Lectures on
Equations Defining Space Curves

By

L. Szpiro

Tata Institute of Fundamental Research

Bombay

1979

Lectures on
Equations Defining Space Curves

By

L. Szpiro

Notes by

N. Mohan Kumar

Published for the

Tata Institute of Fundamental Research, Bombay

Springer–Verlag

Berlin Heidelberg New York

1979

Author

L. Szpiro
Ecole Normale Superieure
Universite de Paris
45, Rue D'Ulm
75230 Paris – FRANCE

ISBN 3-540-09544-6. Springer-Verlag Berlin. Heidelberg. New York
ISBN 0-387-09544-6. Springer-Verlag New York. Heidelberg. Berlin

Printed by N. S. Ray at The BOOK CENTRE Limited
Sion East, Bombay 400 022 and published by H. Goetze
Springer-Verlag, Heidelberg, West Germany

Printed In India

Introduction

THESE NOTES ARE the outcome of a series of lectures I gave in the winter of 1975–76 at the Tata Institute of Fundamental Research, Bombay. The object of the research, we –D. FERRAND, L. GRUSON, C. PESKINE and I–started in Paris was, roughly speaking *to find out the equations defining a curve in projective 3-space* (or in affine 3-space or of varieties of codimension two in projective n-space.) I took the opportunity given to me by the Mathematics Department of T.I.F.R, to try to put coherently the progress made by the four of us since our paper [11]. Even though we are scattered over the earth now, (RENNES, LILLE, OSLO and BOMBAY:) these notes should be considered as the result of common of all of us. I have tried in the quick description of the chapters to obey the "Redde Caesari quae sunt Caesaris."

Chapter 1 contains certain prerequisites like duality, depth, divisors etc. and the following two interesting facts:

i) An example of a reduced curve in $\mathbb{P}^3$ with no *imbedded* smooth deformation (an improvement on the counter example "6.4" in [11] which was shown to me by G. Ellingrud from Oslo who also informed me that it can be found in M. Noether [10]).

ii) A proof that every locally complete intersection curve in $\mathbb{P}^3$ can be defined by four equations.

Chapter 2 is my personal version of the theory of conductor for a curve. A long time ago, O. Zariski asked me what my understanding of Gorenstein's theorem was and this chapter is my answer; even though it contains no valuations and I wonder if it will be to the taste of Zariski. In it I first recall classical facts known since Kodaira, through duality. The three main points are as follows:

If X is a smooth surface, projective over a field k, C, a reduced irreducible curve on $X, \overline{X} \xrightarrow{g} X$, a finite composition of dilations, such that the proper transform $\overline{C}$ of C on $\overline{X}$ is smooth, one has:

a) the conductor $\underline{f}$ is related to dualizing sheaves by

$$\underline{f} = g_* \omega_{\overline{C}} \otimes \omega_C^{-1}$$

b) Gorenstein's theorem is a simple consequence of $H^1(\mathbb{P}^2, O_{\mathbb{P}^2}) = 0$.

c) Regularity of the adjoint system is equivalent to $H^1(\overline{X}, O_{\overline{X}}(\overline{C})) = 0$.

   We conclude the chapter with a counter-example which is new in the litterature:

d) A curve C on a surface X over a field of characteristic $p \geq 5$, such that

   i) $O_X(C)$ is ample.

   ii) Kodaira vanishing theorem holds i.e. $H^1(X, O_X(-C)) = 0$.

   iii) Regularity of the adjoint does not hold.

   i.e. $$H^1(\overline{X}, O_{\overline{X}}(-\overline{C})) \neq 0.$$

   We also give the proof - shown to us by Mumford - that such a situation cannot occur in zero characteristic; i.e. $H^1(X, O_X(-C)) \simeq H^1(\overline{X}, O_{\overline{X}}(-\overline{C}))$ over characteristic zero fields.

   Chapter 3 contains two classical theorems by Castelnuovo. These theorems have been dug out of the litterature by L. Gruson. My only effort was to write them down (with Mohan Kumar). The point, in modern language, is to give bounds for Serre's vanishing theorems in cohomology, in terms of the degree of the given curve in $\mathbb{P}^3$. The two results are the following:

> If C is a smooth curve in $\mathbb{P}^3$, J its sheaf of ideals and d its degree, then
> a) $H^2(\mathbb{P}^3, J(n)) = 0 \qquad n \geq \frac{d-1}{2}$
> b) $H^1(\mathbb{P}^3, J(n)) = 0 \qquad n \geq d - 2$

The reader who is interested in equations defining a curve canonically embedded may read the version of Saint-Donat [14] of Petri's theorem, in which coupling the above results with some geometric arguments, he gets the complete list of equations of such a curve. (In general they are of degree 2, but here we only get that the degree is less than or equal to three.)

In Chapter 4 we give an answer to an old question of Kronecker (and Severi): a local complete intersection curve in affine three space is set theoretically the intersection of two (algebraic) surfaces. We also give the projective version of D. Ferrand: a local complete intersection curve in $\mathbb{P}^3$ is set-theoretically the set of zeroes of a section of a rank two vector bundle. Unfortunately such vector bundles may not be decomposable. The main idea - which is already in [11], example 2.2 - is that if a curve C is "liee" to itself by a complete intersection, then the ideal sheaf of the curve C in O_X is - upto a twist - the dualising sheaf ω_C of C ([11], Remarque 1.5). Starting from that, we construct an extension of O_C by ω_C, with square of ω_C zero, and then a globalisation of a theorem of R. Fossum [3] finishes the proof. The globalisation is harder in the case of D. Ferrand. It must be said that the final conclusion in $\mathbb{A}^3$ has been made possible by Murthy-Towker [9] (and now Quillen-Suslin [12]) theorem on triviality of vector bundles on $\mathbb{A}^3$. Going back to rank two-vector bundles on $\mathbb{P}^3$ we have now three ways of constructing them:

- Horrock's
- Ferrand's
- and by projection of a canonical curve in $\mathbb{P}^3$

It will be interesting to know the relations between these families. We take this opportunity to ask the following question: Can one generalise Gaeta's theorem (for e.g. [11] Theorem 3.2) in the following way:

Is every smooth curve in $\mathbb{P}^3$ liee by a finite number of "liaisons" to a scheme of zeroes of a section of a rank two-vector bundle?[1]

Or as R. Hartshorne has suggested: "What are the equivalence classes of curves in $\mathbb{P}^3$, modulo the equivalence relation given by "liaison". A start in this direction has been taken by his student. A. Prabhakar Rao (Liaison among curves in Projective 3-space, Ph.D. Thesis [13]).[1]

I have news from Oslo, saying that L. Gruson and C. Peskine are starting to understand the mysterious chapter 3 of Halphen's paper [5]. I hope they will publish their results soon. These works and the yet unpublished notes of D. Ferrand on self-liaison would be a good piece of knowledge on curves in 3-space.

N. Mohan Kumar has written these notes and it is a pleasure for me to thank him for his efficiency, his remarks and his talent to convert the "franglais" I used during the course to "good English". The reader should consider all the "gallissisms" as mine and the "indianisms" him. It has been a real pleasure for me to work with him and to drink beer with him in Bombay - a city which goes far beyond all that I had expected, in good and in bad. I thank the many people there who gave me the opportunity of living in India and also made my stay enjoyable - R. Sridharan, M.S. Narasimhan, R.C. Cowsik, S. Ramanan and surprisingly Okamoto from Hiroshima University. The typists of the School of Mathematics have typed these manuscripts with care and I thank them very much. I also thank Mathieu for correcting the orthographic mistakes and Rosalie - Lecan for the documentation she helped me with.

[1]these questions have now been answered by A.P. Rao (the first negatively) in his paper: "Liaisons among curves in $\mathbb{P}^3$" Inventiones Math. 1978.

Contents

Chapter 1

Preliminaries

1

IN THIS CHAPTER, which we offer as an introduction, one will not 1
find many proofs. The aim is to state clearly some concepts so that we can speak rigorously of the different ideals defining a projective embedded variety. We give also the duality theorems and some of their consequences (finiteness, vanishing and Riemann-Roch theorem for curves), notions which play the role of 'Completeness of a linear system' or 'Specialness of a divisor'. The reader will find complete proofs of two intersecting facts:

(i) There exists a curve in $\mathbb{P}^3$ with no imbedded smooth deformation.

(ii) Every curve in $\mathbb{P}^3$ which is locally a complete intersection can be defined by four equations.

For simplicity we throughout assume that the base field is algebraically closed.

A *graded ring* A is a ring of the form:

$$A = A_0 \oplus A_1 \oplus A_2 \oplus \dots,$$

such that A_0 is a ring, A_i's are all A_0-modules and $A_i.A_j \subset A_{i+j}$. Any $f \in A_i$ for some i is said to be a *homogeneous element* of A. An ideal I of A

is said to be a graded ideal, if $\Sigma f_i \in I$, with $f_i \in A_i$ (i.e. f_i homogeneous) then $f_i \in I$.

Assume A_0 is a field and also A is generated by A_1 over A_0. A_1 a finite dimensional vector space over A_0. We define $X = \operatorname{Proj} A$ as follows: Set theoretically $X = \{$All graded prime ideals of $A \neq A_1 \oplus A_2 \oplus \ldots\}$.

We will give X a scheme structure, by covering X by affine open
2 sets: Let $f \in A_1$. Then,

$$A_f = (A_f)_0[T, T^{-1}], \qquad (*)$$

where $(A_f)_0 = \left\{ \dfrac{g}{f^n}\; g \in A_n \right\}$. (degree 0 elts. in A_f.)

$(A_f)_0$ is clearly a ring with identity.
$(*)$ is got by mapping T to f and T^{-1} to f^{-1} in A_f. Denote by X_f the set $\{p \in X / f \notin p\}$, clearly there is a canonical bijection

$$X_f \leftrightarrow \operatorname{Spec}(A_f)_0.$$

Transferring the scheme structure to X_f and verifying that this structure is compatible as we vary $f \in A_1$, we get a scheme structure on X.

Example. 1. Let $A = K[X_0, X_1 \ldots X_n]$ be polynomial ring in $n+1$ variables graded in the natural way: $A_0 = k.A_1 =$ vector space of dimension $n+1$ with $X_0, \ldots, X_n$ as generators i.e. $A_1 =$ set of all homogeneous linear polynomials in X_i's. $A_n =$ set of all homogeneous polynomials in

$$X_i\text{'s of } \deg n.$$

Then $\operatorname{Proj} A = \mathbb{P}^n$, the projective space of dimension n.

2. Let I be any ideal of A generated by homogeneous polynomials $\{f_1, \ldots, f_n\}$. Then $A' = A/I$ is a graded ring $X = \operatorname{Proj} A'$ and $\operatorname{Proj} A = \mathbb{P}^n$: X is the closed subvariety of $\mathbb{P}^n$ defined by equations $(f_1, \ldots, f_n)$.

$M = \bigoplus_{n \in \mathbb{Z}} M_n$ is said to be a *graded A-module* over the graded ring $A = \bigoplus_{i \geq 0} A_i$ if M is an A-module and $A_i.M_n \subset M_{n+i}$.

If M ia a graded A-modale we can associate a sheaf $\tilde{M}$ to M over $X = \operatorname{Proj} A$ as follows: Over X_f we define the sheaf to be $(M_f)_0$ where $(M_f)_0$ is the set of degree zero elements of M_f. It is a 3
module over $(A_f)_0$. [Recall that $X_f = \operatorname{spec}(A_f)_0$]. One can check that this defines a sheaf over X.

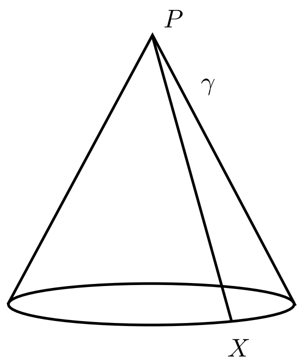
REMARK. $\tilde{A} = O_X$.

If M is a graded A-module, we define $M(n)$ to be the graded A-module given by, $M(n)_k = M_{n+k}$. We denote $\widetilde{M(n)}$ by $\tilde{M}(n)$. In particular.

$$\widetilde{A(n)} = \tilde{A}(n) = O_X(n).$$

If F is any sheaf on X, we denote by $F(n)$ the sheaf $F \otimes_{O_X} O_X(n)$. If $X = \operatorname{Proj} A$, then $Y = \operatorname{spec} A$ is defined to be a cone over X.

et P denote the point in Y corresponding to the special maximal ideal $A_1 \oplus A_2 \oplus \ldots$ P is defined as the vertex of the cone Y over X.



Let I be any graded ideal of A. Then A/I is a graded ring. Denote by Z, the scheme $\operatorname{Proj} A/I$. It can be easily checked that the canonical map $A \longrightarrow A/I$ induces a closed immersion $Z \to X = \operatorname{Proj} A$. Conversely given a closed subscheme Z of X, we can find a graded ideal $I \subset A$, such that the canonical map $\operatorname{Proj} A/I \longrightarrow X$ is an isomorphism of $\operatorname{Proj} A/I$ with Z. Then we say that I ideally defines Z in X. But this I is not completely determined by Z. One can check that if I is any ideal defining Z, then so does Im^n where m corresponds to the special maximal ideal of A. (i.e. it corresponds to the vertex of the given cone)

Since we are assuming that $A_0 = k$ is a field and A is generated by A_1 over k, where A_1 is finite dimensional over k, we have a graded ring
4 *homomorphism,*

$$R = k[X_0, X_1, \ldots X_n] \longrightarrow A,$$

which is surjective. [Polynomial rings have the canonical grading]. The kernel is a graded ideal J in R. So we have a closed immersion $X = \operatorname{Proj} A \hookrightarrow \operatorname{Proj} R = \mathbb{P}^n_k$.

Thus all the schemes we have considered are closed subschemes in some $\mathbb{P}^n_k$ (In particular they are all projective).

REMARK. *We have already seen that J need not be unique. But if X is reduced and if we insist that R/J is also reduced then J is unique. [Take J = root ideal of any ideal defining X].*

If $X = \operatorname{Proj} R/J$, i.e. J is some ideal of R defining X then using (*) one can verify that $\operatorname{spec} R/J - [P]$ is uniquely determined. In other words any ideal J which defines X ideally determines the corresponding cone everywhere except the vertex.

Examples. 1. Let $R = k[X_0, X_1]$. So $\operatorname{Proj} R = \mathbb{P}^1_k$.

Let $J_1 = (X_0)$ and $J_2 = (X_0^2, X_0X_1)$. Then $\operatorname{Proj} R/J_1 \cong \operatorname{Proj} R/J_2$. $J_1 \underset{\neq}{\supset} J_2$. Note that $(R/J_1)_P$ is Cohen-Macaulay ($\because$ depth$(R/J_1)_P = 1$) and depth$(R/J_2)_P = 0$, where P is the vertex.

2. Take the imbedding of $\mathbb{P}^1$ in $\mathbb{P}^3$ given by: $(x_0, x_1) \longrightarrow (x_0^3, x_0^2x_1, x_0x_1^2, x_1^3)$. Then an ideal defining the image in $\mathbb{P}^3$ is $J = (X_0X_3 - X_1X_2, X_0X_2 - X_1^2, X_1X_3 - X_2^2)$. We see that the variety is not a complete intersection and the vertex of the cone is also not a complete
5 intersection. We will show now how properties of the vertex affect the variety itself.

PROPOSITION 1.1. Let $A = k \oplus A_1 \oplus A_2 \oplus \ldots$ be a graded ring where A_1 is a finitely generated vector space over k generating A as a graded k-algebra. Let P be the vertex. Then

i) A_P is $R_i \Longrightarrow \operatorname{Proj} A$ is R_i

ii) A_P is $S_i \Longrightarrow \operatorname{Proj} A$ is S_i

iii) A_P is a complete intersection $\Longrightarrow$ Proj A is locally completely intersection

iv) A_P is a *U.F.D* $\Longrightarrow$ A is factorial

Proof. Assume that A_P has #. Let # denote any of the properties (i), (ii), (iii). Since Proj A is covered by open sets of the type Spec $A_{(f)_\circ}$, $f \in A_1$, it suffices to prove that # holds for each one of them.

So let $p \in$ Spec $A_{(f)_\circ}$. We want to show that $(A_{(f)_\circ} p$ has # But $(A_{(f)_\circ p}$ has # $\Longleftrightarrow A_{(f)_\circ}[T, T^{-1}]_{p[T,T^{-1}]}$ has #.

As we have already seen $A_{(f)_\circ}[T, T^{-1}] \simeq A_f$ and then $p[T, T^{-1}]$ will correspond to a prime ideal qA_f, ($q \hookrightarrow A$ a homogeneous prime ideal.) So it suffices to show that # holds for $A_{f(qA_f)}$. Now q is contained in P, since q is homogeneous and $A_{f(qA_f)} \simeq A_{P_{(qAP)}}$. The result then follows from the fact that A_P has # and hence any localization of A_P also has #. As iv) is easy we leave it to the reader. ◻

2 Cohomology of Coherent Sheaves:

Let $R = k[X_0, \ldots, X_n]$. Then to any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$, one can associate a graded R-module F of finite type. This correspondence is not unique. But given a graded R-module F, we can associate to it a unique sheaf on $\mathbb{P}^n$. For the definition of cohomology and results on cohomology we refer the reader to *FAC* by J.P. Serre and local cohomology by A. Grothendieck. We denote by $O_{\mathbb{P}^n}(1)$ the line bundle got by hyperplane in $\mathbb{P}^n$. **6**

REMARK. $O_{\mathbb{P}^n}(-1) = \operatorname{Hom}(O_{\mathbb{P}^n}(1), O_{\mathbb{P}^n}) = (O_{\mathbb{P}^n}(1))^v$.

PROPOSITION 2.1. There is an exact sequence for any graded R-module M

$$0 \longrightarrow H^0_P(M) \longrightarrow M \longrightarrow \bigoplus_{m\in\mathbb{Z}} H^0(\mathbb{P}^n, \tilde{M}(m)) \longrightarrow H^1_P(M) \longrightarrow 0,$$

$$\text{and} \qquad H^{i+1}_P(M) = \bigoplus_{m\in\mathbb{Z}} H^i(\mathbb{P}^n, \tilde{M}(m)) \; i \geq 1.$$

Proof. This statement is almost the same as Prop. 2.2 in LC. Putting $X = \operatorname{Spec} R$ and $P = Y$ in that result we get

$$0 \longrightarrow H^0_P(M) \longrightarrow M \longrightarrow H^0(\operatorname{Spec} R - \{P\}, \tilde{M}) \longrightarrow H^1_P(M) \longrightarrow 0,$$

where $\tilde{M}$ is the sheaf defined by M on $\operatorname{Spec} R - P$ and

$$H^{i+1}_P(M) \cong H^i(\operatorname{Spec} R - P, \tilde{M}), i > 0.$$

So we only have to check that $H^i(\operatorname{Spec} R - P, \tilde{M}) \cong \bigoplus_m H^i(\mathbb{P}^n, M(m))$, for every i, canonically. We have a map $\operatorname{Spec} R - P \xrightarrow{p} \mathbb{P}^n$ which is a surjection and an affine map. So

$$H^i(\mathbb{P}^n, p_* \tilde{M}) \longrightarrow H^i(\operatorname{Spec} R - P, \tilde{M}).$$

So we want to show that,

$$H^i(\mathbb{P}^n, p_* \tilde{M}) = \bigoplus_m H^i(\mathbb{P}^n, M(m)), \quad \forall_i.$$

7

But one checks that

$$p_* M = \bigoplus_m \tilde{M}(m)$$

canonically and the result follows. □

Example. $\tilde{M} = O_C$, where C is a reduced curve in $\mathbb{P}^n . M = R/J.J$ is an ideal defining C.

$$R/J = \oplus H^0(\mathbb{P}^n, R/J(m)) = \oplus H^0(\mathbb{P}^n, O_C(m))$$

where $O_C(m) = O_C(1)^{\otimes m}$ and $O_C(1) = O_{\mathbb{P}^n}(1)/C$

$$\oplus H^0(\mathbb{P}^n, O_C(m)) = \oplus H^0(C, O_C(m)).$$

Claim. *The map $R/J \longrightarrow \oplus H^0(C, O_C(m))$ is injective if J is the biggest ideal defining C.*

From the above exact sequence, we get

if $R/J \longrightarrow \oplus H^0(C, O_C(m))$ *is injective*
then $H^0_P(R/J) = 0$*i.e.* $\mathrm{depth}_P\, R/J \geq 1$. *(By Theorem 3.8 of LC.)*
then P is not an imbedded component
hence J is the biggest ideal defining C.

