contained in the submodule $U = U(\mathfrak{t}) \bigotimes_{U(\mathfrak{b}_{\mathfrak{t}})} (C \otimes \mathbb{C}_{\lambda-\delta_0})$. Now applying (7.2), we obtain an injection $B^\chi \hookrightarrow F \otimes M(\nu + \lambda)$. By Proposition 3.9, B and hence B^χ admits a nondegenerate $\mathfrak{t}$-invariant form. Clearly $B^\chi = U^\chi$ and so B^χ is a summand of U. Thus we may extend the $\mathfrak{t}$-invariant form on B^χ
39 to U and then by Proposition 3.6 to a form on $F \otimes M(\nu + \lambda)$. Taking the complement of B^χ we find that B^χ is a summand of $F \otimes M(\nu + \lambda)$. Applying (7.3) with $\mu = \nu + \lambda$ and tensoring by F, we obtain a resolution:

$$0 \to \mathscr{D}_d \to \ldots \to \mathscr{D}_0 \to \tau(F \times M(\nu + \lambda)) \to 0 \tag{7.5}$$

where the maps $\mathscr{D}_i \to \mathscr{D}_{i-1}$ are linear combinations of inclusion maps. Since B^χ is a summand of $F \otimes M(\nu + \lambda)$, the resolution (7.5) is the direct sum of two resolutions with one resolution being the resolution (7.4). This proves (7.1).

7.6 Corollary. to (7.1). *Let $\underline{\lambda} \in \mathfrak{L}$ and assume $M(\lambda')$ is irreducible. Let B_s, $s \in \mathfrak{w}_{\mathfrak{t}}$, be a lattice above $L(\underline{\lambda})$ and, for $0 \le i \le d$, put $\beta_i = \sum_{\ell(s)=i} B_s$. Then there is a resolution*

$$0 \to \beta_d \to \ldots \to \beta_0 \to \tau L(\underline{\lambda}) \to 0.$$

As preparation for a $\mathfrak{t}$-multiplicity formula, we need:

7.7 Lemma. *Let $A \in \mathscr{O}$ and assume that $M(\lambda)$ is irreducible. For $\mu \in \mathfrak{h}_0^*$, put $\mu_1 = (\mu, 0) \mid_t$ and $B = A \otimes M(\lambda)$. Then we have the following dimension formula for the weight spaces of the $\mathfrak{n}_{\mathfrak{t}}$-invariants in B:*

$$\dim B_{\mu_1}^{\mathfrak{n}_{\mathfrak{t}}} = \dim A_{\mu-\lambda+\delta_0}.$$

Proof. Since B admits a nondegenerate $\mathfrak{t}$-invariant form (cf. Proposition 3.9), Lemma (7.7) follows from the decomposition $B = A \otimes 1 \oplus n_{\mathfrak{t}}^- B$ and the equivalences $n_{\mathfrak{t}}^+ v = 0 \Leftrightarrow \langle n_{\mathfrak{t}}^+ v, B\rangle = 0 \Leftrightarrow \langle v, n_{\mathfrak{t}}^- B\rangle = 0$. □

7.8 Proposition. *Assume $M(\lambda)$ is irreducible. Let $B = A \otimes M(\lambda) \in \mathscr{O} \otimes (\lambda)$. For $\mu \in \mathfrak{h}_0^*$ P_0-dominant integral, put $\mu_1 = (\mu, 0) \mid_{\mathfrak{t}}$ and let F be the irreducible finite dimensional $\mathfrak{t}$-module with extreme weight μ_1. Then the multiplicity of F in the $(\mathfrak{g}, \mathfrak{t})$-module τB is given by:* **40**

$$\dim \mathrm{Hom}_{\mathfrak{t}}(F, \tau B) = \sum_{s \in \mathfrak{w}_0} (-1)^{d-\ell(s)} \dim A_{s(\mu+\delta_0)-\lambda}.$$

Proof. By assumption ν is P_0-dominant, so μ_1 is P_1-dominant. Since $M(\mu_1 + \delta_{\mathfrak{t}})$ is projective (cf. Lemma 7 [12]) we obtain from (7.1) the formula

$$\dim \mathrm{Hom}_{\mathfrak{t}}(F, \tau B) = \sum_{0 \le i \le d} (-1)^i \dim(\beta_i^{\mathfrak{n}_{\mathfrak{t}}})_{\mu_1}. \tag{7.9}$$

□

From Proposition 4.11 [15] we have the isomorphisms:

$$(B_s^{\mathfrak{n}_{\mathfrak{t}}})_{\mu_1} \sim (B^{\mathfrak{n}_{\mathfrak{t}}})_{t_0 s^{-1}(\mu_1+\delta_{\mathfrak{t}})-\delta_{\mathfrak{t}}}, \quad \forall s \in \mathfrak{w}_{\mathfrak{t}}, \tag{7.10}$$

where t_0 is the unique element of maximal length in $\mathfrak{w}_\mathfrak{t}$. Combining (7.9) and (7.10), we have:

$$\dim \mathrm{Hom}_\mathfrak{t}(F, \tau B) = \sum_{s \in \mathfrak{w}_\mathfrak{t}} (-1)^{\ell(s)} \dim B^{\mathfrak{n}_\mathfrak{t}}_{t_0 s^{-1}(\mu_l + \delta_\mathfrak{t}) - \delta_\mathfrak{t}}. \tag{7.11}$$

Now replacing $t_0 s^{-1}$ by s and using (7.7), we reach the final form of the multiplicity formula, (7.8).

41 **7.12 Corollary.** *Let* $\underline{\lambda} = (\lambda, \lambda') \in \mathfrak{L}$ *and assume* $M(\lambda')$ *is irreducible. Then*

$$\dim \mathrm{Hom}_\mathfrak{t}(F, \tau L(\underline{\lambda})) = \sum_{s \in _0} (-1)^{d - \ell(s)} \dim L(\lambda)_{s(\mu + \delta_0) - \lambda'}.$$

7.13 Corollary. *Let* $\alpha \in P_0$ *be simple and assume* $\lambda'_\alpha = 0$ *and* $\lambda_\alpha \in \mathbb{N}^*$. *Then* $\tau L(\underline{\lambda}) = 0$.

Note: See (9.14) for a sharper result.

Proof. $\ell(s_\alpha s) = \ell(s) \pm 1$ and the dimensions of weight spaces of $L(\lambda)$ are invariant under multiplication by S_α. Therefore by (7.12). $\mathrm{Hom}_\mathfrak{t}(F, \tau L(\underline{\lambda})) = 0$ for all finite dimensional $\mathfrak{t}$-modules F. This implies $\tau L(\underline{\lambda}) = 0$. □

7.14 Corollary. *Let* $\underline{\lambda} = (\lambda, \lambda') \in \mathfrak{L}$ *and assume* $M(\lambda')$ *is irreducible. Put* $\nu = (\underline{\lambda}) \mid_\mathfrak{t}$. *Then, for any finite dimensional* $\mathfrak{t}$*-module* F,

$$\dim \mathrm{Hom}_\mathfrak{t}(F, \tau M(\underline{\lambda})) = \dim \mathrm{Hom}_\mathfrak{t}(\mathbb{C}_\nu, F).$$

Proof. We may assume F is irreducible with -$P_\mathfrak{t}$ highest weight $\mu_1 = (\mu, 0) \mid_t$, $\mu \in \mathfrak{h}^*0$. Using the BGG resolution of F (7.3), we have:

$$\dim F_{t_0\nu} = \dim \mathrm{Hom}_\mathfrak{t}(\mathbb{C}_{t_0\nu}, F) = \sum_{s \in \mathfrak{w}_\mathfrak{t}} (-1)^{\ell(s)} \dim M(st_0(\mu_1 - \delta_\mathfrak{t}))_{t_0\nu}. \tag{7.15}$$

□

Now replacing μ_1 by $t_0\mu_1$ in (7.8), we obtain

$$\dim \operatorname{Hom}_{\mathfrak{t}}(F, \tau M(\underline{\lambda})) = \sum_{s \in \mathfrak{w}_0} (-1)^{\ell(s)} \dim M(\lambda)_{s(\mu-\delta_0)-\lambda'}. \tag{7.16}$$

42

Next define the partition function on $\mathfrak{h}_0^*$ as follows. Let Q be a positive system of roots for Δ_0. If $Q = \{\alpha_1, \dots \alpha_r\}$ and $\xi \in \mathfrak{h}_0^*$ define $\mathscr{P}_Q(\xi)$ to be the number of $\mathfrak{n}$-tuples $(a_1, \dots, a_n)$ with $a_i \in \mathbb{N}$ and $\xi = a_1\alpha_1 + \cdots + a_n\alpha_n$. From the elementary properties of Verma modules, we have:

$$\begin{aligned}
\dim M(\lambda)_{s(\mu-\delta_0)-\lambda'} &= \mathscr{P}_{-P_0}(s(\mu-\delta_0) - \lambda' - \lambda + \delta_0)\\
\dim M(st_0(\mu_l - \delta))_{t_0\nu} &= \mathscr{P}_{-P_0}(t_0(\lambda+\lambda') - st_0(\mu-\delta_0) + \delta_0)\\
&= \mathscr{P}_{P_0}(\lambda + \lambda' - t_0 st_0(\mu-\delta_0) - \delta_0)\\
&\qquad \text{since } t_0P_0 = -P_0\\
&= \mathscr{P}_{-P_0}(t_0 st_0(\mu-\delta_0) - \lambda - \lambda' + \delta_0).
\end{aligned}$$

Since $\ell(t_0 st_0) = \ell(s)$ and $\dim F_\nu = \dim F_{t_0\nu}$, these formulae yield, when inserted into (7.15) and (7.16), the equivalence of dimensions (7.14).

Corollary 7.14 is a reciprocity formula for the $(\mathfrak{g}, \mathfrak{t})$ modules $\tau M(\underline{\lambda})$. In the context of induced representations this same formula is called the Frobenius Reciprocity Theorem (cf. (8.3)).

Chapter 8

The principal series modules

In this section we give the definition of the principal series modules and list some of their basic properties. Our notation will differ from that in [8] (cf. § § 9.3, 9.4) by a slight shift in the parameter. **43**

We retain the notation of § 6. Put $Q = P_0 \times \{0\} \cup \{0\} \times (-P_0)$. Then Q is a positive system for Δ and $\theta Q = -Q$. As usual let $\delta_Q = \frac{1}{2} \sum_{\alpha \in Q} \alpha$. If $n_Q^{\pm}$ and $\mathfrak{b}_Q$ denote the associated nilpotent and Borel subalgebras, then $n_Q^{\pm} = n_0^{\pm} \times n_0^{\mp}$ and $\mathfrak{b}_Q = \mathfrak{h} \oplus n_Q^{+}$. Let $\mathfrak{n}_{Q,0}$ be the real subalgebra of $\mathfrak{g}_0$ such that $i(\mathfrak{n}_{q,0}) = \mathfrak{n}_Q \cap i(\mathfrak{g}_0)$. One may check that $\mathfrak{g}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_{Q,0}$ is an Iwasawa decomposition of $\mathfrak{g}_0$. For $\underline{\lambda} \in \mathfrak{h}^*$, let $\bar{M}(\underline{\lambda})$ denote the Verma module with Q-highest weight $\underline{\lambda} - \delta_Q$.

8.1 Definition. *For $\underline{\lambda} \in \mathfrak{h}^*$, define $X(\underline{\lambda})$ to be the submodule of $U(\mathfrak{t})$-locally finite vectors in the algebraic dual of $\bar{M}(-\underline{\lambda})$. $X(\underline{\lambda})$ is called the principal series module with parameter $\underline{\lambda}$.*

8.2 Remarks. (i) $X(\underline{\lambda})$ is naturally isomorphic to the set of $U(\mathfrak{t})$ - locally finite vectors in coinduced $\mathfrak{g}$-module $\mathrm{Hom}_{U(\mathfrak{b}_Q)}(U(\mathfrak{g}), \mathbb{C}_{\underline{\lambda}+\delta_Q})$.

(ii) $X(\underline{\lambda})$ admits an infinitesimal character $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ parametrized by the $\mathfrak{w}$-orbit of λ.

8.3 (Frobenius Reciprocity)

Let F be a finite dimensional $\mathfrak{k}$-module. Then, if $\nu = \underline{\lambda}\,|_{\mathfrak{t}}$

(i) $\dim \mathrm{Hom}_{\mathfrak{k}}(F, X(\underline{\lambda})) = \dim \mathrm{Hom}_{\mathfrak{t}}(\mathbb{C}_\nu, F)$.

(ii) $X(\underline{\lambda}) = 0$ if ν is not integral.

44 **8.4 Corollary.** *If F is irreducible with extreme weight ν, then F occurs in $X(\underline{\lambda})$ with multiplicity one and all other $\mathfrak{k}$-modules in $X(\underline{\lambda})$ have extreme weights with norms strictly greater than the norm of ν.*

Recall now the translation functors of § 5. Let $\underline{\lambda} = (\lambda, \lambda')$, $\underline{\mu} = (\mu, \mu')$ be elements of $\mathfrak{h}^*$ with $\underline{\lambda}$ integral. Let $\varphi = \varphi^{\underline{\lambda}}_{\underline{\lambda}+\underline{\mu}}$ and $\psi = \psi^{\underline{\lambda}+\underline{\mu}}_{\underline{\lambda}}$ be the Zuckerman translation functors. From Proposition 5.3 on Verma modules we obtain by duality:

8.5

(i) $\psi(X(\underline{\lambda} + \underline{\mu})) = X(\underline{\lambda})$

(ii) if $\underline{\lambda}$ and $\underline{\lambda} + \underline{\mu}$ are equisingular, then $\varphi(X(\underline{\lambda})) = X(\underline{\lambda} + \underline{\mu})$.

