Lectures On
Irregularities Of Distribution

By

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Tata Institute of Fundamental Research,
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Preface

These lectures were given at the Tata Institute of Fundamental Research, Bombay, in the Fall of 1972. Excellent notes were taken by T. N. Shorey.

The theory of Irregularities of Distribution began as a branch of Uniform Distributions, but is of independent interest. The papers appearing in 1922 of Harday an Littlewood [8] and of Ostrowski [16] on fractional parts of sequences α, $2\alpha, \ldots$, may be regraded as forerunners of the general theory. The first papers dealing with the distribution of general sequences $x_1, x_2, \ldots$ are due to T. Van Aardenne Ehrenfest [1, 2] in 1945, 49, and K. F. Roth [19] in 1954.

In these lectures I restricted myself to distribution problems with a geometric interpretation. Unfortunately, because of lack of time, it was not possible to include the important results of K. F. Roth on irregularities of distribution of integer sequences with respect to arithmetic progressions.

December, 1973

Wolfgang M. Schmidt
Boulder, Colorado

Contents

Chapter 1

A Quantitative Theory of Uniform Distribution

1 The Uniform Distribution of a sequence in an interval or in a cube

1

Denote by U the unit interval $0 \leq x \leq 1$, and by I any sub-interval of U. (We shall allow open, closed, half open intervals, as well as single points and the empty set ϕ). Denote the length of I by $|I|$. Let $x_1, x_2, \ldots$ be a sequence of numbers in U. For every interval I, put

$$z(n, I) = \sum_{\substack{1 \leq i \leq n \\ x_i \in I}} 1.$$

The sequence $x_1, x_2, \ldots$ is called *uniformly distributed*, if for every I we have the asymptotic relation

$$z(n, I) \approx n|I|,$$

that is, if $z(n, I)/(n|I|)$ tends to 1 as n goes to infinity. Set

$$D(n, I) = z(n, I) - n|I|,$$
$$\triangle(n) = \sup_{I} |D(n, I)|,$$

where the supermum is taken over all the sub-intervals of U. The function $\triangle(n)$ is called the *discrepancy* function.

Let $\mathfrak{C}$ be a finite collection of sub-intervals of U with $\phi\epsilon\mathfrak{C}, U\epsilon\mathfrak{C}$.

For an arbitrary I, write

$$\delta_{\mathfrak{C}}(I) = \min_{\substack{I_1, I_2 \epsilon \mathfrak{C} \\ I_1 \subseteq I \subseteq I_2}} (|I_2| - |I_1|)$$

Further set

$$\delta_{\mathfrak{C}} = \sup_I \delta_{\mathfrak{C}}(I),$$

2 where the supermum is taken over all the sub-intervals of U, and put

$$\Delta_{\mathfrak{C}}(n) = \sup_{I\epsilon\mathfrak{C}} |D(n, I)|.$$

We claim that

$$\delta(n) \leq \Delta_{\mathfrak{C}}(n) + n\delta_{\mathfrak{C}} \tag{1.1}$$

The proof is as follows. Let I be arbitrary. Since $\mathfrak{C}$ is a finite collection, there exist intervals $I_1, I_2\epsilon\mathfrak{C}$ with

$$I_1 \subseteq I \subseteq I_2 \quad \text{and} \quad |I_2| - |I_1| \leq \delta_{\mathfrak{C}}.$$

Now

$$\begin{aligned} D(n, I) &= z(n, I) - n|I| \\ &\leq z(n, I_2) - n|I_2| + n(|I_2| - |I|) \\ &\leq D(n, I_2) + n\delta_{\mathfrak{C}} \\ &\leq \triangle_{\mathfrak{C}}(n) + n\delta_{\mathfrak{C}}. \end{aligned}$$

A lower bound for $D(n, I)$ may be proved similarly, so that $|D(n, I)| \leq \triangle_{\mathfrak{C}}(n) + n\delta_{\mathfrak{C}}$.

Since this is true for every I, we get (1.1).

Lemma 1A. *A sequence is uniformly distributed if and only if*

$$\triangle(n) = o(n)^1.$$

[1]The notation $g(n) = o(f(n))$ means that $g(n)/f(n) \to 0$ as $n \to \infty$.

Proof. $\triangle(n) = o(n)$ implies that $D(n, I) = o(n)$ for any I, which is equivalent to $z(n, I) \approx n|I|$ for any I. Hence the sequence is uniformly distributed. $\square$

To go in the opposite direction, it will be convenient to introduce the symbol $\dot{<}$. The notation $x \dot{<} \beta$ we shall mean $x < \beta$ if $\beta \neq 1$, and $x \leq \beta$ if
$\beta = 1$. For a positive integer, h, denote by $\mathfrak{C}_h$ the collection of 3
sub-intervals of U of the type $\frac{u}{2^h} \leq x \dot{<} \frac{v}{2^h}$ with integers u, v. Since $\mathfrak{C}_h$ is a finite collection of intervals, the uniform distribution implies that $\triangle_{\mathfrak{C}_h}(n) = o(n)$. Further observe that $\delta_{\mathfrak{C}_h} \leq 2.2^{-h}$. Thus for any given $\epsilon > 0$ we can choose h with $\delta_{\mathfrak{C}_h} < \frac{\epsilon}{2}$. Using (1), we obtain

$$\triangle(n) \leq \triangle_{\mathfrak{C}_h}(n) + \frac{\epsilon}{2} n.$$

In view of $\triangle_{\mathfrak{C}_h}(n) = o(n)$, we get $\triangle(n) \leq \epsilon n$ for large n. This completes the proof of Lemma 1A.

We are interested in sequences which are very well uniformly distributed, i.e. which have $\triangle(n) \ll f(n)$[1] where $f(n)/(n)$ tends to zero very rapidly. The following is an example of such a sequence. (See also Theorem 1D.)

Every real number $\mathfrak{I}$ may uniquely be written as a sum

$$\mathfrak{I} = [\mathfrak{I}] + \{\mathfrak{I}\},$$

where $[\mathfrak{I}]$, the "integer part of $\mathfrak{I}$", is an integer, and where$\{\mathfrak{I}\}$, the "fractional part of $\mathfrak{I}$", satisfies $0 \leq \{\mathfrak{I}\} < 1$.

Theorem 1B*. [2] *[12] Suppose α is a real number which has bounded partial denominators in its continued fraction expansion. Then the sequence $\{\alpha\}, \{2\alpha\}, \{3\alpha\}, \ldots$ has*

$$\triangle(n) \ll \log n.$$

[1]The notation $g(n) \ll f(n)$ (which is due to Vinogradov) means that $|g(n)/f(n)|$ is bounded as a function of n.

[2]Theorems with an attached star are not proved in these lectures.

The following result shows that it is not possible to improve Theorem 1B* (except for giving an explicit value for the constant implies by $\ll$). Recall that the notation

$$f(n) = \Omega(g(n))$$

4 means that $f(n)/g(n)$ does *not* tend to 0 as $n \to \infty$.

Theorem 1C*. *[8]-[16] For an arbitrary real number α, the sequence*

$$\{\alpha\}, \{2\alpha\}, \dots$$

has

$$\triangle(n) = \Omega(\log n).$$

We now shall study the discrepancy function of the sequence of Vander Corput:

$$\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{5}{8}, \frac{3}{8}, \frac{7}{8}, \frac{1}{16}, \dots.$$

The method of constructing the above sequence is illustrated by the scheme

$$\begin{array}{l}
\frac{1}{2}, \\
\frac{1}{4}, \frac{1}{4} + \frac{1}{2}, \\
\frac{1}{8}, \frac{1}{8} + \frac{1}{2}, \frac{1}{8} + \frac{1}{4}, \frac{1}{8} + \frac{1}{4} + \frac{1}{2}. \\
\dots\dots\dots\dots\dots\dots\dots\dots
\end{array}$$

Theorem 1D. *[29] The above sequence has*

$$\triangle(n) \ll \log n.$$

Proof. Call $I \subseteq U$ an *elementary interval* if it is of the type $\frac{w}{2^k} \le x \mathrel{\dot{<}} \frac{w+1}{2^k}$, where w is an integer. For a positive integer h, denote by $\mathfrak{C}$ the collection of sub-interval of U of the type $\frac{u}{2^h} \le x \mathrel{\dot{<}} \frac{v}{2^h}$, with integers u, v. Note that $\delta_{\mathfrak{C}_h} \le 2^{1-h}$. □

We shall show in a moment that

$$|D(n, I)| \leq 1| \tag{1.2}$$

for every elementary interval I. 5

It is readily seen by induction on h that an interval of $\mathfrak{C}_h$ is the union of not more than $2h$ disjoint elementary intervals. So (1.2) implies that

$$\triangle_{\mathfrak{C}_h}(n) \leq 2h,$$

since the function $D(n, I)$ is additive (i.e., $D(n, I) = D(n, I) + D(n, I_2)$, if $I = I_1 \cup I_2$ and $I_1 \cap I_2 = \phi$). Using (1.1). we get

$$\triangle(n) \leq 2h + n2^{1-h}.$$

Setting $h = [\log_2 n]$[3] $+1$, we obtain

$$\triangle(n) \leq 2(\log_2 n + 1) + 2 \leq \frac{2}{\log 2} \log n + 4 \ll \log n.$$

It remains to prove (1.2). Let I be the arbitrary elementary interval $\frac{w}{2^k} \leq x \dot{<} \frac{w+1}{2^k}$. Write the integer n in the dyadic scale,

$$n = a_t \ldots a_0.$$

Notice that Vander Corput's sequence $x_1 = \frac{1}{2}, x_2 = \frac{1}{4} \ldots$ has

$$x_n = 0. \quad a_0 \ldots a_t \qquad \text{(in the dyadic scale).}$$

It follows that $x_n \epsilon I$ precisely if w above has the dyadic expansion $w = a_0 a_1 \ldots a_{k-1}$. Thus for fixed w, we have $x_n \epsilon I$ precisely if n lies in a certain fixed residue class $\mod 2^k$. Hence

$$D(n, I) = |z(n, I) - \frac{n}{2^k}| \leq 1.$$

This completes the proof of Theorem 1D.

