Lectures on
Forms of Higher Degree

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Foreward

THE FOLLOWING lecture notes are based on my lectures at the Tata Institute of Fundamental Research from the middle of January to early March of 1978. My objective was not to give a systematic exposition of the theory of forms of higher degree but only to introduce young mathematicians to a new approach to this area. It is my view that although this theory is still at an early stage, it has all the features of becoming an important branch of Mathematics. If the reader finds these lecture notes intelligible, then it is largely due to the painstaking efforts of Professor Raghavan. I would like to thank him for his excellent collaboration to bring the notes into this final form.

9 March 1978

Jun-Ichi Igusa
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ONE OF THE principal objectives of modern number theory must be to develop the theory of forms of degree more than two, to the same satisfactory level in which the theory of quadratic forms is found today as the cumulative work of several eminent mathematicians and especially of C.L. Siegel. The importance of forms of higher degree as a serious number-theoretic object was pointed out already by Gauss ([11], § 266) as follows: “...il suffira d’avoir recommandé ce champ vaste à l’attention des géomètres, où ils pourront trouver un très-beau sujet d’exercer leurs forces, et les moyens de donner à l’Arithmétique transcendent de très-beaux développements”. It is unbelievable and highly remarkable how young Gauss was able to predict the existence of a full-fledged theory of quadratic forms as it obtains today.

With a view to give some idea as to what may constitute such a theory of forms of higher degree, we shall describe two problems. Before we state the first one, let us fix the notation. Let $f(x) = f(x_1, x_2, \ldots, x_n)$ be a form, i.e., a homogeneous polynomial degree $m \geq 2$ in $n$ variables $x_1, \ldots, x_n$ with coefficients in the ring $\mathbb{Z}$ of integers. Let $p$ be a fixed prime, $e$ a non-negative integer and $u$ an integer such that the g.c.d. $(u, p^e)$ is 1. Let us define the generalized Gaussian sum

$$F^e(u/p^e) = \frac{1}{p^{ne}} \sum_{\xi \mod p^e} ((u/p^e)f(\xi))$$

where $\xi \in \mathbb{Z}^n$ and $e(\ ) = \exp(2\pi \sqrt{-1})$. The first problem is to develop a theory of such exponential sums associated with $f$ and $p$. In this direction, we have the following general theorem:
there exists an integer $e_0 \geq 0$ such that for every $e \geq e_0$, $F^*(u/p^e)$ is a fixed linear combination of expression of the form

$$X(u)(p^e)^{-\lambda}(\log p^e)^j$$

with $X$ equal to a Dirichlet character having a power of $p$ as its conductor, $0 \leq j < n$ and $\text{Re}(\lambda)$ being a positive rational number.

If now we arrange $\text{Re}(\lambda)$ for lambda occurring above in ascending order as

$$0 < \lambda_1 < \lambda_2 < \cdots$$

(where $\lambda_1, \lambda_2, \ldots$ are invariants of the affine hypersurface determined by the zeros of $f$), then for any $\epsilon > 0$, we have, for every $e \geq 0$,

$$F^*(u/p^e) \leq c(p^e)^{-\lambda_1+\epsilon}$$

where $c$ is a constant independent of $e$ and $u$.

A conjecture in this connection states that except for a finite number of primes $p$ depending on $f$ and $e$, we can replace the constant $c$ above by $1$, if $\lambda_1 > 1$.

In all the cases where $F^*(u/p^e)$ can be closely examined or calculated, this conjecture has been verified. Indeed, in the particular case of interest when the projective hypersurface $S$ defined by $f(x) = 0$ is non-singular, the conjecture is valid with $\lambda_1 = n/m$ (and further $0$ instead of $\epsilon$). The proof of the conjecture in this case depends on Deligne’s solution of the Riemann-Weil hypothesis for zeta functions associated with non-singular projective varieties defined over finite fields.

In order to deal with the conjecture when $S$ may not be non-singular, one approach may be to isolate the relevant part of Deligne’s work used to prove the conjecture and examine whether the assumption of $S$ being non-singular may be dropped.

We now state the second problem we referred to earlier in the context of a theory of forms of higher degree. It is the following well-known conjecture:

if $m = 3$, then $f(x) = 0$ has a non-trivial integral solution provided that $n > 9$. 
Davenport ([8]) has proved a similar statement with \( n > 15 \) in place of the condition \( n > 9 \). To bring down 15 to 9 by improving Davenport’s method seems to be almost impossible. We have recently found a new approach to this conjecture in the non-singular case; this is a direct generalization of the method used by Weil to establish the Minkowski-Hasse theorem for quadratic forms connecting the global representability with local representability. Weil’s method depends on a certain poisson formula and the use of the “metaplectic group” (which is a covering of \( SL_2 \)). We have indeed a generalization of such a poisson formula to forms of higher degree. If our approach does really settle the conjecture for forms of higher degree, then we shall be rewarded with a generalization of the beautiful relation between quadratic forms and modular functions. We believe in the existence of a new satisfactory theory of forms of higher degree and hope that the solution of the conjectures above will occupy important positions in such a theory.
Chapter 1
A Theory of Mellin Transformations

IN THIS CHAPTER, we develop a theory of Mellin transformation for the multiplicative group of an arbitrary local field. Although that is the only case in which we shall be interested, we shall start by recalling the definition of the Mellin transformation in the general case.

We shall assume results in the theory of locally compact abelian groups and Fourier transforms.

1 Generalities

1.1

Let $G$ be a locally compact abelian group and $G^*$, the Pontrjagin dual of $G$ consisting of the group $\text{Hom}(G, \mathbb{C}^*)$ of all continuous homomorphisms of $G$ into the group $\mathbb{C}^*$ of complex numbers of absolute value 1. For $g \in G$ and $g^* \in G^*$, we write $\langle g, g^* \rangle = g^*(g)$.

Let $dg$ be the Haar measure on $G$ (unique upto a positive scalar factor) and $L^1(G)$ the space of complex-valued functions which are integrable on $G$. For $F$ in $L^1(G)$, the Fourier transform $F^*$ is defined by

$$F^*(g^*) = \int_G F(g) \langle g, g^* \rangle \, dg \quad \text{for} \quad g^* \in G^*;$$
1. A Theory of Mellin Transformations

then $F^*$ belongs to the space $L^\infty(G^*)$, which is the completion relative to the uniform norm of the space of complex-valued continuous functions on $G^*$ with compact support.

Let $\Lambda(G)$ be the space of continuous functions $F$ in $L^1(G)$ whose Fourier transform $F^*$ is in $L^1(G^*)$; then $F \mapsto F^*$ is a bijection of $\Lambda(G)$ on $\Lambda(G^*)$. The Haar measure $dg^*$ on $G^*$ can be normalized so that $(F^*)^*(g) = F(-g)$ for every $F \in \Lambda(G)$ and $g \in G$. This is just the Fourier inversion theorem and the measure $dg^*$ is said to be the dual of $dg$.

1.2

Let $\Omega(G) = \text{Hom}(G, \mathbb{C}^\times)$, the group of continuous homomorphisms of $G$ into the group $\mathbb{C}^\times$ of non-zero complex numbers. An element $\omega$ of $\Omega(G)$ is called a quasicharacter of $G$. The fact that $\mathbb{C}^\times = \mathbb{R}_+^\times \times \mathbb{C}^\times$ (direct product) induces a corresponding decomposition for any $\omega$ in $\Omega(G)$. Indeed, let $r(g) = |\omega(g)|$ for any $g \in G$ and then $g \mapsto \omega(g)/r(g)$ is in $G^*$ so that for every $g$ in $G$ we have

$$\omega(g) = r(g)(g, g^*)$$

for a unique $g^*$ in $G^*$.

For any complex-valued function $F$ on $G$ and any $\omega$ in $\Omega(G)$, let us define

$$Z(\omega) = \int_G F(g) \omega(g) \, dg$$

where, for the moment, we say nothing about the existence of the integral and merely remark that we have a function $Z$ defined on a certain subset of $\Omega(G)$. We call the map $M : F \mapsto Z$ the Mellin transformation.

Substituting (1) in (2), we have

$$Z(\omega) = \int_G F(g) \, r(g)(g, g^*) \, dg$$

and the Fourier inversion theorem yields the following
1. Generalities

**Theorem 1.2.** If \( dq^* \) is the dual of the measure \( dq \) on a locally compact abelian group \( G \), \( \omega \in \Omega(G) \) and \( F : G \rightarrow \mathbb{C} \) are given, then under the assumption that the function \( g \mapsto \langle F(g)r(g), g^* \rangle \) is in \( \Omega(G) \), the Mellin transform is defined and further

\[
F(g)r(g) = \int_{G^*} Z(\omega) \langle g, -g^* \rangle \, dg,
\]

i.e.

\[
F(g) = \int_{G^*} Z(\omega) \omega(g)^{-1} \, dg.
\]

1.3

If we take \( G = \mathbb{R}^+ \), then \( \Omega(G) \) consists of \( \omega(x) = x^s \) with \( s = \sigma + it \) (\( i = \sqrt{-1} \)), \( s \) and \( t \) being real. Now \( r(x) = x^\sigma \) and \( \langle x, t \rangle = x^t = e^{ity} \) with \( y = \log x \) (or equivalently \( x = e^y \)). The map \( x \mapsto y = \log x \) gives a bijection \( \mathbb{R}^+ \rightarrow \mathbb{R} \). Now

\[
Z(s) = \int_0^\infty F(x) x^s \, d\log x = \int_{-\infty}^\infty F(e^y) e^{\sigma y} \, e\left(\frac{1}{2\pi}yt\right) \, dy.
\]

The Lebesgue measure \( dy \) on \( \mathbb{R} \) is its own dual, relative to \( (y, y') \mapsto e(yy') \). Thus Fourier’s inversion theorem gives, under the assumption that \( F(x)x^\sigma \in \Omega(\mathbb{R}^+) \) (or equivalently that \( F(e^y)e^{\sigma y} \in \Omega(\mathbb{R}) \))

\[
F(y)e^{\sigma y} = \int_{-\infty}^\infty Z(s) \, e\left(-\frac{1}{2\pi}yt\right) \frac{1}{2\pi} \, dt
\]

i.e.

\[
F(x) = \frac{1}{2\pi} \int_{-\infty}^\infty Z(s) x^{-\sigma - it} \, dt = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} Z(s) x^{-s} \, ds
\]

which is just the assertion of the Theorem above for \( G = \mathbb{R}^+ \).
1. Theory of Mellin Transformations

1.4

In the special case $G = \mathbb{R}_+^\times$ discussed above, the fact that $\Omega(\mathbb{R}_+^\times) \cong \mathbb{C}$ plays an important role; we are thereby in a position to use the theory of functions of one complex variable. We would like to know if we have, in a general situation, $G^* \subset \Omega(G)$ with $G^*$ being a real Lie group (possibly without separability) and $\Omega(G)$, its complexification. This is, in fact, true if and only if $G$ is compactly generated (i.e. generated by a compact neighbourhood of 0). From the structure of locally compact abelian groups, we know that $G$ is compactly generated if and only if there exists a unique maximal compact subgroup $C$ with $G \cong C \times \mathbb{R}^m \times \mathbb{Z}^n$. But then $G^* \cong C^* \times \mathbb{R}^m \times (\mathbb{C}_1)^n$ where $C^*$ is discrete and $\Omega(G) \cong C^* \times \mathbb{C}^m \times (\mathbb{C}^\times)^n$. In this situation, one can expect to find a good theory of Mellin transforms. As a matter of fact, we shall take $G$ to be the group $K^\times$ of non-zero elements of a local field $K$, where, by a local field, we mean $\mathbb{R}$ or $\mathbb{C}$ or a finite extension of Hensel’s field $\mathbb{Q}_p$ of $p$-adic numbers or the field of quotients of the ring of formal power-series with a finite coefficient field. In the case of $\mathbb{R}$-fields $K$ (i.e. $K = \mathbb{R}$ or $\mathbb{C}$), we have $K^\times \cong K^\times_1 \times \mathbb{R}$ where $K^\times_1$ is the compact group of elements of $K$ of absolute value 1. In the last two cases which we refer to as $p$-fields, $K^\times \cong K^\times_1 \times \mathbb{Z}$ where $K^\times_1$ is a similarly defined compact group in $K$; here $K$ is not always compactly generated while $K^\times$ is compactly generated.

2 Asymptotic Expansions

Keeping in view some applications, later on, like the determination of the asymptotic behaviour of certain integrals naturally associated over fields with forms of higher degree and of the nature of the so-called “local singular series” for such forms, we now introduce a general notion of asymptotic expansions.

Let $Y$ be a topological space and $x_\infty$, a point of the closure $\overline{X}$ of a sub-space $X$ of $Y$ such that $X$ is separable at $x_\infty$ (i.e. the system of neighbourhoods at $x_\infty$ has a countable base). Let $\phi_0, \phi_1, \phi_2, \ldots$ be a given sequence of complex-valued functions on $X$ such that for every $k \geq 0$, we have
2. Asymptotic Expansions

(i) \( \phi_k(x) \neq 0 \) for all \( x \) in \( X \) which are different from \( x_\infty \) and sufficiently close to \( x_\infty \) and

(ii) \( \phi_{k+1}(x) = O(\phi_k(x)) \) as \( x \) tends to \( x_\infty \), i.e., for given \( \epsilon > 0 \), \( |\phi_{k+1}(x)| \leq \epsilon |\phi_k(x)| \) for \( x \) in \( X \) close enough to \( x_\infty \). We then say that a complex valued function \( f \) on \( x \) has an asymptotic expansion as \( x \) tends to \( x_\infty \), if there exists a sequence \( \{a_n\}_{n \geq 0} \) of complex numbers such that for every \( k \geq 0 \) and all \( x \) in \( X \) close to \( x_\infty \),

\[
f(x) = \sum_{i=0}^{k} a_i \phi_i(x) + O(\phi_{k+1}(x))
\]

i.e.

\[
|f(x) - \sum_{i=0}^{k} a_i \phi_i(x)| \leq C |\phi_{k+1}(x)|
\]

for a constant \( C > 0 \) independent of \( x \). In symbols we denote this by

\[
f(x) \approx \sum_{k=0}^{\infty} a_k \phi_k(x) \text{ as } x \to x_\infty
\]

The symbol \( O \) and \( o \) are in the sense of Hardy-Littlewood. For the given sequence \( \{\phi_n\} \), condition (i) clearly implies the uniqueness of \( a_0, a_1, a_2, \ldots \) and therefore an asymptotic expansion for \( f \) is unique, if it exists. Further, it is sufficient to assume that (3) holds for a cofinal set of natural numbers \( k_1 < k_2 < \ldots \)

Let \( Y = \mathbb{R}^n \) and \( X \), an open set in \( \mathbb{R}^n \). Taking \( x_1, \ldots, x_n \) as coordinates in \( \mathbb{R}^n \), let for \( \alpha = (\alpha_1, \ldots, \alpha_n) \) with non-negative integers \( \alpha_1, \ldots, \alpha_n \), \( D^\alpha \) denote the differential operator \( \frac{\partial^{\alpha_1 + \cdots + \alpha_n}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \). If, in the foregoing, \( f, \phi_0, \phi_1, \phi_2, \ldots \) are infinitely differentiable and if, further, for every \( \alpha \),

\[
(D^\alpha f)(x) \approx \sum_{k=0}^{\infty} a_k (D^\alpha \phi_k)(x) \text{ as } x \to x_\infty
\]

where we assume that \( D^\alpha \phi_k \) vanish for \( 0 \leq k \leq k_\alpha \) and for \( x \) close enough to \( x_\infty \), then we say that the asymptotic expansion of \( f \) as \( x \) tends to \( x_\infty \) is termwise differentiable.

**Remarks.** It may happen that the \( a_k's \) in (4) depend on a parameter. Then we can talk of the uniformity of the asymptotic expansion relative to such
1. A Theory of Mellin Transformations

Let us illustrate the notion of asymptotic expansion for \( f \), with some examples.

**Example 1.** Take \( X = Y = \mathbb{R} \), \( x_\infty = 0 \), \( f \in C^\infty(\mathbb{R}) \) and \( \phi_k(x) = x^k \) for \( k = 0, 1, 2, \ldots \). Then the Maclaurin expansion for \( f \) at \( x = 0 \) gives just an asymptotic expansion for \( f \) as \( x \) tends to 0, namely

\[
f(x) \approx \sum_{k=0}^{\infty} \frac{f_k(0)}{k!} x^k \quad \text{as} \quad x \to 0
\]

(Note that \( \approx \) does not, in general, imply equality). Further, the asymptotic expansion is termwise differentiable, as is obvious.

This example is quite simple but all the important features of an asymptotic expansion are incorporated here. We now give a slightly more complicated example.

**Example 2.** Let \( 0 \leq \lambda_0 < \lambda_1 < \lambda_2 < \ldots \) be a monotonic increasing sequence of non-negative real numbers with no finite accumulation point and let \( m_0, m_1, m_2, \ldots \) be a sequence of natural numbers. Take \( Y = \mathbb{R} \), \( X = \mathbb{R}_+^* \), \( x_\infty = 0 \) and

\[
x^{\lambda_0} (\log x)^{m_0-1}, \ldots, x^{\lambda_i} (\log x)^{m_i-1}, \ldots, x^{\lambda_i}, \ldots
\]

as \( \phi_0(x), \phi_1(x), \ldots \). It is easy to check that condition (i) and (ii) are fulfilled. We can thus talk of the following asymptotic expansion for a function \( f \) as \( x \) tends to 0, namely

\[
f(x) \approx \sum_{k=0}^{m_k} \sum_{m=1}^{m_i} a_{k,m} x^{\lambda_k} (\log x)^{m-1} \quad \text{as} \quad x \to 0
\]

If we set

\[
R_k(x) = f(x) - \sum_{i=0}^{k} \sum_{m=1}^{m_i} a_{i,m} x^{\lambda_i} (\log x)^{m-1}
\]
3. The Classical Case

3.1 The Statement of a Theorem

We are interested in explicitly defined function spaces on which the Mellin transformation is an isomorphism. We begin with the classical case where $G$ is the multiplication group $\mathbb{R}_+^\times$ of positive real numbers. Let $0 \leq \lambda_0 < \lambda_1 < \ldots$ be a strictly increasing sequence of non-negative real numbers with no finite limit point and let $m_0, m_1, m_2, \ldots$ be a sequence of natural numbers. We introduce two function spaces $F$ and $Z$ as follows. The space $F$ is defined as the set of all complex-valued functions $F$ on $\mathbb{R}_+^\times$ such that (i) $F \in C^\infty(\mathbb{R}_+^\times)$, (ii) $F(x)$ behaves like a Schwartz function as $x$ tends to $\infty$ i.e., $F(x) = o(x^\rho)$ as $x \to \infty$ for every $\rho$ and every $n \geq 0$ and (iii) $F(x) \approx \sum_{k=0}^{\infty} a_k x^{k_1} (\log x)^{m_1-1}$ as $x \to 0$ (with constants $a_k, m_1$) and this asymptotic expansion is termwise differentiable. $F$ is clearly nonempty, since the function $F = 0$ is in $F$. If $\lambda_n = n$ for every $n \geq 0$, then $F(x) = e^{-x}$ (for $x \geq 0$) is in the corresponding $F$. The space $Z$ consists of all meromorphic functions $Z(s)$ of the complex variable $s$ for which

1. $Z(s)$ has poles at most at the points $-\lambda_0, -\lambda_1, -\lambda_2, \ldots$,

2. $Z(s) = \sum_{m=1}^{m_1} \frac{b_{k,m}}{s - \lambda_k}$ for suitable constants $b_{k,m}$ is holomorphic in a neighbourhood of the point $s = -\lambda_k$ and
3. for every polynomial $P(s)$ in $s$, the function $P(s)Z(s)$ is bounded in every vertical strip $B_{\sigma_1, \sigma_2} = \{s = \sigma + ti \in \mathbb{C} \mid \sigma_1 \leq \sigma \leq \sigma_2\}$ for $-\infty < \sigma_1 < \sigma_2 < \infty$ after deleting small neighbourhoods of $-\lambda_k$ for every $k$ from $B_{\sigma_1, \sigma_2}$. Under the correspondence $M$ between $\mathcal{F}$ and $\mathcal{Z}$ in Theorem 3.1 below, Euler’s gamma function $\Gamma(s)$ corresponds in $\mathcal{Z}$ to the function $F(x) = e^{-x}(x \geq 0)$ in $\mathcal{F}$ when $\lambda_n = n$ for every $n \geq 0$.

We shall now state and prove a theorem which has many good applications.

**Theorem 3.1.** We have $M : \mathcal{F} \rightarrow \mathcal{Z}$. More precisely, for any $F$ in $\mathcal{F}$,

$$(MF)(s) = \int_{0}^{\infty} F(x)x^{s} \, d\log x$$

defines a holomorphic function on $\mathbb{C}_+ = \{s = \sigma + ti \in \mathbb{C} \mid \sigma > 0\}$ and its meromorphic continuation belongs to $\mathcal{Z}$. Conversely, if $Z$ is in $\mathcal{Z}$, then

$$(M^{-1}Z)(x) = \frac{1}{2\pi i} \int_{\sigma - \infty i}^{\sigma + \infty i} Z(s)x^{-s} \, ds$$

gives rise to a function in $\mathcal{F}$ independently of $\sigma$ for $\sigma > 0$. Moreover

$$b_{k,m} = (-1)^{m-1}(m-1)! \, d_{k,m} \quad (5)$$

for every $k$ and $m$.

3.2

Before we commence the proof of Theorem 3.1, let us make a comment on its statement but, what is more important, namely, also a few remarks to simplify later arguments for the proof of the theorem.

For a moment, denote $M^{-1}Z$ in the statement of Theorem 3.1 by $NZ$ instead. As soon as we know that $M(\mathcal{F}) \subset \mathcal{Z}$ and $N(\mathcal{Z}) \subset \mathcal{F}$, then $NM$ is identity on $\mathcal{F}$ and $MN$ is identity on $\mathcal{Z}$. This follows from Fourier’s inversion theorem. Therefore we can write $N = M^{-1}$. 
Let $D = \sum_{k=0}^{I} p_k(x) \frac{d^k}{dx^k}$ be a differential operator with coefficients $p_k(x)$ which are polynomials in $x$ having complex coefficients. We say that $D$ is homothety-invariant if, for every $f \in C^\infty(\mathbb{R}_+^2)$ and $\lambda > 0$, we have $D(f(\lambda x)) = (Df)(\lambda x)$ for all $x$. Clearly $D$ is such an operator, if and only if $p_k(x) = c_k x^k$ for every $k$. The space of such homothety-invariant differential operators is spanned over $\mathbb{C}$ by $x^k \frac{d^k}{dx^k}$ ($k = 0, 1, 2, \ldots$). On the other hand, we have

$$\left(x \frac{d}{dx}\right)^2 = \left(x \frac{d}{dx}\right) \left(x \frac{d}{dx}\right) = x \frac{d}{dx} + x^2 \frac{d^2}{dx^2}$$

and generally, if

$$\left(x \frac{d}{dx}\right)^n = \sum_{i=1}^{n} c_{n,i} x^i \frac{d^i}{dx^i},$$

then

$$\left(x \frac{d}{dx}\right)^{n+1} = \sum_{i=1}^{n+1} c_{n+1,i} x^i \frac{d^i}{dx^i}$$

where $c_{n+1,i} = ic_{n,i} + c_{n,i-1}$. Thus the space of the homothety-invariant differential operators above is just the $\mathbb{C}$-span of the differential operators $1, x \frac{d}{dx}, (x \frac{d}{dx})^2, \ldots$

We now remark that $\mathcal{F}$ is stable under homothety-invariant differential operators (with polynomial coefficients). For this, it suffices to prove, in view of the foregoing, that $\mathcal{F}$ is invariant under $x \frac{d}{dx}$ and in turn to verify condition (iii) for $xF'$ whenever $F \in \mathcal{F}$. Now

$$F(x) \approx \sum_{k=0}^{\infty} \sum_{j=1}^{m_k} a_{k,j} x^k (\log x)^{j-1}$$

for $F \in \mathcal{F}$ as $x \to 0$ implies that

$$F'(x) \approx \sum_{k=0}^{\infty} \sum_{j=1}^{m_k} a'_{k,j} x^{k-1} (\log x)^{j-1}$$
1. A Theory of Mellin Transformations

where \( a'_{k,j} = \lambda_k a_{k,j} + ja_{k,j+1} \) for \( 1 \leq j \leq m_k - 1 \) and \( a'_{k,m_k} = \lambda_k a_{k,m_k} \).

Thus, for every \( F \in \mathcal{F} \),

\[
x F'(x) \approx \sum_{k=0}^{\infty} \sum_{j=1}^{m_k} a'_{k,j} x^k (\log x)^{j-1} \quad \text{as} \ x \to 0
\]

and so \( x F' \in \mathcal{F} \).

For \( \mathcal{Z} \), we have a corresponding property, namely that \( \mathcal{Z} \) is stable under multiplication by polynomials in \( s \). This is quite obvious, since for \( z \in \mathcal{Z} \), multiplication by polynomials only serves to improve the situation with regard to conditions (1) and (2), while there is nothing new to be verified regarding condition (3).

We assert further that for \( F \in \mathcal{F} \),

\[
M(x F'(x))(s) = (-1)^k (s + k - 1) \ldots s (MF)(s)
\]

for every \( s \in \mathbb{C}_+ \) and \( k \geq 1 \). For \( k = 1 \), the formula is clear since

\[
M(x F'(x))(s) = \int_0^\infty x F'(x) x^s \, d\log x
\]

\[
= F(x) x^s \bigg|_0^\infty - s \int_0^\infty F(x) x^s \, d\log x
\]

\[
= -s(MF)(s), \quad \text{since the first term is 0.}
\]

Assuming (6) with \( k - 1 \) in place of \( k \) (\( \geq 2 \)), we have again, from integration by parts,

\[
M(x^k F'(x))(s) = F^{(k-1)}(x) x^{s+k-1} \bigg|_0^\infty - (s + k - 1)M(x^{k-1} F'(x))(s)
\]

\[
= 0 + (-1)^k (s + k - 1) \ldots s (MF)(s).
\]

For the sake of completeness, we include the following well-known lemma on the interchange of differentiation and integration.

**Lemma 3.2.** Let \( X \) be a locally compact measure space with \( dx \), a Borel measure on \( X \) and let, for an interval \( I = (a, b) \subset \mathbb{R} \), \( f(x, t) \) be a continuous function on \( X \times I \) satisfying the conditions:
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\[
f(x, t) \in L^1(X, dx) \text{ for every } t \in I,
\]

\[
\frac{\partial}{\partial t} f(x, t) \text{ is continuous and } \\
|\frac{\partial}{\partial t} f(x, t)| \leq \phi(x) \text{ for a } \phi \in L^1(X, dx) \text{ for every } t \in I.
\]

Then

\[
\frac{d}{dt} \int_X f(x, t) dx = \int_X \frac{\partial}{\partial t} f(x, t) dx.
\]

**Proof.** The hypotheses imply the existence of the integrals \((\phi(t) \overset{\text{def}}{=} \int_X f(x, t) dx)\) and \((\psi(t) \overset{\text{def}}{=} \int_X \frac{\partial}{\partial t} f(x, t) dx).\) On the other hand, for every \(t, t' \in I,\) we have

\[
f(x, t') = f(x, t) + (t' - t) \frac{\partial}{\partial t} f(x, \tau) \text{ for some } \tau \text{ between } t \text{ and } t'.
\]

Therefore

\[
\left| (\phi(t') - \phi(t))/ (t' - t) - \psi(t) \right| \leq \int_X \left| \frac{\partial}{\partial t} f(x, \tau) - \frac{\partial}{\partial t} f(x, t) \right| dx
\]

and the lemma follows from Lebesgue’s theorem. \(\square\)

### 3.3 Proof of Theorem 3.1

(i) First we show that for every \(F \in \mathcal{F}, Z = MF\) is in \(Z.\)

Let \(s = \sigma + ti \in \mathbb{C}_+\) and \(0 < \sigma_1 \leq \sigma \leq \sigma_2 < \infty.\) Taking \(\epsilon, n\) with \(0 < \epsilon < \sigma_1 \text{ and } n > \sigma_2,\) define

\[
\phi(x) = \begin{cases} 
 x^{\sigma_1 - \epsilon} \max_{0 < x \leq 1} (x^\epsilon |F(x)|), & 0 < x \leq 1 \\
 x^{\sigma_2 - n} \max_{x \leq 1} (x^n |F(x)|), & x > 1
\end{cases}
\]

Then it is easy to verify that \(\phi \in L^1(\mathbb{R}_+^+, d \log x)\) and further \(\phi\) dominates \(F(x)x^\epsilon.\) Therefore the integral defining \(Z(s)\) converges absolutely and the integrand being holomorphic, it follows, in view of the arbitrary nature of \(\sigma_1 \text{ and } \sigma_2,\) that \(Z(s)\) is holomorphic in \(\mathbb{C}_+.\) Since for every \(\sigma, \sigma_2 \text{ with } \sigma \leq \sigma_2 < \infty, F(x)x^\epsilon\) is dominated by a function in
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$L^1((1,\infty), d\log x)$, say the restriction of the function $\phi$ above to $(1,\infty)$, it follows similarly that \( \int F(x)x^s\,d(\log x) \) is indeed an entire function of $s$. Now for every $k$ and $\rho < \lambda_k+1$,

\[
R_k = F(x) - \sum_{i=0}^k \sum_{j=1}^{m_i} a_{i,j} x^{\lambda_i}(\log x)^{j-1}
\]

= \( o(x^\rho) \) by condition(3)' \hspace{1cm}(7)

Therefore in \((0,1]\), $R_k(x)x^\sigma$ is dominated by the function

\[
\psi(x) \overset{\text{def}}{=} x^{\sigma_1+\rho} \max_{0<x\leq 1} x^{-\rho} |R_k(x)|
\]

which clearly belongs to $L^1((0,1]\,d(\log x))$ for every $\sigma_1$ with $-\lambda_k+1 < \sigma_1 \leq \sigma$ and $-\sigma_1 < \rho < \lambda_k+1$. Thus, for similar reasons as above, \( \int R_k(x)x^s\,d(\log x) \) is holomorphic in $s = \sigma + ti$ for $\sigma > -\lambda_k+1$. Finally, for any $\lambda \geq 0$ and $\sigma > 0$, we have

\[
\int_0^1 x^{\sigma+\lambda}d(\log x) = \frac{1}{s+\lambda}
\]

Differentiating the integral $j-1$ times with respect to $s$ for $j \geq 1$, we get (in view of Lemma 3.2) that

\[
\int_0^1 x^{\sigma+\lambda}(\log x)^{j-1}d(\log x) = \frac{(-1)^{j-1}(j-1)!}{(s+\lambda)^j}
\]

Putting together all the facts above and using (7) and (8), we have

\[
Z(s) = \left\{ \int_0^1 + \int_{1}^{\infty} F(x)x^s d(\log x) \right\}
\]

= \( \sum_{i=0}^k \sum_{j=1}^{m_i} b_{i,j} \frac{1}{(s+\lambda_i)^j} + \int_0^1 R_k(x)x^s d(\log x) + \int_0^{\infty} F(x)x^s d(\log x) \) \hspace{1cm}(9)
with $b_{ij} = (-1)^{j-1}(j-1)!a_{ij}$, where the second term in (9) represents a holomorphic function of $s = \sigma + ti$ for $\sigma > -\lambda_{k+1}$ and the third term is an entire function of $s$. Since $k$ is arbitrary, we have thus verified conditions (1) and (2) for $Z$ to belong to $\mathcal{Z}$. The existence of a function in $L^1((1, \infty))$ dominating the integrand shows that the third term in (9) is bounded for $s$ in any vertical strip $B_{\sigma_1, \sigma_2}$; for a similar reason, the second term is also bounded in $B_{\sigma_1, \sigma_2}$ (with $k$ chosen correspondingly) while the first term is bounded, if we delete therefrom neighbourhoods of the points $-\lambda_0, -\lambda_1, -\lambda_2, \ldots$. Thus condition (3) for $Z$ to be in $\mathcal{Z}$ is fulfilled with $P(s) \equiv 1$. To verify condition (3) for arbitrary $P(s)$ we may assume, without loss of generality, that

$$P(s) = \sum_{i=0}^{n} (-1)^{i} a_{i} s(s+1) \ldots (s+i-1)$$

with $a_i$ in $\mathbb{C}$. Then working with $\sum_{i=0}^{n} a_i x^i F^{(i)}(x)$ in place of $F$ above, as is legitimate, we may, in view of the remarks in §3.2, conclude that $P(s)Z(s)$ is bounded in every vertical strip $B_{\sigma_1, \sigma_2}$.

(ii) Let us now prove the converse that for every $Z \in \mathcal{Z}$, $F = M^{-1}Z \in \mathcal{F}$.

Let, for $0 < \sigma_1 < \sigma_2$, $L$ denote the boundary (traversed anticlockwise) of the rectangle in the $s$-plane with vertices at $\sigma_2 + t_0 i$, $\sigma_1 + t_0 i$, $\sigma_1 - t_0 i$, $\sigma_2 - t_0 i$ (for $t_0 > 0$). Let $x > 0$; then by Cauchy’s theorem,

$$\int_{L} Z(s)x^{-s} \, ds = 0,$$

since the integrand is holomorphic in $\mathbb{C}_+$. But, in view of the growth condition satisfied by $Z$ in vertical strips, the integral taken over the horizontal sides of $L$ tends to 0 as $t_0$ tends to infinity. Thus

$$\int_{\sigma_1 - t_0 i}^{\sigma_1 + t_0 i} Z(s)x^{-s} \, ds = \int_{\sigma_2 - t_0 i}^{\sigma_2 + t_0 i} Z(s)x^{-s} \, ds$$

enabling us to conclude that, for $x > 0$,

$$F(x) = \frac{1}{2\pi i} \int_{\sigma - t_0 i}^{\sigma + t_0 i} Z(s)x^{-s} \, ds$$

(10)
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is defined independently of $\sigma > 0$. The growth condition (3) satisfied by $Z$ ensures the absolute convergence of the integrals involved above. Using Lemma 3.2, we obtain for $k \geq 1$, that

$$x^k F(x) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} (-1)^k s(s+1) \cdots (s+k-1)Z(s)x^{-s} \, ds$$  \hspace{1em} (11)

Condition (3) for $Z$ again guarantees the absolute convergence of the integral and we conclude that $F \in C^\infty(R_+)$ and further $F$ behaves at infinity like a Schwartz function, since (11) implies that

$$|x^{\sigma+k}F(x)| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |s(s+1) \cdots (s+k-1)Z(s)| \, dt < \infty.$$  \hspace{1em} (12)

We are left with proving condition (iii) for $F$ to be in $\mathcal{F}$. Choose $\rho$ such that $\lambda_k < \rho < \lambda_{k+1}$ and let, for $t_0 > 0$, the contours $L_1$ and $L_2$ be defined respectively as the boundary of the rectangle with vertices at $\sigma + t_0i\, , \, -\rho + t_0i\, , \, -\rho - t_0i\, , \, \sigma - t_0i$ and the union of the line segments $\{\sigma - ti \mid t_0 \leq t < \infty\}, \{u - t_0i \mid -\rho \leq u \leq \rho\}, \{-\rho + ti \mid -t_0 \leq t \leq t_0\}, \{u + t_0i \mid -\rho \leq u \leq \sigma\}$ and $\{\sigma + ti \mid t_0 \leq t < \infty\}$. While the contour $L_1$ is traversed in the counter-clockwise direction, the contour $L_2$ is covered clockwise from $\sigma - i\infty$. From (11), we have

$$F(x) = \frac{1}{2\pi i} \left( \int_{L_1} + \int_{L_2} \right) Z(s)x^{-s} \, ds.$$  \hspace{1em} (11)

The integration over $L_1$ gives, by Cauchy’s theorem, the sum of the residues of the integrand at the poles inside $L_1$. Condition (3) for $Z$ implies that the contribution to the integral over $L_2$ from the horizontal segments tends to $0$ as $t_0$ tends to $\infty$. Making $t_0$ tend to $\infty$, we obtain for $F(x)$ the expression

$$\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} Z(s)x^{-s}ds = \frac{1}{2\pi i} \int_{-\rho-i\infty}^{-\rho+i\infty} Z(s)x^{-s}ds + S$$  \hspace{1em} (12)
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where \( S \) is the sum of the residues of \( Z(s)x^{-s} \) at the points \( s = -\lambda_0, -\lambda_1, -\lambda_2, \ldots \). The residue at \( s = -\lambda_i \) is just the coefficient of \((s + \lambda_i)^{-1}\) in the power-series expansion at \(-\lambda_i\) of the function

\[
x^{\lambda_i}e^{-(s+\lambda_i)\log x} \sum_{m=1}^{m_i} b_{i,m}(s + \lambda_i)^{-m}
\]

and is seen to be equal to \( \sum_{m=1}^{m_i} x^{\lambda_i}(\log x)^{m-1} a_{i,m} \) with \( a_{i,m} \) precisely as given by (5). In view of this, the integral on the right hand side of (12) is just \( R_k(x) \) introduced earlier. However, the integral can be directly majorised by

\[
\frac{1}{2\pi i} x^\rho \int_{-\infty}^{\infty} |Z(-\rho + ti)| dt;
\]

we thus see for every \( k \geq 1 \), that \( R_k(x) = O(x^\rho) \) where \( \rho \) is arbitrary but subject to the condition that \( \lambda_k < \rho < \lambda_{k+1} \). Consequently, \( R_k(x) = o(x^\rho) \) for every \( \rho < \lambda_k+1 \) and for every \( k \geq 1 \).

We have finally to prove the termwise differentiability of the asymptotic expansion of \( F(s) \) as \( x \to 0 \). By (11) with \( k = 1 \) and our remarks in §3.2, \(-sZ(s)\) is given in \( \mathbb{Z} \) and condition (2) is fulfilled for it with

\[
b'_{k,m} = \begin{cases} 
\lambda_k b_{k,m} - b_{k,m+1} & \text{for } 1 \leq m \leq m_k - 1 \\
\lambda_k b_{k,m} & \text{for } m = m_k
\end{cases}
\]

in place of \( b_{k,m} \). This follows from the fact that

\[
-sZ(s) = (\lambda_k - (s + \lambda_k))Z(s) = (\lambda_k - (s + \lambda_k)) \sum_{m=1}^{m_k} \frac{b_{k,m}}{(s + \lambda_k)^m} + \text{a function holomorphic at } -\lambda_k.
\]

Applying the arguments above to \(-sZ(s)\) in place of \( Z(s) \), we obtain, as indicated, the function \( xF'(x) \) in place of \( F(x) \) and further

\[
xF'(x) \approx \sum_{k=0}^{\infty} \sum_{m=1}^{m_k} \tilde{a}_{k,m} x^{\lambda_k}(\log x)^{m-1}
\]

where \( \tilde{a}_{k,m} = (-1)^{m-1} b'_{k,m} / (m - 1)! \). But using (5), we see that \( \tilde{a}_{k,m} \) is the same as \( d_{k,m}' \) defined in §3.2. From our remarks in §3.2 this
means precisely that the asymptotic expansion of \( F(x) \) as \( x \to 0 \) can be differentiated termwise once. Iteration of this procedure (using (6)), gives us that termwise differentiation is valid any number of times and Theorem 3.1 is completely proved.

4 The Case of \( \mathbb{R} \)-fields

4.1

In this section, we shall prove the analogues of Theorem 3.1 for the cases when \( G \) is multiplicative group \( K^\times (= K \setminus \{0\}) \) of an \( \mathbb{R} \)-field \( K \), i.e, for \( K = \mathbb{R} \) or \( \mathbb{C} \).

We first fix some notation applicable to any local field \( K \). For \( a \in K \), we define the modulus \( |a|_K \) of \( a \) by

\[
|a|_K = \begin{cases} 
\text{the rate of change of the measure in } K \text{ under } x \to ax \\
\text{for } x \in K \text{ and } a \neq 0 \\
0, \text{ for } a = 0.
\end{cases}
\]

It is well-known that \( |a|_\mathbb{R} = |a| \) and \( |a|_\mathbb{C} = |a|^2 \) where \( |\cdot| \) denotes the usual absolute value in \( \mathbb{R} \) or \( \mathbb{C} \).

We had introduced in §1.2 the group \( \Omega(K^\times) \) of quasicharacters of \( K \) as the group (with the compact open topology) of continuous homomorphisms of \( K^\times \) into \( \mathbb{C}^\times \). Then \( \Omega(K^\times)^0 \), the connected component of the identity, consists of \( \omega_s \) for \( s(= \sigma + ti) \in \mathbb{C} \) defined by \( \omega_s(x) = |x|_K^s \) for every \( x \in K^\times \). For \( \omega \in \Omega(K^\times) \), the associated quasicharacter \( |\omega(x)| \) is given by \( |x|_K^{\sigma(\omega)} = \omega_{\sigma(\omega)}(x), \sigma(\omega) \) being in \( \mathbb{R} \). We also set \( \Omega_+(K^\times) = \{ \omega \in \Omega(K^\times) | \sigma(\omega) > 0 \} \) and for an \( \mathbb{R} \)-field \( K \), introduce the angular component \( \text{ac}(x) \) for \( x \in K^\times \) by \( \text{ac}(x) = x/|x| \). For \( x \in \mathbb{R}^\times \), we have \( \text{ac}(x) = \text{sgn}(x) \), the usual sign-function on \( \mathbb{R} \).

4.2 The Case \( K = \mathbb{R} \)

Let us assume, as before, that \( 0 \leq \lambda_0 < \lambda_1 < \ldots \) is a given strictly increasing sequence of non-negative real numbers with no finite accumulation point and \( \{m_k\}_{k \geq 0} \) a given sequence of natural numbers. Correspondingly we define for \( K = \mathbb{R} \) (or \( G = \mathbb{R}^\times \)) the function-spaces \( F \).
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and $\mathcal{Z}$ as follows. The space $\mathcal{F}(= \mathcal{F}(\mathbb{R}^\times))$ is defined as the set of all complex-valued functions $F$ such that

(i) $F \in C^\infty(\mathbb{R}^\times)$,

(ii) $F$ behaves like a Schwartz function of $x$ as $|x|$ tends to infinity i.e. $F^{(n)} = o(|x|^{-\rho})$ as $|x| \to \infty$ for every $\rho$ and every $n \geq 0$, and

(iii) $F(x) \approx \sum_{k=0}^{\infty} \sum_{m} a_{k,m}(\sgn x)|x|^k(\log |x|)^{m-1}$ as $|x| \to 0$ and this asymptotic expansion is termwise differentiable. We write

$$a_{k,m}(u) = a_{k,m,0} + ua_{k,m,1}$$

for $u = +1$ or $-1$; this may be regarded as the (trivial) Fourier expansion of functions on the group $\{1, -1\}$.

The group $\Omega(\mathbb{R}^\times) = \{\omega_s(\sgn p); s \in \mathbb{C}, p = 0, 1\}$ consists of two copies of $\mathbb{C}$ indexed by $p = 0$ or $1$.

The space $\mathcal{Z}(= \mathcal{Z}(\Omega(\mathbb{R}^\times)))$ is defined as the set of all complex-valued functions $Z$ on $\Omega(\mathbb{R}^\times)$ such that

1. $Z(\omega_s(\sgn p))$ is meromorphic on $\Omega(\mathbb{R}^\times)$ with poles at most for $s = -\lambda_0, -\lambda_1, -\lambda_2, \ldots$,

2. $Z(\omega_s(\sgn p)) - \sum_{m=1}^{m_0} \frac{b_{m,s,p}}{s + \lambda_m}$ is holomorphic for $s$ close to $-\lambda_k$, for every $k \geq 0$ and

3. for every polynomial $P \in \mathbb{C}[s]$ and every $\sigma_1, \sigma_2$, the function $p(s)Z(\omega_s(\sgn p))$ is bounded for $s$ in a vertical strip $B_{\sigma_1, \sigma_2}$, with neighbourhoods of the points $-\lambda_0, -\lambda_1, \ldots$ removed therefrom.

The following theorem is an immediate consequence of Theorem 3.1 and the definitions given above.

**Theorem 4.2.** We have a bijective correspondence $M_\mathbb{R}$ (abbreviated as $M$) between $\mathcal{F} = \mathcal{F}(\mathbb{R}^\times)$ and $\mathcal{Z} = \mathcal{Z}(\Omega(\mathbb{R}^\times))$. More precisely, if $F \in \mathcal{Z}$, then

$$(MF)(\omega) = \int_{\mathbb{R}^\times} F(x)\omega(x) \, d^\times x \quad \text{with} \quad d^\times x = \frac{dx}{2|x|_{\mathbb{R}}}$$
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defines a holomorphic function on $\Omega_+(\mathbb{R}^\times)$ which has a meromorphic continuation belonging to $\mathbb{Z}$. Conversely, for $z \in \mathbb{Z}$, the integral
\[
(M^{-1}Z)(x) = \sum_{p=0,1} \left( \frac{1}{2\pi i} \int_{\sigma-\infty}^{\sigma+\infty} Z(\omega_s (\text{sgn})^p)|x|^{-s} ds \right) (\text{sgn} x)^{-p}
\]
defines a function $F$ in $\mathcal{F}$ independently of $\sigma$, for $\sigma > 0$. Furthermore
\[
b_{k,m,p} = (-1)^{m-1}(m-1)! a_{k,m,p}
\]
for every $k, m$ and $p$.