The next two properties of principal series modules are much deeper results from the theory of semisimple Lie algebras. The first is the fundamental result of Harish-Chandra:

8.6 Theorem (The subquotient theorem)**.** *Each irreducible admissible $(\mathfrak{g}, \mathfrak{k})$-module is isomorphic to a Jordan-Hölder factor of some principal series module.*

8.7 Proposition. *Let $s \in \mathfrak{w}_0$, $(\lambda, \lambda') \in \mathfrak{h}^*$. Then $X(\lambda, \lambda')$ and $X(s\lambda, s\lambda')$ have the same Jordan-Hölder factors occurring with the same multiplicity.*

The easiest proof of (8.7) is given by first checking that $X(\lambda, \lambda')$ is isomorphic to the set of $\mathfrak{k}$-finite vectors for a principal series represen-
45 tation of G, say $\bar{X}(\lambda, \lambda')$, and then showing that $\bar{X}(\lambda, \lambda')$ and $\bar{X}(s\lambda, s\lambda')$ have the same distribution character. For more details the reader may consult [31] or [32]. There is also a Lie algebraic proof of (8.7) given in [28].

Chapter 9

The character formula for $Z(\underline{\lambda})$

In this section we describe the isomorphisms between images of Verma modules under τ and the principal series modules. We retain the notation of § 6 and § 7. **46**

9.1 Theorem. *Let* $\underline{\lambda} = (\lambda, \lambda') \in \mathfrak{L}$ *and assume* $M(\lambda')$ *is irreducible. Then* $\tau M(\underline{\lambda})$ *and* $X(\underline{\lambda})$ *are isomorphic.*

A weaker result in a more general setting has been given in [14].

Proof. For $m \in \mathbb{N}$, put $\mu = \lambda' - m\delta_0$ and recall from § 5 the functor $\psi = \psi_{(\lambda,\lambda')}^{(\lambda,\mu)}$. By Proposition (5.3) and (8.5), we have:

$$\psi M(\lambda, \mu) \simeq M(\lambda, \lambda'), \quad \psi X(\lambda, \mu) \simeq X(\lambda, \lambda'). \qquad (*)$$

By $(*)$ and the fact that τ commutes with ψ, it is sufficient to prove (9.1) for the parameter (λ, μ). Taking $n >> 0$, and replacing (λ, λ') by (λ, μ) we may assume:

$$(Re\lambda')_\alpha << 0, \quad \forall \alpha \in P_0. \qquad (\dagger)$$

For any positive system R, set $\mathfrak{L}'(R) = \{\xi \in \mathfrak{L}' \mid \mathrm{Re}\,\xi \text{ is } R\text{-dominant}\}$. Let $\mathscr{P}(\xi)$ be the proposition that $X(\xi)$ and $\tau M(\xi)$ are isomorphic. □

9.2

Let R be a positive system, ξ, μ elements of $\mathfrak{L}'(R)$ with $\xi - \mu$ integral. Then $\mathscr{P}(\xi)$ is true $\Leftrightarrow$ $\mathscr{P}(\mu)$ is true.

Proof. Choose R-dominant integral $\sigma, \gamma \in \mathfrak{h}^*$ with $\xi - \mu = \sigma - \gamma$. Put $\zeta = \sigma + \mu = \xi + \gamma$, $\psi_1 = \psi_\xi^{\xi+\gamma}$ and $\psi_2 = \psi_\sigma^{\sigma+\mu}$. Since τ commutes with ψ_1 and ψ_2, we obtain by Proposition (5.3) and (8.5):

$$\mathscr{P}(\xi) \text{ is true } \Leftrightarrow \mathscr{P}(\zeta) \text{ is true } \Leftrightarrow \mathscr{P}(\mu) \text{ is true.}$$

47 □

9.3

Let R_0 be a positive system for Δ_0 and let β be a simple root in R_0 with $-\beta \in P_0$. Put $R_0' = s_\beta R_0$, $R = R_0 \times \{0\} \cup \{0\} \times (-P_0)$ and $R' = (s_\beta, l)R$. If $\underline{\xi} \in \mathfrak{L}'(R)$ and $\underline{\mu} \in \mathfrak{L}'(R')$ with $\underline{\xi} - \underline{\mu}$ integral, then $\mathscr{P}(\underline{\xi})$ true $\Rightarrow$ $\mathscr{P}(\underline{\mu})$ is true.

Write $\underline{\xi} = (\xi, \xi')$ and $\underline{\mu} = (\mu, \mu')$. By (9.2) we may translate ξ and μ and then ξ' and μ' so that we have:

9.4

(i) $\operatorname{Re} \xi_\alpha >> 0, \quad \forall \alpha \in R_\circ \setminus \{\beta\}$

(ii) $\operatorname{Re} \xi_\beta \leq 2$

(iii) $\mu - \xi = 4\delta_0$

(iv) $\mu' = \xi'$

(v) $\operatorname{Re} \xi'_\alpha << -|\operatorname{Re} \xi_\alpha|, \quad \forall \alpha \in P_0$.

The proof of (9.3) is based on two lemmas which we now describe.

Let F_0 be the finite dimensional $\mathfrak{g}_0$-module with highest weight $4\delta_0$ and let F be the $\mathfrak{g}$-module $F_0 \otimes \mathbb{C}$ where $\mathbb{C}$ is the trivial $\mathfrak{g}_0$-module. Choose a $\mathfrak{b}$-module filtration of F, $F = F_r \supset \ldots \supset F_0 = 0$, and let $\gamma_i \in \mathfrak{h}_0^*$ be determined by isomorphisms $F_i/F_{i-1} \simeq \mathbb{C}_{(\gamma_i, 0)}$, $1 \leq i \leq r$.

The $(\gamma_i, 0)$ are the weights of F with multiplicity. This flag of $\mathfrak{b}$-modules induces a $\mathfrak{g}$-module flag.

$$F \otimes M(\underline{\xi}) = B = B_r \supset \ldots \supset B_0 = \{0\} \text{ with } B_i/B_{i-1} \simeq M(\underline{\xi} + (\gamma_i, 0)) \tag{9.5}$$

Note that $B_1 \simeq M(\underline{\mu})$. For any central character χ, let a subscript χ **48**
denote the generalized eigen subspace.

9.6 Lemma. *Let χ denote the central character of $M(\underline{\mu})$ and let m equal the dimension of the F_0 weight space for the weight $s_\beta\mu - \xi$. Then there exists an exact sequence $0 \to M(\underline{\mu}) \to B_\chi \to L \to 0$ where L is isomorphic to a sum of m copies of $M(s_\beta, \mu, \mu')$.*

Proof. For $1 \le i \le r$, if $\gamma_i + \xi$ lies in the $\mathfrak{w}_0$-orbit of μ, then by (9.4) we conclude that either $\gamma_i = \mu - \xi = 4\delta_0$ or $\gamma_i = s_\beta\mu - \xi$. In the first case γ_i is the highest weight of F_0; and so, $i = 1$. This shows that (9.5) induces a filtration on B_χ and gives a short exact sequence as in (9.6) with L admitting a filtration $L_m \supset \ldots \supset L_0 = 0$ with $L_i/L_{i-1} \simeq M(s_\beta, \mu, \mu')$. However, L is a free $U(n^-)$-modules with cyclic highest weight space; and so, L is a sum of m copies of $M(s_\beta, \mu, \mu')$. This proves (9.6). □

We have an analogous result for principal series modules. Put $D = F \otimes X(\underline{\xi})$.

9.7 Lemma. *Let notation be as in (9.6). Then there exists a short exact sequence, $0 \to X(\mu) \to D_\chi \to N \to 0$ where N is isomorphic to a sum of m copies of $X(s_\beta\mu, \mu')$. Moreover, if $\nu = \underline{\mu}\,|_{\mathfrak{t}}$, then the $\mathfrak{t}$-module with extreme weight ν occurs in D_χ with multiplicity one and is contained in the image of $X(\underline{\mu})$.*

Proof. Recall that $X(\underline{\xi})$ is the space of $\mathfrak{t}$-finite vectors in the algebraic dual of $\overline{M}(\underline{\xi})$. Let F^* be the algebraic dual of F. Our filtration of F **49**
induces a $\bar{\mathfrak{b}}$-module dual filtration $F^* = F_r^* \supset \ldots \supset F_0^* = \{0\}$ with $F_i^*/F_{i-1}^* \simeq \mathbb{C}_{(-\gamma_{r-i+1}, 0)}$. Now $0 \times \mathfrak{g}_0$ acts by zero on F; and so, this filtration is also a $\mathfrak{b}_Q$-module filtration. Inducing from $\mathfrak{b}_Q$ up to $\mathfrak{g}$ we obtain the flag

$$F^* \otimes \overline{M}(-\underline{\xi}) = C = C_r \supset \ldots \supset C_0$$

$$= \{0\} \text{ with } C_i/C_{i-l} \simeq \overline{M}(-\xi - \gamma_{r-i+1}, -\xi'). \qquad (9.8)$$

By duality we obtain a flag $\{D_i\}_{0\leq i\leq r}$ on D, with $D_i/D_{i-l} \simeq X(\xi + \gamma_i, \xi')$. Now arguing as in the proof of (9.6) but for the generalized eigen subspace of C dual to χ and passing by duality to D_χ, we obtain the short exact sequence in (9.7). Then Frobenious reciprocity (8.3) completes the proof of (9.7). ◻

9.9 Corollary. to (9.6). *The following sequence is exact.*

$$0 \to \tau M(\underline{\mu}) \to \tau B_\chi \to \tau L \to 0.$$

Proof. B_χ is an element of the $\mathfrak{t}$-semisimple category $\mathscr{O} \otimes M(\xi')$ (cf. (3.9)). ◻

We now complete the proof of (9.3). Let T be an isomorphism $T : \tau M(\underline{\xi}) \xrightarrow{\sim} X(\underline{\xi})$ and let S be the isomorphism which is the restriction of $1\otimes T$ to $(F\otimes\tau M(\underline{\xi}))_\chi$. Then $S : \tau B_\chi \xrightarrow{\sim} D_\chi$. Put $\nu = \underline{\mu} |_t$ then by (9.7) and (6.9), S injects the subspace $\tau M(\underline{\mu})$ into $X(\underline{\mu})$. But $\overline{(s_\beta\mu, \mu')} \in \mathfrak{L}(R)$ and by (9.2, $\mathscr{P}(s_\beta\mu, \mu')$ is true. This implies that L and N are isomorphic; and so, by (9.7) and (9.9) the injection of $\tau M(\mu)$ into $X(\underline{\mu})$ must be an isomorphism. This completes the proof of (9.3).

50 Next we prove the theorem for one chamber.

$$\text{If} \quad \underline{\lambda} \in \mathfrak{L}'(-P), \quad \text{then} \quad \mathscr{P}(\underline{\lambda}) \quad \text{is true.} \qquad (9.10)$$

We begin with a preliminary result on minimal $\mathfrak{t}$-modules in $(\mathfrak{g}, \mathfrak{t})$-modules. For any integral $\mu \in \mathfrak{t}^*$, let F_μ be the irreducible finite dimensional $\mathfrak{t}$-module with extreme weight μ.

9.11 Definition. *Let $\mu \in \mathfrak{t}^*$ be integral and let A be a $(\mathfrak{g}, \mathfrak{t})$-module. Then F_μ is called a weak minimal $\mathfrak{t}$-type of A if (i) there exists $T \in$* $\mathrm{Hom}_{\mathfrak{t}}(F_\mu, A)$*, $T \neq 0$ (ii) for $\beta \in P_{\mathfrak{t}}$,* $\mathrm{Hom}_{\mathfrak{t}}(F_{\mu-\beta}, \mathfrak{p}\cdot T(F_\mu)) = 0$ *($\mathfrak{g} = \mathfrak{t}\oplus\mathfrak{p}$ is the Cartan decomposition) and (iii)* $\dim \mathrm{Hom}(F_\mu, T(F_\mu) + \mathfrak{p}\cdot T(F_\mu)) = 1$.

9.12 Theorem. *Let A be a $(\mathfrak{g}, \mathfrak{t})$-module with weak minimal $\mathfrak{t}$-type F_μ. Assume $\mu_\beta << 0$ for all $\beta \in P_{\mathfrak{t}}$. Then there exists an element $\lambda \in \mathfrak{h}^*$ with $\lambda |_{\mathfrak{t}} = \mu$ and a nonzero $\mathfrak{g}$-module map S with $S : \tau M(\lambda) \to A$.*

Both (9.11) and (9.12) are formulations in our notations of results from section six of [13].

We may assume λ' satisfies (†). Then by (8.3), $(\lambda, \lambda')\,|_{\mathfrak{t}}$ is a weak minimal $\mathfrak{t}$-type of $X(\underline{\lambda})$ and so by (9.12) there exists $\underline{\nu} = (\mu, \mu')$ and a nonzero $\mathfrak{g}$-module map $T : \tau M(\underline{\gamma}) \to X(\underline{\lambda})$. We claim $\underline{\gamma} = \underline{\lambda}$. By (9.12), $\nu + \nu' = \lambda + \lambda'$. Both $\mathfrak{g}$-modules must have the same $Z(\mathfrak{g})$-character, so $\underline{\lambda}$ and $\underline{\gamma}$ lie in the same $\mathfrak{w}$-orbit. Write $(\nu, \nu') = (r\lambda, s\lambda')$. Then $\lambda + \lambda' = \nu + \nu' = r\lambda + s\lambda'$. But by (†), $\lambda + \lambda'$ is -P_0-dominant while $r\lambda + s\lambda'$ is -sP_0-dominant. Therefore, $s = 1$ and $\mu' = \lambda'$. But then $\nu = \lambda$ and $\underline{\lambda} = \underline{\gamma}$. Thus $T : \tau M(\underline{\lambda}) \to X(\underline{\lambda})$. By assumption $M(\underline{\lambda})$ is irreducible
and so by (6.8), $\tau M(\underline{\lambda})$ is irreducible. Therefore T is an injection and, by **51**
the multiplicity formulae (7.14) and (8.3), T must be an isomorphism. This proves (9.10).