Now we generalise to k dimensions. Points in k - dimensional space will be written as $\underline{\underline{x}} = (x_1, \ldots, x_k)$. Denote by U^k the unit cube consist- 6

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[3]$\log_2 x$ is the logarithm of x with base 2.

ing of $\underline{\underline{x}} = (x_1, \ldots, x_k)$ with $x_i \epsilon U (1 \leq i \leq k)$.

A set $B \subseteq U^k$ will be called a *box* if it is a Cartesian product $I_1 \times \ldots \times I_k$ of intervals $I_1, \ldots, I_k$. If, for example, $I_j = [a_j, b_j], 1 \leq j \leq k$, then B consists of $\underline{\underline{x}} = (x_1, \ldots, x_k)$ with $a_j \leq x_j \leq b_j (j = 1, \ldots, k)$.

Let $\underline{\underline{x}}_1, \underline{\underline{x}}_2, \ldots$ be a sequence of points in U^k. Given a box B, put

$$z(n, B) = \sum_{\substack{1 \leq i \leq n \\ \underline{\underline{x_i}} \epsilon B}} 1.$$

The sequence $\underline{\underline{x}}_1, \underline{\underline{x}}_2, \ldots$ is called *uniformly distributed,* if for every box B we have the asymptotic relation

$$z(n, B) \approx n|B|,$$

where $|B|$ denote the volume of B. Set

$$D(n, B) = z(n, B) - n|B|,$$
$$\triangle(n) = \sup_B |D(n, B)|,$$

where the supremum is taken over all the boxes B in U^k. Here $\triangle(n)$ is called the *discrepancy* function.

Let $\mathfrak{C}$ be a finite collection of boxes with $\phi \epsilon \mathfrak{C}, U^k \epsilon \mathfrak{C}$. For an arbitrary box B, define

$$\delta_{\mathfrak{C}}(B) = \sup_{\substack{B_1, B_2 \epsilon \mathfrak{C} \\ B_1 \subseteq B \subseteq B_2}} (|B_2| - |B_1|),$$
$$\delta_{\mathfrak{C}} = \sup_B \delta_{\mathfrak{C}}(B),$$

7 where the supremum is taken over all the boxes B in U^k, and put

$$\triangle_{\mathfrak{C}}(n) = \sup_{B \epsilon \mathfrak{C}} |D(n, B)|.$$

We remark that the the inequality (1.1) can be established in this general set-up, and Lemma 1A can also be generalised.

Given a point $\underline{\underline{x}} = (x_1, \ldots, x_k)$, write

$$\underline{\underline{x}} = (\{x_1\}, \ldots \{x_k\}).$$

Let $\underline{\underline{\alpha}}$ be arbitrary in k-dimensional space, and consider the sequence $\{\underline{\underline{\alpha}}\}, \{2\underline{\underline{\alpha}}\}, \ldots$. One would like to have points $\underline{\underline{\alpha}}$ fro which this sequence has $\triangle(n) \ll (\log n)^k$. If $k > 1$, it is not known whether such an $\underline{\underline{\alpha}}$ exists, and hence there is no satisfactory analogue of Theorem 1B. It is not even known if an $\underline{\underline{\alpha}}$ exists for $k > 1$ such that $\triangle(n) \ll (\log n)^{k+1}$. However, it was proved [20] that for "almost every $\underline{\underline{\alpha}}$" (in the sense of Lebesgue measure) $\triangle(n) \ll (\log n)^{k+1+\epsilon}$ for every $\epsilon > 0$.

Like for Theorem 1B*, no satisfactory analogue of Theorem 1C* is known. For an analogue of Theorem 1D, we now turn to the sequence of Hammersley [7].

Let $p_1, \ldots, p_k$ be integers greater than 1 and relatively prime in pairs (i.e., $(p_i, p_j) = 1$ whenever $i \neq j$). (The simplest choice is to take $p_1, \ldots, p_k$ to be the first k primes). For a positive integer n, write

$$\begin{aligned} n &= a^{(p_1)}_{w_1} \ldots a^{(p_1)}_0 \qquad \text{(in the scale of} p_1),\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ n &= a^{(p_k)}_{w_k} \ldots a^{p_k}_0 \qquad \text{(in the scale of} p_k). \end{aligned}$$

Put

$$\begin{aligned} x_{n1} &= 0.a^{(p_1)}_0 \ldots a^{(p_1)}_{w_1} \qquad \text{(in the scale of} p_1)\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ x_{nk} &= 0.a^{(p_k)}_0 \ldots a^{(p_k)}_{w_k} \qquad \text{(in the scale of} p_k). \end{aligned}$$ **8**

Set

$$\underline{\underline{x}}_n = (x_{n1}, \ldots, x_{nk}) \qquad (n = 1, 2, \ldots)$$

The sequence $\underline{\underline{x}}_1, \underline{\underline{x}}_2$ so constructed is called *Hammersley's sequence*. (For $k = 1$, $p_1 = 2$ it gives Van der Corput's sequence).

Theorem 1E. *[6] Hammersley's sequence has*

$$\triangle(n) \ll (\log n)^k.$$

Proof. Let $\mathfrak{C}_h^j (1 \leq j \leq k)$ be the collection of intervals in U of the type $\frac{u}{p_j^h} \leq x \dot{<} \frac{v}{p_j^h}$, with integers u, v. Let $\mathfrak{C}_h = \mathfrak{C}_h^1 \times \ldots \times \mathfrak{C}_h^k$, i.e. let $\mathfrak{C}_h$ be the collection of all the boxes $B = I_1 \times \ldots \times I_k$ with $I_j \epsilon \mathfrak{C}_h^j (j = 1, \ldots, k)$. □

We claim that

$$\delta_{\mathfrak{C}_h} \leq \delta_{1_{\mathfrak{C}_h}} + \ldots + \delta_{k_{\mathfrak{C}_h}}. \tag{1.3}$$

Let $B = I_1 \times \ldots \times I_k$ be an arbitrary box. Choose intervals

$$I_1^1, \ldots, I_1^k; I_2^1, \ldots, I_2^k$$

such that

$$I_1^i \subseteq I_j \subseteq I_2^j, I_1^j, I_2^j \epsilon \mathfrak{C}_h^j (j = 1, \ldots, k).$$

and

$$|I_2^j| - |I_1^j| \leq \delta_{\mathfrak{C}_h^j} (j = 1, \ldots, k).$$

9 Set $B_1 = I_1^1 \times \ldots \times I_1^k$ and $B_2 = I_2^1 \times \ldots \times I_2^k$. Then $B_1 \subseteq B \subseteq B_2$ and $B_1, B_2 \epsilon \mathfrak{C}_h$.

Further observe that

$$\delta_{\mathfrak{C}_h}(B) \leq |B_2| - |B_1| \leq \delta_{\mathfrak{C}_h^1} + \ldots + \delta_{\mathfrak{C}_h^k}.$$

Since this is true for any box $B \epsilon \mathfrak{C}_h$, we obtain (1.3).

Notice that $\delta_{\mathfrak{C}_h^j} \leq 2p_j^{-h} (1 \leq j \leq k)$, and so (1.3) gives

$$\delta_{\mathfrak{C}_h} \leq 2(p_1^{-h} + \ldots + p_k^{-h}) \leq 2^{1-h}k. \tag{1.4}$$

An interval of the type $\frac{w}{p_j^t} \leq x \dot{<} \frac{w+1}{p_j^t}$, where w is an integer, is called an *elementary* p_j *-type interval*. A box $B = I_1 \times \ldots \times I_k$ is called an elementary box if each I_j is an *elementary* p_j -type interval.

Let $B = I_1 \times \ldots \times I_k$ be an elementary box with I_j given by

$$\frac{w_j}{p_j^{t_j}} \leq x < \frac{w_j + 1}{p_j^{t_j}} (1 \leq j \leq k).$$

Thus $\underline{\underline{x}}_n \epsilon B$ if and only if

$$\frac{w_j+1}{p_j^{t_j}} \leq x_{nj} < \frac{j^{w+1}}{p_j^{t_j}} (1 \leq j \leq k).$$

Since $x_{nj} < 1$ for the Hammersley sequence, we may replace $\dot{<}$ by $<$ here. For fixed j, the inequalities above determine t_j digits in the expansion of x_{nj} in the scale of p_j, hence determines t_j digits in the expansion of n in the scale of p_j. Thus for fixed j, the inequalities hold if and if n lies in a fixed residue class modulo $p_j^{t_j}$. Hence by the Chinese Remainder Theorem, $\underline{\underline{x}}_n \in B$ precisely if n lies in a fixed residue class modulo $p_1^{t_1} \dots p_k^{t_k}$.

Hence 10

$$D(n,B) = |z(n,B) - \frac{n}{p_1^{t_1} \dots p_k^{t_k}}| \leq 1. \tag{1.5}$$

As is easily seen by induction on h, every interval I of $\mathfrak{C}_h^i$ is a disjoint union of not more than $2(p_i-1)h < 2p_i h$ elementary p_i-type intervals.