**Proof.** By introducing suitable definitions, we shall reduce ourselves to the situation in §3. For the given $F$ and for $x \in \mathbb{R}^\times$, let us define
\[
F_p(x) = \frac{1}{2} (F(x) + (-1)^p F(-x)), \quad p = 0, 1.
\]
Then $F(ux) = F_0(x) + u F_1(x)$ for $u = \pm 1$ and therefore, for every $x \in \mathbb{R}^\times$, we have
\[
F(x) = F_0(|x|) + (\text{sgn} x) F_1(|x|).
\]
Similarly, for $Z$ defined on $\Omega(\mathbb{R}^\times)$, let us introduce a function $Z_p$ on $\mathbb{C}$ for $p = 0, 1$ by the prescription
\[
Z_p(s) = Z(\omega_s (\text{sgn})^p) \quad \text{for } s \in \mathbb{C}
\]
Then, in a purely formal fashion, we have
\[
M_{\mathbb{R}}F = Z \iff MF_p = Z_p \text{ for } p = 0 \text{ and } 1.
\]
In fact, let $M_{\mathbb{R}}F = Z$. Then, by the definition of $Z_p$,
\[
Z_p(s) = \int_{\mathbb{R}^\times} F(x)|x|^{-s} (\text{sgn} x)^p \frac{dx}{2|x|} = \int_0^\infty \left( \frac{1}{2} (F(x) + (-1)^p F(-x))x^s d(\log x) \right)
\]
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$= (MF_p)(s)$

for $p = 0, 1$. Conversely, let $MF_p = Z_p$ for $p = 1, 2$. Then, any $\omega$ in $\Omega(\mathbb{R}^\infty)$ being of the form $\omega_p(sgn)^p$ for $p = 0$ or $1$ and $s$ in $\mathbb{C}$, $Z(\omega) = Z_p(s)$ for a unique $p = 0$ or $1$. But $Z_p(s) = (MF_p)(s)$ and we have only to retrace our steps in the foregoing to conclude that $Z(\omega) = (M_{\mathbb{R}}F)(\omega)$.

Finally, looking at the relations (14) - (16) above between $F$ and $F_0$, $F_1$ and between $Z$ and $Z_0$, $Z_1$, we conclude from the definitions of the various function spaces that

$$F \in \mathcal{F}(\mathbb{R}^\infty) \Rightarrow F_0, F_1 \in \mathcal{F}$$

$$\Rightarrow Z_0, Z_1 \in \mathcal{Z} \quad \text{by Theorem 3.1}$$

$$\Rightarrow Z \in \mathcal{Z}(\Omega(\mathbb{R}^\infty))$$

Similarly, we have

$$Z \in \mathcal{Z}(\Omega(\mathbb{R}^\infty)) \Rightarrow Z_0, Z_1 \in \mathcal{Z}$$

$$\Rightarrow F_0, F_1 \in \mathcal{F} \quad \text{by Theorem 3.1}$$

$$\Rightarrow F \in \mathcal{F}(\mathbb{R}^\infty).$$

From the uniqueness of the coefficients in asymptotic expansions and the definitions (13) and (14), it follows that for $p = 0, 1$, the coefficients $a_{k,m,p}$ correspond to $F_p(x)$ in $\mathcal{F}$ (for $\mathbb{R}^\infty$) in the same way as $a_{k,m}$ to $F \in \mathcal{F}(\mathbb{R}^\infty)$ for every $k, m$. Similarly from (16), the coefficients $b_{k,m,p}$ feature in the expansion of $Z_p(s)$ at $s = -\lambda_k$ for $p = 0, 1$. The last assertion of the theorem now follows from Theorem 3.1 together with the fact that, for $p = 0, 1$, the function $F_p$ and $Z_p$ correspond to each other (under the correspondence $M$ in §3.1).

4.3 The Case $K = \mathbb{C}$

We shall now prove an analogue of Theorem 3.1 for the complex case.

Let, as before, $0 \leq \lambda_0 \leq \lambda_1 < \lambda_2 < \ldots$ be a strictly increasing sequence of non-negative real numbers with no finite accumulation point and $\{m_k\}_{k \geq 0}$ a sequence of natural numbers. Let $\mathcal{F}$ be the associated space of complex-valued functions $F$ on $\mathbb{C}^\infty$ such that
(i) \( F \in \mathbb{C}^\infty(\mathbb{C}^\times) \), (ii) \( F(x) \) behaves like a Schwartz function as \(|x|_C| \to \infty \), namely,
\[
\frac{\partial^{a+b} F}{\partial x^a x^b}(x) = o(|x|^{-\rho}_C)
\]
as \(|x|_C| \to \infty \), for every \( \rho > 0 \) and for every \( a, b \geq 0 \) and (iii) \( F(x) \approx \sum_{k=0}^{\infty} \sum_{m=1}^{m_k} a_{k,m} \log |x|_C^{m-1} \) as \(|x| \to 0 \) with \( a_{k,m} \in \mathbb{C}^\infty(\mathbb{C}^\times) \) for every \( k, m \), is an asymptotic expansion which is termwise differentiable and uniformly in \( \log |x|_C \) even after termwise differentiation (in the sense that for every \( k \geq 0 \) and for every \( \rho < \lambda_k + 1 \) and for given \( \epsilon > 0 \), there exists \( \delta \in \mathbb{C} \) independent of \( \log |x|_C \) such that for \(|x|_C < \delta\), we have
\[
|R_k(x)| = \left| \sum_{i=0}^{m_k} \sum_{m=1}^{m_k} a_{i,m} \log |x|_C^{m-1} \right| \leq \epsilon |x|_C^\rho
\]
and further, similar inequalities hold even after the termwise differentiation of the asymptotic expansion.

Since for every \( k, m \) we have \( a_{k,m} \in \mathbb{C}^\infty(\mathbb{C}^\times) \), we have the Fourier expansion \( a_{k,m}(u) = \sum_{p \in \mathbb{C}} a_{k,m,p} u^p \).

Putting \(|x|_C = r \) and \( u = \log |x|_C \) for \( x \in \mathbb{C}^\times \), we have \( x = r e^{i \theta} \) and
\[
d^nx = \frac{dx}{2\pi |x|_C} = d(r) d\theta.
\]

It is easy to check that
\[
\frac{\partial}{\partial r} = \frac{1}{2} \left( x \frac{\partial}{\partial x} + \bar{x} \frac{\partial}{\partial \bar{x}} \right) \quad \text{and} \quad \frac{\partial}{\partial \theta} = 2\pi i \left( x \frac{\partial}{\partial x} - \bar{x} \frac{\partial}{\partial \bar{x}} \right) \quad (17)
\]

Suppose \( D \) is a differential operator of the form
\[
\sum_{0 \leq a, b \leq n} p_{a,b}(x, \bar{x}) \frac{\partial^{a+b}}{\partial x^a \bar{x}^b}
\]
with polynomials \( p_{a,b} \) in \( x, \bar{x} \) having coefficients in \( \mathbb{C} \). We call \( D \) homothety-invariant, if for every \( F \in \mathbb{C}^\infty(\mathbb{C}^\times) \) and \( t \in \mathbb{C}^\times \), we have \((DF)(tx) = D(F(tx)) \) identically in \( x \). The space of such homothety-invariant differential operators is easily seen to be spanned over \( \mathbb{C} \) by differential
operators of the form \((x\frac{\partial}{\partial x})^a(x\frac{\partial}{\partial x})^b\) for \(a, b \geq 0\) in \(\mathbb{Z}\) and hence, in view of (17), the differential operators \(D_{a,b} = \frac{1}{(2\pi)^b} r^a \frac{\partial^{a+b}}{\partial r^a \partial \theta^b}\) for \(a, b \geq 0\) in \(\mathbb{Z}\) also generate the above space over \(C\).

For \(F \in C^\infty(\mathbb{C}^\times)\) and for fixed \(r = |x|_C\), we have the (absolutely convergent) Fourier expansion

\[F(x) = \sum_{p \in \mathbb{Z}} F_p(r)u^p.\]  

We shall characterise \(F\) being in \(\mathcal{F}(\mathbb{C}^\times)\) in terms of an alternative set of conditions involving Fourier coefficients \(F_p\) in (18) as follows:

(I) \(F \in C^\infty(\mathbb{C}^\times) \iff \{F_p\}_{p \in \mathbb{Z}}\) as in I above and further, for every \(a, b \geq 0\) in \(\mathbb{Z}\) and \(r_1, r_2\) with \(0 < r_1 < r_2 < \infty\), we have

\[\sup_{p \in \mathbb{Z}, r_1 \leq r \leq r_2} |p^b F_p^{(a)}(r)| < \infty\]  

Moreover, when these equivalent conditions hold, the Fourier expansion (18) is termwise differentiable, i.e.

\[r^{-a}D_{a,b}F(x) = \frac{1}{(2\pi)^b} \frac{\partial^{a+b}}{\partial r^a \partial \theta^b} F(x) = \sum_{p \in \mathbb{Z}} p^b F_p^{(a)}(r)u^p\]  

for every \(a, b \geq 0\) in \(\mathbb{Z}\)  \(\ldots \) (20)

(II) \(F \in C^\infty(\mathbb{C}^\times)\) and \(F(x)\) behaves like a Schwartz function as \(r \to \infty \iff \{F_p\}_{p \in \mathbb{Z}}\) is as in I above and further, for every \(a, b \geq 0\) and every \(s\), we have

\[\sup_{p \in \mathbb{Z}, r \geq 1} |r^s p^b F_p^{(a)}(r)| < \infty\]  

(III) \(F \in C^\infty(\mathbb{C}^\times)\) and the asymptotic expansion \(F(x) \approx \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} a_{k,m}(u) r^k (\log r)^{m-1}\) as \(r \to 0\) is termwise differentiable uniformly for \(u\) in \(\mathbb{C}_1^\times \iff \{F_p\}_{p \in \mathbb{Z}}\) is as in I above; for every \(p\), there exists an asymptotic expansion \(F_p(x) \approx \sum_{k=0}^{\infty} \sum_{m=1}^{m_p} a_{k,m,p} r^{-k} (\log r)^{m-1}\) as \(r \to \infty\) \(\ldots \)
0 which is termwise differentiable and further, for every $k \geq 0$, $a, b \geq 0$ in $\mathbb{Z}$ and $\sigma > -\lambda_{k+1}$, we have
\[
\sup_{p \in \mathbb{Z}, 0 < r \leq 1} |r^{\sigma+a} p^b R_{k,p}(r)| < \infty \tag{22}
\]
where, for $x \in \mathbb{C}^\times$, 
\[
R_k(x) \overset{\text{def}}{=} F(x) - \sum_{i=0}^k \sum_{m=1}^{m_i} a_{i,m}(u) r^i (\log r)^{m-1} = \sum_{p \in \mathbb{Z}} R_{k,p}(r) u^p
\]
with
\[
R_{k,p}(r) = F_p(r) - \sum_{i=0}^k \sum_{m=1}^{m_i} a_{i,m,p} r^i (\log r)^{m-1}.
\]
In (21), we may replace $r^{\sigma+p} F_p(r)$ by $r^{\sigma}(D_{a,b} F)_p(r)$ and similarly, in (22), $r^{\sigma+a} p^b R_{k,p}(r)$ by $r^{\sigma}(D_{a,b} R_k)_p(r)$.

We now give the proof of assertions, I, II, III above.

**Proof of I.** We shall use Lemma 3.2 in the following special form: namely if, for every $p \in \mathbb{Z}$, $f_p(t)$, $f'_p(t)$ are continuous on $I = (a, b) \subset \mathbb{R}$ such that
\[
\sum_{p \in \mathbb{Z}} |f_p(t)| < \infty, |f'_p(t)| \leq c_p, \sum_{p \in \mathbb{Z}} c_p < \infty \text{ for every } t \in I,
\]
then
\[
\frac{d}{dt} \left( \sum_{p \in \mathbb{Z}} f_p(t) \right) = \sum_{p \in \mathbb{Z}} f'_p(t) \tag{23}
\]
If $F \in C^\infty(\mathbb{C}^\times)$, then, for $r^{-a} D_{a,b} F$, we have corresponding to (18), the Fourier expansion
\[
(r^{-a} D_{a,b} F)(x) \overset{\text{def}}{=} \frac{1}{(2\pi i)^b} \frac{\partial^{a+b} F}{\partial r^a \partial \theta^b}(x) = \sum_{p \in \mathbb{Z}} \tilde{F}_p(r) u^p \tag{24}
\]
converging absolutely (and hence uniformly for $u$ in $\mathbb{C}^\times_1$), for every $a, b, \geq 0$ in $\mathbb{Z}$. Therefore, for every $p \in \mathbb{Z}$ we have
\[
\tilde{F}_p(r) = \int_0^1 r^{-a} D_{a,b} F(x) e(-p \theta) d\theta \text{ by definition of } \tilde{F}_p
\]
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$$p^b \int_0^1 \frac{\partial^a}{\partial r^a} F(x) e^{-p\theta} d\theta, \quad \text{on integrating by parts } b \text{ times,}$$

$$= p^b \frac{d^a}{dr^a} \left( \int_0^1 F(x) e^{-p\theta} d\theta \right), \quad \text{using (23)}$$

$$= p^b F^{(a)}(r) \text{, using (23)}$$

Thus, for the expression on the left hand side of (19), we have the bound

$$\sup_{r_1 \leq |r| \leq r_2} |r^{-a} D_{a,b} F(x)| \text{ which is certainly finite. We have therefore proved (19) and (20).}$$

To prove the reverse implication in I, we use (19) with $b + 2$ in place of $b$ and the fact that $\sup_{r_1 \leq r \leq r_2} |F^{(a)}_0(r)| \leq c < \infty$ for every $a, b \geq 0$ in $\mathbb{Z}$.

Then, for a constant $c'$ which may be taken to satisfy $c' \geq c$, we have

$$\sum_{p \in \mathbb{Z}} p^b F^{(a)}(r) \leq c'(1 + 2 \sum_{p=1}^\infty p^{-2}) < \infty \quad \text{for} \quad r_1 \leq r \leq r_2.$$ 

Applying (23) a sufficient number of times, we get $(r^{-a} D_{a,b} F)(x) = \sum_{p \in \mathbb{Z}} p^b F^{(a)}_p(r) u^p$.

**Proof of II.** It is easy to check that any $F$ in $C^\infty(\mathbb{C}^\times)$ behaves like a 27 Schwartz function as $|x|_\infty \to \infty$ if and only if, for every $\sigma$ and every $a, b, \geq 0$ in $\mathbb{Z}$, we have

$$\sup_{r \geq 1} \left| r^{-\sigma} \frac{\partial^{a+b} F}{\partial r^a \partial \theta^b}(x) \right| < \infty$$

However, the last condition implies (21) for every $\sigma$ and every $a, b \geq 0$ in $\mathbb{Z}$ in view of (20). Conversely, in can be deduced by assuming (21) with $(b + 2)$ in place of $b$, just as above.

**Proof of III.** A given $F$ in $C^\infty(\mathbb{C}^\times)$ has an asymptotic expansion as $r \to 0$ which is termwise differentiable uniformly in $u$, if and only if, for
every $a, b, k \geq 0$ in $\mathbb{Z}$ and every $\sigma > -\lambda_{k+1}$,

$$\sup_{0 < |x| \leq 1} \left| \rho' \left( \frac{1}{(2\pi i)^b} \rho^a R_k \frac{\partial^{a+b} R_k}{\partial r^a \partial \theta^b}(x) \right) \right| < \infty.$$ 

But proceeding exactly as above (in the proof of I) the last condition is seen to be equivalent to (22) being valid for every $a, b, k \geq 0$ in $\mathbb{Z}$ and every $\sigma > -\lambda_{k+1}$.

Finally, let us remark that $\mathcal{F}(\mathbb{C}^x)$ is stable under homothety-invariant differential operators, as is clear from the foregoing.

We may now proceed to define the function-space $\mathcal{Z}(\Omega(\mathbb{C}^x))$. First, we observe that $\Omega(\mathbb{C}^x) = \{ \omega_s(ac)^p; s \in \mathbb{C}, p \in \mathbb{Z} \}$ consists of a countable number of copies of $\mathbb{C}$ indexed by $p$ in $\mathbb{Z}$. The space $\mathcal{Z}(\Omega(\mathbb{C}^x))$ is defined as the set of all complex-valued functions $Z$ such that

1. $Z$ is meromorphic on $\Omega(\mathbb{C}^x)$ with poles at most when $s = -\lambda_0, -\lambda_1, -\lambda_2, \ldots$.

2. $Z(\omega_s(ac)^p) - \sum_{m=1}^m \frac{b_{k,m,p}}{(s+\lambda_k)^m}$ is holomorphic for $s$ close enough to $-\lambda_k$, where $b_{k,m,p}$ are constants and $k \geq 0$ is arbitrary in $\mathbb{Z}$; and

3. for any $p \in \mathbb{Z}$ and $s$ belonging to any vertical strip $B_{\sigma_1, \sigma_2}$ with neighbourhoods of the points $-\lambda_0, -\lambda_1, -\lambda_2, \ldots$ removed therefrom and for every polynomial $P(s, p)$ in $s$ and $p$ with coefficients in $\mathbb{C}$ we have

$$|P(s, p)Z(\omega_s(ac)^p)| \leq c''$$

for a constant $c''$ depending on $P, Z, \sigma_1, \sigma_2$ and the neighbourhoods of $-\lambda_0, -\lambda_1, -\lambda_2, \ldots$ removed but neither on $s$ nor on $p$.

For $Z \in \mathcal{Z}(\Omega(\mathbb{C}^x))$, let us define the functions $Z_p$ by $Z_p(s) = Z(\omega_s(ac)^p)$ for $s \in \mathbb{C}$ and $p \in \mathbb{Z}$. We may thus look upon $Z$ as a collection of functions $Z_p$ for $p \in \mathbb{Z}$. It is clear that $\mathcal{Z}(\Omega(\mathbb{C}^x))$ is stable under $Z_p \mapsto QZ_p$ where $Q$ is an arbitrary polynomial in $p$ and $s$, for every $p \in \mathbb{Z}$. We are now in a position to state and prove
Theorem 4.3. We have a bijective correspondence $M_C$ between $\mathcal{F}(\mathbb{C}^\times)$ and $\mathbb{Z}(\Omega(\mathbb{C}^\times))$. More precisely, for any $F$ in $\mathcal{F}(\mathbb{C}^\times)$,

$$(M_C F)(\omega) = \int_{\mathbb{C}^\times} F(x) \omega(x) d^\times x \quad \text{with} \quad d^\times x = \frac{dx}{2\pi|x|}$$

(26)

defines a holomorphic function on $\Omega_+(\mathbb{C}^\times)$ and its meromorphic continuation is in $\mathbb{Z}(\Omega(\mathbb{C}^\times))$. Conversely, for any $Z \in \mathbb{Z}(\Omega(\mathbb{C}^\times))$ and $x \in \mathbb{C}^\times$

$$(M_C^{-1} Z)(X) = \sum_{p \in \mathbb{Z}} \left\{ \frac{1}{2\pi i} \int_{\sigma-\infty}^{\sigma+\infty} Z(\omega_s(ac)^p)|x|^{-s} ds \right\} ac(x)^{-p}$$

defines a function in $\mathcal{F}(\mathbb{C}^\times)$ independently of $\sigma$ for $\sigma > 0$. Moreover, for every $k \leq 0$, $m \leq 1$ and $p$ in $\mathbb{Z}$, we have

$$b_{k,m,p} = (-1)^{m-1}(m-1)!a_{k,m-p}.$$ 

Proof. As in the case of $\mathbb{R}^\times$, we do fall back on Theorem 3.1 for the proof of this theorem; however, an additional complication is caused here by the maximal compact subgroup $\mathbb{C}^\times_1$ of $\mathbb{C}^\times$ being infinite, unlike the case of $\mathbb{R}^\times$ and forces us to seek, every time, estimates which are uniform with regard to $p$.

From I, II, III above, we know, for $F \in \mathcal{F}(\mathbb{C}^\times)$, that $F_p \in \mathcal{F}$ (for $\mathbb{R}^\times$) for every $p \in \mathbb{Z}$ and further, $D_{a,b} F \in \mathcal{F}(\mathbb{C}^\times)$ for every $a, b \geq 0$ in $\mathbb{Z}$. Substituting (18) into (26), with $d^\times x = d(\log r)d\theta$), we have (on integrating first with respect to $\theta$), $(M_C F_p) = M(F_p)$. On the other hand, let us define, for $Z_p$, the function $Z_p^\beta$ by

$$Z_p^\beta(s) = (-1)^{a+b}(s + a - 1) \ldots s.p^\beta Z_p(s)$$

(27)

for $a, b \geq 0$ in $\mathbb{Z}$ and let $D = D_{a,b}$. Again, substituting now (24) into (26) with $DF \in \mathcal{F}(\mathbb{C}^\times)$ in place of $F$, we have similarly,

$$(M_C(DF))(s) = (M(r^\mu \tilde{F}_p))(s) = (-p)^{b}(s + a - 1) \ldots s(M(F_p))(s)$$

by (20)
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\[ = (-1)^{a+b} p^b (s + a - 1) \ldots s(M(F-p))(s). \]

Thus, if \( Z = M_C F \), then

\[ Z_p^\# = (M_C (DF))_p. \quad (28) \]

Let us assume now that \( F \in \mathcal{F}(C^\infty) \) and show that \( M_C F \) gives a function in \( Z(\Omega(C^\infty)) \). From above, \( F_p \in \mathcal{F}(for \, \mathbb{R}_+^c) \) for every \( p \in \mathbb{Z} \) and therefore \( (M_C F)_p = M(F-p) \in \mathbb{Z} \) for \( \mathbb{R}_+^c \) by Theorem 3.1. On the other hand, for \( \sigma > -\lambda_{k+1} \), we have from (9),

\[ (MF_p)(s) = \sum_{i=0}^{k} \sum_{m=1}^{m_i} \frac{(-1)^{m-1}(m-1)!a_{i,m,p}}{(s + \lambda_i)^m} \]

\[ + \int_0^1 R_{k,p}(r)r^d(d\log r) + \int_{1}^{\infty} F_p(r)r^d(d\log r). \]

Now \( a_{i,m}(u) = \sum_{p \in \mathbb{Z}} a_{i,m,p} u^p \) for \( u \in C^\infty_1 \) and therefore, \( |a_{i,m,p}| \leq ||a_{i,m}||_\infty \) (the supremum norm), for every \( p \). Thus

\[ \left| \sum_{i=0}^{k} \sum_{m=1}^{m_i} \frac{(-1)^{m-1}(m-1)!a_{i,m,p}}{(s + \lambda_i)^m} \right| \leq \sum_{i=0}^{k} \sum_{m=1}^{m_i} (m-1)||a_{i,m}||_\infty \epsilon^{-m} \text{ for } s + \lambda_i \geq \epsilon > 0. \]

Further, by (22), \( |R_{k,p}(r)\epsilon^r| \leq c_1 r^{\sigma_1 - \sigma_0} \) with \( c_1 = \sup_{0<r \leq 1, p \in \mathbb{Z}} |r^{\sigma_0} R_{k,p}(r)| \) for \(-\lambda_{k+1} < \sigma_0 < \sigma_1 \leq \sigma \) and we have consequently

\[ \left| \int_0^1 R_{k,p}(r)r^d(d\log r) \right| \leq \frac{c_1}{\sigma_1 - \sigma_0} \]

Again, using (21), we have

\[ |F_p(r)\epsilon^r| \leq \sup_{r \geq 1, p \in \mathbb{Z}} |r^n F_p(r)|. r^{\sigma_1 - n} = c_2 r^{\sigma_1 - n}, \quad \text{say} \]
leading to
\[ \int_{1}^{\infty} f_p(r)r^s d(\log r) \leq \frac{c_2}{n - \sigma_2}, \text{ for } \sigma \leq \sigma_2 < n. \]

Putting these together, we conclude that for every \( p \in \mathbb{Z} \), \(((MZF)_p)(s)\) is bounded for \( s \) in vertical strips \( B_{\sigma,1,\sigma_2} \) with neighbourhoods of \( -\lambda_0, -\lambda_1, -\lambda_2, \ldots \) removed therefrom and for \( -\lambda_{k+1} < \sigma_1 < \sigma < \infty \) and all \( k \geq 0 \).

To prove the corresponding assertion for \( P(s,p)((MZF)_p)(s) \) with arbitrary polynomials \( P(s,p) \), we may assume, without loss of generality, that \( P(s,p) = (-1)^{a+b} p^b(s + a - b) \ldots s \) for some \( a, b \geq 0 \) in \( \mathbb{Z} \). Now we can work with \( D_{a,b}F \) instead of \( F \) and the required assertion follows in view of (28). The verification for \( MZF \) to belong to \( \mathcal{Z} \) is nothing but checking the conditions 1), 2), 3) for \((MZF)_p\) which are all satisfied as seen above.

Conversely, let us assume that \( Z \) is in \( \mathcal{Z} \). Then, for every \( p \in \mathbb{Z} \), we have \( Z_p \in \mathcal{Z} \) (for \( \mathbb{R}_+^\lambda \)) and by Theorem 3.1, \( F_p \overset{\text{def}}{=} M^{-1}(Z_p) \) is in \( \mathcal{F} \) (for \( \mathbb{R}_+^\lambda \)). Therefore, for \( \sigma > 0 \),
\[
2\pi r^\sigma F_p(r) = \int_{-\infty}^{\infty} Z_{-p}(\sigma + ti)r^{-ti} dt.
\]

On the other hand, there exists a constant \( c_3 > 0 \) depending on \( \sigma \) such that, for every \( t \in \mathbb{R} \) and \( p \in \mathbb{Z} \),
\[
\max(1, |t|^2)|Z_p(\sigma + ti)| \leq c_3,
\]
and hence we have
\[
2\pi r^\sigma |F_p(r)| \leq \int_{-\infty}^{\infty} |Z_p(\sigma + ti)| dt \leq 4c_3.
\]
Replacing \( Z \) by \( Z^\delta \) for which \((Z^\delta)_p = Z^\delta_p \) given by (27) from every \( p \in \mathbb{Z} \), we obtain for \( \sigma > 0 \) and every \( D_{a,b} \) that
\[
\sup_{p \in \mathbb{Z}, r > 0} |r^\sigma(D_{a,b}F)_p(r)| < \infty.
\]
But if we choose $\sigma$ with $-\lambda_{k+1} < \sigma < -\lambda_k$ then we have, by our arguments concerning (12) in §3.3, that

$$2\pi r^\sigma R_{k,p}(r) = \int_{-\infty}^{\infty} Z_{-p}(\sigma + ti) dt.$$ 

On the other hand, there exists a constant $c_4$ depending on $\sigma$ such that

$$\max(1, |t|^2) |Z_{-p}(\sigma + ti)| \leq c_4$$

for every $t \in \mathbb{R}$ and every $p \in \mathbb{Z}$. This gives first

$$2\pi r^\sigma |R_{k,p}(r)| \leq 4c_4,$$

and, as before, on replacing $Z$ by $Z^\sharp$, we get

$$\sup_{p \in \mathbb{Z}, r \geq 0} |r^\sigma (D_{a,b} R_k)_p(r)| < \infty$$

for every $D_{a,b}$ with $a, b \geq 0$ in $\mathbb{Z}$ and every $\sigma$ with $-\lambda_{k+1} < \sigma < -\lambda_k$. This implies (22) and putting all these together, we conclude from I, II, III that $M_{-1}^{\infty} C \subseteq F(\mathbb{C}^\times)$.

The last assertion of Theorem 4.3 follows from the fact that $(M_C F)_p = M(F_{-p})$ for every $p \in \mathbb{Z}$ and from the corresponding assertion of Theorem 3.1.

Our theorem is now completely proved. \hfill $\square$

**REMARK.** The above theory is essentially one-dimensional, since $\Omega(K^\times)$ for $K = \mathbb{R}, \mathbb{C}$ (or even a $p$-field) is a one-dimensional complex Lie group and we have used function theory only in the case of one complex variable. To generalise our results to general compactly generated abelian groups, there seems to be another problem, namely, that of guessing the right kind of asymptotic expansions to be employed, for which perhaps one needs the powerful intuition of a good applied mathematician!
5 The Case of $p$-fields

5.1 We deal now with a bijective correspondence between function spaces $\mathcal{F}$ and $\mathcal{Z}$ associated, in a similar manner, with $p$-fields which, as we recall, are just finite algebraic extensions of Hensel’s field $\mathbb{Q}_p$ of $p$-adic numbers or fields $\mathbb{F}((t))$ of Laurent series in one variable $t$ with coefficients in a finite field $\mathbb{F}_q$.

We first recall certain well-known facts about $p$-fields. Let $K$ be a $p$-field, $R = \{ x \in K ; |x|_k \leq 1 \}$, the maximal compact subring of $K$ and $P = \{ x \in K ; |x|_k < 1 \}$ the (unique) maximal ideal of $R$. Then the residue field $R/P$ is a finite field $\mathbb{F}_q$ consisting of $q$ elements (with $q$ equal to some power of $p$ in the former case and equal to the cardinality of $\mathbb{F}$ in latter case). In fact, if $K = \mathbb{F}((t))$, then $R = F[[t]]$, the ring of power-series in $t$ over $\mathbb{F}$, $P = t F[[t]]$, and further $\mathbb{F}$ and $R/P$ are isomorphic and have the same cardinality, say $q$, so that we can write $\mathbb{F}_q$ instead of $\mathbb{F}$ in this case.

We take $dx$ to be a Haar measure on $K$ which is so normalised as to make the measure $m(R)$ of $R$ equal to 1. Now $R = \bigcup_a (P + a)$, the disjoint union of finitely many cosets $P + a$ of $R$ modulo $P$ and $m(P + a) = m(P)$ for every $a$. Therefore we have

$$1 = m(R) = \sum_{a \mod P} m(P + a) = [R : P]m(P)$$

i.e., $m(P) = q^{-1}$.

More generally, for any $e \in \mathbb{Z}$, we have

$$m(P^e) = \begin{cases} 1/[R : P^e] = 1/([R : P][P : P^2] \ldots [P^{e-1} : P^e]) & \text{for } e \geq 0 \\ [P^e : R] = [P^e : P^{e+1}] \ldots [P^{-1} : R] & \text{for } e < 0 \end{cases}$$

$$= q^{-e}$$

For $a \in K^\times$, we define the order $\text{ord}(a)$ of $a$ by $aR = P^{\text{ord}(a)}$. Then $|a|_K$ introduced in \S4 as the rate of change of measure in $K$ under $x \mapsto ax$, is just $m(aK)/m(K) = m(P^{\text{ord}(a)})$, i.e., $|a|_K = q^{-\text{ord}(a)}$. 
The group $R^\times$ of units in $R$ is precisely $R \setminus P = \{ x \in K; |x|_K = 1 \}$ and $m(R^\times) = m(R) - m(P) = 1 - q^{-1}$.

We define $d^\times x$ as the Haar measure on $K$, with the obvious normalisation, namely, the one that makes the measure of $R^\times$ to be 1. It is now easy to see that

$$d^\times x = \frac{1}{1 - q^{-1}} \frac{dx}{|x|_K}$$

From the definition of $\text{ord}(a)$ for $a \in K^\times$ above, it is clear that we have an exact sequence

$$\{ 1 \} \rightarrow R^\times \xrightarrow{j} K^\times \xrightarrow{\text{ord}} \mathbb{Z} \rightarrow \{ 0 \}$$

where $j$ is the inclusion map and “ord” is the map $a \mapsto \text{ord}(a)$. For any $e$ in $\mathbb{Z}$ the complete inverse image in $K^\times$ under the “ord” map i.e., $\{ a \in K; |a|_k = q^{-e} \}$ is a compact open subset of $K$ and called the $e^{th}$ wall (around 0).

We choose a cross-section for the map “ord”:

$$K^\times \rightarrow \mathbb{Z}$$

above, by selecting an element $\pi$ in $K^\times$ with $\text{ord}(\pi) = 1$. It is to be noted that $\pi$ is not unique but once chosen, it is fixed once for all.

For every $x \in K^\times$, we define the angular component $\text{ac}(x)$ as $x\pi^{-\text{ord}(x)}$ in $R^\times$ and again the definition is not intrinsic in view of the ambiguity about the choice of $\pi$.

The map $x \mapsto (\text{ord}(x), \text{ac}(x))$ gives, from above, an isomorphism $K^\times \cong \mathbb{Z} \times R^\times$. Thus $\Omega(K^\times) \cong \mathbb{C}^\times \times (R^\times)^*$ under the isomorphism $\omega \mapsto (z, X)$ where, for $e \in \mathbb{Z}$ and $u \in R^\times$, we have $\omega(\pi^e u) = q^e X(u)$. We could have written instead, for any $x = \pi^e u \in K^\times$, $\omega(x) = |x|_k X(\text{ac}(x))(= q^e X(u))$. But $|x|_k$ for $x \in K^\times$ always belongs to the set $\{ q^e; e \in \mathbb{Z} \}$ and we have therefore to take $s$ only modulo $2\pi i / \log q$. As a result, the right parameter to be taken is actually $z = q^{-s}$ and not $s$. As stated above, $\Omega(K^\times)$ consists of countably many copies of $\mathbb{C}^\times$ indexed by $(R^\times)^*$, the discrete dual of $R^\times$ (which now corresponds to the indexing set $\mathbb{Z}$ in the case of $K = \mathbb{C}$). For any function $Z$ on $\Omega(K^\times)$ and for $\omega \in \Omega(K^\times)$ with $\omega(\pi^e u) = q^e X(u)$ as mentioned above, we write $Z_\chi(\omega)$ for $Z(\omega)$. Let $\Omega_+(K^\times) = \{ \omega \in \Omega(K^\times); \sigma(\omega) > 0 \}$.

We now proceed to define the spaces $\mathcal{F}(K^\times)$ and $\mathcal{Z}(\Omega(K^\times))$ for $p$-fields $K$. While doing so, we guided by the existence of accepted ana-
5. The Case of $p$-fields

logues, for $p$-fields, of $C^\infty$ functions behaving like Schwartz functions on $\mathbb{R}$-fields and also by the subsequent applications we have in mind.

5.2

Let $\Lambda = \{\lambda \in \mathbb{C}; \lambda \text{ modulo } 2\pi i / \log q; \Re \lambda \geq 0\}$ be a given finite set and $\{m_k; k \in \Lambda\}$ be a given set of natural numbers. Let $\mathcal{F}(K^\times)$ be the space of complex-valued functions $F$ on $K^\times$ such that

(i) $F$ is locally constant i.e, for any $x \in K^\times$, there exists a natural number $n$ depending on $x$ such that $F(x) = F(y)$ for all $y$ with $y - x \in xP^n$,

(ii) $F(x) = 0$ for $|x|_K \gg 1$ (i.e, for all sufficiently large $|x|_K$), and

(iii) for $|x|_K \ll 1$ (i.e, for all sufficiently small $|x|_K$), we have

$$F(x) = \sum_{\lambda \in \Lambda} \sum_{m = 1}^{m_i} a_{\lambda,m}(ac(x))|x|^2 \frac{\log |x|_K}{m - 1}$$

(29)

with locally constant functions $a_{\lambda,m}$ on $\mathbb{R}^\times$.

Since $a_{\lambda,m}(u)$ are locally constant on $R^\times$, we may write

$$a_{\lambda,m}(u) = \sum_{X \in (R^\times)^*} a_{\lambda,m,X}(u)$$

(30)

as a finite Fourier series, with constant coefficients $a_{\lambda,m,X}$.

For any $F \in \mathcal{F}(K^\times)$, we know that its restriction to the $e^{th}$ wall $p^e \setminus p^{e+1}$ is locally constant and therefore we have, for $u \in R^\times$, the finite Fourier series

$$F(\pi^e u) = \sum_{X \in (R^\times)^*} c_{e,X}(u)$$

(31)

with constant coefficients $c_{e,X}$. The set $\{X \in (R^\times)^*; c_{e,X} \neq 0 \text{ for some } e \text{ in } \mathbb{Z}\}$ is finite. Indeed, for all $e < 0$in $\mathbb{Z}$ with $|e|$ large, we have $F(\pi^e u) = 0$ for all $u$ and therefore $c_{e,X} = 0$ for all such $e$. From (29) and (30) we have, for all sufficiently large $e$,
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\[ c_{e,x} = \sum_{\lambda \in \Lambda} \sum_{m=1}^{M_{\lambda}} a_{\lambda,m,X} q^{-\lambda e}(-e \log q)^{m-1} \]  

(32)

and the number of \( a_{\lambda,m} \) involved and hence from (30), the number of \( X \) with \( a_{\lambda,m,X} \neq 0 \) is finite. For the remaining finitely many \( e \) again, the number of \( X \) with \( c_{e,X} \neq 0 \) is finite, since (31) is a finite sum. The space \( \mathcal{Z}(\Omega(K^\times)) \) is defined as the set of complex-valued functions \( Z \) on \( \Omega(K^\times) \) for which

1. for every \( X \in (R^\times)^* \), there exist constants \( b_{\lambda,m,X} \) such that

\[ Z_X(z) - \sum_{\lambda \in \Lambda} \sum_{m=1}^{M_{\lambda}} b_{\lambda,m,X} \left( \begin{array}{c} 1 - q^{-\lambda e} \\ (1 - q^{-\lambda e})^m \end{array} \right) \]

is a polynomial in \( z \) and \( z^{-1} \) with complex co-efficients, and

2. for almost all \( X \), \( Z_X(z) \) is identically zero.

Let, for every \( m \geq 1 \), the numbers \( e_{m,1}, \ldots, e_{m,m} \) be defined by the following identity in \( t \):

\[ t^{m-1} = \sum_{j=1}^{m} e_{m,j} \frac{(t + j - 1)(t + j - 2)\ldots(t + 1)}{(j - 1)!} \]  

(33)

Actually \( e_{m,j} \) are integers and further, for \( m = 1, 2 \ldots \)

\[ e_{m,m} = (m - 1)!; \quad e_{i,j} = 0 \quad \text{for} \quad i < j, \]  

(34)

5.3

We may now state and prove

**Theorem 5.3.** We have an isomorphism \( M_K \) (or briefly, \( M \)) between the spaces \( \mathcal{F}(K^\times) \) and \( \mathcal{Z}(\Omega(K^\times)) \) for any \( p \)-field \( K \). More precisely, for any \( F \in \mathcal{F}(K^\times) \),

\[ (M_K F)(\omega) = \int_{K^\times} F(x) \omega(x) d^\times x \]
defines a function on $\Omega_+(K^\infty)$ and its meromorphic continuation is in $\mathcal{Z}_+(K^\infty)$. Conversely, if $Z$ is in $\mathcal{Z}(\Omega(K^\infty))$, then $(M_K^{-1}Z)(x)$ is given by

$$ \sum_{x \in (R^\times)^{\nu}} (\text{Residue}_{z=0}(Z_{\lambda}(z)z^{-\text{ord}(x)-1}))X(ac(x))^{-1} $$

(35)

and defines a function in $\mathcal{F}(K^\infty)$. Moreover, we have

$$ b_{\lambda,m,X} = \sum_{j=m}^{m_1} e_{jm} (-\log q)^{j-1} a_{\lambda,lX}^{-1} $$

(36)

for every $\lambda, m$ and $X$, where $e_{jm}$ are given by (33).

Proof. First, let $F$ be given in $\mathcal{F}(K^\infty)$ and $\omega \in \Omega(K^\infty)$, so that $\omega(\pi^\epsilon u) = \varepsilon'X(u)$ and $|\varepsilon| = q^{-\sigma(\omega)} < 1$. Now

$$ (M_KF)_X(z) = \int_{K^\infty} F(x)\omega(x)d^\infty x $$

$$ = \sum_{\varepsilon \in \mathbb{Z}} \int_{R^\times} F(\pi^\epsilon u)\varepsilon'X(u)d^\infty u $$

(37)

$$ = \sum_{\varepsilon \in \mathbb{Z}} c_{E^{-1}}\varepsilon' $$

(38)

on substituting (31) and using the orthogonality relations $\int_{R^\times} X'(u)X(u) d^\infty u$ is 1 or 0 according as $X' = X^{-1}$ or not. All the steps used above are justified, recalling that $|\varepsilon| < 1$.

As remarked earlier, the number of $X$ in $(R^\times)^{\nu}$ with the property that $c_{E,X} \neq 0$ for some $e$ (depending on $X$) in $\mathbb{Z}$ finite. Hence, expect for finitely many $X$, we have $(M_KF)_X = 0$. In view of (37), (38), (31) and (32), we have

$$ (M_kF)_X(z) = \sum_{\varepsilon \in \mathbb{Z}} c_{E^{-1}}\varepsilon' + \sum_{e=e_0}^{\infty} \left( \sum_{\lambda \in \mathbb{A}} \sum_{m=1}^{m_1} a_{\lambda,m,X^{-1}} q^{-le}(-e\log q)^{m-1} \right) \varepsilon' $$

where $\sum_{\varepsilon \in \mathbb{Z}}$ is a finite summation and $e_0 \geq 0$ in $\mathbb{Z}$ depends on $F$. Therefore, writing $\equiv$ to imply equality modulo $\mathbb{C}[z, z^{-1}]$, we have
From (31) and (38), by reversing the process, we get, for $M_k$:

$$(M_k F)_\chi(z) \equiv \sum_{e=0}^{\infty} \left( \sum_{l \in \Lambda} \sum_{m=1}^{m_l} a_{l,m,\chi^{-1}} q^{-\lambda_l} (-e \log q)^{m-1} \right) z^e$$

$$= \sum_{e=0}^{\infty} \left( \sum_{l \in \Lambda} \sum_{m=1}^{m_l} a_{l,m,\chi^{-1}} q^{-\lambda_l} (-\log q)^{m-1} \sum_{j=1}^{m} e_{m,j} \left( \frac{e + j - 1}{j - 1} \right) \right) z^e,$$

using (33)

$$= \sum_{l \in \Lambda} \sum_{j=1}^{m_l} \left( \sum_{m=j}^{m_l} e_{m,j} (-\log q)^{m-1} a_{l,m,\chi^{-1}} \right) \sum_{e=0}^{\infty} \left( \frac{e + j - 1}{j - 1} \right) (q^{-A} z)^e,$$

on reversing the order of the summations over $j$ and $m$ in the preceding line and then interchanging them with the summation over $e$. We remark that this is justified $|z| < 1$ and the summations over $m, j$ and $A$ are all finite, using (36) to define $b_{l,m,\chi}$ and the fact that $|q^1| z < 1$, we have, finally.

$$(M_k F)_\chi(z) \equiv \sum_{l \in \Lambda} \sum_{j=1}^{m_l} b_{l,j,\chi} \frac{z^j}{(1 - q^{-A} z)^j}$$

and therefore $M_k F$ is indeed in $Z(\Omega(K^\chi))$.

Conversely, let $Z$ be given in $Z(\Omega(K^\chi))$, so that, for every $X \in (R^\chi)^*$, we have $Z_X(z) = \sum_{e \in \zeta^e} d_{X,e} z^e$, where $0 < |z| < 1$ and further, in view of conditions (1) and (2) satisfied by $Z$, we have the following:

$$\{X \in (R^\chi)^*; d_{X,e} \neq 0 \text{ for some } e \in \zeta \} \text{ is a finite set, } d_{X,e} = 0 \text{ for}$$

$$e < 0 \in \zeta \text{ with } |e| \text{ large and } d_{X,e} = \sum_{l \in \Lambda} \sum_{m=1}^{m_l} b_{l,m,\chi} \left( \frac{e + m - 1}{m - 1} \right) q^{-\lambda_l}$$

for all large $e > 0$ in $\zeta$. (39)

From (31) and (38), by reversing the process, we get, for $e \in \zeta$ and $u \in R^\chi$ that

$$(M_k^{-1} Z)(\pi^e u) = \sum_X d_{X,e} \chi(u)^{-1}$$

$$= \sum_{X \in (R^\chi)^*} (\text{Residue}_{z=0}(Z_X(z)z^{-e-1})) \chi(u)^{-1}. \quad (40)$$

It is now clear that $M_k^{-1} Z$ satisfies conditions (i) and (ii) defining $\mathcal{F}(K^\chi)$. 

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Also, for any given \(X\) and \(\lambda\), equations (36) for \(1 \leq m \leq m_1\) considered as a set of equations in the \(m_1\) unknowns \(a_{\lambda,j}X^{-1}\) \((1 \leq j \leq m_1)\) are solvable, in view of (34). From (40) and (38), we have, for all large \(e > 0\) in \(\mathbb{Z}\),

\[
(M_K^{-1}Z)(\pi^e u) = \sum_{\lambda \in \Lambda} \left\{ \sum_{j=1}^{m_1} a_{\lambda,j}(u)^{-e} \left( \sum_{m=1}^{e} e_{j,m} \left( \frac{e + m - 1}{m - 1} \right) \right) \right\} \left( \frac{e + m - 1}{m - 1} \right) X(u)^{-1},
\]

using the solvability of equations (36) with \(1 \leq m \leq m_1\) for \(a_{\lambda,j}X^{-1}\). Again, reversing the order of the summations over \(j\) and \(m\) and then using (30) and (33), we have, for all large \(e > 0\) in \(\mathbb{Z}\),

\[
(M_K^{-1}Z)(\pi^e u) = \sum_{\lambda \in \Lambda} \left\{ \sum_{j=1}^{m_1} a_{\lambda,j}(u)^{-e} \left( \sum_{m=1}^{e} e_{j,m} \left( \frac{e + m - 1}{m - 1} \right) \right) \right\} \left( \frac{e + m - 1}{m - 1} \right) \pi^{-e} X(u)^{-1} - \sum_{\lambda \in \Lambda} \left\{ \sum_{j=1}^{m_1} a_{\lambda,j}(u) \right\} \left( \frac{e + m - 1}{m - 1} \right) X(u)^{-1},
\]

Thus, condition (29) is proved for \(M_K^{-1}Z\) with \(|x|_K \ll 1\), implying that \(M_K^{-1}Z\) is in \(\mathcal{F}(K^\times)\).