9.13

$\mathscr{P}(\underline{\lambda})$ is true for all $\underline{\lambda} \in \mathfrak{L}$ with $M(\lambda')$ irreducible.

By (†), $\underline{\lambda} \in \mathfrak{L}(S)$ where S is a positive root system of the form $S = R \times \{0\} \cup \{0\} \times -P_0$, R a positive system of Δ_0. We proceed by induction on $m = |S \cap P|$. If $m = 0$ then $\mathscr{P}(\underline{\lambda})$ is true by (9.10). If $m \neq 0$, then there exists $\beta \in R$ simple with $\beta \in P_0$. Put $S' = (s_\beta, 1)S$. Then for $n \in \mathbb{N}$, $n >> 0$, $\underline{\lambda} + n\delta_{S'} \in \mathfrak{L}(S')$ and $|S' \cap P| = |S \cap P| - 1$. So by induction $\mathscr{P}(\underline{\lambda} + n\delta_{S'})$ is true. By (9.3), $\mathscr{P}(\underline{\lambda})$ is true. This proves (9.13) and completes the proof of Theorem 9.1.

Using (9.1) we now describe a sharper vanishing theorem than Corollary 7.13.

9.14 Proposition. *Let* $\underline{\lambda} \in \mathfrak{L}$ *and assume* $M(\lambda')$ *is irreducible. Assume* $\underline{\lambda}$ *does not satisfy (6.5). Then* $\tau L(\underline{\lambda}) = 0$.

Proof. Let Δ^0 be the sub root system of Δ_0 which stabilizes λ'. Let $\mathfrak{w}^0$ be the corresponding stabilizer of λ' in $\mathfrak{w}_0$ and put $P^0 = P_0 \cap \Delta^0$. P^0 is a positive system for Δ^0. Let μ be the unique element in the $\mathfrak{w}^0$ orbit of λ which is -P^0-dominant. Then, putting $\underline{\mu} = (\mu, \lambda')$, we have the inclusion $M(\underline{\mu}) \hookrightarrow M(\underline{\lambda})$. These modules are elements of the $\mathfrak{t}$-semisimple category $\mathscr{O} \otimes M(\lambda')$. By (9.1) and (8.7), the inclusion $M(\underline{\mu}) \hookrightarrow M(\underline{\lambda})$

induces an isomorphism $\tau M(\underline{\mu}) \xrightarrow{\sim} \tau M(\underline{\lambda})$. By the exactness of τ on $\mathscr{O} \otimes M(\lambda')$, $\tau(M(\underline{\lambda})/M(\underline{\mu}))' = \overline{0}$. $L(\underline{\lambda})$ occurs in $M(\underline{\lambda})/M(\underline{\mu})$; and so by exactness of τ, $\tau L(\underline{\lambda}) = \overline{0}$. □

52 We complete this section with a character formula for $Z(\underline{\lambda})$. For $\underline{\nu} \in \mathfrak{L}$, let $E(\underline{\nu})$ be the distribution character of the principal series representation of G whose K-finite vectors are $\mathfrak{g}$-isomorphic to $X(\underline{\nu})$. Similarly, for $\underline{\lambda}$ with $M(\lambda')$ irreducible, let $\Theta(\underline{\lambda})$ denote the distribution character of any admissible representation of G with K-finite vectors $\mathfrak{g}$-isomorphic to $\tau L(\underline{\lambda})$. For any module A in $\mathscr{O}$, let chA denote the formal character of A (cf. [24]).

9.15 Proposition. *Fix integers* $m(s\lambda)$, $s \in \mathfrak{w}_\lambda$, *such that* $ch\ L(\lambda) = \sum_{s\lambda \in \mathfrak{w}_\lambda \cdot \lambda} m(s\lambda) chM(s\lambda)$. *Then*

$$\Theta(\underline{\lambda}) = \sum_{s\lambda \in \mathfrak{w}_\lambda \cdot \lambda} m(s\lambda)\ E(s\lambda, \lambda').$$

Proof. By Proposition 3.9, τ is exact on $\mathscr{O} \otimes M(\lambda')$. This exactness of τ implies that τ induces a linear map of Grothendieck groups. So by (9.1), the formula for $chL(\lambda)$ becomes under the map $\tau(\cdot \otimes M(\lambda'))$ the formula (9.15). □

An alternate and more general definition of minimal $\mathfrak{k}$-type has been given by Vogan [34]. The reader may wish to compare (9.12) with Theorem 3.14 in [34].

Chapter 10

Determination of the irreducible admissible $(\mathfrak{g}, \mathfrak{k})$-modules

Here we combine the results of the last four sections to describe the equivalence classes of irreducible admissible $(\mathfrak{g}, \mathfrak{k})$-modules. In turn this will give a classification of the infinitesimal equivalence classes of topologically completely irreducible representations of G. **53**

10.1 Lemma. *Let $\lambda, \nu, \mu \in \mathfrak{h}_0^*$. Assume* $\operatorname{Re}\nu$ *is -P_ν-dominant, that $\lambda + \nu$ and $\mu + \nu$ are integral, and that μ lies below λ in the Bruhat ordering. Let S be the stabilizer of ν in $\mathfrak{w}_0$. Then $\|\mu + \nu\| \geq \|\lambda + \nu\|$ and equality holds if and only if μ and λ lie in the same S-orbit.*

Proof. By assumption choose $d \in \mathbb{N}$ and elements $\alpha_0, \ldots, \alpha_{d-1} \in P_0$ with $\mu_{i+1} = s_{\alpha_i}\mu_i$, $0 \leq i \leq d-1$, and $\mu = \mu_d < \ldots < \mu_0 = \lambda$. Now $\|\mu_{i+1} + \nu\| = \|\mu_i + \nu - m\alpha_i\|$ with $m = (\mu_i)_{\alpha_i} \in \mathbb{N}^*$. Then $\|\mu_{i+1} + \nu\|^2 = \|\mu_i + \nu\|^2 + \langle \mu_i + \nu - \frac{1}{2}m\alpha_i, -2m\alpha_i \rangle = \|\mu_i + \nu\|^2 + \langle \operatorname{Re}\nu, -2m\alpha_i \rangle$. Since $\mu_i + \nu$ is integral, $\nu_{\alpha_i} \in \mathbb{N}$ and we may replace $\operatorname{Re}\nu$ by ν. Also since $\operatorname{Re}\nu$ is -P_ν-dominant, $\|\mu_{i+l} + \nu\| \geq \|\mu_i + \nu\|$. Moreover, equality holds if and only if $s_{\alpha_i} \in S$. So, $\|\mu + \nu\| \geq \|\lambda + \nu\|$ and λ and μ lie in the same S-orbit if and only if equality holds. $\square$

10.2 Corollary. *Let* $\underline{\lambda} = (\lambda, \lambda') \in \mathfrak{L}$ *and assume* $\underline{\lambda}$ *satisfies (6.5). Assume* μ *lies below* λ *in the Bruhat ordering. Then* $\|\mu + \lambda'\| > \|\lambda + \lambda'\|$.

Proof. By (6.5), if $Q = P_0 \cap \{\alpha \mid \lambda'_\alpha = 0\}$ then λ is -Q-dominant. S is
54 generated by the s_α, $\alpha \in Q$, and so, with notation as in the proof of (6.1)
with $\lambda' = \nu$, $\lambda_{\alpha_0} \in \mathbb{N}^*$ implies $\alpha_0 \notin Q$. Then $\|\mu + \lambda'\| \geq \|\mu_1 + \lambda'\| > \|\lambda + \lambda'\|$. This proves (10.2). □

10.3 Definition. *For* $\underline{\lambda} \in \mathfrak{L}$*, let* $\hat{X}(\underline{\lambda})$ *denote the unique subquotient of* $X(\underline{\lambda})$ *which contains the unique irreducible* $\mathfrak{t}$*-submodule with extreme weight* $\underline{\lambda}|_{\mathfrak{t}}$.

Note that for $\underline{\lambda} = (\lambda, \lambda')$, the norm of $\underline{\lambda}\,|_{\mathfrak{t}}$ is equal to $\|\lambda + \lambda'\|$.

10.4 Lemma. *For all* $s \in \mathfrak{w}$*,* $\hat{X}(\underline{\lambda})$ *and* $\hat{X}(s\underline{\lambda})$ *are isomorphic.*

Proof. By (8.7), $X(\underline{\lambda})$ and $X(s\underline{\lambda})$ have the same Jordan-Hölder factors. □

10.5 Proposition. *Assume* $\underline{\lambda} \in \mathfrak{L}$ *and satisfies (6.5). Then* $Z(\underline{\lambda})$ *and* $\hat{X}(\underline{\lambda})$ *are isomorphic. Moreover, if* $\underline{\lambda}$ *and* $\underline{\mu}$ *lie in the same* $\mathfrak{w}_{\mathfrak{t}}$*-orbit and both satisfy (6.5), then* $Z(\underline{\lambda})$ *and* $Z(\underline{\lambda})$ *are isomorphic.*

Proof. The second assertion follows from the first by (10.4). Using Theorem 9.1 we now prove the first assertion by induction on the length of a Jordan-Hölder series for $M(\lambda)$. If $M(\lambda)$ is irreducible then by (9.1), $\hat{X}(\underline{\lambda}) = X(\underline{\lambda})$ and the proof is complete. Assume $M(\underline{\lambda})$ is reducible. By (9.1), $Z(\underline{\lambda})$ occurs as a Jordan-Hölder factor of $X(\underline{\lambda})$; and so, we need only show that $Z(\underline{\lambda})$ contains the $\mathfrak{t}$-submodule with extreme weight $\underline{\lambda}\,|_{\mathfrak{t}}$. □

Any Jordan-Hölder factor of $X(\underline{\lambda})$ has by (9.1) the form $\tau L(\mu, \lambda')$ where $L(\mu)$ is a factor of $M(\lambda)$. If $\mu \neq \lambda$, then by (10.2), $\|\mu + \lambda'\| > \|\lambda + \lambda'\|$. Since $\|\mu + \lambda'\|$ is the smallest norm of an extreme weight of
any $\mathfrak{t}$-submodule of $X(\mu, \lambda')$ (cf. (8.4)), the $\mathfrak{t}$-submodule with extreme
55 weight $\underline{\lambda}\,|_{\mathfrak{t}}$ does not occur in $\tau M(\mu, \lambda')$ or in $\tau L(\mu, \lambda')$. Therefore, this
$\mathfrak{t}$-submodule occurs in $Z(\underline{\lambda})$ and the proof is complete.

10.6 Proposition. *Let* $\underline{\lambda} = (\lambda, \lambda')$ *and assume* $M(\lambda')$ *is irreducible. Then* $\tau L(\underline{\lambda})$ *is irreducible or zero according as* $\underline{\lambda}$ *satisfies (6.5) or not.*

Proof. If $\underline{\lambda}$ satisfies (6.5), then by (10.5), $\tau L(\underline{\lambda})$ is irreducible. If $\underline{\lambda}$ does not satisfy (6.5) then by (9.14), $\tau L(\underline{\lambda})$ is zero. □

10.7 Proposition. *Every irreducible admissible* $(\mathfrak{g}, \mathfrak{t})$*-module is isomorphic to some* $Z(\underline{\lambda})$ *where* $\underline{\lambda} \in \mathfrak{L}$ *and* $\underline{\lambda}$ *satisfies (6.5).*

Proof. By the subquotient theorem (8.6) combined with (8.7) and (9.1), if A is an irreducible admissible $(\mathfrak{g}, \mathfrak{t})$-module then A is isomorphic to $\tau L(\underline{\lambda})$ for some $\underline{\lambda}$ with $M(\lambda')$ irreducible. Lemma (10.6) implies that $\underline{\lambda}$ satisfies (6.5). □

Recall Definition (6.6).

10.8 Theorem. *The map* $\underline{\lambda} \mapsto Z(\underline{\lambda})$ *induces a bijection of* $\mathfrak{w}_{\mathfrak{t}}$*-orbits in* $\mathfrak{L}$ *and the set of equivalence classes of irreducible admissible* $(\mathfrak{g}, \mathfrak{t})$*-modules.*

Proof. By convention (cf. § 6) we choose any element of the $\mathfrak{w}_{\mathfrak{t}}$-orbit of $\underline{\lambda}$ which satisfies (6.5). Then by (10.5), the equivalence class of $Z(\underline{\lambda})$ is independent of this choice. Moreover, by (10.7), the induced map in the theorem is surjective. We now prove injectivity of the map.

Let $\underline{\mu}, \underline{\lambda} \in \mathfrak{L}$ both satisfying (6.5) and assume $Z(\underline{\lambda})$ and $Z(\underline{\mu})$ are **56**
isomorphic. They must have the same infinitesimal character so there are $t, s \in \mathfrak{w}_0$ with $\mu = t\lambda$ and $\mu' = s\lambda'$. Let $L_1 = \{\alpha \in \Delta_0 \mid \operatorname{Re} \lambda'_\alpha < 0\}$, $L_0 = \{\alpha \in \Delta_0 \mid \operatorname{Re} \lambda'_\alpha = 0\}$. Then $L = L_1 \cup L_0$ is the set of roots of a parabolic subalgebra and any positive root system L_0^+ of L_0 gives a positive system $L_1 \cup L_0^+$ of Δ_0. Using this, let R be any positive system of Δ_0 with $\alpha \in R \Rightarrow \operatorname{Re} \lambda'_\alpha \leq 0$, and $\lambda'_\alpha = 0 \Rightarrow \operatorname{Im} \lambda'_\alpha \leq 0$. Let $N_0 = \{\alpha \in \Delta_0 \mid \lambda'_\alpha = 0\}$ and $N_1 = \{\alpha \in R \mid \alpha \notin N_0\}$. Then $N = N_1 \cup N_0$ is the set of roots of another parabolic subalgebra of $\mathfrak{g}_0$ and any positive root system N_0^+ of N_0 gives a positive system $N_1 \cup N_0^+$ of Δ_0. Let B denote the set of simple roots of R and write $B = B' \cup B^{\sim}$ where B' is the set of simple roots of $R \cap N_0$. Let $B^{\sim} = \{\alpha_1, \ldots, \alpha_t\}$ and let $\{\gamma_1, \ldots, \gamma_t\}$ be the set of dual roots to $B^{\sim}$ orthogonal to B'; i.e., $\langle \gamma_i, \alpha_j \rangle = \delta_{ij}$, $1 \leq i$,

$j \leq t$, and $\langle \gamma_i, B' \rangle = 0$. Put $\lambda^{\sim} = \lambda' - a \sum_{1 \leq i \leq t} \gamma_i$, $a \in \mathbb{N}$, $a >> 0$. Since λ' and $\lambda^{\sim}$ have the same stabilizer in $\mathfrak{w}_0$, the Zuckerman functors applied to $Z(\underline{\lambda})$ and $Z(\underline{\mu})$ give the isomorphism $Z(\lambda, \lambda^{\sim}) \simeq Z(\mu, s\lambda^{\sim})$.