Claim. *If C is a smooth curve,*

$$R/J \longrightarrow \bigoplus_m H^0(C, O_C(m))$$

is surjective if and only if C is arithmetically normal.
By the above exact sequence

- $R/J \longrightarrow \oplus H^0(C, O_C(m))$ *is injective and surjective* **8**
- $H^0_P(R/J) = H^1_P(R/J) = 0\, \mathrm{depth}_P\, R/J \geq 2$ *by Th. 3.8 of LC*

since $\mathrm{Spec}\, R/J - [P]$ *is normal we have to check normality only at P. Since P is of codim 2 in* $\mathrm{Spec}\, R/J, (R/J)_P$ *is normal by Serre's criterion.*
i.e. C is arithmetically normal

3 Vanishing Theorem and Duality

Vanishing Theorem (Serre). Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$ Then for all $i > 0, H^i(\mathbb{P}^n, \mathcal{F}(m)) = 0$ and $H^0(\mathbb{P}^n, \mathcal{F}(m))$ generate $\mathcal{F}(m)$ as $O_{\mathbb{P}^n}$ module for $m \gg 0$.

Duality Theorem. Let $\mathcal{F}$ be a locally free sheaf on $\mathbb{P}^n$, of finite type. $\omega_{\mathbb{P}^n} = \overset{n}{\Lambda}\Omega_{\mathbb{P}^n/k}$, where $\Omega^1_{\mathbb{P}^n/k}$ is the sheaf of differentials. So $\omega_{\mathbb{P}^n} = O_{\mathbb{P}^n}(-n-1)$. Then $H^i(\mathbb{P}^n, \mathcal{F}) \times H^{n-i}(\mathbb{P}^n, \mathcal{F} \otimes \omega) \longrightarrow H^n(\mathbb{P}^n, \omega) \simeq k$ is a perfect pairing.

Duality on a Locally Cohen-Macaulay Curve C**:** Let $\mathcal{F}$ be a locally sheaf of finite rank on $C \hookrightarrow \mathbb{P}^n$. Then,

$H^i(C, \mathcal{F}) \times H^{1-i}(C, \mathcal{F}^\vee \otimes \omega_C) \xrightarrow{\sim} H^1(C, \omega_C)$ is a perfect pairing, with $\omega_C = \underline{\mathrm{Ext}}^{n-1}_{\mathbb{P}^n}(O_C, \omega_{\mathbb{P}^n})$.

1. If X is smooth, $\omega_X = \overset{\max}{\Lambda}\, \Omega^1_{X/k}$.

2. Let X and Y be equidimensional locally Cohen-Macaulay varieties with $X \hookrightarrow Y$. If c is the codimension of X in Y, then

$$\omega_X = \underline{\mathrm{Ext}}^c(O_X, \omega_Y).$$

COROLLARY. *If X and Y are as above with X a divisor on Y, then $(L \otimes \omega_Y)_{|X} = \omega_X$ where L is the line bundle associated to the divisor X.*

9 3. Let X and Y be equidimensional locally Cohen-Macaulay varieties with a finite surjective morphism $X \xrightarrow{f} Y$. Then

$$f_*\omega_X = \underline{\mathrm{Hom}}_{O_Y}(f_*O_X, \omega_Y).$$

4. Let Y be a locally Cohen-Macaulay variety. $\mathbb{P}^n_Y = \mathbb{P}^n \underset{k}{\times} Y, \mathbb{P}^n_Y \longrightarrow Y$ be the projection. Then

$$\omega_{\mathbb{P}^n_Y} = \pi^*\omega_Y \otimes O_{\mathbb{P}^n_Y}(-n-1).$$

The above results can be found in either of the following: *FAC* by J.P. Serre or Introduction to Grothendieck Duality Theory, A. Altman and S. Kleiman.

Example 1. If C is a local complete intersection ω_C is a line bundle.

Example 2. If C is smooth $\omega_C = \Omega^1_{C/k}$.

4 Divisors, Line Bundles and Riemann-Roch Theorem.

Let X be a scheme, O_X its structures sheaf, K_X the sheaf of meromorphic functions on X, O^*_X the sheaf of units of O_X, K^*_X the sheaf of units of K_X. We call a collection $[f_i, U_i]$ where U_i's are affine open in X and $f_i \in \Gamma(U_i, K^*)$, *Cartier Divisor* on X if $f_i f_j^{-1} \in \Gamma(U_i \cap U_j, O^*_X)$.

Example. If we take $f_i = 1, \quad \forall_i$, then this line bundle is precisely O_X.

For a more detailed description of divisors and line bundles the reader can refer to almost any of the Geometry books; for instance D. Mumford, "Lectures on Curves on an Algebraic Surface".

Let $C \longrightarrow \mathbb{P}^n$ be a locally complete intersection-curve in $\mathbb{P}^n$. Let C be reduced and connected. Let L be a line bundle on C.

$$\chi(L) = \dim H^0(C, L) - \dim H^1(C, L).$$

$p = \dim H^1(C, O_C) = \dim H^0(C, \omega_C)$ is called the *arithmetic genus*. We define degree of line bundle to be,

$$d^\circ L = \chi(L) - \chi(O_C).$$

10

Riemann-Roch Theorem. *$d^\circ L$ is additive i.e. $d^\circ(L \otimes M) = d^\circ L + d^\circ M$.*

Let $D \hookrightarrow C$ be a closed subschem locally defined by one equation which is a non-zero divisor. Then D is an effective Cartier divisor. If J_D is its ideal we have the following exact sequence

$$0 \longrightarrow J_D \longrightarrow O_C \longrightarrow O_D \longrightarrow 0.$$

and J_D is a line bundle.

DEFINITION. $J_D = O_C(-D)$.

$$d^\circ O_C(1) = d^\circ(C), \quad \textit{if} \quad C \longrightarrow \mathbb{P}^3 \quad \textit{is a curve}.$$

If C is a proj *Curve of* deg d *in* $\mathbb{P}^2$, *then,*

$$p = \frac{(d-1)(d-2)}{2}$$

For a proof of this fact see 'Elements of the Theory of Algebraic Curves' by A. Seidenberg, pp. 118. If g is the geometric genus of C, then $p \geq g$.

Bezout's Theorem. *If $X, Y \hookrightarrow \mathbb{P}^n$* proj *varieties imbedded in $\mathbb{P}^n$* $\dim X = r, \dim Y = n - r$ *and* $\dim(X \cap Y) = 0, \deg X = d_1, \deg Y = d_2$, *then* $d^\circ(X \cap Y) = d_1 d_2$.

Example. A curve in $\mathbb{P}^3$ with no imbedded smooth deformation.

Smmoth deformation. We define a smooth imbedded deformation of a scheme $Y \xrightarrow{\text{closed}} \mathbb{P}^n$ as: S an irreducible parametrizing family, X a scheme. Such that there exists a commuting diagram,

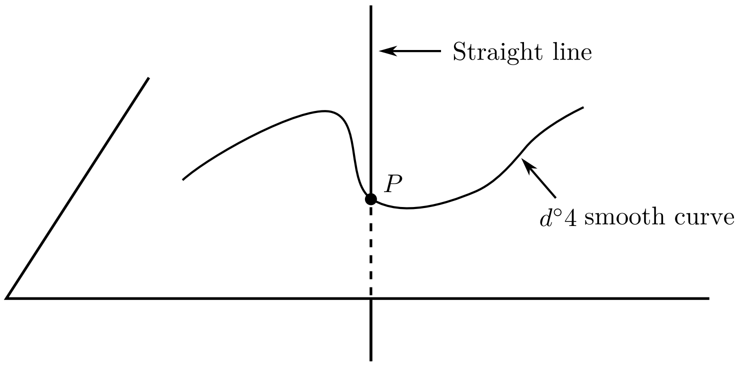
$$\begin{array}{ccc} X & \xrightarrow{f} & S \\ \downarrow & \nearrow & \\ \mathbb{P}^n \underset{k}{\times} S & & \end{array}$$

11 where f is flat, $X(s_\circ) \hookrightarrow \mathbb{P}^n_{k(s_\circ)}$ is the given morphism $Y \longrightarrow \mathbb{P}^n$.

We know by results of M. Schaps (Asterisque 1973 $n°$ 7–8 colloque surles singularities en geometrie analytyque, Cargese 1972) that given any locally Cohen-Macaulay curve in $\mathbb{P}^3$ it has a smooth deformation. More precisely given a locally Cohen-Macaulay curve $C \hookrightarrow \mathbb{P}^3$, there exists a scheme X, a discrete henselian valuation ring V and a line bundle L on X, such that we have a morphism flat $X \hookrightarrow \operatorname{Spec} V$ with the following properties.

If $s_\circ \in \operatorname{Spec} V$ is the closed point of $\operatorname{Spec} V$, $s \in \operatorname{Spec} V$, the generic point, then $X(s_\circ) = C, X(s)$ is smooth and $L(s_\circ) = O_C(1)$. To have an imbedded smooth deformation we must also have $L(s)$ very ample. Here we construct an example such that $L(s)$ is not very ample for any choice made.

Let C be a plane smooth curve of deg 4. L a line in $\mathbb{P}^3$ not in the same plane as C and intersecting C exactly at one point P, transversally. Let $\mathcal{Z} = CUL$. We claim $\mathcal{Z}$ has no smooth deformations in $\mathbb{P}^3$.



If one has Flatness of $X \longrightarrow \operatorname{Spec} V$ as above then $d^\circ(\mathcal{Z})$ is constant on the family X and for generic s, arithmetic genus of $X(s)$ = arithmetic genus of $X(s_\circ) = p$ and $\chi(O_{\mathcal{Z}}(n)) = nd^\circ(\mathcal{Z}) - p + 1$ by $R-R$. Moreover here we have $d^\circ(\mathcal{Z}) = 5$. There exists an exact sequence **12**

$$0 \longrightarrow O_Z \longrightarrow O_C \oplus O_L \longrightarrow O_P \longrightarrow 0.$$

So

$$0 \longrightarrow H^0(OZ) \longrightarrow H^0(O_C \oplus O_L) \longrightarrow H^0(O_P) \longrightarrow H^1(O_Z) - -- \\ \longrightarrow H^1(O_C \oplus (O_L) \longrightarrow 0.$$

So $\dim H^1(O_Z) = \dim H^1(O_C \oplus O_L) = p$ because $H^1(O_L) = 0$. Since C is a plane curve of deg 4,

$$p_C = H^1(O_C) = \frac{(4-1)\,(4-2)}{2} = 3,$$

hence $p = 3$.

$$\chi(O_Z(n)) = 5n - 3 + 1 = 5n - 2.$$

Now we claim that if $\mathcal{Z}'$ is a smooth deformation, then $\mathcal{Z}'$ is a plane curve. Since $d\mathcal{Z}' = 5$ and $p = 3$

$$p = \frac{(5-1)\,(5-2)}{2} = 6 \neq 3.$$

So this will give us the required contradiction. Let J be the ideal of $\mathcal{Z}$ We want to,

$$0 \longrightarrow J(1) \longrightarrow O_{\mathbb{P}}(1) \longrightarrow O_{\mathcal{Z}}(1) \longrightarrow 0.$$

$$0 \longrightarrow H^0(J(1)) \longrightarrow H^0(O_{\mathbb{P}^3}(1)) \longrightarrow H^0(O_{\mathcal{Z}'}(1)) \dim H^0(O_{\mathbb{P}^3}(1)) = 4.$$

$$\dim(H^0(O_{\mathcal{Z}'}(1))) - \dim H^1(O_{\mathcal{Z}'}(1)) = \deg O_{\mathcal{Z}'}(1) - p + 1$$

$$\dim(H^0(O_{\mathcal{Z}'}(1)) - \dim H^1(O_{\mathcal{Z}'}(1)) = 5 - 3 + 1 = 3.$$

13 So if we show that $H^1(O_{\mathcal{Z}'}(1)) = 0$ we are through. But $H^1(\mathcal{Z}', O_{\mathcal{Z}'}(1)) = H^0(\mathcal{Z}', \omega_{\mathcal{Z}'}(-1)), d^\circ\omega, = 2p-2 = 4, d^\circ(\omega_{\mathcal{Z}'}(-1) = d^\circ(\omega_{\mathcal{Z}'}) - d^\circ(\delta_{\mathcal{Z}'}(1)) = 4 - 5 = -1. H^0(C, \mathcal{L}) = 0$ if $\mathcal{L}$ is a line bundle on a smooth connected curve and $d^\circ \mathcal{L} < 0$.

PROPOSITION 1.2. Let $C \hookrightarrow \mathbb{P}^3$ be an equidimensional curve which is locally a complete intersecion. Then there exist F_1, F_2, F_3, F_4 forms in $R = k[X_0, X_1, X_2, X_3]$ such that $\mathrm{Proj}(R/\underline{F}) = C$ where $\underline{F} = (F_1, F_2, F_3, F_4).R$.

Proof. We prove the result by choosing F_i's successively as follows.

1. Choose F_1 such that C is a divisor on the surface defined by zeros of F_1.

2. F_2 is chosen such that (F_1, F_2) generate the ideal of C generically. So ideal generated by (F_1, F_2) is equal to the ideal of C at almost all points of C.

3. We choose F_3, such that (F_1, F_3) generate the ideal of C at those finite number of points of C where (F_1, F_2) is not equal to the ideal of C and $V(F_1, F_2, F_3)$ has no one dimensional components other than that of C.

4. F_4 is chosen such that it does not pass through the zero dimensional components of $V(F_1, F_2, F_3)$.

We use the following eassy lemmas:

1. **Serre**. Let M be a vector bundle of rank r on a variety X. Let $V \subset H^0(X, M)$ be a finite dimensional vector space and assume

(a) $r > \dim X$

(b) For all closed points $x \in X$, the map from V to $M \otimes k(x)$ is surjective.

Then there is an open non empty zariski set of elements $s \in V$ whose image in every space $M \otimes k(x)$ is non-zero.
See: Lectures on Curves on an Algebraic surface by D. Mumford pp. 148. □

II. Lemma of Devissage (i). Let $I, J_i, i = 1, \dots n$ be ideals of a ring A and at most two of the J_i's are not prime. If $I \subset \underset{i=1}{\cup} J_i$ then $I \subset J_i$ for some i. 14

II. Lemma of Devissage (ii). If E is a vector space over a field k and $\{E_i\}_{i \in I}$ sub-vector spaces of E with $\# I < \# k$, then $E = UE_i \Rightarrow E = E_i$ for some i.

1. **Existence of F_1.** Let I be the ideal sheaf of the curve. Then I/I^2 is a vector bundle of rank 2 over O_C by assumption. The exact sequence

$$I(n) \xrightarrow{\varphi} I/I^2(n) \longrightarrow 0 \quad \text{gives a map,}$$
$$H^0(\mathbb{P}^3, I(n)) \xrightarrow{\varphi} H^0(\mathbb{P}^3), I/I^2(n)) = H^0(C, I/I^2(n)),$$

   for every n. Since for large enough $n, I/I^2(n)$ is generated by sections and φ is surjective we conclude that $\Im\varphi$ generates $I/I^2(n)$ for large enough n. Since rank $I/I^2 > \dim C$, by (I) we have $F_1 \in H^0(\mathbb{P}^3, I(n))$ such that $\varphi(F_1)$ is nowhere vanishing on C. So C lies on the surface Y defined by $F_1 = 0, \varphi(F_1)$ is nowhere vanishing on C implies that it can be considered as one of the two generators of I at every point of C. So $I/(F_1)$ which is a sheaf on Y is locally one-generated and it is clear that the generator is not a zero divisor at those point. So C is a divisor on F_1.

2. Choose F_2 as the local equation of C on Y at some point. So in a neighbourhood of that point (F_1, F_2) generate $I.i.e.$ (F_1, F_2) is generically equal to I.

3. Let $P_1, P_2 \ldots P_m$ be all the points of C where $(F_1, F_2) \subsetneq I_{P_i}$. Let $V \subset H^0(Y, \frac{I(n)}{(F_1)})$ for large enough n such that V generates $\frac{I(n)}{(F_1)}$. Let us denote $\frac{I_{P_i}}{F_1}$ by $\overline{I}_{P_i}$. Then we have a natural map $V \xrightarrow{\alpha} \bigoplus\limits_{i=1}^{m} \overline{I}_{P_i/m_i I_{P_i}}$ where m_i corresponds to the maximal ideal of O_{P_i}. Since V generates $\frac{I}{(F_1)}(n)$ at every point, the composite maps,

$$V \xrightarrow{\alpha} \bigoplus_{i=1}^{m} \overline{I}_{P_i}/m_i \overline{I}_{P_i} \xrightarrow{\beta_i} \overline{I}_{P_i/m_i \overline{I}_{P_i}}$$

15 are all surjective. So $\ker(\beta_i \circ \alpha) \subsetneq V$ for every i. By III, $\bigcup\limits_{i=1} \ker(\beta_i \circ \alpha) \subsetneq V$. So in the natural Zariski topology on V, there exists a non-empty open set $V - \bigcup\limits_{i=1}^{m} \ker(\beta_i \circ \alpha) = W$ such that for any $s \in W, \beta_i \circ \alpha(s) \neq 0, \quad \forall_i$.

Let us denote by $R_1, R_2, \ldots R_t$ the one dimensional components of $V(F_1, F_2)$, other than that of C.

Again we have the map

$$V \xrightarrow{\alpha'} \bigoplus_{i=1}^{t} O_{R_i/m_{R_i}} \quad \text{induced from} \quad O \longrightarrow O_{R_i}$$

Since image of V in $O_{R_i/m_{R_i}}$ generates $O_{R_i/m_{R_i}}$ as $O_{R_i/m_{R_i}}$-vector spaces, we find that as before there exists a non-empty open set A in V such that for any $u \in A, \alpha'(u)$ has all components non-zero.

Since A and W are non-empty open subsets of V, which is irreducible, $A \cap W \neq \phi$. Let $s \in A \cap W$. $s \in H^0(Y, \frac{I(n)}{F_1})$. So s comes from a homogeneous polynomial $F_3 \in I$. Since $\beta_i \circ \alpha(s) \neq 0$ and $\overline{I}_{P_i}$ is one generated we see that $\beta_i \circ \alpha(s)$ is generated by $\beta_i \circ \alpha(s) i.e.\ (F_1, F_3)_{P_i} = I_{P_i}\ \forall P_i$. Again we see that since $\alpha'(s) \neq 0$ in every one of the $O_{R_i/m_{R_i}}, (F_1, F_3)_{R_i} = O_{R_i} i.e.\ R_i$'s are not components of this variety. So we have,

$$(F_1, F_2, F_3)_P = I_P \quad \text{at every} \quad P \in C.$$

The only irreducible components of $V(F_1, F_2, F_3)$ are components of C and finitely many closed points.

4. We choose $F_4 \in I$ which does not vanish at these zero dimensional components. So one has F_1, F_2, F_3, F_4 which generate I on $\mathbb{P}^3$.

REMARK. *There exist smooth curves (in particular, locally complete intersection) in* $\mathbb{P}^3$ *which are not ideally defined by three equations. See: Corollary 7.2, Liaison des varietes algebriques. I, by C. Peskine and L. Szpiro. Inventiones Math. (26, 1974.* 16

Chapter 2

The Theory of Adjoint Systems

THIS CHAPTER DEALS with the conductor $\underline{f}$ of a reduced irreducible **17**
curve C with respect to its normalisation $\bar{C}$. The material is fairly classical, but we have developed it from the point of view of duality. We will show that there is a very simple relation between the conductor and the sheaf of differential (or the dualising sheaf), namely

$$\underline{f} = \varphi_* \omega_{\bar{C}} \otimes \omega_C^{-1},$$

where φ is the canonical morphism $\bar{C} \longrightarrow C$.