Remark 1. In (29), we have assumed (while defining \(\mathcal{F}(K^\times)\)) that the functions \(a_{\lambda,m}\) on \(R^\times\) are locally constant. However, we shall now show that this is actually a consequence of \(F\) being locally constant on \(K^\times\). For this, it suffices to prove that the functions

\[
|x|_K^\lambda (\log |x|_K)^{m-1}\]

for \(\lambda \in \Lambda\) and \(1 \leq m \leq m_1\), on \(|x|_K = q^{-e}; e\) sufficiently large in \(\mathbb{Z}\) are linearly independent over \(\mathbb{C}\).

If possible, let \(a_{\lambda,m,1}\) be complex constants such that

\[
\sum_{\lambda \in \Lambda} \sum_{m=1}^{m_1} a_{\lambda,m,1} |x|_K^\lambda (\log |x|_K)^{m-1} = 0
\]

(41)
for all \(|x_k| \ll 1\). Then, proceeding exactly as in the proof of the first half of Theorem 5.3, we obtain \(b_{\lambda,m,1}\) related to \(a_{\lambda,m,1}\) as in (36) with \(X = 1\) such that
\[
\sum_{\lambda \in \Lambda} \sum_{m=1}^{m_{\lambda}} \frac{b_{\lambda,m,1}}{(1-q^{-\lambda}z)^m} \equiv 0 \pmod{\mathbb{C}[z,z^{-1}]}.
\] (42)

Since, on the left side, \((1-q^{-\lambda}z)^{-m}\) has a pole \(q^{-\lambda}\) and the points \(q^{-\lambda}\) for \(\lambda \in \Lambda\) are all distinct, (42) necessarily implies that \(b_{\lambda,m,1}\) are all 0. Therefore, in view of (36) with \(X = 1\), the constants \(a_{\lambda,m,1}\) in (41) are all 0.

**Remark 2.** Formula (35) for the inverse transform \(M^{-1}_K\) looks, on the face of it, quite different from its counterpart in the case of \(\mathbb{R}\)-fields. But the analogy will be obvious, if we rewrite it, for \(\sigma > 0\), as
\[
(M^{-1}_KZ)(x) = \sum_{X \in (\mathbb{R}^+)^\times} \left( \frac{\log q}{2\pi i} \int_{\sigma - \frac{\pi}{\log q}}^{\sigma + \frac{\pi}{\log q}} Z_X(q^{-s}) |x_k|^{-s} \, ds \right) X(ac(x))^{-1}
\]
where now \(\pi = 3.14\ldots\) (the length of the circumference of a circle of unit diameter!). In fact, this is immediate from (35), since \(Z_X(z)^{1-e^{-1}}\) is holomorphic in \(0 < |z| < 1\) and for \(\sigma > 0\), we have
\[
\text{Residue } z=0(Z_X(z)^{1-e^{-1}}) = \frac{1}{2\pi i} \int_{|z|=q^{-\sigma}} Z_X(z)z^{-\sigma} d(\log z)
\]
\[
= \frac{\log q}{2\pi i} \int_{\sigma - \frac{\pi}{\log q}}^{\sigma + \frac{\pi}{\log q}} Z_X(q^{-s}) |x_k|^{1-e^{-1}} \, ds
\]
(on setting \(z = q^{-s}\)).
APPENDIX

POISSON FORMULA OF HECKE TYPE

This appendix is based on a preprint entitled “On a generalization of the Fourier transformation” by T. Yamazaki [55], where the special case corresponding to $x = 1 - \frac{1}{m}$ for $m = 2, 4, 6, \ldots$ below has been discussed. The Poisson formula that we shall derive here goes back, in its simplest form, to Hecke [14]. Our generalisation, besides being of interest on its own, may be expected to provide, together with a suitable enlargement of this theory so as to cover all the local fields, the ‘metaplectic group’ to be associated with diagonal forms of $m \geq 2$. In this connection, we should also refer to a series of papers by T. Kubota where he has generalised the Fourier transformation and proved a poisson formula for his transformation (See [30]).

1 The Unitary Operator $W$

We recall Theorem 3.1 on the bijective correspondence between the spaces $\mathcal{F}$ and $\mathcal{Z}$ associated with $\mathbb{R}^\times_+$.

Choosing $a_k = k, m = 1$ for $k = 0, 1, 2 \ldots$, let us remark the space for $\mathbb{R}^\times_+$ can be characterised also as the space of $F \in C^\infty(\mathbb{R}^\times_+)$ such that $F(x)$ behaves like a Schwartz function as $x \to \infty$ and further $F \in C^\infty([0, \infty))$ by defining $F(0) = \lim_{x \to 0} F(x)$. The last condition is seen to be equivalent to $F$ in $C^\infty(\mathbb{R}^\times_+)$ having a termwise differentiable asymptotic expansion $F(x) \approx \sum_{n=0}^{\infty} a_n x^n$ as $x \to 0$; one has merely to define $F(0) = a_0$, in order to show that conditions (iii) for $F$ in $\mathcal{F}$ implies “$FC^\infty([0, \infty))$”.

The spaces $\mathcal{F}$ can also be described as the space of complex-valued functions obtained by restricting to $\mathbb{R}^\times_+$, Schwartz functions on $\mathbb{R}$ (i.e, $C^\infty$ functions $G(x)$ on $\mathbb{R}$ which behave like Schwartz functions as $|x| \to \infty$); this can be seen by using, for example, a theorem of H. Whitney [54] that $C^\infty$ functions on closed subsets of $\mathbb{R}^n$ admit $C^\infty$ extensions to the whole of $\mathbb{R}^n$.

The space $\mathcal{Z}$ for $\mathbb{R}^\times_+$ can be described alternatively as the family of complex-valued functions $Z(s)$ on $\mathbb{C}$ such that $Z(s)(\Gamma(s))^{-1}$ is an entire
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function of \( s \) and further, for every polynomial \( P \in \mathbb{C}[s] \), the function \( P(s)Z(s) \) is bounded in any vertical strip \( B_{\sigma_1,\sigma_2} \) with neighbourhoods of the points \( 0, -1, -2, \ldots \) removed therefrom. This is clear, since \( 1/\Gamma(s) \) is an entire function of \( s \) with simple zeros at \( 0, -1, -2, \ldots \)

For any \( a > 0 \), the function \( F(x) = e^{-ax} \) is clearly in \( \mathcal{F} \) and further, \( M(e^{-ax})(s) = \int_0^\infty e^{-ax}x^s d\log x = a^{-s}\Gamma(s) \) (43)

almost by the definition of \( \Gamma(s) \). More generally, for any \( \tau \) in \( \mathbb{C} \) with \( \text{Im}(\tau) > 0 \), we get, by the principle of analytic continuation, from (43), that

\[
M(e^{i\tau x})(s) = (-i\tau)^{-s}\Gamma(s)
\]

where \((-i\tau)^{-s} = e^{-s\text{Log}(-i\tau)}\) with \( \text{Log} \) denoting the principal branch of the logarithm. We also recall the following asymptotic formula for the gamma-function: namely,

\[
|r(s)| = (2\pi)^{\frac{1}{2}}|t|^{\sigma-\frac{1}{2}} \exp\left(-\frac{\pi}{2}|t|\right)(1 + o(1)) \quad \text{(44)}
\]

for \( s = \sigma + ti \) in any vertical strip \( B_{\sigma_1,\sigma_2} \), as \( |s| \to \infty \).

Let us fix, once for all, a real number \( x > 0 \). Then, from (44), it is immediate that

\[
\left|\frac{\Gamma(s)}{\gamma(x-s)}\right| = |t|^{2\sigma-x}(1 + o(1)) \quad \text{(45)}
\]

for \( s = \sigma + ti \) in a vertical strip \( B_{\sigma_1,\sigma_2} \).

For any \( Z \) in \( \mathcal{F} \), we define \( Z^\gamma \) by

\[
Z^\gamma(s) = \Gamma(s) \frac{Z(x-s)}{\Gamma(x-s)}
\]

Then \( Z(s)/\Gamma(s) = z(x-s)/\Gamma(x-s) \) therefore, an entire function. For any polynomial \( P \), we have \( P(s)Z(s) \) bounded in vertical strips with neighbourhoods of \( 0, -1, -2, \ldots \) removed therefrom and (45) then implies the same property for \( Z^\gamma \) instead of \( Z \). Thus \( Z^\gamma \) is again in \( \mathcal{Z} \) and moreover it is trivial to check that \((Z^\gamma)^\gamma = Z\).
1. The Unitary Operator $W$

For any $Z$ in $\mathcal{Z}$, if we define
\[
||Z||^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \mathcal{Z} \frac{1}{2} x^2 + ti \right|^2 dt
\]
then the growth condition on $Z$ implies $||Z|| < \infty$ and it is clear that $|| \cdot ||$ is a norm in $\mathcal{Z}$. Since the left hand side of (45) is 1 for $s = \frac{1}{2} x + ti$, it is easily seen that $||Z^s|| = ||Z||$ for every $Z$ in $\mathcal{Z}$. We can define, similarly, for $F$ in $\mathcal{F}$,
\[
||F||^2 = \int_0^\infty |F(x)|^2 x^4 d\log x.
\]
The integral is finite since $x > 0$ and $F$ behaves like a Schwartz function at infinity. We have again a norm in $\mathcal{F}$ given by $|| \cdot ||$ but, as we shall see, it is the same as one obtained by transporting to $\mathcal{F}$ the norm from $\mathcal{Z}$ under $M^{-1}: \mathcal{Z} \to \mathcal{F}$.

We assert that $||F||^2 = ||MF||^2$. In fact, if $Z = MF$, then
\[
Z \left( \frac{1}{2} x + ti \right) = \int_0^\infty F(x) x^\frac{3}{2}, x^4 d\log x
\]
is just the Fourier transform of $F(x) x^\frac{3}{2}$. Since $\frac{1}{2\pi} dt$ and $d\log x$ are dual measures, the Plancherel theorem gives
\[
(||MF||^2) = ||Z||^2 = ||F||^2.
\]

We now define an operator $W: \mathcal{F} \to \mathcal{F}$ as the composite of the operators $M, Z \mapsto Z^s$ and $M^{-1}$; more explicitly, we have, for any $F$ in $\mathcal{F}$,
\[
WF = M^{-1}(MF)^\times.
\]
(46)

From $(Z^s)^\times$, it is immediate that $W^2 F = F$. Further
\[
||WF|| = ||M^{-1}(MF)^\times|| = ||(MF)^\times|| = ||MF|| = ||F||
\]

Thus $W$ is a unitary operator of order 2 on the pre-Hilbert space $\mathcal{F}$. 
We shall make use of the following formula, later on: namely, for any \( F \in \mathcal{F} \),

\[
(WF)(0) = \frac{(MF)(\bar{x})}{\Gamma(\bar{x})}
\]

(47)

This is easy to prove; in fact, if \((WF)(x) \approx a'_0 + a'_1 x + \ldots \) as \( x \to 0 \), then, by Theorem 3.1, \( \frac{(M(WF))(s)}{\Gamma(s)} - \frac{a'_0}{s} \) is holomorphic at \( s = 0 \) which implies that \( \lim_{s \to 0} \frac{(M(WF))(s)}{\Gamma(s)} = a'_0 = WF(0) \). But \( \frac{(MF)(\bar{x} - s)}{\Gamma(\bar{x} - s)} \) and taking limits on both sides as \( s \to 0 \), we see that (47) is immediate.

**Remarks.** One can give another proof of (47) by using the relation

\[
(WF)(x) = \int_0^\infty k(xy) \frac{F(y) y^{\bar{x}}}{x} \log y
datax
\]

where \( k(x) = \sum_{n=0}^\infty \frac{(-x^n)^n}{n! (n+\bar{x})} = x^{\bar{x}/2(\bar{x}-1)} J_{\bar{x}-1}(2x^{1/2}) \) and \( J_{\bar{x}-1} \) is the usual Bessel function of order \( \bar{x} - 1 \).

It should be stressed that \( \bar{x} \) is involved in the definition of \( W \).

2 A Poisson Formula

Consider a triple \( \{\lambda, \bar{x}, \gamma\} \), where \( \lambda \) and \( \bar{x} \) are positive real numbers and \( \gamma \) is a complex number. Following Hecke, we say that a complex-valued function \( \phi \) on \( \mathbb{C} \) is a function of signature \( \{\lambda, \bar{x}, \gamma\} \) if

1) \( (s - \bar{x})\phi(s) \) is an entire function of \( s \) and further \( \phi \) is (at most) of polynomial growth in any vertical strip \( B_{\sigma_1, \sigma_2} \) for every \( \sigma_1 < \sigma_2 \).

2) \( R(s) = \frac{1}{2\pi i} \int \Gamma(s) \phi(s) ds \) satisfies the function equation \( R(s) = \gamma R(\bar{x} - s) \), and

3) for all sufficiently large \( \sigma \), \( \phi(s) \) is represented by an absolutely convergent Dirichlet series \( \sum_{n=1}^\infty \frac{\phi_n}{n^\sigma} \).
Our condition (1) above is a variant of Hecke’s assumption \([14]\):  
1) \((s - \chi)\phi(s)\) is an entire function genus. Conditions 1), 2), 3) above are together equivalent to conditions 1)', 2), 3). With regard to 3), we should mention that there exist Dirichlet series e.g., \(\sum(-1)^n/(\sqrt{n}(\log n)^s)\) convergent for all values of \(s\), but never absolutely convergent. If there exists a function \(\phi \neq 0\), of signature \(\{\lambda, \chi, \gamma\}\), then condition (2) above implies \(\gamma^2 = 1\), as is obvious on applying \(s \mapsto \chi - s\) to the functional equation.

**Theorem.** Suppose that \(\phi \neq 0\), is a functional of signature \(\{\lambda, \chi, \gamma\}\) and further, let  
\[
a_0 = \gamma \left(\frac{A}{2\pi}\right)^{\chi} \Gamma(\chi) \text{Residue}_{s=\chi} \phi(s).
\]
Then we have, for every \(F\), in \(\mathcal{F}\), the Poisson formula  
\[
\sum_{n=0}^{\infty} a_n(WF) \left(\frac{2\pi n}{\lambda}\right) = \gamma \sum_{n=0}^{\infty} a_nF \left(\frac{2\pi n}{\lambda}\right)
\]
Proof. From (46) and the functional equation for \(R(s)\), we get  
\[
\frac{(MF)(s)}{\Gamma(s)} R(s) = \frac{(MF)(\chi - s)}{\Gamma(\chi - s)} \gamma R(\chi - s)
\]
i.e.
\[
(MWF)(s) \left(\frac{A}{2\pi}\right)^{\chi} \phi(s) = \gamma(MF)(\chi - s) \left(\frac{\lambda}{2\pi}\right)^{\chi - s} \phi(\chi - s) \quad (48)
\]
Since \(\phi\) has got at most a simple pole at \(s = \sigma + ti = \chi\) and holomorphic everywhere else, we see, in view of Theorem 3.1 applied to \(WF\) (with \(\lambda_k = k, m_k = 1\) for all \(k \geq 0\)) that the left side of (48) is holomorphic if \(s \neq \chi, 0, -1, -2, \ldots\). For similar reasons, the right side of (49) is holomorphic for \(\chi - s \neq 0, -1, -2, \ldots\), i.e., for \(s \neq 0, \chi, \chi + 1, \chi + 2, \ldots\). Thus the function of \(s\) represented by (48) is holomorphic except for \(s = 0, \chi\) at most. Moreover, from condition (1) satisfied by \(\phi\) and an application of Theorem 3.1 to \(F\) and \(WF\) again, we see that, even after multiplication by a polynomial \(P(s)\), both sides of (48) remain bounded in any given vertical strip \(B_{\sigma,\epsilon,\epsilon_2}\) with neighbourhoods of the
poles above removed therefrom. Thus, we have, in particular, for any \( \sigma \) with \( 0 < \sigma < \xi \), that

\[
\frac{1}{2\pi i} \int_{\sigma - \infty}^{\sigma + \infty} (M(WF))(s) \left( \frac{\lambda}{2\pi} \right)^s \phi(s) \, ds = \frac{\gamma}{2\pi i} \int_{\sigma - \infty}^{\sigma + \infty} (MF)(\xi - s) \left( \frac{\lambda}{2\pi} \right)^{\xi - s} \phi(\xi - s) \, ds, \tag{49}
\]

both integrals converging absolutely. On the left side of (49), we can shift the line integration from \( \text{Re}(s) = \sigma \) to \( \text{Re}(s) = \sigma_0 \) for any sufficiently large \( \sigma_0 > 0 \), provided that we take into account the residue of the integrand at \( s = \xi \). The nice growth conditions satisfied by the integrands (in vertical strips) make, as before, the integrals on horizontal segments of fixed length tend to zero, as the segments recede to infinity on either side of the real axis. In this manner, we obtain that the left hand side of (49) is equal to

\[
\frac{1}{2\pi i} \int_{\sigma_0 - \infty}^{\sigma_0 + \infty} (M(WF))(s) \sum_{n=1}^{\infty} a_n \left( \frac{\lambda}{2\pi n} \right)^s \, ds - \left( M(WF)(\xi) \left( \frac{\lambda}{2\pi} \right)^{\xi} \text{Res}_{s=\xi} \phi(s) \right)
\]

which is the same as \( \sum_{n=1}^{\infty} a_n (WF) \left( \frac{2\pi n}{\lambda} \right) - \gamma a_0 F(0) \). We have used here, the relation (47) with \( WF \) in place of \( F \), the definition of \( a_0 \) and the fact that \( \gamma = \pm 1 \); further, we have interchanged the integration on the line \( \text{Res} = \sigma_0 \) and the summation over \( n \) as justifiable without difficulty and also appealed to Theorem 3.1. By applying entirely similar arguments to the right hand side of (49), we see that it is equal to \( \gamma \left( \sum_{n=1}^{\infty} a_n F \left( \frac{2\pi n}{\lambda} \right) - \gamma a_0 (WF)(0) \right) \) and our theorem is completely proved. \( \square \)
3 Relation with Hecke’s Theory

As we have remarked earlier, the theorem proved above, in a special but quite typical case, goes back to Hecke whose theory of the correspondence between Dirichlet series with functional equations and modular forms is well-known. We now make the connection of our theorem with Hecke’s explicit.

For a complex number \( \tau \) with \( \text{Im}(\tau) > 0 \), let us set \( F(x) = e^{ix\tau} \) for \( x \in \mathbb{R}_+ \). Then we know, from above, that \((MF)(s) = \Gamma(s)(-i\tau)^{-s}\) and this gives us

\[
(W(e^{ix\tau}))(x) = (M^{-1}(\Gamma(s)(M(e^{ix\tau}))(\xi - s)))(x)
= (M^{-1}(\Gamma(s)(-i\tau)^{-(\xi-s)}))(x)
= (M^{-1}(\Gamma(s)(-i\tau)^{-x}(\frac{i}{\tau}^{-s}))(x)
= (i\tau)^{-x}e^{-ix/\tau}.
\]

Thus, taking \( F(x) = e^{ix\tau} \) in the theorem above, we get the relation

\[
\sum_{n=0}^{\infty} a_n e^{-\frac{n\lambda \tau}{\lambda^2}} = \gamma(-i\tau)^{X} \sum_{n=0}^{\infty} a_n e^{\frac{nt}{\lambda}}
\]

which becomes more transparent, on being restated as follows. Namely, under Hecke’s correspondence

\[
\phi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \mapsto f(\tau) = \sum_{n=0}^{\infty} a_n e^{\frac{n\tau}{\lambda}}
\]

with \( a_n = \gamma \left( \frac{\lambda}{\pi^2} \right)^X \Gamma(\xi) \text{Residue}_{x=\xi} \phi(s) \), the functional equation \( R(s) = \gamma R(x - s) \) of \( R(s) = \left( \frac{\lambda}{\pi^2} \right)^X \Gamma(s) \phi(s) \) corresponds to the transformation-law \( f \left( \frac{-1}{\tau} \right) = \gamma(-i\tau)^{-x}f(\tau) \) which merely says that the function \( f(\tau) \) obviously holomorphic in \( \tau \) for \( \text{Im}(\tau) > 0 \) behaves like a modular form of weight \( X \) under the transformations from the subgroup of \((PSL_2(\mathbb{R})) \) generated by \( \tau \mapsto \tau + \lambda \) and \( \tau \mapsto -1/\tau \). This was indeed the starting point of Hecke theory (15).
1. A Theory of Mellin Transformations

We go back now, to the general case starting from an arbitrary \( F \) in \( \mathcal{F} \) and proceed to give an interpretation of the set-up above as follows. For any \( u \) in \( \mathbb{R} \), let us denote by \( \xi(u) \), the multiplication by \( e^{iux} \) of any function in \( \mathcal{F} \) and further define \( \eta(u) \) on \( u \) on \( \mathcal{F} \) by

\[
\eta(u)F = W\xi(-u)WF
\]

for any \( F \). Let \( M_p(\mathbb{R}) \) (respectively \( M_p(\mathbb{Z}) \)) denote the subgroup of the group of unitary automorphisms of \( \mathcal{F} \) generated by \( \xi(u) \) and \( \eta(u) \) as \( u \) varies over \( \mathbb{R} \) (respectively over \( \mathbb{Z} \)). Finally, let us fix a function \( \phi \neq 0 \) of signature \( \{ \lambda, \overline{\lambda}, \gamma \} \) and define for any \( F \) in \( \mathcal{F} \), a function \( \theta_F \) on \( M_p(\mathbb{R}) \) by

\[
\theta_F(g) = \sum_{n=0}^{\infty} a_n(gF) \left( \frac{2\pi n}{\lambda} \right) \quad \text{for} \quad g \in M_p(\mathbb{R})
\]

Then, for any \( g_0 \) in \( M_p(\mathbb{Z}) \),

\[
\theta_F(g_0 g) = \theta_F(g) \quad \text{for every} \quad g \in M_p(\mathbb{R}).
\]

Also, it can be proved that \( w = e\left(\frac{\overline{\lambda}}{4}\right)W \) belongs to \( M_p(\mathbb{R}) \). Moreover, if \( (M_p)'(\mathbb{Z}) \) denotes the subgroup of \( M_p(\mathbb{R}) \) generated by \( \xi(u) \) for all \( u \) is \( \mathbb{R} \) and \( w \), then

\[
\theta(g'_o g) = X(g'_o)\theta_F(g) \quad (g \in M_p(\mathbb{R}))
\]

for every \( g'_o \) in \( (M_p)'(\mathbb{Z}) \), where \( X \) is a character of \( M_p(\mathbb{Z}) \). All the assertions above except the one about \( w \) being in \( M_p(\mathbb{R}) \), are consequences of the Poisson formula.

The group \( M_p(\mathbb{R}) \) is known as the Metaplectic group and our interpretation above becomes significant in view of the following fact that can be proved: namely lat \( \mu \) denote the cyclic subgroup of \( \mathbb{C}_1^{\infty} \) generated by \( e(\overline{x}) \); then \( \mu \) is contained in \( M_p(\mathbb{R}) \) and under the correspondence

\[
\xi(u) \mapsto \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad \eta(u) \mapsto \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}
\]

We have

\[
M_p(\mathbb{R})/\mu \sim PSL_2(\mathbb{R}) or SL_2(\mathbb{R})
\]

according as \( \lambda \) is or is not the quotient of an integer by an integer.
Remark. By using the same function \( k(x) \) as in the remarks at the end of §1 above, we have

\[
(\eta(t)F)(x) = (it)^{-x}e^{ix/t} \int_{0}^{\infty} k\left(\frac{xy}{t^2}\right)e^{iy/t}F(y)y^x d\log y
\]

for every \( t \neq 0 \) in \( \mathbb{R} \) and every \( F \) in \( \mathcal{F} \); this integral representation of \( (\eta(t)F)(x) \) can be used to determine the structure of \( M_p(\mathbb{R}) \) stated above.
Chapter 2
Dual Asymptotic Expansions

This chapter is essentially concerned with asymptotic expansions which are dual to the asymptotic expansions mentioned in chapter I, in a certain sense. We shall deal with the following two questions about the spaces \( F \) and \( Z \) with the additional assumption that \( \lambda_0 > 0 \) in the case of \( \mathbb{R} \)-fields \( K \) and \( \text{Re}(\lambda) > 0 \) for every \( \lambda \in \Lambda \) for \( p \)-fields \( K \); namely,

(i) characterising the space \( (\omega^{-1}F)^* = \{ \text{the Fourier transform } (\omega^{-1}F)^* F \text{ for } F \in F \} \) and

(ii) giving necessary and sufficient conditions for the function \( \omega^{-1}F \) to have its Fourier transform as well in \( L^1 \), in terms of \( \omega^{-1}F \) and \( Z = MF \).

The reasons for our careful discussion of asymptotic expansions in chapters I and II will become evident when we deal, later on, with the local arithmetic theory of forms of higher degree and a Poisson formula of the Siegel-Weil type for such forms. At this stage, by way of pointing out just one reason, we shall merely highlight the following important problem. For locally compact abelian groups \( X \) and \( G \) with a continuous map \( f : X \to G \), Weil has defined, for any \( \phi \) in the space \( \mathcal{S}(X) \) of Schwartz-Bruhat functions on \( X \) and any \( g^* \) in the dual \( G^* \) of \( G \),

\[
F^*_\phi(g^*) = \int_X \phi(x)(f(x), g^*) \, dx
\]

where \( dx \) is the Haar measure on \( X \) and \( (f(x), g^*) = g^*(f(x)) \). It is still...
a fundamental problem to determine precisely when, for a given \( f \), the function \( F^*_\phi \) (clearly uniformly continuous on \( G^* \)) actually belongs to \( L^1(G^*) \) for every \( \phi \) in \( \mathcal{S}(X) \). In his well-known paper [52], Weil calls this the “condition A” and clarifies its meaning for quadratic forms \( f \), in the case of local fields and adele-spaces. For forms of higher degree, one can similarly give necessary and sufficient conditions involving an invariant associated with the surface determined by \( f \), at least in the case when \( G = K \).

\section{1 Fourier Transforms of Quasi-characters}

We shall quickly recall well-known results concerning Fourier transforms of quasi-characters.

\subsection{1.1 Notation}

Let \( X \) be a finite-dimensional vector space over a local field \( K \) with \( \psi \), a nontrivial character of \( K \). Let \([x, y]\) be a symmetric non-degenerate \( K \)-bilinear form \( X \times X \), \(|dx|\) denote a Haar measure on \( X \) and let \( X \) be identified with its dual \( X^* \).

For every \( \phi \) in \( L^1(X) \), we define

\[ \phi^* (x) = \int_X \phi(y)\psi([x, y]) \ |dy|. \]

If \( \Lambda(X) \) is the space of continuous functions \( \phi \) in \( L^1(X) \) such that \( \phi^* \) is also in \( L^1(X) \), then the measure \(|dx|\) can be normalised uniquely in such a manner that \((\phi^*)^*(x) = \phi(-x)\) for every \( \phi \) belonging to \( \Lambda(X) \). We shall then call \(|dx|\) the self-dual measure relative to \( \psi([x, y]) \).

If \( \epsilon(\cdot) = exp(2\pi \sqrt{-1} \cdot) \), then for \( \psi \), we have the standard choice

\[ \phi(x) = \begin{cases} \epsilon(x) & \text{for } x \in K = \mathbb{R} \\ \epsilon(2Re(x)) & \text{for } x \in K = \mathbb{C} \end{cases} \]

In the case of \( p \)-fields \( K \), we take a \( \psi \) which is 1 on \( R \) and non-trivial on \( P^{-1} \). Further, if \([x, y] = xy\) for \( x, y \in K \), then \(|dx|\) is the same as the
measure $dx$ in chapter I, namely

$$dx = \begin{cases} 
\text{usual measure for } K = \mathbb{R} \\
\text{twice the usual measure for } K = \mathbb{C} \\
\text{the measure with } M(R) = 1 \text{ for } p\text{-fields } K.
\end{cases}$$

1.2 The Space $\mathcal{S}(X)$ of Schwartz-bruhat Functions on $X$ and the Space $\mathcal{S}'(X)$ of Tempered Distributions on $X$

The space $\mathcal{S}(\mathbb{R}^n)$ of Schwartz functions on $\mathbb{R}^n$ consists of all $C^\infty$ functions from $\mathbb{R}^n$ to $\mathbb{C}$ such that for every linear differential operator $D = \sum a_{\alpha\beta} x^\alpha D^\beta$ with $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $D^\beta = \frac{\partial^{\beta_1 + \cdots + \beta_n}}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}}$, $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \geq 0$ in $\mathbb{Z}$, $a_{\alpha\beta} \in \mathbb{C}$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we have $||D\phi||_\infty = \sup_{x \in \mathbb{R}^n} |D\phi(x)| < \infty$. We topologize $E = \mathcal{S}(\mathbb{R}^n)$ with respect to the family of seminorms $\phi \mapsto ||D\phi||_\infty$, as $D$ varies over all such linear differential operators with polynomial coefficients. Thus, a sequence $\{\phi_n\}_{n \geq 0}$ in $E$ tends to 0 if (and only if), for every $D$ as above, the sequence $\{|D\phi_n||_\infty\}_n$ of real numbers tends to 0 as $n \to \infty$. The same topology can also be obtained as follows. Namely, we rearrange $\{|(x^\alpha D^\beta)(\phi)|\}_{\alpha, \beta}$ as $c_1(\phi), c_2(\phi), \ldots$, and define

$$||\phi|| = \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{c_m(\phi)}{1 + c_m(\phi)}.$$

Then we have

(i) $||\phi|| = ||-\phi|| \geq 0$ with equality only when $\phi = 0$;

(ii) $||\phi + \psi|| \leq ||\phi|| + ||\psi||$ for every $\phi, \psi$ in $E$; and

(iii) the scalar multiplication $(\lambda, \phi) \mapsto \lambda\phi$ from $\mathbb{C} \times E$ to $E$ is continuous.

While (i) and (ii) make $E$ a metric space with $||\phi - \psi||$ as the distance between $\phi$ and $\psi$ and $||\ ||$ being fairly close to a norm, (iii) in addition, makes $E$ a topological vector space. It is direct verification to check that
2. Dual Asymptotic Expansions

$E$ is complete. The dual $E'$ consisting of all complex-valued $\mathbb{C}$-linear continuous maps of $E$ is called the space of tempered distributions on $X$; since $E$ satisfies (i), (ii), (iii) and is complete, by the Banach-Steinhaus theorem.

We now take up the definition of $\mathcal{S}(X)$, $\mathcal{S}(X)'$ for the case of $p$-fields $K$. Let $G$ be a locally compact abelian group. We say that $G$ has arbitrarily large (respectively arbitrarily small) compact open subgroups $H$, if for every compact subset $C$ of $G$ (respectively neighbourhood $V$ of $0$) there exists such an $H$ containing $C$ (respectively contained in $V$). Whenever $G$ has arbitrarily large and arbitrarily small compact open subgroups, we define the Schwartz-Bruhat space $\mathcal{S}(G)$ as the space of all complex-valued functions $\varphi$ on $G$ which are locally constant (i. e, constant in some neighbourhood of every point) and which have, in addition, compact support. If, for any subset $S$ of $G$, we denote by $\varphi_S$ the characteristic function of $S$, then $\mathcal{S}(G)$ is just the $\mathbb{C}$-span of the functions $\varphi_{a+H}$ with a varying over $G$ and $H$ over subgroups described above. We may take $\mathcal{S}(G)'$ as just the algebraic dual of $\mathcal{S}(G)$, which consists of all $\mathbb{C}$-linear function on $\mathcal{S}(G)$; it is not necessary to topologies $\mathcal{S}(G)$, in order to define the dual space $\mathcal{S}(G)'$.

Both in the case of $\mathbb{R}$-fields and $p$-fields $K$, the Fourier transformation $\varphi \mapsto \varphi^*$ gives an automorphism of $\mathcal{S}(X)$ and hence an automorphism $T \mapsto T^*$ of the dual $\mathcal{S}(X)'$, if we set $T^*(\varphi) = T(\varphi^*)$ for every $\varphi \in \mathcal{S}(X)$.

The following two definitions are needed for our purposes, later on.

Let $\varphi \in L^1_{\text{loc}}(X)$ i. e, $\varphi$ is a complex-valued locally integrable function on $X$ which has, in addition, at most polynomial growth (at infinity) whenever $K$ is an $\mathbb{R}$-field. We can then define a tempered distribution $T_\varphi \in \mathcal{S}(X)'$ by simply setting, for every $\phi$ in $\mathcal{S}(X)$,

$$T_\varphi(\phi) = \int_X \phi(x) \varphi(x) \, dx$$

(50)

For the case of $p$-fields, only the convergence of the integral needs to be verified and this is taken care of by $\phi$ having compact support and $\varphi$ being locally integrable. For $\mathbb{R}$-fields, the convergence of the integral at infinity is made possible by $\phi$ being Schwartz function and $\varphi$ being
at most of polynomial growth there; as for the continuity, it is easy to check that $T_\psi(\phi) \to 0$ as $\phi \to 0$ in $\mathcal{S}(X)$.

We remark that under the imbedding $\phi \to T_\phi$, $\mathcal{S}(X)$ is dense in $\mathcal{S}'(X)$.

Given $T$ in $\mathcal{S}(X)'$, $U$ open in $X$ and $f \in L^1_{\text{loc}}(U)$, we say that $T = f$ in $U$, if

$$ T(\phi) = \int_U \phi(x)f(x) \, |dx| \quad (51) $$

for every $\phi$ in $\mathcal{S}(X)$ with support of $\phi$ contained in $U$.

1.3

In the case when $K$ is a $p$-filed, we shall have occasion to use the following facts frequently.

We shall assume $\psi$ to be standard i.e., $\psi$ is 1 on $R$ and $\psi$ is non-constant on $P^{-1}$. For any $X(R^\times)$, we denote by $e_X$ the smallest natural number $e$ such that $X$ is 1 on the set $1 + P^e$; thus, by definition, we have always $e_X \geq 1$. We assert that the following formulae are valid:

$$ \int_{R^\times} X(u)(\pi^{-e}u) \, du = \begin{cases} 1 - 1/q & \text{for } e \leq 0 \text{ and } X = 1 \\ 1 - 1/q & \text{for } e \leq 1 \text{ and } X = 1 \\ 0 & \text{for } e > 1 \text{ and } X = 1 \\ 0 & \text{for } e \neq e_X \text{ and } X \neq 1 \end{cases} \quad (52) $$

Formulae (52) for $X = 1$ are straightforward. We shall therefore prove only that integral is 0 for $X \neq 1$ and $e \neq e_X$. First, if $e > e_X$, then the integral is the same as

$$ \sum_{a \in R^\times, \mod p^{-1}P^{-1}} \int_{P^{-1}} X(a(1 + x)) \psi(\pi^{-e}a(1 + x)) \, dx $$

$$ = \sum_{a \in R^\times, \mod P^{-1}} X(a) \psi(\pi^{-e}a) \int_{P^{-1}} \psi(\pi^{-e}x) \, dx $$

The last integral over $P^{-1}$ is 0, since $P^{-1} \ni x \to \psi(\pi^{-e}x)$ is a non-
trivial character. Similarly, if \( e < e_X \) and \( e_X \geq 2 \), then the integral on the left side of (52) is equal to

\[
\sum_{a \in R^\times \mod P^{e_X}} \int_X \chi(a(1 + x)) \psi(\pi^{-e} a(1 + x)) \, dx
\]

and the integral over \( P^{e_X} \) is 0 since \( P^{e_X} \ni x \rightarrow X(1 + x) \) is a non-trivial character of \( P^{e_X} \). Finally, if \( e < e_X = 1 \), the left hand side of (52) is just \( \int_X \chi(u) du = 0 \), since \( X \neq 1 \).

We have the following supplement to (52). For any \( X \) (possibly equal to 1) in \( (R^\times)^* \), let us define

\[
g_X = \int_{R^\times} X(u) \psi(\pi^{-e_X} u) du \tag{53}
\]

Then, trivially, we have

\[
\overline{g_X} = X(-1) g_{X^{-1}} \tag{54}
\]

Moreover, it is clear from (52) that

\[
g_1 = -q^{-1} \tag{55}
\]

If we write \( u = a + \pi^{e_X} v \) and let \( a \) run over representatives of \( R^\times \) modulo \( P^{e_X} \) and \( v \) over \( R \), then we can rewrite (53) as

\[
g_X = q^{-e_X} \sum_{a \in R^\times \mod P^{e_X}} \chi(a) \psi(\pi^{-e_X} a) \tag{56}
\]

Thus \( q^{e_X} g_X \) is a standard Gaussian sum and for \( X \neq 1 \), we know indeed that

\[
|g_X|^2 = q^{-e_X} \tag{57}
\]
1.4

We are now in a position to recall the following well-known table of Fourier transforms of quasi-characters; for $K = \mathbb{R}$, one may refer to M. J. Lighthill [32] or I. M. Gel’fand and G. E. Shilov ([12]).

For any quasicharacter $\omega$ in $\Omega_+(K)$, i.e., with $\sigma(\omega) > 0$, the function $\omega\omega^{-1}$ is in $L^1_{\text{loc}}(K)$ as may be verified at once from the convergence of the integral $\int |x|^{p-1} \, dx$ taken over a compact neighbourhood of 0. Thus $T_{\omega\omega^{-1}}$ in the sense of (50) is defined and therefore, also $(T_{\omega\omega^{-1}})^\infty$, although the latter is not, in general, of the form $T\varphi$ for some $\varphi$. But if we restrict it to $K^\times$, then

$$(T_{\omega\omega^{-1}})^\infty = b(\omega)\omega^{-1}\text{ on } K^\times$$

for a constant $b(\omega)$ explicitly described below: namely, let $\psi$ be standard and let $d = 1/2$ or 1 according as $K = \mathbb{R}$ or $\mathbb{C}$-field $K$ and for $\omega = \omega_s(ac)^p$, we have

$$b(\omega) \overset{\text{defn}}{=} b_p(s) = |p|^{d(1-2s)} \frac{\Gamma(ds + |p|/2)}{\Gamma(d(1-s) + |p|/2)}$$

and for a $p$-field $K$, we have, for $\omega = \omega_sX$,

$$b(\omega) \overset{\text{defn}}{=} b_X(s) = \begin{cases} \frac{1}{1-q^{-1}} & \text{for } X = 1 \\ g_X q^s X & \text{for } X \neq 1 \end{cases}$$

A transparent proof for the complete table (59) can be found in Weil’s exposé ([53]) and we shall give a brief outline of his method.

The natural action of $K^\times$ on $K$ induces the action of $K^\times$ on $\mathcal{F}(K)$ given by $\phi \mapsto g\phi$ with $(g\phi)(x) = \phi(g^{-1}x)$ and hence, by duality, the action on $\mathcal{F}(K)'$ given by $T \mapsto gT$ with

$$(gT)(\phi) = T(g^{-1}\phi), \text{ i.e., } (gT)(\phi(x)) = T(\phi(gx))$$

for $\phi$ in $\mathcal{F}(K)$. It can be immediately verified that for $T = T_{\omega\omega^{-1}}$ (in the sense of (50)) with $\omega \in \Omega_+(K^\times)$, we have

$$(gT)(\phi) = \int_K \phi(gx)(\omega\omega^{-1})(x) \, dx$$
\begin{equation}
= \omega(g)^{-1} \int_k \phi(x)(\omega \omega_1^{-1})(x)dx \mid \text{using } x \mapsto g^{-1}x \tag{60}
= \omega(g)^{-1}T(\phi)
\end{equation}

for every \( g \in K^\times \) and \( \phi \in \mathcal{S}(K) \).

For a given \( \rho \in \Omega(K^\times) \), suppose \( \Delta \in \mathcal{S}'(K^\times) \) satisfies the condition
\( g\Delta = \rho(g)^{-1}\Delta \) for every \( g \) in \( K^\times \). Then we shall write \( \Delta \in P(K, K^\times; \rho) \).

Taking the restriction of \( \Delta \) to \( K^\times \) i.e., by requiring \( \phi \) in \( \Delta(\phi) \) to have
support contained in \( K^\times \), we get a distribution on \( K^\times \), which is relatively
invariant under the action of \( K^\times \) (i.e., picking up a factor \( \rho(g) \) under the
multiplication by \( g \) in \( K^\times \)). Then, by a method similar to the proof of
the uniqueness of a relatively invariant measure (on \( K^\times \)), we can show
that
\[ \Delta \text{ on } K^\times = c(\rho \omega_1^{-1})(x) \, dx \tag{61} \]
for a complex constant \( c \) which may be 0. For instance, this proof goes
as follows in the case of a \( p \)-field \( K \): if we put
\[ \Delta'(\phi) = \Delta(\rho^{-1}\phi) \]
for every \( \phi \) in \( \mathcal{S}(K) \) with support of \( \phi \) contained in \( K^\times \), then \( \Delta' \)
becomes even invariant under the action of \( K^\times \). Since, for every \( e \geq 1 \), we have
\[ \varphi_{1+p^e}(x) = \sum_{g \in 1+p^e, \mod p^{e+1}} \varphi_{1+p^{e+1}}(g^{-1}x) \]
we obtain, by the invariance of \( \Delta' \), that
\[ q^e \Delta'(\varphi_{1+p^e}) = q^{e+1} \Delta'(\varphi_{1+p^e} + 1) = \text{a constant } c', \text{ say;} \]
then, for every \( a \) in \( K^\times \) and \( e \geq 1 \), we have
\[ \Delta'(\varphi_{a(1+p^e)}) = c' q^{-e} = c' \int_{K^\times} \varphi_{a(1+p^e)}(x) \frac{dx}{|x|_k} \]
Then implies
\[ \Delta' = c' \frac{dx}{|x|_k} ie., \Delta = c'(\rho \omega_1^{-1})(x) \, dx \text{ in } K^\times. \]
1. Fourier Transforms of Quasi-characters

Coming back to $\triangle \in P(K, K^\times; \rho)$, we see, from the simple relation $(g^{-1}\phi)^* = \omega_1(g)^{-1}g^{\phi^*}$ for every $\phi$ in $\mathcal{S}(K)$, that

$$\triangle \in P(K, K^\times; \rho) \Rightarrow \triangle^* \in (K, K^\times, \rho^{-1}\omega_1)$$

Taking $\triangle = T_{\omega\omega^{-1}}$ for $\omega \in \lambda_e(K^\times)$ and $\omega$ instead of $\rho$, we can see, in view of (60) and (61), that the relation (58) is valid. The function $b(\omega)$ can be determined by evaluating both sides of (58) at a suitable $\phi$. For example, in the case of a $p$-field $K$, we can take for $\phi$, the characteristic function of $1 + peX$ and then we get the expression for $b_X(s)$ in (59).

We shall give some details in the case of a $p$-field $K$: if $\phi_E$ denotes the characteristic function of a compact open subset $E$ of $K$, then, for any $a$, in $K$, and $e$ in $Z$, we have

$$\varphi_{a+pe}(x) = \int_{a+pe} \psi(xy)dy = \psi(ax) \int_{pe} \psi(xy)dy$$

$$= \psi(ax) \begin{cases} m(pe) \text{ if } x \in pe \\ 0 \text{ otherwise}, \end{cases}$$

i.e., $\varphi_{a+pe}(x) = q^{-e}\psi(ax)\varphi_{p^{-e}}(x)$ (62)

If $\sigma(\omega) > 0$ and $X = 1$, then for $\phi = \varphi_{1+p}$, we obtain, using (52), that

$$b(\omega) = \int_{p^{-1}} \psi(x)|x|_{K}^{\frac{s-1}{2}} dx = \sum_{e=-1}^{\infty} q^{-es} \int_{pe} \psi(pxu)du$$

$$= -q^{-1} + \frac{1-q^{-1}}{1-q^{-s}} \sum_{e=0}^{\infty} q^{-es}$$

$$= \frac{1-q^{-1}}{1-q^{-s}}$$

If, on the other hand, $X \neq 1$, then for $\phi = \varphi_{1+p\epsilon X}$, we get again, in view of (52), that

$$b(\omega) = \int_{p^{\epsilon X}} \psi(x)|x|_{K}^{\frac{s-1}{2}} X(ax(x)) \, dx$$
2. Dual Asymptotic Expansions

\[
= \sum_{e=e_x}^{\infty} q^{-es} \int_{\mathbb{R}^e} \psi(\pi^e u)X(u)du
\]

\[
= q^{es} \widehat{gX}
\]

1.5

For a moment, we shall write \( X \) for \((ac)^p\) as well and correspondingly denote \( b_p(s) \) as \( b_X(s) \). If \( \text{Re}(s) > 0 \), then \( \varphi(x) = X(ac(x)) |x|^{s-1} (\log |x|_K)^{m-1} \) is locally integrable on \( K \) (and at most of polynomial growth at infinity for \( \mathbb{R} - \text{fields } K \)). Further, for the Fourier transform of \( \varphi \) (which we shall identify with the distribution \( T\varphi \)), we have the relation

\[
(X(ac(x)) |x|^{s-1} \log |x|_K)^{m-1} \quad \text{on } K^\times = \quad (63)
\]

\[
= \sum_{j=1}^{m} (-1)^{j-1} \binom{m-1}{j-1} \frac{b_X(s)}{ds^{m-j}} X(ac(x))^{-1} |x|^{s} (\log |x|_K)^{j-1}
\]

62 (in the sense of the definition at the end of 1.2). The proof of (63) is as follows. If \( \sigma(\omega) > 0 \), then

\[
\left( \frac{d^{m-1}}{ds^{m-1}}(\omega \omega^{-1}) \right)^* = \frac{d^{m-1}}{ds^{m-1}}(\omega \omega^{-1})^*
\]

and moreover, the left hand side of (63) is the same as that of (64); this is an identity in \( \mathcal{S}'(K) \). If now we restrict the right hand side of (64) to \( K^\times \), then in view of (58), it becomes equal to

\[
\frac{d^{m-1}}{ds^{m-1}}(b(\omega \omega^{-1}))(x) = \frac{d^{m-1}}{ds^{m-1}}(b_X(s)X(ac(x))^{-1} |x|^{s}_K);
\]

on applying the Leibnitz formula, it coincides with the right hand side of (52).