Let T be the positive system $-N_1 \cup (N_0 \cap -P_0)$. Then by (6.5) and the choice of $\lambda^{\sim}$, $\lambda + \lambda^{\sim}$ is T-dominant. Also, by (6.5), μ is $(sN_0 \cap -P_0)$-dominant; and so, if $T' = -N_1 \cup (N_0 \cap -s^{-1}P_0)$ then $s^{-1}\mu + \lambda^{\sim}$ is T'-dominant. However, both $\lambda + \lambda^{\sim}$ and $\mu + s\lambda^{\sim}$ are extreme weights of the minimal $\mathfrak{k}$-submodule of $Z(\lambda, \lambda^{\sim})$. Therefore, they lie in the same $\mathfrak{w}_0$-orbit; i.e., $\lambda + \lambda^{\sim} = r(\mu + s\lambda^{\sim})$. Thus we have $\lambda + \lambda^{\sim} = rs(s^{-1}\mu + \lambda^{\sim})$. However T and T' are positive system contained in $-N$ and $\lambda + \lambda^{\sim}$ is T-dominant while $s^{-1}\mu + \lambda^{\sim}$ is T'-dominant. Therefore rs must be an element of the Weyl group of N_0. Thus $rs\lambda' = \lambda'$; and so, $\lambda + \lambda' = r\mu + \lambda'$.
57 Then $\lambda = r\mu$ and, from above, $\mu = t\lambda$. So $r^{-1}\lambda = t\lambda$ and $r^{-1}\lambda' = s\lambda'$. This proves $\underline{\lambda}$ and $\underline{\mu}$ lie in the same $\mathfrak{w}_{\mathfrak{k}}$-orbit; and this completes the proof. □

For our special case of $(\mathfrak{g}, \mathfrak{k})$-modules, we can now give a proof of Theorem 5.2 (iii) and (iv) for this category $\mathfrak{U}$ of admissible $(\mathfrak{g}, \mathfrak{k})$-modules. Let notation be as in (5.2) except that we shall replace λ by $\underline{\lambda}$, μ by $\underline{\mu}$ and write $\underline{\lambda} = (\lambda, \lambda')$, $\underline{\mu} = (\mu, \mu')$.

10.9 Proposition. *Assume $\underline{\lambda} + \underline{\mu}$ satisfies (6.5). The conditions*

(i) *$\underline{\lambda}$ satisfies (6.5)*

(ii) *If $Q = \{\alpha \in P_0 \mid \lambda_\alpha = 0\}$ then $\lambda + \mu$ is -Q-dominant*

are necessary and sufficient for $Z(\underline{\lambda})$ to be nonzero and to have $\psi Z(\underline{\lambda} + \underline{\mu}) = Z(\underline{\lambda})$. If either condition does not hold then $\psi Z(\underline{\lambda} + \underline{\mu}) = 0$.

Proof. Put $A = \psi Z(\underline{\lambda} + \underline{\mu})$. If (ii) does not hold then by (5.3), $\psi L(\underline{\lambda} + \underline{\mu}) = 0$. Since ψ and τ commute, $A = \tau(0) = 0$. If (ii) holds but (i) does not then, by (5.3), $A \simeq \tau L(\underline{\lambda})$ and by (9.14) (whose proof does not rely on (5.2) (iii) or (iv)), $\psi L(\underline{\lambda}) = 0$. This shows $A = 0$ unless both conditions hold. □

Now assume both conditions. Then $A \simeq \tau L(\underline{\lambda})$, by (5.3). To complete the proof we shall show that $\tau L(\underline{\lambda})$ contains with multiplicity one the $\mathfrak{t}$-submodule with extreme weight $\underline{\lambda}\mid_{\mathfrak{t}}$. We know $\tau M(\underline{\lambda}) = X(\underline{\lambda})$ and by (8.4), $X(\underline{\lambda})$ contains this $\mathfrak{t}$-module with multiplicity one. Using the exactness of τ on $\mathscr{O}\otimes M(\lambda')$, for some $\underline{\xi}$ below or equal to $\underline{\lambda}$ in the Bruhat ordering, $\tau L(\underline{\xi})$ contains this $\mathfrak{t}$-module. But $\tau L(\underline{\xi})$ is a quotient of $X(\underline{\xi})$; and so, by (8.4), $\|\xi + \xi'\| \leq \|\lambda + \lambda'\|$, where $\underline{\xi} = (\xi, \xi')$. Since $\underline{\lambda}$ satisfies (6.5) and $\underline{\xi}$ is below or equal to $\underline{\lambda}$, $\xi' = \lambda'$. Then applying (10.2), we have $\underline{\xi} = \underline{\lambda}$. This proves $\tau L(\underline{\lambda})$ is not zero and completes the proof.

10.10 Proposition. (i) *For irreducible $A \in \mathfrak{U}$, ψA is either irreducible or zero.* 58

(ii) *If $A \in \mathfrak{U}_{\lambda}$ is irreducible then there exists $B \in \mathfrak{U}_{\lambda+\mu}$ irreducible with $\psi B = A$.*

Proof. To avoid a circular argument we shall make use of the preceding results only for regular parameter. First, we assume $\underline{\lambda} + \underline{\mu}$ is regular. We may assume that $\underline{\lambda} + \underline{\mu}$ satisfies (6.5) and also, by (10.7), that $A \simeq Z(\underline{\lambda} + \underline{\mu})$. Now by the proof of (10.9), ψA is either zero or contains a $\mathfrak{t}$-submodule with multiplicity one. From the Zuckerman article [38], ψA is primary and thus ψA is irreducible. For $B \in \mathfrak{U}_{\lambda}$ and irreducible, by [38], $\psi\varphi B$ is primary with irreducible factors isomorphic to B. ψ is exact so choose any Jordan-Hölder factor A of φB such that $\psi A \neq 0$. Then by (i), $\psi A \simeq B$. This completes the proof for $\underline{\lambda} + \underline{\mu}$ regular. This case of $\underline{\lambda} + \underline{\mu}$ regular is sufficient for all applications of (10.10) in the previous sections of these notes. Now, for $\underline{\lambda} + \underline{\mu}$ general, we may apply (10.7) in the above argument. This argument now proves (10.10) without restriction on $\underline{\lambda} + \underline{\mu}$. □

10.11 Remark. By combining (10.8) and (10.9) we have actually strengthened the results asserted in (5.2). For all parameters $\underline{\lambda}$ and $\underline{\lambda} + \underline{\lambda}$, (10.9) describes the image $\psi(A)$ with A irreducible.

Chapter 11

An application to highest weight modules

Our main purpose in these notes is to describe the category of admissible $(\mathfrak{g}, \mathfrak{k})$-modules by defining a correspondence between highest weight modules and admissible $(\mathfrak{g}, \mathfrak{k})$-modules. The working hypothesis here is that the category of highest weight modules is more tractable than the category of admissible $(\mathfrak{g}, \mathfrak{k})$-modules. Usually, we expect results for highest weight modules to give new results for admissible $(\mathfrak{g}, \mathfrak{k})$-modules. The present section is an exception to this rule. Here we use the vanishing theorem $\tau L(\underline{\lambda}) = 0$ for certain $\underline{\lambda}$ to conclude various skew-symmetry properties for coefficients of the character of $L(\lambda)$, $\underline{\lambda} = (\lambda, \lambda')$. 59

For any module $A \in \mathscr{O}$, let chA denote the formal character of A (cf. [24]). Fix $\lambda \in \mathfrak{h}_0^*$ and write

$$ch\ L(\lambda) = \sum m(s\lambda) ch\ M(s\lambda) \tag{11.1}$$

with the sum taken over the $\mathfrak{w}_\lambda$ orbit of λ.

11.2 Proposition. *Let α be a simple root of P_λ. Assume $\lambda_\alpha \in \mathbb{N}^*$. Then $m(s_\alpha s\lambda) + m(s\lambda) = 0$.*

The reader should compare (11.2) with Satz 2.12 in [24] which gives for regular λ and which an additional hypothesis on α, the identity $m(ss_\alpha\lambda) + m(s\lambda) = 0$.

Proof. Choose λ' with $\lambda + \lambda'$ integral, λ' -P_λ-dominant and such that $\lambda'_\beta = 0, \beta \in P_0 \Rightarrow \beta = \alpha$. Set $\underline{\lambda} = (\lambda, \lambda')$ and note that $P_{\lambda'} = P_\lambda$. For $\underline{\nu} \in \mathfrak{L}$, let $E(\underline{\nu})$ denote the global distribution character of G for the prin-
60 cipal series representation with $\mathfrak{g}$-module of K-finite vectors isomorphic to $X(\underline{\nu})$. Let $\Theta(\underline{\lambda})$ denote the distribution character of any admissible representation of G with subspace of K-finite vectors $\mathfrak{g}$-isomorphic to $\tau L(\underline{\lambda})$. Since the category $\mathcal{O} \otimes M(\lambda')$ is $\mathfrak{t}$-semisimple (Proposition 3.9), τ is exact on $\mathcal{O} \otimes M(\lambda')$. This exactness of τ implies that τ induces a linear map of Grothendieck groups. So the formula (11.1) becomes under τ:

$$\Theta(\underline{\lambda}) = \sum m(s\lambda)E(s\lambda, \lambda') \tag{11.3}$$

By our choice of λ', $\{\pm\alpha\}$ is the set of roots orthogonal to λ'; and so, $E(s_\alpha s\lambda, \lambda') = E(s\lambda, \lambda')$, $\forall s \in \mathfrak{w}_\lambda$. Moreover, these identities generate all the linear relations among the $E(s\lambda, \lambda')$. By assumption $\lambda'_\alpha \in \mathbb{N}^*$; and so, $\underline{\lambda}$ does not satisfy (6.5). This implies by (10.6), $\Theta(\underline{\lambda}) = 0$; and thus, $m(s_\alpha s\lambda) + m(s\lambda) = 0$. □

Let B_λ denote the simple roots of P_α. Choose a subset $B^0_\lambda \subset B_\lambda$, be the positive system generated by B^0_λ and let $\mathfrak{w}^0_\lambda$ be the Weyl group of B^0_λ.

11.4 Proposition. *Let $\lambda, \mu, \nu \in \mathfrak{h}^*_0$. Assume μ is -P^0_λ-dominant and assume $\lambda = w\mu$ for some $w \in \mathfrak{w}^0_\lambda$. If ν is -P^0_λ-dominant, then $L(\nu)$ does not occur as a subquotient of $M(\lambda)/M(\mu)$.*

Proof. Choose λ' with $\lambda + \lambda'$ integral, λ' -P_λ-dominant and with $B^0_\lambda = \{\alpha \in B_\lambda \mid \lambda'_\alpha = 0\}$. Put $\underline{\lambda} = (\lambda, \lambda')$ and $\underline{\mu} = (\mu, \lambda')$. Now, by assumption on λ, $\underline{\lambda}$ and $\underline{\mu}$ lie in the same $\mathfrak{w}$-orbit; and so, $\tau M(\underline{\lambda}) \simeq \tau M(\underline{\lambda})$. By the exactness of τ on $\mathcal{O} \otimes M(\lambda')$, $\tau(M(\underline{\lambda})) = 0$ and, for any subquotient A
61 of $M(\underline{\lambda})/M(\underline{\mu})$, $\tau A = 0$. If $L(\nu)$ occurs as a subquotient of $M(\lambda)\backslash M(\mu)$, then $\tau L(\nu, \lambda') = 0$. By (10.6), (ν, λ') must not satisfy (6.5); and so, ν is not -P^0_λ-dominant.

The reader should consult section 4.5 in [3] as well as Satz 2.12 in [24] for identities relating multiplicities of irreducible highest weight modules in Verma modules. □

Chapter 12

Concepts from homological algebra

In this section, let $\mathscr{O}$ denote the category $\mathscr{O}$ for the data $(\mathfrak{t}, \mathfrak{t}, P_{\mathfrak{t}})$. Let 62
$\mathfrak{n}$ denote the category of $\mathfrak{g}$-modules whose underlying $\mathfrak{t}$-modules lie in $\mathscr{O}$. Using standard concepts from homological algebra we will define a collection of left derived functors on $\mathfrak{n}$. In the next section, we will relate these to an inverse for the lattice functor τ.

12.1 Lemma. *If P is a projective object in $\mathscr{O}$, then $U(\mathfrak{g}) \bigotimes_{U(\mathfrak{t})} P$ is projective in $\mathfrak{n}$.*

The proof of (12.1) is elementary.

For $\mu \in \mathfrak{t}^*$, let $M(\mu)$ be the $\mathfrak{t}$-Verma module with highest weight $\mu - \delta_{\mathfrak{t}}$ and put $U(\mu) = U(\mathfrak{g}) \bigotimes_{U(\mathfrak{t})} M(\mu)$.