Hence every box B of $\mathfrak{C}_h$ is the union of not more than $(2p_1h)\dots(2p_kh) = (2h)^k p_1 \dots p_k$ elementary boxes. We therefore see form (1.1), (1.4) and (1.5) that

$$\triangle(n) \leq (2h)^k p_1 \dots p_k + n2^{1-h}k.$$

Setting $h = [\log_2 n] + 1$, we get

$$\triangle(n) \leq (2\log_2 n + 2)^k p_1 \dots p_k + 2k \ll (\log n)^k.$$

2 Roth's Theorem

Assume $k = 1$. One can ask if there exists a sequence with $\triangle(n) \ll 1$. Van der Corput [29] conjectured that $\triangle(n) \ll 1$ is impossible. Aardenne-Ehrenfest [1] proved the conjecture. Later [2] she further showed that $\triangle(n) = \Omega(\log\log n / \log\log\log n)$. Then K. F. Roth [19] improved it to $\Omega(\sqrt{\log n})$ and W. M. Schmidt [25] to $\Omega(\log n)$.

When k is arbitrary, $\triangle(n) = \Omega((\log n)^{k/2})$, which is contained in the following theorem of K. F. Roth [19].

Theorem 2A. *Let $\underline{\underline{x}}_1, \underline{\underline{x}}_2, \ldots$ be a sequence in U_k. If $N > c_k$, then there exists an n, $1 \leq n \leq N$, such that*

$$\triangle(n) > c'_k(\log N)^{k/2}. \tag{2.1}$$

Here $c_k > 0, c'_k > 0$ are constants depending only on k.

Remark. It follows from Theorem 2A that there exist infinitely many
11 n satisfying $\triangle(n) > c'_k(\log n)^{k/2}$. If $n, 1 \leq n \leq N$, satisfies (2.1), then $n \geq \triangle(n) \geq c'_k(\log N)^{k/2}$, and so $n \to \infty$ as $N \to \infty$. Hence $\triangle(n) = \Omega((\log n)^{k/2})$.

Let $\underline{\underline{p}}_1, \ldots, \underline{\underline{p}}_N$ be N points in U^k. If A is a measurable subset of U^k with measure $\mu(A)$, put $Z(A)$ for the number of $i(1 \leq i \leq N)$ for which $\underline{\underline{p}}_i \epsilon A, D(A) = Z(A) - N(A) - N\mu(A)$.

Theorem 2B. *There exists a box B with*

$$|D(B)| > d'_k(\log N)^{(k-1)/2} \qquad \textit{if} \quad N > d_k.$$

Here $d_k > 0$, $d'_k > 0$ are constants depending only on k.

Roth observed: *The case k* of Theorem 2A is equivalent to the case $(k+1)$ of Theorem 2B. We shall prove this equivalence for $k = 1$, as the proof for arbitrary k is similar. We shall first show that

THEOREM 2B with $k = 2$ implies THEOREM 2A with $k = 1$.

BY Theorem 2B with $k = 2$, there exists for large N a box B satisfying

$$|D(B)| > d'_2\sqrt{\log N}. \tag{2.2}$$

Introduce the notations $B(x, y) = [0, x] \times [0, y]$, $B(x-, y) = [0, x) \times [0, y]$, $B(x, y-) = [0, x] \times [0, y)$ and $B(x-, y-) = [0, x) \times [0, y)$ for boxes. Put

$$Z(x, y) = Z(B, (x, y))$$
$$D(x, y) = D(B(x, y)) = Z(x, y) - Nxy.$$

Similarly define $Z(x-, y), D(x-, y)$, etc. Assume at the moment that the box B of (2.2) is of the type

$$B : (\zeta, \eta), a < \zeta \leq b, c < \eta \leq d.$$

Then

$$Z(B) = Z(b,d) - Z(a,d) - Z(b,c) + Z(a,c).$$

and therefore **12**

$$D(B) = D(b,d) - D(a,d) - D(b,c) + D(a,c).$$

Since $|D(B)| > d_2' \sqrt{logN}$, there exists a point (x_0, y_0) (with $x_0 = a$ or b, $y_0 = c$) such that

$$|D(x_o, y_0)| > \frac{d_2'}{4} \sqrt{\log N}.$$

If B is of some other type, we may come up with, say,

$$|D(x_o-, y_0)| > \frac{d_2'}{4} \sqrt{\log N}.$$

If $D(x_0-, y_0) > 0$, then $D(x_0, y_0) \geq D(x_0-, y_0) > \frac{d_2'}{4} \sqrt{\log N}$. If $D(x_0-, y_0) < 0$, choose $\hat{x}_0$ with $x_0 - \frac{1}{N}(|D(x_0-, y_0)| - \frac{d_2'}{4} \sqrt{\log N}) < \hat{x}_0 < x_0$. Then $|D(\hat{x}_0, y_0)| \geq -D(\hat{x}_0, y_0) = N\hat{x}_0 y_0 - Z(\hat{x}_0, y_0) \geq N x_0 y_0 - Z(x_0-, y_0) - Ny_0(x_0 - \hat{x}_0) > |D(x_0-, y_0)| - (|D(x_0-, y_0))| - \frac{d_2'}{4} \sqrt{\log N}) = \frac{d_2'}{4} \sqrt{\log N}$.

In this way one sees that there is always on x_0, y_0 with $|D(x_0, y_0)| > \sqrt[c]{\log N}$ with $x = d_2'/4$.

Let $x_1, \ldots, x_N$ be in $U^1 = U$. Apply what we just said to the N points $\underline{\underline{p}}_1 = (x_1, \frac{1}{N}), \ldots, \underline{\underline{p}}_N = (x_N, \frac{N}{N})$. Let (x_0, y_0) be such that

$$|D(x_0, y_0)| > \sqrt[c]{\log N}.$$

Let I be the interval $0 \leq x \leq x_0$. Put $n = [Ny_0]$. Then $1 \leq n \leq N$ is large.

For if we had $n = 0$, then $y_0 < 1/N$, whence $1 > |Nx_0 y_0| = |D(x_0, y_0)| > \sqrt[c]{\log N}$. which is not possible for large N. Observe that

$$\begin{aligned} z(n, I) &= \text{the number of i with} 1 \leq i \leq n, x_i \epsilon I, \quad \text{h}ence \\ &= \text{the number of i for which } \underline{\underline{p}}_i = (x_i, \frac{i}{N}) \qquad \text{has} \end{aligned}$$

$$
\begin{aligned}
&0 \le x_i \le x_0, \frac{i}{N} \le \frac{n}{N}, \quad \text{hence} \\
&= Z\left(x_0, \frac{n}{N}\right) \\
&= Z(x_0, y_0), \quad \text{since} \frac{n}{N} \le y_0 < \frac{n+1}{N}.
\end{aligned}
$$

13 Further

$$
\begin{aligned}
|D(n, I)| &= |z(n, I) - n|I|| \\
&= |Z(x_0, y_0) - [Ny_0]x_0| \\
&\ge |Z(x_0, y_0) - Nx_0y_0| - |Nx_0y_0 - [Ny_0]x_0| \\
&\ge |D(x_0, y_0)| - 1 > \frac{c}{2}\sqrt{\log N}, \text{ if}
\end{aligned}
$$

N is large. Thus $\triangle(n) > \frac{c}{2}\sqrt{\log N}$.

We next are going to show that THEOREM 2A with $k = 1$ implies THEOREM 2B with $k = 2$.

Let $\underline{\underline{p}}_1 = (x_1, y_1), \ldots, \underline{\underline{p}}_N = (x_N, y_N)$ be any N points in U^2. We may assume, without loss of generality, that

$$y_1 \le y_2 \ldots \le y_N.$$

We construct new points

$$\underline{\underline{p}}^*_1 = (x_1, \frac{1}{N}), \ldots, \underline{\underline{p}}^*_N = (x_N, \frac{N}{N}).$$

Apply Theorem 2A, with $k = 1$, to the points $x_1, \ldots x_N$; there exists an integer n, $1 \le n \le N$, such that

$$\triangle(N) > c'_1 \sqrt{\log N}.$$

i. e., there exists an interval I with

$$|z(n, I) - n|I|| > c'_1 \sqrt{\log N}.$$

Suppose I is, say, $a < x \le b$. If we define $z(n, x)$ as the number of i in $1 \le i \le n$ with $0 \le x_i \le x$, then

$$z(n, I) - n|I| = (z(n, b) - nb) - (z(n, a) - na).$$

So there exists an x_0 (namely $x_0 = a$ or $x_0 = b$) with

$$|z(n, x_0) - nx_0| > \frac{c_1'}{2}\sqrt{\log N}.$$

Even if I is of any other type, we can conclude that there exists an x_0 with the above property. 14

Let

$$M = \sup_{(x,y)\epsilon U^2} |Z(x,y) - Nxy|,$$
$$M^* = \sup_{(x,y)\epsilon U^2} |Z^*(x,y) - Nxy|,$$

where $Z^*(x,y)$ is the number of $i(1 \leq i \leq N)$ with $\underline{\underline{p}}_i^* \epsilon B(x,y)$. Put $y_0 = \frac{n}{N}$.

Note that $Z^*(x_0, y_0) = z(n, x_0)$ and

$$|Z^*(x_0, y_0) - Nx_0y_0| = |z(n, x_0) - nx_0| > \frac{c_1'}{2}\sqrt{\log N}.$$

Hence

$$M^* > \frac{c_1'}{2}\sqrt{\log N}.$$

If we can show that $|M - M^*| \leq 3M$, then we are through, since then $M \geq \frac{1}{4}M^* \geq \frac{c_1'}{8}\sqrt{\log N}$. So *what remains to be proved is that* $|M - M^*| \leq 3M$.