If we take \( X = 1, s = 1, m = 1 \) in (63), the left hand side becomes just \( \delta \) or the Dirac distribution \( \delta_0 \) with \( \{0\} \) as support while the right hand side is 0 since \( b_1(1) = 0 \), by (59); thus \( \delta_0 = 1 \) on \( K^\times \), which is indeed true!
2. The Space \((\omega_1^{-1}F)^*\)

2.1 Statement of a Theorem

(A) Given a strictly increasing sequence \(\{\lambda_k\}_{k \geq 0}\) of positive real numbers with no finite accumulation point and a sequence \(\{m_k\}_{k \geq 0}\) of natural numbers, the space \((\omega_1^{-1}F)^* = \{(\omega_1^{-1}G); G \in F\}\) in the case of an \(\mathbb{R}\)-field \(K\), consists of all complex-valued \(C^\infty\) functions \(F^\# \) on \(K\), with a termwise differentiable uniform asymptotic expansion of the form

\[
F^\#(x) \approx \sum_{k=0}^{\infty} \sum_{m=1}^{m_k} a_{k,m}^\#(ac(x))|x|_K^{-\lambda_k} \left(\log |x|_K\right)^{m-1} \tag{65}
\]

as \(|x|_K \to \infty\) with \(a_{k,m}^\#\) denoting a \(C^\infty\) function on \(K^\times\). Furthermore, if \(F^\#\) is the same as the Fourier transform \(F^*\) of \(F \in \omega_1^{-1}F\) which has the termwise differentiable asymptotic expansion

\[
F(x) \approx \sum_{k=0}^{\infty} \sum_{m=1}^{m_k} a_{k,m}(ac(x))|x|_K^{-\lambda_k-1} \left(\log |x|_K\right)^{m-1} \tag{66}
\]

as \(|x|_K \to 0\), then (65) is the termwise Fourier transform of the expansion (66).

(B) Given a finite set \(\Lambda = \{\lambda \mod 2\pi i / \log q; \text{Re}(\lambda) > 0\}\) and natural numbers \(m_\lambda\) for every \(\lambda\) in \(\Lambda\), the space \((\omega_1^{-1}F)^* = \{(\omega_1^{-1}G); G \in F\}\) in the case of a \(p\)-field \(K\), consists of all complex-valued locally constant functions \(F^\#\) on \(K\) such that

\[
F^\#(x) = \sum_{\lambda \in \Lambda} \sum_{m=1}^{m_\lambda} a_{\lambda,m}^\#ac(x)|x|_K^{1-\lambda_k} \left(\log |x|_K\right)^{m-1} \tag{67}
\]

for all sufficiently large \(|x|_K\). Further, if \(F^\#\) is the Fourier transform \(F^*\) of an \(F \in \omega_1^{-1}F\) for which

\[
F(x) = \sum_{\lambda \in \Lambda} \sum_{m=1}^{m_\lambda} a_{\lambda,m}(ac(x))|x|_K^{1-\lambda_k-1} \left(\log |x|_K\right)^{m-1} \tag{68}
\]
for all sufficient small $|x|_K$, then (67) is obtained from (68) by taking the Fourier transform termwise.

(C) In both the cases $(A)$ and $(B)$, the inverse of the map $F \mapsto F^*$ from $\omega_1^{-1}\mathcal{F}$ to $(\omega_1^{-1}\mathcal{F})^*$ is given by

$$F(x) = \lim_{r \to \infty} \int_{|y|_K \leq r} F^*(y) \psi(-xy) dy, \quad x \in K^\times$$

(69)

REMARKS. For the proof of part $(A)$ of Theorem 2.1 in the case $k = \mathbb{R}$, the reader may refer to [32], Chapter 4, §4.2, §4.3. A partial form of $(A)$ for $K = \mathbb{R}$ is the principal result proved by Lighthill who has shown that for $F$ in $\omega_1^{-1}\mathcal{F}$ the Fourier transform $F^*$ has an asymptotic expansion of the form (65) as $|x| \to \infty$, without discussing its termwise differentiability, however. In this context, we merely remark that the differentiability of $F^*$ is immediate and the termwise differentiability of the asymptotic expansion for $F^*(x)$ as $|x|_K \to \infty$ follows from the stability of $\omega_1^{-1}\mathcal{F}$ and $(\omega_1^{-1}\mathcal{F})^*$ under any homothety-invariant differential operator. In fact, for $K = \mathbb{R}$, we get $(DF)^* = -F^* - DF^*$ with $D = x \frac{d}{dx}$ and therefore, the asymptotic expansions of $-(DF^*)(x)$ as $|x| \to \infty$ is obtained simply as the sum, taken termwise, of the asymptotic expansion of $F^*(x)$ and $(DF)^*(x)$ as $|x| \to \infty$; we can easily verify that this is the same as the asymptotic expansion of $F^*(x)$ as $|x| \to \infty$ being termwise differentiable once and by repeated applications of $D$, the asymptotic expansion of $F^*(x)$ is seen to be termwise differentiable (i.e., any number of times). The proof of part $(A)$ for the case $K = \mathbb{C}$ is left as an exercise. We therefore give only the proof for $(B)$ and our proof given below for $(B)$ is an adaptation to the case of $p$-fields, of Lighthill’s proof mentioned above; actually, in the case of $p$-fields, we can also give a proof entirely avoiding the concept of distributions and using the properties of the integral considered in [32].

Before we proceed to consider assertions $(B)$ and $(C)$ of Theorem 2.1, let us prove the following lemma.

Lemma 2.1. For arbitrary $s$ in $\mathbb{C}$ and $e_o$ in $\mathbb{Z}$, let
Proof of assertions (B) and (C) of Theorem 2.1.

2. The Space \((\omega^{-1}F)^*\)

\[
\varphi(x) = \begin{cases} 
X(ac(x)|x|^{-1}(\log |x|)^{m-1} & \text{for ord } (x) < e_0 \\
0 & \text{for ord}(x) \geq e_0
\end{cases}
\]

where \(X \in (R^\infty)\). Then \(\varphi^\ast = 0\), in \(K \setminus p^{-e}X^e\).

Proof. If \((a + P^e) \cap P^{-eX-e_0+1} \neq \phi\), then ord\(a) < -e_X - e_0 + 1; for otherwise, ord\(a) \geq -e_X - e_0 + 1 = t, say and \((a + P^e) \cap P^t = (a + P^e) \cap (a + P^t) = a + P_{max(t)} \neq \phi\).

Every \(\phi\) in \(C(K)\) with support contained in \(K \setminus P^{-eX-e_0+1}\) is a linear combination of functions of the form \(\varphi_{a,P^e}\) with \((a + P^e) \cap P^{-eX-e_0+1} = \phi\). Therefore, in order to prove the lemma, it suffices to show that \(T^\ast_\psi(\varphi_{a,P^e}) = 0\) for \(a + P^e\) disjoint with \(P^{-eX-e_0+1}\). And, indeed we have then

\[
T^\ast_\psi(\varphi_{a,P^e}) \overset{\text{dfn}}{=} T_\psi(\varphi_{a,P^e}) \\
= q^{-e} \int_{P^{-e}} \varphi(x)\psi(ax)\,dx, \quad \text{by (62)}
\]

\[
= q^{-e} \sum_{-e \leq j < e_0} q^{-j} (-j \log q)^{m-1} \int_{K^e} X(u)\psi(a\pi^j u) du.
\]

Since ord\(a) < -e_X - e_0 + 1\) from above, we have ord\((a\pi^j) = ord(a) + j \leq \pi^j, -e_X - e_0 + 1 + e_0 - 1 = -e_X\). Therefore, by (52), each one of the last mentioned integrals vanishes and the lemma is proved.

Proof of assertions (B) and (C) of Theorem 2.1

We do not prove assertion (B) in its entirety. We shall only start from any \(F\) in \(\omega^{-1}F\), with the expansion (63) for sufficiently small \(|x|_K\), say, for ord\((x) \geq e_0\) and show that (67) holds for the Fourier transform \(F^\ast\) of \(F\) in \(L^1(K)\) and further \(F\) has compact support. For any function \(f\) in \(L^1(K)\) with compact support, we see trivially that \(f^\ast\) is locally constant; in fact, for some \(t \geq 1\), we know that \(f\) vanishes outside \(P^{-t}\) and

\[
f^\ast(x + z) = \int_{P^{-t}} f(y)\psi((x + z)y)\,dy = \int_{P^{-t}} f(y)\psi(xy)\psi(zy)\,dy = f^\ast(x)
\]
for \( z \in P^1 \) and every \( x \in K \). Thus \( F^* \) is locally constant on \( K \).

Let us define \( \varphi_0 : K \to \mathbb{C} \) by

\[
\varphi_0(x) = \begin{cases} 
\text{Right hand side of (68)} & \text{for ord } (x) < e_0 \\
0 & \text{for ord } (x) \geq e_0
\end{cases}
\]

Then \( \varphi_0 \in L^1_{\text{loc}}(K) \) and by Lemma 2.1, \( \varphi_0^\circ \) vanishes for all sufficiently large \( |x|_K \). If, now, we define the function \( \phi \) by setting \( \phi(0) = 0 \) and for \( x \in K^* \), \( \phi(x) \) by

\[
F(x) = (\text{right hand side of (68)}) + \varphi_0(x) + \phi(x), \quad (70)
\]

then clearly \( \phi(x) \) is again locally constant and furthermore, since \( \phi(x) = F(x) \) for ord\( (x) < e_0 \), \( \phi \) has compact support, so that \( \phi \in \mathcal{S}(K) \). Applying the Fourier transform to (70), we get, on using (63), that

\[
F^* = \left( \sum_{j \in \Lambda} \sum_{m=1}^{m_j} a_{j,m}(ac(x))|x|_k^{j-1}(\log |x|_K)^{m-1} \right)^* + \varphi_0^* + \phi^*
\]

\[
= \sum_{j \in \Lambda} \sum_{m=1}^{m_j} a_{j,m}(ac(x))|x|_k^{j-1}(\log |x|_K)^{m-1}
\]

for all sufficiently large \( |x|_K \), where

\[
a_{j,m}(u) = \sum_{X \in (R^n)^{r}} \left( \sum_{j=m}^{m_j} (-1)^{m-1} \binom{j-1}{m-1} a_{j,i,x} \frac{d^{j-m}b_X(\lambda)}{d\lambda^{j-m}} \right) X^{-1}(u) \quad (71)
\]

Thus \( F^* \) has the expansion (67), for all large \( |x|_K \), that is obtained from (68) by termwise application of Fourier transform. This also proves that any \( F^\circ \) in \((\omega_1^{-1}F)^*\) has an expression of the type (67) for all sufficiently large \( |x|_K \).

It now remains for us to prove assertion (C) of Theorem 2.1 again only partly. We first remark that the integral in (69) exists since \( F^* \) is \( C^\infty \) or locally constant, we shall prove (69) only for a \( p \)-field \( K \). The right hand side of (69) is just

\[
\lim_{e \to 0^+} \int_{P^1} \psi(-xy) \left( \int_{K} \psi(yz) F(z) dz \right) dy = \lim_{e \to 0^+} \int_{K} \left( \int_{P^1} \psi((z-x)y) dy \right) F(z) dz
\]
2. The Space \((\omega_1^{-1}F)^*\)

\[
= \lim_{e \to 0} q^e \int_{x+Pe} F(z)dz,
\]

since the inner integral over \(P^{-e}\) is \(q^e\) or 0 according as \(z-x \in P^e\) or otherwise. But, for \(x \neq 0\), \(F\) is constant on \(x+P^e\) for sufficiently large \(e\) and therefore the limit above is just \(F(x)\).

**REMARKS.** (1) For any \(F\) in \(\omega_1^{-1}F\) as above with the asymptotic expansion \((66)\) or \((68)\) as \(|x|_K \to 0\), we have, for the Fourier transform \(F^*\) a corresponding asymptotic expansion \((65)\) or \((67)\) as \(|x|_K \to \infty\).

As remarked in \((71)\) for the case of \(p\)-fields, the Fourier coefficients \(a_{\lambda,m,X}, a_{\lambda,m,X}^\#\) respectively of the coefficients \(a_{\lambda,m}(u), a_{\lambda,m}^\#(u)\) in the asymptotic expansion of \(F\) and \(F^*\), are related as follows: namely, for every \(\lambda\) and \(1 \leq m \leq m_1\),

\[
a_{\lambda,m,X}^\# = (-1)^{m-1} \sum_{j=m}^{m_1} (-1)^{j-m} \frac{d^{j-m}}{ds^{j-m}}(\lambda)a_{\lambda,j,X}. \tag{71}'
\]

If we fix \(\lambda\) and \(X\) in \((K^r)^*\) and arrange \(a_{\lambda,j,X}(1 \leq j \leq m_1)\) and \((-1)^{m-1} a_{\lambda,m,X}^\#(1 \leq m \leq m_1)\) as \(M_1\)-rowed column vectors, then the coefficient-matrix corresponding to the relations \((71)\)' becomes an upper triangular matrix of \(m_1\) rows and columns, with all the diagonal entries to \(b_X(\lambda)\) and the entries just above the diagonal equal to \(b'_{X}(\lambda), 2b'_{X}(\lambda), \ldots (m_1-1)b'_{X}(\lambda)\). Therefore, if \(b_X(\lambda) = 0\), then certainly \(a_{\lambda,m,X}^\#\) is 0 and moreover, \(a_{\lambda,1,X}\) disappears from the expression for \(a_{\lambda,m,X}^\#\) for \(1 \leq m < m_1\). Thus the asymptotic expansion of \(F^*(x)\) as \(|x|_K \to 0\) although \(F\) is determined by \(F^*\), on \(K^*\) in view of \((69)\). For \(F(x) = e^{-x^2}\), the asymptotic expansion of \(F^*(= F)^*\) as \(|x| \to \infty\) is just \(F^*(x) \approx 0\) while \(F(x) \approx \sum_{n=0}^{\infty} (-\pi)^n x^{2n}/n!\) as \(|x| \to 0\) tends to zero!

For an \(\mathbb{R}\)-field \(K\) and \(X = (ac)^n\), we know from \((59)\), that \(b_X(s) = 0\) if and only if \(s = 1 + \frac{1}{2^s} + \frac{1}{2^t}\) for \(n = 0, 1, 2 \ldots\); for such \(s, b_{X}^\#(s) \neq 0\). Thus the rank of the \(m_1\)-rowed matrix above is \(m_1 - 1\). If \(K\) is a \(p\)-field, then again from \((59)\), \(b_X(s) = 0\), if and only if \(X = 1, q^s = q\); in this case, \(b_{X}^\#(s) \neq 0\). Hence, at least for \(p\)-fields of characteristic 0, the rank of the above-mentioned matrix is \(m_1 - 1\).
2. Dual Asymptotic Expansions

(2) Let us define $m_K$ to be $2, 2\pi$ or $1 - q^{-1}$ according as $K = \mathbb{R}, \mathbb{C}$ or a $p$-field so that, in all cases, we get

$$d^*x \text{ (the normalised Haar measure on } K) = \frac{dx}{m_K|x|_K}$$

For $\lambda_0 > 0$ in the case of $\mathbb{R}$-fields and $\text{Re}(\lambda) > 0$ for all $\lambda$ in the case of $p$-fields, let for any $F \in \omega_1^{-1}F$, $F^*$ be its Fourier transform and $Z = M(m_K \omega_1 F)$. If we accept Theorem 2.1 then $F^*$ determines $F$ on $K^*$ by (69) and $F$ in turn, determines $Z$. Similarly, if we start from $Z$, then $F, F^*$ are determined. Thus any one of $F, F^*, Z$ can be chosen arbitrarily in $\omega_1^{-1}F, (\omega_1^{-1}F)^*$ and $Z$ respectively and the other two are determined.

But, we shall seldom start from $F^*$, so that the lack of a complete proof of Theorem 2.1 in these lectures will not present any discontinuity.

3 The Space $(\omega_1^{-1}F)^* \cap L^1(K)$

3.1

Let $0 < \lambda_0 < \lambda_1 < \ldots$, $\text{Re}\lambda > 0$ for $\lambda \in \Lambda$ and $M_k$ be as in §2.1 and let $F \in \omega_1^{-1}F$ and $F^*$, the Fourier transform of $F$. Then although $F$ is in $L^1(K)$, $F^*$ may not be in $L^1(K)$. We proceed to characterise the subspace $(\omega_1^{-1}F)^* \cap L^1(K)$ by the following

**Theorem 3.1.** For $F \in \omega_1^{-1}F$, the following assertions are equivalent:

(a) $F^* \in L^1(K)$ :

(b) $F(0) = \lim_{|x| \to 0} F(x)$ exists;

(c) $Z_\chi(s)(\chi \neq 1), (s + 1)Z_\chi(s) \
\frac{Z_\chi(q^{-s})(\chi \neq 1), (1 - q^{-(s+1)})Z_\chi(q^{-s})}{\text{are holomorphic}}$ for $\text{Re}(s) \geq -1$;

(d) $F(0) = \frac{1}{m_k} \begin{cases} 
\lim_{s \to -1} (s + 1)Z_\chi(s) \\
\lim_{s \to -1} (1 - q^{-(s+1)})Z_\chi(q^{-s}) 
\end{cases}$

and $F^* \in L^{-1}(K)$. 
To prove the theorem, we will show that $a) \Rightarrow b) \Rightarrow c) \Rightarrow d)$. We first need to prove a lemma, to take care of the proof in the case of $p$-fields,

**Lemma 3.1.** Let $f(z)$ be meromorphic in $|z| \leq r$ for $r > 0$, and let $f(z) = \sum_{n \in \mathbb{Z}} s_n z^n$ be its Laurent expansion around $z = 0$. Then $a = \lim_{n \to \infty} r^n s_n$ exists and is finite if and only if $f(z) - \frac{a'}{z^{a'}}$ is holomorphic in $0 < |z| \leq r$ for a constant $a'$, in which case $a' = a$.

**Proof.** We merely sketch a proof of this lemma which is rather elementary. If $g(z) = \sum_{n \in \mathbb{Z}} t_n z^n$ is a Laurent expansion of $g$ meromorphic in $0 < |z| \leq r$, then $g$ is holomorphic in $0 < |z| \leq r$ if and only if $\lim_{n \to \infty} |t_n|/n < r^{-1}$; if $\lim_{n \to \infty} |t_n|/n < r^{-1}$ then clearly $\lim_{n \to \infty} r^n t_n = 0$. Thus $f(z) - \frac{a}{z^{a'}}$ is holomorphic in $0 < |z| \leq r$, then $a < \infty$ and $a' = a$.

Now, if $a < \infty$, then $f$ is holomorphic in $0 < |z| < r$. Let us refer to “$a = \lim_{n \to \infty} r^n s_n$ exists and is finite” as the property $P$ of $f$. If $f$ has property $P$, then so does $Rf$, for any polynomial $R$ in $\mathbb{C}[z]$. We now assert that $f$ cannot have a pole at any $a \neq r$ on $|z| = r$; for otherwise, $(1 - a^{-1})^{-1} = \sum_{n=0}^{\infty} a^{-n} z^n$ will have the property $P$, which gives a contradiction. Moreover, $f$ can have at most a simple pole at $z = r$; for, otherwise, $(1 - z/r)^{-2} = \sum_{n=0}^{\infty} (n + 1) r^{-n} z^n$ will also have the property $P$, leading to a contradiction. This proves the lemma. \[\square\]

**Proof of Theorem 3.1**

a) $\Rightarrow b)$. This is straightforward. In fact, if $G, G^*$ are dual groups with dual measures $dg, dg^*$, then, for $F_0$ in $L^{-1}(G; dg)$ with its Fourier transform $F_0$ in $L^{-1}(G^*; dg^*)$, we have $F_0(g) = (F_0^*)^*(-g)$ at every $g$ where $F_0$ is continuous. Thus, in our situation, $F(x) = (F^*)^*(-g)$ at every $g$ where $F_0$ is continuous. Thus, in our situation, $F(x) = (F^*)^*(-x)$ for $x \in K^*$ and $F(0) = \lim_{|x| \to 0} F(x)$ exists, since $(F^*)^*$ is continuous.

b) $\Rightarrow c) \Rightarrow d)$. We know that $F$ has an asymptotic expansion as in (66) as $|x| \to 0$ with $\lambda_0 > 0$; it follows that $F(0) = \lim_{|x| \to 0} F(x)$ exists if and only if $\lambda_k \geq 1$ for every $k$ and, in addition, when
λ_k = 1 for some k, then \( m_k = 1, a_k,1 = a_k,1(1) \). This is equivalent to saying that \( F(x) = F(0) + \sum_{\lambda_k \neq 1} \sum_{m=1}^{m_k} a_k,m(ac(x))|x|_{K}^{\lambda_k-1} \log |x|_K |m-1. \) In view of Ch. I §4, this is equivalent to \( Z_p(s) \) for every \( p \neq 0 \) and \( Z_0(x) - \frac{mK_0(0)}{s+1} \) being holomorphic for \( Re(s) \geq -1 \), where now \( Z = M_0(m_0 \omega_1 F) \), since \( (m_0 \omega_1 F)(x) \approx m_0 F(0)|x|_K + m_0 \sum_{\lambda_k \neq 1} \sum_{m=1}^{m_k} a_k,m(ac(x))|x|_{K}^{\lambda_k} \log |x|_K |m-1 \) as \( |x|_K \to 0 \) and further \( b_{k,m,p} = (-1)^{m-1}(m-1)!a_{k,m,-p} \). This, in turn, implies that \( F^*(x) \approx \sum_{\lambda_k \neq 1} \sum_{m=1}^{m_k} a_k,m(ac(x))|x|_{K}^{\lambda_k} \log |x|_K |m-1 \) as \( |x|_K \to \infty \), in view of Theorem 2.1 above, (71), and the fact that \( b_0(1) = 0 \). Such an asymptotic expansion clearly implies that \( F^* \) is in \( L^1(K) \).

b) \( \Rightarrow c) \Rightarrow d) \) (for a \( p \)-field \( K \)) If we write \( F(\pi^2 u) = \sum_{\lambda \in \Lambda} c_{e,\lambda} \chi(u) \) for \( e \in \mathbb{Z} \) and \( u \in \mathbb{R}^* \), then \( (m_0 \omega_1 F)(\pi^2 u) = m_0 \sum_{\lambda \in \Lambda} (q^{-1}c_{e,\lambda}) \chi(u) \) and in view of (38), we have \( Z_\chi(z) = m_0 \sum_{e \in \mathbb{Z}} c_{e,\chi-1}(q^{-1}z)^e \). Now b) is equivalent to “\( \lim_{z \to \infty} c_{e,\chi} \) is equal to \( F(0) \) for \( \chi = 1 \) and to 0 for \( \chi \neq 1 \)”. This, in turn, is equivalent to \( Z_\chi(z) \) for every \( \chi \neq 1 \) and \( Z_1(z) - \frac{mK_0(0)}{1-q^{-1}z} \) being holomorphic in \( 0 < |z| \leq q \), because of Lemma 3.1. This, again, is equivalent to the fact that

\[
F(x) = F(0) + \sum_{\lambda \in \Lambda} \sum_{m=1}^{m_k} a_k,m(ac(x))|x|_{K}^{\lambda-1} \log |x|_K |m-1
\]

for all sufficiently small \( |x|_K \). Just as above, this new implies

\[
F^*(x) = \sum_{\lambda \in \Lambda} \sum_{m=1}^{m_k} a_k,m(ac(x))|x|_{K}^{\lambda-1} \log |x|_K |m-1
\]

for all large enough \( |x|_K \), so that finally \( F^* \) is in \( L^1(K) \). This completes the proof Theorem 3.1.
Chapter 3
Local Arithmetic Theory of Forms of Higher Degree

As indicated by the title, we shall apply the results of the first two chapters to study the arithmetic of forms $f$ of higher degree considered over local fields. Broadly speaking, many facts that we shall discover do represent generalizations to the case when $f$ has degree greater than 2, of earlier results; however, several striking new aspects are thrown up, which, as is not surprising, could not have been expected for quadratic forms. These will be recognized clearly, as we proceed further.

1 Three Functions: $F_\phi$, $F^*_\phi$, $Z_\phi$

We shall use the notation in Chapter II, §1, with the modification that $X = K^n$ for a local field $K$, $[x, y] = x_1y_1 + \cdots + x_ny_n$ for $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ in $X$. We shall use the standard $\psi$, so that the self-dual measure $|dx|$ becomes the product of the usual measure on $K$. We choose a non-constant polynomial $f(x) = f(x_1, \ldots, x_n)$ in $K[x_1, \ldots, x_n]$, since the case when $f$ is a constant, is trivial.

The following well-known lemma will be useful to establish the continuity of a function that we shall define by means of an integral.

Lemma. Let $(X, dx)$ be a measure space, $T$ a metric space (or just separable at every point) and $f(x, t) : X \times T \to \mathbb{C}$ be such that $f$ is continuous
in $t$ for any fixed $x$ and locally integrable in $x$ for any fixed $t$ and, further, $|f(x, t)| \leq \varphi(x)$ for a $\varphi$ in $L^1(X, dx)$, for every $t$. Then $\int_X f(x, t)dx$ defines a continuous function of $t$.

**Proof.** is immediate, in view of Lebesgue’s theorem. □

### 1.1 Definition of $Z_\varphi$

For $\omega \in \Omega_+(K^\times)$ extended by continuity to $K$ and $\varphi \in \mathcal{S}(K)$, the integral

$$Z_\varphi(\omega) = \int_X \varphi(x)|\omega(f(x))|dx$$

defines a holomorphic function on $\Omega_+(K^\times)$; in fact, for

$$0 \leq \sigma(\omega) \leq \sigma_2 < \infty, \quad |\omega(f(x))| = |f(x)|^{\sigma(\omega)} < \max(1, |f(x)|_K)^{\sigma_2}$$

and $\varphi(x)\max(1, |f(x)|_K)^{\sigma_2}$ is in $L^1(X)$.

### 1.2 Definition of $F^*_\varphi$

With $f$ in $K[x_1, \ldots, x_n]$ chosen above, we define, for any $\varphi$ in $\mathcal{S}(X)$, the function $F^*_\varphi$ on $K$ by

$$F^*_\varphi(i^*) = \int_X \varphi(x)|\psi(i^* f(x))|dx \quad \text{for} \quad i^* \in K.$$

Applying the lemma above to $f(x, t) = \varphi(x)|\psi(tf(x))$ and $\varphi(x) = |\varphi(x)|$ it follows that $F^*_\varphi$ is continuous on $K$ (and indeed, bounded and uniformly continuous). The definition of $F^*_\varphi$ is deceptively simple but it is a rather difficult function to investigate.

### 1.3 The Measure $|\theta_i|$ (continued)

The critical set $C_f$ for $f$ is the set of points $a$ in $X$ at which the gradient grad $f$ of $f$ is 0. If, therefore, we define $U(i) = f^{-1}(i)\setminus C_f$ for $i \in K$, then $a$ is in $U(i)$ if and only if $a \in X$ satisfies $f(a) = i$ and further at least
one partial derivative of \( f \), say \( \frac{\partial f}{\partial x_k} \), does not vanish at \( a \). Since \( f(x) = i \) on \( U(i) \), we have the relation

\[
\frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n = 0.
\]

Taking the exterior product of both sides with the \((n-2)\) - form \( dx_1 \wedge \cdots \wedge \hat{dx}_j \wedge \cdots \wedge dx_n \) (where \( \hat{\cdot} \) indicates that the differentials \( dx_j, dx_k (j < k) \) have been omitted), we obtain

\[
(-1)^{j-1} \frac{\partial f}{\partial x_j} dx_1 \wedge \cdots \wedge \hat{dx}_k \wedge \cdots \wedge dx_n + (-1)^{k-2} \frac{\partial f}{\partial x_k} dx_1 \wedge \cdots \wedge \hat{dx}_j \wedge \cdots \wedge dx_n = 0
\]

Thus \( \theta_i(x) = (-1)^{k-1} \left( \frac{\partial f}{\partial x_k} \right)^{-1} dx_1 \wedge \cdots \wedge \hat{dx}_k \wedge \cdots \wedge dx_n |_{U(i)} \) is a well-defined non-vanishing regular \((n-1)\) - form around \( a \in U(i) \) and thereby giving rise to a global regular and non-vanishing \((n-1)\) - form on \( U(i) \); this is just “the residue of \((f(x) - i)^{-1} dx \) along \( U(i) \)” and we denote it still by \( \theta_i \). Now \( \theta_i \) induces on \( U(i) \) a Borel measure \( |\theta_i| \) such that for every continuous function \( \varphi \) on \( X \) with compact support disjoint with \( C_f \), we have

\[
\int_X \varphi(x) dx = \int_K \left( \int_{U(i)} \varphi(\theta_i) \right) |d\theta_i|.
\]  

Furthermore, for every such \( \varphi \), the function \( i \mapsto \int_{U(i)} \varphi(\theta_i) \) is continuous on \( K \). The proof depends on the implicit function theorem for \( X \). Let us assume, after re-ordering the subscripts if necessary, that the partial derivative \( \frac{\partial f}{\partial x_n} \) does not vanish at a given point \( a = (a_1, \ldots, a_n) \) in \( U(i) \). Since

\[
\frac{\partial (x_1, \ldots, x_{n-1}, f(x))}{\partial (x_1, \ldots, x_{n-1}, x_n)}(a) = \frac{\partial f}{\partial x_n}(a) \neq 0,
\]

we may use \( x_1, \ldots, x_{n-1}, f(x) \)
as local coordinates at \(a\). If \(x' = (x_1, \ldots, x_{n-1})\) and \(a' = (a_1, \ldots, a_{n-1})\), then, by the implicit function theorem, there exists a convergent power-series \(g(x', t)\) in \(x' - a'\), \(t - i\) with coefficients in \(K\), such that, around \(a\), \(f(x) = t\) if and only if \(x_n = g(x', t)\). Applying a partition of unity, if necessary, we may suppose that \(\varphi\) has its support “near \(a\)”. Then for \(t\) close to \(i\), we have, by the very definition of \(|\theta|\), that

\[
\int_{U(i)} \varphi[\theta] = \int_{K^{n-1}} \varphi(x', g(x', t)) \left| \frac{\partial f}{\partial x_n}(x', g(x', t)) \right|^{-1} dx' |dt|.
\]  

The lemma stated at the beginning of §1 ensures that the right hand side of (73) is continuous in \(t\). Also, our assertions are independent of the manner in which the partition of unity is chosen. Now integration on \(K\) gives

\[
\int_{K} \left( \int_{U(i)} \varphi[\theta] \right) |dt| = \int_{K^{n}} \varphi(x', g(x', t)) \left| \frac{\partial (x', x_n)}{\partial (x', t)} \right|^{-1} dx' |dt|
\]

\[
= \int_{X} \varphi(x) dx
\]

implying (72).

**1.4 Definition of \(F_{\varphi}\)**

For the sake of simplicity, let us assume, in the sequel, that \(C_f \subset f^{-1}(0)\) (equivalently, \(U(i) = f^{-1}(i)\) for every \(i \in K^\times\)). If \(f(x)\) is homogeneous and if the characteristic of \(K\) does not divide \(m\), the degree of \(f\), then this assumption is fulfilled; in fact, for \(a = (a_1, \ldots, a_n) \in C_f\), we have

\[
0 = a_1 \frac{\partial f}{\partial x_1}(a) + \cdots + a_n \frac{\partial f}{\partial x_n}(a) = mf(a)
\]

which is impossible unless \(f(a) = 0\). Our assumption is not serious when the characteristic of \(K\) is 0; for, then, by a theorem of Bertini, \(C_f\) is contained in finitely many fibres \(f^{-1}(i)\) which can be reduced, after
applying a suitable partition of unity to the situation where there is only one fibre and therefore (by translation), we may assume without loss of generality that \( C_f \subset f^{-1}(0) \). For \( i \in K^n \), we define

\[
F_\phi(i) = \int_{U(i)} \phi|_\theta.
\]

Then if \( \phi \) has compact support disjoint with \( C_f \), then we know from above that \( F_\phi \) is continuous on \( K^n \); if, however, \( \phi \) has compact support which is not disjoint with \( C_f \), we may apply a partition of unity to isolate points on our fibre and conclude that \( F_\phi \) is continuous. Thus, at least for the case of \( p \)-fields, the continuity of \( F_\phi \) is clear. However, if \( K \) is an \( \mathbb{R} \)-field and if \( \phi \) in \( \mathcal{S}(X) \) does not have compact support, even the convergence of the integral above is not clear, not to speak of the continuity of \( F_\phi \). We therefore take up the typical case \( K = \mathbb{R} \) and show that the integral defining \( F_\phi(i) \) is absolutely convergent for every \( \phi \) in \( \mathcal{S}(\mathbb{R}^n) \) and further, that \( F_\phi \) is continuous on \( \mathbb{R}^n \). We may assume after a translation, that \( C_f \subset f^{-1}(a) \) for some \( a \neq 0 \) and that \( i = 0 \) in (73). In order to prove the continuity of \( F_\phi \) at \( i(=0) \), it is enough to show that for any \( \epsilon > 0 \), there exists \( r = r(\epsilon) > 0 \) ensuring that

\[
\int_{f^{-1}(b), \|x\| \geq r} |\phi| |\theta_b| < \epsilon
\]

for every \( b \) with \( |b| \leq \rho < |a| \), where, for \( x = (x_1, \ldots, x_n) \in X, \|x\| = (x_1^2 + \cdots + x_n^2)^{1/2} \); the rest is clear from the case of compact support. We now appeal to the following lemma due to Hörmander ([17]): for any polynomial \( P(x_1, \ldots, x_n) \in \mathbb{R}[x_1, \ldots, x_n] \), there exist constants \( c, \alpha > 0, \beta \geq 0 \), such that

\[
|P(x)| \geq c \cdot \frac{\text{dis}(x, V(P))^\alpha}{\max(1, \|x\|^\beta)} \quad \text{for every} \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n
\]

where \( V = V(P) \) is the variety of the real zeros of \( P \) and \( \text{dis}(x, V) \) is the distance of \( x \) from the (closed) set \( V \). The critical set \( C_f \) is precisely the set of common zeros of \( \frac{\partial f}{\partial x_i} (1 \leq i \leq n) \) and hence the same as \( V(P_0) \) for
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\[ P_0(x) = \sum_{i=1}^{n} \left( \frac{\partial f}{\partial x_i} \right)^2. \]

We now assert that

\[ \eta = \inf_{f(x) \leq \rho} \{ \max(1, \|x\|)^{\gamma} P_0(x)^{1/2} \} > 0 \]  \hspace{1cm} (75)

for a constant \( \gamma \geq 0 \). By Hörmander’s inequality, we have

\[ P_0(x) \geq c \frac{\text{dis}(x, C_f)^\alpha}{\max(1, \|x\|)^\beta} \text{ for every } x \in X. \]  \hspace{1cm} (76)

We claim that (75) holds for \( \gamma = (1/2)((m-1)\alpha + \beta) \), \( m \) being the degree of \( f \). Otherwise, let \( S \) be a sequence of points \( x \) with \( \|f(x)\| \leq \rho \) such that

\[ \lim_{x \in S} \max(1, \|x\|)^{\gamma} P_0(x) = 0 \]

implying incidentally that \( \|x\| \to \infty \). This means, in view of (76), that

\[ \lim_{x \in S} \|x\|^{(m-1)\alpha} (\text{dis}(x, C_f))^{\alpha} < c^{-1} \cdot \lim_{x \in S} \|x\|^{2\gamma} P_0(x) = 0 \]  \hspace{1cm} (77)

We may, therefore, assume that \( \|x\| \geq 1 \) and \( \|x\|^{m-1} \text{dis}(x, C_f) \leq 1 \) for every \( x \) in \( C_f \). On the other hand, for every \( x \), there exists \( y = (y_1, \ldots, y_n) \in C_f \) such that \( \|x - y\| = \text{dis}(x, C_f) \) and we have

\[
|f(y) - f(x)| \leq \sum_{k=1}^{m} \sum_{i_1, \ldots, i_k} \frac{1}{k!} \left| \frac{\partial^k f(x)}{\partial x_{i_1} \ldots \partial x_{i_k}} (y_{i_1} - x_{i_1}) \ldots (y_{i_k} - x_{i_k}) \right|
\]

\[
\leq \sum_{k=1}^{m} c_k \|x\|^{m-k} \|y - x\|^k \text{ with constants } c_k
\]

\[
\leq c' \|x\|^{m-1} \|x - y\| \text{ for a constant } c'
\]

\[
\to 0 \quad \text{by (77)}
\]

But this contradicts

\[ |f(y) - f(x)| \geq |f(y)| - |f(x)| \geq |a| - \rho > 0, \]

for every \( x \) in \( S \). Thus (75) is valid with the chosen \( \gamma \) and

\[ f^{-1}([-\rho, \rho]) \subset \bigcup_{k=1}^{n} W_k \]
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where $W_k = \{x \in X; \max(1, ||x||)^{\gamma} \left| \frac{\partial f}{\partial x_k}(x) \right| \geq \eta/\sqrt{n} \}$; in other words, on $f^{-1}([-\rho, \rho])$ at least one of the partial derivatives goes to 0 at most like $||x||^{-\gamma}$ and not too rapidly. On the other hand, since $\phi \in \mathcal{S}(\mathbb{R}^n)$, there exists a constant $c'' > 0$ such that for every $x = (x_1, \ldots, x_n) \in X$ and correspondingly $x' = (x_1, \ldots, \hat{x}_k, \ldots, x_n)$ in $\mathbb{R}^{n-1}$ with $||x'||^2 = ||x||^2 - x_k^2$, we have

$$\max(1, ||x'||^n) \max(1, ||x||)^{y+1} |\phi(x)| < c''$$

for $1 \leq k \leq n$. Therefore, for the left hand side of (74), we have the upper estimate

$$\sum_{k=1}^{n} \int_{f^{-1}(0) \cap W_k} |\phi(x)| [1/|\partial f(x)/\partial x_k|] dx'$$

$$\leq \frac{\sqrt{n}}{\eta} \sum_{k=1}^{n} \int_{f^{-1}(0) \cap W_k} \max(1, ||x||)^{\gamma} |\phi(x)| dx'$$

$$\leq \frac{\sqrt{n}}{\eta} mnc'' r^{-1} \int_{\mathbb{R}^{n-1}} \max(1, ||x'||)^{-n} dx' = O(r^{-1}), \text{ as } r \to \infty.$$ 

This is sufficient to take care of the continuity of $F_\phi$.

1.5 The Relations Between $F_\phi, F_\phi^*$ and $Z_\phi$

If $\sigma(\omega) > 0$, then, for $\phi \in \mathcal{S}(K)$, we have, by definition,

$$Z_\phi(\omega) = \int_{f^{-1}(0)} \phi(x) \omega(f(x)) dx$$

$$= \int_{K} \left( \int_{U_i(\omega)} \phi(\theta_i) \omega(i) d\theta \right)$$

$$= M_K(m_K \omega_1 F_\phi(\omega), \text{ in a formal sense.})$$

For every $i^* \in K$, we have, on the other hand,
\[ F_\phi(i^\ast) = \int_{X \setminus f^{-1}(0)} \phi(x)\psi(i^\ast f(x))|dx|, \text{ since } f^{-1}(0) \text{ has measure 0}, \]
\[ = \int_{K^\times} \left( \int_{U(i)} \phi(\theta_i^\ast) \psi(i^\ast \theta_i^\ast)|d\theta_i| \right) \]
\[ = \int_{K^\times} F_\phi(i)\psi(i^\ast)|d\theta_i| \]
\[ = \int_{K^\times} F_\phi(i)\psi(i^\ast)|d\theta_i|, \text{ extending } F_\phi \text{ to } K. \]

i.e. \( F_\phi^\ast \) is just the Fourier transform \( (F_\phi)^\ast \) of \( F_\phi \).

We note that both the integrals considered above are absolutely convergent:

\[ \int_{K^\times} |F_\phi(i)\omega(i)| |d\theta_i| \leq \int_X |\phi(x)| |f(x)|^{\sigma(\omega)}_K |dx| < \infty, \]
\[ \int_{K^\times} |F_\phi(i)\psi(i^\ast)| |d\theta_i| \leq \int_X |\phi(x)| |dx| < \infty. \]

### 1.6 Statement of a Theorem

We now state the first substantial theorem, in our theory, for forms of higher degree.

**Theorem 1.6.** Suppose that the characteristic of \( K \) is 0 and that \( C_f \subset f^{-1}(0) \). Then there exists a set of \( \lambda \)'s with Re \( \lambda > 0 \) such that \( Z_\phi \) is in \( \mathcal{Z} \), \( F_\phi \) is in \( \omega_1^{-1}\mathcal{F} \) and \( F_\phi^\ast \) is in \( (\omega_1^{-1}\mathcal{F})^\ast \). In particular, \( Z_\phi(\omega) \) defined by (78) has a meromorphic continuation to the whole of \( \Omega(K^\times) \) and further, \( F_\phi(i) \) (respectively \( F_\phi^\ast(i^\ast) \)) possesses an asymptotic expansion as \( |i|_K \to 0 \) (respectively \( |i^\ast|_K \to \infty \)).

The major part of the proof of this theorem which will be given in the next two sections, is to show that \( Z_\phi \) is in \( \mathcal{Z} \); once that is proved, it will follow that \( F_\phi \) is in \( \omega_1^{-1}\mathcal{F} \) and hence \( F_\phi^\ast \) in \( (\omega_1^{-1}\mathcal{F})^\ast \). Indeed, together
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with the absolute convergence of the integral for $Z_\phi(\omega)$ mentioned above and the continuity of $F_\phi$, the fact that $Z_\phi$ is in $\mathcal{Z}$ will enable us to apply the Fourier inversion theorem and conclude that

$$F_\phi = m_K^{-1} \omega_1^{-1} M^{-1}(Z_\phi)$$

The rest of the assertions of Theorem 1.6 now follows from the theorems proved in Chapter I.

**REMARKS.**

1) The assumption that $C_f \subset f^{-1}(0)$ in Theorem 1.6 cannot be dropped. Consider, for example, the case when $n = 2, f(x) = 1 + x_1 x_2$ and $\phi$ is the characteristic function of $V = P^{e_1} \times P^{e_2}$ with $e_1, e_2$ in $\mathbb{Z}$ satisfying $e_1 + e_2 \geq 1$. Then $C_f = \{0\} \subset f^{-1}(1)$ and with our former notation,

$$Z_\chi(z) = \int_\mathbb{V} \chi(1 + x_1 x_2)dx$$

$$= \sum_{j \geq e_1} (1 - q^{-1}) \int_{1 + p^{j+e_2}} \chi(u)du$$

$$= q^{-\max(e_1 + e_2, e_1)}$$

$$\neq 0$$

for every $\chi$. This violates condition (2) defining $\mathcal{Z}(\Omega(K^r))$.

2) If $K$ has characteristic $p > 0$, then the condition “$p$ does not divide the degree of $f$” is necessary (although not sufficient) to ensure that $Z_\phi$ is in $\mathcal{Z}$. Consider now, for example, the case when $n = 1, f(x) = x^p$ and $\phi$ is the characteristic function of $R$. Then, for $|z| < q^{1/p}$, we can show that

$$Z_\chi(z) = \begin{cases} 
\frac{1-q^{-1}}{1-q^{-p}} & \text{if } \chi^p = 1 \\
0 & \text{if } \chi^p \neq 1.
\end{cases}$$

But the number of $\chi$ in $(R^*)^s$ with $\chi^p = 1$ is infinite; in fact, for the finite group $R/(1 + P^{pj})$ with $j \geq 1$, the cardinality of the cokernel of the $p^n$ power map is $q^{(p-1)j}$ and therefore, the number of $\chi$ in $(R^*)^s$ with $\chi^p = 1$ and $e_\chi \leq pj$ is $q^{(p-1)j}$. Again condition (2) in the definition of $\mathcal{Z}$ is violated by $Z_\phi$. 

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1.7 Gaussian Sums and Singular Series

For any \( p \)-field \( K \) and \( e \in \mathbb{Z} \), let us denote the \( n \)-fold product \( P^e \times \ldots \times P^e \) of \( P^e \) by \( (P^e)^{(n)} \) and for \( e = 0, \mathbb{R}^{(n)} \) by \( X^0 \). If \( f(x) \in \mathbb{R}[x_1, \ldots, x_n] \) and \( \phi \) is the characteristic function of \( X^0 \), we simply write \( F^*, F^*_\phi, Z_\phi \) respectively. Then, without any further assumption on \( f \), we have, for \( i^* = \pi^e u \) with \( e \geq 0 \) in \( \mathbb{Z} \) and \( u \) in \( \mathbb{R}^* \), the following relation:

\[
F^*(i^*) = q^{-ne} \sum_{\xi \mod P^e} \psi(i^* f(\xi))
\]

which is immediate on writing any \( x \in X^0 \) as \( \xi + \pi^e y \) with \( \xi \) running over \( X^0 \mod (P^e)^{(n)} \) and \( y \) in \( X^0 \). The sum over \( \xi \) modulo \( (P^e)^{(n)} \) is usually called a generalized Gaussian sum. Although this relation is quite straightforward, the interesting fact is that, once we accept Theorem 1.6, then we get

\[
q^{-ne} \sum_{\xi \mod P^e} \psi(i^* f(\xi)) = \sum_{\lambda, m} a_{\lambda, m}(ac(i^*))|i^*|_K^{-1}(\log |i^*|_K)^{m-1}
\]

is a fixed linear combination of functions of the form

\[
\chi(ac(i^*))|i^*|_K^{-1}(\log |i^*|_K)^{m-1}
\]

for all sufficiently large \( e = -\text{ord}(i^*) \). Such a “stable behaviour” of a generalized Gaussian sum is quite remarkable.