12.2 Corollary. *If $\mu \in \mathfrak{t}^*$ is dominant, then $U(\mu)$ is projective in $\mathfrak{n}$.*

Proof. $U(\mu) = U(\mathfrak{g}) \bigotimes_{U(\mathfrak{t})} M(\mu)$ and, since μ is dominant, $M(\mu)$ is projective (cf. Lemma 7 [12]). ◻

Fix a set $\psi \subseteq \mathfrak{t}^*$ of dominant integral elements. For any $\mathfrak{t}$-module L, put $L' = \sum L_\mu^{n_{\mathfrak{t}}}$ with the sum over μ in ψ.

12.3 Definition. (a) *A complex in the category* $\mathfrak{n}, \ldots A_i \to \ldots \to A_0 \to A \to 0$*, is called a* ψ*-resolution of* A *if (i)* A_i *is projective,* $i \in \mathbb{N}$*, (ii) for* $i \in \mathbb{N}$*,* A_i *is generated over* $U(\mathfrak{g})$ *by the subspace* A'_i *and (iii)* $\ldots A'_i \to \ldots \to A'_0 \to A' \to 0$ *is exact.*

63 (b) *A* ψ*-resolution is called special if each* A_i *is the formal direct sum of* $\mathfrak{g}$*-modules isomorphic to* $U(\mu)$ *for* $\mu \in \psi$*.*

12.4 Lemma. *Every object* A *in* $\mathfrak{n}$ *admits a special* ψ*-resolution.*

Proof. Let $A_0 = U(\mathfrak{g}) \bigotimes_{U(\mathfrak{b}_\mathfrak{t})} A'$ and let d_0 be the unique $\mathfrak{g}$-module map of A_0 to A which extends the inclusion $A' \hookrightarrow A$. Assume $d_i : A_i \to A_{i-1}$ has been defined (here $A_{-1} = A$). Let K_i = kernal d_i and put $A_{i+1} = U(\mathfrak{g}) \bigotimes_{U(\mathfrak{b}_\mathfrak{t})} K'_i$. As above let d_{i+1} be the unique $\mathfrak{g}$-module map of A_{i+1} to A_i which extends the inclusion $K_i \hookrightarrow A_i$. Then $d_{i+1} : A_{i+1} \to K_i$ and $A'_{i+1} \to K'_i \to 0$ is exact. By induction we have constructed a special ψ-resolution of A. □

Let A, B be $\mathfrak{g}$-modules in $\mathfrak{n}$ and $\varphi : A \to B$ a $\mathfrak{g}$-module map. Let $A_* \to A$ and $B_* \to B$ be ψ-resolutions of A and B respectively. For any $\mathfrak{g}$-module let $C^\sim$ be the $\mathfrak{g}$-submodule of C generated by C'. Note that $\varphi A^\sim \subseteq B^\sim$. We now show that φ induces a map φ_*.

12.5

φ induces a map $\varphi_* : A_* \to B_*$ which commutes with the boundary maps.

The map $B_0 \to B^\sim$ is surjective and since A_0 is projective there is a map φ_0 which makes the following diagram commutative.

$$\begin{array}{ccccc} B_0 & \longrightarrow & B^\sim & \longrightarrow & 0 \\ \uparrow{\scriptstyle \varphi_0} & & \uparrow{\scriptstyle \varphi} & & \\ A_0 & \longrightarrow & A^\sim & & \end{array} \qquad (12.6)$$

64 Let L_i equal the kernel of $d_i : B_i \to B_{i-1}$ and K_i equal the kernel $d_i : A_i \to A_{i-1}$. Note that by (12.6), $\varphi_0 K_0 \subseteq L_0$. Assume for $i \in \mathbb{N}^*$, φ_i

has been defined where $\varphi_i : A_i \to B_i$ and $\varphi_i K_i \subseteq L_i$. The map $B_{i+1} \to L_i^{\sim}$ is surjective by (12.3) (iii); and so, by projectivity of A_{i+1}, there exists a map φ_{i+1} which makes the following diagram commutative.

$$\begin{array}{ccccc} B_{i+1} & \longrightarrow & L_i^{\sim} & \longrightarrow & 0 \\ \uparrow{\scriptstyle \varphi_{i+1}} & & \uparrow{\scriptstyle \varphi_i} & & \\ A_{i+1} & \longrightarrow & K_i^{\sim} & & \end{array} \tag{12.7}$$

Note that (12.7) implies $\varphi_{i+1} : K_{i+1} \to L_{i+1}$. By induction we obtain the map φ_* of complexes, $\varphi_* : A_* \to B_*$. This proves (12.5).

We next recall the standard notion of homotopy. A map of complexes, $\Sigma_* : A_* \to B_*$, is said to be of degree j if $\Sigma_i : A_i \to B_{i+j}$, $i \in \mathbb{N}$. Unless stated otherwise a chain map will be a map φ_* of complexes of degree zero. For two chain maps φ_* and ψ_*, we say these maps are homotopic if there exists a degree one chain map $\Sigma_* : A_* \to B_*$ with $\varphi_i - \psi_i = d_{i+1} \circ \Sigma_i - \Sigma_{i-1} \circ d_i$, $i \in \mathbb{N}$. The degree one map Σ_* is called the homotopy.

12.8 Lemma. *Let A and B be $\mathfrak{g}$-modules in $\mathfrak{n}$, φ a map $\varphi : A \to B$ and let $A_* \to A$, $B_* \to B$ be ψ-resolutions. Let φ_* and ψ_* be any two chain maps induced by φ. Then φ_* and ψ_* are homotopic.*

For projective resolutions, (12.8) is a standard result. In our case of ψ-resolutions, the standard proof works (cf. Theorem 4.1 [23]) with modules C replaced by submodules $C^{\sim}$ as in the construction of φ_* given **65**
above.

Let $\mathfrak{U}$ be a category of modules with enough projectives. Let T be an additive functor defined on the full subcategory $\mathscr{P}$ whose objects are the projective objects in $\mathfrak{U}$. We now define a set of functors T_i, $i \in \mathbb{N}$, on $\mathfrak{U}$ called the left derived functors of the functor T on $\mathscr{P}$. For an object A in $\mathfrak{U}$, let $A_* \to A$ be a projective resolution of A. Then TA_* is a complex and we define T_iA to be the i^{th} homology group of the complex; i.e., $T_iA = \text{kernel } Td_i/\text{image } Td_{i+1}$. Since any two projective resolutions are homotopic and since T is additive, the module T_iA is independent of the choice of the projective resolution. For $i \in \mathbb{N}$, T_i is a covariant functor on $\mathfrak{U}$.

We would like to apply this technique to our setting. To do this we begin by defining a functor σ on a special full subcategory of $\mathfrak{n}$. Let $\mathscr{S}$ be the full subcategory of $\mathfrak{n}$ whose objects are formal direct sums of modules $U(\mu)$, $\mu \in \psi$. Let t_0 denote the maximal element of $\mathfrak{w}_{\mathfrak{t}}$. Then, for $\mu \in \psi$, $M(t_0\mu)$ is the unique irreducible submodule of the $\mathfrak{t}$-Verma module $M(\mu)$. Let $\sigma U(\mu)$ be the submodule of $U(\mu)$ given by the inclusion $U(\mathfrak{g}) \bigotimes_{U(\mathfrak{t})} M(t_0\mu) \hookrightarrow U(\mathfrak{g}) \bigotimes_{U(\mathfrak{t})} M(\mu) = U(\mu)$. Extend σ linearly to a map on objects in $\mathscr{S}$. Next we prove:

$$\sigma \text{ is an additive functor on } \mathscr{S}. \tag{12.9}$$

Proof. For objects A and B in $\mathscr{S}$ and for $\varphi \in \mathrm{Hom}(A, B)$, we define $\sigma\varphi$ to be the restriction of φ to the submodule $\sigma A \subset A$. Assume $\varphi(\sigma A) \subset \sigma B$. Then clearly σ takes the identity map to the identity map and also σ commutes with compositions. Thus to prove (12.9) we need only
66 check $\varphi(\sigma A) \subset \sigma B$. From the definition of $\mathscr{S}$ we need only check, for $\mu, \nu \in \psi$.

$$\text{if } \varphi : U(\mu) \to U(\nu) \text{ then } \varphi(\sigma U(\mu)) \subset \sigma U(\nu). \tag{12.10}$$

□

For $s \in \mathfrak{w}_{\mathfrak{t}}$, put $U_s = U(\mathfrak{g}) \bigotimes_{U(\mathfrak{t})} M(s\mu)$, $V_s = U(\mathfrak{g}) \bigotimes_{U(\mathfrak{t})} M(s\nu)$. Now $\varphi : U_1 \to V_1$. Assume $\varphi : U_s \to V_s$ and assume α is a simple root in $P_{\mathfrak{t}}$ with $(s\mu)_\alpha \in \mathbb{N}^*$. Put $\mathfrak{t}^\alpha$ equal to the reductive subalgebra of $\mathfrak{t}$, $\mathfrak{t}^\alpha = t \oplus \mathfrak{t}_\alpha \oplus \mathfrak{t}_{-\alpha}$. The module $M(s\nu)/M(s_\alpha s\nu)$ is locally $U(\mathfrak{t}^\alpha)$-finite; and so, $V_s/V_{s_\alpha s}$ is also. The cyclic vector for $U_{s_\alpha s}$ generates under $U(\mathfrak{t}^\alpha)$ an irreducible Verma $\mathfrak{t}^\alpha$-module. Therefore the induced map $U_{s_\alpha s} \to V_s/V_{s_\alpha s}$ must be zero. This implies $\varphi : U_{s_\alpha s} \to V_{s_\alpha s}$. By induction we have $\varphi : U_r \to V_r$ for all $r \in \mathfrak{w}_{\mathfrak{t}}$. For $r = t_0$, we have proved (12.10) which completes the proof of (12.9).

For convenience, we list a corollary to the proof of (12.10):

12.11 Corollary. *Let A_s and B_s, $s \in \mathfrak{w}_{\mathfrak{t}}$, be lattices of $\mathfrak{g}$-modules and $\varphi : A_1 \to B_1$ a $\mathfrak{g}$-module map. Then, for each $s \in \mathfrak{w}_{\mathfrak{t}}$, $\varphi A_s \subseteq B_s$.*

12.12 Definition. *Let A be an object in $\mathfrak{n}$ and by (12.4) let $A_* \to A$ be a special ψ-resolution of A. Define $\sigma_i A$ to be the i^{th} homology group of the complex σA_*. By (12.9), σ is an additive functor on $\mathscr{S}$; and so, each σ_i is a covariant functor on $\mathfrak{n}$. Also, using (12.8), $\sigma_i A$ is independent of the choice of ψ-resolution A_* of A. We call the covariant functor σ_i the i^{th} left derived functor of σ.*

We conclude this section by showing that the σ_i, $i \in \mathbb{N}$, preserve **67**
central character.

12.13 Proposition. *Let $A \in \mathfrak{n}$ and let χ be a character of $Z(\mathfrak{g})$. Assume for some $n \in \mathbb{N}^*$, $a(z) = (1 - \chi(z))^n$ annihilates A for $z \in Z(\mathfrak{g})$. Then, for $i \in \mathbb{N}$ and $z \in Z(\mathfrak{g})$, $a(z)$ annihilates $\sigma_i(A)$.*

Proof. Let $A_* \to A$ be a special ψ-resolution of A. Then multiplication by $a(z)$ and by zero on A_* are homotopic. For $T \in \operatorname{Hom}(A_*, A_*)$, σT is restriction to σA_*. Therefore, $\sigma(a(z)) = a(z)$; and so, multiplication by $a(z)$ and by zero are homotopic on σA_*. Thus, they induce the same maps on homology groups. This proves (12.13). □

Chapter 13

The category of admissible $(\mathfrak{g}, \mathfrak{k})$-modules

The results of this section are an application of the functors defined in § 12 in the setting of semisimple complex Lie groups. Here we analyze several cases where the functor σ_0 defined in § 12 is a natural inverse to the lattice functor τ. 68

We will use the notation set up in § 6 - § 10. Fix $\underline{\lambda} \in \mathfrak{L}$, $\underline{\lambda} = (\lambda, \lambda')$ and let $\mathfrak{U}$ denote the category of all admissible $(\mathfrak{g}, \mathfrak{k})$-modules with generalized $Z(\mathfrak{g})$-character parametrized by the $\mathfrak{w}$-orbit of $\underline{\lambda}$. Assume $\underline{\lambda}$ satisfies (6.5) and let $\mathfrak{B}$ denote the category of $\mathfrak{g}$-modules which are finitely generated, have $L(s\lambda, \lambda')$, $s \in \mathfrak{w}_0$, as irreducible objects, and are semisimple $\mathfrak{k}$-modules.

13.1

The reader can check that the map, $B \mapsto B/(0, n_0^-)B$, induces a natural equivalence of $\mathfrak{B}$ onto the category of finitely generated $\mathfrak{g}_0$-modules which are $U(\mathfrak{b}_0)$-locally finite, have generalized $Z(\mathfrak{g}_0)$-character with orbit $\mathfrak{w}_0 \cdot \lambda$ and have generalized $\mathfrak{h}_0$-weights ν with $\nu + \lambda'$ integral.

13.2 Theorem. *Let $\underline{\lambda} = (\lambda, \lambda') \in \mathfrak{L}$ with $M(\lambda')$ irreducible and λ' regular. Then the functor τ gives a natural equivalence; $\tau : \mathfrak{B} \xrightarrow{\sim} \mathfrak{U}$.*

Using translation functors (cf. Proposition 5.2), we may translate the parameter λ' and thus assume

$$\operatorname{Re}\lambda'_\alpha << 0 \quad \text{for all} \quad \alpha \in P_0. \tag{13.3}$$

We establish several preliminary results before proving (13.2).