From this fact and the computation of the dualising sheaf for the blowing up of a point on a smooth surface, we get that, adjoint curves are those, whose equation is in the 'conductor'. The classical theorem of Gorenstein (adjoints of degree $m-3$ cut out on C, the complete canonical system of $\bar{C}$) is an easy consequence of the previous considerations. So is the local formula $n = 2\delta$. This point of view introduces very naturally Kodaira's vanishing theorem. When one tries to have a theorem similar to Gorenstein's, for a surface which is not $\mathbb{P}^2$. We recommend to the interested reader, Pathologies III [8] of D. Mumford, which contains a very nice proof of the regularity of the adjoint system for a normal surface in characteristic zero. In §10 we give an example (char $p > 0$)

of an ample divisor satisfying Kodaira's vanishing theorem, but not the regularity of the adjoint system.

1 Intersection Multiplicities

18 Let X be a projective, irreducible and non-singular surface. Let L_1 and L_2 be two line bundles on X. Then we define the *intersection multiplicity* of L_1 and L_2 denoted by $(L_1.L_2)$ as

$$(L_1.L_2) = \chi(O_X) - \chi(L_1^{-1}) - \chi(L_2^{-1}) + \chi(L_1^{-1} \otimes L_2^{-1}).$$

If $L_2 = O_X(D)$ for some divisor D, then $(L_1, L_2) = \deg(L_{1/D})$. Let $L_1 = O_X(D_1)$ and $L_2 = O_X(D_2)$ such that D_1 and D_2 have no common components. Then,

$$(L_1.L_2) = \sum_{Q \in X} \text{length} \quad (O_{D_1 \cap D_2, Q}).$$

The pairing $(- \cdot -)$ is bilinear.

2 Blowing up a Closed Subscheme

Let X be any scheme and Y a closed subscheme of X. Let J be the sheaf of ideals of Y in O_X.

DEFINITION. *The blowing up of X with centre Y is the scheme $Z =$* $\mathrm{Proj}(O_X \oplus J \oplus J^2 \oplus \ldots)$.

We have a canonical morphism, $f : Z \longrightarrow X$ where f is an isomorphism of $Z - f^{-1}(Y)$ and $X - Y$. The blowing up has the following universal property. The image $J.O_Z$ of $f^* J$ in O_Z is locally defined by one equation which is a non-zero divisor. If W is a scheme and $g : W \longrightarrow X$ a morphism such that the image of $g^* J$ in O_W is isomorphic to $O_W(-D)$ for some divisor D, then g factors through Z, i.e. there exists a unique morphism $h : W \longrightarrow Z$ such that the following diagram is commutative

$$\begin{array}{ccc} W & \longrightarrow & X \\ {\scriptstyle h}\downarrow & \nearrow_{f} & \\ Z & & \end{array}$$

and the image of $h*(J.O_Z)$ in O_W is equal to $O_W(-D)$. We recall the definitions of g^* and g_*. **19**

Let $g : X \longrightarrow Y$ be a morphism of schemes. Let F be any pre-sehaf on X. We define a pre-sehaf on Y called g_*F as follows: $g_*F(U) = F(g^{-1}(U))$ for any open set in Y. One can check that if F is a sheaf, g_*F is also a sheaf.

Let G be a pre-sheaf on Y. Consider the functor $F \longrightarrow \mathrm{Hom}(G, g_*F)$, from the category of pre-sheaves on X to the category of sets. If this functor is representable, we call the pre-sheaf, g^*G.

$$\mathrm{Hom}(F, g^*G) = \mathrm{Hom}(G, g_*F) \quad \text{for every pre-sheaf } F \text{ on } X.$$

One can verify that in our case of O_X-modules this functor is representable by an O_X-module "associated to the following pre-sheaf, for U open in $Y, g^{-1}(U) = \Gamma(U, G) \underset{\Gamma(U, O_y)}{\otimes} \Gamma(g^{-1}(U), O_X)$

DEFINITION. *The subscheme of Z defined by the sheaf of ideal* $(J.O_Z)$ *is called the exceptional divisor on Z under the blowing up. It is equal to* $f^{-1}(Y)$.

Let X be a non-singular, irreducible scheme of dimension 2. Let P be a closed point on X. Let $\underline{M}$ denote the sheaf of ideals defining P in O_X. Let X' be the blown up of X at P. So

$$X' = \mathrm{Proj}(O_X \oplus \underline{M}^2 \oplus \ldots)$$

Let E be the exceptional divisor. So

$$\begin{aligned} E &= \mathrm{Proj}((O_X \oplus \underline{M}^2 \oplus \ldots) \otimes \mathrm{Spec}\, O_X/\underline{M}) \\ &= \mathrm{Proj}(O_X/\underline{M} \oplus \underline{M}/\underline{M}^2 \oplus \underline{M}^2/M_3 \oplus \ldots) \end{aligned}$$

Since X is non-singular surface and $\underline{M}$ is a maximal ideal, we see that

$$E = \mathrm{Proj}(k[X, Y] = \mathbb{P}^1_k.$$

20

where k is equal to the residue field of P. Hence

$$O_X(E)_{/E} \quad \text{is equal to} \quad O_{\mathbb{P}^1}(-1)$$

and the self intersection $(E.E) = -1$.

REMARK. *X' is irreducible and non-singular.*

3 Canonical Divisor of the Blown up

PROPOSITION. *Let X be a non-singular irreducible quasi-projective variety and P any closed point of X. Let X_1 be the blown up of X at P, and E the exceptional divisor. Let $f : X_1 \longrightarrow X$ be the blowing up map. If one denotes by $\omega_X(resp.\ \omega_{X_1})$ the dualising sheaf $X(resp.\ of X_1)$ one has*

$$\omega_{X_1} = f^*\omega_X \underset{O_{X_1}}{\otimes} O_{X_1}(E).$$

Proof. Let L denote the line bundle $\omega_{X_1} \underset{O_{X_1}}{\otimes} (f^*\omega_X)^{-1}$ Since f is an isomorphism from $X_1 - E$ to $X - P$, we see that $L_{/X_1-E} \simeq O_{X_1-E}$. Since X_1 is non-singular $L = O_{X_1}(D)$ for some Weil divisor D on X_1. Let $D = D_1 + \alpha E$ where D_1 does not have E as a component. Since $O_{X_1}(D)_{/X_1-E}$ is the trivial line bundle we see that the divisor D_1 restricted to $X_1 - E$ is principal, i.e. there exists a rational function g on X_1 such that $\mathrm{div}(g)_{/X_1-E} = D_{1/X_1-E}$. So we see that $D_1 - \mathrm{div}(g)$ has only E as component and hence $D - \mathrm{div}(g)$ also has only E as its component. So we get that $L = O_{X_1}(D) = O_{X_1}(D - \mathrm{div}(g)) = O_{X_1}(nE)$ for some integer n. Thus

$$\omega_{X_1} \simeq f^*\omega_X \otimes O_{X_1}(nE).$$

Now

$$O_{\mathbb{P}^1}(-2) = \omega_E = \omega_{X_1} \underset{O_{X_1}}{\otimes} O_{X_1}(E) \underset{O_{X_1}}{\otimes} O_E$$

$$= f^*\omega_X \underset{O_{X_1}}{\otimes} O_{X_1}((1+n)E) \underset{O_{X_1}}{\otimes} O_E.$$

Since $E = f^*(P)$ and ω_X is trivial on E, we see that $f^*\omega_X \underset{O_{X_1}}{\otimes} O_E =$ **21**
O_E So

$$O_{\mathbb{P}^1}(-2) = O_{X_1}((1+n)E) \underset{O_{X_1}}{\otimes} O_E = O_{\mathbb{P}^1}(-1(1+n)).$$

Therefore $1+n=2$ or $n=1$, *i.e.* $\omega_{X_1} = f^*\omega_X \underset{O_{X_1}}{\otimes} O_{X_1}(E)$. □

4 Proper Transform

Let R be a regular local ring of dimension 2. Let g be a non-zero element in the maximal ideal M or R. We will calculate the Hilbert-Samuel function of the ring $A = R/_{(g)}$.

Let us denote the maximal ideal of A by $\overline{M}$. $\overline{M} = M/_{gR}$. Let $g \in M^r - M^{r+1}$. Then

$$\overline{M}^n = (M/gR)^n = M^n/gR \cap M^n.$$

Since $g \in M^n$, for $n < r$,

$$\overline{M}^n = M^n/gR.$$

So, for $n < r$,

$$\overline{M}^n/\overline{M}^{n+1} = M^n/gR \Big/ M^{n+1}/gR = M^n/M^{n+1}.$$

For $n > r$ we claim that $gR \cap M^n = g.M^{n-r}$. Clearly $gM^{n-r} \subset gR \cap M^n$. Let $g.h \in M^n$, with $h \in M^p - M^{p+1}$ Since the graded ring of R is a domain, we see that $p + r \geq n$ *i.e.* $p \geq n - r$. So $h \in M^{n-r}$ So for $n > r$, we have

$$\overline{M}^n/\overline{M}^{n+1} = M^n/gM^{n-r} \Big/ M^{n+1}/gM^{n-r+1}$$

We have an exact sequence of $k(= R/_M = A/_{\overline{M}})$ vector spaces:

$$0 \longrightarrow \frac{gM^{n-r}}{gM^{n-r+1}} \longrightarrow M^n/M^{n+1} \longrightarrow M^n/gM^{n-r} \Big/ M^{n+1}/gM^{n-r+1} \longrightarrow 0.$$

Since $\frac{gM^{n-r}}{gM^{n-r+1}} \simeq M^{n-r}/M^{n-r+1}$ we get, **22**

$$\ell(\overline{M}^n/\overline{M}^{n+1}) = \ell(M^n/M^{n+1}) - \ell(M^{n-r}/M^{n-r+1})$$
$$= (n+1) - (n-r+1) = r.$$

For $n < r$, $\ell(\overline{M}^n/_{\overline{M}^{n+1}}) = \ell(M^n/_{M^{n+1}}) = n+1$. So, for $n < r$,

$$\ell(A/_{\overline{M}^n}) = \frac{n(n+1)}{2}$$

and for $n > r$

$$\ell(A/_{\overline{M}^n}) = \frac{r(r+1)}{2} + (n-r).r = rn - \frac{r(r-1)}{2}$$

Thus we see that the Hilbert-Samuel polynomial is $P(n) = rn - \frac{r(r-1)}{2}$ and,

$$\ell(A/_{\overline{M}^n}) = P(n) \quad \text{for every} \quad n > r-1.$$

Let A be a local ring of dimension d. Let the Hilbert-Samuel polynomial of A be,

$$P(n) = \frac{r}{d!}n^d + \ldots$$

DEFINITION. *The multiplicity of the local ring A is defined to be the integer r.*

Coming back to our case, let D be an effective divisor on the non-
23 *singular irreducible surface X. Let P be any closed point on X and let g define D in $O_{X,P}$*

DEFINITION. *We define multiplicity of D at P denoted by $m_P(D)$, as the multiplicity of the local ring $O_{X,P/(g)}$. We have seen that in this case,*

$$m_P(D) = r$$

where $g \in M^r - M^{r+1}$, M the maximal ideal of $O_{X,P}$. Let $X_1 \xrightarrow{f} X$ be the blowing up of X at the point P and E the exceptional divisor, $f^*(D)$ is the divisor $f^{-1}(D) + rE$.

DEFINITION. *The divisor* $D_1 = f^*(D) - m_P(D).E$ *is defined as the proper transform of* D *under the blowing up* f.

D_1 **is the blown up of** D **at** P. *Let* $M \subset O_X$ *be the sheaf of ideals corresponding to the point* P. *Let* $m_P(D) = r$ *so that* $O_X(-D) \subset M^r$ *and* $O_X(-D) \not\subset M^{r+1}$. *Denote by* $\overline{M}$, *the sheaf of ideals* $M/_{O_X(-D)}$ *in* O_D. *With this notation we see that,*

$$\begin{aligned} f^*(D) &= \mathrm{Proj}((O_X \oplus M \oplus M^2 \oplus \ldots) \otimes O_D) \\ &= \mathrm{Proj}(O_D \oplus M/_{O_X(-D)M} \oplus M^2/_{O_X(-D)M^2} \oplus \ldots) \\ &= \mathrm{Proj}(O_D \oplus M^r/O_X(-D)M^r \oplus M^{r+1}/O_X(-D)M^{r+1} \oplus \ldots) \\ &= \mathrm{Proj}\, S. \\ rE &= \mathrm{Proj}(O_X \oplus M \oplus M^2 \oplus \ldots) \otimes O_X/_{M^r} \\ &= \mathrm{Proj}(O_X/M^r \oplus M/M^{r+1} \oplus M^2/M^{r+2} \oplus \ldots) \\ &= \mathrm{Proj}(O_X/M^r \oplus M^r/M^{2r} \oplus M^{r+1}/M^{2r+1} \oplus \ldots) \\ &= \mathrm{Proj}\, T. \end{aligned}$$

Blown up of D *at* P *is given by,* **24**

$$\begin{aligned} \overline{D} &= \mathrm{Proj}(O_D \oplus \overline{M} \oplus \overline{M}^2 \oplus \ldots) \\ &= \mathrm{Proj}(O_D \oplus \overline{M}^r \oplus \overline{M}^{r+1} \oplus \ldots) \\ &= \mathrm{Proj}(O_X/O_X(-D) \oplus M^{r+1}/O_X(-D) \oplus M^{r+1}/O_X(-D) \cap M \oplus \ldots) \\ &= \mathrm{Proj}\, U. \end{aligned}$$

We have surjections $S \longrightarrow T$ *and* $S \longrightarrow U$, *both graded ring homomorphisms. Also the intersection of the kernels of these is the zero ideal. This implies that,*

$$f^*(D) = rE + \overline{D}.$$

So $D_1 = \overline{D}$ *i.e.* D_1 *is the blown up of* D *at* P.

REMARK. $(E, D_1) = r$. *Since* $(E, f^*(D)) = 0$ *we have,*

$$0 = (rE + D_1), E) = r(E, E) + (D_1, E)$$

i.e.

$$(D_1, E) = -r(E, E) = r.$$

One can easily see that the coordinates of the points of intersections
25 *of E and D_1 in $\mathbb{P}^1 = E$ are the directions of the tangents of D at P.*

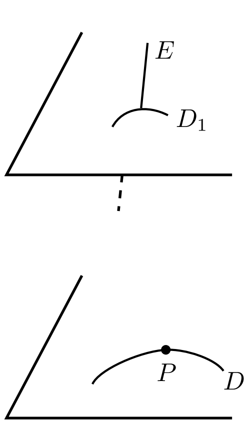


Fig 1.

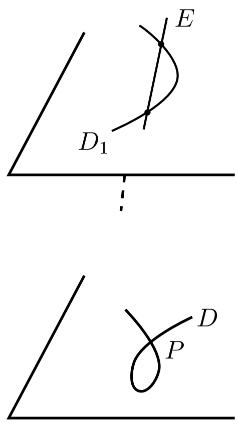


Fig 2.

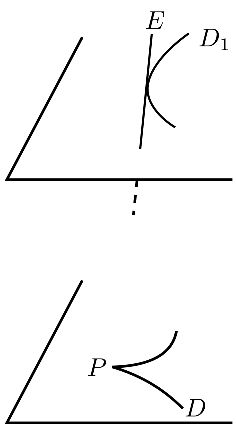


Fig 3.

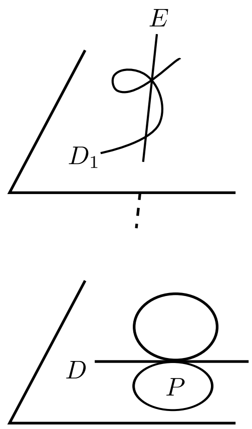


Fig 4.

5 Conductor

Let $\varphi : X \to Y$ be any morphism. We define the conductor $\underline{f}$ of this morphism as the O_Y-module, 26

$$\underline{f} = \mathrm{Hom}_{O_Y}(\varphi_* O_X, O_Y)$$

We note that when φ is birational,

$$\underline{f} = Ann(\varphi_* O_X / _{O_X})$$

In case ω_Y is a line bundle, we see that,

$$\begin{aligned}\underline{f} &= \mathrm{Hom}_{O_Y}(\varphi_* O_X, O_Y)\\ &= \mathrm{Hom}_{O_Y}(\varphi_* O_X, \omega_Y) \underset{O_Y}{\otimes} \omega_Y^{-1}\\ &= \varphi_* \omega_X \underset{O_Y}{\otimes} \omega_Y^{-1}\end{aligned}$$

So going back to our case, let X be a non-singular irreducible surface. Let D be a reduced and irreducible curve on X. Let P be a point on X and X_1, the blown up of X at P and $\varphi : X_1 \longrightarrow X$ the canonical map. Let D_1 be the proper transform of D. We denote restriction of φ to D_1 also by φ. Let E be the exceptional divisor and $r = m_P(D)$. So we have $\varphi : D_1 \longrightarrow D$ is a finite birational map and ω_D (the dualising sheaf of D) is a line bundle. Let $\underline{f}$ be the conductor of φ. Then we have

$$\underline{f} = \varphi_* \omega_{D_1} \otimes_{O_D} \omega_D^{-1}$$

Now, if ω_{X_1}, is the dualising sheaf of $X_1, \omega_{D_1} = O_{X_1}(D_1) \underset{O_{X_1}}{\otimes} \omega_{X_1} \underset{O_{X_1}}{\otimes} O_{D_1}$ because D_1 is a divisor on X_1. So

$$\begin{aligned}\omega_{D_1} &= O_{X_1}(\omega^* D - rE) \underset{O_{X_1}}{\otimes} \varphi^* \omega_X \underset{O_{X_1}}{\otimes} O_{X_1}(E) \underset{O_{X_1}}{\otimes} O_{D_1}\\ &= \varphi^*(O_X(D)) \underset{O_{X_1}}{\otimes} O_{X_1}(-(r-1)E) \underset{O_{X_1}}{\otimes} \varphi^* \omega_X \underset{O_{X_1}}{\otimes} O_{D_1}\end{aligned}$$

where ω_X is the dualising sheaf on X. Since $O_{D_1} = \varphi^*(O_D) \underset{O_{X_1}}{\otimes} O_{D_1}$ we 27 have,

$$\begin{aligned}\omega_{D_1} &= \varphi^*(O_X(D) \underset{O_X}{\otimes} \omega_X) \underset{O_{X_1}}{\otimes} O_{X_1}(-(r-1)E) \underset{O_{X_1}}{\otimes} \varphi O_D \underset{O_{X_1}}{\otimes} O_{D_1} \\ &= \varphi^*(O_X(D) \underset{O_X}{\otimes} \omega_X \otimes O_D) \underset{O_{X_1}}{\otimes} O_{X_1}(-(r-1)E) \otimes O_{D_1} \\ &= \varphi^*\omega_D \underset{O_{D_1}}{\otimes} O_{D_1}(-(r-1)E). \\ \omega_{D_1} &= \varphi^*\omega_D \underset{O_{D_1}}{\otimes} O_{D_1}(-(r-1)E\end{aligned}$$

Claim.

$$\underline{f} = \varphi_*(O_{D_1}(-(r-1)E)).$$

We have a canonical injection,

$$0 \longrightarrow O_{D_1}(-(r-1)E) \longrightarrow O_{D_1}$$

This gives two injections of O_D-modules,

$$0 \longrightarrow \varphi_* O_{D_1}(-(r-1)E) \underset{O_D}{\otimes} \omega_D \longrightarrow \varphi_* O_{D_1} \underset{O_D}{\otimes} \omega_D$$

and

$$0 \longrightarrow \varphi_*(O_{D_1}(-(r-1)E) \underset{O_{D_1}}{\otimes} \varphi^*\omega_D) \longrightarrow \varphi_*\varphi^*\omega_D$$

28 We have a canonical O_D-module homomorphism $\omega_D \longrightarrow \varphi_*\varphi^*\omega_D$ which in turn gives a $\varphi_* O_{D_1}$ homomorphism,

$$\omega_D \underset{O_D}{\otimes} \varphi_* O_{D_1} \longrightarrow \varphi_*\varphi^*\omega_D \underset{O_D}{\otimes} \varphi_* O_{D_1}$$