We also recall that for \( i \) in \( \mathbb{R} \),

\[
N_e(i) = \text{the number of } \xi \mod P^e \text{ with } f(\xi) \equiv i \pmod{P^e},
\]

also plays an important role in the classical theory. If \( f \) is homogeneous of degree \( m \), \( i \) is in \( \mathbb{R}\{0\} \) and the characteristic of \( K \) does not divide \( m \), then we will see that

\[
q^{-(n-1)e}N_e(i) \text{ is independent of } e, \text{ for } e \geq 2 \text{ord}(mi) + 1. \quad (79)
\]

Further, for \( i \neq 0 \) in \( \mathbb{R} \), we have the relation

\[
F(i) = \text{the above “stable quotient” } q^{-(n-1)e}N_e(i) \quad (80)
\]
which is known as the singular series associated with \( f \) and \( i \). The proof of (79) is given in the appendix to this chapter. However, if we assume that \( K \) has characteristic 0 and \( C \subset f^{-1}(0) \)

then (even for \( f \) not necessarily homogeneous) we can use Theorem 1.6 to derive, at one stroke, the relations (79) in perhaps a less precise form and (80). The proof is as follows. In fact, the theorem implies that \( F_\phi \) and, in particular, \( F = F_\phi \) for \( \phi = \varphi_{X^0} \) is in \( \omega_1^{1/F} \) and hence, locally constant on \( K^X \). Thus there exists \( e > \text{ord}(i) \) such that \( F(t) = F(i) \) for all \( t \in i + P^e \). If now, we take \( \varphi \) to be the characteristic function of the compact open set \( X^0 \cap f^{-1}(i + P^e) \), the right hand side of (72) is just

\[
\int_K \left( \int_{U(t)} \varphi(\theta_t) \right) |dt| = \int_{i+P^e} \left( \int_{U(t) \cap X^0} |\theta_t| \right) |dt|
\]

\[
= \int_{i+P^e} F(t)|dt| \quad \text{by definition 73}
\]

\[
= F(i)q^{-e}, \quad \text{since } F \text{ is constant on } i + P^e.
\]

But the left hand side of (72) is merely

\[
m(X^0 \cap f^{-1}(i + P^e)) = m(\{ \xi \in X^0; f(\xi) \equiv i \pmod{P^e} \})
\]

\[
= N_e(i) \cdot m(\pi P^e X^0)
\]

\[
= q^{-me} N_e(i).
\]

Hence there exists \( e > \text{ord}(i) \) such that

\[
F(i) = N_e(i)/q^{(n-1)e}
\]

and this relation persists even if we replace \( e \) by a larger integer.
2 Preparations for the Proof of the “Main Theorem”

2.1

We shall prove the “main theorem” i.e. Theorem 1.6 only for $p$-fields $K$ and merely give an outline of the proof when $K$ is an $\mathbb{R}$-field. First we explain the concept of a resolution of (the singularities of) the surface defined by $f(x) = 0$ in a general set-up.

Let $k$ be an arbitrary field, $X$ an (absolutely) irreducible non-singular algebraic variety defined over $k$ and $D$, a positive $k$-rational divisor of $X$. Then a $k$-resolution

$$h : Y \rightarrow X$$

of $D$ is defined by the following three conditions:

(i) $h$ is an everywhere regular birational map of an irreducible non-singular algebraic variety $Y$ to $X$, with both $Y$ and $h$ defined over $k$, such that $h$ can be written as a product of monoidal transformations each with an irreducible non-singular centre;

(ii) $h^{-1}$ is regular and hence biregular, at every simple point $a$ of $D$; and

(iii) for every $b$ in $Y$, all the irreducible components of $h^*(D)$ passing through $b$ are defined over the field $k(b)$ and further are mutually transversal at $b$.

Although this may appear to be much too general, we shall deal in our applications only with the cases when $X$ is either an affine or a projective space and $D$, the divisor defined by a polynomial. Also, that part of condition (i) which requires $h$ to be a product of monoidal transformations may be replaced by the weaker condition that $h$ is “projective”, if our immediate objective is only to prove Theorem 1.6. However, the full force of condition (i) will be used only later on. It should also be remarked that the monoidal transformations cannot, in general, be required to be defined over $k$ itself.
We also consider the following situation. Suppose \( D' \) is the sum of irreducible subvarieties of codimension 1 in \( X \) which are defined over \( k \) and further mutually transversal at every point of \( X \); in our applications, \( D' \) will be either 0 or irreducible (and non-singular). Moreover, let condition (iii) above be satisfied with \( h'(D + D') \) in place of \( h'(D) \). Then we say that \( h \) is a \( k \)-resolution of \((D, D')\). If \( h \) is a \( k \)-resolution of \((D, D')\), then \( h \) is also a \( K \)-resolution of \((D, D')\) for any extension field \( K \) of \( k \). Moreover, for any \( k \)-open subset \( U \) of \( X \), if \( Y_U = h^{-1}(U) \) and \( D_U \) (respectively \( D'_U \)) are the restrictions of \( D \) (respectively \( D' \)) to \( U \), then \( h \) gives rise to a \( k \)-resolution \( Y_U \to U \) of \((D_U, D'_U)\). These assertions are quite obvious. But what is important is that a \( k \)-resolution of \((D, D')\) always exists, if \( k \) has characteristic 0; and this is indeed a consequence of Hironaka’s fundamental theorem ([16]).

2.2

If \( h \) is a \( k \)-resolution of \((D, D')\), then every irreducible component of \( h^*(D + D') \) is non-singular. Let \( E \) be an irreducible component of \( h^*(D) \); then we have

\[
h^*(D) = \sum_E N_E \cdot E
\]

where the integer \( N_E \) is defined as follows. If \( b \) is any point of \( E \) and if \( D \) is defined locally by the equation \( g = 0 \) around \( a = h(b) \), then the multiplicity of \( E \) in \( g \circ h = 0 \) is denoted by \( N_E \); this does not depend on the choice of \( b \) in \( E \). We say that the \( k \)-resolution \( h \) is tame if the characteristic of \( k \) does not divide any \( N_E \).

We need to define another component for our datum. Let us choose local coordinates \( x_1, \ldots, x_n \) around \( a \) in \( X \) and local coordinates \( y_1, \ldots, y_n \) around \( b \) in \( Y \). Then the multiplicity of \( E \) in the local divisor defined by \( \frac{\partial}{\partial(x_1, \ldots, x_n)} = 0 \) depends only on \( E \) and is denoted by \( \nu_E - 1 \). We call the pair \((N_E, \nu_E)\) the numerical datum of \( h \) along \( E \).

We observe that \( \nu_E \geq 1 \) and further, that \( \nu_E = 1 \) if and only if \( h \) is biregular at every point of \( E \) not contained in any other component of \( h^*(D) \). If \( E_1, E_2, \ldots \) are the irreducible components of \( h^*(D) \) passing through any point \( b \) of \( Y \), then the cardinality of the set \( \{E_i\} \) is at most
equal to the dimension $n$ of $X$. If we rewrite the numerical datum of $h$ along $E_i$ as $(N_i, \nu_i)$, then we call $\{(N_i, \nu_i)\}$ the numerical data of $h$ at $b$.

Suppose that $K$ is an extension field of $k$ and $h : Y \to X$ is a $k$-resolution of $(D, D')$ as above. For any $K$-rational point $b$ of $Y$ (i.e. for $b \in Y_K$), let $x_1, \ldots, x_n$ be local coordinates around $a = h(b)$ on $X$ which are defined over $k$ and let $D$ be defined locally around $a$ by the equation $g = 0$. Further, let us write $dx = dx_1 \wedge \ldots \wedge dx_n$. We can then find local coordinates $y_1, \ldots, y_n$ around $b$ on $Y$ which are defined over $K$ and satisfy $y_1(b) = \ldots = y_n(b) = 0$, such that we have

$$g \cdot h = \epsilon \prod_i y_i^{N_i}$$

$$h^*(dx) = \eta \prod_i y_i^{\nu_i - 1} \cdot dy$$

(81)

around $b$, with functions $\epsilon, \eta$ on $Y$ that are defined over $K$ and invertible in the local ring of $Y$ at $b$. If $K$ is a local field, we shall write $X, Y, \ldots$ etc. for $X_K, Y_K, \ldots$, consistent with our previous notation.

### 2.3 An Example

Suppose that $k$ has characteristic 0 and $D$ is an absolutely irreducible curve containing the origin $0$, in a $k$-open subset $X$ of the affine plane; let $0$ be the only singular point of $D$ and further, $D$ absolutely analytically irreducible at 0. It is well-known that there exists then a unique minimal $k$-resolution of $D$. Let $E_i, i = 1, 2, \ldots, I$ be the exceptional curve “created” at the stage of the $i$th quadratic transformation. (We remark that such an ordering of the exceptional curves is used only here).

Then the graph of the function

$$\varphi(i) = \nu_i / N_i \text{ for } 1 \leq i \leq 1$$

has been examined in our paper [25]. In the first approximation, the graph looks as in the following figure.
The (strict) minimum value of \( \varphi \), say \( \lambda \), has a remarkably simple meaning. If \( f(x) = 0 \) is a local equation for \( D \) around 0 and \( f_m(x) \) the leading form in \( f(x) \), then after replacing \( x_1 \) by \( x_1 + cx_2 \) for some \( c \) in \( k \), if necessary, we may suppose that \( f_m(0, 1) \neq 0 \). Then we can solve the equation \( f(x) = 0 \) by a power-series \( x_2 = g(x_1) \) in \( x_1^{1/m} \). Let \( x_1^r \) be the first non-integral power of \( x \) appearing in \( g(x_1) \) with non-zero coefficient. Then the so-called “first characteristic exponent” \( r \) satisfies \( r > 1 \) and further,

\[
\lambda = \frac{1}{m} \left( 1 + \frac{1}{r} \right).
\]

In particular, for \( f(x) = x_1^n - x_2^m \) with \( m, n \) coprime and \( 1 < m < n \), we have \( r = n/m \) and hence, \( \lambda = \frac{1}{m} + \frac{1}{n} \).

2.4

The following lemma will prove useful later, for example, in settling the rationality of \( Z_\varphi(\omega) \).

**Lemma 2.4.** For a p-filed \( K \), let \( \omega \) be in \( \Omega(K^c) \) with \( \omega(\pi^i u) = \zeta^i \chi(u) \) as before and further, let \( \nu + N\sigma(\omega) > 0 \) (or, equivalently \( |q^{-\nu} z^N| < 1 \)) for given \( N, \nu \) in \( \mathbb{Z} \). Then, for any \( c \) in \( K \) and \( e \) in \( \mathbb{Z} \), we have
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\[ \int_{c+\mathbb{P}^e} \omega(t^N|_K)^{-1}|dt| = \begin{cases} 
(1 - q^{-1}) q^{-\nu c N} / (1 - q^{-\nu} c N), & \text{if } \chi^N = 1 \text{ and } c \in \mathbb{P}^e \\
0, & \text{if } \chi^N \neq 1 \text{ and } c \in \mathbb{P}^e \\
\chi^N (c \pi^{-\ord(c)} q^{-(\nu-1) \ord(c) - e \ord(c) N}), & \text{if } \chi^N = 1 \\
0, & \text{if } \chi^N \neq 1 \text{ on } 1 + c^{-1} \mathbb{P}^e \text{ and } c \notin \mathbb{P}^e 
\end{cases} 
\]

(82)

Proof. Let \( e_0 = \ord(c) \) and \( u_0 = c \pi^{-e} \). Then, for \( c \in \mathbb{P}^e \), the left hand side of (82) is clearly the same as

\[ \sum_{i \in c} (q^{-\nu} c N)^i \int_{K^e} \chi^N(u) du = \begin{cases} 
(1 - q^{-1}) (q^{e N})^c, & \text{for } \chi^N = 1 \\
0, & \text{if } \chi^N \text{ is non-trivial.}
\end{cases} 
\]

If, however, \( c \notin \mathbb{P}^e \), then the left hand side of (82) is obviously equal to \( \omega(c)^N|_{K^e}^{|c|^{-1}} \int_{1+c^{-1} \mathbb{P}^e} \chi^N(u) du \) on applying \( t \mapsto ct \) and this is then seen to coincide with the expression on the right hand side. \( \square \)

2.5

Let \( m \) denote a positive integer and let us denote the map \( x \mapsto x^m \) in \( K \) by \([m]\). Thus, for any subset \( S \) of \( K \), we have \( S^{[m]} = \{s^m \mid s \in S\} \). The following lemma gives a very precise description of the images under \([m]\) of certain subsets of \( R^e \) in a \( p \)-field \( K \).

**Lemma 2.5.** If \( m \) is a natural number not divisible by the characteristic of a \( p \)-field \( K \), then, for every integer \( i > \ord(m) \), we have

\[ (1 + P^i)^{[m]} = 1 + m P^i. \]

Proof. Our proof depends on the following simple version of Hensel’s lemma. For any \( g(x) \) in \( R[x_1, \ldots, x_n] \) and \( a \) in \( X = K^n \), let \( \mathcal{H} = \mathcal{H}(a) \) be defined by

\[ \mathcal{H} = \mathcal{H}(a) = \min \left\{ \ord \left( \frac{\partial g}{\partial x_1}(a) \right), \ldots, \ord \left( \frac{\partial g}{\partial x_n}(a) \right) \right\} \]
so that \( \mathcal{X}(a) = \infty \) if and only if \( a \) is in \( C_f \). If for \( a \) in \( X^0 \setminus C_f \), we have \( g(a) \equiv 0 \pmod{P^2 + 1} \), then there exists \( b \) in \( X^0 \) such that
\[
g(b) = 0 \quad \text{and} \quad b \equiv a \pmod{g(a)P^{-1}}
\] (83)

In particular, every simple point \( a \pmod{P} \) on \( g(x) \equiv 0 \pmod{P} \) can be “lifted” to a point \( b \) of the variety given by \( g(x) = 0 \), in \( X^0 \). For the sake of completeness, we give a quick proof of (83). If \( e_0 = \text{ord}(g(a)) \), then \( e_0 \geq 2 \mathcal{Y} + 1 \); let us exclude the trivial case when \( e_0 = \infty \). We can construct, as follows, a sequence \( a^{(0)} = a, \quad a^{(1)}, \quad a^{(2)}, \ldots \) in \( X^0 \) such that
\[
g(a^{(i)}) \equiv 0 \pmod{P^{e_0+i}} \quad \text{and} \quad a^{(i)} \equiv a^{(i-1)} \pmod{P^{e_0+i-1} + \mathcal{X}}
\] (84)

for \( i \geq 1 \). For \( i = 0 \), the first condition in (84) is already true while the second condition is non-existent. Assume (84) to have been proved for some \( i \geq 0 \) and set
\[
a^{(i+1)} = a^{(i)} + P^{e_0+i-\mathcal{X}} t
\]
with an unknown \( t = (t_1, \ldots, t_n) \) in \( X^0 \). Then we have
\[
g(a^{(i+1)}) \equiv 0 \pmod{P^{e_0+i+1}} \equiv -\pi^{(e_0+i)} g(a^{(i)}) \pmod{P}
\]
in view of \( e_0 \geq 2 \mathcal{Y} + 1 \), on using the Taylor expansion of \( f \). Now the definition of \( \mathcal{X} \) enables us to establish (84) for \( i + 1 \) instead of \( i \) and (83) follows from the completeness of \( K \).

We proceed to derive Lemma 2.5 from our version of Hensel’s lemma. It is clear that, for \( i \geq \text{ord}(m) \), we have \( (1 + P^i)^m \subset 1 + mP^i \). In order to prove the reverse inclusion for \( i > \text{ord}(m) \), we have only to start from any element \( 1 + mP^i a \) of \( 1 + mP^i \) with \( a \in R \) and set
\[
g(x) = (mP^i)^{-1}((1 + P^i x)^m - (1 + mP^i a))
\]

Then \( g(x) \equiv x - a \pmod{P} \) and \( a \pmod{P} \) is a simple point on \( g(x) \equiv 0 \pmod{P} \). Therefore, we have, from above, an element \( b \) of \( R \) with \( g(b) = 0 \), proving the desired inclusion and the lemma.
3 Proof of Theorem 1.6 for \( p \)-Fields

### 3.1 Reformulation of the Theorem for \( p \)-Fields

Let \( K \) be a \( p \)-field and then, by our identification of \( X \), \( Y \) etc. with \( X_K \), \( Y_K \) etc., \( X \) is the same as \( K^n \). Instead of assuming that \( C_f \subset f^{-1}(0) \), let us make the weaker assumption that, for the given \( \phi \) in \( \mathcal{S}(X) \), its support \( \text{Supp}(\phi) \) satisfies the condition

\[
C_f \cap \text{Supp}\phi \subset f^{-1}(0).
\]  

(85)

Let us recall that \( Z_\phi(\omega) = \int_X \phi(x)\omega(f(x))|dx| \) where \( \omega \in \Omega_+(K^\times) \) with \( \omega(x) = z^{\text{ord}}(x)\chi(x^{1\text{ord}}(x)) \) for \( x \in K^\times \). We shall prove that, if a tame resolution \( h : Y \to X \) for the hypersurface \( D \) defined by \( f(x) = 0 \) exists, then

\[
Z_\phi(\omega) \text{ is a rational function of } z, \text{ holomorphic in } 0 < |z| \leq 1 \quad (86)
\]

and further, that

\[
Z_\phi(\omega) \text{ vanishes identically in } z, \text{ for almost all } \chi \in (R^\times)^n \quad (87)
\]

Moreover, it will turn out that \( \Lambda \subset \{ \lambda \in \mathbb{C}; \lambda \text{ mod } 2\pi i / \log q, \lambda N_\lambda \equiv \nu(\text{mod } 2\pi i / \log q) \text{ i.e. } q^{\nu(\lambda)} = q^{\nu(\lambda)} \} \) and \( m_\lambda \leq n \) for every \( \lambda \in \Lambda \). Both the assumption 85 and the tameness of \( h \) are essential for the validity of 87. (See Remarks 1, 2 Ch. III, §1.6). Moreover, for our proof, we need to resolve only the \( K \)-rational singularities of \( D \); no further singularities come into the picture or rather, need to be resolved. As remarked already in §2.1, the latter part of condition (i) for \( h \) is invoked only to ensure that \( h \) is a proper map.

### 3.2 Reduction of the Proof

For any \( b \) in \( Y = Y_K \), we get, in view of 81, local coordinates \( y_1, \ldots, y_n \) on \( Y \) defined over \( K \) such that \( y_1(b) = \ldots = y_n(b) = 0 \) and

\[
f \circ h = \epsilon \prod_i y_i^{N_i}
\]
3. Proof of Theorem 1. 6 for $p$-Fields

$$h^*(dx) = \eta \prod_i y_i^{\nu_i-1} dy \quad (88)$$

with invertible $K$-analytic functions $\epsilon, \eta$ around $b$ and the characteristic of $K$ dividing no $N_i$. We can modify any $y_i$ in $88$ by throwing in a factor such as an invertible $K$-analytic function around $b$ and a fortiori, a factor from $K^\times$. In $88$ if, for example, the product $\prod_i y_i^{N_i}$ is empty, it is to be interpreted as $1$, as when no component passes through $b$. Let

$$m_0 = \max_b \max_i \{\text{ord}(N_i)\} \quad (89)$$

where we first take the maximum over the finitely many natural numbers $N_1, N_2, \ldots$ and then take the maximum over all $b$ in $h^{-1}(\text{Supp}(\phi))$; it is clear that $m_0$ exists.

If $f(a) \neq 0$ for $a = h(b)$, then the assumption $85$ enables us to include $f(x) - f(a)$ among the coordinate functions $y_1, \ldots, y_n$ so that, in this case, we may take

$$f \circ h(= \epsilon) = \epsilon(b)(1 + y_1), \quad f(a) = \epsilon(b)$$

$$h^*(dx) = \eta dy \quad (88)'$$

instead of $88$. In the general case, we have still,

$$f \circ h = \epsilon \prod_i y_i^{N_i}, \quad \epsilon = \epsilon(b) \left(1 + \sum_{|\alpha| \geq 1} A_\alpha y^\alpha\right), \quad \epsilon(b) \neq 0$$

$$h^*(dx) = \eta \prod_i y_i^{\nu_i-1} dy, \quad \eta = \eta(b) \left(1 + \sum_{|\alpha| \geq 1} B_\alpha y^\alpha\right), \quad \eta(b) \neq 0 \quad (88)''$$

where, for $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_1 \geq 0$ in $\mathbb{Z}$, we have written $y^\alpha$ for $y_1^{\alpha_1} \cdots y_n^{\alpha_n}$, $A_\alpha$ for $A_{\alpha_1, \ldots, \alpha_n}$, $B_\alpha$ for $B_{\alpha_1, \ldots, \alpha_n}$ and $|\alpha|$ for $\alpha_1 + \cdots + \alpha_n$. Both the power-series above converge in a neighbourhood $(P_{\nu_0})^{(\alpha)}$ of the origin for some $\nu_0 \geq 0$ and therefore $\lim A_\alpha \pi^{\nu_0|\alpha|} = \lim B_\alpha \pi^{\nu_0|\alpha|} = 0$ as $|\alpha| \to \infty$. Replacing, if necessary, each $y_i$ by $\pi^\nu y_i$ for a suitably large integer $\nu$, we may suppose, without loss of generality, that all the coefficients $A_\alpha, B_\alpha$ are already in $R$. Thus, for all $c$ close to $b$ (i.e. with all $y_i(c)$ in $P$ for example), we have, in $88)'$

$$|\eta(c)|_K = |\eta(b)|_K. \quad (90)$$
We now assert that it is possible to choose an open neighbourhood $U$ of $b$ sufficiently small so as to satisfy the two conditions:

(i) $Y \supset U \hookrightarrow (P^{m_0+1}(n) \subset K(n)$, via $c \mapsto (y_1(c), \ldots, y_n(c))$ from $U$ to $K(n)$.

(ii) $\phi \circ h$ is constant on $U$.

This is possible since $\phi$ is locally constant, $\text{Supp}(\phi)$ is compact and $h$ is a proper map. In fact, we can express $\phi$ as a finite linear combination of characteristic functions of disjoint compact open subsets $W$ of $X$ which are of the form $a' + (P^e)(n)$ and $\phi \circ h$ is constant on each compact open set $h^{-1}(W)$. We can now cover the compact open set $h^{-1}(\text{Supp}(\phi))$ by a finite number of $U$'s, say $U_1, U_2, \ldots$ and by the standard process of taking $U_1, U_2 \setminus U_1, U_3 \setminus (U_1 \cup U_2)$, etc., we may assume that $U_1, U_2, \ldots$ are already disjoint and non-empty. After imbedding each of these in $K^n$ and decomposing them into disjoint cosets modulo $P^{e_0}$ for a fixed $e_0 \geq m_0 + 1$, we call the resulting finitely many compact open sets $V_1, V_2, \ldots$ where each $V_i$ is of the form $b' + (P^{e_0})(n) \subset (P^{m_0+1})(n)$, (for $m_0$ defined by (89)). In view of (88), we now see that $Z(\phi(\omega))$ is a finite linear combination of integrals $Z(\omega)$ extended over $V = V_1, V_2, \ldots$, where

$$Z(\omega) = \int_V (\phi \circ h)\omega(\xi \circ h)|h^*(dx)|$$

$$= \phi(h(b))\omega(\xi(b))|\eta(b)|_K \int_V \chi \left(1 + \sum_{|\alpha| \geq 1} A_\alpha y^\alpha\right) \prod_i \omega(y_i)^N[y_i]^{\nu_i-1}|dy|$$

and the factor outside the integral in (91) is evidently a constant multiple of $z^{\text{ord}(\xi(b))}$. 


3. Proof of Theorem 1.6 for p-Fields

3.3

Before we proceed further, we first isolate the case when \( f(a) = (f \circ h)(b) \neq 0 \). We have then, by (88)′

\[
\int_V (\phi \circ h) \omega (f \circ h) [h^* (dx)] = c' q^{-(n-1)e_0} \int_{b'_1 + P^{e_0}} \chi(1 + y_1) dy_1 = c'' \int_{1 + P^{e_0}} \chi(t) dt \text{ since } 1 + b'_1 \in R^x
\]

\[
= \begin{cases} c''' & \text{if } e_\chi \leq e_0 \\ 0 & \text{if } e_\chi > e_0 \end{cases}
\]

where \( c', c'', c''' \) are constants. Thus, \textit{upto a constant factor}.

\[
Z(\omega) = \begin{cases} \text{ord}(f(a)) & \text{if } e_\chi \leq e_0 \\ 0 & \text{if } e_\chi > e_0 \end{cases} \tag{92}
\]

in view of (91) since \( f(a) = e(b) \).

3.4

Suppose now \( b \) is such that \( f(a) = (f \circ h)(b) = 0 \) (i.e. there exists \( i \) such that \( E_i \) passes through \( b \)). We try to pull out \( \chi \left( 1 + \sum_{|a| \geq 1} A_{\alpha} y^a \right) \) from inside the integral in (91) over \( V \cong b' + (P^{e_0})^{(n)} \). We set \( e = \max\{e_0, e_\chi\} \) and decompose \( (P^{e_0})^{(n)} \) into cosets modulo \( P^e \); namely, let

\[
(P^{e_0})^{(n)} = \bigsqcup_{b''} b'' + (P^e)^{(n)}
\]

so that

\[
V \cong \bigsqcup_c c + (P^e)^{(n)} \tag{93}
\]

with \( c = b' + b'' \). Now

\[ y \in c + (P^e)^{(n)} \subset (P^m)^{(n)} \Rightarrow y \equiv c \pmod{P^e} \]
3. Local Arithmetic Theory of Forms of Higher Degree

\[ y \equiv c \pmod{P^e \chi} \]
\[ 1 + \sum_{|\alpha| \geq 1} A_\alpha y^\alpha \equiv 1 + \sum_{|\alpha| \geq 1} A_\alpha c^\alpha \pmod{P^e \chi} \]
\[ \chi \left( 1 + \sum_{|\alpha| \geq 1} A_\alpha y^\alpha \right) = \chi \left( 1 + \sum_{|\alpha| \geq 1} A_\alpha c^\alpha \right) \]

since both the series are \( \equiv 1 \pmod{P^{m_0+1}} \) and, a fortiori, \( \equiv 1 \pmod{P} \).

The integral in (91) is thus a sum of finitely many terms (indexed by \( c = (c_1, \ldots, c_n) \)) which, up to a factor \( q^{-r} \) (with \( r \) equal to the number of missing coordinate functions \( y_j \) in (88)) are of the form

\[ \chi \left( 1 + \sum_{|\alpha| \geq 1} A_\alpha c^\alpha \right) \prod_i \int_{c_i + P^e} \omega(t)^{N_i} |t|^{v_i - 1} d|t|. \]

Therefore, by Lemma 2.4, it is the quotient of a polynomial in \( z \) and \( z^{-1} \) by \( \prod_i (1 - q^{-v_i} z^{N_i}) \) for \( |z| < 1 < \min q^{v/N_i} \). As a result, \( Z(\omega) \) is a rational function of \( z \), holomorphic in \( 0 < |z| \leq 1 \). Putting this together with (92), our assertion (86) is proved as well as the remark on the composition of \( A \).

3.5 Proof of Assertion (87)

It is sufficient to show that

\[ Z_\phi(\omega) = 0 \quad \text{for} \quad e_\chi > m_0 + e_0 \quad (94) \]

since the number of \( \chi \in (R^\times)^r \) with \( e_\chi \leq m_0 + e_0 \) is finite. We shall assume (94) to be false and arrive at a contradiction; note that \( f(a) = 0 \) in view of (92). If \( e = \max(e_0, e_\chi) \), then clearly, \( e = e_\chi \). Now since \( Z_\phi(\omega) \neq 0 \), we must have

\[ \int_{c_i + P^e} \omega(t)^{N_i} |t|^{v_i - 1} d|t| \neq 0 \]

for some \( c \) and some \( i \), at least. By Lemma 2.4 above, we have

\[ \chi^{N_i} = 1 \pmod{P^e} \quad \text{for} \quad c_i \in P^e, \]
\[ \chi^N \equiv 1 \text{ on } 1 + c_i^{-1} P e \text{ for } c_i \notin P e. \]

Since \( c_i \in P^{e_{m_0 + 1}} \), \( c_i^{-1} P e \supset P^{e_{m_0} - 1} P e \), so that \( 1 + P e^{e_{m_0} - 1} \subset 1 + c_i^{-1} P e \) and both are clearly in \( R^\times \). Thus \( \chi^N_\xi = 1 \text{ on } 1 + P e^{e_{m_0} - 1} \), implying that \( \chi = 1 \text{ on } (1 + P e^{e_{m_0} - 1})[N] \) which is the same as \( 1 + N_i P e^{e_{m_0} - 1} \) by Lemma 2.5 provided that \( e - m_0 - 1 > \text{ord}(N_i) \). But the last condition is indeed fulfilled, since \( e - m_0 - 1 = e_\xi - m_0 - 1 \geq 1 + m_0 + e_0 - m_0 - 1 = e_0 \geq m_0 + 1 > \text{ord}(N_i) \) for every \( i \) by 94 and 89. Since \( \text{ord}(N_i) + e - m_0 - 1 \leq m_0 + e - m_0 - 1 = e_\xi = 1 \), we have \( \chi = 1 \text{ on } 1 + P e^{e_{m_0} - 1} \) while \( e_\xi > m_0 + e_0 \geq 2m_0 + 1 \geq 1 \), giving the desired contradiction from the definition of \( e_\xi \). We have therefore proved 94 and indeed, as a result, that \( Z_\xi \) is in \( Z \). Theorem 1.6 is completely proved for the case of \( p \)-fields, in view of the remarks immediately following the statement of the theorem.

**REMARK.** On a close examination of the proof of the rationality of \( Z_\phi(\omega) \) above, it may be seen that neither the tameness of the resolution nor condition 85 is required to be assumed.

It may be of interest to exhibit the rationality of a related function. Suppose that \( f(x) \) is an arbitrary polynomial in \( R[x_1, \ldots, x_n] \). Consider the power-series

\[
P_0(z) = \sum_{e=0}^{\infty} N_e \cdot (q^e z)^e = 1 + N_1 q^e z + N_2 q^{2e} z^2 + \cdots
\]

where \( N_e = N_e(0) \) is the number of \( \xi \) in \( X^0 \), modulo \( P e \) such that \( f(\xi) \equiv 0 \text{mod}(P e) \). (See Chapter III, §17). Since, trivially, we have \( N_e \leq q^e \) for every \( e \geq 0 \), it follows that \( P_0(z) \) is holomorphic in the unit disc \( |z| < 1 \). It has been conjectured in [6] (Page 47, Problem 9), that \( P_0(z) \) represents a rational function of \( z \). We shall now establish the validity of this conjecture. Let us decompose \( X^0 \) as \( X^0 = \left( \bigsqcup_{e \geq 0} E_e \right) \bigsqcup E_{e_\infty} \), the disjoint union of \( E_e = X^0 \cap f^{-1}(p^e R^\times) \) for \( e \geq 0 \) and of \( E_{e_\infty} = X^0 \cap f^{-1}(0) \). Then, on \( E_e \), \( |f|_K^e \) is always equal to \( z^e \) and further

\[
m(E_e) = m(X^0 \cap f^{-1}(P^e)) - m(X^0 \cap f^{-1}(P^{e+1})) = N_e q^{-ne} - N_{e+1} q^{-ne}.
\]
Therefore we get
\[ Z_1(z) = \int_{X^0 \backslash E_\omega} |f(x)|^dK |dx| = \sum_{e=0}^\infty \{ N_e(q^{-n} z)^e - N_{e+1}(q^{-n} z)^{e+1} z^{-1} \} = P_0(z) - (P_0(z) - 1) z^{-1}, \]
implying that
\[ P_0(z) = \frac{1 - z Z_1(z)}{1 - z} \]
which is indeed a rational function of \( z \) from the considerations above.

4 Proof of Theorem 1.6 for \( \mathbb{R} \)-Fields

4.1 Reformulation of the Theorem for \( \mathbb{R} \)-Fields

Let \( X = K^n \) for an \( \mathbb{R} \)-field \( K \) and for a given \( \phi \) in \( \mathcal{S}(X) \) and the chosen \( f \), let us assume, as before, that
\[ \text{Supp}(\phi) \cap C_f \subset f^{-1}(0). \]
For \( \omega = \omega_1(ac)^p \in \Omega(K^X) \) with \( \text{Re}(s) > 0 \), we recall the definition
\[ Z_\phi(\omega) = \int_X \phi(x) \omega(f(x)) |dx|. \]
Using the same kind of resolution as in the case of \( p \)-fields, we shall prove that
\[ Z_\phi(\omega) \] has a meromorphic continuation to the entire \( s \)-plane, having poles at most at the points \( -\lambda = -(v_E/N_E + r/(2dN_E)) \) for \( r = 0, 1, 2, \ldots \) and of order at most \( n = \dim X \), notation being the same as in §2 above.

(cf. [2], [3]). Moreover, we shall prove that, for every polynomial \( P_0(s, p) \) and every vertical strip \( B_{\sigma_1, \sigma_2} \),
\[ P_0(s, p)Z_\phi(\omega) \] is uniformly bounded for \( s \) belonging to \( B_{\sigma_1, \sigma_2} \) with neighbourhoods of the poles of \( Z_\phi(\omega) \) removed therefrom, for \( p \) in \( \mathbb{Z} \).

Both [25] and [96] will together ensure that \( Z_\phi \in \mathcal{Z}(\Omega(K^X)) \) and our theorem for \( \mathbb{R} \)-fields will follow, once again, in view of the remarks immediately following its statement in §1.6.
4. Proof of Theorem 1.6 for $\mathbb{R}$-Fields

4.2 Reduction of the Proof

We shall give an outline of the proof of 95 and 96 only for the case when $\phi$ has compact support: the general case will be dealt with later.

Using the same kind of $K$-resolution $h : Y \to X$ for the hypersurface defined by $f(x) = 0$ as in the case of $p$-fields, we have, for any $b$ in $Y$, local coordinates $y_1, \ldots, y_n$ centered at $b$ such that

$$f \circ h = \varepsilon \prod_i y_i^{N_i}, \quad h^*(dx) = \eta \prod_i y_i^{\nu_i - 1} dy$$

for invertible $\varepsilon, \eta$ $K$-analytic around $b$. If $a = h(b)$ and $f(a) \neq 0$, then we have, as in (88)',

$$f \circ h = f(a)(1 + y_1), \quad h^*(dx) = \eta dy.$$ (97)

If $f(a) = 0$, we have, on replacing $y_1$ by $(\pm \varepsilon)^{1/N_1} y_1$ for example,

$$f \circ h = \delta \prod_{1 \leq i \leq n} y_i^{N_i}, \quad \delta = \begin{cases} \pm 1 & \text{for } K = \mathbb{R} \\ 1 & \text{for } K = \mathbb{C} \end{cases}$$

instead of the first relation in 97. The numerical data of the resolution at $b$ are given by $\{N_i, \nu_i\}$.

Again $h^{-1}(\text{Supp } \phi)$ is compact and we choose a finite covering by subsets which are homeomorphic (under the mapping by the $n$ coordinate functions) to the polydisc $C$ in $K^n$ defined by $|y_i| < 1$ for $1 \leq i \leq n$. We choose a smooth partition of unity subordinate to the finite covering and proceeding exactly as in § 3, we see that $Z_{\phi}(\omega)$ becomes a finite sum of expressions $Z(\omega)$ where

$$Z(\omega) = \begin{cases} \omega(f(a)) \int_{K^n} \psi(y) \omega(1 + y_1) dy_1 & (98)' \\ \int_{K^n} \psi(y) \prod_i \omega(y_i)^{N_i} |y_i|^{\nu_i - 1} dy \end{cases} (98)''$$

depending on the two cases above and further $\psi$ is a $C^\infty$-function with support contained in the polydisc $C$. In (98)' we may assume that $N_i \geq 1$ for $1 \leq i \leq n$, after reducing $n$ if necessary.
4.3
The integral in (98) may be rewritten as
\[
\int_{\mathbb{K}} \varphi(t) \omega(t) \, dt
\]
where now \( \varphi \) is a \( C^\infty \) function with support contained in the disc \(|t - 1| < 1\) and hence not containing 0. By the special case of our theory discussed in Chapter I, §§3-5, we may conclude that, in this case, \( Z(\omega) \) is actually an entire function function of \( s \) and further satisfies the growth condition 96.

4.4
We shall prove that the integral in (98) represents a meromorphic function with poles of order at most \( n \) and all the poles of the form
\[
-\frac{1}{N_1} \left( v_1 + \frac{r}{2d} \right), \quad r = 0, 1, 2, \ldots : 1 \leq i \leq n
\]
with \( d = 1/2 \) for \( \mathbb{K} = \mathbb{R} \) and 1 for \( \mathbb{K} = \mathbb{C} \) and further, that \( Z(\omega) \) is bounded for \( s \) in \( B_{\sigma_1,\sigma_2} \) from where neighbourhoods of the poles have been removed. The last assertion will imply (96), in view of the familiar argument employed in Chapter I, §§ 4-5, in such a context: we could have well worked with \( D \psi \) instead of \( \psi \), where \( D \) is a suitable homothety-invariant differential operator, say \( D = y_1^k \frac{\partial^k}{\partial y_1^k} \) or \( r_1^k \frac{\partial^k}{\partial r_1^k} \frac{\partial^\ell}{\partial \theta_1^\ell} \) for \( y_1 = \sqrt{r_1} + (\theta_1) \) according as \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \).

4.5
We first prove a simple lemma that will enable us to invoke an induction argument. By a \( C^\infty \) function \( g(z_1, \ldots, z_n) \) on \( \mathbb{C}^n \), we mean, as usual, a \( C^\infty \) function of the 2\( n \) variables \( z_1, \overline{z}_1, \ldots, z_n, \overline{z}_n \) (i.e. of \( \text{Re}(z_1), \text{Im}(z_1) \), etc.). For \( \mathbb{K} = \mathbb{R} \), we write \( m_j(z) \) for \( z^j \) and for \( \mathbb{K} = \mathbb{C} \), \( j = (j', j'') \) with \( j', j'' \geq 0 \) in \( \mathbb{Z} \), we write \( m_j(z) \) instead of \( z^{j'} \overline{z}^{j''} \) and \( |j| \) for \( j' + j'' \).

Lemma 4.5. For an \( \mathbb{R} \)-field, let \( \psi(z_1, \ldots, z_n) \in C^\infty(K^n) \) and \( t = (t_1, \ldots, t_n) \in \mathbb{Z}^n \) with all \( t_i \geq 0 \). Then there exist \( C^\infty \) functions
\[
A_{k,j}(z) = A_{k,j}(z_1, \ldots, z_k, \ldots, z_n), B_{j_1,\ldots,j_n}(z)
\]
4. Proof of Theorem 1.6 for \( \mathbb{R} \)-Fields

for \( |j_k| = t_k, \ 1 \leq k \leq n \) such that

\[
\psi(z) = \sum_{k=1}^{n} \sum_{|j| < t_k} A_{k,j}(z_1, \ldots, z_n) m_j(z_k) + \sum_{|j|=t_k} B_{j_1, \ldots, j_k}(z) \prod_{r=1}^{n} m_r(z_r) \tag{100}
\]

Proof. If we fix \( z' = (z_1, \ldots, z_{n-1}) \) in \( K^{n-1} \) as a parameter and apply Taylor’s theorem to \( \psi(z) \), then we get

\[
\psi(z) = \sum_{|j| < t_n} A_{n,j}(z_1, \ldots, z_{n-1}, z_n) m_j(z_n) + \sum_{|j|=t_n} \psi_{1,j}(z) m_j(z_n)
\]

where \( A_{n,j} \in C^\infty(K^{n-1}) \) and \( \psi_{1,j} \in C^\infty(K^n) \). Next, we regard \( z'' = (z_1, \ldots, z_{n-2}, z_n) \) in \( K^{n-1} \) as a parameter and apply Taylor’s theorem to each \( \psi_{1,j}, \) obtaining

\[
\psi_{1,j}(z) = \sum_{|j'| < t_{n-1}} A_{n,j,j'}(z'') m_{j'}(z_{n-1}) + \sum_{|j'|=t_{n-1}} \psi_{2,j,j'}(z) m_{j'}(z_{n-1})
\]

with \( C^\infty \) functions \( A_{n,j,j',}\psi_{2,j,j'} \). We now denote \( \sum_{|j|=t_n} A_{n,j,j'}(z'') m_{j'}(z_n) \) as \( A_{n-1,j'}(z'') \). Iteration of this procedure yields the lemma.

We take the case when \( K = \mathbb{R} \) and substitute, in the integral \([98]'\) the expression given by \([100]\) for \( \psi(y) \) with \( t_1 = \ldots = t_n = t \) large enough to satisfy the conditions

\[
N_i r_1 + v_i + t > 0 \quad \text{for} \quad 1 \leq i \leq n. \tag{101}
\]

Then we get

\[
Z(\omega) = \sum_{k=1}^{n} \sum_{j=0}^{t-1} A_{k,j}(y_1, \ldots, y_k, \ldots, y_n) \prod_{i=1}^{n} \omega(y_i)^{N_i} |y_i|^{v_i-1} |dy| + \int B_{t-\ldots}(y) \prod_{i=1}^{n} \omega(y_i)^{N_i} |y_i|^{v_i-1} |dy| \tag{102}
\]

the integrations being performed only over the compact support of \( \psi \) in the polydisc \( C \). Since the integrand in the second integral in \([102]\) is
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majorised by \((\sup_{y \in C} |B_t(y)|) \prod_{i=1}^{n} |y_i|^{N_i \sigma_1 + \nu - 1}\) which is in \(L^1(C)\) in view of \([101]\) the second integral in \([102]\) represents a holomorphic function of \(s\) bounded in \(B_{\sigma_1, \sigma_2}\). On the other hand, an easy computation gives

\[
\int_{-1}^{1} |y_k|^{N_k s + \nu_k + j - 1} (\text{sgn} y_k)^{N_k p + j} dy_k = \frac{1 + (-1)^{N_k p + j}}{N_k s + \nu_k + j}
\]

(103)

and using induction on \(n\), it turns out the first integral in \([102]\) is a meromorphic function of \(s\) with a pole of order at most \(n\), possibly at points of the form \([99]\) with \(r = 0, 1, \ldots, t\) and further bounded in \(B_{\sigma_1, \sigma_2}\) from which the neighbourhoods of the poles have been removed. Thus our assertions \([95]\), \([96]\) are verified for \(Z(\omega)\) instead of \(Z(\phi(\omega))\) and therefore for \(Z(\phi(\omega))\) as well.

4.6

In the case \(K = \mathbb{C}\), we use entirely similar arguments choosing \(t_1 = t_2 = \ldots = t_n = t\) in Lemma \([4.5]\) (for \(K = \mathbb{C}\)) large enough so that

\[N_i \sigma_1 + \nu_i + t/2 > 0 \quad \text{for} \quad 1 \leq i \leq n.\]

Now the presence of poles of order at most \(n\) for \(Z(\omega)\) at the points of the form \([99]\) is made possible by the contribution from \([102]\) which is a linear combination of the “principal parts”, namely

\[
\prod_{1 \leq j \leq n} (N_j s + \nu_j + (k'_j + k''_j)/2)^{-1}
\]

where \(k'_j, k''_j\) are \(\geq 0\) in \(\mathbb{Z}\), such that

\[k'_j + k''_j < t, \quad k'_j - k''_j + N_j p = 0, \quad 1 \leq j \leq n.\]

(104)

Incidentally \([104]\) implies that \(|p| \leq N_j|p| < t\) and therefore \(Z(\phi(\omega))\) is holomorphic in the right half-plane \(\sigma(\omega) \geq \sigma_1, \text{ for almost all } p\). This remarkable fact seems to be the counterpart in the case of \(\mathbb{R}\)-fields of the assertion \([87]\) in §3.1.
5. Strongly Non-Degenerate Forms

4.7

We have so far assumed that \( \phi \) in \( \mathcal{S}(X) \) has compact support. In the general case we compactify \( X \) by imbedding it in a projective space \( X^\# = X \cup H_\infty \) with \( H_\infty \), a hyperplane at infinity. If \((f^\#)_0\) is the divisor of zeros of the extension \( f^\# \) of \( f \) to \( X^\# \), we use a \( K \)-resolution of \((f^\#)_0, H_\infty \).

We no longer have the guarantee that \( N_i \) are all positive but we have the compensating factor that \( \mathcal{S}(X_K) \) can be identified with the subspace of \( C_\infty \) functions on \( X^\# \) which are “infinitely divisible by the local equation of \( H_\infty \)” (i.e. locally representable as a multiple of any given large power of \( |g|_K \) where \( g = 0 \) is a local equation for \( H_\infty \)); this makes everything work perfectly well here and we leave the verification of the details as an exercise. The proof of Theorem 1.6 for \( \mathbb{R} \)-fields \( K \) is now complete.

5 Strongly Non-Degenerate Forms

We shall now study the implications of the preceding sections in a less general but perhaps the most important and typical situation to be considered.

5.1

Let \( k \) be an arbitrary field. We call a homogeneous polynomial \( f \) of degree \( m \geq 2 \) in \( k[x_1, \ldots, x_n] \) strongly non-degenerate if \( C_f = \{0\} \) or equivalently, if the projective hypersurface defined by \( f(x) = 0 \) is non-singular. If, for some extension \( K \) of \( k \), the set of \( K \)-rational points of the critical set \( C_f \) for (homogeneous) \( f \) in \( k[x_1, \ldots, x_n] \) consists only of \( \{0\} \), then we call \( f \) strongly non-degenerate over \( K \).