Observe that for every $(x,y)\epsilon U^2$,

$$|Z(x,y) - Z^*(x,y)| \leq \max(t, t'),$$

where t denotes the number of i with $y_i \leq y < \frac{i}{N}$ and t' the number of i with $\frac{i}{N} \leq y < y_i$. It suffices to prove that $\max(t,t') \leq 3M$, since

$$\Big||Z(x,y) - Nxy| - |Z^*(x,y) - Nxy|\Big| \leq \Big|Z(x,y) - Z^*(x,y)\Big|.$$

From the definition of M, at most $2M$ numbers y_i can be equal. (If more than 2M were equal, either $Z(1,0) > 2M$ whence $|Z(1,0) - N.1.0| >$

$2M$, which is impossible. Or $Z(1, y)$ would have a 'jump' greater than $2M$ for some $y > 0$, hence $Z(1, y) - Ny$ would have a 'jump' $> 2M$ and $\sup\limits_{0 \leq y \leq 1} |Z(1, y) - Ny| > M$, which contradicts the definition of M). It follows that

$$|Z(1, y_i) - i| \leq 2M - 1 \qquad (1 \leq i \leq N).$$

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By the definition of M,

$$|Z(1, y_i) - ny_i| \leq M \qquad (1 \leq i \leq N).$$

Combining the above two inequalities, we obtain

$$|Ny_i - i| \leq 3M - 1 \qquad (1 \leq i \leq N) \tag{2.3}$$

By the definition of t,

$$y_{j+1} \leq \ldots y_{j+t} \leq y < \frac{j+1}{N} \ldots < \frac{j+t}{N}.$$

Observe that

$$j + t = j + 1 + (t - 1) > Ny + (t - 1), y_{j+t}N \leq Ny.$$

Combining these inequalities, we obtain

$$|(j + t) - Ny_{j+t}| \geq (t - 1).$$

This last inequality together with (2.3) yields $t \leq 3M$. Similarly $t' \leq 3M$. Hence $\max(t, t') \leq 3M$. This completes the proof.

More generally, one sees that Theorem 2A for a particular k is equivalent to Theorem 2B with $k + 1$. Since Theorem 2B is trivial for $k = 1$, the Theorem 2A and 2B are equivalent.

Let $uuby_1, uuby_2, \ldots$ be Hammersley's sequence in $(k - 1)$ -dimensional space.

This sequence has $\triangle(n) \ll (\log n)^{k-1}$. The points $\underline{\underline{x}}_1 = (\underline{\underline{y}}, \frac{1}{N}), \underline{\underline{x}}_2 = (\underline{\underline{y}}_2, \frac{2}{N}), \ldots, \underline{\underline{x}}_N = (\underline{\underline{y}}_N, 1)$ lie in U^k. An argument given above shows that

$$|D(B)| \leq \overline{c}_k (\log N)^{k-1}$$

for every box B with sides parallel to the axes.

Theorem 2C (Roth). *Let* $\underline{\underline{p}}_1, \ldots, \underline{\underline{p}}_N$ *be* N *points in* U^k. *Put* $Z(x_1, \ldots, x_k)$ *for the number of* $i, 1 \leq i \leq N$, *for which* $\underline{\underline{p}}_i$ *lies in the box* $B(x_1, \ldots, x_k)$ *consisting of points* $\underline{\underline{p}} = (p_1, \ldots, p_k)$ *with* $0 \leq p_j \leq x_j (j = 1, \ldots, k)$. 16

If $N > e_k$, then

$$\int_{U^k} \cdots \int (Z(x_1, \ldots, x_k) - N x_1 \ldots x_k)^2 dx_1 \ldots dx_k > e'_k (\log N)^{k-1}.$$

Here $e_k > 0, e'_k > 0$ are constants depending only on k.

In particular, there exists a k-tuple $(x_1, \ldots, x_k)$ with

$$|Z(x_1, \ldots, x_k) - N x_1 \ldots x_k| > \sqrt{e'_k} (\log N)^{(k-1)/2}.$$

Hence Theorem 2C implies Theorem 2B.

[Added in June 1976: Recently W. M. Schmidt in a manuscript "Irregularities of Distribution X" extended Theorem 2C by showing that for $r > 1$,

$$\int_{U^k} \cdots \int \left| Z(x_1, \ldots, x_k) - N x_1 \ldots x - k \right|^r dx_1 \ldots dx_k > e'_{kr} (\log N)^{(k-1)r/2}.$$

Moreover, for $k > 1$,

$$\int_{U^k} \cdots \int \left| Z(x_1, \ldots, x_k) - N x_1 \ldots x_k \right| dx_1 \ldots dx_k$$
$$> e''_k \log \log N / \log \log \log N.]$$

3 Proof of Roth's Theorem

We shall prove Theorem 2C. For convenience we shall restrict ourselves to the case $k = 2$. We have to show that

$$\int_0^1 \int_0^1 (Z(x, y) - Nxy)^2 dx dy > e'_2 \log N$$

if N is large. For $x \epsilon U$, write

$$x = \sum_{j=0}^{\infty} \beta(x) 2^{-j} \quad \text{(dyadic expansion)},$$

where $\beta_j(x)$ is 0 or 1. Assume that $\beta_j(x)$ never equals 1 for all $j \geq j_0$, except for $x = 1$. Then the dyadic expansion for x is unique. Put

$$\psi_r(x) = (-1)^{\beta_r(x)} \quad (r = 1, 2, \ldots).$$

(The $\psi_r(x)$ are called 'Rademacher - functions'.)

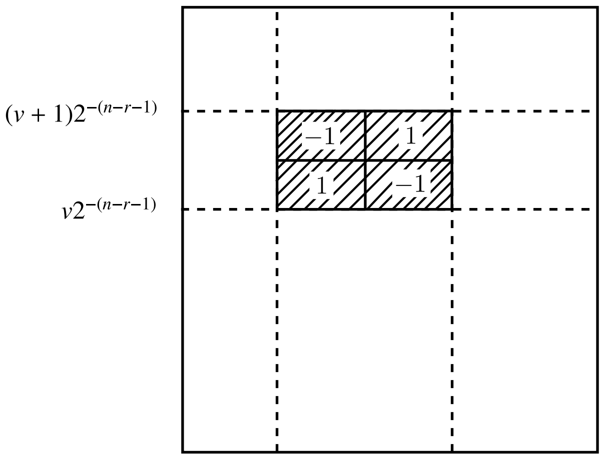
17 Let $\underline{\underline{p}}_1 = (x_1, y_1), \ldots, \underline{\underline{p}}_N = (x_N, y_N)$ be any N points in U^2. Let $n > 1$ be an integer. Further assume that r is an integer with $0 < r < n$.

For $(x, y) \epsilon U^2$, set

$$F_r(x, y) = \begin{cases} 0, & \text{if there is a } \underline{\underline{p}}_i = (x_i, y_i) \text{with} \\ & \psi_1(x_i) = \psi_1(x), \ldots, \psi_{r-1}(x_i) = \psi_{r-1}(x), \\ & \psi_1(y_i) = \psi_1(y), \ldots \psi_{n-r-1}(y_i) = \psi_{n-r-1}(y). \\ & \psi_r(x)\psi_{n-r}(y), \quad \text{otherwise .} \end{cases}$$

We explain the function $F_r(x, y)$ with the help of the following diagram.

U^2 :



Here u, v are integers. $F_r(x, y) = 0$, if there is a point $\underline{\underline{p}}_i$ in the shaded "big" rectangle A. If there is no point $\underline{\underline{p}}_i$ in the A, then $F_r(x, y) = \pm 1$

in the pattern shown, i. e. , it is $+1$ and -1 in two of the four "small" rectangles in A.[4]

Let y be fixed and let I be an interval whose end points are integral multiples of $2^{-(r-1)}$. Then

$$\int_I F_r(x,y)dx = 0.$$

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This is true because in any interval

$$u2^{-(r-1)} < x < (u+1)2^{-(r-1)},$$

either $F_r(x,y) = 0$, or $F_r(x,y) = \pm 1$ in sub - intervals of equal length.

Similarly fix x and take an interval I whose end points are integral multiples of $2^{-(n-r-1)}$. Then

$$\int_I F_r(x,y)dy = 0.$$

Define $F(x,y)$ for $(x,y)\epsilon U$ by

$$F(x,y) = \sum_{0<r<n} F_r(x,y).$$

The proof of Theorem 2C depends on the following lemmas.

Lemma 3A.

$$\int_0^1 \int_0^1 xyF(x,y)dxdy \geq (n-1)2^{-2n}(2^{n-2} - N).$$

Proof. It is sufficient to show that

$$\int_0^1 \int_0^1 xyF_r(x,y)dxdy \geq 2^{-2n}(2^{n-2} - N)\,(0 < r < n).$$

□

[4]These rectangles include their left and their lower edge, but not their right and upper edge, except when they adjoin the right or upper edge of U^2

Let B be a rectangle of the form $u2^{-(r-1)} \leq x \dot{<} (u+1)2^{-(r-1)}$, $v2^{-(n-r-1)} \leq y \dot{<} (v+1)2^{-(n-r-1)}$. Denote the centre of this rectangle by (ζ, η).