69 Let $\wedge\mathfrak{t}$ denote the exterior algebra of $\mathfrak{t}$ and put $\psi_0 = \{(s\lambda,\lambda')\} \mid_{\mathfrak{t}} -2\delta_{\mathfrak{t}} + \xi \mid s \in \mathfrak{w}_0,\ \xi$ a weight of $\wedge\mathfrak{t}$ }. With t the maximal element of $\mathfrak{w}_{\mathfrak{t}}$ and with t' denoting the affine shift $t'\mu = t(\mu+\delta_{\mathfrak{t}}) - \delta_{\mathfrak{t}}$, $\mu \in \mathfrak{t}^*$, we put $\psi = t'\psi_0$. By (13.3), the elements of ψ are $P_{\mathfrak{t}}$-dominant; and so, we have a set of covariant functors σ_i, $i \in \mathbb{N}$, defined by this choice of ψ. Recall the category $\mathfrak{n}$ of § 12 and note $\mathfrak{U} \subseteq \mathfrak{n}$.

13.4 Proposition. *Assume* $\underline{\mu} = (s\lambda,\lambda')$ *for some* $s \in \mathfrak{w}_0$. *Then*

$$\sigma_i X(\underline{\mu}) = \begin{cases} M(\underline{\mu}) & \text{if } i = 0 \\ 0 & \text{if } i \geq 1. \end{cases}$$

Proof. For $0 \leqq j \leqq d = \dim(\mathfrak{b}/\mathfrak{b}_{\mathfrak{t}})$ and $\nu = \underline{\mu} \mid_{\mathfrak{t}} -2\delta_{\mathfrak{t}}$, put $L_j = \wedge^j(\mathfrak{b}/\mathfrak{b}_{\mathfrak{t}}) \otimes \mathbb{C}_\nu$ and $E_j = U(\mathfrak{g}) \bigotimes_{U(\mathfrak{b}_{\mathfrak{t}})} L_j$. Proposition 5.9 of [15] gives a resolution

$$0 \to E_d \to \ldots \to E_0 \to M(\underline{\mu}) \to 0. \tag{13.5}$$

Write $M = M(\underline{\mu})$ and let M_s, $s \in \mathfrak{w}_{\mathfrak{t}}$, and $E_{j,s}$, $s \in \mathfrak{w}_{\mathfrak{t}}$, be lattices above M and E_j respectively. By applying completion functors we obtain from (13.5), for $s \in \mathfrak{w}_{\mathfrak{t}}$,

$$0 \to E_{d,s} \to \ldots \to E_{0,s} \to M_s \to 0. \tag{13.6}$$

Theorem 5.7 of [13] asserts that (13.6) is exact. For $s = l$, (13.6) is a ψ-resolution of M_1. □

By (13.3) each $\mathfrak{g}$-module E_i is isomorphic to a module of the form $U(\mathfrak{g}) \bigotimes_{U(\mathfrak{t})} D_i$ with D_i a sum of irreducible Verma $\mathfrak{t}$-modules with highest

70 weights in ψ_0. In turn each $E_{i,1}$ is isomorphic to a sum of modules of

type $U(\xi)$, $\xi \in \psi$ (cf. § 12); and so, we have $\sigma E_{i,1} = E_i$, $0 \leq i \leq d$. Since (13.6) for $s = 1$ is a ψ-resolution of M_1, we obtain:

$$\sigma_0 M_1 = M(\underline{\mu}), \sigma_i M_1 = 0, \quad i \geq 1. \tag{13.7}$$

We now claim:

$$\sigma_i M_s = 0, \quad i \in \mathbb{N}, \quad s \neq 1. \tag{13.8}$$

From (13.3) and Proposition 4.13 of [15], M_s does not have any $n_{\mathfrak{t}}$-invariants of weight ν, $\nu \in \psi$; and so, $0 \to M_s \to 0$ is a special ψ-resolution of M_s. Clearly (13.8) follows from this.

Using (7.1), we have a resolution:

$$0 \to \mathfrak{m}_d \to \ldots \to \mathfrak{m}_0 \to X(\underline{\mu}) \to 0$$

where $\mathfrak{m}_i = \sum\limits_{\ell(s)=i} M_s$. By (13.7) and (13.8) we have computed $\sigma_i \mathfrak{m}_j$, $j, i \in \mathbb{N}$. Since $\sigma_i \mathfrak{m}_j = 0$ for $i \in \mathbb{N}$, $j \geq 1$, we obtain $\sigma_i \mathfrak{m}_0 \simeq \sigma_i X(\underline{\mu})$, $i \in \mathbb{N}$. Now formulae (13.7) proves (13.4).

For any $\mathfrak{g}$-module A, put A' (resp. $A^{\sim}$) equal to the subspace of $n_{\mathfrak{t}}$-invariants of weight ν, $\nu \in \psi$ (resp. $\nu \in \psi_0$).

13.9 Lemma. *For $A \in \mathfrak{B}$, let A_s be a lattice above A. Then the surjection $A_1 \to \tau(A)$ induces a bijection $A'_1 \xrightarrow{\sim} \tau(A)'$.*

Proof. By (13.3), for $s \neq 1$, $A'_s = 0$. Since elements of ψ are dominant $A'_1 \to \tau(A)'$ is surjective. This implies the induced map is a bijection. □

13.10 Lemma. *Assume (13.3). Let $B \in \mathfrak{B}$ and let B_s be a lattice above* 71
B and let $E_* \to B_1$ be a special ψ-resolution of B_1. Then $\sigma E_* \to B$ is a complex and $\sigma E_0^{\sim} \to B^{\sim}$ is surjective.

Proof. By (12.11), $\sigma E_* \to B$ is a complex. By definition of ψ-resolution $E'_0 \to B'_1$ is surjective. By Proposition 4.13 [15], this implies $\sigma E_0^{\sim} \to B^{\sim}$ is surjective. □

13.11 Lemma. *Let $A, B \in \mathfrak{B}$ (resp. $\mathfrak{U}$) and $\varphi : A \to B$. Then φ is injective (resp. surjective) if and only if the restriction $\varphi : A^{\sim} \to B^{\sim}$ (resp. $\varphi : A' \to B'$) is injective (resp. surjective).*

Proof. By (3.9), both $\mathfrak{U}$ and $\mathfrak{B}$ are $\mathfrak{t}$-semisimple categories. So, for $C \in \mathfrak{U}$, $D \in \mathfrak{B}$, $C \neq 0$, $D \neq 0$ implies $C' \neq 0$, $D^{\sim} \neq 0$. This implies the assertions of (13.11). □

13.12 Lemma. *Let $A, B \in \mathfrak{U}$ and $E_* \to A$ be a special ψ-resolution. If φ is a $\mathfrak{g}$-module map $\varphi : E_0 \to B$ with $\varphi \operatorname{Im} E_1 = 0$, then φ induces a map φ' with the following commutative diagram:*

$$\begin{array}{ccc} E_0 & \xrightarrow{\varphi} & B \\ \downarrow & \nearrow_{\varphi'} & \\ A & & \end{array}$$

Proof. E_0 is isomorphic to a sum of modules $U(\mu)$, $\mu \in \psi$, and so, E_0 admits a unique maximal $U(\mathfrak{t})$-locally finite quotient. Let D denote the maximal $U(\mathfrak{t})$-locally finite quotient of $E_0 / \operatorname{Im} E_1$. Since $A \in \mathfrak{U}$, the map $E_0 \to A \to 0$ induces a map $D \to A$. Clearly the map gives an isomorphism $D' \xrightarrow{\sim} A'$; and so, by (13.11) the map is an isomorphism
72 $D \xrightarrow{\sim} A$. Since $B \in \mathfrak{U}$, φ induces a map of D to B and, with $D \simeq A$, we obtain a map $\varphi' : A \to B$ with the commutative diagram (13.12). □

13.13 Proposition. *Assume (13.13). Then $\tau : \mathfrak{B} \to \mathfrak{U}$ is a natural equivalence of categories and $\sigma_0 : \mathfrak{U} \to \mathfrak{B}$ is the natural inverse of τ. When restricted to $\mathfrak{U}$, σ_i is exact, for $i \in \mathbb{N}$, and $\sigma_i = 0$ for $i \in \mathbb{N}^*$.*

Proof. For $i \in \mathbb{N}$, put $\gamma_i = \sigma_i \circ \tau$ and $\nu_i = \tau \circ \sigma_i$. We first check that: □

13.14

γ_0 is naturally equivalent to the identity functor on $\mathfrak{B}$.

Let $B \in \mathfrak{B}$ and let $E_* \to \tau(B)$ be a special ψ-resolution of $\tau(B)$. Using (13.9), and letting B_s be a lattice above B, E_* also gives a special ψ-resolution of B_1. Now apply (12.11) to obtain the complex $\sigma E_* \to B$. This map induces a map $t_B : \gamma_0 B \to B$. One may easily check that the correspondence $B \mapsto t_B$ is natural; i.e., for $B, C \in \mathfrak{B}$, $\varphi : B \to C$ then the following diagram is commutative:

$$\begin{array}{ccc} B & \xrightarrow{\varphi} & C \\ \uparrow{\scriptstyle t_B} & & \uparrow{\scriptstyle t_C} \\ \gamma_0 B & \xrightarrow[\gamma_0\varphi]{} & \gamma_0 C \end{array}$$

To prove (13.14) we must check that t_B is an isomorphism for each $B \in \mathfrak{B}$. First we check that $\gamma_i B \in \mathfrak{B}$. Clearly $\gamma_i B$ is a semisimple t-module by construction and so it is sufficient to check this for irreducible B, say $B = L(s\mu, \lambda')$ with μ -$P_{\lambda'}$-dominant. We proceed by induction on $\ell(s)$.

For $s = 1$, (13.4) gives the result. If $\ell(s) > 0$ then put $M = M(s\lambda, \lambda')$ **73**
and let $0 \to J \to M \to B \to 0$ be the short exact sequence induced by the natural map of M onto B. Since τ is exact, we obtain the long exact sequence.

$$\to \gamma_i J \to \gamma_i M \to \gamma_i B \to \gamma_{i-1} J \to \ldots \to \gamma_0 M \to \gamma_0 B \to 0.$$

By the induction hypothesis $\gamma_i J \in \mathfrak{B}$ while (13.4) gives $\gamma_i M \in \mathfrak{B}$. Therefore $\gamma_i B \in \mathfrak{B}$; and so, γ_i maps $\mathfrak{B}$ into $\mathfrak{B}$ for all i.

For any $B \in \mathfrak{B}$ we have an isomorphism of $U(\mathfrak{g})^{\mathfrak{t}}$-modules, $B^{\sim} \simeq B'_1$; and so, we have

$$B^{\sim} \simeq E'_0 / \operatorname{Im} E'_1 \simeq \sigma E^{\sim}_0 / \operatorname{Im} E^{\sim}_1. \tag{13.15}$$

Combining (13.15) and (13.10), $\gamma_0 B^{\sim}$ and $B^{\sim}$ are isomorphic by t_B; and then, by (13.11), t_B is an isomorphism. This completes the proof of (13.14).

Since the identity functor is exact, γ_0 is exact. This fact and (13.4) now easily imply that $\gamma_i = 0$ for $i \geq 1$. In turn this shows that for each irreducible object A in $\mathfrak{U}$, $\sigma_i A = 0$, $i \in \mathbb{N}^*$. So $\sigma_i = 0$ on $\mathfrak{U}$, $i \in \mathbb{N}^*$, and thus, σ_0 is exact. Next we claim:

13.16

ν_0 is naturally equivalent to the identity on $\mathfrak{U}$.

For $A \in \mathfrak{U}$, let $E_* \to A$ be a special ψ-resolution of A. Then we have the exact sequence $\sigma E_1 \to \sigma E_0 \to \sigma_0 A \to 0$. Let C_s, $s \in \mathfrak{w}_\mathfrak{t}$, be the lattice above $\sigma_0 A$. By applying completion functors to the exact sequence we obtain the complex: $E_1 \to E_0 \to C_1$. Let φ denote the composition of the map $E_0 \to C_1$ with the quotient map $C_1 \to \nu_0 A \to 0$. Then by (13.12), φ induces a map φ' of A to $\nu_0 A$. Put $s_A = \varphi'$. To prove (13.16) we must that s_A is an isomorphism for all $A \in \mathfrak{U}$. For A isomorphic to a principal series $X(\underline{\mu})$ with $\underline{\mu} = (\mu, \lambda')$, we may choose the special ψ-resolution (13.6) for $\overline{s} = \overline{1}$. Since (13.5) splits as a $\mathfrak{t}$-module sequence (cf. Theorem 6.17 [15]), we find that s_A gives an isomorphism of A' onto $\nu_0 A'$. Then (13.11) implies that s_A is an isomorphism. Using the exactness of ν_0 and the five lemma we conclude that s_A is an isomorphism for irreducible A and then for all A in $\mathfrak{U}$. This proves (13.16) and completes the proof of (13.13). **74**

Proof of Theorem (13.2). The remark before (13.3) and Proposition (13.13) combine to prove (13.2).

Chapter 14

Preunitary pairings

Let σ denote the conjugate linear antiautomorphism of $U(\mathfrak{g})$ which 75
equals $(-1)\cdot$ identity on the real form $i(\mathfrak{g}_0)$ of $\mathfrak{g}$. Recall remark (3.13) and note that all invariant sesquilinear pairings and forms will be given with respect to this choice of σ. In this section we begin the study of which irreducible admissible $(\mathfrak{g}, \mathfrak{k})$-modules admit invariant Hermitian forms with definite signature.

14.1 Definition. *An irreducible admissible* $(\mathfrak{g}, \mathfrak{k})$*-module A will be called unitarizable if A admits a nonzero invariant Hermitian form which is positive definite.*

T' definition is suggested by the fact: A satisfies (14.1) if and only if A is isomorphic to the set of K-finite vectors in an irreducible unitary representation of G.