Suppose that \( n > 2 \) and \( f(x) \) is strongly non-degenerate: then \( f(x) \) is absolutely irreducible, as is not hard to see. In this situation, the quadratic transformation with centre at the origin 0 gives a \( k \)-resolution for the hypersurface defined by \( f(x) = 0 \). We now present some details concerning this well-known fact.

Let \( x_1, \ldots, x_n \) be coordinates in an affine \( n \)-space \( X \) over \( k \) and let \( w_1, \ldots, w_n \) be homogeneous coordinates in a projective \((n - 1)\)-space...
Let us consider, in the product \( X \times W \), the subvariety \( Y \) defined by the equations

\[ x_iw_j - x_jw_i = 0 \quad \text{for} \quad 1 \leq i \leq j \leq n. \]

Then the quadratic transformation \( h : Y \to X \) with centre 0 is just the projection \( X \times W \to X \) restricted to \( Y \). If \( W_1 \) denotes the affine \((n-1)\)-space in \( W \), defined by \( w_1 \neq 0 \), then \( y_i = w_i/w_1 \) for \( 1 \leq i \leq n \) form coordinates in \( W_1 \). Let \( Y_1 \) denote the intersection of \( Y \) with the affine \((2n-1)\)-space \( X \times W_1 \) with coordinates \( x_1, \ldots, x_n, y_2, \ldots, y_n \); then \( Y_1 \) is defined by

\[ x_i - x_1y_i = 0 \quad \text{for} \quad 1 < i \leq n. \]

Therefore, if we put \( y_1 = x_1 \), then \( Y_1 \) becomes an affine \( n \)-space with coordinates \( y_1, y_2, \ldots, y_n \) and further \( Y \) is covered by \( Y_1 \) together with \( Y_2, \ldots, Y_n \) which are defined in a similar manner. In particular, \( Y \) is non-singular and if we put

\[ E = h^{-1}(0) = \{0\} \times W \subset Y, \]

then \( h \) gives a \( k \)-isomorphism i.e. a biregular map of \( Y \setminus E \) to \( X \setminus \{0\} \), defined over \( k \). We observe that \( h^*(dx) \) is given in \( Y_1 \) by

\[ h^*(dx) = dy_1 \wedge d(y_1y_2) \wedge \ldots \wedge d(y_1y_n) = y_1^{n-1}dy. \]

Moreover, \( f \circ h \) is given in \( Y_1 \) by

\[ f \circ h = y_1^m f(1, y_2, \ldots, y_n) \]

and \( f(1, y_2, \ldots, y_n) = 0 \) gives a local equation in \( Y_1 \), of an irreducible non-singular hypersurface in \( Y \), say \( E' \) (this being just the “proper transform” of the hypersurface given by \( f(x) = 0 \)). We see that \( E' \) and \( E \) are transversal to each other at every point of \( Y \). Therefore \( h \) gives a \( k \)-resolution of the hypersurface defined by \( f(x) = 0 \) and is tame, if the characteristic of \( k \) does not divide \( m \). The numerical data of \( h \) along \( E' \) and \( E \) are \((1, 1)\) and \((m, n)\) respectively.
5. Strongly Non-Degenerate Forms

5.2

Let us apply our theory to the case when \( k \) is a local field \( K \), identifying \( X \) with \( X_K \) etc., as usual. Then, for a \( p \)-field \( K \) and every \( \phi \) in \( \varphi(X) \), we have

\[
Z_\phi(\omega) = \frac{P_{\phi,\chi}(z)}{(1 - z/q)(1 - z^n/q^m)}
\]

with \( P_{\phi,\chi}(z) \) in \( \mathbb{C}[z, z^{-1}] \). Therefore, if the characteristic of \( k \) does not divide \( m \) and if \( n > m \), then we get

\[
F_\phi^*(i^*) = \sum_{\lambda} d_{\lambda}^*(ac(i^*))|i^*|_{K}^{-1}
\]

for all sufficiently large \( |i^*|_K \); here, \( \lambda \) satisfies \( q^{m\lambda} = q^n \) (We steer clear of the case \( n = m \), just to avoid multiple poles at \( z = q \). See Ch. II, Theorem 3.1). Now (105) implies, for every \( i^* \) in \( K \), that

\[
|F_\phi^*(i^*)| \leq c \max(1, |i^*|_K)^{-n/m}
\]

(106) with a constant \( c > 0 \). (This follows just from the numerical data and involves no computation). The same estimate can be seen to be valid for the case of \( \mathbb{R} \)-fields as well, when \( \phi \) has compact support; actually, even for arbitrary \( \phi \) in \( \mathcal{S}(X) \) in this case, we can draw the same conclusion, by using the same quadratic transformation as in §5.1 but extended to a projective space in which the affine space \( X \) is imbedded. All this enables us to assert that \( F_\phi^* \) is an \( L^1 \)-function on \( K \) and hence \( F_\phi(0) \) exists for \( n > m \) and the characteristic of \( K \) not dividing \( m \). We shall see later that \( F_\phi(0) \) is given by

\[
F_\phi(0) = \int_{U(0)} \phi|\theta_0|.
\]

We have thus clarified “Condition (A)” of Weil ([52]) referred to earlier (see Ch. II), in the case of strongly non-degenerate forms over local fields. However, if we wish to examine the same relative to a global field, we need, at the present moment, to appeal to the theorem of Deligne ([9]). We shall now explain this point in some detail.
Let us start from a strongly non-degenerate form $f(x_1, \ldots, x_n)$ of degree $m(\geq 2)$ with coefficients in a global field $k$, assuming, as above, that $n > m$ and that $m$ is not divisible by the characteristic of $k$. Let $K$ denote a completion of $k$ with respect to a non-archimedean valuation $v$ on $k$. If we exclude a finite number of $v$’s depending on $f(x)$, then, for every $i^*$ in $K$, we have

$$|F^*(i^*)| \leq \max(1, |i^*|_K)^{-n/m}$$

In (108), we have written $F^*$ for $F_{\psi_0}$ corresponding to the characteristic function $\psi_0$ of $X^0(= R(n))$; further the constant $c$ in (106) is replaced by 1 in (108). We give a sketch of the proof of (108) as follows.

Let $f(x)$ be in $R[x_1, \ldots, x_n]$, without loss of generality, and let $mR = R$; moreover, let the reduction $\bar{f}(x)$ of $f(x)$ modulo $P$ be strongly non-degenerate. Writing $i^*$ in $K^\times$ as $\pi^{-e}u$ with $e \geq 0$ in $\mathbb{Z}$ and $u$ in $R^\times$, an elementary argument gives us

$$F^*(i^*) = \begin{cases} q^{n[-e/m]}, & \text{for } e \not\equiv 1 \bmod m; \\ q^{-m(1+e-1)/m} \sum_{t \bmod P} \psi(\pi^{-1}ut)N_1(t), & \text{for } e \equiv 1 \bmod m. \end{cases}$$

For the first case in (109), we have immediately

$$|F^*(i^*)| \leq (q^e)^{-n/m} = |i^*|_K^{-n/m}.$$  

But, in the second case, such an inequality holds if and only if

$$q^{-n} \sum_{t \bmod P} \psi(\pi^{-1}ut)N_1(t) \leq q^{-n/m}$$

On the left side of (110), the expression inside the sign for absolute value is merely the (generalized) Gaussian sum

$$q^{-n} \sum_{\xi \bmod P} \psi(\pi^{-1}uf(\xi))$$

and both sides of 110 are equal for $m = 2$. In the general case, the problem is to estimate the number of $\mathbb{F}_q$-rational points on $\bar{f}(x) = 0$ or on
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Let \( f(x) = 7^{n_1} \) with \( 7 \in \mathbb{P}^r_q \), both the hypersurfaces being non-singular. We can indeed apply Deligne’s theorem ([9]) to carry out such an estimation; for the details, we may refer the reader to our paper [21].
For any $p$-field $K$, we had introduced in §1.7, the functions $F = F_\phi$, $F^* = F_\phi^*$ corresponding to the characteristic function $\phi$ of $R^m$, $f(x) \in R[x_1, \ldots, x_n]$ and the standard character $\psi$ on $K$. If

$$i^* = \pi^{-e}u \quad \text{with} \quad e \geq 0 \quad \text{in} \quad \mathbb{Z} \quad \text{and} \quad u \in R^\times,$$

we had the relation

$$F^*(i^*) = q^{-ne} \sum_{\xi \mod p^e} \psi(i^* f(\xi))$$

where the right hand side is essentially a generalized Gaussian sum and on the basis of Theorem 1.6, we had referred to its “stable behaviour” for all sufficiently large $e = -\text{ord}(i^*)$, namely, its being a fixed linear combination of expressions of the form $\chi(ac(i^*))/K^{1/2} \log |i^*|^j$.

In the simplest non-trivial case where $f(x) = x^2$ and $2 \in R^\times$, we have

$$F^*(i^*) = \begin{cases} |i^*|^{-1/2}_K & \text{for even} \quad e = -\text{ord}(i^*) \geq 0 \\ q^{1/2} \chi(ac(i^*))/K^{1/2} & \text{for odd} \quad e = -\text{ord}(i^*) > 0 \end{cases} \quad (111)$$

where $\chi$ is the non-trivial character of order 2 on $R^\times$. Moreover, $(q^{1/2} \chi)^2 = \chi(-1) = \pm 1$ (cf. Relation [57]).

We shall give a quick sketch of the proof of 111. First $e_\chi = 1$, in view of the relation $(1 + P)^{1/2} = 1 + P$ from Lemma 2.5. Now it is easy to see that

$$F^*(i^*) = \sum_{r \geq 0} q^{-r} \int_{R^\times} \psi(\pi^{-(e-2r)}ac(i^*)u^2) |du|,$$

but, for any continuous function $\varphi$ on $R^\times$, we have

$$\int_R \varphi(x^2) dx = \int_{R^\times} \varphi(x)(1 + \chi(x)) dx.$$
Taking \( \varphi(u) = \psi(\pi^{-(e-2)}ac(i^*)u) \), we now get
\[
F^*(i^*) = \begin{cases} 
(1 - q^{-1}) \sum_{r \geq e/2} q^{-r} - q^{-(e+1)/2 \nu(\pi^r)}q^{-(e-1)/2}g(x) & \text{for } e \text{ odd} \\
(1 - q^{-1}) \sum_{r \geq e/2} q^{-r} & \text{for } e \text{ even}
\end{cases}
\]
on using [52] and this proves [111].

We now give the proof of assertion [79] which is well-known; namely, for homogeneous \( f(x) \) of degree \( m \) in \( R[x_1, \ldots, x_n] \) and \( i \neq 0 \) in \( R \),
\[
q^{-e(n-1)}N_e(i) \text{ is independent of } e, \text{ for } e \geq 1 + 2 \text{ord}(mi).
\]

For any \( a \) in \( X \), let us define \( \mathcal{X} = \mathcal{X}(a) \) as \( \min_{1 \leq i \leq n} \left\{ \text{ord} \left( \frac{\partial f}{\partial x_i}(a) \right) \right\} \). Then \( \mathcal{X} \) is a \( \mathbb{Z} \)-valued locally constant function on \( \mathcal{X}(C_f) \). For \( a \) in \( X^0 = R^n \) and \( b \) in \( X \) with \( b \equiv a \pmod{P^{\mathcal{X}(a)+1}} \), we have \( \frac{\partial f}{\partial x_i}(b) \equiv \frac{\partial f}{\partial x_i}(a) \pmod{P^{\mathcal{X}(a)+1}} \) for every \( i \) and therefore \( \mathcal{X}(a) = \mathcal{X}(b) \). Further, it is immediate, on using the Taylor expansion of \( f \), that
\[
a \equiv b \pmod{P^{\mathcal{X}(a)}}, \quad e \geq 2\mathcal{X}(a) \Rightarrow f(a) \equiv f(b) \pmod{P^e}.
\]

Let us observe that, in view of the foregoing,
\[
E(e, \mathcal{X}) = \{ a \pmod{P^e}; f(a) \equiv 0(\pmod{P^e}), \mathcal{X}(a) = \mathcal{X} \},
\]
\[
E'(e, \mathcal{X}) = \{ a \pmod{P^{e-\mathcal{X}}}; f(a) \equiv 0(\pmod{P^e}), \mathcal{X}(a) = \mathcal{X} \}
\]
are well-defined for \( e \geq \mathcal{X} + 1 \) and \( e \geq 2\mathcal{X} + 1 \) respectively. Therefore, if \( e \geq 2\mathcal{X} + 1 \), both are well-defined and the canonical map
\[
X^0/\pi^eX^0 \to X^0/\pi^{e-\mathcal{X}}X^0
\]
gives rise to a surjective map from \( E(e, \mathcal{X}) \) to \( E'(e, \mathcal{X}) \) such that each fibre has the cardinality \( [\pi^{e-\mathcal{X}}X^0; \pi^eX^0] = q^{e-\mathcal{X}} \) independent of \( e \). Similarly, the natural map
\[
X^0/\pi^{e-\mathcal{X}+1}X^0 \to X^0/\pi^{e-\mathcal{X}}X^0
\]
induces a surjective map $E'(e + 1, \mathcal{X}) \to E'(e, \mathcal{X})$; actually $f(a + \pi^{e-\mathcal{X}} x) \equiv 0 \pmod{P^{e+1}}$ if and only if $\sum_{1 \leq i \leq n} \pi^{e-\mathcal{X}} \frac{\partial f}{\partial x_i} (a) x_i \equiv -\pi^{-e} f(a) \pmod{P}$ and the linear congruence has exactly $q^{n-1}$ solutions $x$ modulo $P$, since one of the elements $\pi^{e-\mathcal{X}} \frac{\partial f}{\partial x_i} (a)$ is in $R^\times$. Thus, in this case, each fibre has $q^{n-1}$ elements. As a consequence, we have, for $e \geq 2 \mathcal{X} + 1$,

$$\text{card } E(e + 1, \mathcal{X}) = q^{n\mathcal{X}} \text{ card } E'(e + 1, \mathcal{X}) = q^{n\mathcal{X}} \text{ card } E'(e, \mathcal{X}) \cdot q^{n-1} = \text{ card } E(e, \mathcal{X}) q^{n-1}$$

where “card” stands for the cardinality. We may therefore conclude that for $e \geq 2 \mathcal{X} + 1$,

$$q^{-e(n-1)} \text{ card } E(e, \mathcal{X}) \text{ is independent of } e. \tag{112}$$

We should perhaps remark that the canonical map $E(e + 1, \mathcal{X}) \to E(e, \mathcal{X})$ is not always surjective.

We now start from a homogeneous $f(x)$ of degree $m$ in $R[x_1, \ldots, x_n]$ and for a fixed $i$ in $R \setminus \{0\}$, use $f(x) - i$ in place of the polynomial $f(x)$ referred to in the preceding paragraph. Then we see that, for $e \geq \text{ord}(mi) + 1$, the set $[\xi \pmod{P^e}; f(\xi) \equiv i \pmod{P^e}]$ is the disjoint union of $E(e, \mathcal{X})$ for $0 \leq \mathcal{X} \leq \text{ord}(mi)$; in fact, for $e \geq \text{ord}(i) + 1$ and $\xi$ in $X^0$ with $f(\xi) \equiv i \pmod{P^e}$, we have $\mathcal{X}(\xi) \leq \text{ord}(mi)$, in view of the relation

$$\sum_{i=1}^{n} \xi_i \frac{\partial f}{\partial x_i} (\xi) = mf(\xi) \equiv mi \pmod{mP^e}.$$ 

We may now conclude from above that

$$q^{-e(n-1)} N_e(i) = \sum_{0 \leq \mathcal{X} \leq \text{ord}(mi)} q^{-e(n-1)} \text{ card } (E(e, \mathcal{X}))$$

is independent of $e \geq 2 \text{ord}(mi) + 1$, by (112).

Next we proceed to prove assertion (80), namely, that the above stable quotient $q^{-e(n-1)} N_e(i)$ for $e \geq 2 \text{ord}(mi) + 1$ which is, therefore, the same
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as \( \lim_{\varepsilon \to \infty} q^{-\varepsilon(n-1)} N_\varepsilon(i) \), coincides with \( \int_{X^0 \cap U(i)} |\theta_i| \). We first observe that if \( \{a_{\nu,r}\} \) is a complete set of representatives \( a_{\nu,r} \) of \( E'(e,\nu) \) with \( f(a_{\nu,r}) = i \), then \( U(i)^0 = X^0 \cap U(i) \) is the disjoint union of \( V_{\nu,r} = \{ a \in U(i)^0; a \equiv a_{\nu,r} (\text{mod} \, P^{e^{-\nu}}) \} \) for varying \( r \) and for \( \nu \) with \( 0 \leq \nu \leq \nu_0 = \text{ord}(m) \).

From the implicit function theorem for analytic functions on complete fields, we have the following: namely, for a \( p \)-field \( K \) and any convergent power-series \( g(x) \) in \( R[[x_1, \ldots, x_n]] \) with \( g(0) = 0 \) and \( \frac{\partial g}{\partial x_n}(0) \neq 0 \) (mod \( P \)), there exists a unique convergent power-series \( \varphi(x') \) in \( x' = (x_1, \ldots, x_{n-1}) \) with coefficients in \( R \) such that \( \varphi(0) = 0 \) and further.

(i) \( g(x', \varphi(x')) = 0 \) for all \( x' \) sufficiently close to 0; and

(ii) whenever \( g(a) = 0 \) for \( a = (a_1, \ldots, a_n) \) close to 0, then \( a_n = \varphi(a') \).

We shall, however, use a refinement of the same, which is not hard to prove: namely, if, in addition, we have \( g(x) = \sum a_x x^\alpha \) (with the usual notation) and \( c_\alpha \equiv (\text{mod} \, P^{e^{-\nu}-1}) \), then \( \varphi(x') \) converges on the whole of \( R^{(n-1)} \) and further satisfies

(i') \( g(x', \varphi(x')) = 0 \) for all \( x' \in R^{(n-1)} \); and

(ii') for any \( a = (a_1, \ldots, a_n) \in R^{(n)} \) with \( g(a) = 0 \), we have \( a_n = \varphi(a') \).

Going back to the proof of \( 80 \), we shall examine closely the set

\[ V = \{ x \in X^0; f(x) = i, x \equiv \xi \pmod{P^{e^{-\nu}}} \} \]

where \( \xi \) is one of the \( a_{\nu,r} \) above. Putting \( x = \xi + \pi^{e^{-\nu}} y \), we see that \( V \) is defined in \( X^0 \) by the condition

\[ 0 = \pi^{-\xi}(f(x) - i) = \sum_{i=1}^{n} \pi^{-\nu} \frac{\partial f}{\partial x_i}(a)y_i + \pi^{-\nu} Q \quad (113) \]

where \( Q \) is a polynomial in \( \pi^{e^{-\nu}} y_1, \ldots, \pi^{e^{-\nu}} y_n \) with coefficients in \( R \) and having no terms of degree \( \leq 1 \). Now \( (113) \) gives an equation of the form

\[ \sum_{i=1}^{n} c_i y_i + \sum_{|\alpha| \geq 2} c_\alpha y_\alpha = 0 \]
where $c_i, c_\alpha$ are all in $R$ and further, $c_\alpha \in P|\alpha|^{-1}$ (since $|\alpha|(e-\nu-1)-e+1 \geq e-2\nu -1 \geq 0$), while at least one of $c_1, \ldots, c_n$, say $c_n$, is not in $P$. By the above-mentioned refinement of the implicit function theorem (applied to $K$ and $\pi^e(f(x) - i)$ instead of $g(x)$), we see that $V$ is the same as \{(y', \varphi(y')); y' \in R^{(n-1)}\}. Now, on $V$,

$$\theta_{i} = (-1)^{n-1} \left( \frac{\partial f}{\partial x_n} \right)^{-1} \left\{ dx_1 \wedge \ldots \wedge dx_{n-1} \right\}_V$$

and therefore

$$|\theta_{i}|_{V} = q^{-e(n-1)+n\nu}|dy'|$$

implying that

$$\int_{V} |\theta_{i}| = q^{-e(n-1)+n\nu}$$

We have, finally from above,

$$\int_{U(\varrho)} |\theta_{i}| = \sum_{0 \leq \nu \leq \varrho_{0}} \sum_{E_{0}} \int_{V_{0}} |\theta_{i}|$$

$$= \sum_{0 \leq \nu \leq \varrho_{0}} q^{-e(n-1)} \text{card}(E'(e, \nu)) q^{n\nu}$$

$$= \sum_{0 \leq \nu \leq \varrho_{0}} q^{-e(n-1)} \text{card}(E(e, \nu))$$

$$= q^{-e(n-1)} N_{e}(i), \text{ for } e \geq 2 \text{ ord}(mi) + 1.$$
Chapter 4
Poisson Formula of Siegel-Weil Type

1 Formulation of a Poisson Formula

In this Chapter, we formulate a Poisson formula of general type which corresponds to forms of higher degree in the same manner as the Siegel-Weil formula ([52]) is related to quadratic forms; the classical Poisson summation formula is again a special case of such a Poisson formula and it is well-known ([47]) that in this classical formula lies embedded a substantial part of the validity of the functional equation of Hecke L-series. Our Poisson formula, as in the case of Weil, is to be regarded as an identity between two tempered distributions on adelized vector spaces arising from global fields. We shall formulate sufficient conditions for the validity of the general Poisson formula. Moreover, we shall use the so-called “adelic language” ([50]), essentially to simplify the presentation.

1.1 Standard Notation

Let $k$ be a global field i.e. an algebraic number field of finite degree over the rational number field or an algebraic function field of one variable with finite constant field. For any non-trivial absolute value $|\cdot|_v$ on $k$, denote by $k_v$ the completion of $k$ with respect to the metric $d(a, b) = |a - b|_v$. 
4. Poisson Formula of Siegel-Weil Type

Let us assume that \( | |_v \) is normalized, as already described. We call \( v \) archimedean or non-archimedean according as \( k_v \) is an \( \mathbb{R} \)-field or a \( p \)-field; archimedean valuations arise only in the case of algebraic number fields. We choose exactly one representative (and indeed normalised as well) in each “equivalence class” of valuations \( v \). By \( S \), we shall mean, in this article, a finite set of valuations \( v \) which always includes the archimedean valuations. For non-archimedean \( v \), we denote the valuation ring by \( R_v \) or \( R \) and its maximal ideal by \( P_v \) or simply \( P \).

1.2 Adelization

A subset \( U \) of an affine space \( X \) of dimension \( n \) is called locally \( k \)-closed (or quasi-\( k \)-affine) if \( U = V \setminus W \) for \( k \)-closed subsets \( V, W \) in the affine space. If \( V \) is defined by the equations \( f_1(x) = \ldots = f_r(x) = 0 \) and \( W \) by \( g_1(x) = \ldots = g_s(x) = 0 \) for \( f_1, \ldots, f_r, g_1, \ldots, g_s \) in \( k[x_1, \ldots, x_n] \), then \( a \) belongs to \( U \) if and only if \( f_1(a) = \ldots = f_r(a) = 0 \) and further \( g_j(a) \) is not equal to zero for at least one \( j \) with \( 1 \leq j \leq s \). If we define \( U_v = U_{k_v} = \{ x \in k_v^n; f_1(a) = \ldots = f_r(a) = 0 \) and \( g_j(a) \) is not in \( R_v \) for at least one \( j \) \} \) is compact. For any \( S \) described in §1.1 above, \( \prod_{v \in S} U_v \) is locally compact and \( \prod_{v \not\in S} U_v \) is compact, so that

\[
U_S = \prod_{v \in S} U_v \times \prod_{v \not\in S} U_v
\]

is locally compact. For two such \( S, S' \) with \( S \subset S' \), clearly \( U_S \) is open in \( U_{S'} \).

For any locally \( k \)-closed \( U \), we define the adelization \( U_A \) of \( U \) by \( U_A = \lim_{S \to} U_S \) and \( U_S \) is open in \( U_A \) for every \( S \). Then \( U_A \) is a locally compact space and it is not hard to see that \( U_A \) depends only on \( U \) (although the requirement that for \( a \in U_v^0 \), \( g_j(a) \) is a unit for at least one \( j \), may look disturbing!). We can show that adelization is a “functor” and, in particular, for any \( k \)-morphism \( j : U \to U' \) of locally \( k \)-closed subsets \( U, U' \) in affine spaces, we have a unique continuous map \( j_A : \)
1. Formulation of a Poisson Formula

$U_A \to U'_A$. The set $U_k$ of $k$-rational points on $U$ can be imbedded in $U_A$ through the diagonal imbedding and it is, in fact, discrete in $U_A$.

1.3 Examples

We shall illustrate the foregoing with some examples.

(i) Let $X$ be the affine $n$-space, so that $X_k \cong k^n$. We denote $X_A$, for $n = 1$, by $k_A$, the ring of $k$-adeles; for any $n$, $X_A = (k_A)^n$ and $X_A/X_k$ is compact. If $n = 1$, $k = \mathbb{Q} \hookrightarrow \mathbb{Q}_A$ and any $t$ in $\mathbb{Q}_A$ is of the form $(t_\infty, t_p, \ldots)$ with $t_\infty \in \mathbb{R}$ and $t_p$ is in the ring $\mathbb{Z}_p$ of $p$-adic integers for almost all (non-archimedean) $p$. If $< t_p >$ denotes the “fractional part” of $t_p$, then $< t_p >= 0$ for almost all $p$ so that $t' = t - \sum_p < t_p >$ in $\mathbb{Q}_A$ has its non-archimedean components $t'_p$ in $\mathbb{Z}_p$ for every $p$. It is now immediate that $\mathbb{Q}_A/\mathbb{Q} \cong \mathbb{R}/ \mathbb{Z} \times \prod_p \mathbb{Z}_p$ and hence, it is compact. The imbedding of $\mathbb{Q}$ as a discrete subgroup of $\mathbb{Q}_A$ corresponds precisely to the imbedding $\mathbb{Z} \hookrightarrow \mathbb{R}$.

(ii) For $f(x) \in k[x_1, \ldots, x_n]$ and $i$ in $k$, we have introduced $U(i)$ as $f^{-1}(i) \setminus C_f$; thus, $a$ belongs to $U(i)$ if and only if $f(a) = i$ and further, $\frac{\partial f}{\partial x_j}(a) \neq 0$ for at least one $j$. Clearly $U(i)$ is a locally $k$-closed subset and hence $U(i)_A$ is defined.

1.4 Tamagawa Measure

Suppose $U$ is a non-singular locally $k$-closed subset of affine $n$-space and $\omega$, an everywhere regular differential form of the highest degree on $U$ vanishing nowhere and moreover, defined over $k$. We shall be dealing only with situations where such a form $\omega$ exists; for example if $U = X$, $\omega(x) = dx_1 \wedge \ldots \wedge dx_n$ in Example (i) above and $\omega(x) = \theta_i(x)$ for $U = U(i)$ in Example (ii).

By a procedure (which may not, however, work always), we can associate to $\omega$, the so-called Tamagawa measure $|\omega|_A$ on $U_A$ as follows. Let us start from the Borel measure $|\omega|_v$ on $U_v$ associated with $\omega$; in particular, on the open set $U_v^0$, we have a measure, say $m_v$. Let...
us assume that the (infinite product) measure \( \otimes_{v \in S} |\omega|_v \) exists on \( \prod_{v \in S} U^0_v \),
this, incidentally, happens to exist if and only if the infinite product
\( \prod_{v \in S} m(U^0_v) \) is absolutely convergent. On the other hand, we always have
the product measure \( \otimes_{v \in S} |\omega|_v \) on the (finite) product \( \prod_{v \in S} U_v \).
Thus, we can define, for every \( S \), the measure
\( |\omega|_A \) on \( U_S = \prod_{v \in S} U^0_v \times \prod_{v \in S} U_v \).
However, we have, still, to remove
the ambiguity that arises in our definition of \( |\omega|_v \) above, due to our not
having fixed a Haar measure on \( k^n \), for \( n = 1, 2, \ldots \). For this pur-
pose, let us proceed as follows. Let us choose a non-trivial character
\( \psi: k_A/k \to C^*_1 \). Then \( k_A \) may be identified with its dual by setting
\( <a, b> = \psi(ab) \) for \( a, b \in k_A \) and further, \( k = (k_A/k)^* \) via the map
\( c \mapsto \psi(ct) \). Let \( j_v: k_v \to k_A \) be the imbedding which sends any \( t \) in
\( k_v \) to the adele with \( t \) as the \( v \)-th component and 0 elsewhere; we set
\( \psi_v = \psi \circ j_v \). On \( X_v = k^n_v \), we take as our measure, the \( n \)-fold product
\( |dx|_v \) of the measure on \( k_v \), which is self-dual relative to \( (t, t') \mapsto \psi_v(tt') \); then, this is the same as the measure on \( X_v = k^n_v \) self-dual relative to
\( ((x_1, \ldots, x_n), (y_1, \ldots, y_n)) \mapsto \psi_v(x_1y_1 + \cdots + x_ny_n) \). Moreover, \( \psi_v = 1 \) on
\( R_v \) and nontrivial on \( P_{v}^{-1} \), for all but finitely many \( v \); further, \( m(X^0_v) = 1 \)
for such \( v \). The measure \( |dx|_A \) always exist on \( X_A \) and has the charac-
teristic property that \( X_A/X_k \) has measure 1; on each open subgroup \( X_S \)
defined above, \( |dx|_A \) is just the product of the measures \( |dx|_v \), in the usual
sense.

1.5 The Schwartz-Bruhat Space \( \mathcal{S}(X_A) \)

If \( k \) is an algebraic function field of one variable with finite constant
field, then \( X_A \) is a locally compact abelian group with arbitrarily large
and small compact open subgroups in the sense of Chapter II, § 1.2. The Schwartz-Bruhat space \( \mathcal{S}(X_A) \) is thus already familiar to us and so
is the dual, \( \mathcal{S}(X_A)' \) of tempered distributions.

Let, on the other hand, \( k \) be an algebraic number field and the degree
\( [k: \mathbb{Q}] \) of \( k \) over \( \mathbb{Q} \) be finite. If \( S_{\infty} \) denotes the set of all archimedean
valuations on \( k \), we write, for the affine space \( X \) of dimension \( n \),
\[
X_{\infty} = \prod_{v \in S_{\infty}} X_v = \mathbb{R}^{n[k\mathbb{Q}]}
\]
\[
X_0 (= X_{\text{finite}}) = \lim_{S \to S_{\infty}} \prod_{v \in S \setminus S_{\infty}} X_v^0.
\]

Let us remark that \( X_0 \) is a locally compact abelian group with arbitrarily large and small compact open subgroups and therefore the Schwartz-Bruhat space \( \mathcal{S}(X_0) \) is already known to us. Moreover, we are also quite familiar with the Schwartz-Bruhat space \( \mathcal{S}(X_{\infty}) \). We now define the Schwartz-Bruhat space associated with \( X_A \) by
\[
\mathcal{S}(X_A) = \mathcal{S}(X_0) \otimes_{\mathbb{C}} \mathcal{S}(X_{\infty});
\]
every element of this space is a finite linear combination of \( \phi_0 \otimes \phi_\infty \) with \( \phi_0 \in \mathcal{S}(X_0) \) and \( \phi_\infty \in \mathcal{S}(X_{\infty}) \). We define the “topological dual” \( \mathcal{S}(X_A)' \) of \( \mathcal{S}(X_A) \) as the space of all \( \mathbb{C} \)-linear functionals on \( \mathcal{S}(X_A) \) such that \( T(\phi_0 \otimes \phi_\infty) \) depends continuously on \( \phi_\infty \) in \( \mathcal{S}(X_{\infty}) \) for every fixed \( \phi_0 \) in \( \mathcal{S}(X_0) \). Let \( \{ T_n \} \) be a sequence contained in \( \mathcal{S}(X_A)' \) such that, for every \( \phi \in \mathcal{S}(X_A) \), \( \lim_{n \to \infty} T_n(\phi) \) exists; denoting this limit by \( T(\phi) \), we see trivially that \( T \) is \( \mathbb{C} \)-linear. Moreover, for every fixed \( \phi_0 \) in \( \mathcal{S}(X_0) \) and any \( \phi_\infty \in \mathcal{S}(X_{\infty}) \), we have \( T(\phi_0 \otimes \phi_\infty) = \lim_{n \to \infty} T_n(\phi_0 \otimes \phi_\infty) \), by the definition of \( T \). Now since \( \phi_\infty \mapsto T_n(\phi_0 \otimes \phi_\infty) \) belongs to \( \mathcal{S}(X_{\infty})' \) for every fixed \( \phi_0 \) in \( \mathcal{S}(X_0) \), the well-known completeness of \( \mathcal{S}(X_{\infty})' \) entails that \( \phi_\infty \mapsto T(\phi_0 \otimes \phi_\infty) \) belongs to \( \mathcal{S}(X_{\infty})' \). Thus \( \mathcal{S}(X_A)' \) is complete. It is called the space of tempered distributions on \( X_A \), as usual.

1.6 Poisson Formula

We are now in a position to formulate the general Poisson formula associated with \( f(x) \in k[x_1, \ldots, x_n] \) and with \( \mathcal{S}(X_A) \). We first make the simple remark that for every \( i^* \in k \), \( \psi(i^* f(x)) \in \mathcal{S}(X_A)' \), if we simply define \( \psi(i^* f(x))(\phi) = \int_{X_A} \phi(x) \psi(i^* f(x))|dx|_A \), for every \( \phi \) in \( \mathcal{S}(X_A) \) and note that the integral converges absolutely.

Let us formulate a series of assumptions.

(PF-1) The infinite sum \( \sum_{i^* \in k} \psi(i^* f(x)) \) belongs to \( \mathcal{S}(X_A)' \), i.e. (equi
4. Poisson Formula of Siegel-Weil Type

valently) the Eisenstein-Siegel series \( \sum_{i \in k} \int_{X_A} \phi(x) \psi(i^* f(x)) |d\lambda_A | \) associated with \( f(x) \) converges absolutely for every \( \phi \) in \( \mathcal{S}(X_A) \).

(PF-2) For every \( i \) in \( k \), the measure \( |\theta_i|_A \) exists on \( U(i)_A \).

(PF-2)' If \( j : U(i)_A \rightarrow X_A \) is induced by \( U(i) \rightarrow X \), then for every \( i \) in \( k \), the global singular series \( j_* (|\theta_i|_A) \) or simply \( |\theta_i|_A \) associated with \( f(x) \) and \( i \) exists in \( \mathcal{S}(X_A) \) i.e. (equivalently) \( \int_{U(i)_A} \phi |\theta_i|_A \) is absolutely convergent for every \( \phi \) in \( \mathcal{S}(X_A) \).

(PF-3) The infinite sum \( \sum_{i \in k} |\theta_i|_A \) belongs to \( \mathcal{S}(X_A)' \).

We say that the Poisson formula holds for \( f(x) \) (relative to \( k \)) if all the assumptions (PF-1), (PF-2)', (PF-2)'', (PF-3) are valid and further

(PF-4) \[
\sum_{i \in k} |\theta_i|_A = \sum_{i^* \in k} \psi(i^* f(x)) \quad (114)
\]

The simplest example of a Poisson formula is furnished by considering the case when \( n = 1 \), \( f(x) = x \) so that \( \phi^*(x) = \int_{k_A} \phi(y) \psi(xy) |d\lambda|_A \).

Then \[114\] reads \[
\sum_{i \in k} \delta(x - i) = \sum_{i^* \in k} \psi(i^* x)
\]

since \( |\theta_i|_A \) is just the Dirac measure supported at \( i \). But the relation above is the same as saying

\[
\sum_{i \in k} \phi(i) = \sum_{i^* \in k} \phi^*(i^*) \quad (115)
\]

for every \( \phi \in \mathcal{S}(k_A) \). As remarked already, there remains, built into the simple-looking classical Poisson formula \[115\], quite a substantial part of the proof of the functional equation of Hecke’s \( L \)-series rephrased in adelic language.
2. Criteria for the Validity of the Poisson Formula and Applications

2.1

Sufficient conditions for the validity of the Poisson formula stated in the last section are provided by the following theorem which is the second substantial theorem in our theory for forms of higher degree.

Theorem 2.1. Let \( f(x) \) denote a homogeneous polynomial of degree \( m \geq 2 \) in \( n \) variables \( x_1, \ldots, x_n \) with coefficients in a global field \( k \) of characteristic not dividing \( m \) and let us assume that a tame \( k \)-resolution of the projective hypersurface defined by \( f(x) = 0 \) exists. Then the Poisson formula holds for \( f(x) \) relative to \( k \) if the following two conditions are satisfied:

(C1) the codimension of \( C_f \) in \( f^{-1}(0) \) \( \geq 2 \) (i.e. equivalently, \( \text{codim} (C_f) \geq 3 \));

(C2) there exist \( \sigma > 2 \) and a finite set \( S \) of valuations \( v \) of \( k \), such that, for every \( i^* \) in \( k_v \backslash R_v \) and \( v \notin S \),

\[
|F_v^*(i^*)| \leq |i^*|_v^{-\sigma}.
\]

where \( F_v^*(i^*) = \int_{X_0} \psi_v(i^* f(x))|dx|_v \).

The proof of this theorem will be given later. We merely remark that there exists a conjecture to replace condition (C2) by a geometric condition; this will be explained later. Condition (C1) is easy to verify and simply means that the hypersurface defined by \( f(x) = 0 \) is irreducible and normal.

2.2 Applications

We enumerate a series of applications of Theorem (2.1), assuming that \( k \) has characteristic 0, just for the sake of simplicity.
A.1) Let $f(x)$ be strongly non-degenerate i.e. let $f(x)$ be homogeneous with the critical set $C_f = \{0\}$. Then $(C_1)$ is equivalent to the condition $n \geq 3$, since $\text{codim}_{f^{-1}(0)}(C_f) = n-1$. Further, from \[108\], we know that, for all but finitely many $v$ and for every $i^* \in k_v \setminus R_v$,

$$|F_v(i^*)| \leq |i^*|^{-n/m} \quad (108)'$$

as a consequence of Deligne’s theorem ([9]); for $\text{ord}(i^*)$ divisible by $m$, actually equality holds in \[108\]'. Thus (C2) is equivalent to $n > 2m$ and the Poisson formula holds for strongly non-degenerate $f(x)$ if $n > 2m$. In any case, the existence of a tame resolution is obvious here and even, in the case of function fields $k$, the Poisson formula holds, provided that $m$ is not divisible by the characteristic of $k$.

A.1)' If $f(x)$ is a non-degenerate quadratic form, we are just in the situation A.1) above with $m = 2$. Thus, the Poisson formula holds for $n > 4$. Incidentally, condition C2) is easy to verify, since

$$|F_v(i^*)|^2 = \int_{X^m \times X^m} \psi_v(i^*(f(x) - f(y))|dx \wedge dy|_v$$

$$= \left( \int_{R_v^m} \psi_v(i^* x_1 y_1) |dx_1 \wedge dy_1 \right)^n$$

$$= |i^*|^{-n}$$

for every $i^*$ in $k_v \setminus R_v$ and for all but finitely many $v$; we may replace $f(x)$ by $\sum_i u_i x_i^2$ with $u_i \in R_v^{\times}$, in the computation above.

A.2) If $f(x)$ is just a homogeneous polynomial of degree $m$ in $n$ variables with coefficient in $k$, then by applying the techniques of Birch and Davenport for forms in many variables ([4]), it can be shown that, for any $\epsilon > 0$,

$$|F_v(i^*)| \leq |i^*|^{-(\text{codim}(C_f)/2m-1(m-1)|k:Q|)+\epsilon}$$
for every \(i^*\) in \(k_v\setminus R_v\) and all but finitely many \(v\). The condition (C1) is already fulfilled if we require that

\[
\text{codim}(C_f) > 2^m(m-1)[k : \mathbb{Q}](\geq 2)
\]

and consequently the Poisson formula holds, subject to this requirement being satisfied. For further details, we refer to our paper [26].

2.3 Further Applications of the Poisson Formula

A.3) Let \(G\) be a "\(k\)-form" of \(SL_{2m}\) (i.e. an algebraic group defined over \(k\) and isomorphic to \(SL_{2m}\) over an algebraic closure \(\overline{k}\) of \(k\)) and let \(\rho : G \to GL(X)\) be a "\(k\)-form" of the \((2m-1)\) - dimensional fundamental representation of \(G\), where \(X\) is a certain affine space. If \(G\) is \(k\)-isomorphic to \(SL_{2m}\), then we may identify \(X\) with the vector space of \(2m\)-rowed skew-symmetric matrices \(x\) on which the action of \(G\) via \(\rho\) is described by \(x \to \rho(g) \cdot x = gx^t g\) for \(g\) in \(G\).

Upto a factor from \(K\times\), there exists a unique invariant of degree \(m\), with coefficients in \(k\); this invariant is precisely the Pfaffian \(Pf(x)\) with its well-known property that \(Pf(gx^t g) = \det(g) Pf(x)\) for any \(g\) in \(GL_{2m}\). In this case, it is known that \(F^*(i^*) = \sum_{i=1}^{m-1} c_i |i^*|_v^{-(2i+1)},\)

where \(c_i = \sum_{1 \leq j \leq m, j \neq i} (1 - q^{-2(j+1)})/(1 - q^{-2(j-i)}).\) In the general case, there exists a unique invariant \(f(x)\) which, for all but finitely many \(v\), coincides with \(Pf(x)\) after a non-singular linear transformation with coefficients in \(R_v\). As a result, the above formula for \(F^*(i^*)\) is valid even in the general case and consequently, we have

\[
|F^*_v(i^*)| \leq |i^*|_v^{-3+\epsilon}
\]

for every \(i^*\) in \(k_v\setminus R_v\) and for all but finitely many \(v\). Further

\[
\text{codim}_{f^{-1}(0)}(C_f) = 5
\]

and thus the Poisson formula is valid in this case. Complete details may be found in our paper [19].
4. Poisson Formula of Siegel-Weil Type

A.4) Let \( G \) be a \( k \)-form of the connected, simply connected, simple group of type \( E_6 \) and let \( \rho : G \to GL(X) \) be a \( k \)-form of the 27-dimensional fundamental representation of \( G \). Then there exists a cubic invariant \( f(x) \) known as “det” (in the theory of Jordan algebras) and unique up to a factor from \( k^* \), as before (28). In this case, it can be shown that

\[
\text{codim}_{f^{-1}(0)}(C_f) = 9 \quad \text{and} \quad |F^*_i(i^*)| \leq |i^*|^{-5+\varepsilon}
\]

and hence the Poisson formula holds. The inequality above is a consequence of the more precise assertion:

\[
F^*_i(i^*) = \frac{1 - q^{-9}}{1 - q^{-4}} |i^*|^{-5} - q^{-4} \frac{1 - q^{-5}}{1 - q^{-4}} |i^*|^{-9} \quad (116)
\]

Formula (116) may be found in [36], § 14.

A.5) Consider the complete polarization \( \det(a, b, c) \) of the cubic form “det” in example A.4) above; the form \( \det(a, b, c) \) is symmetric trilinear in \( a, b, c \) and \( \det(a, a, a) = 3! \det(a) \). Taking a non-degenerate symmetric bilinear form \( Q(a, b) \) with coefficients in \( k \), on the same space, define \( a \times b, a^\# \) by

\[
Q(a \times b, c) = \det(a, b, c), \quad a^\# = (1/2)(a \times a).
\]

Then, for a suitable choice of \( Q(a, b) \), we have

\[
(a^\#)^\# = \det(a) \cdot a.
\]

Let \( X \) be the \((2(27 + 1) - 1) = 56\) dimensional space consisting of \( x = (a, b, \alpha, \beta) \) where \( a, b \) are elements of the 27-dimensional space mentioned in A.4) and \( \alpha, \beta \) are in \( \mathbb{C} \). After Freudenthal, we take the quartic

\[
f(x) = Q(a^2, b^2) + \alpha \det(b) + \beta \det(a) - (1/4)(Q(a, b) - \alpha \beta)^2.
\]

The group of automorphisms of \( f(x) \) has two connected components and the connected component \( G \) of the identity is a \( k \)-form
of the connected, simply connected simple group of type $E_7$ and corresponds to its “first fundamental representation”. Conversely, for a $k$-form of type $E_7$ with its 56-dimensional fundamental representation defined over $k$, there exists only one invariant, namely $f(x)$, up to a factor from $K^\times$.

For the quartic $f(x)$ above, we have

\[
\operatorname{codim}_{f^{-1}(0)}(C_f) = 10 \quad \text{and} \quad |F_v(i^*)| \leq |i^*|^{-5+1/2+\epsilon}
\]

and therefore, the Poisson formula holds in this case. The estimate for $F_v(i^*)$ follows from the more precise relation:

\[
F_v(i^*) = \frac{(1 - q^{-14})(1 - q^{-18})}{(1 - q^{-4})(1 - q^{-17})}\gamma(i^*)|i^*|^{-5-1/2}
\]

\[
- q^{-4}\frac{(1 - q^{-10})(1 - q^{-14})}{(1 - q^{-4})(1 - q^{-8})}\gamma(i^*)|i^*|^{-9-1/2}
\]

\[
+ (q^{-14} + c(i^*))|i^*|^{-14},
\]

where

\[
\gamma(i^*) = \begin{cases} 
1, & \text{for } \operatorname{ord}(i^*) \text{ even} \\
\chi(-\pi^{-\operatorname{ord}(i^*)}i^*)q^{1/2}g_\chi, & \text{for } \operatorname{ord}(i^*) \text{ odd}
\end{cases}
\]

with $\chi \neq 1, \chi^2 = 1$ and

\[
c(i^*) = \begin{cases} 
q^{-13}\frac{(1 - q^{-1})(1 + q^{-13})(1 - q^{-14})}{(1 - q^{-9})(1 - q^{-17})}, & \text{for } \operatorname{ord}(i^*) \text{ even} \\
q^{-17-1/2}\frac{(1 - q^{-1})(1 + q^{-9})(1 - q^{-14})}{(1 - q^{-9})(1 - q^{-17})}\gamma(i^*), & \text{for } \operatorname{ord}(i^*) \text{ odd}.
\end{cases}
\]

For further details, one may refer to [27].