Since $\varphi_*\varphi^*\omega_D$ is a $\varphi_* O_{D_1}$-module we have a canonical $\varphi_* O_{D_1}$ - homomorphism

$$\varphi_*\varphi^*\omega_D \underset{O_D}{\otimes} \varphi_* O_{D_1} \longrightarrow \varphi_*\varphi^*\omega_D$$

The composite gives us a canonical homomorphism,

$$\omega_D \underset{O_D}{\otimes} \varphi_* O_{D_1} \longrightarrow \varphi_*\varphi^*\omega_D.$$

Since φ is an affine morphism, this is an isomorphism. So we have a diagram:

$$\begin{array}{ccccc} 0 \longrightarrow & \varphi_* O_{D_1}(-(r-1)E) \underset{O_D}{\otimes} \omega_D & \longrightarrow & \varphi_* O_{D_1} \underset{O_D}{\otimes} \omega_D \\ & \downarrow & & \downarrow \wr \\ 0 \longrightarrow & \varphi_* \omega_{D_1} & \longrightarrow & \varphi_* \varphi^* \omega_D \end{array}$$

Since φ is affine, one sees that $\varphi_* O_{D_1}(-(r-1)E) \underset{O_D}{\otimes} \omega_D$ and $\varphi_* \omega_{D_1}$ coincide locally and hence globally. i.e.

$$\varphi_* O_{D_1}(-(r-1)E) \underset{O_D}{\otimes} \omega_D = \varphi_* \omega_{D_1}.$$

So

$$\underline{f} = \varphi_* \omega_{D_1} \underset{O_D}{\otimes} \omega_D^{-1}, \underline{f} = \varphi_* O_{D_1}(-(r-1)E).$$

From this formula we get, **29**

$$\ell(\varphi_* O_{D_1}/\underline{f}) = \ell(\varphi_* O_{D_1})/_{\varphi_* O_{D_1}(-(r-1)E)}).$$

We have an exact sequence,

$$0 \longrightarrow O_{D_1}(-(r-1)E) \longrightarrow O_{D_1} \longrightarrow O_{D_1/O_{D_1}(-r-1)E)} \longrightarrow 0.$$

Since φ is affine, φ_* is an exact functor. So we have,

$$0 \longrightarrow \varphi_* O_{D_1}(-(r-1)E) \longrightarrow \varphi_* O_{D_1} \longrightarrow \varphi_*(O_{D_1}/O_{D_1}(-(r-1)E)) \longrightarrow 0$$

i.e.

$$\begin{aligned} \varphi_* O_{D_1/\varphi_* O_{D_1}}(-(r-1)E) &= \varphi_*(O_{D_1}/O_{D_1}(-(r-1)E)) \\ \ell(\varphi_* O_{D_1/\underline{f}}) &= \ell(\varphi_*(O_{D_1}/O_{D_1}(-(r-1)E)) \\ &= \dim_K H^\circ(D, \varphi_*(O_{D_1}/O_{D_1}(-(r-1)E)) \\ &= \dim_k H^\circ(D_1, O_{D_1}/O_{D_1}(-(r-1)E)) \\ &= \deg O_{D_1}((r-1)E) \\ &= (r-1).d^\circ O_{D_1}(E) \end{aligned}$$

$$= r(r-1).$$

This gives a cohomology exact sequence

$$\begin{aligned} 0 \longrightarrow H^{\circ}(D, \varphi_* \omega_{D_1}) \longrightarrow H^{\circ}(D < \omega_D) \longrightarrow H^{\circ}(D, O_{D/\underline{f}}) \\ \longrightarrow H^1(D, \varphi_* \omega_{D_1}) \longrightarrow H^1(D, \omega_D) \longrightarrow 0 \end{aligned}$$

30 Since φ is affine, $H^1(D, \varphi_* \omega_{D_1}) = H^1(D_1, \omega_{D_1})$, and by duality, $H^1(D_1, \omega_{D_1}) = H^{\circ}(D_1, O^{\nu}_{D_1})$ and $H^1(D, \omega_D) = H^{\circ}(D, O^{\nu}_D)$. So we get an exact sequence

$$\begin{aligned} 0 \longrightarrow H^{\circ}(D_1, \omega_{D_1}) \longrightarrow H^{\circ}(D, \omega_D) \longrightarrow H^{\circ}(D, O_{D/\underline{f}}) \\ \longrightarrow H^{\circ}(D_1, O^{\nu}_{D_1}) \longrightarrow H^{\circ}(D, O^{\nu}_D) \longrightarrow 0 \end{aligned}$$

Since D and D_1 are reduced and irreducible, the last two terms are one dimensional over k and hence isomorphic. Therefore we have an exact sequence,

$$0 \longrightarrow H^{\circ}(D_1, \omega_{D_1}) \longrightarrow H^{\circ}(D, \omega_D) \longrightarrow H^{\circ}(D, O_{D/\underline{f}}) \longrightarrow 0$$

5.1 Gorenstein Theorem for the Blowing up

Let the situation be as before.

THEOREM. *(local Gorenstein theorem for a blowing up). Let D be a reduced irreducible divisor on a regular projective surface over an algebraically closed field. Let D_1 be the blowing up of D at a closed point, then:*

$$2.\ell(\varphi_* O_{D_1}/O_D) = 2.\ell(O_{D/\underline{f}}) = \ell(\varphi_* O_{D_1/\underline{f}}).$$

Proof. We will prove the first equality. The second equality follows because

$$\ell(\varphi_* O_{D_1/\underline{f}}) = \ell(\varphi_* O_{D_1}/O_D) + \ell(O_{D/\underline{f}}).$$

We have an exact sequence,

$$0 \longrightarrow \underline{f} \longrightarrow O_D \longrightarrow O_{D/\underline{f}} \longrightarrow 0,$$

31 which gives after tensoring by φ_D, an exact sequence,

$$0 \longrightarrow \underline{f} \underset{O_D}{\otimes} \omega_D \longrightarrow \omega_D \longrightarrow O_{D/\underline{f}} \longrightarrow 0.$$

Since $\underline{f} \otimes \omega_D^{\cdot} = \varphi_* \omega_{D_1}$ we have an exact sequence,

$$0 \longrightarrow \varphi_* \omega_{D_1} \longrightarrow \omega_D \longrightarrow O_{D/\underline{f}} \longrightarrow 0.$$

Therefore

$$\ell(O_{D/\underline{f}}) = \dim H^\circ(D, \omega_D) - \dim H^\circ(D_1, \omega_{D_1}) \qquad \text{(A)}$$

Again we have an exact sequence,

$$0 \longrightarrow O_D \longrightarrow \varphi_* O_{D_1} \longrightarrow \varphi_* O_{D_1}/O_D \longrightarrow 0$$

which gives a cohomology exact sequence,

$$0 \longrightarrow H^\circ(D, O_D) \longrightarrow H^\circ(D_1, O_{D_1}) \longrightarrow H^\circ(D, \varphi_* O_{D_1}/O_D)$$
$$H^\circ(D, \omega_D^\nu) \longrightarrow H^\circ(D_1, \omega_{D_1}^\nu) \longrightarrow 0.$$

Further the first two terms are one-dimensional and hence isomorphic to each other. So we have,

$$0 \longrightarrow H^\circ(D, \varphi_* O_{D_1}/O_D) \longrightarrow H^\circ(D, \omega_D^\nu) \longrightarrow H^\circ(D_1, \omega_{D_1}^\nu) \longrightarrow 0$$

is exact. Therefore

$$\ell(\varphi_* O_{D_1}/O_D) = \dim H^\circ(D, \omega_D) - \dim H^\circ(D_1, \omega_{D_1}). \qquad \text{(B)}$$

Using (A) and (B) we get 32

$$\ell(O_{D/\underline{f}}) = \ell(\varphi_* O_{D_1/O_D}).$$

□

COROLLARY. *Let $\overline{M}$ be the sheaf of ideals of O_D defining P. Then*

$$\underline{f} = \overline{M}^{r-1}.$$

Proof. We have a map $\varphi^*\overline{M}^j \longrightarrow O_{D_1}$ for every j and the image of the map is $O_{D_1}(-jE)$. So we have a map $\varphi_*\varphi^*\overline{M}^{r-1} \longrightarrow \varphi_*O_{D_1}(-(r-1)E)$. But $\varphi_*O_{D_1}(-(r-1)E) = \underline{f}$ and is an ideal in O_D. So we have the map $\varphi_*\varphi^*\overline{M}^{r-1} \longrightarrow \underline{f} \subset O_D$. The map $\overline{M}^{r-1} \subset O_D$ is canonical and hence $\overline{M}^{r-1} \longrightarrow \underline{f}$ is an inclusion. But by the Gorenstein theorem,

$$\ell(O_D/\underline{f}) = \frac{r(r-1)}{2}$$

and by the computation of the Hilbert-Samuel function,

$$\ell(O_D/\overline{M}^{r-1}) = \frac{r(r-1)}{2}.$$

This implies that $\overline{M}^{r-1} = \underline{f}$. □

REMARK. *By the above corollary, D_1 is also the blowing up of $\underline{f}$.*

6 Resolution of Singularities of a Reduced Irreducible Curve on a Non-Singular Surface

Let D be a reduced irreducible curve on a non-singular surface X and P be any point on X. Let X_1 be the blown up of X at P and D_1, the proper transform of D.

$$\varphi : X_1 \longrightarrow X$$
$$\varphi : D_1 \longrightarrow D$$

33 **Claim.** *The canonical morphism, $O_D \longrightarrow \varphi_*O_{D_1}$ is an isomorphism if and only if either D is non-singular at P or D does not pass through P.*

The homomorphism $O_D \longrightarrow \varphi_*O_{D_1}$ is an isomorphism if and only if $\ell(\varphi_*O_{D_1}/O_D) = 0$. The condition $\ell(\varphi_*O_{D_1}/O_D) = 0$ is equivalent to $\frac{r(r-1)}{2} = 0$, where $r = m_P(D)$. This is equivalent to $r = 0$ or $r = 1$, i.e. either D does not pass through P or D is non-singular at P.

Let D be any reduced irreducibel curve on a regular surface X. Suppose that the normalisation $\overline{D}$ of D is finite over D. If P is a point on D where D is singular, we blow up X at P and let X_1 be the blown up of X and D_1 the proper transform of D. Again if Q is a singular point on D_1 we blow up X_1 at Q and let X_2 be the blow up of X_1 and D_2, the proper transform of D_1. We can continue this process to get a chain,

$$D_s \xrightarrow{\varphi_s} D_{s-1} \longrightarrow D_2 \xrightarrow{\varphi_2} D_1 \xrightarrow{\varphi_1} D.$$

Since all these maps are finite and birational, we have a map $\overline{D} \longrightarrow D_s$ where $\overline{D}$ is the normalisation of D. It is clear that this process stops because the length $\varphi_* O_{\overline{D}}/O_{D_s}$ decreases and is bounded by the length of $\varphi_* O_D/O_D$

EXAMPLES. (1) Consider the curve C given by $Y^n Z - X^{n+1}$ in $\mathbb{P}^2$, where (X, Y, Z) are the homogeneous coordinates. $P = (0, 0, 1)$ is the only singular point of C. If C is the normalisation of C, one easily sees that, $\ell(\varphi_* O_C/O_C)$ is equal to $\frac{n(n-1)}{2}.m_P(C) = n$ and so if we blow up C at P to get a curve C_1, we get $\ell(\varphi_* O_{C_1}/O_C) = \frac{n(n-1)}{2}$ **34**
So C_1 is isomorphic to $\overline{C}$ *i.e.* C gets desingularised in one blowing up. (cf. Fig. 3)

(2) Take a curve with ordinary singularities, i.e. each branch at any point is non-singular and it has distinct tangents for each branch. Since in the blowing up the number of points on the exceptional divisor corresponds to these tangents, the curve is non-singular at those points. So be blowing up the singular points once we get the desingularisation (cf. Fig. 2)

Now we will calculate the conductor of D_i over D in terms of the relative conductors. Let $\underline{f}$ be the conductor of the map $D_i \xrightarrow{\varphi_1 \circ \ldots \circ \varphi_i} D$ and $\underline{f}_j$ the conductor of the map $D_j \longrightarrow D_{j-1}$ Then we claim:

$$\underline{f} = \underline{f}_1 . \varphi_{1*} \underline{f}_2 . (\varphi_1 \circ \varphi_2)_* \underline{f}_3 \ldots (\varphi_1 \circ \varphi_2 \ldots \circ \varphi_{i-1})_* \underline{f}_1 .$$

There is a canonical inclusion of the ideal sheaf on the right hand side into the ideal sheaf $\underline{f}$. So to prove equality one only has to show it locally.

So we have a sequence of semilocal rings. $A_\circ \subset A_1 \subset A_2 \subset \ldots \subset A_i$ the corresponding morphism of Spec A_j's are finite and birational. Let the same letters denote the conductors. We have $\underline{f}_1.\underline{f}_2 \ldots \underline{f}_i \subset \underline{f}.\underline{f}$ is an ideal of $A_\circ$. Let $a \in \underline{f}$. So $a.A_i \subset A_\circ$.

Since A_i's are semilocal, relative conductors are principal above, i.e. $\underline{f}_j = t_j A_j$ where t_j. belongs to $\underline{f}_j$

We will show that at_1^{-1} is an element of the relative conductor of $A_1 \subset A_i$ and by induction we will be through. We therefore want to show
35 that $aA_i \subset t_1 A_1$. Since $A_1 \subset A_i$ and $aA_i \subset A_\circ$, we see that any element of a A_i annihilates $A_1/A_\circ$ and hence $aA_i \subset t_1 A_1 = \underline{f}_1$. *i.e.* $at_1^{-1}.A_i \subset A_1$. So by induction we see that $at_1^{-1}.t_2^{-1} \ldots t_{i-1}^{-1}.A_i \subset A_{i-1}$ *i.e.* $at_1^{-1}t_2^{-1} \ldots t_{i-1}^{-1}$ belongs to $\underline{f}_i$, which implies

$$a \quad \text{belongs to} \quad \underline{f}_1 \ldots \underline{f}_1$$
$$\text{i.e. } \underline{f} = \underline{f}_1 \ldots \underline{f}_i.$$

REMARK. *D_i can also be obtained by blowing up D at the conductor on the map $D_i \longrightarrow D$.*

DEFINITION. *Let D be a reduced irreducible curve. The set of infinitely near points of D is equal to the disjoint union of closed points of curves D which can be obtained from D by a finite sequence of blowing-ups of closed points.*

7 Adjoint Systems

Let X be a non-singular surface, L a line bundle on X. Let D be a reduced irreducible curve on X. Let $\bar{D}$ be its normalisation. Let $\underline{f}$ be the conductor of the map $\bar{D} \longrightarrow D$. So that $\underline{f}$ is an ideal sheaf in O_D. Since there is a surjection $O_X \longrightarrow O_D$, there exists a unique ideal sheaf $\underline{F}$ in O_X such that $O_{X/\underline{F}} = O_{D/\underline{f}}$. ($\underline{F}$ is the inverse image of $\underline{f}$ in O_X).

DEFINITION. *Let L be a line bundle on X. An effective divisor D_1 on X is an adjoint of type L with respect to D if $O_X(D_1) = L$ and the corresponding element in $H^\circ(X, L)$ is in the sublinear system $H^\circ(X, \underline{F} \otimes L)$.*

PROPOSITION. *Let X be a regular surface and D, a reduced irreducible divisor on X. Suppose that the normalisation $\overline{D}$ of D is finite over D. A divisor D' on X is an adjoint of D of type L if and only if the* 36
following conditions are satisfied.

i) $O_X(D') = L$

ii) For any finite chain of blowing-ups of closed points,

$$X_s \longrightarrow X_{s-1} \longrightarrow \ldots \longrightarrow X_1 \longrightarrow X_\circ = X,$$

if we denote by $f_{s,j}$ the morphism from X_s to $X_j (j < s)$ then for any point Q in X_s,

$$m_Q(f^*_{s,\circ}D' + f^*_{s,1}E_1 + \cdots + f^*_{s,s-1}E_{s-1} + E_s) \geq m_Q(f^*_{s,\circ}D) - 1$$

where $(*)$ denotes the total transform and E_j, the last exceptional divisor of X_j.

The proof is simple and is carried out by induction on s.

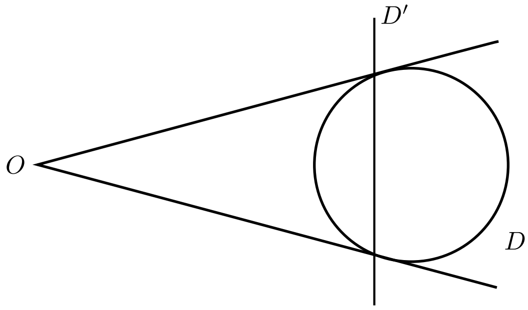
REMARKS. *1. If D' satisfies the property that*

$$m_Q(D'_s) \geq m_Q(D_s) - 1,$$

where D_s and D'_s are proper transforms of D and D' in X_s, then D' is an adjoint of D. This can be easily seen by showing that this condition is stronger than the condition in the Proposition.

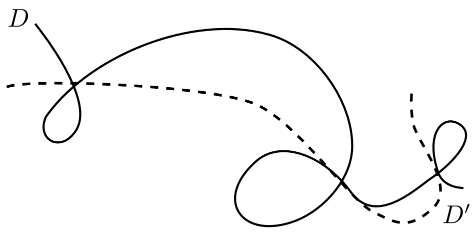
2. If all singularities of D become desingularized at the first blowing up, then the two conditions are actually equivalent.

EXAMPLE 1. Polar Curves.



37 Let D be a projective plane integral curve of characteristic zero. Let 0 be a point in the plane with homogeneous co-ordinates a_0, a_1, a_2. If F is the homogeneous equation of D the polar curve D' of D with respect to 0 is the curve with the equation $\sum a_i \frac{\partial F}{\partial X_i}$ The polar curve is an adjoint of type $O_{\mathbb{P}^2}(m-1)$ where m is the degree of D.

EXAMPLE 2. Suppose that D has only ordinary double points as singularities. Then a curve D' is an adjoint of D if and only if it passes through all singular points of D.



38 **Exercise .** (Max Noether theorem: $AF + BG$). Let F and G be homogeneous polynomials defining two curves (F) and (G) in $\mathbb{P}^2$. Suppose that (F) is reduced and irreducible and is not a component of (G). Then a homogeneous polynomial H is of the form $AF + BG$, where A and B are polynomials, if for any infinitely near point Q of (F), $m_Q(F.H) \geq m_Q(F.G)$.

(Hint:Prove that there exists such a B which defines an adjoint curve of (F))

8 Gorenstein Theorem

THEOREM. *Let C be an integral curve in $\mathbb{P}^2$ of degree m. Then the adjoint curves of C of type $O_{\mathbb{P}^2}(m-3)$ cut out on C, the complete canonical system of $\bar{C}$, where $\bar{C}$ is the normalisation of C.*

Proof. Let $\underline{f}$ be the conductor of $\bar{C} \longrightarrow C$ and $\underline{F}$ the inverse image of $\underline{f}$ under the map $O_{\mathbb{P}^2} \longrightarrow O_C$. Then by the previous proposition we know that the adjoints of type $O_{\mathbb{P}^2}(m-3)$ are zeros of section of

$\underline{F}(m-3)$. So the theorem states that the image of $H^\circ(\mathbb{P}^2, \underline{F}(m-3)) \longrightarrow H^\circ(C, \omega_C)$ under the canonical map is equal to the image of $H^\circ(\bar{C}, \omega_{\bar{C}})$ in $H^\circ(C, \omega_C)$.