From the definition of σ, σ equals the identity on $\mathfrak{k} \cap \mathfrak{h}_{\mathbb{R}}$ and (-1) identity on $\mathfrak{p} \cap \mathfrak{h}_{\mathbb{R}}$. Let $^-$ denote conjugation on $\mathfrak{h}$ (resp. $\mathfrak{h}^*$) with respect to the real form $\mathfrak{h}_{\mathbb{R}}$ (resp. $\mathfrak{h}^*_{\mathbb{R}}$). Then

$$(H, H')^\sigma = \overline{(H', H)},\ (\lambda, \lambda')^\sigma = \overline{(\lambda', \lambda)},\ H, H' \in \mathfrak{h}_0, \lambda, \lambda' \in \mathfrak{h}_0^*. \quad (14.2)$$

From (14.2), we have:

$$\sigma(n^\pm) = n^\mp. \quad (14.3)$$

For an admissible $(\mathfrak{g}, \mathfrak{k})$-module A, let A^σ denote the $\mathfrak{h}$-locally finite subspace of the algebraic dual of A. Give A^σ the conjugate complex structure and define the action of $\mathfrak{g}$ on A^σ by:

$$(x \cdot \varphi)(a) = \varphi(x^\sigma \cdot a),\ x \in \mathfrak{g},\ \varphi \in A^\sigma,\ a \in A. \tag{14.4}$$

76 Then A^σ is a $\mathfrak{g}$-module and is called the conjugate dual to A.

14.5 Lemma. *Let $\underline{\lambda}$, $\underline{\mu} \in \mathfrak{h}^*$. Then $L(\underline{\lambda})$ and $L(\underline{\mu})$ admit a nonzero invariant sesquilinear pairing if and only if $\underline{\mu} = \underline{\lambda}^\sigma$. Moreover, if $\underline{\mu} = \underline{\lambda}^\sigma$ then the space of invariant sesquilinear pairings is one dimensional.*

Proof. By (14.2), $\mathbb{C}_{\underline{\lambda}}$ and $\mathbb{C}_{\underline{\mu}}$ admit an $\mathfrak{h}$-invariant pairing if and only if $\underline{\mu} = \underline{\lambda}^\sigma$. By Proposition (3.6), we obtain (14.5) for modules $M(\underline{\lambda})$ and $\overline{M}(\underline{\mu})$. However by cyclicity of the highest weight space it follows that every pairing of $M(\underline{\lambda})$ and $M(\underline{\lambda})$ is the pull back of a pairing of $L(\underline{\lambda})$ and $L(\underline{\mu})$. This proves (14.5). □

14.6 Lemma. *Let A be an irreducible $(\mathfrak{g}, \mathfrak{k})$-module and let $I(A)$ be the vector space of invariant sesquilinear forms on A. Then $\dim_{\mathbb{C}} I(A) \leq 1$. Also, if this dimension is one then there exists a Hermitian form φ on A.*

Proof. Let B be an irreducible admissible $(\mathfrak{g}, \mathfrak{k})$-module and let B^σ denote the $\mathfrak{g}$-module of $U(\mathfrak{k})$-locally finite vectors in the algebraic dual of B. Give B^σ the conjugate complex structure and action (14.4). Then B^σ is an irreducible $\mathfrak{g}$-module and any pairing φ of A and B corresponds to a $\mathfrak{g}$-module map $\overline{\varphi} : A \mapsto B^\sigma$ where $\overline{\varphi}(a)(b) = \varphi(a, b)$, $a \in A$, $b \in B$. Now the first part of (14.6) follows by Schur's lemma.

For any invariant form φ, put $\varphi_1(a, b) = \varphi(a, b) + \overline{\varphi(b, a)}$, $\varphi_2(a, b) = \varphi(a, b) - \overline{\varphi(b, a)}$ $a, b \in A$. Then φ_1 and φ_2 are also invariant. Moreover, φ_1 and $\sqrt{-1}\varphi_2$ are Hermitian. This completes the proof of (14.6). □

77 We now give a necessary and sufficient condition for an irreducible admissible $(\mathfrak{g}, \mathfrak{k})$-module to admit an invariant Hermitian form.

14.7 Proposition. *Let $\underline{\lambda} \in \mathfrak{L}$ and assume $\underline{\lambda}$ satisfies (6.5). Then $Z(\underline{\lambda})$ admits an invariant Hermitian form if and only if $\underline{\lambda}$ and $\underline{\lambda}^\sigma$ lie in the same $\mathfrak{w}_{\mathfrak{k}}$-orbit.*

Proof. Let (6.5)′ denote the condition (6.5) with the roles of μ and μ' interchanged. Note that if $\underline{\lambda}$ satisfies (6.5) then $\underline{\lambda}^\sigma$ satisfies (6.5)′. The proof of (10.5) applies equally well under the hypothesis (6.5)′ in place of (6.5). Therefore,

$$\tau L(\underline{\lambda}^\sigma) \text{ is isomorphic to } \hat{X}(\underline{\lambda}^\sigma). \tag{14.8}$$

By (14.3), let φ be a nonzero invariant sesquilinear pairing of $L(\underline{\lambda})$ and $L(\underline{\lambda}^\sigma)$. Applying τ, $\tau\varphi$ is an invariant sesquilinear pairing of $Z(\underline{\lambda})$ and $\hat{X}(\underline{\lambda}^\sigma)$. Both of these modules are irreducible; and so, $Z(\underline{\lambda})$ admits an invariant sesquilinear form $\Leftrightarrow$ $Z(\underline{\lambda})$ and $\hat{X}(\underline{\lambda}^\sigma)$ are isomorphic $\Leftrightarrow$ $\underline{\lambda}$ and $\underline{\lambda}^\sigma$ lie in the same $\mathfrak{w}_\mathfrak{t}$ -orbit (cf. (10.4), (10.5), (10.8)). Lemma (14.6) completes the proof. ◻

The question of positive definiteness of Hermitian forms is much deeper than that resolved by (14.7). In the remaining portion of this section we relate the question of definiteness of invariant forms on modules $Z(\underline{\lambda})$ to a property of pairing of $L(\underline{\lambda})$ and $L(\underline{\lambda}^\sigma)$.

For $\mu \in \mathfrak{t}^*$, let $U(\mu) = U(\mathfrak{g}) \bigotimes_{U(\mathfrak{t})} M(\mu)$. If $\mu, \nu \in \mathfrak{t}^*$ and $M(\nu)$ is a sub Verma module of $M(\mu)$ then the inclusion $M(\nu) \hookrightarrow M(\mu)$ induces an inclusion $U(\nu) \hookrightarrow U(\mu)$.

14.9 Definition. *Let A and B be $\mathfrak{g}$-modules and assume that φ is an invariant sesquilinear pairing of A and B. We call φ a preunitary pairing of A and B if the following properties hold:* 78

(i) *there exists $\mu \in \mathfrak{t}^*$ which is $-P_\mathfrak{t}$-dominant integral and regular and maps $S : U(\mu) \to A$, $T : U(\mu) \to B$.*

(ii) *if $\overline{\varphi}$ is the "pull back" form on $U(\mu)$ given by $\overline{\varphi} = (S(\ ), T(\ ))$, then $\overline{\phi}$ has a nonzero positive semidefinite restriction to the span of the subspaces of $\mathfrak{n}_\mathfrak{t}$-invariants of weight ξ with $\xi + 2\delta_\mathfrak{t}$ $-P_\mathfrak{t}$-dominant.*

14.10 Proposition. *Let $\underline{\lambda} \in \mathfrak{L}$ and assume $\underline{\lambda}$ satisfies (6.5) and $Z(\underline{\lambda})$ admits an invariant Hermitian form. Then $Z(\underline{\lambda})$ is unitarizable if and only if $L(\underline{\lambda})$ and $L(\underline{\lambda}^\sigma)$ admit a preunitary pairing.*

Proof. For any $\mathfrak{k}$-module A, let A' (resp. $A^{\sim}$) denote the span of the subspaces of $n_{\mathfrak{k}}$-invariants of weight ξ with ξ $P_{\mathfrak{k}}$-dominant integral (resp. $\xi + 2\delta_{\mathfrak{k}}$ -$P_{\mathfrak{k}}$ dominant integral). Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition for $\mathfrak{g}$ and let $S(\mathfrak{p})$ denote the symmetric tensor algebra of $\mathfrak{p}$. Then for $\mu \in \mathfrak{k}^*$, $S(\mathfrak{p}) \otimes M(\mu)$ and $U(\mu)$ are isomorphic as $\mathfrak{k}$-modules. Let t_0 be the element of $\mathfrak{w}_{\mathfrak{k}}$ of maximal length and let $\mu \in \mathfrak{k}^*$ be $P_{\mathfrak{k}}$-dominant integral. Then from Proposition 4.13 [15], there is an isomorphism:

$$U(\mu)' \xrightarrow{\sim} U(t_0\mu)^{\sim}. \tag{14.11}$$

Moreover, if ψ is any invariant sesquilinear form on $U(t_0\mu)$ then by successive completions we obtain a form ψ_1 on $U(\mu)$. From (14.11) and (3.14) we find that: □

14.12

79 ψ restricted to $U(t_0\mu)^{\sim}$ is positive semidefinite (positive definite) if and only if ψ_1 restricted to $U(\mu)'$ is positive semidefinite (positive definite).

First assume $Z(\underline{\lambda})$ is unitarizable. Then by (14.7), $\underline{\lambda}$ and $\underline{\lambda}^{\sigma}$ lie in the same $\mathfrak{w}_{\mathfrak{k}}$-orbit. Let φ be a nonzero invariant sesquilinear pairing of $L(\underline{\lambda})$ and $L(\underline{\lambda}^{\sigma})$ given by (14.5). Then $\tau\varphi$ is an invariant sesquilinear pairing of $Z(\underline{\lambda})$ and $\tau L(\underline{\lambda}^{\sigma})$. By (14.8) and (14.7), these modules are isomorphic; and so, $\tau\varphi$ is an invariant sesquilinear form on $Z(\lambda)$. By (14.6), we may assume that $\tau\varphi$ is Hermitian and positive definite since $Z(\underline{\lambda})$ is unitarizable. Let L_s (resp. L_s^{σ}), $s \in \mathfrak{w}_{\mathfrak{k}}$, be a lattice above $L(\underline{\lambda})$ (resp. $L(\underline{\lambda}^{\sigma})$). Let z be any nonzero $\mathfrak{k}$-highest weight vector in $Z(\underline{\lambda})$. Since $Z(\underline{\lambda})$ is an admissible $(\mathfrak{g}, \mathfrak{k})$-module the weight $\mu - \delta_{\mathfrak{k}}$ of z is dominant integral. Choose maps $S : U(\mu) \to L_1$, $T : U(\mu) \to L_1^{\sigma}$ which when composed with the projections onto $Z(\underline{\lambda})$, map the canonical cyclic vector of weight μ onto z. We now claim:

$$S : U(t_0\mu) \to L(\underline{\lambda}), \quad T : U(t_0\mu) \to L(\underline{\lambda}^{\sigma}). \tag{14.13}$$

Let U_s, $s \in \mathfrak{w}_{\mathfrak{k}}$, be a lattice above $U(t_0\mu)$. Then $U_1 = U(\mu)$. We now prove $S : U_s \to L_s$ for all $s \in \mathfrak{w}_{\mathfrak{k}}$ by induction on the length of s. For $s = 1$, this is (14.13). Assume $s \neq 1$ and choose $\alpha \in P_{\mathfrak{k}}$ simple with $\ell(s_\alpha s) = \ell(s) - 1$. Then by the induction hypothesis, $S : U_{s_\alpha s} \to L_{s_\alpha s}$.

Consider the induced map of U_s into $L_{s_\alpha s}/L_s$. The former module is generated by an irreducible infinite dimensional Verma $\mathfrak{a}^{(\alpha)}$-module (cf. § 4) while the latter module is $U(\mathfrak{a}^\alpha)$-locally finite. This implies that the induced map is zero; and so, $S : U_s \to L_s$. This proves the first part of (14.13). The same argument applies to prove the second.

Let ψ denote the pull back of φ to a form on $U(t_0\mu)$; i.e., $\psi =$ **80**
$\varphi(S(\,), T(\,))$. By (14.13), ψ is defined. Moreover, if ψ_1 is the pull back of $\tau\varphi$ to $U(\mu)$ then one checks easily that ψ_1 can be defined by successive completions beginning with ψ. Since $Z(\underline{\lambda})$ is unitarizable, $\tau\varphi$ is positive definite and thus ψ_1 is positive semidefinite and nonzero when restricted to $U(\mu)$. Then (14.12) states that ψ is positive semidefinite and nonzero when restricted to $U(t_0\mu)^{\sim}$. This means that φ is a preunitary pairing of $L(\underline{\lambda})$ and $L(\underline{\lambda}^\sigma)$.

We now prove the converse. Assume φ is a perunitary pairing of $L(\underline{\lambda})$ and $L(\underline{\lambda}^\sigma)$. From (14.9) we let $\mu \in \mathfrak{t}^*$ be $P_{\mathfrak{t}}$-dominant integral and regular and let $S : U(t_0\mu) \to L(\underline{\lambda})$, $T : U(t_0\mu) \to L(\underline{\lambda}^\sigma)$ be given by (14.9). To conform with the notation above we let S and T also denote the extensions of S and T to the lattice above $U(t_0)$ obtained by applying completion functors. If ψ is the pull back of φ to $U(t_0\mu)$ and if ψ_1 is the pull back of $\tau\varphi$ to $U(\mu)$, then ψ_1 is obtained from ψ by successive completions. So by (14.12), ψ_1 is positive semidefinite and nonzero when restricted to $U(\mu)'$. But then $\tau\varphi$ is nonzero and positive semidefinite; and since $Z(\underline{\lambda})$ is irreducible, $\tau\varphi$ must be positive definite. Therefore $Z(\underline{\lambda})$ is unitarizable.

Chapter 15

Unitary representations and relative Lie algebra cohomology

Although in the preceding section the question of unitarizability of $Z(\underline{\lambda})$ **81**
is reduced to a question of pairings for highest weight modules, the determination of the unitarizable irreducible admissible $(\mathfrak{g}, \mathfrak{k})$-module remains an open question. However, with certain restrictions on either the group or the class of representations, solutions do exist. In this section we summarize without proof these partial results.