3 The Siegel Formula

In this section, we shall explain the significance of the Poisson formula in the simplest case and then derive the Siegel formula due to Weil, as a consequence of the Poisson formula for quadratic forms.
4. Poisson Formula of Siegel-Weil Type

REMARK. Except for the case A.1) and hence also A.1)', we have assumed that \( k \) has characteristic 0. But actually, G.R. Kempf has explicitly constructed, in the fall of 1974, \( k \)-resolutions for the hypersurface defined by \( f(x) = 0 \), in the cases A.3), A.4) and A.5), for an arbitrary field \( k \). Since conditions C1), C2) are valid in the case of function fields as well, we see, by incorporating the results of Kempf, that the Poisson formula holds, if the characteristic of \( k \) does not divide \( m! \) in the case A.3) and if \( k \) has characteristic different from 2, 3 in the cases A.4) and A.5). However, we might mention that, without any restriction on the characteristic of \( k \), the Poisson formula is valid in the case A.3).

3.1 Statement of the Siegel Formula

Let \( k \) be a global field of characteristic (different from 2), \( X \) the affine space and let \( f(x) \) be a non-degenerate quadratic form in \( n \geq 3 \) variables \( x_1, \ldots, x_n \) with coefficients in \( k \). Let \( G \) be the special orthogonal group \( \text{SO}(f) \) of \( f \), consisting of all linear transformations on \( x_1, \ldots, x_n \) which leave \( f(x) \) invariant and have determinant 1. Let \( dg \) be an invariant differential form on \( G \) of (maximal) degree \( \frac{n(n-1)}{2} \), defined over \( k \).

Then, for all but finitely many valuations \( v \) of \( k \), we have

\[
m(G_v^0) = \int \frac{|dg|_v}{G^0_v} = \begin{cases} 
\prod_{1 \leq i \leq (n-1)/2} (1 - q^{-2i}), & \text{for } n \text{ odd} \\
\prod_{1 \leq i < n/2} (1 - q^{-2i}) \cdot (1 - \chi(d)q^{-n/2}), & \text{for } n \text{ even}
\end{cases}
\]

where \( \chi^2 = 1 \), \( \chi \neq 1 \) and the discriminant \( d \) of \( f(x) \) is defined to be \((-1)^{(n-1)/2} \det(t_{ij})\), if \( f(x, y) = f(x + y) - f(x) - f(y) \) is written as

\[
\sum_{i,j=1}^{n} t_{ij} x_i x_j \quad \text{with } t_{ij} = t_{ji}.
\]

In particular, (for \( n \geq 3 \)) we see, from above, that \( \prod_{v \in S} m(G_v^0) \) is absolutely convergent, where \( S \) is the finite set of exceptional \( v \) mentioned above; thus the (adelic) measure \( |dg|_A = \bigotimes_v |dg|_v \) exists. The Siegel formula (for the special orthogonal group \( G \)) due to Weil ([52]) states that

\[
(1/2) \int_{G_A/G_k} \left( \sum_{\xi \in X_k} \phi(g\xi) \right) |dg|_A = \phi(0) + \sum_{r \in k \setminus X_k} \int \phi(x)\psi(i^r f(x))dx|_A \quad (117)
\]
3. The Siegel Formula

for every $\phi$ in $\mathcal{O}(X_A)$ and $n > 4$ (with $\psi \in (k_A/k)^\times$ chosen already).

3.2

In the course of deriving the Siegel formula, we shall freely use the following theorems.

(i) Minkowski-Hasse theorem: $U(i)_A \neq \phi \Rightarrow U(i)_k \neq \phi$ for every $i \in k$ (for every $n \geq 1$).

(ii) Witt’s theorem: $\xi_i \in U(i)_k(\neq \phi) \Rightarrow G_k \cdot \xi_i = U(i)_k$ for every $i \in k$ and further $G_A \cdot \xi_i = U(i)_A$ (if $n \geq 3$).

(iii) For $n = 3, 4$, $\tau_k(G) \overset{\text{def}}{=} \int_{G_A/G_k} |dg|_A = 2$.

(iv) For the twisted Fourier transformation $^\ast$ on $X_A$ defined by $\tilde{\phi}(x) = \int_{X_A} \phi(y)\psi(f(x,y))|dy|_A$, we have

$$\psi(i^x f(x))^\ast = \psi((-i^x)^{-1}f(x))$$

for $n \geq 1$ and every $i^x \in k^\times$, where $f(x,y) = f(x+y) - f(x) - f(y)$.

Among the theorems mentioned above, (i) and (ii) are well-known. For the proof of (iii), we refer to Theorem 3.7.1 in the lectures [50] of Weil. For the proof of (iv), we refer to Théorème 2 and Théorème 5 in the paper [51] of Weil and Theorem 3 in our paper on “Harmonic analysis and theta functions”, Acta Math. 120(1968), 187-222.

3.3 Tamagawa Number

We recall that, on any algebraic group $G$ defined over an arbitrary field $k$, there exists a left $G$-invariant differential form $dg \neq 0$, which is of degree equal to the dimension of $G$ and further, defined over $k$; we shall simply call such a form a gauge form on $G$. A gauge form is automatically everywhere regular and non-vanishing on $G$. More generally, suppose that $U$ is a non-singular algebraic variety defined over $k$ with a $k$-rational point $\xi$, such that $G$ acts transitively on $U$ through $(g, x) \mapsto g \cdot x$ and
4. Poisson Formula of Siegel-Weil Type

Further, the action of $G$ is defined over $k$; then the stabilizer $H$ at $\xi$ is an algebraic subgroup of $G$ defined over $k$. In such a case, we simply write $G/H = U$ and call $U$ an algebraic homogeneous space defined over $k$. A $G$-invariant differential form $\theta \neq 0$ on $U$ which is of degree equal to the dimension of $U$ and defined over $k$, is called a gauge form on $U$; such a form does not exist, in general. But if it exists, then any other gauge form on $U$ differs from $\theta$ only by a scalar factor from $k^\times$. We observe that if $G$ is the special orthogonal group $SO(f)$ of a quadratic form $f(x)$ over $k$, then $\theta$ is a gauge form on $U_i$ for every $i$ in $k$.

Let now $k$ be a global field and let $k_A, k^\times_A$ be its adele and idele groups respectively, as before. For any idele $t = (\ldots, t_v, \ldots) \in k^\times_A$, define its modulus $|t|_A$ as the rate of measure change in $k_A$ under the multiplication by $t$; then $|t|_A = \prod_v |t_v|_v$. For any $c \in k^\times$, multiplication by $c$ gives an automorphism of the compact group $k_A/k$; hence, for $c \in k^\times$, we have $|c|_A = 1$. This fact (well-known as the product formula for global fields) has a far-reaching implication: namely, if a gauge form $\theta$ exists on $U = G/H$ and if, further, the Tamagawa measure $|\theta|_A$ also exists on $U_A$, then, for any other gauge form $\omega$ on $U$, $|\omega|_A$ (exists and) is the same as $|\theta|_A$, since $\omega = c\theta$ for $c \in k^\times$ and $|\omega|_A = |c|_A|\theta|_A = |\theta|_A$. Thus $|\theta|_A$ depends only on $U$ and $k$. Furthermore, if $dg, dh$ are the gauge forms on $G, H$ and if $G_A \cdot \xi = U_A$, then we will get $G_A/H_A = U_A$ and $|dg|_A, |dh|_A, |\theta|_A$ automatically “match together” in the sense that

$$\int_{G_A} \varphi(g)dg|_A = \int_{U_A} \left( \int_{H_A} \varphi(gh)dh|_A \right) |\theta(g \cdot \xi)|_A$$

(118)

for any continuous function $\varphi$ on $G_A$ with compact support. These are the excellent properties of the Tamagawa measures.

On the other hand, for any $\gamma$ in $G_k$, we have $d(\gamma y) = c \cdot dg$ with $c \in k^\times$ and hence $|dg|_A$ is right $G_k$-invariant. Therefore, an intrinsic measure $\tau_k(G)$ can be introduced as

$$\tau_k(G) = m(G_A/G_k) = \int_{G_A/G_k} |dg|_A$$
and $\tau_k(G)$ is called the Tamagawa number of $G$ relative to $k$ (usually under the assumption that $\tau_k(G) < \infty$). Theorem (iii) in § 3.2 (that we have assumed) merely states that, for $n = 3$ or $4$, we have $\tau_k(\text{SO}(f)) = 2$. Furthermore, if $\mathbb{G}_a$ denotes the affine line considered as an additive algebraic group (defined over the prime field), then we have $\tau_k(\mathbb{G}_a) = m(k_A/k) = 1$. In order to proceed further, we shall prove the following elementary lemma.

**Lemma 3.2.** Suppose that a locally compact group $G$ splits as a semi-direct product of $H$ by $G'$: $G = G' \cdot H$. Let $dg$, $dg'$, $dh$ denote left-invariant measures on $G$, $G'$, $H$ respectively such that $dg = dg' \otimes dh$. Let $\Gamma, \Gamma', \Delta$ denote discrete subgroups of $G$, $G'$, $H$ respectively such that $\Gamma = \Gamma' \cdot \Delta$. Assume that $dg$, $dg'$, $dh$ are also right-invariant under $\Gamma$, $\Gamma'$, $\Delta$ respectively and further that, for every $\gamma$ in $\Gamma'$, $d(\gamma h \gamma^{-1}) = dh$. Then we have

$$\int_{G/\Gamma} \varphi(g) dg = \int_{G'/\Gamma'} \left( \int_{H/\Delta} \varphi(g'h) dh \right) dg'$$

for every continuous function $\varphi$ on $G/\Gamma$ with compact support.

*Proof.* We can find a continuous function $\varphi_0$ on $G$ with compact support such that

$$\varphi(g) = \sum_{\gamma \in \Gamma} \varphi_0(\gamma g)$$

for every $g$ in $G$. Similarly, if we define $\varphi_1(g)$ as $\sum_{\delta \in \Delta} \varphi_0(g \delta)$ then we get

$$\int_{G/\Gamma} \varphi(g) dg = \int_{G} \varphi_0(g) dg = \int_{G'} \left( \int_{H} \varphi_0(gh) dh \right) dg'$$

$$= \int_{G'/\Gamma'} \left( \sum_{\gamma \in \Gamma'} \int_{H/\Delta} \varphi_1(g' \gamma' h) dh \right) dg'.$$

If we apply to the inner integral the measure-preserving automorphism $h \to \gamma^{-1} h \gamma'$ of $H/\Delta$, then (119) is immediate. $\square$
4. Poisson Formula of Siegel-Weil Type

We now go back to our previous notation and denote by $G$, $G'$, $H$ algebraic groups defined over a global field such that $G$ is the semi-direct product of $H$ by $G'$. Let $dg$, $dg'$, $dh$ denote gauge forms on $G$, $G'$, $H$. Then we can apply Lemma 3.2 to $G_A$, $G'_A$, $H_A$, $G_k$, $G'_k$, $H_k$, $|dg|_A$, $|dg'|_A$, $|dh|_A$ instead of $G$, $G'$, $H$, $\Gamma$, $\Gamma'$, $\Delta$, $\Delta'$, $dg$, $dg'$, $dh$ respectively. In fact, all the conditions in the lemma are satisfied. Since we can replace $\varphi$ in (119) by any non-negative measurable function on $G/\Gamma$, we get, on putting $\varphi = 1$, the formula

$$\tau_k(G) = \tau_k(G') \tau_k(H)$$  \hspace{1cm} (120)

Since $\tau_k(\mathbb{G}_a) = 1$, formula (120) implies that $\tau_k(G) = \tau_k(G')$, whenever $H$ is $k$-isomorphic to a direct product of $\mathbb{G}_a$.

3.4 Proof of the Siegel Formula

We proceed to prove the Siegel formula given by (117).

Applying induction on $n$, let us assume that the Tamagawa number of the special orthogonal group of any non-degenerate quadratic form over $k$ in $n'$ variables with $3 \leq n' < n$ is 2; in spite of the fact that $n > 4$ (as stated after formula (117)), our induction hypothesis is quite legitimate by virtue of Theorem (iii) assumed in §3.2.

With the same notation as in Theorem (ii) assumed in §3.2, we have clearly

$$X_k \{0\} = \bigcup_{i \in \mathbb{k}} U(i)_k = \bigcup_{i \in \mathbb{k}} G_k \cdot \xi_i$$ \hspace{1cm} (121)

where the accent (over the symbol for disjoint union) indicates that $i$ runs only over the subset of $k$ which ensures $U(i)_k \neq \phi$. For any such $i$, we set

$$H_i = \text{the stabiliser of } G \text{ at } \xi_i.$$

We then assert that $\tau_k(H_i) = 2$ for every such $i$. The proof runs as follows. If $i \neq 0$, let $X' = \{x \in X; f(\xi_i, x) = 0\}$ and $f'$ the restriction of $f$ to $X'$. Then we have $SO(f') = H_i$ and since $X'$ has dimension $n - 1 \geq 4$, we see that $\tau_k(H_i) = 2$, by the induction hypothesis. Let now $i = 0$. Then choosing $\eta$ from $X_k$ with $f(\xi_0, \eta) = 1$ and putting $\eta_0 = -f(\eta)\xi_0 + \eta$, we obtain $\eta_0 \in X_k$, $f(\eta_0) = 0$ and $f(\xi_0, \eta_0) = 1$. Let us
now take $X'$ to be the subspace of $X$ defined by $f(\xi_0, x) = f(\eta_0, x) = 0$ and $f'$ to be the restriction of $f$ to $X'$. Then, as in the previous case, $f'$ is non-degenerate and $H_0$ can be verified to be the semidirect product by $\text{SO}(f')$ of the group $H$ of matrices of the form

$$
\begin{pmatrix}
1 & -f'(c) & -cT' \\
0 & 1 & 0 \\
0 & c & 1_{n-2}
\end{pmatrix}
$$

where $c$ is an arbitrary $(n-2)$-rowed column, $f'(x') = 1/2'x'T'x'$ for $x' \in X'$ and $1_{n-2}$ is the $(n-2)$-rowed identity matrix. Since $H$ is isomorphic to the $(n-2)$-fold product of $\mathbb{G}_a$ and since $n-2 \geq 3$, it follows by applying (120), that $\tau_k(H_0) = \tau_k(\text{SO}(f')) = 2$, in view of the induction hypothesis once again.

For $\xi_i \in U(i)_k$, the map $g \mapsto \xi_i$ of $G$ to $U(i)$ induces the continuous injection $G_A/(H_i)_A \to U(i)_A$ and Witt’s theorem assumed in §3.2 guarantees the surjectivity of the latter map. Thus we have a homeomorphism

$$G_A/(H_i)_A \cong U(i)_A$$

(in view of a well-known theorem in the theory of topological groups).

We are now ready to start the proof of the Siegel formula. The left hand side of (117) is the same as

$$
\tau_k(G)\phi(0) + \int_{G_A/G_k} \left( \sum_{\xi \in X_k \setminus \{0\}} \phi(g\xi) \right) |dg|_A,
$$
on pulling out the term corresponding to $\xi = 0$. By (121), we have

$$
\sum_{\xi \in X_k \setminus \{0\}} \phi(g\xi) = \sum_{i \in k} \sum_{\xi \in U(i)_k} \phi(g\xi) = \sum_{i \in k} \sum_{\xi \in U(i)} \phi(g\xi) = \sum_{i \in k} \sum_{\gamma \in G_k \mod (H_i)_k} \phi(g\gamma \xi_i).
$$

Therefore, integrating over $G_A/G_k$, we get

$$
\int_{G_A/G_k} \left( \sum_{\xi \in X_k \setminus \{0\}} \phi(g\xi) \right) |dg|_A = \sum_{i \in k} \int_{G_A/(H_i)_k} \phi(g\xi_i)|dg|_A$$
we have used here the fact that integrating over $G_A/(H_i)_A$ with respect to $|dg|_A$ is the same as integrating over $(H_i)_A/(H_i)_k$ relative to $|dh|_A$ coming from a gauge form $dh$ on $H_i$ and then over $G_A/(H_i)_A = U(i)_A$ relative to $|\theta|_A$. We recall now that $\tau_k(H_i) = 2$ for every $i$ in $k$ such that $U(i)_k \neq \phi$.

In view of the Minkowski-Hasse theorem, $U(i)_k = \phi$ implies $U(i)_A = \phi$ and hence the integral over $U(i)_A$ is 0; hence we may legitimately remove the accent over the symbol of summation in (122). In this way, we get, for $n > 4$ and $\phi$ in $\mathcal{S}(X_A)$, that

$$
\int_{G_A/G_k} \left( \sum_{\xi \in X_k} \phi(g\xi) \right) |dg|_A = \tau_k(G)\phi(0) + 2 \sum_{i \in k} \int_{U(i)_A} \phi|_A
$$

$$
= \tau_k(G)\phi(0) + 2 \sum_{i \in k} \int_{X_A} \phi(x)\psi(i^* f(x))|dx|_A. 
$$

The second step in (123) is a consequence of the Poisson formula [114]. On the other hand, for $\Psi(x) = \phi(gx)$, we have

$$
\hat{\Psi}(x) = \int_{X_A} \phi(gy)\psi(f(x, y))|dy|_A
$$

$$
= \int_{X_A} \phi(gy)\psi(f(gx, gy))|dy|_A \quad (y \rightarrow g^{-1}y)
$$

$$
= \int_{X_A} \phi(y)\psi(f(gx, y))|dy|_A
$$

$$
= \hat{\phi}(gx).
$$

The usual Poisson formula implies that

$$
\sum_{\xi \in X_k} \phi(g\xi) = \sum_{\xi \in X_k} \Psi(\xi) = \sum_{\xi \in X_k} \hat{\Psi}(\xi) = \sum_{\xi \in X_k} \hat{\phi}(g\xi). \quad (124)
$$
Therefore, the left hand side of (123) is invariant if $\phi$ is replaced by $\hat{\phi}$.

By Theorem (iv) assumed in § 3.2, we have

$$\int_{X_A} \hat{\phi}(x)\psi(-i^{r^{-1}}f(x))dx|_{A} = \int_{X_A} \phi(x)\psi(-i^{r^{-1}}f(x))dx|_{A}$$

(125)

for every $i^r$ in $k^\times$. The invariance under $\phi \to \hat{\phi}$ of the right hand side of (123) implies, in view of (125), the invariance under $\phi \to \hat{\phi}$ of

$$\tau_k(G)\phi(0) + 2 \int_{X_A} \phi(x)|dx|_{A} = \tau_k(G)\phi(0) + 2\hat{\phi}(0).$$

Hence $\tau_k(G)$ is necessarily equal to 2 and the Siegel formula is proved.

REMARK. In the course of proving the Siegel formula, we have tacitly used the fact that $\tau_k(G) < \infty$. However, following H. Ariturk (11), the finiteness of $m(G_A/G_k)$ can be established inductively as follows. In view of Theorem (iii) assumed in § 3.2 the induction hypothesis entails that the Tamagawa number of the special orthogonal group of a non-degenerate quadratic form over $k$ in $n'$ variables with $3 \leq n' < n$ is not only finite but, in fact, equal to 2. Now we can certainly find $\varphi \neq 0$ in $\mathcal{S}(X_A)$ such that $\varphi \geq 0$ and $\varphi(0) = 0$; it is clear then that $\varphi(0) > 0$. Moreover, as is well-known, we can also choose $\psi$ in $\mathcal{S}(X_A)$ such that $\varphi \geq 0$ and, further, $|\varphi(x)| \leq \varphi(x)$ for every $x$ in $X_A$. By going through the previous argument for such $\varphi$, we have

$$\int_{G_A/G_k} \left( \sum_{\xi \in X_k} \varphi(g\xi) \right)|dg|_{A} = \int_{G_A/G_k} \left( \sum_{\xi \in X_k \setminus \{0\}} \varphi(g\xi) \right)|dg|_{A}$$

$$= \sum_{i \in k} \tau_k(H_i) \int_{U(i)_{A}} \varphi(\theta)|_{A}$$

$$= 2 \sum_{i \in k} \int_{U(i)_{A}} \varphi(\theta)|_{A} \quad \text{(by induction)}$$

$$< \infty,$$
since the Poisson formula holds for $f(x)$ and hence (PF-3) is valid.

On the other hand,

$$\int_{G_A/G_k} \left| \sum_{\xi \in X_k \setminus \{0\}} \hat{\phi}(g \xi) \right| dg \leq \int_{G_A/G_k} \left( \sum_{\xi \in X_k \setminus \{0\}} \varphi(g \xi) \right) dg \leq 2 \sum_{\xi \in \Omega_A} \varphi|\theta|_{A}$$

in a similar manner. In other words, $\sum_{\xi \in X_k} \phi(g \xi)$ and $\sum_{\xi \in X_k \setminus \{0\}} \hat{\phi}(g \xi)$ for the chosen $\phi$ are both in $L^1(G_A/G_k)$ and hence the same is true of their difference as well. Thus the constant function $\hat{\phi}(0) > 0$ is in $L^1(G_A/G_k)$ which implies that $m(G_A/G_k) < \infty$.

### 4 Other Siegel Formulas

#### 4.1 General Comments

In order that we may be able to carry over the method of proof for the Siegel formula for orthogonal groups to other cases, we need to have, besides the Poisson formula (114), suitable generalizations of Theorems (i), (ii), (iii) and (iv) assumed in § 3.2. In the situations covered by A.3), A.4), and A.5) in § 2.3, we have, indeed, $U(i) \neq \phi$ for every $i$ in $k$; therefore, there is no need to look for an analogue of Theorem (i) assumed in § 3.2. Theorem (ii) assumed in § 3.2 can be replaced by the following general theorem (stated without proof).

**Theorem 4.1.** Let $U = G/H$ denote an algebraic homogeneous space defined over a global field $k$ and assume that both $G$ and $H$ are connected and simply connected. Then $U_k$ decomposes into finitely many disjoint $G_k$-orbits $G_k \xi_\alpha$ and $U_A$ becomes the disjoint union of $G_A$-orbits $G_A \cdot \xi_\alpha$ for the same $\xi_\alpha$, with each $G_A \cdot \xi_\alpha$ open in $U_A$.

We wish to emphasise that in the cases covered by A.3), A.4) and A.5) in § 2.3 every $G$-orbit in $X$ defined over $k$ satisfies the conditions...
in the above theorem. Even in the case of quadratic forms corresponding to A.1’ of §[2] the stabilisers turn out to be connected and simply connected, if we use Spin($f$) and $G$ instead of SO($f$). A systematic exposition (with references) of this subject can be found in [20]. We recall, in this context, a conjecture of Weil which states that $\tau_k(G) = 1$ for any connected simply connected algebraic group $G$ defined over a global field $k$. Since the stabilisers (other than $G$) have smaller dimension, we can hope to verify this conjecture inductively at least in some cases.

We have already explained the nature of the analogue in the general case of Theorem (iii) of § 3.2. As for Theorem (iv), we remark that a generalization of the formula $\psi(i^* f(x)) = \psi(-i^* f(x))$ is not known and may even not exist. (In this connection, we refer to our paper [24]).

Therefore, following Mars ([36]), we proceed as follows. We choose a symmetric (or skew-symmetric) non-degenerate bilinear form $Q(x, y)$ on $X \times X$ with coefficients in $k$, such that $G$ becomes invariant under $g \mapsto g^{-1}$, where, for any $g$ in $GL(X)$, we define the adjoint $g^{-1}$ by the relation $Q(gx, y) = Q(x, gy)$. Such a choice of $Q(x, y)$ is always possible; for example, we may take $Q(x, y) = f(x, y)$ in A.1’ of §[2] As in that case, we define the (twisted) Fourier transformation $\phi \mapsto \hat{\phi}$ by

$$\hat{\phi}(x) = \int_{X_A} \phi(x)\psi(Q(x, y))d\gamma_A$$

for $x \in X_A$.

Then, for every $g$ in $GL(X)_A$, we have

$$\sum_{\xi \in X_k} \phi(g\xi) = |\det(g)|_A^{-1} \sum_{\xi \in X_k} \hat{\phi}(g^{-1}\xi);$$

this is just the usual Poisson formula. Since the automorphism $g \mapsto g^{-1}$ of $G$ gives rise to measure-preserving homeomorphism of $G_A/G_k$ with itself, we see that

$$I(\phi) = \int_{G_A/G_k} \left( \sum_{\xi \in X_k} \phi(g\xi) \right)dg_A$$

satisfies the relation

$$I(\phi_t) = |t|^{\frac{n}{2}}_A I(\hat{\phi}_{t-1})$$

(126)
where \( t \in k_+^* \) and \( \phi_t(x) = \phi(tx) \). For \( t = 1 \), this relation reduces to the invariance of \( I(\phi) \) under the Fourier transformation \( \phi \to \hat{\phi} \). The idea of Mars is that, without looking for an analogue of Theorem (iv) or rather the invariance under \( \phi \to \hat{\phi} \) of

\[
\sum_{i^* \in k^* \times A} \int_{X} \phi(x) \psi(i^* f(x)) |dx|_A,
\]

one may consider, instead, the relation

\[
\lim_{|t|_A \to \infty} I(\phi_t) = \lim_{|t|_A \to \infty} |d|_A^m I(\hat{\phi}_{t^{-1}})
\]

after expressing both sides of (126) by “orbital integrals” and incorporating the Poisson formula. For instance, if we now go back to the case of a quadratic form \( f(x) \) in \( n > 4 \) variables, then (127) will yield either the relation \( \tau_k(G)\phi(0) = 2\phi(0) \) or the relation \( \tau_k(G)\phi(0) = \phi(0) \) for every \( \phi \) in \( \mathscr{S}(X_A) \), according as \( G = \text{SO}(f) \) or \( \text{Spin}(f) \). (In the latter case, we need, of course, to assume correspondingly, that \( \tau_k(\text{Spin}(f)) = 1 \) for \( n = 3, 4 \)).

### 4.2 Other Siegel Formulas

We now indicate, in some detail, how one may obtain a Siegel formula in the general case; actually in the cases covered by A.3), A.4) of §2.3 such a generalisation exists. Using Theorem 4.1, we have, for every \( \phi \) in \( \mathscr{S}(X_A) \),

\[
\int_{G_A/G_1} \left( \sum_{\xi \in U_k} \phi(g\xi) \right) |dg|_A = \int \in \phi|\theta_U|_A
\]

where \( \theta_U \) is a gauge form on the homogeneous space \( U \) and further, \( \psi(x) = \tau_k(H_\alpha) \) if \( x \) is in \( G_A \cdot \xi_\alpha \). In all cases, we know that \( U(i) \) is a \( G \)-orbit for every \( i \) in \( k \) and further, \( f^{-1}(0) \) is a union of finitely many \( G \)-orbits. From (128), we see that

\[
I(\phi) = \tau_k(G)\phi(0) + \sum_{i \in k} \int \in \phi|\theta_V|_A + \sum_{i \in k} \int \in \phi|\theta_\cup|_A
\]
4. Other Siegel Formulas

letting $V$ run over all $G$-orbits in $f^{-1}(0)$ other than $U(0)$ and $\{0\}$. Further, we also have the Poisson formula

$$\sum_{i \in k} \int_{U(i)} \phi \theta_i \phi = \sum_{i \in k} \int_{X_A} \phi(x) \psi(i^* f(x)) \, dx |A|.$$

Let us define the “completed Eisenstein-Siegel series” $E(\phi)$ for $\phi$ in $S(X_A)$ by

$$E(\phi) = \phi(0) + \sum_{V} \int_{V} \phi \theta_V \phi + \sum_{i \in k} \int_{X_A} \phi(x) \psi(i^* f(x)) \, dx |A|,$$

where $V$ runs over the same $G$-orbits as in (129). Then a (conjectural) Siegel formula in the general case, is the simple (looking) relation that, for every $\phi$ in $H(X_A),$

$$I(\phi) = E(\phi). \quad (130)$$

In the case covered by A.3) of § 2.3, it is known that $\tau_k(G) = \epsilon(x) = 1$ and therefore, the Siegel formula (130) is indeed valid. As for A.4) and A.5), we apply the method of Mars explained in § 4.1. Then we get $\tau_k(G) = \epsilon(x) = 1$ as far as A.4) is concerned and therefore the Siegel formula (130) is valid here. Similarly, we can uphold the Siegel formula also for A.5), provided, however, that $\epsilon(x) = 1$ i.e. if the above-mentioned conjecture of Weil is verified for the stabilisers.

REMARK. For any $u \in k_A$, $\phi(x) \rightarrow \psi(uf(x))\phi(x)$ represents a unitary operator of $H(X_A)$. The unitary operators of $H(X_A)$ defined by

$$\phi(x) \rightarrow \Psi(x) = \psi(uf(x))\phi(x) \rightarrow \phi(-x)$$

which are obtained by conjugating unitary operators above with respect to $\phi \rightarrow \phi$, generate, together, with them, what might be called a meta-plactic group associated with $f(x)$. Let us denote this group by $Mp_A$ and its subgroup obtained by restricting $u$ to $k$, by $Mp_k$. Then $I(\phi)$ is invariant under the action of $Mp_k$. Therefore, if the Siegel formula (130) is valid, then the correspondence

$$Mp_A \ni S \rightarrow E(S\phi) \in \mathbb{C}$$
defines a continuous function on \( M_{pA} \) which is left invariant or “automorphic” under \( M_{pk} \). In the case when \( f(x) \) is a quadratic form and \( Q(x, y) = f(x, y) \), \( M_{pA} \) is just the metaplectic group of Weil; the invariance of \( E(S\phi) \) under \( M_{pk} = SL_2(k) \) gives the adelic version of the automorphic behaviour of the classical Eisenstein series.

### 4.3 A Final Comment

In the case covered by A.1) of § 2.3 i.e. when \( f(x) \) is a strongly non-degenerate form of degree \( m \geq 2 \) in \( n > 2m \) variables with \( m \) not divisible by the characteristic of \( k \), there is no Siegel formula in the strict sense. In this case, we still take

\[
E(\phi) = \phi(0) + \sum_{\ell \in K_A} \int X_A \phi(x) |\ell^* f(x)| dx \mid_A
\]

as the completed Eisenstein-Siegel series but there is no group over which we can take the average of

\[
I_0(\phi) = \sum_{\xi \in X_A} \phi(\xi).
\]

However, at least a subgroup of what might be called a metaplectic group is readily available. In fact, we convert the product \( P = G_a \times G_m \) into an algebraic group by defining

\[
(u, t)(u', t') = (u + tmu', tt').
\]

(Here \( G_m \) is the multiplicative group \( G_a \setminus \{0\} \)). For \( (u, t) \in P_A = k_A \times k_A^\times \), and \( \phi \) in \( \mathcal{S}(X_A) \), we define

\[
((u, t) \cdot \phi)(x) = |tm^{1/2}| \psi(uf(x))\phi(tx).
\]

Then \( (I_0 - E)((u, t) \cdot \phi) \) becomes a \( P_k \)-invariant continuous function on \( P_A \) and we can show that it vanishes at every “\( k \)-rational boundary point” of \( P_A \) in the following sense:

\[
(I_0 - E)((u, t) \cdot \phi) = O(|t|_A^{m-n/2}) \text{ as } |t|_A \to \infty
\]
and moreover,

\[(I_0 - E)((u, t) \cdot \phi) = O(|t|^{n/2-m}) \text{ as } |t|_A \to 0\]

but subject to the restriction that \((u + i^*)r^{-m}\) remains in a compact subset of \(k_A\) for some \(i^*\) in \(k\). If we specialise \(f\) to be a quadratic form and \(k\) to be \(\mathbb{Q}\), the theorem above gives the well-known behaviour of theta series as the variable in the complex upper half-plane approaches the rational points on the real axis. In the general case, the theorem above suggests that \((I_0 - E)((u, t) \cdot \phi)\) remains bounded as \(|t|_A\) tends to 0. We have observed that this conjecture will have as its consequence a generalisation of the Minkowski-Hasse theorem (namely \(U(i)_A \neq \phi\) implying that \(U(i)_k \neq \phi\)). For the details, we refer to our paper [23].

5 Siegel’s Main Theorem for Quadratic Forms

As it is well-known, the main theorem of Siegel in the analytic theory of quadratic forms is equivalent to the existence of an identity between a certain weighted average of theta series and an Eisenstein series, when one considers non-degenerate integral quadratic forms in more than four variables. We shall see in this article how this identity can be derived from the Siegel formula for quadratic forms due to Weil, quite explicitly in the positive-definite case.

5.1 Siegel’s Main Theorem

For any matrix \(M\), we denote its transpose by \(^tM\) and its determinant (whenever it makes sense) by \(\det M\). If \(M\) is a real \(n\)-rowed symmetric non-singular matrix, we say that \(M\) is of signature \((p, q)\) if exactly \(p\) eigenvalues of \(M\) are positive (and consequently \(q = n - p\) eigenvalues are negative); \(M\) is called positive-definite or indefinite according as \(q = 0\) or \(pq > 0\).

Two \(n\)-rowed integral symmetric matrices \(T, T'\) are said to be in the same (equivalence) class (respectively narrow class) if \(T = ^tAT'\) for an integral matrix \(A\) of determinant ±1 (respectively 1). We say that two non-singular \(n\)-rowed integral symmetric matrices \(T, T'\) of the same
signature *belong to the same genus*, if, for every prime number \( p \), there exists a matrix \( A_p \) with entries in the ring \( \mathbb{Z}_p \) of \( p \)-adic integers and of determinant 1, such that \( T = t^p A T' A^p \); we remark that this notion of a genus is the same as that of Siegel, in view of the fact every symmetric matrix \( M \) over \( \mathbb{Z}_p \) admits an \( A \) over \( \mathbb{Z}_p \) such that \( t^A M A^t = M \) and \( \det A = -1 \) (See [40], I, Hilfssatz 19]. Clearly any two \( T, T' \) as above which are in the same class, belong to the same genus as well; thus the genus of \( T \) splits into classes. From the reduction theory of quadratic forms due to Minkowski, Hermite and Siegel, it is known that the genus of any non-singular integral symmetric matrix \( T \) consists of only a finite number of classes and therefore, only finitely many narrow classes. For the narrow classes, we choose a complete set of representatives say \( T_1 (= T), T_2, \ldots, T_r \). We may further assume that if, for one of these \( T_i \), there exists no integral matrix \( A \) of determinant \( -1 \) with \( t^A T_i A^t = T_i \), then, along with \( T_i \), the matrix \( T_i^* = D T_i D \) also occurs among the \( r \) representatives for a fixed integral \( D \) of determinant \( -1 \); we may take, for example, \( D \) to be a diagonal matrix with \(-1\) as its first diagonal element and \( 1 \) as the remaining diagonal elements. To each \( T_i, 1 \leq i \leq r \), we associate the quadratic form \( f_i(x) = 1/2^x T_i x \), where \( x \) now denotes the \( n \)-rowed column with elements \( x_1, \ldots, x_n \); further, we write \( f(x) \) instead of \( f_1(x) \).

The special orthogonal group \( \text{SO}(f) \) of \( f(x) = 1/2^x T \), consists of all matrices \( A \) with \( t^A T A = T \) and \( \det A = 1 \). It acts, in a natural fashion, on the affine \( n \)-space \( X \). Denoting \( \text{SO}(f) \), for the present, by \( G \), it is known that \( G \) is a semi-simple linear algebraic group (for \( n > 2 \)) defined over \( \mathbb{Q} \); the adele group \( G_A \) has the property that \( G_A/G_Q \) has finite measure. We assume that the measure \( \mu \) on \( G_A \) is normalized so that

\[ \mu(G_A/G_Q) = 1. \quad (131) \]

The Siegel formula for orthogonal groups due to Weil may now be stated once again: namely, for \( n > 4 \) and for any \( \phi \) in \( \mathcal{S}(X_A) \), we have

\[ \int_{G_A/G_Q} \left( \sum_{g \in X_Q} \phi(g \xi) \right) d\mu(g) = \phi(0) + \sum_{i \in \mathbb{Q}} \int_{X_A} \phi(x) \psi(i^* f(x)) dx_A. \quad (132) \]
Let $T$ be positive-definite, unless otherwise stated and let $T_1, \ldots, T_r$ be the above-mentioned representatives of the narrow classes in the genus of $T$. Corresponding to each $f_i(x) = 1/2^i xT_ix$ and a complex variable $\tau$ with $\text{Im}(\tau) > 0$, let us define the theta series $v_i$ by

$$v_i(\tau) = \sum_{\xi \in \mathbb{Z}^n} e(\tau f_i(\xi)).$$

It is not hard to show that this series converges absolutely, uniformly when $\text{Im} \tau \geq \epsilon > 0$. Thus $v_i(\tau)$ is a holomorphic function of $\tau$ for $\text{Im}(\tau) > 0$ and furthermore, $v_i(\tau)$ depends only on the class of $T_i$. For $T_i$ and $T_i^*$, in particular, we have the same $v_i(\tau)$. Let $e_i$ (respectively $e_i^+$) denote the number of integral matrices $A$ of determinant $\pm 1$ (respectively 1) such that $AT_iA = T_i$; then $e_i$ is obviously finite and at least equal to 2. Moreover, $e_i = \delta_i e_i^+$ where $\delta_i = 2$ or 1 according as $T_i$ admits an integral $A$ with $AT_iA = T_i$, det $A = -1$ or otherwise. With our understanding above, only one of $T_i$ or $T_i^* = T_i[D]$ occurs among the $r$ representatives if $\delta_i = 2$, while both of them occur for $\delta_i = 1$.

The analytic formulation of the main theorem of Siegel for representation of integers by $T$ for $n > 4$ may now be given: namely,

$$\left( \sum_{i=1}^{r} v_i(\tau)/e_i^+ \right) / \left( \sum_{i=1}^{r} 1/e_i^+ \right) = 1 + \frac{e(n/8)}{\sqrt{\det T}} \sum_{c,d} e^{-n/2} \left( \sum_{\xi \mod c} e\left(-\frac{d}{c} f(\xi)\right)\right)(ct+d)^{-n/2} \quad (133)$$

where, on the right hand side, the summation is over all pairs of coprime integers $c, d$ with $c \geq 1$ and further, with $cd f(\xi) \in \mathbb{Z}$ for all $\xi \in \mathbb{Z}^n$. In view of our remarks above, it is not difficult to check that the left hand side of (133) is the same as the “analytic genus-invariant” of $T$ in the sense of Siegel; formula (133) is thus the same as formula (83) in the above-mentioned fundamental paper ([40], I).

5.2

We wish to prove that the Siegel formula (132) for positive-definite integral quadratic forms yields the analytic identity (133), if we merely
specialise \( \phi \) in (132) suitably. But first, let us refer to another familiar notion of “classes in a genus” associated with an arbitrary linear algebraic group over a global field.

Let \( G \) be a linear algebraic group defined over a global field \( k \) of characteristic 0. The adele group \( G_A \) contains only finitely many distinct double cosets modulo \( G_\Omega, G_k \) where \( G_\Omega \) is just the open subgroup denoted earlier by \( G_{S,\infty} \):

\[
G_A = \bigcup_{i=1}^{r} G_\Omega g_i G_k
\]

(134)

The number \( r = r_k(G) \) is known sometimes as the class number of \( G \).

We shall prove, in § 5.6, that

\[
r_\mathbb{Q}(SL_n) = 1
\]

(135)

Actually, more generally, even \( r_k(SL_n) = 1 \). If \( G \subset GL(V) \) for a vector space \( V \) of dimension \( n \), then \( G_A \) acts in a natural fashion, on the “lattices” in \( V_k \); the orbit of a “lattice” under \( G_A \) (respectively \( G_k \)) constitutes the genus (respectively class) of the “lattice” and the number of classes in a genus is finite.

Going back to the \( n \)-rowed integral positive-definite matrix \( T \) once again, we take \( G = SO(f) \), for the associated \( f(x) = 1/2x^T x \). From (134) and (135), we have

\[
(SL_n)_A = (SL_n)_{S,\infty} \cdot (SL_n)_{\mathbb{Q}}.
\]

(135)′

Thus any \( g \) in \( SO(f)_A \subset (SL_n)_A \) can be written as

\[
g = A \cdot C^{-1}, \quad A \in (SL_n)_{S,\infty}, \quad C \in (SL_n)_\mathbb{Q}.
\]

(136)

Since \( 'CTC = 'g^{-1}Atg^{-1}A = 'ATA \), it follows that \( 'ATA \) is in the genus of \( T \); its class depends only on the double coset \( G_\Omega gG_\mathbb{Q} \). Conversely, if \( T' \) is in the genus of \( T \), then, in \( (GL_n)_A \), \( T' = 'ATA \) for some \( A \) in \( (SL_n)_{S,\infty} \); by the Minkowski-Hasse theorem, we know then that \( T' = 'CTC \) for some \( C \) in \( (SL_n)_\mathbb{Q} \) and thus \( T' \) corresponds to the double coset \( G_\Omega(AC^{-1})G_\mathbb{Q} \). Therefore, for our \( r \) (narrow) class-representatives
5. Siegel’s Main Theorem for Quadratic Forms

$T_1, \ldots, T_r$ above, we may take $T_i = A_i T C_i$ as in (136) and the double coset decomposition 134 for $SO(f)_A$. From (131), (134) and the invariance of $\mu$, we obtain

$$1 = \sum_{i=1}^{r} \mu(G_\Omega g_i G_\Omega / G_\Omega)$$

$$= \sum_{i=1}^{r} \mu(g_i^{-1} G_\Omega g_i G_\Omega / G_\Omega)$$

$$= \sum_{i=1}^{r} \mu(g_i^{-1} G_\Omega g_i / (G_\Omega \cap g_i^{-1} G_\Omega g_i))$$

$$= \sum_{i=1}^{r} \mu(G_\Omega / (g_i G_\Omega g_i^{-1} \cap G_\Omega))$$

$$= \mu(G_\Omega) \sum_{i=1}^{r} 1/e_i^+$$

in view of the fact that $g_i G_\Omega g_i^{-1} \cap G_\Omega \cong A_i^{-1} G_\Omega A_i \cap C_i^{-1} G_\Omega C_i = SO(f)_\mathbb{Z}$ is of order $e_i^+$. Thus

$$\mu(G_\Omega) = 1/\left(\sum_{i=1}^{r} 1/e_i^+\right). \quad (137)$$

5.3

For $\phi \in \mathcal{S}(X_A)$ in (132), we take $\phi = \phi_0 \otimes \phi_\infty$ in $\mathcal{S}(X_0) \otimes \mathcal{S}(X_\infty)$, following the notation of Chapter IV, §1.5, with

$$\phi_0 = \text{the characteristic function of } \prod_{p} X_p^0$$

and

$$\phi_\infty(x_\infty) = e(f(x_\infty) \tau).$$

The left hand side of (132) becomes just

$$\sum_{i=1}^{r} \int_{G_\Omega / (g_i G_\Omega g_i^{-1} \cap G_\Omega)} \left(\sum_{\xi \in X_\infty} \phi(g \xi, \xi)\right) d\mu(g) \quad (138)$$
4. Poisson Formula of Siegel-Weil Type

The same arguments that led us to the proof of (137). The inner-most series in (138) is the same as
\[ \sum_{\xi \in X} \phi(g' A_i \xi) \] with \( g' = (g'_0, g'_\infty) \) in \( G_\Omega \). If now we use the definition of \( \phi \), then the last-mentioned series is seen to be precisely
\[ \sum_{\xi \in \mathbb{Z}^n} e(f(g'_\infty A_i\xi)) = \sum_{\xi \in \mathbb{Z}^n} e(f(\xi)) = v_i(\tau). \]

It is now immediate that (138) is exactly \( \mu(G_\Omega) \sum_{i=1}^{\nu_i(\tau)} e^{i} \) and therefore, by (137), the left hand side of (132) is the same as that of (133).

5.4

What remains to be proved then is only that (132) has the same right hand side as (133), for \( \phi \) as chosen as above. Since \( \phi(0) = 1 \), we are reduced to showing that the series on the right hand side of (132) is identical with the series over \((c, d)\) in (133).

Since \((k_A/k)^* \approx k\), the right hand side is independent of the choice of \( \psi \) and therefore we may assume that
\[ \psi_p : \mathbb{Q}_p \to \mathbb{Q}_p / \mathbb{Z}_p \to \mathbb{R} / \mathbb{Z} \approx C_1^* \quad \text{and} \]
\[ \psi_\infty (x_\infty) = e(-x_\infty). \]

Any \( i^* \) in \( \mathbb{Q} \) may be written as \(-d/c\) with coprime \( c, d \) in \( \mathbb{Z} \) and further \( c \geq 1 \). The integral over \( X_A \) in (132) becomes
\[ \left( \prod_p \int_{X_p^0} \psi_p \left( \frac{-d}{c} f(x) \right) |dx|_p \right) \cdot \int_{X_\infty} e(f(x)(\tau + d/c)) |dx|_\infty. \tag{139} \]

The integral over \( X_p^0 \) above may be rewritten as
\[ \sum_{\xi \mod c \mathbb{Z}_p} \int_{c \mathbb{Z}_p} \psi_p \left( \frac{-d}{c} f(\xi + x) \right) |dx|_p. \tag{140} \]
Since $\psi_p\left(-\frac{d}{c} f(\xi + x)\right) = \psi_p\left(-\frac{d}{c} f(\xi)\right)\psi_p\left(-\frac{d}{c} f(x)\right)$, the integral over $c\mathbb{Z}_p^n$ in (140) is (upto the factor $\psi_p\left(-\frac{d}{c} f(\xi)\right)$ equal to

$$|c|^n_p \int_{\mathbb{Z}_p^n} \psi_p(-cd f(x))|dx|_p = \begin{cases} |c|^n_p & \text{if } cd f(x) \text{ is } \mathbb{Z}_p \text{-valued on } \mathbb{Z}_p^n \\ 0, & \text{otherwise} \end{cases}$$

on noting that $\psi_p(-cd f(x))$ is a character of $\mathbb{Z}_p^n$. Hence the infinite product over primes $p$ in (139) is precisely

$$c^{-n} \sum_{\xi \mod c} \epsilon\left(-\frac{d}{c} f(\xi)\right)$$

assuming, of course, the $cd f(x)$ is $\mathbb{Z}$-valued on $\mathbb{Z}^n$. On the other hand, the integral over $X_\infty$ in (139) is clearly equal to

$$\int_{\mathbb{R}^n} \epsilon\left(\frac{1}{2}(Mx)(Mx)(\tau + d/c)\right)|dx|_\infty \text{ with } \epsilon^i MM = T$$

$$= |\det M|^{-1} \left(\int_{\mathbb{R}} \epsilon\left(\frac{1}{2}(\tau + d/c)\right) d\tau\right)^n$$

$$= (\det T)^{-1/2} \left(\frac{1}{\sqrt{i(\tau + d/c)}}\right)^n$$

$$= (\det T)^{-n/2} \epsilon(n/8) c^{n/2}(c\tau + d)^{-n/2}$$

(142)

Taking into account (139), (140), (141) and (142), it is clear that our assertion at the beginning of §5.4.4 is established. With this, the proof of the fact that the Siegel formula (132) implies (133) is also complete.