We have a commuting diagram of exact sequences as follows:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & O_{\mathbb{P}^2}(-m) & = & O_{\mathbb{P}^2}(-m) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & F & \longrightarrow & O_{\mathbb{P}^2} & \longrightarrow & O_{C}/\underline{f} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \underline{f} & \longrightarrow & O_C & \longrightarrow & O_{C}/\underline{f} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Tensoring the diagram by $O_{\mathbb{P}^2}(m-3)$ we get a commutative diagram 39
of exact sequences:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & O_{\mathbb{P}^2}(-3) & = & O_{\mathbb{P}^2}(-3) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \underline{F}(m-3) & \longrightarrow & O_{\mathbb{P}^2}(m-3) & \longrightarrow & O_{C}/\underline{f} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \underline{f}(m-3) & \longrightarrow & O_C(m-3) & \longrightarrow & O_{C}/\underline{f} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Since C is a divisor on $\mathbb{P}^2$,

$$\begin{aligned}\omega_C &= (\mathcal{O}_{\mathbb{P}^2}(m) \otimes \mathcal{O}_{\mathbb{P}^2}(-3))/\mathcal{O}_C \\ &= \mathcal{O}_C(m-3)\end{aligned}$$

Again $\varphi_*\omega_{\bar{C}} = \omega_C \otimes \underline{f} = \underline{f}(m-3)$. So we have a diagram:

$$\begin{array}{ccccc}
 & & 0 & & 0 \\
 & & \downarrow & & \downarrow \\
 & & \mathcal{O}_{\mathbb{P}^2}(-3) & = & \mathcal{O}_{\mathbb{P}^2}(-3) \\
 & & \downarrow & & \downarrow \\
0 & \longrightarrow & \underline{F}(m-3) & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(m-3) \\
 & & \downarrow & & \downarrow \\
0 & \longrightarrow & \varphi_*\omega_{\bar{C}} & \longrightarrow & \mathcal{O}_C(m-3) = \omega_C \\
 & & \downarrow & & \downarrow \\
 & & 0 & & 0
\end{array}$$

40 Thus we get a diagram of exact sequences:

$$\begin{array}{ccccc}
0 & \longrightarrow & H^\circ(\mathbb{P}^2, \underline{F}(m-3)) & \longrightarrow & H^\circ(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m-3)) \\
 & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^\circ(C, \varphi_*\omega_{\bar{C}}) & \longrightarrow & H^\circ(C, \omega_C) \\
 & & \downarrow & & \\
 & & H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3)) & &
\end{array}$$

Since $H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3)) = 0$, the map

$$H^\circ(\mathbb{P}^2, \underline{F}(m-3)) \longrightarrow H^\circ(C, \varphi_*\omega_{\bar{C}}) = H^\circ(\bar{C}, \omega_{\bar{C}})$$

is a surjection, which proves the theorem. $\square$

9 Regularity of the Adjoint System

Let C be an integral curve on a smooth projective surface X over an algebraically closed field. Let $\bar{C}$ be the normalisation of C and $\underline{f}$ the conductor. Let $\underline{F}$ be the inverse image of $\underline{f}$ under the map $O_X \longrightarrow O_{\bar{C}}$. So we have a map $\underline{F} \longrightarrow O_{\bar{C}}$. Therefore we have a map,

$$\underline{F} \otimes O_X(C) \otimes \omega_X \longrightarrow O_C \otimes O_X(C) \otimes \omega_X = \omega_C,$$

which in turn gives a map,

$$H^\circ(X, \underline{F} \otimes \omega_X(C)) \longrightarrow H^\circ(C, \omega_C).$$

Let V be the image of this map. We denote by q the irregularity of X:

$$q = \dim H^1(X, O_X) = \dim H^1(X, \omega_X).$$

DEFINITION. *The adjoint system is called regular if*

$$\dim H^\circ(\bar{C}, \omega_{\bar{C}}) - \dim V = q.$$

PROPOSITION . *If the adjoint system of C is regular then $H^1(X,$ 41
$O_X(-C)) = 0$.*

Proof. We have commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & O_X(-C) & = & O_X(-C) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \underline{F} & \longrightarrow & O_X & \longrightarrow & O_{C/\underline{f}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \underline{f} & \longrightarrow & O_C & \longrightarrow & O_{C/\underline{f}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

which gives a commutative diagram of exact sequences, by tensoring with $\omega_X(C)$.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \omega_X & = & \omega_X & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \underline{F} \otimes \omega_X(C) & \longrightarrow & \omega_X(C) & \longrightarrow & O_{C/\underline{f}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \underline{f} \otimes \omega_C & \longrightarrow & \omega_C & \longrightarrow & O_{C/\underline{f}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Taking cohomologies we have the following diagram of exact sequences:

$$
\begin{array}{ccccc}
 & & 0 & & 0 \\
 & & \downarrow & & \downarrow \\
 & & H^{\circ}(X,\omega_X) & = & H^{\circ}(X,\omega_X) \\
 & & \downarrow & & \downarrow \\
0 & & H^{\circ}(\underline{F}\otimes\omega_X(C)) & \longrightarrow & H^{\circ}(X,\omega_X(C)) \\
 & & \downarrow & & \downarrow \\
0 & & H^{\circ}(\bar{C},\omega_{\bar{C}}) & \longrightarrow & H^{\circ}(C,\omega_C) \\
 & & \downarrow & & \downarrow \\
 & & H^1(X,\omega_X) & = & H^1(X,\omega_X) \\
 & & & & \downarrow \\
 & & & & H^1(X,\omega_X(C)) \\
 & & & & \downarrow \\
 & & & & H^1(C,\omega_C) \\
 & & & & \downarrow \\
 & & & & H^2(X,\omega_X)) \\
 & & & & \downarrow \\
 & & & & H^2(X,\omega_X(C)) \\
 & & & & \downarrow \\
 & & & & 0
\end{array}
$$

42

First $H^2(X,\omega_X(C)) = H^{\circ}(X,O_X(-C)) = 0$. Again $H^1(C,\omega_C^{\nu}) = H^{\circ}(C,O_C) = k$ and $H^2(X,\omega_X^{\nu}) = H^{\circ}(X,O_X) = k$, and since the map $H^1(C,\omega_C) \longrightarrow H^2(X,\omega_X)$ is a surjection it must be an isomorphism. So the map $H^1(X,\omega_X) \longrightarrow H^1(X,\omega_X(C))$ is surjection.

The regularity of the adjoint system implies that the map $H^{\circ}(\bar{C}, \omega_{\bar{C}}) \longrightarrow H^1(X,\omega_X)$ is surjective. So we see that the map $H^{\circ}(C,\omega_C) \longrightarrow$

, $H^1(X, \omega_X)$ is surjective. The exactness of the sequence implies that $H^1(X, \omega_X(C)) = 0$, i.e.

$$H^1(X, O_X(-C)) = 0.$$

43 There are much stronger results in this direction in characteristic zero by Kodaira [2] and others. □

THEOREM. (Kodaira). *Let X be a Kählarian variety, L an ample line bundle on X. Then $H^i(X; L^{-1}) = 0$ for $i = 0, 1, \ldots \dim X - 1$.*

The reader can find similar results and an introduction to the above theorem in D. Mumford [3]. There are several results in this direction, for e.g. see [1], [4], [5], [6] and [7]. But note that all these results are for characteristic zero.

10 A Difference Between Characteristic Zero and Characteristic p

In this section, as before, X is a smooth connected surface, projective over a field k, C a reduced irreducible curve on $X, \overline{C}$, the normalisation of C and $\overline{X}$, the surface obtained after a finite number of dilations of points from X, such that the proper transform of C on X is $\overline{C}$. We will denote by $\underline{F}$, the sheaf of adjoint curves to C.

PROPOSITION. *Let $X' \longrightarrow X$ be the blowing up of a closed point P of X, C', the proper transform of $C, \underline{F}'$, the sheaf of adjoint curves to C. There is a canonical isomorphism,*

$$H^1(X', \underline{F}' \otimes \omega_X(C')) \longrightarrow H^1(X, \underline{F} \otimes \omega_X(C)).$$

In other words, $H^1(X, \underline{F} \otimes .\omega_X(C))$ is a birational invariant (equal to $H^1(\overline{X}, \omega_{\overline{X}}(\overline{C})) \simeq H^1(\overline{X}, O_{\overline{X}}(-C))$.

Proof. Let M be the sheaf of ideals defining P in X and r, the multiplicity of C at P. By the Leray spectral sequence for the morphism,
44 $g : X' \longrightarrow X$, it will be enough to prove that

$$g_*(\underline{F}' \otimes \omega_{X'}(C')) = \underline{F} \otimes \omega_X(C) \tag{i}$$

and

$$R^1 g_*(\underline{F}' \otimes \omega_{X'}(C')) = 0 \tag{ii}$$

We know that, $\omega_{X'} = g^* \omega_X \otimes O_{X'}(E)$, where E is the exceptional divisor.

$$O_{X'}(C') = g^* O_X(C) \otimes O_{X'}(-rE).$$

So to prove (ii), it is enough to have

$$R^1 g_*(\underline{F}' \otimes O_{X'}(-(r-1)E)) = 0 \tag{ii}$$

If $\underline{f}'$ is the conductor of C', one has the exact sequence,

$$0 \longrightarrow O_{X'}(-C') \longrightarrow \underline{F}' \longrightarrow \underline{f}' \longrightarrow 0$$

Tensoring this by $O_{X'}(-(r-1)E)$, one gets

$$\begin{aligned} 0 \longrightarrow O_{X'}(-C' - (r-1)E) &\longrightarrow \underline{F}' \otimes O_{X'}(-(r-1)E) \\ \longrightarrow \underline{f}' \otimes O_{X'}(-(r-1)E) &\longrightarrow 0 \end{aligned} \tag{*}$$

to be exact. Since the morphism $C' \longrightarrow C$ is affine, from the long exact sequence for derived functors, we see that it is sufficient to prove,

$$R^1 g_*(O_{X'}(-C' - (r-1)E)) = 0 \tag{iii}$$

But $R^1 g_*$ commutes with base change, since it is the last non-zero cohomology functor. So it will be enough to prove that,

$$O_{X'}(-C' - (r-1)E)_{/E} \simeq O_{\mathbb{P}^1}(-1).$$

This is evident, because C'/E is equal to r points counted with multiplicity and hence $O_{X'}(-C')/E = O_{\mathbb{P}^1}(-r)$ and $O_{X'}(-(r-1)E)/E = O_{\mathbb{P}^1}(r-1)$.

It remains to prove (i). By the adjunction formula, it is enough to **45**
prove,

$$g_*(\underline{F}' \otimes O_{X'}(-(r-1)E)) = \underline{F}.$$

By (*), it is sufficient to prove that,

$$g_*(\underline{f}' \otimes O_{X'}(-(r-1)E)) = \underline{f} \tag{iv}$$

and

$$g_*(O_{X'}(-C' - (r-1)E)) = O_X(-C) \tag{v}$$

Let $\bar{g} : \bar{X} \longrightarrow X'$ be the composite of dilations and let $h = g \circ \bar{g}$ one has,

$$\begin{aligned} \underline{f} &= h_* \omega_{\bar{C}} \otimes \omega_C^{-1} = g_*(\bar{g}_* \omega_{\bar{C}}) \otimes \omega_C^{-1} \\ &= g_*(\underline{f}' \otimes \omega_C \otimes g^* \omega_C^{-1}). \end{aligned}$$

We have seen in § 5, that $\omega_{C'} = g^* \omega_C \otimes O_{C'}(-(r-1)E)$ which when substituted in the above expression gives (iv).

To prove (v), consider the exact sequence,

$$0 \longrightarrow O_{X'}(-C' - (r-1)E) \longrightarrow O_{X'}(-(r-1)E) \longrightarrow O_{C'}(-(r-1)E) \longrightarrow 0.$$

Applying g_* and since $R^1 g_*(O_{X'}(-C' - (r-1)E) = 0$, we see that, $0 \longrightarrow g_* O_{X'}(-C' - (r-1)E) \longrightarrow g_* O_{X'}(-(r-1)E) \longrightarrow g_* O_{C'}(-(r-1)E) \longrightarrow 0$ is exact.

If we denote by $\bar{M}$, the ideal sheaf of P in C, this sequence is same as

$$0 \longrightarrow g_* O_{X'}(-C' - (r-1)E) \longrightarrow M^{r-1} \longrightarrow \bar{M}^{r-1} \longrightarrow 0.$$

46 and the map we have from M^{r-1} to $\bar{M}^{r-1}$ is the canonical map. Therefore, since $O_X(-C) \hookrightarrow M^{r-1}$, one has, $g_*(O_{X'}(C' - (r-1)E)) = O_X(-C)$, proving (v), and thus proving the Proposition.

The next proposition proves that $H^1(X, \omega_X(C))$ is a birational invariant in characteristic zero. □

PROPOSITION. *Assume, in addition to the notations in the beginning of this section, that char $k = 0$. Let $X' \longrightarrow X$ be a dilation with centre P and C' the proper transform of C. Then the canonical map,*

$$H^1(X', \omega_{X'}(C')) \to H^1(X, \omega_X(C)) \quad \textit{is an isomorphism}.$$

Proof. By duality, we need only prove that the canonical map,

$$H^1(X, O_X(-C)) \longrightarrow H^1(X', O_{X'}(-C'))$$

is an isomorphism.

Because C and C' are connected, we have exact sequences,

$$0 \longrightarrow H^1(X, O_X(-C)) \longrightarrow H^1(X, O_X) \longrightarrow H^1(C, O_C)$$

and

$$0 \longrightarrow H^1(X', O_{X'}(-C')) \longrightarrow H^1(X', O_{X'}) \longrightarrow H^1(C', O_{C'}).$$

So $H^1(X, O_X(-C))$ $(resp.\ H^1(X', O_{X'}(-C'))$ is the tangent space at identity of the connected component of the kernel of the map $\alpha : \mathrm{Pic}^\circ(X) \longrightarrow \mathrm{Pic}^\circ(C)(resp.\ \alpha : \mathrm{Pic}^\circ(X') \longrightarrow .\ \mathrm{Pic}^\circ(C'))$. Call $K(resp.\ K')$ the Kernel of $\alpha(resp.\ \alpha)$. We have a commutative diagram of exact sequences

$$\begin{array}{ccccccc}
0 & \longrightarrow & K & \longrightarrow & \mathrm{Pic}^\circ(X) & \longrightarrow & \mathrm{Pic}^\circ(C) \\
 & & \downarrow & & \| & & \downarrow \\
0 & \longrightarrow & K' & \longrightarrow & \mathrm{Pic}^\circ(X') & \longrightarrow & \mathrm{Pic}^\circ(C')
\end{array}$$

The map $K \longrightarrow K'$ is an inclusion and $\mathrm{Pic}^\circ(C) \longrightarrow \mathrm{Pic}^\circ(C')$ is a **47** surjection. Thus $K'/K = \mathrm{Ker}(\mathrm{Pic}^\circ C \longrightarrow \mathrm{Pic}^\circ C')$. Bur we know that $\mathrm{Pic}^\circ C$ is an extension of $\mathrm{Pic}^\circ C'$ by an affine group, and K'/K is an abelian variety. Thus K'/K is discrete.

Thus the tangent map at identity of the connected components of K and K' is an isomorphism, since we are in characteristic zero, i.e. $H^1(X, O_X(-C)) \longrightarrow H^1(X', O_{X'}(-C'))$ is an isomorphism.

Now we show that the above proposition is not valid in characteristic $p > 0$. ◻

A counter example in characteristic $p > 0$. For this counter example we make use of a surface constructed by J.P. Serre in his Mexico Lecture

[15] Let k be a field of characteristic $p > 5$, and G the group generated by the 4×4 matrix,

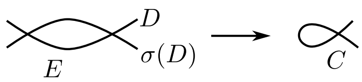
$$\sigma = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Order of σ is p, and G acts on $\mathbb{P}^3_k$. G has only one fixed point. So take a smooth invariant hypersurface Y in $\mathbb{P}^3$, which does not pass through this fixed point. The surface $X = Y^G$ is then smooth, and $r : Y \longrightarrow X$ is an etale cover of degree p. This cover corresponds to a non-trivial element $\alpha \in H^1(X, O_X)$ which is fixed by the Frobenius. (In fact Serre proves in the quoted paper, that α generates $H^1(X, O_X)$ as a k-vector space.)

48 Since Y is invariant under G, the natural isomorphism,

$$H^\circ(\mathbb{P}^3, O_{\mathbb{P}}(1)) \longrightarrow H^\circ(Y, O_Y(1))$$

is a G-morphism Since G does not fix a general hyper-plane, it is easy to see that a generic curve D cut out by a hyper-plane on Y is smooth, irreducible and $\sigma(D) \neq D$. Take $E = \bigcup_{n=0}^{p-1} \sigma^n(D)$ Then E is connected, since D is ample. Let $C = E^G$. Then C is an irreducible curve on X, birationally isomorphic to D. We claim that

i) C is singular 

ii) $H^1(X, O_X(-C)) = 0$.

iii) $H^1(X, O_{\overline{X}}(-\overline{C})) \neq 0$, where as usual $\overline{X}$ is obtained from X by successive blowing ups of points and C, the proper transform of $\overline{C}$ on X is non-singular.

Proof. i) Every point of intersection of $\sigma^i(D)$'s gives a singular point of C.

ii) $H^1(X, O_X(-C))$ is the kernel of the map $H^1(X, O_X) \longrightarrow H^1(C, O_C)$. But since $\alpha \in H^1(X, O_X)$ generates $H^1(X, O_X)$ and since the image of α in $H^1(C, O_C)$ corresponds to the cover $E \longrightarrow C$, which is non-trivial, $H^1(X, O_X(-C)) = 0$.

iii) First note that blowing up a point P on X is equivalent to the following: blow up Y at all point of $\pi^{-1}(P)$. Then G still acts on this surface and so take the quotient. So we could blow up Y at all points of intersection of $\sigma^i(D)$'s sufficiently many times **49**
to obtain a surface Y, so that $\bar{E}$ the proper transform of E has become a disjoint union of non-singular irreducible curves on Y. Let $X = Y^G$. Then it is easy to see that $\bar{E}^G = \bar{C}$ where $\bar{C}$ is the smooth model of C. Now the cover $\bar{Y} \longrightarrow \bar{X}$ is still etale and hence gives a non-trivial element $\beta \in H^1(X, O_X)$. But then image of β in $H^1(C, O_C)$ is trivial since the corresponding cover $\bar{E} \longrightarrow \bar{C}$ is trivial. Thus $\beta \in H^1(\bar{X}, O_{\bar{X}}(-\bar{C}))$ and hence $H^1(\bar{X}, O_{\bar{X}}(-\bar{C})) \neq 0$. This proves our claim.

□

REMARK. *1. $O_{\bar{X}}(\bar{C})$ is not ample because a non-zero element of $H^1(\bar{X}, \bar{O}_X)$ which is stable under Frobenius and belongs to $H^1(X, \bar{O}_{\bar{X}}(-\bar{C}))$ gives a non-zero element in $H^1(\bar{X}, O_{\bar{X}}(-p^n\bar{C}))$ for any $n \in \mathbb{N}$. So $H^1(\bar{X}, O_{\bar{X}}(-m\bar{C}))$ is not zero for $m \gg 0$, which contradict Zariski-Enriques-Severi lemma.*

2. In a way similar to the counter example above, one can construct another class of counter-examples.

Let Y be any abelian surface with p-rank positive. Then there exists a subgroup G of order p in Y. Let $X = Y^G$ As before a general ample curve D on Y can be shown to be smooth, irreducible and not fixed by G. Let C be the image of such a curve in X. Then all the arguments we had before can be carried out for the surface X and the curve C, to give another example.

Chapter 3

Castelnuovo's Theorems

1

IN THIS CHAPTER we will be proving two theorems of Castelnuovo. **50**
In the proof we will use a theorem of Bertini which we will not prove (cf. for example Hartshorne's book).

THEOREM. (Bertini) *Let C be a smooth curve embedded in $\mathbb{P}^r$, $r \geq 3$. Also assume that C is not contained in any hyperplane in $\mathbb{P}^r$. Then for generic hyperplane H of $\mathbb{P}^r$, $H \cap C$ is reduced and any r points of $H \cap C$ generate H as a linear space.*

REMARK. *If C is contained in $\mathbb{P}^3$, where C is a smooth integral curve, Bertini's theorem implies that, there exists a point in $\mathbb{P}^3$, such that the projection of C from this point onto a curve C' in $\mathbb{P}^2$ is a birational isomorphism and C' has only ordinary double points as singularities.*

We will now prove a lemma, which will be the lemma for our proofs of Castelnuovo's theorems.