In the first such result we restrict the infinitesimal character of the module.

15.1 Theorem. *Let Z be an irreducible admissible $(\mathfrak{g}, \mathfrak{k})$-module having regular integral infinitesimal character. Then Z is unitarizable if and only if there exists a parabolic subgroup Q of G and a one dimensional unitary representation σ of Q with Z equivalent to the $\mathfrak{g}$-module of K-finite vectors in the representation of G unitarily induced from σ and Q to G.*

It is somewhat surprising that the proof of (15.1) follows by only a computation in $\mathfrak{h}^*$ and application of the Dirac operator inequality. A proof of (15.1) is given in [18]. We have two corollaries to (15.1). The

first involves the relative Lie algebra cohomology groups for the pair $(\mathfrak{g}, \mathfrak{k})$. If F is an irreducible finite dimensional $\mathfrak{g}$-module and Z is an irreducible admissible $(\mathfrak{g}, \mathfrak{k})$-module then we let $H^*(\mathfrak{g}, \mathfrak{k}; F \otimes Z)$ denote the relative Lie algebra cohomology groups with coefficients in $F \otimes Z$ as defined in [4]. Chapter I. A version of Shapiro's lemma due to P. Delorme (cf. [4] III 3.3) gives a formula for computing the relative
82 cohomology groups with coefficients in induced representations. By combining this formula and (15.1), we obtain the following vanishing theorem:

15.2 Theorem. *Assume G is simple and Z is unitarizable and nontrivial. Then $H^r(\mathfrak{g}, \mathfrak{k}; F \otimes Z) = 0$ for all integers $r < r_G$ where r_G is given in Table 1 below.*

Table 1.

Type of G	r_G	*Type of G*	r_G
A_ℓ	ℓ	E_6	16
B_ℓ	$2\ell - 1$	E_7	27
C_ℓ	$2\ell - 1$	E_8	57
D_ℓ	$2\ell - 2$	F_4	15
		G_2	5

As corollaries to (15.2) we obtain results for a co-compact discrete subgroup Γ of G. For any representation of Γ on A, let $H^*(\Gamma, A)$ denote the cohomology groups of Γ with values in A.

15.3 Corollary. *For any finite dimensional $\mathfrak{g}$-module F and integer $r < r_G$, there is a natural isomorphism*

$$H^r(\mathfrak{g}, \mathfrak{k}; F) \xrightarrow{\sim} H^r(\Gamma, F).$$

Let $\mathbb{C}$ denote the trivial representation of $\mathfrak{g}$.

15.4 Corollary. *Assume Γ has no elements of finite order different from the identity: Let X_u be the simply connected compact symmetric space dual to G/K. For integers $p, 0 \leq p < r_G$, the p^{th} Betti numbers of $\Gamma \backslash G/K$ and X_u are equal and equal to* $\dim H^p(\mathfrak{g}, \mathfrak{k}; \mathbb{C})$.

83 For a more detailed discussion of the results above the reader should consult [18] and [4].

The vanishing theorem (15.2) has been proved in the general setting of a real semisimple Lie group [4], [39]. In the general setting the constant r_G is replaced by the split rank of G. For a type of generalization of (15.1) to a general real semisimple Lie group the reader should consult the recent article of Vogan and Zuckerman [35].

As a second corollary to (15.1) we note that if Z satisfies the hypotheses of (15.1) then the distribution character of Z is completely determined. It is the character of a representation induced from a one dimensional representation of a parabolic subgroup.

Lastly we consider general results for special cases. For the cases where the rank of $\mathfrak{g}_0$ is one or two or $\mathfrak{g}_0 \simeq s\ell(4)$ or $s\ell(5)$, Duflo [10], [11], has determined the unitarizable admissible irreducible $(\mathfrak{g}, \mathfrak{k})$ - modules. This work for $s\ell(n)$, $n = 2, 3, 4, 5$, suggests a generalization of (15.1). If $\mathfrak{g}_0$ is of type A_ℓ then we might expect that every unitarizable irreducible admissible $(\mathfrak{g}, \mathfrak{k})$-module is equivalent to a representation induced from a quasi character of a parabolic subgroup of G. For the other types the description in [10] already shows that the picture will be more complicated. There are unitary representations which are not induced.

Chapter 16

Connections with the derived functors introduced by Zuckerman

We begin this section with a description of certain right derived functors **84**
introduced by Zuckerman [40]. Following this we consider connections between these derived functors and the lattice functor τ.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{C}$ and let $\mathfrak{t}$ and $\mathfrak{t}$ be subalgebras of $\mathfrak{g}$ with $\mathfrak{g} \supset \mathfrak{t} \supset t$. Also assume both $\mathfrak{t}$ and t are reductive in $\mathfrak{g}$. For any Lie algebras $\mathfrak{a}$, $\mathfrak{b}$ with $\mathfrak{a} \subset \mathfrak{b}$, let $C(\mathfrak{b}, \mathfrak{a})$ denote the category of $\mathfrak{b}$-modules which are $\mathfrak{U}(\mathfrak{a})$-locally finite and semisimple as $\mathfrak{a}$-modules. Let S be the additive covariant functor defined on $C(\mathfrak{g}, \mathfrak{t})$ by letting $S(X)$ be the maximal subspace of $U(\mathfrak{t})$-locally finite vectors in X, $X \in C(\mathfrak{g}, \mathfrak{t})$. The categories $C(\mathfrak{t}, t)$ and $C(\mathfrak{g}, \mathfrak{t})$ admit sufficiently many injective objects to insure that any object has an injective resolution. We may thus define the right derived functors of S. Let $X \in C(\mathfrak{t}, t)$ and let $0 \to X \to X_0 \to X_1 \to \ldots$ be an injective resolution of X. Then $0 \to SX_0 \to SX_1 \to \ldots$ is a complex and we define $R^iS(X)$ to be the i^{th} cohomology group of this complex. If $X \in C(\mathfrak{g}, \mathfrak{t})$ then by taking an injective resolution in $C(\mathfrak{g}, \mathfrak{t})$ and applying S, we obtain another set of right derived functors which we denote by $R^i_{\mathfrak{g}}S$. From Lemma 3.1 [19], we find that each injective object in $C(\mathfrak{g}, \mathfrak{t})$ remains injective when considered as an object of $\mathcal{C}(\mathfrak{t}, \mathfrak{t})$. From this we have

16.1 Lemma. *For $X \in \mathcal{C}(\mathfrak{g}, \mathfrak{t})$, let $X_{\mathfrak{t}}$ denote the $\mathfrak{t}$-module underlying X. Then, for $i \in \mathbb{N}$, the $\mathfrak{t}$-module underlying $R^i_{\mathfrak{g}}S\,(X)$ is isomorphic to*
85 *$R^i S\,(X_{\mathfrak{t}})$. This lemma shows that the functors $R^i S$ may be employed to determine the $\mathfrak{t}$-module structure of the $\mathfrak{g}$-modules $R^i_{\mathfrak{g}}S\,(X)$, $X \in \mathcal{C}(\mathfrak{g}, \mathfrak{t})$. The next step is then the determination of $R^i S\,(M)$ for a Verma $\mathfrak{t}$-module M. Assume t is a CSA for $\mathfrak{t}$ and $P_{\mathfrak{t}}$ is a positive system of roots. Let $\delta_{\mathfrak{t}}$ equal half the sum of positive roots and let $\mathfrak{w}_{\mathfrak{t}}$ denote the Weyl group. For $\mu \in \mathfrak{t}^*$, let $M(\mu)$ be the Verma $\mathfrak{t}$-module with highest weight $\mu - \delta_{\mathfrak{t}}$ and if μ is integral, let F_μ be the irreducible finite dimensional $\mathfrak{t}$-module with extreme weight μ. Put $d = \frac{1}{2}\dim \mathfrak{t}/t = \operatorname{card} P_{\mathfrak{t}}$.*

16.2 Proposition. *Let $\mu \in t^*$ be antidominant and let $s \in \mathfrak{w}_{\mathfrak{t}}$. Then*

$$R^i S\,(M(s\mu)) = \begin{cases} F_{\mu+\delta_{\mathfrak{t}}} & \text{if } \mu \text{ is regular and} \\ & \quad i = d + \ell(s) \\ 0 & \text{otherwise} \end{cases}$$

An elementary proof of (16.2) is given in [19] (cf. Proposition 6.3 and Theorem 4.3 in [19]) where it is also shown that (16.2) is equivalent to the Bott-Borel-Weil Theorem.

We now specialize to the setting of these notes. Let $\mathfrak{g}$, $\mathfrak{t}$ and t be as in § 6. Using (16.1) and (16.2), the arguments in [15] as well as those in § 5 through § 10 in these notes can be carried out when the lattice functor τ is replaced by the "middle dimension" derived functor $R^d_{\mathfrak{g}}S$, $d = \frac{1}{2}\dim \mathfrak{t}/t$. For example, recalling the principal series modules, the precise analogue of Theorem 9.1 is:

86 **16.3 Theorem.** *Let $\underline{\lambda} = (\lambda, \lambda') \in \mathfrak{L}$ and assume $M(\lambda')$ is irreducible. Then*

$$R^i_{\mathfrak{g}}S\,(M(\underline{\lambda})) = \begin{cases} X(\underline{\lambda}) & \text{if } i = d \\ 0 & \text{if } i \neq d. \end{cases}$$

Assume $M(\lambda')$ is irreducible. The category $\mathcal{O} \otimes M(\lambda')$ is $\mathfrak{t}$-semisimple by (3.9); and so, by (16.1) each $R^i_{\mathfrak{g}}S$ is an exact functor on $\mathcal{O} \otimes M(\lambda')$. This fact, (16.3) and the arguments of section ten combine to prove:

16.4 Theorem. (a) *For $i \in \mathbb{N}$, $i \neq d$, the functor $R^i_{\mathfrak{g}}S$ is zero on $\mathscr{O} \otimes M(\lambda')$.*

(b) *$R^d_{\mathfrak{g}}S$ is an exact functor on $\mathscr{O} \otimes M(\lambda')$.*

(c) *For $\underline{\lambda} \in \mathfrak{L}$ satisfying (6.5),*

$$R^d_{\mathfrak{g}}S(L(\underline{\lambda})) \simeq Z(\underline{\lambda}) = \tau L(\underline{\lambda}).$$

From these results one might guess that the results of section thirteen also remain true with τ replaced by $R^d_{\mathfrak{g}}S$. However, if true a different proof must be supplied. A related question is whether or not τ and $R^d_{\mathfrak{g}}S$ are naturally equivalent on some special category of $\mathfrak{g}$-modules. The results of these notes suggest that the category should contain $\mathscr{O} \otimes M(\lambda')$ for $M(\lambda')$ irreducible.

Lastly we give a rank two example to show that the category referred to above cannot be too large. For this example, let $\mathfrak{g} = \mathfrak{k} = s\ell(3, \mathbb{C})$. Fix μ dominant integral and regular and let α and β be the simple roots of the positive system $P_{\mathfrak{k}}$. Put $M = M(s_\alpha \mu)$, $L = L(s_\alpha \mu)$, $M' = M(s_\alpha s_\beta s_\alpha \mu)$ and note that M' is an irreducible Verma module.

16.5 Lemma. *There exists a nontrivial extension of M' by M.* **87**

Proof. Let Ext^* denote the right derived functors of Hom on the category $\mathscr{O}$. Consider the short exact sequence $0 \to J \to M \to L \to 0$ and apply $\mathrm{Ext}^*(M', \cdot)$. We obtain the long exact sequence.

$$\cdots \to \mathrm{Hom}(M', L) \to \mathrm{Ext}^1(M', J) \to \mathrm{Ext}^1(M', M) \to \cdots.$$

Clearly $\mathrm{Hom}(M', L) = 0$; and so, to prove the lemma it is sufficient to check that $\mathrm{Ext}^1(M', J) \neq 0$. Now consider the short exact sequence $0 \to M' \to J \to B \to 0$ where $B = L(s_\alpha s_\beta \mu) \oplus L(s_\beta s_\alpha \mu)$. Applying $\mathrm{Ext}^*(M', \cdot)$ to this short exact sequence we obtain:

$$\cdots \to \mathrm{Ext}^1(M', J) \to \mathrm{Ext}^1(M', B) \to \mathrm{Ext}^2(M', M') \to \cdots.$$

However, Delorme has shown that $\mathrm{Ext}^j(M(s\mu), L(r\mu)) = 0$ if $j > \ell(s) - \ell(r)$. Therefore, $\mathrm{Ext}^2(M', M') = 0$; and so, to see that $\mathrm{Ext}^1(M',$

$J) \neq 0$ it is sufficient to check $\mathrm{Ext}^1(M', B) \neq 0$. But the algebraic dual of $M(s_\alpha s_\beta \mu) \oplus L(s_\beta s_\alpha \mu)$ gives a nonzero extension of M' by B (cf. section three). This shows $\mathrm{Ext}^1(M', B) \neq 0$. So, as noted above, $\mathrm{Ext}^1(M', M) \neq 0$ and this completes the proof. □

16.6 Proposition. *Let E be ta nontrivial extension of M' by M as above. Then*

$$\tau E = 0 \quad R^i S(E) = \begin{cases} F_{\mu-\delta} & \text{if } i = 3 \text{ or } 5 \\ 0 & \text{otherwise}. \end{cases}$$

88 *Proof.* Since E is a nontrivial extension, the dimensions of the spaces of $n_{\mathfrak{t}}$-invariants of weight $s_\alpha s_\beta s_\alpha \mu - \delta$ and $s_\alpha \mu - \delta$ are both one. Then, by Proposition 4.13 [15], $\tau E = 0$. □

From the short exact sequence for E we obtain the long exact sequence

$$\cdots \to R^i S(M) \to R^i S(E) \to R^i S(M') \to R^{i+1} S(M) \to \cdots .$$

Now (16.6) follows from the formulae (16.2).

In this example $d = 3$; and so, $\tau E \neq R^d S(E)$.

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