5.5

If we consider an $n$-rowed integral symmetric matrix $T$ of signature $(p, q)$ with $0 < q = n - p < n$, then Siegel’s main theorem takes the following form. For the special orthogonal group $G = \text{SO}(f)$ with $f(x) = 1/2 x^T x$, $G_\infty$ is no longer compact; $G_\mathbb{Z}$ is no more finite. One
now considers the space $\mathcal{H}$ of all the majorants $M$ of $T$, which are just $n$-rowed symmetric positive-definite matrices such that $M^{-1}SM^{-1}$ is equal to the $n$-rowed identity matrix. The group $G_\infty$ acts on $\mathcal{H}$ via the maps $M \mapsto g_\infty M g_\infty$ for $g_\infty \in G_\infty$ and indeed transitively; the stabiliser of any point in $\mathcal{H}$ is a maximal compact subgroup of $G_\infty$. Let $\mu$ be a measure on $\mathcal{H}$ such that $\mu(\mathcal{H}/G_Z) = 1$. The main theorem of Siegel ([42]) now reads: for $n > 4$,

$$
\int_{\mathcal{H}/G_Z} \left( \sum_{\xi \in \mathbb{Z}^n} e(f(\xi)\text{Re}(\tau) + i\xi \cdot M \xi \text{Im}(\tau)) \right) d\mu(M)
$$

$$
= 1 + |\det T|^{-1/2} \otimes \left( \frac{p - q}{8} \right) \sum_{c,d} c^{-n/2}
$$

$$
\left( \sum_{\xi \mod c} \otimes \left( -\frac{d}{c} f(\xi) \right) (c\tau + d)^{-p/2}(c\tau + d)^{-q/2} \right)
$$

where $c, d$ run over all pairs of coprime integers with $c \geq 1$ and $cd f(\xi) \in \mathbb{Z}$ for all $\xi \in \mathbb{Z}^n$. This formula can again be derived from (132), with a proper choice of $\phi$; actually we may take $\phi = \phi_0 \otimes \phi_\infty$, where $\phi_0$ is, as before, the characteristic function of $\prod_p X^0_p$ but $\phi_\infty(x_\infty) = \exp(-2\pi x_\infty M x_\infty)$.

5.6

We now give an indirect but hopefully instructive proof of the assertion (135) used in the course of our arguments in §5.2. First, we give an outline of the proof of the fact that the Tamagawa number

$$
\tau_k(SL_n) = \int_{(SL_n)\lambda/(SL_n)k} |dg|_\lambda = 1 \quad (143)
$$

for any $n \geq 1$ and any global field $k$. Since the assertion (143) is clear for $n = 1$, assume that it has been proved with $r \leq n - 1$ in place of $n$, for $n > 1$. Let us now take $G$ to be $SL_n$ and $X$ to be the same as the space $M_{n,1}$ of $n$-rowed vectors. For any field $K$, we have

$$
X_K \backslash \{0\} = G_K \cdot e_1
$$
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where now $e_1$ is the unit vector with 1 as its first element and 0 everywhere. The stabiliser $H$ in $G$ of the vector $e_1$ is a semi-direct product of $SL_{n-1}$ and the $(n-1)$-fold product of the additive group $G_a$. Hence the induction hypothesis yields $\tau_k(H) = 1$. For any $\phi$ in $\mathcal{S}(X_A)$, it can be shown that

$$
\int_{G_A/G_k} \left( \sum_{\xi \in X_k} \phi(g\xi) \right) |dg|_A = \tau_k(G)\phi(0) + \tau_k(H)\phi^*(0)
$$

(144)

where $\phi^*$ is the Fourier transform of $\phi$. The left hand side of (144) can be seen to be invariant under $\phi \mapsto \phi^*$, in view of the (classical) Poisson formula (115). But then the invariance of the right hand side under $\phi \mapsto \phi^*$ implies that $\tau_k(G) = \tau_k(H) = 1$, which establishes (143).

Coming back to the proof of (135)', we need, in view of (133), only to show that the measure of

$$(SL_n)_{S,\infty}(SL_n)_{Q}/(SL_n)_{Q} = \prod_p SL_n(Z_p) \times SL_n(R)/SL_n(\mathbb{Z})$$

is precisely 1. It is known that

$$\mu(SL_n(Z_p)) = (1 - p^{-n}) \ldots (1 - p^{-2})$$

and therefore

$$\mu \left( \prod_p SL_n(Z_p) \right) = 1 / \prod_{i=2}^{n} \zeta(i)$$

where $\zeta$ is the Riemann zeta function. In order to complete the proof of (135)', we have only to show that

$$\mu(G_R/G_Z) = \mu(H_R/H_Z) \cdot \zeta(n).$$

(145)

Now, analogous to (144), we also have the relation

$$\int_{G_R/G_Z} \left( \sum_{\xi} \phi_{\infty}(g\xi) \right) |dg|_\infty = \mu(H_R/H_Z)\phi^*_{\infty}(0)$$

(146)
140

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for \( \phi_\infty \) in \( \mathcal{S}(X_\infty) \), where \( \xi \) runs over all “primitive vectors” i.e. over all the elements of the orbit \( SL_n(\mathbb{Z})e_1 \) of the vector \( e_1 \) above. We may assume that \( \phi_\infty \geq 0, \phi_\infty \neq 0 \), so that \( \phi_\infty^*(0) > 0 \). If we replace \( \phi_\infty(x) \) in \( (146) \) by \( \phi_\infty(tx) \) for \( t > 0 \), then we obtain, instead,

\[
\int_{G_\mathbb{R}} \left( \sum_{\xi} \phi_\infty(gt\xi) \right) |dg|_\infty = \mu(H_\mathbb{R}/H_\mathbb{Z}) r^{-n} \phi_\infty^*(0). \quad (147)
\]

Taking \( t = 1, 2, 3, \ldots \) and summing up both sides of \( (147) \) over \( t \), we get

\[
\int_{G_\mathbb{R}/G_\mathbb{Z}} \left( \sum_{\xi \in X_\mathbb{R}\{0\}} \phi_\infty(g\xi) \right) |dg|_\infty = \mu(H_\mathbb{R}/H_\mathbb{Z}) \zeta(n) \phi_\infty^*(0). \quad (148)
\]

Replacing \( \phi_\infty(x) \) in \( (148) \) by \( \phi_\infty(t'x) \) for \( t' > 0 \) and letting \( t' \) tend to 0, we have

\[
\int_{G_\mathbb{R}/G_\mathbb{Z}} \left( \int_{X_\mathbb{R}} \phi_\infty(gx) dx \right) |dg|_\infty = \mu(H_\mathbb{R}/H_\mathbb{Z}) \zeta(n) \phi_\infty^*(0).
\]

But now the left hand side is the same as \( \mu(G_\mathbb{R}/G_\mathbb{Z}) \phi_\infty^*(0) \) and 145 is proved.

6 Proof of the Criterion for the Validity of the Poisson Formula

We devote this section to giving an outline of the proof of Theorem 2.1.

Let us recall that \( k \) is a global field, \( \psi \), a non-trivial character of \( k\mathbb{A}/k, f(x) \in k[x_1, \ldots, x_n] \) a homogeneous polynomial of degree \( m \geq 2 \). We had further assumed that \( m \) is not divisible by the characteristic of \( k \) and that a tame \( k \)-resolution of the projective hypersurface defined by \( f(x) = 0 \) exists. Further, the two criteria stated are:

(C1) \( r = \text{codim}_{f^{-1}(0)} C_f \geq 2 \); and
(C2) there exist $\sigma > 2$ and a finite set $S$ of valuations $v$ of $k$, such that, for every $i^*$ in $k_v \setminus R_v$ and $v \not\in S$, we have

\begin{equation}
|F^v(i^*)| = \left| \int_{x^0} \psi_v(i^* f(x)) dx|_v \right| \leq |i^*|_v^{-\sigma}
\end{equation}

We have to show that (C1) and (C2) imply the validity of (PF-1), . . . , (PF-3) and (PF-4).

### 6.1 Geometric Part of the Proof

Let $D$ be a positive divisor on an irreducible non-singular algebraic variety $X$ and $h : Y \rightarrow X$ be a $K$-resolution of $D$ for any arbitrary field $K$. Let $\{(N_i, v_i)\}_i$ denote the numerical date of $h$ at a point $b$ in $Y$, in the sense of Chapter III, § 2.2. We say that $h$ has property $(P_0)$ at $b$, if $v_i \geq N_i$ for every $i$ and if, further, the equality holds for at most one $i$. If, moreover, in case such equality holds, say $v_{i_0} = N_{i_0}$, we have $v_{i_0} = N_{i_0} = 1$, then $h$ is said to have property $(P)$ at $b$. It may happen that $h$ has property $(P_0)$ at every point of $Y_K$ but does not have property $(P)$ at some point of $Y_K$. We have, however, the following

**Lemma 6.1.** If $h$ (is tame and) has property $(P_0)$ everywhere, then $h$ has necessarily property $(P)$ everywhere.

**Proof.** By restricting $h$ to a Zariski open neighbourhood of $h^{-1}(h(b))$ for a point $b$ in $Y$, we may assume that $X$ is affine and hence that $D = (g)$ for some function $g$ regular everywhere on $X$; moreover, we may suppose that there exists a “gauge form” $dx$ on $X$, namely an everywhere regular differential form of degree $n = \dim(X)$ on $X$, vanishing nowhere. \hfill \Box

If the lemma is false, then, for the numerical data $\{(N_i, v_i)\}_i$ at some point in $Y$, we have $v_i > N_i$ for $i \neq i_0$ and $v_{i_0} = N_{i_0} \geq 2$; without loss of generality, let $i_0 = 1$. Thus there exists an irreducible component $E_1$ of $h^*(D)$ with $v_1 = N_1 \geq 2$ and $N_1(= N_{E_1})$ not divisible by the
characteristic of $K$, since $h$ is tame. We shall construct a regular $(n - 1)$-form $\theta \neq 0$ on $E_1$, as follows. At an arbitrary point $b$ on $E_1$, there exist local coordinates $y_1, \ldots, y_n$ of $Y$ centred at $b$ such that

$$g \circ h = \epsilon \prod_i y_i^{N_i}, \quad h^*(dx) = \eta \prod_i y_i^{y_i-1} dy_1 \wedge \ldots \wedge dy_n$$

with $\epsilon, \eta$ regular and non-vanishing around $b$ and $y_1 = 0$ being the local equation for $E_1$. Then, for the $(n - 1)$-form

$$\Theta = \left( N_1 \epsilon + y_1 \frac{\partial \epsilon}{\partial y_1} \right)^{-1} \eta \prod_{i>1} y_i^{y_i-N_i-1} dy_2 \wedge \ldots \wedge dy_n$$

on $Y$ regular along $E_1$, we have

$$h^*(dx) = d(g \circ h) \wedge \Theta.$$

Also, locally around $b$, the restriction $\theta$ of $\Theta$ to $E_1$, i.e.

$$\theta = \text{the restriction to } E_1 \text{ of } (N_1 \epsilon)^{-1} \eta \prod_{i>1} y_i^{y_i-N_i-1} dy_2 \wedge \ldots \wedge dy_n$$

is well-defined (i.e. independent of the choice of local coordinates), different from 0 and regular on $E_1$.

We recall that $h$ can be factored into monoidal transformations each having irreducible non-singular centre and $E_1$ is “created” at one such stage, say $h' : Y' \rightarrow X'$; we have tacitly used the fact that $\nu_1 \geq 2$ and hence $h$ is not biregular along $E_1$. Let $Z$ denote the centre of $h'$ and let $E' = (h')^{-1}(Z)$; then $E'$ is irreducible and non-singular and the restriction of $Y \rightarrow Y'$ to $E_1$ gives a birational morphism $g' : E_1 \rightarrow E'$. Let $\theta'$ denote the image of $\theta$ under $g'$. Then $\theta'$ is different from 0 and everywhere regular on $E'$; if $\theta'$ is not regular on $E'$, then $g^{-1}(C')$, for any component $C'$ of its polar divisor, becomes a component of the polar divisor of $\theta$, contradicting the regularity of $\theta$. If $\pi = h'|_{E'}$, then $\pi : E' \rightarrow Z$ makes $E'$ into a fibre bundle with the projective space $\mathbb{P}_{t-1}$ of dimension $t - 1$ as fibre; here $t$ = (codimension of $Z$) $\geq 2$. Choosing $b'$ in $E'$ at which $\theta'$ does not vanish and putting $a' = \pi(b')$, we choose a local gauge form $dz$ on $Z$ around $a'$ and write $\theta' = \pi^*(dz) \wedge \rho$ with
a $(t - 1)$ form $\rho$ on $E'$ regular along $F = \pi^{-1}(a')$. This is possible and further, the restriction $\rho_F$ of $\rho$ to $F$ is well-defined, different from 0 and regular on $F$. But $F$ being isomorphic to $\mathbb{P}_{r-1}$, there is no regular form other than 0. This contradiction proves the lemma.

The following lemma is quite elementary.

**Lemma 6.2.** Let $f(x)$ be a homogeneous polynomial of degree $m$ in $n$ variables $x_1, \ldots, x_n$ with coefficients in (an arbitrary field) $k$. Let $X_0$ (respectively $X$) denote the projective (respectively affine) spaces with $(x_1, \ldots, x_n)$ as its homogeneous (respectively affine) coordinates and $X^\sharp$, the projective space with $(1, x_1, \ldots, x_n)$ as its homogeneous coordinates. Let $H_\infty = X^\sharp \setminus X$ and $f^\sharp$ denote the rational function on $X^\sharp$ defined by $f(x)$. Let us assume that a $k$-resolution $h_0 : Y_0 \to X_0$ of the projective hypersurface defined by $f(x) = 0$ exists. Then $h_0$ gives rise to a $k$-resolution $h^\sharp : Y^\sharp \to X^\sharp$ of $((f^\sharp)_0, H_\infty)$ such that the numerical data of $h^\sharp$ are the numerical data of $h_0$ at some point of $Y_0$ possibly augmented by $(m, n)$; here $(f^\sharp) = (f^\sharp)_0 - mH_\infty$ is the divisor of $f^\sharp$ on $X^\sharp$.

**Proof.** Clearly $(1, x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n)$ gives a rational map $\varphi : X^\sharp \to X_0$ over $k$, regular except at $(1, 0, \ldots, 0)$. If $g : Z \to X^\sharp$ is the quadratic transformation centred at $(1, 0, \ldots, 0)$, then, $h_1 = \varphi \circ g : Z \to X_0$ is a $k$-morphism. If $Y^\sharp$ is the fibre product $Y_0 \times_{X_0} Z = \{(y, z) \in Y_0 \times Z : h_0(y) = h_1(z)\}$, define $h^\sharp : Y^\sharp \to X^\sharp$ as the composite of $h_0 \times 1 : Y^\sharp \to Z$ and $g : Z \to X^\sharp$. For $X_0$ (respectively $Y_0$), we have $k$-open coverings by $X_1, \ldots, X_n$ (respectively $Y_1, \ldots, Y_n$) where $X_1$ is the affine space obtained from $X_0$ by letting $x_1 = 0$ define the hyperplane at infinity and $Y_i = h_0^{-1}(X_i)$ for $1 \leq i \leq n$. Let $A_i$ (respectively $\mathbb{P}_{1,i}$) be the affine line (respectively projective line) with coordinate ring $k[1/x_i]$ (respectively homogeneous coordinate ring $k[t, tX_i]$ with a new variable $t$) for $1 \leq i \leq n$. Then $X, A_i \times X_i (1 \leq i \leq n)$ form a $k$-open covering of $X^\sharp$ and $\mathbb{P}_{1,i} \times X_i (1 \leq i \leq n)$ form a $k$-open covering of $Z$. For $Y^\sharp$, we have a $k$-open covering by $\mathbb{P}_{1,i} \times Y_i$.

We show that $h^\sharp$ has the required property. Let $a^\sharp = h^\sharp(b^\sharp)$ for a point $b^\sharp$ of $Y^\sharp$. We have only to show that the irreducible components of $(h^\sharp)^* \left( ((f^\sharp)_0 + H_\infty) \right)$ are defined over $k(b^\sharp)$ and mutually transversal at $b^\sharp$ and that the numerical data of $h^\sharp$ at $b^\sharp$ are the same as those of $h_0$ at a point.
of $Y_0$ except possibly for the addition of $(m, n)$. By changing the indices, we may assume that $b^\sharp \in \mathbb{P}_{1,1} \times Y_1$; thus we may write $b^\sharp = (a_1, b)$ with $a_1$ in the universal field or $a_1 = \infty$ and $b$ in $Y_1$. Let

$$u_2 = x_2/x_1, \ldots, u_n = x_n/x_1.$$ 

Then there exist local coordinates $(v_2, \ldots, v_n)$ of $Y_0$ centred at $b$ and defined over $k(b)$ such that

$$f(1, u_2, \ldots, u_n) = \epsilon \prod_{i > 1} y_i^{N_i},$$

$$du_2 \wedge \ldots \wedge du_n = \eta \prod_{i > 1} y_i^{\nu_i-1} \cdot dy_2 \wedge \ldots \wedge dy_n$$

where $\epsilon, \eta$ are regular and non-vanishing around $b$. First, let $a^\sharp \in H_\infty$ i.e. $a_1 \neq \infty$. Then we put $y_1 = x_1 - a_1, y_2 = v_2, \ldots, y_n = v_n$ and $(y_1, \ldots, y_n)$ form local coordinates of $Y^\sharp$ centred at $b^\sharp$ and clearly defined over $k(b^\sharp)$ such that

$$f(x_1, \ldots, x_n) = \epsilon(y_1 + a_1)^m \prod_{i > 1} y_i^{N_i}$$

$$dx = \eta(y_1 + a_1)^{\nu-1} \prod_{i > 1} y_i^{\nu_i-1} \cdot dy.$$ 

Since $(x_1, \ldots, x_n)$ form local coordinates on $X^\sharp$ around $a^\sharp$ and $f(x_1, \ldots, x_n) = 0$ gives a local equation for $(f^\sharp)_0$, $h^\sharp$ has the required property at $b^\sharp$. On the other hand, let $a_1 = \infty$, then setting $y_1 = 1/x_1, y_2 = v_2, \ldots, y_n = v_n$, we see that $(y_1, u_2, \ldots, u_n)$ (respectively $(y_1, \ldots, y_n)$) form local coordinates on $X^\sharp$ (respectively $Y^\sharp$) around $a^\sharp$ (respectively centred at $b^\sharp$) and defined over $k(b^\sharp)$. Moreover, $f(1, u_2, \ldots, u_n) = 0$ (respectively $y_1 = 0$) gives local equations for $(f^\sharp)_0$ (respectively $H_\infty$); also we have

$$f(1, u_2, \ldots, u_n) = \epsilon \prod_{i > 1} y_i^{N_i},$$

$$dy_1 \wedge du_2 \wedge \ldots \wedge du_n = \eta \prod_{i > 1} y_i^{\nu_i-1} \cdot dy.$$
This finally establishes the required property of $h^\#$ at $b^\#$.

We remark that the addition of $(m, n)$ to the numerical data results if and only if $a_1 = 0$ i.e. if and only if $a^\# = h^\#(b^\#)$ has $(1, 0, \ldots, 0)$ as its homogeneous coordinates.

### 6.2 Key Lemmas

Let $f(x)$ be a polynomial in $x_1, \ldots, x_n$ with coefficients in a global field $k$, such that $C_f \subset f^{-1}(0)$. Defining $X, X^\# = X \cup H_\infty$ and $f^\#$ (the rational function extending $f$ to $X^\#$) as in Lemma lem6.2, let us assume that a tame $k$-resolution $h^\# : Y^\# \to X^\#$ of $((f^\#)_0, H_\infty)$ exists. Then the restriction of $h^\#$ to $f = f^{-1}(X)$ gives a tame $k$-resolution $h : Y \to X$ of the hypersurface $f(x) = 0$. In the following lemma, $\mathcal{D}(X_v)$ stands for the space of Schwartz-Bruhat functions on $X_v$ with compact support; thus $\mathcal{D}(X_v) = \mathcal{S}(X_v)$ for non-archimedean $v$.

**Lemma 6.3.** For given $b$ in $Y_v$, choose $\phi_v \geq 0$ in $\mathcal{D}(X_v)$ with $\phi_v(h(b)) > 0$ and (i) assume that existence of $F_{\phi_v}(0)$ or (ii) make the stronger assumption that

$$F_{\phi_v}(0) = \int_{U(0)_v} \phi_v|\theta_0|_v. \quad (149)$$

Then $h$ has property $(P_0)$ or property $(P)$ at $b$ depending on (i) or (ii) above. Conversely, if $h$ has property $(P)$ at every point of $Y_v$, then $(149)$ holds for every $\phi_v$ in $\mathcal{D}(X_v)$; further, if $h^\#$ has property $(P)$ at every point of $Y^\#_v$, then $(149)$ holds for every $\phi_v$ in $\mathcal{S}(X_v)$.

**Proof.** Using the notation of Chapter III, § 3.2, there exist, for the $k_v$-resolution $h : Y_v \to X_v$, local coordinates $y_1, \ldots, y_n$ of $Y$ centred at $b$ and defined over $k_v$ such that

$$f \circ h = \epsilon \prod_i y_i^{N_i}, \quad h^*(dx) = \eta \prod_i y_i^{N_i-1} \cdot dy$$

with $\epsilon, \eta$ regular and non-vanishing around $b$. Now, $\epsilon, \eta, y_1, \ldots, y_n$ give rise to $k_v$-analytic functions on an open neighbourhood, say $V$, of $b$ in $Y_v$. 


By choosing $V$ smaller still, if necessary, we may assume that already $\epsilon \eta$ is non-vanishing on $V$ and also that, for the given $\phi_v$.

$$|\epsilon(y)| v|\eta(y)|, \phi_v(h(y)) \geq c > 0$$

for a suitable constant $c$ independent of $y$ in $V$ and $s$ with $-1 \leq s \leq 0$.

Then, for all such $s$,

$$Z_{\phi_v}(\omega_s) \geq c \int_V \prod_i |y_i|^{N_i s + \nu_i - 1} |dy|_v$$  \hspace{1cm} (150)

On the other hand, from the existence of $F_{\phi_v}(0)$, we can conclude, in view of Chapter II, § 3.1, that the function $(s + 1)Z_{\phi_v}(\omega_s)$, which is positive for $-1 < s \leq 0$, is also continuous and bounded in this range. On being multiplied by $(s + 1)$, the right hand side of (150) is asymptotic to a constant multiple of $(s + 1)/ \prod_i (N_i s + \nu_i)$ as $s \to -1$ from the right.

Therefore, $v_i \geq N_i$ for every $i$ and the equality $v_i = N_i$ can hold for at most one $i$. In other words, the existence of $F_{\phi_v}(0)$ entails that $h$ has property $(P_0)$ at $b$. \hfill \square

We shall show next that (149) implies $v_{i_0} = N_{i_0} = 1$. For that purpose, we shall first obtain an expression for $F_{\phi_v}(0)$ without assuming (149). We take an arbitrary $b'$ from $Y_v$ at which $\phi_v(b') > 0$; then what we have shown for $b$ is applicable to $b'$. For the sake of simplicity, we shall write $b$ again for $b'$ and use the same notation as above. By changing indices, we may suppose that $i_0 = 1$. Let $\epsilon_1 \left( N_1 \epsilon + y_1 \frac{\partial \epsilon}{\partial y_1} \right)^{-1} \eta$, locally around $b$ in $Y_v$. Then, for every $i \neq 0$ in $k$, we have, $h^*(\theta_i) = \epsilon_1^{\prod_j y_j^{-N_j - 1}} dy_2 \wedge \ldots \wedge dy_n$ to $(f \circ h)^{1-1}(i)$. Furthermore, if we denote $E_1$ by $E$, then the everywhere regular differential form $\theta$ on $E$ (obtained by restriction as in Lemma (6.1)) exists. If $E_v \neq \phi$, then $E$ and $\theta$ are both defined over $k_v$. Moreover, $\theta$ gives rise to a positive Borel measure $|\theta|_v$ on $E_v$, with $E_v$ as its exact support.

We may take the neighbourhood $V$ of $b$ above small enough so that $\epsilon_1 \neq 0$ on $V$. Then for every locally constant or $C^\infty$ function $\varphi_0$ on $V$
with compact support, we have
\[
\lim_{i \to 0} \int_{(f \circ h)^{-1}(i)} \varphi_0 h^*(\theta_0) = \int_{E \cap V} \varphi_0 |\theta_0|.
\]

We know that \( h \) is biregular at every point of \( h^{-1}(U(0)) \); if \( \nu = N_1 = 1 \) as long as only points of \( V \) are considered, we have
\[
h^{-1}(U(0)) = E \cup \text{Hyperplanes defined by } y_j = 0 \text{ for } N_j \geq 1.
\]

Hence we get
\[
\int_{E \cap V} \varphi_0 |\theta_0| = \int_{h^{-1}(U(0))} \varphi_0 h^*(\theta_0).
\]

We take a finite covering of the preimage, under \( h \), of the support of \( \phi \), by means of open sets \( V \) as above. Choosing a partition of unity \( (p_v)_V \) subordinate to this covering, we apply the observation above to each
\[
\phi = (\phi \circ h)p_v \text{ and obtain}
\]
\[
F_{\phi}(0) = \int_{U(0)} \phi|\theta_0| + \sum_{\nu \geq 2} E \int (\phi \circ h)|\theta_0|.
\]

But \( |\theta_0| \) has \( E \) as its exact support and therefore, the validity of (149) now implies that no \( E \) with \( \nu_E = N_E \geq 2 \) can pass through a point \( b' \) where \( \phi(b') > 0 \). Therefore (149) implies that \( h \) has property (P) at \( b \).

Conversely, suppose that \( h \) has property (P) at every point of \( Y \). Then certainly (149) holds for every \( \phi \) in \( D(X) \). If, further, \( h \) has property (P) at every \( b^d \) in \( Y_d \), then we choose local coordinates \( y_1, \ldots, y_n \) on \( Y_d \) centred at \( b^d \) and defined over \( k_v \) such that, upto a regular and non-vanishing function around \( b^d \), \( f^d \circ h^d \) becomes a product of (possibly negative) powers of \( y_1, \ldots, y_n \) and further, \( (h^d)^*(dx) \) is another such power-product in \( y_1, \ldots, y_n \) multiplied by a local gauge form around \( b^d \).

Suppose, for example, the exponent of \( y_j \) in \( f^d \circ h^d \) is negative. Then the “infinite divisibility” of \( \phi \) by a local equation for \( H_{\infty} \) implies the infinite divisibility of \( \phi \circ h^d \) by \( y_j \). Thus the component with \( y_j = 0 \) as a local equation becomes negligible and (149) holds for every \( \phi \) in \( D(X_v) \).
In the sequel, we shall assume as in the beginning of § (6), that \( k \) is a global field, \( f(x) \) in \( k[x_1, \ldots, x_n] \) is homogeneous of degree \( m \geq 2 \) and not divisible by the characteristic of \( k \) and further, a tame \( k \)-resolution \( h_0 \) of the projective hypersurface defined by \( f(x) = 0 \) exists. We shall also use the notation of Lemma (6.2).

**Lemma 6.4.** If \( F^* \) is in \( L^1(k_v) \) for all but finitely many valuations \( v \) of \( k \), then \( h^\sharp \) has property \((P)\) everywhere.

**Proof.** Since \( h_0 \) is tame by assumption, all irreducible components of the total transform on \( Y_0 \) of the projective hypersurface defined by \( f(x) = 0 \) are defined over the separable closure \( k_s \) of \( k \). In view of the quasi-compactness of the Zariski topology and the Zariski density of the set of \( k_s \)-rational points on any irreducible variety defined over \( k_s \), we can choose a finite number of \( k_s \)-rational points of \( Y_0 \) such that the set of the numerical data at these points contains the numerical data at every point of \( Y_0 \). Let \( k_1 \) denote the extension field of \( k \) obtained by adjoining (the coordinates of) all the finitely many points mentioned above; then \( k_1 \) is a finite separable extension of \( k \) and as a result, there exist infinitely many non-archimedean valuations on \( k_1 \) with degree 1 relative to \( k \). (This is a consequence of the fact that the zeta function \( \zeta_k(s) \) of a global field is holomorphic for \( \text{Re}(s) > 1 \) and has a pole at \( s = 1 \)). We can choose one such valuation so that its restriction, say \( w \), to \( k \) is not one of the finitely many valuations excluded in the hypothesis; then the field \( k_1 \) is contained in \( k_w \) and \( F^*_w \) is in \( L^1(k_w) \). We have thus achieved a situation where the numerical data of \( h^\sharp \) at any point of \( Y^\sharp \) turn out to be also the numerical data of \( h \) at some point of \( Y_w \cap h^{-1}(X_0^0) \).

Since \( F^*_w \) is in \( L^1(k_w) \), \( F_w(0) \) exists and by Lemma 6.3, \( h \) has property \((P_0)\) at every point of \( Y_w \cap h^{-1}(X_0^0) \). But, then, by the construction, \( h^\sharp \) has property \((P_0)\) everywhere and hence, by Lemma 6.1, has property \((P)\) everywhere and the proof is complete. \( \square \)

As a consequence, by the last assertion in Lemma 6.3, (149) holds good, implying that \( F^*_{\phi_v} \) is in \( L^1(k_v) \) for every \( \phi_v \) in \( \mathcal{S}(X_v) \) and for every valuation \( v \) of \( k \).
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6.3 Standard Lemmas

For a non-archimedean valuation \( v \) of \( k \) and the characteristic function \( \phi_v \) of \( X^0_v \), we shall denote \( F_{\phi_v} \) simply by \( F_v \); then \( F^*_v = F_v^\ast \) is the Fourier transform of \( F_v \). Further, if \( F_v^\ast \) is in \( L^1(k_v) \), then

\[
F_v^\ast(\sigma \lambda) = \int_{k_v} F_v(\sigma \lambda \xi) \psi_v(-\sigma \lambda \xi) \left| \frac{d\sigma \lambda \xi}{v} \right|,
\]

for every \( \sigma \) in \( k_v \). In the following two lemmas, we shall assume that \( v \) is non-archimedean, \( \psi_v = 1 \) on \( R_v \), \( \psi_v \) is non-constant on \( P^{-1}_v \) and further \( f(x) \in R_v[x_1, \ldots, x_n] \). Anyway, these conditions hold good for all but finitely many \( v \).

**Lemma 6.5.** If there exists \( \sigma > 1 \) such that \( |F_v^\ast(\sigma \lambda)| \leq |\sigma \lambda|^{-\sigma} \) for every \( \sigma \) in \( k_v \), then, for every \( \sigma \) in \( R_v \), we have

\[
|F_v(\sigma \lambda) - 1| \leq c, q_v^{-(\sigma-1)}
\]

for a constant \( c \) depending only on \( \sigma \).

**Proof.** By our assumptions on \( v \) above, \( F_v^\ast = 1 \) on \( R_v \). Since \( X^0_v \) has measure 1 and since \( F_v^\ast \) is in \( L^1(k_v) \) in view of the hypothesis, we have

\[
F_v(\sigma \lambda) - 1 = \int_{k_v \setminus R_v} F_v^\ast(\sigma \lambda \xi) \psi_v(-\sigma \lambda \xi) \left| \frac{d\sigma \lambda \xi}{v} \right|
\]

for every \( \sigma \) in \( R_v \). Thus

\[
|F_v(\sigma \lambda) - 1| \leq \int_{k_v \setminus R_v} |\sigma \lambda|^{-\sigma} \left| \frac{d\sigma \lambda \xi}{v} \right|
\]

\[
= (1 - q_v^{-1})(1 - q_v^{-(\sigma-1)})^{-1}(\sigma-1) q_v
\]

\[
\leq (1 - 2^{-(\sigma-1)})^{-1} q_v^{-(\sigma-1)}
\]

and the lemma is proved. \( \square \)
For $i \in \mathbb{R}_v$, we recall that $U(i)_v^0$ is a compact subset of $U(i)_v$ defined as follows. Namely, if $i \neq 0$, then $U(i)_v^0$ is just $U(i)_v \cap X_0^0$ and for $i = 0$, it is the subset of $U(i)_v \cap X_0^0$, where, in addition, grad$_x f \neq 0( \mod P_v)$. We also recall that $r$ is the codimension of $C_f$ in $f^{-1}(0)$.

**Lemma 6.6.** Under the same assumptions as in LEMMA [6.5], we have, for every $i$ in $\mathbb{R}_v$,

$$\left| \int_{U(i)_v^0} |\theta|_v - 1 \right| \leq c'(q_v^{-(\sigma-1)} + q_v^{-r})$$

(151)

with a constant $c'$ independent of $i$ and $v$.

**Proof.** Since the left hand side of (151) is at most equal to $|F_v(i)| + 1 \leq c + 2$ (in view of Lemma [6.5]), we may exclude some more finitely many valuations while proving this lemma. We may, in particular, assume that $m \neq 0( \mod P_v)$. If $N_v(i)$ (respectively $N_v^0(i)$) is the number of $\xi$ in $X_0^0$ modulo $P_v^e$ such that $f(\xi) \equiv i( \mod P_v^e)$ (respectively $f(\xi) \equiv i( \mod P_v^e)$ and grad$_x f \neq 0( \mod P_v^e)$), for $e = 0, 1, 2, \ldots$ then

$$q_v^{-(\sigma-1)} N_v(i) = \int_{P_v^e} F_v^*(i^*) \psi_v(-i^*) |d\theta|_v.$$

As in the proof of Lemma [6.5] we have

$$|q_v^{-(\sigma-1)} N_v(i) - 1| \leq c q_v^{-(\sigma-1)}$$

for a constant $c$ independent of $i$ and $v$. But it is elementary to prove that $N_v(i) = N_v^0(i)$ for $i \notin P_v$ and further that $q_v^{-(\sigma-1)} (N_v(i) - N_v^0(i))$ is bounded uniformly for $i$ in $\mathbb{R}_v$ and all $v$ (151). Also, we have $q_v^{-(\sigma-1)} N_v^0(i) = \int_{U(i)_v^0} |\theta|_v$ for all but finitely many $v$. Putting these together, the estimate (151) follows. \hfill \Box

**Lemma 6.7.** Suppose that, for every $v$ in $S_{\infty}$, $\sigma_v > 0$ and further that, for every $i^*$ in $k_v$ and every $\phi_v$ in $\mathcal{S}(X_v)$, we have

$$|F_{\phi_v}(i^*)| \leq c(\phi_v) \max(1, |i^*|_v)^{-\sigma_v}.$$
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with a constant $c(\phi_v) > 0$. Let us write

$$X_\infty = \prod_{v \in S_\infty} X_v, \quad |dx|_\infty = \bigotimes_{v \in S_\infty} |dx|_v, \quad k_\infty = \prod_{v \in S_\infty} k_v$$

and let $C_\infty$ denote a compact subset of $\mathcal{S}(X_\infty)$. Then there exists a constant $c \geq 0$ depending on $C_\infty$ such that

$$\left| \int_{X_\infty} \left( \prod_{v \in S_\infty} \psi_v(i^* f(x)) \right) \phi_\infty(x) |dx|_\infty \right| \leq c \prod_{v \in S_\infty} \max(1, |i^*_v|_v)^{-\sigma_v}$$

(152)

for every $i^* = (i^*_v)_v$ in $k_\infty$ and every $\phi_\infty$ in $C_\infty$.

Proof. We first remark that, on the left hand side of (152), $\phi_\infty$ is not necessarily of the form $\bigotimes_{v \in S_\infty} \phi_v$ (with $\phi_v$ in $\mathcal{S}(X_v)$); this is just the “raison d'être” for this lemma.

If we put

$$T_{i^*}(\phi_v) = \max(1, |i^*_v|_v)^{\sigma_v} \cdot F_{\phi_v}(i^*_v),$$

then we get a subset $\{T_{i^*}\}$ of $\mathcal{S}(X_v)'$ parametrized by $k_v$. The hypothesis means precisely that this subset is “bounded” i.e. $|T_{i^*}(\phi_v)| \leq c(\phi_v)$ for every $i^*$ in $k_v$. The proof of the lemma now follows from the two well-known facts given below:

(i) For any bounded subset $B$ of $\mathcal{S}(\mathbb{R}^p)'$ and any compact subset $C$ of $\mathcal{S}(\mathbb{R}^p)$, we have

$$\sup_{T \in B, \phi \in C} |T(\phi)| < \infty;$$

(ii) For bounded subsets $B, B'$ respectively of $\mathcal{S}(\mathbb{R}^p)'$, $\mathcal{S}(\mathbb{R}^q)'$, the set $\{T \otimes T'; T \in B, T' \in B'\}$ is bounded in $\mathcal{S}(\mathbb{R}^{p+q})'$.

Lemma 6.8. Let $\sigma_v > 1$ for all $v$ and further, for all but finitely many $v$, let $\sigma_v \geq \sigma > 2$. Then

$$\sum_{i \in k_\Lambda} \prod_v \max(1, |i^*_v + j^*_v|_v)^{-\sigma_v}$$

is bounded as $j^*$ varies over $k_\Lambda$. 

□
Proof. Since \( k_A/k \) is compact, we may restrict \( j^* \) to a compact subset of \( k_A \). If \( v \) is non-archimedean and if \( j^*_v \) is in \( R_v \), then

\[
\max(1, |i^* + j^*_v|_v) = \max(1, |i^*_v|_v)
\]

for every \( i^* \) in \( k_v \). We may therefore assume \( j^* = 0 \) in the foregoing. In the special case, when \( k = \mathbb{Q} \) and \( \sigma_v = \sigma(> 2) \) for all \( v \), we have

\[
\prod_v \max(1, |i^*_v|_v) = \max(c, |d|)
\]

for \( i^* = d/c \) with \( c \geq 1 \) and \( (c, d) = 1 \). By a simple calculation, it can be shown that

\[
\sum_{i^* \in k} \prod_v \max(1, |i^*_v|_v)^{-\sigma} = \zeta(\sigma)(4\zeta(\sigma - 1) - \zeta(\sigma)) < \infty
\]

where \( \zeta(s) \) is the Riemann zeta function. For the case of any global field \( k \), the proof can be found in [19], page 187 and [21], page 226. \( \square \)

6.4 Proof of Theorem 2.1

We are now ready to prove that the criteria (C1) and (C2) imply the validity of the Poisson formula. From (C1), (C2) and Lemma 6.6, we see that (PF-2)' is valid i.e. the restricted product measure \( |\theta|_A \) exists on \( U(i)_A \) for every \( i \) in \( k \). Further, by Lemma 6.5, (PF-2)'' is valid i.e. the image measure under \( U(i)_A \rightarrow X_A \) is tempered. On the other hand, for every \( \phi \) in \( \mathcal{S}(X_A) \) and every \( i^* \) in \( k_A \), we set

\[
F^*_\phi(i^*) = \int_{X_A} \phi(x)\psi(i^* f(x))|dx|_A.
\]

We recall that \( \phi \) is a finite sum of functions of the form \( \phi_0 \otimes \phi_\infty \), in our earlier notation. We restrict \( \phi_\infty \) to compact subset \( C_\infty \) of \( \mathcal{S}(X_\infty) \); then there exists a constant \( c \geq 0 \) depending on \( \phi_0 \) and \( C_\infty \) such that

\[
\sum_{i^* \in k} |F^*_\phi(i^* + j^*)| \leq c \sum_{i^* \in k} \prod_v \max(1, |i^* + j^*_v|_v)^{-\sigma} < \infty
\]
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for every \( j^* \) in \( k_A \), as a consequence of (C2) and Lemmas 6.3, 6.4, 6.7, 6.8. This establishes the validity of (PF-1) i.e. the absolute convergence of the Eisenstein-Siegel series. Since \( \sum_{i^* \in k} F^*_\phi(i^* + j^*) \) represents a continuous periodic function of \( j^* \), we get the Fourier expansion

\[
\sum_{i^* \in k} F^*_\phi(i^* + j^*) = \sum_{i^* \in k} c^*_\phi(i) \psi(i j^*)
\]  

(153)

where

\[
c^*_\phi(i) = \int_{k_A} F^*_\phi(j^*) \psi(-i j^*) |dj|^A,
\]

provided that the series on the right hand side of (153) converges absolutely. This is certainly the case if the series is just a finite sum.

If \( \phi \) is of the form \( \bigotimes_v \phi_v \) with \( \phi_v \) in \( \mathcal{S}(X_v) \) for every \( v \) and further \( \phi_v \) equal to the characteristic function of \( X_0^0 \) for all but finitely many \( v \), then (C2) together with Lemmas 6.3, 6.4, 6.5 implies that

\[
c^*_\phi(i) = \prod_v \int_{k_v} F^*_\phi_v(i^*) \psi_v(-i i^*) |di|^v \]

(154)

provided that (PF-3) holds i.e. the right hand side of (154) is absolutely convergent. As remarked earlier, this is certainly true, if the series is a
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finite sum, as, for example, in the case of $\phi$ having compact support. Let us now consider the general case of $\phi$ in $\mathcal{S}(X_A)$, not necessarily having compact support. For any such $\phi$, we know that there exists $\mathcal{F} \geq 0$ in $\mathcal{S}(X_A)$ such that $|\phi(x)| \leq \varphi(x)$ for every $x$ in $X_A$. It is sufficient therefore to show that the right hand side of (152) converges for every $\phi \geq 0$ in $\mathcal{S}(X_A)$. Let us therefore assume that $\phi \geq 0$ and moreover, take $\phi$ to be of the form $\phi_0 \otimes \phi_\infty$ where $\phi_0$ is the characteristic function of a large compact open subset of $X_0$ and $\phi_\infty \geq 0$ in $\mathcal{S}(X_\infty)$. We can easily find a monotone increasing sequence $\{\phi_\infty,n\}_n$ of non-negative $C^\infty$ functions on $X_\infty$ with compact support which converges to $\phi_\infty$ in $\mathcal{S}(X_\infty)$; then $\{\phi_\infty,n\}_n \cup \phi_\infty$ is a compact subset of $\mathcal{S}(X_\infty)$. We put $\phi_n = \phi_0 \otimes \phi_\infty,n$ for $n = 1, 2, 3, \ldots$; then every $\phi_n$ is in $\mathcal{S}(X_A)$ and has compact support. Further the monotone increasing sequence $0 \leq \phi_1 \leq \phi_2 \leq \ldots$ converges to $\phi$. Therefore, for a constant $c \geq 0$, we have, in view of Lemma 6.7,

$$\sum_{i \in k} \int_{U(i)_A} \phi_n|\theta_i|_A \leq \sum_{i' \in k} |F_{\phi_n}(i')| \leq c < \infty$$

for $n = 1, 2, 3, \ldots$. Since the left hand side is the integral of $\phi_n$ over the (disjoint) union of $U(i)_A$ for all $i$ in $k$, we obtain, by Lebesgue’s theorem on a monotone increasing sequence of non-negative functions, that

$$\sum_{i \in k} \int_{U(i)_A} \phi|\theta_i|_A = \lim_{n \to \infty} \sum_{i \in k} \int_{U(i)_A} \phi_n|\theta_i|_A \leq c.$$ 

The proof of the validity of the Poisson formula is now complete.

**REMARK.** We observe that the proof of our theorem becomes much simpler in the function-field case, since the consideration of $X_\infty$ disappears. We might, moreover, mention that we could have applied to the left hand side of (153), the result of Weil on page 7 of [52] and thereby avoided the use of the monotone sequence $\{\phi_n\}_n$.

### 6.5 A Conjecture

We conclude this series of lectures by stating a conjecture: namely, if, with the notation of Lemma 6.2, we assume that the resolution
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$h^\#: Y^\# \to X^\#$ of $((f^\#)^0, H_{\infty})$ is tame and further that $\lambda_1 = \min_{\nu E \geq 2} (\nu E / N_E) > 1$, then for any $\sigma < \lambda_1$, we have

$$|F'_{\phi_v}(i^\#)| \leq c(\phi_v) \max(1, |i^\#|_v)^{-\sigma}$$

for every $\phi_v$ in $\mathcal{S}(X_v)$, all $i^\#$ in $k_v$ and all $v$. It is very likely that we can take $c(\phi_v) = 1$ for the characteristic function $\phi_v$ of $X^0_\nu$, for all but finitely many $v$. If this conjecture is true, then condition (C2) can be replaced by the simple geometric condition $\lambda_1 > 2$. 

Bibliography