Suppose that no point $\underline{\underline{p}}_i$ is in B. Applying the substitution $x = \zeta + x', y = \eta + y'$, we obtain

$$\begin{aligned}
\int_R \int xyF_r(x,y)dxdy \\
&= \int_{-2^{-r}}^{2^{-r}} \int_{-2^{-(n-r)}}^{2^{-(n-r)}} (\zeta + x')(\eta + y') \text{ sign } x' \text{ sign } y' dx' dy' \\
&= \left(\int_{-2^{-r}}^{2^{-r}} (\zeta + x) \text{ sign } x dx \right) \left(\int_{-2^{-(n-r)}}^{2^{-(n-r)}} (\eta + y) \text{ sign } y dy \right)
\end{aligned}$$

19 (Here sign $x = 1$ if $x > 0, = -1$ if $x < 0, = 0$ if $x = 0$). Observe that

$$\begin{aligned}
\int_{-2^{-r}}^{2^{-r}} (\zeta + x) \text{ sign } x dx &= \zeta \int_{-2^{-r}}^{2^{-r}} \text{ sign } x dx + \int_{-2^{-r}}^{2^{-r}} x \text{ sign } x dx \\
&= 0 + 2 \int_0^{2^{-r}} x dx = 2^{-2r}.
\end{aligned}$$

Similarly,

$$\int_{-2^{-(nir)}}^{2^{-(n-r)}} (\eta + y) \text{ sign } y dy = 2^{-2(n-r)}.$$

Hence

$$\int \int_B xyF_r(x,y)dxdy = 2^{-2n} \ (0 < r < n).$$

If $\underline{\underline{p}}_i \epsilon B$, then $\int \int_B xyF_r(x,y)dxdy = 0$. The total number of boxes B as above is $2^{r-1}2^{n-r-1} = 2^{n-2}$. The number of boxes containing *no* point $\underline{\underline{p}}_i$ is $\geq (2^{n-2} - N)$. Hence

$$\int_0^1 \int_0^1 xyF_r(x,y)dxdy \geq 2^{-2n}(2^{n-2} - N) \, (0 < r < n).$$

This proves Lemma 3A.

Lemma 3B.
$$\int_0^1 \int_0^1 F^2(x,y)dxdy \leq (n-1).$$

Proof. We have

$$\int_0^1 \int_0^1 F^2(x,y)dxdy = \sum_{0<r<n} \int_0^1 \int_0^1 F_r^2(x,y)dx\,dy$$
$$+ 2 \sum_{0<r_1<r_2<n} \int_0^1 \int_0^1 F_{r_1}(x,y)F_{r_2}(x,y)dx\,dy.$$

□

Clearly the first sum on the right hand side is $\leq (n-1)$, since $F_r^2(x,y)$ is ≤ 1.

We shall show that the second sum on the right hand side is equal to zero. For this, it suffices to show that for every r_1, r_2 with $0 < r_1 < r_2 <$ **20**
n,
$$\int_0^1 \int_0^1 F_{r_1}(x,y)F_{r_2}(x,y)dx\,dy = 0.$$

Let y be fixed. Let I be an interval for x, of the type $u2^{-(r_2-1)} \leq x \dot{<} (u+1)2^{-(r_2-1)}$. In this interval, $F_{r_1}(x,y)$ is constant and $F_{r_2}(x,y)$ is either identically zero or $+1, -1$ in sub-intervals of equal length. Therefore
$$\int_I F_{r_1}(x,y)F_{r_2}(x,y)dx = 0.$$
and since U is the disjoint union of intervals I, we get
$$\int_0^1 F_{r_1}(x,y)F_{r_2}(x,y)dx = 0.$$

This holds for every y in U. Hence $\int_0^1 \int_0^1 F_{r_1}(x,y)F_{r_2}(x,y)dx\,dy$, and the proof of Lemma 3B is complete.

Lemma 3C. *Let $\underline{\underline{p}}_i = (x_i, y_i)$ be any of the given N points. Then*
$$\int_{x_i}^1 \int_{y_i}^1 F(x,y)dx\,dy = 0\,(1 \leq i \leq N).$$

Proof. It is enough to show that for every r, $0 < r < n$,

$$\int_{x_i}^{1}\int_{y_i}^{1} F_r(x,y)dx\,dy = 0 \quad (1 \le i \le N).$$

$\square$

Let X be the least integer multiple of $2^{-(r-1)}$ which is $\ge x_i$ and Y the least integer multiple of $2^{-(n-r-1)}$ which is $\ge y_i$. Then

$$\int_{x_i}^{1}\int_{y_i}^{1} F_r(x,y)dx\,dy = \int_{x_i}^{X}\int_{y_i}^{Y}\ldots + \int_{X}^{1}\int_{y_1}^{Y} \ldots + \int_{x_i}^{X}\int_{Y}^{1}\ldots + \int_{X}^{1}\int_{Y}^{1}\ldots$$

21 In the domain of the first integral, $F_r(x,y) = 0$, because it is contained in a box B of the form $u2^{-(r-1)} \le x \dot{<} (u+1)2^{-(r-1)}, v2^{-(n-r-1)} \le y \dot{<} (v+1)2^{-(n-r-1)}$ which contains $\underset{=}{p}_i$. In the second integral the end points for integration x are integer multiples of $2^{-(r-1)}$. Therefore $\int_X^1 F_r(x,y)dx = 0$. So the second integral is also zero. An arguments similar to that just given for the second integral, with the values of x, y interchanged, shows that the third integral is zero. In the last integral we may integrate in either order, and in either case the inner integral is zero. Hence Lemma 3C is proved.

Proof of Theorem 2C. As mentioned earlier, we shall restrict ourselves to case $k = 2$ for the proof of the Theorem. Let $\underset{=}{p}_1 = (x_1, y_1), \ldots, \underset{=}{p}_N = (x_N, y_N)$ be any N points in U^2. Observe that

$$\begin{aligned}&\int_0^1\int_0^1 Z(x,y)F(x,y)dx\,dy\\ &\quad = \int_0^1\int_0^1 \left(\sum_{\substack{i \text{ with}\\ x_i \le x\\ y_i \le y}} 1\right) F(x,y)dx\,dy = \sum_{i=1}^{N}\int_{x_i}^{1}\int_{y_i}^{1} F(x,y)dx\,dy = 0,\end{aligned}$$

by Lemma 3C. Therefore

$$\int_0^1 \int_0^1 (Nxy - Z(x,y))F(x,y)dxdy$$
$$= N \int_0^1 \int_0^1 xyF(xy)dx\,dy \geq (n-1)N2^{-2n}(2^{n-2} - N),$$

by Lemma 3A. This holds for any integer $n > 1$. Suppose now that

$$2^{n-2} > N.$$

By the inequality just derived and by Schwarz' inequality.

$$(n-1)^2(N2^{-2n}(2^{n-2} - N))^2$$
$$\leq \left(\int_0^1 \int_0^1 (Nxy - Z(x,y))^2 dxdy\right)\left(\int_0^1 \int_0^1 F^2(x,y)dx\,dy\right)$$
$$\leq (n-1) \int_0^1 \int_0^1 (Z(x,y) - Nxy)^2 dx\,dy,$$

in view of Lemma 3B. Hence **22**

$$\int_0^1 \int_0^1 (Z(x,y) - Nxy)^2 dx\,dy \geq (n-1)(N2^{-2n}(2^{n-2} - N))^2.$$

Now choose n with

$$2^3 N < 2^n \leq 2^4 N.$$

Using $2^{n-2} \geq 2N$, $2^{-n} \geq 2^{-4}N^{-1}$ and $n \geq \log_2 N + 3$, we obtain

$$\int_0^1 \int_0^1 (Z(x,y) - Nxy)^2 dx\,dy$$
$$\geq (\log_2 N)2^{-16} = \frac{2^{-16}}{\log_2} \log N.$$

4 A Theorem of Davenport

Theorem 4A [3]. *Suppose that θ is any irrational number with bounded partial quotients in its expansion as a simple continued fraction.*

Let M be a large integer. Put $N = 2M$. Consider the N points

$$(\{\pm t\theta\}, \frac{t}{m}), t = 1, \ldots, M.$$

Then with these N points, we have

$$\int_0^1 \int_0^1 (Z(x, y) - Nxy)^2 dx\, dy \leq c(\theta) \log N.$$

Here $c(\theta)$ is a positive constant depending only on θ.

This shows that Theorem 2C is best possible if $k = 2$. If $k > 2$, we do not know if Theorem 2C is best possible. However Davenport
23 remarked that one could obtain an analogue of Theorem 4A for $k > 2$, if there existed a $(k - 1)$ tuple $\theta_1, \ldots, \theta_{k-1}$ of real numbers with

$$\left|\theta_1 - \frac{p_1}{q}\right| \ldots \left|\theta_{k-1} - \frac{p_{k-1}}{q}\right| > \frac{c(\theta_1, \ldots, \theta_{k-1})}{q^k},$$

for all integers $p_1, \ldots, p_{k-1}$, $q > 0$. (For $k = 3$, this is equivalent to the falsity of a well-known conjecture of Littlewood.) Note that when $k = 2$, the above inequality reduces to $\left|\theta - \frac{p}{q}\right| > \frac{c(\theta)}{q^k}$, for all integers p, q; which is equivalent to saying that θ has bounded partial quotients in its expansion as a simple continued fraction.