LEMMA 1. *Let S be a set of $(r-1)k+1$ points in $\mathbb{P}^{r-1}$ such that any r points of S generate $\mathbb{P}^{r-1}$. Then, the canonical map $H^\circ(\mathbb{P}^{r-1}, O_{\mathbb{P}^{r-1}}(k)) \longrightarrow H^\circ(S, O_S(k)$ is surjective.*

Proof. The statement of the lemma is equivalent to proving that for any P in S, there exists a homogeneous polynomial F of degree k (in the r

variables corresponding to the homogeneous co-ordinates of $\mathbb{P}^{r-1}$) such that F vanishes at every point of S other than P and $F(P)$ is not equal to zero. By the hypothesis, the subspace generated by any $(r-1)$ points in S
51 is a hyperplane. We partition $S-P$ into k parts consisting of $(r-1)$ points each. So there exists homogeneous linear polynomials $(F_i) 1 \leq i \leq k$, which define these k hyperplanes. So $F_i(P) \neq 0$ for every i and for any $Q \in S - \{P\}$, there exists some F_j such that $F_j(Q) = 0$. Taking $F = \prod_{i=1}^{k} F_i$, we see that our requirements are met. □

COROLLARY 1. *If $S' \subset S, S$ as above, then the canonical map $H^\circ(\mathbb{P}^{r-1}, O_{\mathbb{P}^{r-1}}(k)) \longrightarrow H^\circ(S', O_{S'})$ is surjective.*

COROLLARY 2. *Let C be a smooth curve of degree d embedded in $\mathbb{P}^r$ and H a hyperplane as in Bertini's theorem (i.e. $H \cap C$ is a finite set of reduced points and any r points of $H \cap C$ generate H). Define*

$$t_n = \operatorname{rank}(H^\circ(\mathbb{P}^r, O_{\mathbb{P}^r}(n)) \longrightarrow H^\circ(C \cap H, O_{C\cap H}(n)).$$

Then $t_n \geq \inf(d, (r-1)n + 1)$.

Proof. If $d \leq (r-1)n + 1$, we take the d points of $H \cap C$ and choose $((r-1)n+1-d)$ points on H such that any r points from these $((r-1)n+1)$ points generate H. (This is always possible, since the d points of $H \cap C$ have this property.) Denote this set by S.S satisfies the conditions of the lemma. Since $C \cap H \subset S$, by Corollary 1, we get $t_n \geq d$.

If $d > (r-1)n + 1$ and if S is a subset of $C \cap H$, consisting of $((r-1)n+1)$ points, the image of $H^\circ(\mathbb{P}^r, O_{\mathbb{P}^r}(n))$ in $H^\circ(C \cap H, O_{C\cap H}(n))$ contains the subspace $H^\circ(S, O_S(n))$ and hence $t_n \geq (r-1)n + 1$. □

2

52

a) First Theorem of Castelnuovo

THEOREM 1. *Let C be a smooth curve embedded in $\mathbb{P}^r, r \geq 3$, and let degree of C be d. Assume that C is not contained in any hyperplane. Define χ as the largest integer such that,*

$$(r-1)\chi + 1 < d.$$

Then $H^1(C, O_C(n)) = 0$ *for every* $n \geq \chi$.

Proof. Choose H, a hyperplane of $\mathbb{P}^r$, as in Bertini's theorem. So $H \cap C$ is reduced and any r points in $H \cap C$ generate H. So by lemma $H^\circ(H, O_H(m)) \longrightarrow H^\circ(C \cap H, O_{C\cap H}(m)) \longrightarrow 0$ is exact for every $m > \chi$.

We have a commutative diagram of exact sequences:

$$\begin{array}{ccccc}
H^\circ(\mathbb{P}^r, O_{\mathbb{P}^r}(m)) & \longrightarrow & H^\circ(H, O_H(m)) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
H^\circ(C, O_C(m)) & \xrightarrow{\varphi} & H^\circ(C \cap H, O_{C\cap H}(m)) & & \\
 & & \downarrow & & \\
 & & 0 & &
\end{array}$$

for $m > \chi$. So we see that φ is surjective for $m > \chi$. Since H is a hyperplane, we have an exact sequence, $0 \longrightarrow (-1) \longrightarrow\longrightarrow_H\longrightarrow 0$, which tensoring by O_C gives an exact sequence,

$$0 \longrightarrow O_C(-1) \longrightarrow O_C \longrightarrow O_{C\cap H} \longrightarrow 0.$$

(The sequence is exact at the left because C is integral and C is not contained in H). So for any m in $\mathbb{Z}$ we get an exact sequence

$$0 \longrightarrow O_C(m-1) \longrightarrow O_C(m) \longrightarrow O_{C\cap H}(m) \longrightarrow 0.$$

The corresponding exact sequence of cohomologies is,

$$0 \longrightarrow H^\circ(O_C(m-1)) \longrightarrow H^\circ(O_C(m)) \xrightarrow{\varphi} H^\circ(O_{C\cap H}(m))$$
$$\longrightarrow H^1(O_C(m-1))H^1(O_C(m)) \longrightarrow 0.$$

53

Since φ is surjective for $m > \chi$, we see that $H^1(O_C(m-1) \xrightarrow{\sim} H^1(O_C(m))$ for $m > \chi$. Since $H^1(O_C(n)) = 0$ for large n, we see that, $H^1 O_C(m)) = 0$ for every $m > \chi$. Q.E.D

Define $d_n = \dim H^1(\mathbb{P}^r, J(n))$ where J is the sheaf of ideals of the curve C (as above) in $\mathbb{P}^r$. □

LEMMA 2. *If $n \geq \chi$, then $d_n \geq d_{n+1}$.*

Proof. Let H be a hyperplane as before. We have an exact sequence,

$$0 \longrightarrow O(-1) \longrightarrow O \longrightarrow O_H \longrightarrow 0.$$

Tensoring by J which is a torsion free 0-module, we get an exact sequence,

$$0 \longrightarrow J(-1) \longrightarrow J \longrightarrow J_H \longrightarrow 0.$$

So for any integer n we have the exact sequence,

$$0 \longrightarrow J(n) \longrightarrow J(n+1) \longrightarrow J_H(n+1) \longrightarrow 0,$$

which in turn gives an exact sequence of cohomologies,

$$H^1(\mathbb{P}^r, J(n)) \xrightarrow{\varphi_n} H^1(\mathbb{P}^r, J(n+1)) \longrightarrow H^1(\mathbb{P}^r, J_H(n+1)) \qquad (*)$$

So cokernel of φ_n is contained in $H^1(\mathbb{P}^r, J_H(n+1))$ for every n.

Tensoring the exact sequence $0 \longrightarrow J \longrightarrow O \longrightarrow O_C \longrightarrow 0$ by $_H$, we get the sequence, $0 \longrightarrow J_H \longrightarrow O_H \longrightarrow O_{C\cap H} \longrightarrow 0$, to be exact because J is a prime ideal. so we get an exact sequence of cohomologies,

$$H^\circ(\mathbb{P}^r, O_H(n+1)) \longrightarrow H^\circ(\mathbb{P}^r, O_{C\cap H}(n+1)) \longrightarrow H^1(\mathbb{P}^r, J_H(n+1)$$
$$H^1(\mathbb{P}^r, O_H(n+1)) \longrightarrow 0, \quad \text{for every} \quad n \in \mathbb{Z}.$$

54 $H^1(\mathbb{P}^r, O_H(n+1) = 0$ for $n \geq 0$. Also by lemma 1,

$$H^\circ(\mathbb{P}^r, O_H(n+1)) \longrightarrow H^\circ(\mathbb{P}^r, O_{C\cap H}(n+1))$$

is surjective for $n \geq \chi$. Thus $H^1(\mathbb{P}^r, J_H(n+1)) = 0$ for $n \geq \chi$. So from (*), we see that φ_n is surjective for $n \geq \chi$. *i.e.* $d_n \geq d_{n+1}$ for $n \geq \chi$. ◻

b) **COROLLARY (M. Noether).** Let C be a smooth curve of genus greater than or equal to 3 and non-hyperelliptic (i.e. ω_C is very ample.) Then the canonical embedding is arithmeticaly normal.

Proof. Since $\dim H^{\circ}(C, \omega_C) = g =$ genus of C, the canonical embedding is $C \longrightarrow \mathbb{P}^{g-1}$ and degree of $C = 2g-2$. If $g = 3$, then $C \hookrightarrow \mathbb{P}^2$ and hence C is a complete intersection and in particular arithmetically normal. So we can assume g is greater than 3. Arithmetic normality is equivalent to $O_C(n)$ being complete for every $n \geq 0$. $O_C(1)$ is complete by hypothesis. Now we will calculate χ for this curve

$$(g-2)\chi + 1 < 2g-2$$
$$\text{i.e.} \qquad \chi < \frac{2g-3}{g-2} = 2 + \frac{1}{g-2}$$

and so $\chi = 2$. By lemma 2, we know that $d_n \geq d_{n+1}$ for $n \geq 2$. So if we show that $d_2 = 0$, then $O_C(n)$ is complete for every $n \geq 0$.

Choosing H as before, we have an exact sequence, $0 \longrightarrow O(-1) \longrightarrow O \longrightarrow O_H \longrightarrow 0$ Therefore, $0 \longrightarrow O_C(1) \longrightarrow O_C(2) \longrightarrow O_{C\cap H}(2) \longrightarrow 0$ is exact. Thus we get a commutative diagram of exact sequences:

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \uparrow & & & & \\
0 & \longrightarrow & H^{\circ}(C, O_C(1)) & \xrightarrow{i} & H^{\circ}(C, O_C(2)) & \xrightarrow{\pi} & H^{\circ}(C\cap H, O_{C\cap H}(2)) \\
 & & \uparrow f & & \uparrow \overline{g} & \nearrow h & \\
0 & \longrightarrow & H^{\circ}(\mathbb{P}^r, O_{\mathbb{P}^r}(1)) & \longrightarrow & H^{\circ}(\mathbb{P}^r, O_{\mathbb{P}^r}(2)) & &
\end{array}$$

rank of $h = t_2$. By Corollary 2 of lemma 1, we have, $t_2 \geq \inf(2g-2, (g-$ 55
$2).2+1) = 2g-3$. Since f is surjective, Image of i is contained in Image of $\overline{g}$. Dimension of Image of $i = g$. $\dim H^{\circ}(C, O_C(2)) = 4g-4-g+1 = 3g-3$ by Riemann-Roch theorem. Thus

$$\text{rank of } h = \text{rank of } \overline{g} - \dim(\operatorname{Im} i) = (\text{rank of } \overline{g}) - g.$$

Hence (rank of $\overline{g}$) $= t_2 + g \geq 2g-3+g = 3g-3$. Thus $\overline{g}$ is surjective and $d_2 = 0$. $\square$

LEMMA 3. *Let χ and d_n be as before. If $n \geq \chi + 1$ then, $d_n > d_{n+1}$ or $d_n = 0$.*

Proof. 1. Choose H as before. Now choose another hyperplane H_1 such that $H \cap H_1 \cap C = \phi\emptyset$. Let $f = 0$ and $g = 0$ be their respective equations. Let J be the sheaf of ideals of C in $\mathbb{P}^r$ and consider the Koszul complex with respect to (f, g):

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J(n-1) & \longrightarrow & J(n) \oplus J(n) & \longrightarrow & J(n+1) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & O_{\mathbb{P}}(n-1) & \longrightarrow & O_{\mathbb{P}}(n) \oplus O_{\mathbb{P}}(n) & \longrightarrow & O_{\mathbb{P}}(n+1) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & O_C(n-1) & \longrightarrow & O_C(n) \oplus O_C(n) & \longrightarrow & O_C(n+1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

The bottom sequence is exact because, f and g have no common zeros on C. So we see that the mapping cone,

$$\begin{array}{ccccccccc}
0 \longrightarrow J(n-1) \longrightarrow & O_{\mathbb{P}}(n-1) \oplus J(n) \oplus J(n) & & \longrightarrow & & O_{\mathbb{P}}(n) \oplus O_{\mathbb{P}}(n) \oplus J(n+1) & \longrightarrow O_{\mathbb{P}}(n+1) \longrightarrow 0 \\
& & \searrow & & \nearrow & & \\
& & & M(n+1) & & & \\
& & \nearrow & & \searrow & & \\
& 0 & & & & 0 &
\end{array}$$

56 is exact. From this we get a complex,

$$H^1(J(n-1)) \longrightarrow H^1(O_{\mathbb{P}}(n-1) \oplus J(n) \oplus J(n))$$
$$\longrightarrow H^1(O_{\mathbb{P}}(n) \oplus O_{\mathbb{P}}(n) \oplus J(n+1)).$$

Denote the homology at the middle by W_{n+1}.

2. **CLAIM**. $W_{n+1} = \mathit{Coker}\,(H^\circ(J(n+1) \oplus O_{\mathbb{P}}(n) \xrightarrow{\alpha} H^\circ(O_{\mathbb{P}}(n+1)).$
We have exact sequences as follows for $n \geq \chi + 1$.

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \uparrow & & & & \\
0 & \longrightarrow & W_{n+1} & \longrightarrow & H^1(M(n+1)) & \longrightarrow & H^1(J(n+1)) & \longrightarrow & 0 \\
 & & & & \uparrow & & & & \\
 & & & & H^1(J(n)+J(n)) & & & & \\
 & & & & \uparrow & & & & \\
 & & & & H^1(J(n-1) & & & &
\end{array}$$

β is surjective because Coker β is contained in, $H^2(J(n-1)) = H^1(C, O_C(n-1)) = 0$ by Castelnuovo's theorem, since $n \geq \chi + 1$.
We obtain the result by using the following lemma: □

LEMMA 3′. *If $n > \chi, d_n = d_{n+1}$ implies $W_{n+1} = 0$ and $d_{n+1} = d_{n+2}$. It is clear that lemma 3′ implies lemma 3.*

Proof. It is clear from the commutative diagrams we have considered that, $H^1(J(n) \oplus J(n)) \longrightarrow H^1(J(n+1))$ is surjective and (image $H^1(J(n-1)) \longrightarrow H^1(J(n) \oplus J(n)))$ is at least d_n-dimensional for $n > \chi$. We have the exact sequence, $H^1(J(n-1)) \xrightarrow{\varphi} H^1(J(n) \oplus J(n)) \longrightarrow H^1(J(n+1)) \longrightarrow$
57 0 if $n > \chi$. $\dim(\operatorname{Im}\varphi) \geq d_n, \dim(\ker\Psi) = 2d_n - d_{n+1}$. If $d_n = d_{n+1}$, then $\dim\ker\Psi = d_n \geq \dim(\operatorname{Im}\varphi) \geq d_n$. Hence $\ker\Psi = \operatorname{Im}\varphi$. Therefore $W_{n+1} = 0$. So we get, $H^\circ(J(n+1)) \oplus O^2_{\mathbb{P}}(n)) \longrightarrow H^\circ(O_{\mathbb{P}}(n+1))$ is

surjective, and hence a commutative diagram of exact sequence:

$$\begin{array}{ccccc}
H^\circ(J(n+1)\oplus O_{\mathbb{P}}(n))\otimes H^\circ(\mathbb{P}, O_{\mathbb{P}}(1)) & \xrightarrow{\alpha\otimes Id} & H^\circ(O_{\mathbb{P}}(n+1))\otimes H^\circ(\mathbb{P}, O_{\mathbb{P}}(1)) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
H^\circ(J(n+2)\oplus O^2_{\mathbb{P}}(n+1)) & \xrightarrow{\alpha'} & H^\circ(O_{\mathbb{P}}(n+2)) & \longrightarrow & 0
\end{array}$$

Therefore α' is surjective and hence $W_{n+2} = 0$. As before we get a complex,

$$H^1(J(n)) \xrightarrow{\varphi} H^1(J(n+1)\oplus J(n+1)) \longrightarrow H^1(J(n+2)) \longrightarrow 0$$

and since $W_{n+2} = 0$, this is exact. $\dim(\operatorname{Im}\varphi) = d_{n+1}$. But $\dim H^1(J(n)) =$

$d_n = d_{n+1}$. Therefore $\dim \operatorname{Im} \varphi = d_{n+1}$ and hence φ is injective. So $d_{n+1} = 2d_{n+1} - d_{n+2}$ *i.e.* $d_{n+1} = d_{n+2}$. □

3

THEOREM 2. (Castelnuovo). *Let C be a smooth irreducible curve embedded in $\mathbb{P}^r, r \geq 3$ and let J be the ideal sheaf of C. Then $H^1(\mathbb{P}^r, J(n)) = 0$ for $n \geq d-2$ where d is the degree of C [i.e. for such $n, O_C(n)$ is a complete linear system on C.]*

Proof. First we will show that it is sufficient to do this when $r = 3$. Any smooth curve in $\mathbb{P}^r$ can be projected from a point to $\mathbb{P}^{r-1}$ isomorphically if $r > 3$. Let $\pi : \mathbb{P}^r - (P) \longrightarrow \mathbb{P}^{r-1}$ be the projection map, so that $C \longrightarrow \pi(C)$ is an isomorphism. Because this is a projection from a point and since P can be chosen such that $d^\circ C = d^\circ \pi(C)[d^\circ$ denotes
58 the degree of the curve]. This can be easily checked if we go through the construction of π and using Bertini. Since $C \simeq \pi(C)$, we see that $O_C \underset{\sim}{\overset{\pi}{\rightarrow}} O_{\pi(C)}$ and for these different embeddings of C and $\pi(C)$, we still have $O_C(n) \underset{\sim}{\overset{\pi}{\rightarrow}} O_{\pi(C)}(n)$. So we have the following commutative diagram, $r > 3$,

$$H^\circ(\mathbb{P}^r, O_{\mathbb{P}^r}(n)) = H^\circ(\mathbb{P}^r - (P), O_{\mathbb{P}^r}(n))$$
$$\longrightarrow H^\circ(C, O_C(n)) \longrightarrow H^1(\mathbb{P}^r, J_C(n)) \longrightarrow 0$$
$$H^\circ(\mathbb{P}^{r-1}, O_{\mathbb{P}^{r-1}}(n)) \longrightarrow H^\circ(\pi(C), O_{\pi(C)}(n))$$
$$\longrightarrow H^1(\mathbb{P}^{r-1}, J_{\pi(C)}(n)) \longrightarrow 0$$

If we have proved the result in $\mathbb{P}^{r-1}$, then $H^\circ(\mathbb{P}^{r-1}, O_{\mathbb{P}^{r-1}}(n)) \longrightarrow H^\circ(\pi(C), O_{\pi(C)}(n))$ is surjective and by commutativity, $H^\circ(\mathbb{P}^r - (P), O_{\mathbb{P}^r}(n)) \longrightarrow H^\circ(C, O_C(n))$ is surjective. So $H^1(\mathbb{P}^r, J_C(n)) = 0$. Thus it is sufficient to prove the result in $\mathbb{P}^3$

So assume $C \longrightarrow \mathbb{P}^3$. If C is already a plane curve we have nothing to prove, since then C is complete intersection. If that is not the case, since C is not contained in any hyperplane, we can project from a ponit in $\mathbb{P}^3$, so that the image curve is birational to C and has only ordinary double points as singularities. (Remark after Bertini's theorem.) Let C' denote the image of C in $\mathbb{P}^2$. Consider cones over C and C'. They are graded rings over k. Cone over $C = A = k[x_0, x_1, x_2, x_3]$, x_i's so chosen that, cone over $C' = B = k[x_0, x_1, x_2]$. Since C' has only ordinary double points we see that x_3 satisfies a degree 2 equation over B.

We have maps, $B_n + x_3 B_{n-1} \longrightarrow A_n$, which gives a graded B - homomorphism, $B + x_3 B \longrightarrow A$.

$$\operatorname{Proj} B = C' \quad \text{and} \quad \operatorname{Proj} A = C.$$

This in turn gives a morphism of $O_{C'}$-sheaves $O_{C'} \oplus O_{C'}(-1) \longrightarrow O_C$.