Proof of Theorem 4A. Define

$$\psi(x) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \text{ is not an integer} \\ 0, & \text{if } x \text{ is an integer.} \end{cases}$$

This function has the Fourier series

$$\psi(x) = \sum_{\nu \neq 0} -\frac{e(\nu x)}{2\pi i \nu},$$

where $e(\nu x) = e^{2\pi i\nu x}$ and the sum is taken over all integer $\nu \neq 0$. Suppose that $0 < x < 1$, $\alpha \epsilon U$. Then it is easy to check that

$$x + \psi(\alpha - x) - \psi(\alpha) = \begin{cases} 1, & \text{if } 0 < \alpha < x, \\ 0, & \text{if } x < \alpha < 1, \\ \frac{1}{2}, & \text{if } \alpha = 0 \text{ or } \alpha = x \text{ or } \alpha = 1 \end{cases} \tag{4.1}$$

Assume that $0 < x < 1$ and $x \neq \{\pm\theta k\}\,(k = 1, 2, \ldots)$. Consider (4.1) with $\alpha = \{\theta k\}$. Then α cannot be equal to 0 or 1 (since θ is irrational). Further $\alpha \neq x$, because of the restriction on x. Using (4.1) and observing that only the first or second alternative may occur and noting that $\psi(x)$ is periodic with period 1, we obtain: *The number of k, $1 \leq k \leq V$, with $\{\theta k\} < x$ equals* 24

$$\sum_{k=1}^{V} (x + \psi(\theta k - x) - \psi(\theta k)).$$

Further, the number of k, $1 \leq k \leq V$, with $\{-\theta k\} < x$, equals

$$\sum_{k=1}^{V} (x + \psi(-\theta k - x) - \psi(-\theta k)).$$

Hence since ψ is odd, the number of k, $1 \leq k \leq V$, with $\{\pm\theta\} < x$, equals[5]

$$2Vx + \sum_{k=1}^{V} (\psi(\theta k - x) + \psi(-\theta k - x)),$$

and this is

$$\begin{aligned} &= 2Vx + \sum_{k=1}^{V} \sum_{\nu \neq 0} \left(\frac{e(\nu(\theta k - x))}{-2\pi i\nu} + \frac{e(\nu(-\theta k - x))}{-2\pi i\nu} \right) \\ &= 2Vx + \sum_{k=1}^{V} \sum_{\nu \neq 0} \frac{1}{2\pi i\nu} (e(\nu x - \nu\theta k) + e(\nu x + \nu\theta k)) \\ &= 2Vx + \sum_{\nu \neq 0} e(\nu x) c_\nu, \end{aligned}$$

[5] k is counted twice if both $\{+\theta k\} < x$ and $\{-\theta k\} < x$.

where
$$c_\nu = \frac{1}{2\pi i\nu}\sum_{k=1}^{V}(e(\nu\theta k) + e(-\nu\theta k)).$$

Suppose that $y\epsilon U$ and that $yM \geq 1$. Put $V = [yM] \geq 1$. Clearly $Z(x,y)$ is the number of points
$$(\{\pm\theta k\}, \frac{k}{M}), (k = 1, \ldots, M) \text{ with } \{\pm\theta k\} < x, k \leq My.$$

But $k \leq My$ is equivalent to $k \leq [My] = V$, so that $Z(x,y)$ is the number of k, $1 \leq k \leq V$, with $\{\pm\theta k\} < x$, and hence is
$$2Vx + \sum_{\nu\neq 0} e(\nu x)c_\nu.$$

25 All this is true provided that $0 < x < 1$, $x \neq \{\pm\theta k\}$, $(k = 1, 2, \ldots)$, and that $y\epsilon U$ and $yM \geq 1$. Note that the countably many exceptional x with $x = \{\pm\theta k\}$ form a set of Lebesgue measure zero. By Parseval's formula.
$$\int_0^1 (Z(x,y) - 2Vx)^2 dx = \sum_{\nu\neq 0}|c_\nu|^2. \tag{4.2}$$

This formula is valid for any y, satisfying $y\epsilon U$ and $yM \geq 1$.

Now we shall estimate $\sum_{\nu\neq 0}|c_\nu|^2$. Since $c_\nu^2 = c_{-\nu}^2$, we have
$$\sum_{\nu\neq 0}|c_\nu|^2 = 2\sum_{\nu=1}^{\infty}|c_\nu|^2 \ll {}^6 \sum_{\nu=1}^{\infty}\frac{1}{\nu^2}\left|\sum_{k=1}^{V}(e(\nu\theta k) + e(-\nu\theta k))\right|^2.$$

We have
$$\left|\sum_{k=1}^{V} e(\nu\theta k)\right| = \left|e(\nu\theta)\frac{e(\nu\theta V) - 1}{e(\nu\theta) - 1}\right| \leq \frac{2}{|e(\nu\theta) - 1|}.$$

Denote by $||\zeta||$ the distance from ζ to the nearest integer. We claim that
$$|e(\zeta) - 1| \gg ||\zeta||.$$

[6]The symbol $\ll$ (introduced by Vinogradov) is used as follows. $A \ll B$ means that $A \leq cB$ with an absolute constant c. $f(n) \ll g(n)$ means that $f(n) \leq cg(n)$ with c independent on n.

It is sufficient to prove this for $|\zeta| \leq \frac{1}{2}$, since both sides of the inequality are periodic with period 1. But for $|\zeta| \leq \frac{1}{2}$ we have

$$|e(\zeta) - 1| = \left| e(\frac{\zeta}{2}) + e(-\frac{\zeta}{2}) \right| = |2 \sin \pi\zeta| \gg |\zeta| = ||\zeta||.$$

We therefore obtain

$$\left| \sum_{k=1}^{V} e(\nu\theta k) \right| \ll \frac{1}{||\nu\theta||},$$

and a similar inequality with $e(\nu\theta k)$ replaced by $e(-\nu\theta k)$. Hence

$$\begin{aligned} \sum_{\nu \neq 0} |c_\nu|^2 &\ll \sum_{\nu=1}^{\infty} \frac{1}{\nu^2} \min \left(V^2, \frac{1}{||\nu\theta||^2} \right) \\ &= \sum_{r=1}^{\infty} \sum_{2^{r-1} \leq \nu < 2^r} \frac{1}{\nu^2} \left(V^2, \frac{1}{||\nu\theta||^2} \right) \qquad (4.3) \\ &= \sum_{r=1}^{\infty} \sum_{2^{r-1} \leq \nu < 2^r} 2^{-2r} \min \left(V^2 \frac{1}{||\nu\theta||^2} \right). \end{aligned}$$

Since θ is irrational and has bounded partial quotients, we have **26**

$$\left| \theta - \frac{\mu}{\nu} \right| > \frac{c_1^{(\theta)}}{\nu^2}$$

for all rational numbers $\frac{\mu}{\nu}$, where $c_1(\theta) > 0$ is a constants depending only on θ. Therefore

$$||\theta\nu|| > \frac{c_1(\theta)}{\nu} (\nu = 1, 2, \ldots). \qquad (4.4)$$

We *claim* that if $s > 0$ is an integer, there is at most one integer ν with

$$s c_1(\theta) 2^{-r} \leq \{\theta\nu\} < (s+1) c_1(\theta) 2^{-r} \qquad (4.5)$$

and

$$2^{r-1} \leq \nu < 2^r.$$

Suppose there were two : $\nu_1 < \nu_2$. Then

$$\|\theta\nu_2 - \theta\nu_1\| \le |\{\theta\nu_2\} - \{\theta\nu_1\}| < \frac{c_1(\theta)}{2^r} < \frac{c_1(\theta)}{\nu_2 - \nu_1}.$$

This contradicts (4.3). Similarly there exists at most one integer ν with

$$sc_1(\theta)2^{-r} \le \{-\theta\nu\} < (s+1)c_1(\theta)2^{-r} \text{ and } 2^{r-1} \le \nu < 2^r.$$

Since $\|\theta\nu\| = \min(\{\theta\nu\}, \{-\theta\nu\})$, there are at most two integer ν with

$$sc_1(\theta)2^{-r} \le \|\theta\nu\| < (s+1)c_1(\theta)2^{-r} \text{ and } 2^{r-1} \le \gamma < 2^r. \tag{4.6}$$

Further note that each ν with $2^{r-1} \le \nu < 2^r$ does satisfy (4.6) with
27 some integer $s > 0$, since otherwise

$$\|\theta\nu\| < c_1(\theta)2^{-r} < \frac{c_1(\theta)}{\nu},$$

which contradicts (4.3).

Ordering the summands in the last inner sum of (4.3) with respect to the s for which (4.5) holds, we see that the bottom line of (4.3) is

$$\begin{aligned}
&\le \sum_{r=1}^{\infty} 2^{-2r} \sum_{s=1}^{\infty} 2 \min\left(V^2, \frac{2^{2r}}{c_1(\theta)^2 s^2}\right) \\
&\ll \sum_{2^r \le V} \sum_{s=1}^{\infty} \frac{1}{s^2} + \sum_{2^r > V} \sum_{s=1}^{\infty} \min\left(\frac{V^2}{2^{2r}}, \frac{1}{s^2}\right) \\
&\ll \log V + V \sum_{2^r > V} \frac{1}{2^r} \\
&\ll \log V + 1 \ll \log M,
\end{aligned}$$

since $V = [yM] \le yM \le M$. Hence by (4.2), (4.3)

$$\int_0^1 (Z(x,y) - 2xV)^2 dx \ll \log M.$$

All this was done under the hypothesis that $yM \le 1$. But the inequality obtained is also true for $yM < 1$, since then $Z(x,y) = 0$ and $V = 0$.

Using the inequality $(a+b)^2 \ll a^2 + b^2$, and recalling that $N = 2M$, we obtain

$$
\begin{aligned}
&\int_0^1 (Z(x,y) - Nxy)^2 dx \\
&= \int_0^1 ((Z(x,y) - 2xV) + (2x(V - My)))^2 dx \\
&\ll \int_0^1 (Z(x,y) - 2xV)^2 dx + 1 \ll \log M \ll \log N.
\end{aligned}
$$

Davenport's Theorem follows on integration with respect to y.