If $Q \in C'$ is a non-singular point, then $O_{C',Q} \longrightarrow O_{C,\pi} - 1_{(Q)}$ is an isomorphism and hence the above map is surjective. If Q is a singular point, we have a sequence, 59

$$0 \longrightarrow O_{C',Q} \longrightarrow O_{C,\pi} - 1_{(Q)} \longrightarrow O_{C,\pi} - 1_{(Q)/O_{C',Q}} \longrightarrow 0$$

But since $\ell(O_{C,\pi} - 1_{(Q)/O_{C',Q}}) = 1(C'$ has only ordinary double points as singularities) and image of $O_{C',Q}(-1)$ is not contained in the image of $O_{C',Q}$ under the given map, we see that the above map is surjective. ◻

Claim. *We have an exact sequence of $O_{C'}$-sheaves,*

$$0 \longrightarrow \underline{f}(-1) \longrightarrow O_{C'} \oplus O_{C'}(-1) \longrightarrow O_C \longrightarrow 0 \qquad (*)$$

where $\underline{f} \subset O_{C'} \subset O_C$ is the conductor sheaf.

This map at graded ring level is as follows: If F denotes the graded ideal corresponding to $\underline{f}$,

$$F_{n-1} \xrightarrow{\varphi} B_n + B_{n-1} \longrightarrow A_n$$

$\alpha \longrightarrow (-x_3\alpha, \alpha)[x_3\alpha \in B_n$ because α is in the conductor.] and $\varphi(\alpha, \beta) = \alpha + x_3\beta$. So it is clear that the sequence (*) is a complex. Also it is clear that φ is an injection. So we have only to check exactness at the middle. We have the complex locally,

$$0 \longrightarrow \underline{f}(-1)_Q \longrightarrow O_{C',Q} \oplus O_{C',Q}(-1) \longrightarrow O_{C,\pi^{-1}(Q)} \longrightarrow 0.$$

Let $(\alpha, \beta) \longrightarrow 0$ i.e.

$$\alpha + x_3\beta = 0 \Longrightarrow \alpha = -x_3\beta \Longrightarrow x_3\beta \in \ C', Q.$$

Since x_3 generates $O_{C,\pi^{-1}(Q)}$ over $O_{C',Q}$ and x_3 is integral of degree 2
60 over $O_{C',Q}$, we see that this implies, $\beta \in \underline{f}(-1)_Q$. *i.e.* $(\alpha, \beta) = (-x_3\beta, \beta)$ with $\beta \in \underline{f}(-1)_Q$. This proves the claim.

$$\begin{aligned} \omega_{C'} &= O_{C'}(d-3) \\ \omega_C &= \mathrm{Hom}_{O_{C'}}(0_C, \omega_{C'}) \\ &= \mathrm{Hom}_{O_{C'}}(O_{C'}, O_{C'}) \otimes \omega_{C'} = \underline{f} \otimes_{O_{C'}} \omega_{C'} = \underline{f} \otimes_{O_{C'}} (d-3). \end{aligned}$$

Therefore $\underline{f}(-1) = \omega_C \otimes_{O_{C'}} O_{C'}(2-d)$.
Tensoring (*) by $O_{C'}(d-2)$ and noting that $\pi^*(O_{C'}(r)) = O_C(r)$, we see that,

$$0 \longrightarrow \omega_C \longrightarrow O_{C'}(d-2) \oplus O_{C'}(d-3) \longrightarrow O_C(d-2) \longrightarrow 0$$

is exact, as $O_{C'}$-modules.
So we have an exact sequence,

$$\begin{aligned} 0 \longrightarrow H^\circ(C, \omega_C) \longrightarrow H^\circ(C, O_{C'}(d-2)) \oplus H^\circ(C', O_{C'}(d-3)) \\ \longrightarrow H^\circ(C, O_C(d-2)) \\ \xrightarrow{\alpha} H^1(C, \omega_C) \xrightarrow{\beta} H^1(C', O_{C'}(d-2)) \oplus H^1(C', O_{C'}(d-3)) \longrightarrow 0 \end{aligned}$$

$H^1(C, O_C(d-2)) = 0$ by Castelnuovo's first theorem: since $d \geq 3$, because C is not a complete intersection, and $O_{C'}(d-3) = \omega_{C'}$, we see that $H^1(C', O_{C'}(d-3)) \neq 0$ and $H^1(C, \omega_C) \simeq k$, because C is smooth and ω_C is the canonical line bundle. So β surjective implies that β is an isomorphism and hence α is the zero map. So we have a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \uparrow & & & & \\ 0 & \longrightarrow & H^\circ(C,\omega_C) & \longrightarrow & H^\circ(C', O_{C'}(d-2)) \oplus H^\circ(C', O_{C'}(d-3)) & \longrightarrow & H^\circ(C, O_C(d-2) & \longrightarrow & 0 \\ & & & & \uparrow & & \uparrow & & \\ & & & & H^\circ(\mathbb{P}^2, O_{\mathbb{P}^2}(d-2)) \oplus H^\circ(\mathbb{P}^2, O_{\mathbb{P}^2}(d-3) & \longrightarrow & H^\circ(\mathbb{P}^3, O_{\mathbb{P}^3}(d-2)) & & \end{array}$$

61 So it follows that, $H^\circ(\mathbb{P}^3, O_{\mathbb{P}^3}(d-2)) \longrightarrow H^\circ(C, O_C(d-2))$ is surjective. i.e. $H^1(\mathbb{P}^3, J(d-2)) = 0$.
Thus by lemma 2, we see that

$$H^1(\mathbb{P}^3, J(n)) = 0 \quad \text{for} \quad n \geq d-2.$$

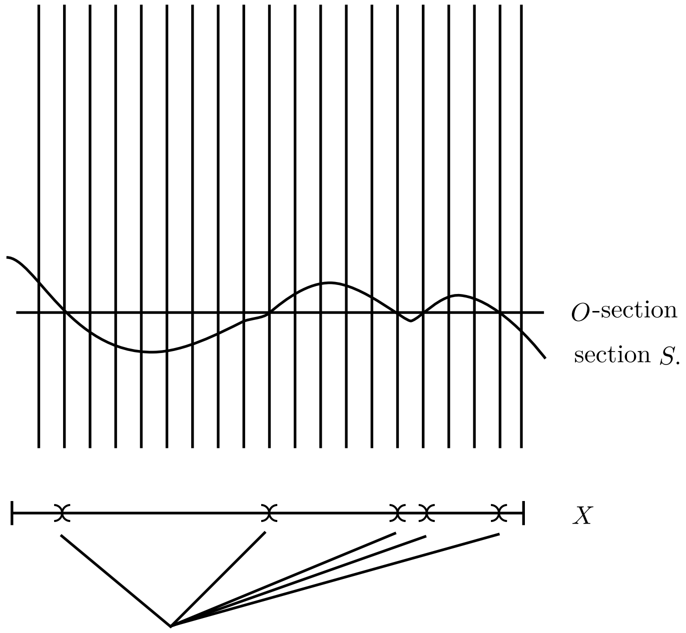
Chapter 4

On Curves which are the Schemes of Zeros of a Section of a rank Two Vector Bundle

1

Let X be a scheme and E a vector bundle on X. Suppose we are given 62
a section of E. i.e. an injection $O_X \xrightarrow{s} E$. Then taking duals we get a morphism, $\check{E} \longrightarrow O_X$. The cokernel of this map defines a closed subscheme Y of X, which we call the *scheme of zeroes of the section of* E. So we have an exact sequence,

$$\check{E} \longrightarrow O_X \longrightarrow O_Y \longrightarrow 0$$



Y = Scheme of zeroes of the section S.

Fig. 1

Now we prove a proposition by Serre, which gives a sufficient condition for a closed sub-scheme of codimension two to be the scheme of zeroes of a section of a rank two vector bundle.

63 **PROPOSITION. (J-P. Serre):** *Let Y be a regular, connected scheme with dualising sheaf ω_Y. Let X be a closed sub-scheme of Y, equidimensional, Cohen-Macaulay and of codimension two. Assume the following conditions are satisfied:*

a) There exists a line bundle L on Y such that,

$$\omega_X = L/X$$

b) $H^2(Y, L^{-1} \otimes \omega_Y) = 0.$

Then X is the scheme of zeroes of a section of a rank two vector bundle E such that $\overset{2}{\bigwedge} E \otimes \omega_Y = L$.

Proof. Let J be the sheaf of ideals of X in Y. If a vector bundle E exists with the above properties, then we have the Koszul complex $0 \longrightarrow \overset{2}{\bigwedge} \check{E} \longrightarrow \check{E} \longrightarrow O_Y \longrightarrow O_X \longrightarrow 0$ with is exact. So the sequence $0 \longrightarrow \overset{2}{\bigwedge} \check{E} \longrightarrow E \longrightarrow J \longrightarrow 0$ is exact. But we want $\overset{2}{\bigwedge} \check{E}$ to be equal to $L^{-1} \otimes \omega_Y$.

Thus we see that the problem is to find an extension of J by $L^{-1} \otimes \omega_Y$, which is a vector bundle i.e. we want an element of $\mathrm{Ext}^1(J, L^{-1} \otimes \omega_Y)$, which gives a locally free extension.

The spectral sequence for Ext's gives an exact sequence,

$$\begin{aligned} \mathrm{Ext}^1(J, L^{-1} \otimes \omega_Y) &\longrightarrow H^\circ(Y, \underline{\mathrm{Ext}}^1(J, L^{-1} \otimes \omega_Y)) \\ &\longrightarrow H^2(Y, \underline{\mathrm{Hom}}(J, L^{-1} \otimes \omega_Y)). \end{aligned} \tag{A}$$

From the exact sequence,

$$0 \longrightarrow J \longrightarrow O_Y \longrightarrow O_X \longrightarrow 0$$

applying the functor $\mathrm{Hom}(\cdot, L^{-1} \otimes \omega_Y)$ and noting that $\mathrm{Ext}^i(O_X, L^{-1} \otimes \omega_Y)$ is zero if $i \neq 2$ (because X is Cohen-Macaulay of codimension 2 in a **64** regular scheme), we see that

$$L^{-1} \otimes \omega_Y \simeq \underline{\mathrm{Hom}}(J, L^{-1} \otimes \omega_Y)$$

and

$$\mathrm{Ext}^1(J, L^{-1} \otimes \omega_Y) \simeq \mathrm{Ext}^2(O_X, L^{-1} \otimes \omega_Y).$$

We know that $\underline{\mathrm{Ext}}^2(O'_X L_{-1} \otimes \omega_Y) = L^{-1} \otimes \omega_X$ and by the assumption a) in the proposition, $L^{-1} \otimes \omega_X = O_X$. Thus we see that $\underline{\mathrm{Ext}}^1(J, L^{-1} \otimes \omega_Y)$ is a monogenic O_Y-Module. This generator corresponds to a section η of $\underline{\mathrm{Ext}}^1(J, L^{-1} \otimes \omega_Y)$ and generates $\underline{\mathrm{Ext}}^1(J, L^{-1} \otimes \omega_Y)$ at every point of Y.

Since $H^2(Y, \mathrm{Hom}(J, L^{-1} \otimes \omega_Y))$ is isomorphic to $H^2(Y, L^{-1} \otimes \omega_Y)$ which is zero by assumption b) from (A) we get that,

$$\mathrm{Ext}^1(J, L^{-1} \otimes \omega_Y) \longrightarrow H^\circ(Y, \underline{\mathrm{Ext}}^1(J, L^{-1} \otimes \omega_Y))$$

is surjective. So there is an element of $\mathrm{Ext}^1(J, L^{-1} \otimes \omega_Y)$ which maps onto η. Let $\check{E}$ be the extension corresponding to this element. We will show that $\check{E}$ is locally free.

We prove the following: Let η be a section of $\underline{\mathrm{Ext}^1}(J, L^{-1} \otimes \omega_Y)$ coming from $\mathrm{Ext}^1(J, L^{-1} \otimes \omega_Y)$. The corresponding extension G is free at a point y in Y if and only if the image of η in $\mathrm{Ext}^1_{O_Y,y}(J_y(L^{-1} \otimes \omega_Y)_y)$ generates it as an $O_{Y,y}$-module. This will prove our assertion that $\check{E}$ is locally free. Assume G is free at y. So we have an exact sequence,

$$0 \longrightarrow (L^{-1} \otimes \omega_Y) = O_{Y,y} \longrightarrow G_y = O_{Y,y} \oplus O_{Y,y} \longrightarrow J_y \longrightarrow 0,$$

which gives rise to an exact sequence,

$$O_{Y,y} = \mathrm{Hom}_{O_{Y,y}}(O_{Y,y}, O_{Y,y}) \longrightarrow \mathrm{Ext}^1_{O_{Y,y}}(J_y, O_{Y,y}) \longrightarrow 0.$$

65

Since image of η corresponds to image of the identity of $O_{Y,y}$ in $\mathrm{Ext}^1_{O_{Y,y}}(J_y, O_{Y,y})$ we see that image of η generates it. Conversely let image of η generate $\mathrm{Ext}^1_{O_{Y,y}}(J_y, O_{Y,y})$ and let G_y be the corresponding extension.

From the exact sequence,

$$0 \longrightarrow O_{Y,y} \longrightarrow G_y \longrightarrow J_y \longrightarrow 0$$

we see that the assumption implies, $\mathrm{Ext}^1(G_y, O_{Y,y}) = 0$. Since Y is regular and X is Cohen-Macaulay of codimension 2, we see that, (Projective dimension of G_y) ≤ 1. So for any $O_{Y,y}$-module $N, \mathrm{Ext}^1(G_y, N) = \mathrm{Ext}^1(G_y, O_{Y,y}) \otimes N = 0$ *i.e.* G_y is projective and hence free.

Thus $\check{E}$ is locally free. From the exact sequence, $0 \longrightarrow L^{-1} \otimes \omega_Y \longrightarrow \check{E} \longrightarrow J \longrightarrow 0$, one sees by additivity of ranks, that rank $\check{E}$ is 2. We have an exact sequence,

$$0 \longrightarrow L^{-1} \otimes \omega_Y \longrightarrow \check{E} \longrightarrow O_Y \longrightarrow O_X \longrightarrow 0.$$

The fact that the Koszul complex

$$0 \longrightarrow \bigwedge^2 \check{E} \longrightarrow \check{E} \longrightarrow O_Y \longrightarrow O_X \longrightarrow 0.$$

66 is also exact implies that $\overset{2}{\wedge}\check{E} \simeq L^{-1} \otimes \omega_Y$. So X is the scheme of zeroes of a section of the rank two vector bundle E with $\overset{2}{\wedge} E \otimes \omega_{Y/X} = \omega_X$. □

REMARK. *Conversely if X and Y are as in the proposition and if X is the scheme of zeroes of a section of a rank two vector bundle E, then one can see that a) is satisfied i.e. $\overset{2}{\wedge} E \otimes \omega_{Y/X} = \omega_X$. But b) may not be satisfied.*

COROLLARY. *Let $Y = \operatorname{Spec} R$ where R is a regular ring. If X and Y satisfy the conditions in the proposition (note that $H^2(Y, \cdot) = 0$, since Y is affine.) then X is a local complete intersection. In addition, if projectives are free over R, then X is a complete intersection.*

As an interesting corollary to Serre's proposition, we will prove the following:

THEOREM. (G. Horrocks). *Give any pair of integers C_1 and C_2, there exists an indecomposable rank two vector bundle E on $\mathbb{P}^3_k$(k infinite) such that $C_1(E) = C_1$ and $C_2(E) = C_2$ (where $C_i(E)$ denote the i^{th} Chern number of E) if and only if $C_1C_2 = 0 \mod 2$*

We recall some definitions and properties of Chern numbers: Let E be a vector bundle on $\mathbb{P}^3$ of rank 2. Let X be a scheme of zeroes of a section of E.

Then $\omega_X = O_X(C_1(E) - 4)$ and $C_2(E) = d^\circ X$.

We also note that,

$$C_1(E(n)) = C_1(E) + 2n$$
$$C_2(E(n)) = C_2(E) + nC_1(E) + n^2,$$

and $C_1(\check{E}) = -C_1(E)$. 67

Proof. (of the theorem). First we will show that $C_1C_2 = 0(\mod 2)$ is a necessary condition for any vector bundle of rank two on $\mathbb{P}^3$. Let E be any vector bundle of rank 2 on $\mathbb{P}^3$. Since $C_1(E)C_2(E) - C_1(E(n)).C_2(E(n)) = 0(\mod 2)$ for any n, it is sufficient to show for large n. Hence we can assume that E has a smooth section, say X. Then

$$\Omega^1_X = \omega_X = O_X(C_1(E) - 4).$$

By Riemann-Roch, $\deg \Omega^1_X = 2g - 2$, and $\deg \omega_X = C_2(E)(C_1(E) - 4)$. So $C_1(E).C_2(E) = 0 (\mod 2)$.

We will now construct indecomposable vector bundles of rank 2 with given Chern numbers.

I Let $a > 0$ be any integer and $m_1, m_2 \ldots m_r$ be integers such that $0 < m_i < a$. Then there exists a vector bundle E of rank 2. Such that $C_1(E) = a$ and $C_2(E) = \sum_{i=1}^{r} m_i(a - m_i)$ and an exact sequence,

$$0 \longrightarrow O_{\mathbb{P}^3}(-a) \longrightarrow E \longrightarrow O_{\mathbb{P}^3} \longrightarrow O_X \longrightarrow 0$$

68 i.e. X is the scheme of zeroes of a section of E. We will prove this by induction on r. If $r = 1$, we take $E = O(m_i) + O(a - m_i)$. Assume we have proved this upto $(r-1)^{th}$ stages so we have a vector bundle E such that

$$C_1(E) = a, C_2(E) = \sum_{i=1}^{r-1} m_i(a - m_i)$$

and an X which is the scheme of zeroes of a section of E.

So $\omega_X = O_X(C_1(E) - 4) = O_X(a - 4)$ and

$$d^\circ(X) = C_2(E).$$

One can easily see that the zeroes of a generic section of $O(m_r) \oplus O(a - m_r)$ does not meet X. So let X_1 be such a scheme. Then $\omega_{X_1} = O_{X_1}(a - 4)$ and $d^\circ X_1 = m_r(a - m_r)$. From this we see that $X \cup X_1$ is locally a complete intersection and $\omega_{X \cup X_1} = O_{X \cup X_1}(a - 4), d^\circ(X \cup X_1) = C_2(E) + m_r(a - m_r) = \sum_{i=1}^{r} m_i(a - m_i)$. Serre's proposition assures that $X \cup X_1$ is the scheme of zeroes of a section of a rank two vector bundle, say F. Then $C_1(F) = a$ and $C_2(F) = \sum_{i=1}^{r} m_i(a - m_i)$ and we have a section as required.

II By the above construction, if $r \geq 2$, then the corresponding vector bundle E is indecomposable. If $r \geq 2$, then by construction, there exists a section, the scheme of zeroes X of which has more than one connected component. Let X be the scheme of zeroes of any section. Then one verifies from the exact sequence corresponding to the section that

$$\dim H^1(\mathbb{P}^3, \check{E}) = \dim H^\circ(X, O_X) - 1.$$

If X has more than one connected component we see that $\dim H^1(\mathbb{P}^3, \check{E}) \geq 1$. *i.e.* $H^1(\mathbb{P}^3, \check{E}) \neq 0$. If E is decomposable, then so is $\check{E}$ and then $H^1(\mathbb{P}^3, \check{E}) = 0$. So if $r \geq 2$, we see that E is indecomposable.

We denote by $\Delta(E)$, then number, $C_1(E)^2 - 4C_2(E)$. Note that $\Delta(E(n)) = \Delta(E)$. If C_1 and C_2 are two integers with $C_1C_2 \equiv 0 \mod 2$, we see that, $C_1^2 - 4C_2 \equiv 0(\mod 4)$ if C_1 is even and $\equiv 1(\mod 8)$ if C_1 is odd. **69**

□

LEMMA. *Let n be a positive integer $\equiv 0(\mod 4)$ or $\equiv 1(\mod 8)$. Then there exists positive integers a, m_1 and m_2 such that, $0 < m_i < a$ and $n = a^2 - 4. \sum_{i=1}^{2} m_i(a - m_i)$.*

Proof. Because of the conditions on n there exists a positive integer N such that $N^2 = n + 8m_1m_2$ where m_i's are some positive integers. Then take $a = 2(m_1 + m_2) + N$.

III Let C_1 and C_2 be any two integers such that $C_1C_2 = 0 \mod 2$. Now we will construct an indecomposable vector bundle of rank 2, whose chern numbers are C_1 and C_2.

By the above lemma, we have an expression, $C_1^2 - 4C_2 = a^{2.} - 4 \sum_{i=1}^{r} m_i(a - m_i)$. With $0 < m_1 < a$ and $r \geq 2$. So by our construction we have a vector bundle E of rank two such that $C_1(E) = a$ and $C_2(E) = \sum_{i=1}^{r} m_i(a - m_i)$. Since $r \geq 2E$ is indecomposable. $C_1 - a = 2m$ for some m. Now $C_1(E(m)) = a + C_1 - a = C_1$ and $\Delta(E(m)) = \Delta(E) = C_1^2 - 4C_2$. Therefore $C_2(E(m)) = C_2$ and $E(m)$ is indecomposable.

□

2 Thickening in a Normal Direction

Let Y be any regular scheme and X a closed subscheme of Y which is a local complete intersection. So the conormal bundle N_X is a vector bundle over X.

Let L be any line bundle over X and let $N_X \longrightarrow L$ be any surjection. Then we have a diagram,

$$\begin{CD} 0 @>>> N_X @>>> O_X \\ @. @VVV @. \\ @. L @. \\ @. @VVV @. \\ @. 0 @. \end{CD}$$

70 where $O_{X-1} = O_{Y/I^2}, I$, the ideal sheaf of X. Let O_Z be the push out. So O_Z is the structure sheaf of a scheme Z which is closed in Y and topologically isomorphic to X. So we have commutative diagram,

$$\begin{CD} 0 @>>> N_X @>>> O_{X_1} @>>> O_X @>>> 0 \\ @. @VVV @VVV @| @. \\ 0 @>>> L @>>> O_Z @>>> O_X @>>> 0 \\ @. @VVV @VVV @. @. \\ @. 0 @. 0 @. @. \end{CD}$$

DEFINITION . *Z is defined as the thickening of X in the direction of the quotient L.*

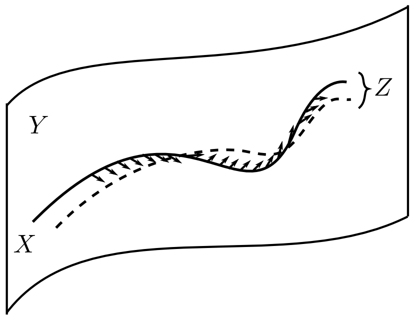


Fig. 2

3 Curves which are Zeros of Sections of Rank Two Vector Bundels in 3-Space

In this section we will prove that curves in a smooth quasiprojective scheme of dimension 3, which are local complete intersection are schemes of zeroes of a section of a rank two vector bundle. **71**

We start with a proposition due to R. Fossum:

PROPOSITION. *Let A be a local ring which is Cohen-Macaulay. Let ω_A be its dualising sheaf. Let B be any algebra extension of A by ω_A with square of ω_A zero. Then B is Gorenstein.*

Proof. We have an exact sequence of B-modules,

$$0 \longrightarrow \omega_A \longrightarrow B \longrightarrow A \longrightarrow 0.$$

Let B be the quotient of a regular local ring R. One can assume without loss of generality that B is complete. Then A is also a quotient of R, in a natural way. Codimension of B and A in R are equal, say r. Let $\dim R = n$. Then $\mathrm{Ext}^i_R(A, R) = 0$ for $i \neq n - r$

$$= \omega_A \quad i = n - r.$$

Since ω_A is the dualising sheaf of A, we see that,

$$\mathrm{Ext}^i_R(\omega_A, R) = 0 \quad \text{for} \quad i \neq n - r$$

$$= A \quad \text{for} \quad i = n - r.$$

Thus applying the functor $\mathrm{Hom}_R(., R)$ to the above exact sequence, we see that,

$$\mathrm{Ext}^i_R(B, R) = 0, \quad \text{for} \quad i \neq n - r.$$

Therefore B is Cohen-Macaulay. Hence $\mathrm{Ext}^{n-r}_R(B, R) = \omega_B$, the dualising sheaf of B. So we get an exact sequence,

$$0 \longrightarrow \mathrm{Ext}^{n-r}_R(A, R) \longrightarrow \mathrm{Ext}^{n-r}_R(B, R) \longrightarrow \mathrm{Ext}^{n-r}_R(\omega_A, R) \longrightarrow 0$$

72 i.e. $0 \longrightarrow \omega_A \longrightarrow \omega_B \longrightarrow A \longrightarrow 0$ is exact.
If x in R is a non-zero divisor in A, then it is also a non-zero divisor in ω_A. Hence it is also a non-zero divisor in B and ω_B. Going modulo such an x, we get an exact sequence.

$$0 \longrightarrow \omega_{A/x\omega_A} \longrightarrow \omega B/x\omega_B \longrightarrow A/_{xA} \longrightarrow 0.$$

But since $\omega_{A/x\omega_A} = \omega_{A/xA}$ and $\omega_{B/x\omega_B} = \omega_{B/xB}$ we have

$$0 \longrightarrow \omega_{A/xA} \longrightarrow \omega_B/_{xB} \longrightarrow A/_{xA} \longrightarrow 0$$

is exact. Continuing this process we can reduce the case to A and B artinian. Since B is Gorenstein if and only if $B/_{(x_1 \ldots x_i)B}$ is Gorenstein for sequence $x_i, \ldots, x_i$ of B, it is sufficient to show that B is Gorenstein in the artinian case. So assume A is an artinian local ring and B an extension as before with $\omega_A^2 = 0$.

Let M_A and M_B be the respective maximal ideals. Since $\mathrm{Hom}_A(k, \omega_A) = \omega_k = k$, we have an exact sequence,

$$0 \longrightarrow k \longrightarrow \mathrm{Hom}_B(k, B) \longrightarrow \mathrm{Hom}_A(k, A) \longrightarrow$$

We want to show that $\mathrm{Hom}_B(k, B) = k$, which will prove that B is Gorenstein. So let $f : k \longrightarrow B$; be any B-module homomorphism. i.e. There exists λ in B such that $\mathrm{Ann}\,\lambda = M_B$. If $\overline{\lambda}$, the image of λ in A is not zero, then $\overline{\lambda}.\omega_A = 0$ since $\omega_A \hookrightarrow M_B$. But $\mathrm{Ann}_A\,\omega_A = (0)$, which implies that $\overline{\lambda} = 0$ *i.e.* λ belongs to ω_A. Thus the map, $k \longrightarrow \mathrm{Hom}_B(k, B)$ is an isomorphism. This proves our assertion.

Now we will state and prove the main theorem. □

73 **THEOREM.** *Let Y be any smooth quasi-projective scheme of dimension 3 over an infinite field. Let X be a closed subscheme of Y, locally a complete intersection and codimension two in Y. Then X is set theoretically the scheme of zeroes of a rank two vector bundle on Y, with multiplicity two.*

Proof. Let L be an ample line bundle on Y. Let ω_Y and ω_X denote the dualising sheaves of Y and X respectively. Let N denote the normal bundle of X in Y. We choose an integer n large enough so that the following are satisfied.

1. $N \otimes \omega_X \otimes L^n$ is generated by sections.

2. $H^1(X, \omega_X \otimes L^n) = 0$

3. $H^2(Y, \omega_Y \otimes L^n) = 0$.

From now on we will call L^n as L. The idea of the proof is that we thicken X along the quotient $\omega_X \otimes L$ (first showing that $\omega_X \otimes L$ is a quotient of N) and then show that the dualising sheaf of the thickened scheme is actually restriction of a line bundle on Y. Then by Serre's proposition we are through.

Since X is a local complete intersection of codimension two in Y, $N \otimes \omega_X \otimes L$ is a vector bundle of rank two over X and it is generated by sections. So by a lemma of Serre, (since basefield is infinite.) it has a nowhere vanishing section, $O_X \longrightarrow N \otimes \omega_X \otimes L$. This implies by taking duals,

$$\check{N} \otimes \omega_X^{-1} \otimes L^{-1} \longrightarrow {}_X$$

is a surjection i.e. , there exists a surjection,

$$\check{N} \longrightarrow \omega_X \otimes L.$$

So as in section 2 we thicken X along this quotient. Let Z be the thickened scheme, and X_1 denote the scheme defined by I^2 in Y, where I is the sheaf of ideals defining X. So we have a commutative diagram, 74

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \check{N} & \longrightarrow & O_{X_1} & \longrightarrow & O_X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \omega_X \otimes L & \longrightarrow & O_Z & \longrightarrow & O_X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

By Fossum's theorem, we see that Z is locally Gorenstein. By corollary of Serre's proposition we see that Z is locally a complete intersection in Y.

By applying $\mathrm{Hom}(., \omega_Y)$ to the bottom exact sequence in the diagram above we get an exact sequence,

$$0 \longrightarrow \mathrm{Ext}^2_{O_Y}(O_X, \omega_Y) \longrightarrow \mathrm{Ext}^2_{O_Y}(O_Z, \omega_Y) \longrightarrow \mathrm{Ext}^2_{O_Y}(\omega_X \otimes L, \omega_Y) \longrightarrow 0$$

$$\text{i.e.} \qquad 0 \longrightarrow \omega_X \longrightarrow \omega_Z \longrightarrow O_X \otimes L^{-1} \longrightarrow 0.$$

This gives an exact sequence,

$$0 \longrightarrow \omega_X \otimes L \longrightarrow \omega_Z \otimes L \longrightarrow O_X \longrightarrow 0.$$

Since $H^1(X, \omega_X \otimes L) = 0$, we see that the section '1' of O_X comes from a section of $\omega_Z \otimes L$. i.e. there exists a homomorphism, $O_Z \longrightarrow \omega_Z \otimes L$ such that the composite. $O_Z \longrightarrow O_X$ is the canonical surjection. So we have a commutative diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \omega_X \otimes L & \longrightarrow & O_Z & \longrightarrow & O_X & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & \omega_X \otimes L & \longrightarrow & \omega_Z \otimes L & \longrightarrow & O_X & \longrightarrow & 0 \end{array}$$

We will show that this morphism is an isomorphism. It is sufficient to show locally. So we have a commutative diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \omega_A = A & \longrightarrow & B & \longrightarrow & A & \longrightarrow & 0 \\ & & & & \downarrow & & \| & & \\ 0 & \longrightarrow & A = \omega_A & \longrightarrow & B & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

75 Since the composite $B \longrightarrow B \longrightarrow A$ is the canonical surjection, the map $B \longrightarrow B$ must be of the form $1 \longrightarrow 1 + r$ where r comes from ω_A. But since ω_A is nilpotent in B, we see that $1 + r$ is a unit in B and hence the map is an isomorphism. So

$$\omega_Z \simeq L^{-1}/Z.$$

Since $H^2(Y, L \otimes \omega_Y) = 0$, by Serre's proposition we see that Z is the scheme of zeros of a rank two vector bundle on Y. Also since the ideal sheaf of Z contains I^2, we see it is of 'multiplicity 2'. This prove the theorem. □

NOTE. In the affine case (*i.e.* $Y = \operatorname{Spec} R$), we do not need the base field to be infinite.

THEOREM 2. (Ferrand, Szpiro). *If $Y = \mathbb{A}^3$, the affine 3-space, then curves which are local complete intersections are set theoretic complete intersection with multiplicity 2.*

Proof. Note that vector bundles are free over $\mathbb{A}^3$. So combining the result with theorem 1, we get the result. □

THEOREM 3. (Ferrand). *If X is a locally complete intersection curve in $\mathbb{P}^3$ (over an infinite field) then it is set theoretically the scheme of zeroes of a section of a rank two vector bundle with multiplicity 2.*

EXAMPLE. The twisted cubic in $\mathbb{P}^3$, which is defined by the equations, $X_\circ X_3 - X_1X_2 = 0, X_\circ X_2 - X_1^2 = 0$ and $X_1X_3 - X_2^2 = 0$ is set theoretically defined by $X_1X_3 - X_2^2$ and $X_\circ(X_\circ X_3 - X_1X_2) - X_1(X_\circ X_2 - X_1^2)$.

REMARK. *S.S. Abhyankar has proved that if a prime ideal $P \subset k[X,$ 76
$Y, Z]$ defines a smooth curve in $\mathbb{A}^3$, then P is generated by 3 elements. (Algebraic Space Curves, S.S. Abhyankar, Montreal Notes.) So P can be generated by the 2×2 determinants of a 3×2 matrix:*

$$\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}$$

Theorem 2 says that, after an automorphism of $\mathbb{A}^3$, the above entries can be chosen so that $d = e$.

Then the equations of the curve, set theoretically, are

$$cf - d^2 = 0 \quad \textit{and} \quad a(af - bd) - b(ad - bc) = 0$$

If Δ_i's are the corresponding minors then $\Delta_I = 0$ and $a\Delta_2 + b\Delta_3 = 0$, are the corresponding equations. In the language of [11] the curve is liee to itself!!

Chapter 5

An Application to Complete Intersections

AS AN APPLICATION of the results we have proved till now, we prove 77
a result on complete intersecions:

THEOREM. *Let X be a smooth closed subscheme of* $\mathbb{P}^n_k$ *of codimension 2. Let us further assume that char* $k = 0$ *and* $n \geq 2d^\circ(X)$. *Then X is a complete intersection.*

Proof. If $n \leq 5$ then $d^\circ X = 1$ or 2. So we see immediately that X is a complete intersection. So we can assume $n \geq 6$. Then by Barth's theorem, $\operatorname{Pic} X = \mathbb{Z}.O_X(1)$. Since X is a local complete intersection, (X is smooth) ω_X is a line bundle and hence $\omega_X = O_X(r)$ for some integer r. By Serre's lemma [Chapter 4] X is the scheme of zeros of a section of a rank two vector bundle on $\mathbb{P}^n$. So we see that the theorem is essentially a criterion for decomposability of a rank two vector bundle on $\mathbb{P}^n$.

i) We will first prove that the integer r above is negative. By Bertini's theorem we can cut X by a linear subspace of $\mathbb{P}^n$ of codimension $(n-3)$ to get a smooth curve C in $\mathbb{P}^3$. Then

$$\omega_C = O_C(r + n - 3).$$

Since $H^{\circ}(C, O_C)^{v} = H^1(C, \omega_C) = H^1(C, O_C(r+n-3))$, we see that

$$H^1(C, O_C(r+n-3)) \neq 0.$$

By Castelnuovo's theorem,

$$r + n - 3 \leq d - 3.$$

Since $n \geq 2d$, we see that r is negative.

ii) Applying Kodaira's vanishing theorem, we see:

$$H^i(X, O_X(m)) = 0, m < 0, 0 \leq i < \dim X = n - 2.$$

78

By duality we get,

$$H^i(X, O_X(m)) = H^{n-2-i}(X, O_X(r-m)).$$

Since r is negative we see that,

$$H^i(XO_X(m)) = 0$$

for every m and $0 < i < \dim X$.

iii) Let Spec A be the cone of X (in the given embedding in $\mathbb{P}^n$.) Let

$$\bar{A} = \mathop{\oplus}_{m} H^{\circ}(X, O_X(m)).$$

We will show that $\bar{A}$ is a Gorenstein ring. By (ii) we see that $\bar{A}$ is Cohen-Macaulay. Since

$$\omega_{\bar{A}} = \mathop{\oplus}_{m} H^{\circ}(X\omega_X(m)) = \oplus H^{\circ}(XO_X(m+r)) = \bar{A}(r),$$

$\bar{A}$ is Gorenstein.

iv) Let $\operatorname{Spec} R = \operatorname{Spec} k[X_\circ, X_1, \ldots X_n]$ be the cone over $\mathbb{P}^n$. So we have a surjection $R \longrightarrow A$ and $\overline{A}$ is a finite A-module and hence a finite R-module. Let s = minimum number of generators of $\overline{A}$ over R. Since $\overline{A}$ is a Cohen-Macaulay module of dimension $n-1$, we have a resolution,

$$0 \longrightarrow R^m \longrightarrow R^t \longrightarrow R^s \longrightarrow \overline{A} \longrightarrow 0.$$

Since $\overline{A}$ is Gorenstein, $s = m$. So the minimal resolution for $\overline{A}$ is,

$$0 \longrightarrow R^s \longrightarrow R^{2s} \longrightarrow R^s \longrightarrow \overline{A} \longrightarrow 0.$$

v) We claim that $s \leq d - 2. d = d^\circ X$.

We can choose X_i's above in such a manner that $\overline{A}$ is integral over,

$$R' = k[X_\circ, \ldots, X_{n-2}]$$

the inclusion $R' \longrightarrow \overline{A}$ is a graded homomorphism. Since $\overline{A}$ is Cohen-Macaulay we see that $\overline{A}$ is locally free over R'. But since $\overline{A}$ is a graded R'-module it is actually free and $rk_{R'}\overline{A} = d$. **79**

Since X can be assumed to be not contained in any hyperplane, we see that the images of X_{n-1} and X_n in $\overline{A}$ form a part of a minimal set of generators of $\overline{A}$ over R'. So we see that the R' module got as the cokernel of, $R \longrightarrow \overline{A}$, is generated by atmost $d-3$ elements. i.e. $\overline{A}$ is generated over R by at most $d-2$ elements. So

$$s \leq d - 2.$$

vi) Now we claim that $s = 1$.

Since outside the closed point (vertex) of R, A and $\overline{A}$ are isomorphic, we have that the $(s-1)\times(s-1)$ minors of the matrix defining the map $R^{2s} \longrightarrow R^s$, defines the vertex of R as its set of zeroes. So the codimension of the scheme defined by the $(s-1)\times(s-1)$ minors of this $s \times 2s$ matrix in R is $n+1$. But from general principles, codimension of the variety defined by the $z \times z$ minors of an $r \times t$ matrix is less than or equal to $(r - z + 1)(t - z + 1)$. So here,

$$n + 1 \leq (2s - (s-1) + 1)(s - (s-1) + 1) = (s+2).2.$$

Since $s \leq d - 2$, we see that, $n + 1 \leq 2d$ which is a contradication to our hypothesis. Therefore $s = 1$.

So we see that the composite map $R \longrightarrow A \longrightarrow \bar{A}$ is surjective. i.e. $A = \bar{A}$. So A is Gorenstein and codimension two in R and hence a complete intersection.

□

Bibliography

[1] ABHYANKAR, S.S., Algebraic Space Curves, University of Montreal, Lecture Notes.

[2] BARTH, W., Transplanting cohomology classes in complex projective space, Amer. J. Math., 92 (1970), 951–967.

[3] FOSSUM, R., et al: The Homological Algebra of Trivial Extensions of Abelian Categories with application to Ring Theory. Preprint series 72/73 -Aarhus Universitet.

[4] GAETA, F.: Quelques progres recents dans Ia classification des varieties algebriques dun espace projectif Deuxieme Collogue de Geometrie Algebrique Liege. C.B.R.M., 145–181 (1952).

[5] HALPHEN, G.-H.: Memoire sur la classification des courbes gauches algebriques, J. Ec. Polyt. 52 (1882), 1–200.

[6] HORROcKS, G: Vector bundles on a punctured spectrum, Proc. Lond. Math. Soc. 14 (1964), 689–713.

[7] KODAIRA, K.,: On a differential-geometric method in the theory of analytic stacks, Proc. Nat. Acad. Sci. USA 39 (1953), 1268–1273.

[8] MUMFORD, D.: Pathologies III, Amer. J. Math. 89 (1967), 94–104.

[9] MURTHY, M.P., and TOWBER, J.: Algebraic vector bundles over $\mathbb{A}^3$ are trivial, Inventiones Math. 24, 173–189 (1974).

[10] NOETHER, M.: Zur Grundlegung der Theorie der Algebracschen Raumeurven, Verlag der komgliehen Akademie der Wissensehaften, Berlin (1883).

81 [11] PESKINE, C. and L. SZPIRO: Liaison des varietes algebriques I, Inventiones Math., 26, 271–302, (1974).

[12] QUILLEN, D.: Projective Modules over Polynomial Rings Inventiones Math., 36, 166–172 (1976).

[13] RAO, A. PRABHAKAR: Liaison among curves in Projective 3-space, Ph.d. Thesis, University of California, Berkeley.

[14] SAINT-DONAT, B.,: On Petri's analysis of the linear system of quadrics through a canonical curve. Math. Ann. 206 (1973), 157–175.

[15] SERRE J-P.: Sur la topologic des varietes algebriques en caracteristique p, Symposium Int. de Topologie Algebraica, Mexico (1958), 24–53.