5 The Correct Order of Magnitude of $\triangle(n)$ in the One-dimensional Case

In this section section, we shall restrict ourselves to the one-dimensional **28**
case $k = 1$. Let $x_1, x_2, \ldots$ be a sequence of points in U. We shall prove that

$$\triangle(n) = \Omega(\log n). \tag{5.1}$$

This is the correct order of magnitude for $\triangle(n)$, since we saw in Section 1 that there exist sequences with $\triangle(n) = 0(\log n)$. Now (5.1) follows from the following

Theorem 5A. *[25] Suppose that $N \geq 1$ is an integer. Then there exists an integer n, $1 \leq n \leq N$, such that*

$$\triangle(n) > \frac{1}{1000} \log N.$$

Note. $\frac{1}{1000}$ can be improved to $\frac{1}{100}$ and even better. This Theorem improves the case $k = 1$ of Theorem 2A. No improvement of the relation $\triangle(n) = \Omega((\log n)^{k/2})$ or of Theorem 2A is known if $k > 1$.

Theorem 5B. *Let $\underset{=}{p}_1, \ldots, \underset{=}{p}_N$ be N points in U^2. Then there is a box B with sides parallel to the axes, with the property that*

$$|D(B)| > \frac{1}{8000} \log N.$$

By the arguments of Section 2, Theorem 5A and 5B are equivalent except for the values of the constants.

Let $x_1, x_2, \ldots$ be a sequence of numbers in U. Suppose at first that

$$0 \le \alpha \le 1.$$

Put

$$z(n,\alpha) = \sum_{\substack{1 \le i \le n \\ 0 \le x_i \le \alpha}} 1$$

29 and

$$D(n,\alpha) = z(n,\alpha) - n\alpha.$$

We extend these definitions to arbitrary α by

$$\begin{aligned} z(n,\alpha) &= z(n,\{\alpha\}) + n[\alpha], \\ D(n,\alpha) &= z(n,\alpha) - n\alpha \\ &= z(n,\{\alpha\}) - n\{\alpha\}. \end{aligned}$$

Then $D(n,\alpha)$ is periodic in α with period 1.

Denote by $\mathscr{T}.\Re, \ldots$ "intervals" of integers. If $\mathscr{T}$ is the interval $a < n \le b$ with integer end points a, b, put $\ell(\mathscr{T}) = b - a$, so that $\ell(\mathscr{T})$ is the number of integers in $\mathscr{T}$. Write

$$g^+(\mathscr{T},\alpha) = \max_{n \epsilon \mathscr{T}} D(n,\alpha), g^-(\mathscr{T},\alpha) = \min_{n \epsilon \mathscr{T}} D(n,\alpha),$$
$$h(\mathscr{T},\alpha) = g^+(\mathscr{T},\alpha) - g^-(\mathscr{T},\alpha).$$

Put

$$\begin{aligned} D(n,\alpha,\beta) &= D(n,\beta) - D(n,\alpha) \\ &= z(n,\beta) - z(n,\alpha) - n(\beta - \alpha), \end{aligned}$$

$$g^+(\Re,\alpha,\beta) = \max_{n \epsilon \Re} D(n,\alpha,\beta), g^-(\Re,\alpha,\beta) = \min_{n \epsilon \Re} D(n,\alpha,\beta).$$

For every pair of intervals, $\Re, \Re'$, put

$$h(\Re,\Re',\alpha,\beta) = \max(0, g^-(\Re,\alpha,\beta) - g^+(\Re',\alpha,\beta), g^-(\Re',\alpha,\beta) - g^+(\Re,\alpha,\beta)).$$

Lemma 5C. *Let $\mathscr{T}$ be an interval of integers and let $\Re, \Re'$ be subintervals of $\mathscr{T}$. Then for any α, β, we have*

$$h(\mathscr{T}, \alpha)+h(,\beta) \geq h(\Re, \Re', \alpha, \beta)+\frac{1}{2}(h(\Re, \alpha)+h(\Re, \beta)+h(\Re', \alpha)+h(\Re', \beta)).$$

Proof. The lemma is trivial if $h(\Re, \Re', \alpha, \beta) = 0$. We may therefore **30**
assume without loss of generality that

$$h(\Re, \Re', \alpha, \beta) = g^{-}(\Re, \alpha, \beta) - g^{+}(\Re', \alpha, \beta) > 0.$$

□

Then for every $n \epsilon \Re$ and for every $n' \epsilon \Re'$ we have

$$D(n, \alpha, \beta) - D(n', \alpha, \beta) \geq h(\Re, \Re', \alpha, \beta),$$

i. e.

$$D(n, \beta) - D(n, \alpha) - D(n', \beta) + D(n', \alpha) \geq h(\Re, \Re', \alpha, \beta). \tag{5.2}$$

We choose $m_\alpha, n_\alpha, m_\beta, n_\beta \epsilon \Re$ with

$$g^{+}(\Re, \alpha) = D(m_\alpha, \alpha), g^{-}(\Re, \alpha) = D(n_\alpha, \alpha),$$
$$g^{+}(\Re, \beta) = D(m_\beta, \beta), g^{-}(\Re, \beta) = D(n_\beta, \beta).$$

Then

(i) $D(m_\alpha, \alpha) - D(n_\alpha, \alpha) = h(\Re, \alpha)$,

(ii) $D(m_\beta, \beta) - D(n_\beta, \beta) = h(\Re, \beta)$.

Similarly choose $m'_\alpha, n'_\alpha, m'_\beta, n'_\beta \epsilon \Re'$ with

(iii) $D(m'_\alpha, \alpha) - D(n'_\alpha, \alpha) = h(\Re', \alpha)$,

(iv) $D(m'_\beta, \beta) - D(n'_\beta, \beta) = h(\Re', \beta)$.

Applying (5.2) with $n = m_\alpha, n' = m'_\beta$, we get

(v) $D(m_\alpha, \beta) - D(m_\alpha, \alpha) - D(m'_\beta, \beta) + D(m'_\beta, \alpha) h(\Re, \Re', \alpha, \beta)$.

Applying (5.2) with $n = n_\beta, n' = n'_\alpha$, we obtain

(vi) $D(n_\beta, \beta) - D(n_\beta, \alpha) - D(n'_\alpha, \beta) + D(n'_\alpha, \alpha) \geq h(\Re, \Re', \alpha, \beta)$.

Adding the equations and inequalities (i) to (iv), we obtain

$$\begin{aligned} D(m'_\alpha, \alpha) - D(n_\alpha, \alpha) + D(m'_\beta, \alpha) - D(n_\beta, \alpha) \\ + D(m_\beta, \beta) - D(n'_\beta, \beta) + D(m_\alpha, \beta) - D(n'_\alpha, \beta) \\ \geq 2h(\Re, \Re', \alpha, \beta) + h(\Re, \alpha) + h(\Re, \beta) + h(\Re', \alpha) + h(\Re', \beta). \end{aligned}$$

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Since

$$h(\mathscr{T}, \alpha) \geq \max(D(m'_\alpha) - D(n_\alpha, \alpha), D(m'_\beta, \alpha) - D(n_\beta, \alpha))$$

and

$$h(\mathscr{T}, \beta) \geq \max(D(m_\beta, \beta) - D(n'_\beta, \beta), D(m_\alpha, \beta) - D(n'_\alpha, \beta)).$$

we finally get

$$\begin{aligned} & 2h(\mathscr{T}, \alpha) + 2h(\mathscr{T}, \beta) \\ & \geq 2h(\Re, \Re', \alpha, \beta) + h(\Re, \alpha) + h(\Re, \beta) + h(\Re', \alpha) + h(\Re', \beta). \end{aligned}$$

This proves Lemma 5C.

Lemma 5D. *Suppose that $s \geq 0, t \geq 1$ are integers, and $\mathscr{T}$ is an interval with*

$$\ell(\mathscr{T}) \geq 6^{s+t}.$$

Then for every β,

$$\frac{1}{6^t}\sum_{j=1}^{6^t} h(\mathscr{T},\beta + j6^{-s-t}) \geq \frac{t}{120}.$$

Remarks.

(i) Only the special case $s = 0$ will be used in the proof of Theorem 5B. The general case will be used in Section 6.

(ii) No special significance is attached to the number 6.

Proof of Lemma 5D. The proof is by induction on t. First take $t = 1$. Put $\ell = \frac{1}{2}6^{s+1}$. Suppose n is an integer. Then

$$\begin{aligned} &D(n+\ell,\beta+6^{-s-1}) - D(n,\beta+6^{-s-1}) - D(n+\ell,\beta) + D(n,\beta)\\ &\quad = a - (n+\ell)(\beta+6^{-s-1}) + n(\beta+6^{-s-1}) + (n+\ell)\beta - n\beta\\ &\quad = a - \ell 6^{-s-1}\\ &\quad = a - \frac{1}{2}, \text{ where a is some integer.}\end{aligned}$$

Hence **32**

$$\left|D(n+\ell,\beta+6^{-s-1}) - D(n,\beta+6^{-s-1}) - D(n+\ell,\beta) + D(n,\beta)\right| \geq \frac{1}{2},$$

which implies that

$$\left|D(n+\ell,\beta+6^{-s-1}) - D(n,\beta+6^{-s-1})\right| + |D(n+\ell,\beta) - D(n,\beta)| \geq \frac{1}{2}.$$

Now if $\ell(\mathscr{T}) \geq 6^{s+1}$, there exists an integer n such that $n, n+\ell \epsilon \mathscr{T}$. Hence

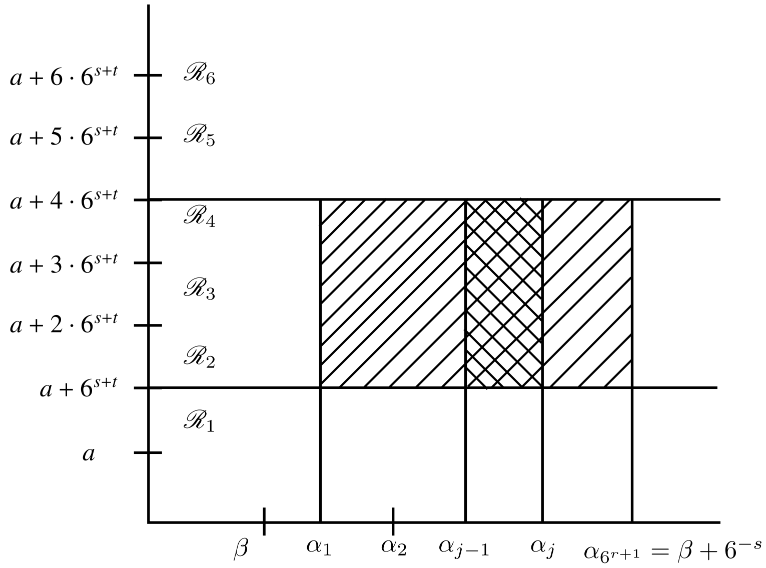
$$h(\mathscr{T},\beta+6^{-s-1}) + h(\mathscr{T},\beta) \geq \frac{1}{2}.$$

This is true for every β. Therefore

$$\frac{1}{6}\sum_{j=1}^{6} h(\mathscr{T},\beta + j6^{-s-1}) \geq \frac{1}{6}.3.\frac{1}{2} = \frac{1}{4} > \frac{1}{120}.$$

The proof of the case $t = 1$ is complete.

We now turn to the induction step from t to $t + 1$. Say $\mathscr{T}$ is the interval $a < n \le b$, with $\ell(\mathscr{T}) \ge 6^{s+t+1}$. Let $\Re_r$ be the intervals $a + (r - 1)6^{s+t} < n \le a + r6^{s+t}(r = 1, \ldots, 6)$. Since $\ell(\mathscr{T}) \ge 6^{s+t+1}$, $\Re_1, \ldots, \Re_6 \le \mathscr{T}$.



33 If the given sequence is $x_1, x_2, \ldots$, construct the points $(x_1, 1)$, $(x_2, 2), (x_3, 3), \ldots$. All these points will be in the "half strip" $0 \le x \le 1$, $y \ge 0$. To account for our periodic extension of $z(n, \alpha)$, construct the "periodic" set of points $(x_1 + m_1, 1)$, $(x_2 + m_2, 2), \ldots$ where $m_1, m_2, \ldots$ run through the integers. Then for $n > 0$, $\alpha > 0$, $z(n, \alpha)$ is the number of points in the rectangle $0 < x \le \alpha$, $0 \le y \le n$. Put

$$\begin{aligned} Z =& z(a + 4.6^{s+t}, \beta + 6^{-s}) - z(a + 4.6^{s+t}, \beta + 6^{-s-t-1}) \\ &- z(a + 6^{s+t}, \beta + 6^{-s}) + z(a + 6^{s+t}, \beta + 6^{-s-t-1}). \end{aligned}$$

Write

$$\alpha_j = \beta + j6^{-s-t-1},$$

and put

$$\begin{aligned} z_j =& z(a+4.6^{s+t},\alpha_j) - z(a+4.6^{s+t},\alpha_{j-1}) \\ &- z(a+6^{s+t},\alpha_j) + z(a+6^{s+t},\alpha_{j-1}) \qquad (j=0,1,2,\ldots). \end{aligned}$$

Z is the number of points in the shaded rectangle of the diagram above. Also z_j is the number of points in the doubly shaded rectangle. Note that the z_j are non-negative integers and that

$$Z = \sum_{j=2}^{6^{t+1}} z_j.$$

We shall consider the following two cases separately :

$$I : Z > \frac{5}{7}6^t$$
$$II : Z \leq \frac{5}{7}6^t.$$

Case I. $Z > \frac{5}{7}6^t$. For every $n\epsilon\Re_5$, $n'\epsilon\Re_1$, the rectangle with vertices (n,α_j), (n',α_j), (n,α_{j-1}), (n',α_{j-1}) contains the (doubly shaded) rectangle with vertices $(a+4.6^{s+t},\alpha_j)$, $(a+6^{s+t},\alpha_j)$, $(a+4.6^{s+t},\alpha_{j-1})$, $(a+6^{s+t},\alpha_{j-1})$, and hence **34**

$$z(n,\alpha_j) - z(n',\alpha_j) - z(n,\alpha_{j-1}) + z(n',\alpha_{j-1}) \geq z_j (2 \leq j \leq 6^{t+1}).$$

Therefore

$$\begin{aligned} D(n,\alpha_{j-1},\alpha_j) - D(n',\alpha_{j-1},\alpha_j) &\geq z_j - (n-n')(\alpha_j - \alpha_{j-1}) \\ &\geq z_j - 5.6^{s+t}6^{-s-t-1} \\ &= z_j - \frac{5}{6}. \end{aligned}$$

So

$$h(\Re_5,\Re_1,\alpha_{j-1},\alpha_j) \geq z_j - \frac{5}{6},$$

whence

$$h(\Re_1,\Re_5,\alpha_{j-1},\alpha_j) \geq \frac{1}{6}z_j,$$

since $h(\Re_1,\Re_5,\alpha_{j-1},\alpha_j)$ is non-negative and z_j is an integer. On using Lemma 5C, we obtain

$$\begin{aligned}&h(\mathscr{T},\alpha_{j-1})+h(\mathscr{T},\alpha_j) \qquad (2\le j\le 6^{t+1})\\ &\ge \frac{1}{6}z_j+\frac{1}{2}(h(\Re_1,\alpha_{j-1})+h(\Re_1,\alpha_j)+h(\Re_5,\alpha_{j-1})+h(\Re_5,\alpha_j))\end{aligned}$$

We shall also require the trivial estimates

$$\begin{aligned}h(\mathscr{T},\alpha_1)&\ge\frac{1}{2}(h(\Re_1,\alpha_1)+h(\Re_5,\alpha_1)),\\ h(\mathscr{T},\alpha_{6^{t+1}})&\ge\frac{1}{2}(h(\Re_1,\alpha_{6^{t+1}})+h(\Re_5,\alpha_{6^{t+1}})),\end{aligned}$$

Taking the sum of all these inequalities, we get

$$2\sum_{j=1}^{6^{t+1}}h(\mathscr{T},\alpha_j)\ge\frac{1}{6}\sum_{j=2}^{6^{t+1}}z_j+\sum_{j=1}^{6^{t+1}}h(\Re_1,\alpha_j)+\sum_{j=1}^{6^{t+1}}h(\Re_5,\alpha_j).$$

35 Note that $\ell(\Re_1)\ge 6^{s+t}$ and

$$\begin{aligned}\sum_{j=1}^{6^{t+1}}h(\Re_1,\alpha_j)&=\sum_{j=1}^{6^{t+1}}h(\Re_1,\beta+j6^{-s-t-1})\\ &=\sum_{i=0}^{5}\left(\sum_{\substack{j=1\\ j\equiv i(mod 6)}}^{6^{t+1}}h(\Re_1,\beta+j6^{-s-t-1})\right).\end{aligned}$$

By induction hypothesis, the inner sum here is $\ge 6^t\frac{t}{120}$, and our double sum is $\ge 6^{t+1}\frac{t}{120}$. Similarly,

$$\sum_{j=1}^{6^{t+1}}h(\Re_5,\alpha_j)\ge\frac{6^{t+1}t}{120}.$$

Hence

$$2\sum_{j=1}^{6^{t+1}} h(\mathscr{T},\alpha_j) \geq \frac{1}{6}Z + 2.6^{t+1}\frac{t}{120}$$
$$> \frac{1}{6}\frac{5}{7.6}6^{t+1} + 2.6^{t+1}\frac{t}{120} > \frac{2.6^{t+1}(t+1)}{120}$$

i. e.,

$$\frac{1}{6^{t+1}}\sum_{j=1}^{6^{t+1}} h(\mathscr{T},\alpha_j) \geq \frac{(t+1)}{120}.$$

Case II. $Z \leq \frac{5}{7}6^t$. For every $n \in \Re_4$, $n' \in \Re_2$, we have

$$z(n,\alpha_j) - z(n',\alpha_j) - z(n,\alpha_{j-1}) + z(n',\alpha_{j-1}) \leq z_j.$$

Therefore

$$\begin{aligned} D(n',\alpha_{j-1},\alpha_j) - D(n,\alpha_{j-1},\alpha_j) &\leq -z_j + (n-n')(\alpha_j,\alpha_{j-1}) \\ &\geq 6^{s+t}6^{-s-t-1} - z_j \\ &= \frac{1}{6} - z_j, \end{aligned}$$

and we have 36

$$h(\Re_2,\Re_4,\alpha_{j-1},\alpha_j) \geq \frac{1}{6} - z_j.$$

By Lemma 5C,

$$\begin{aligned} &h(\mathscr{T},\alpha_{j-1}) + h(\mathscr{T},\alpha_j). \qquad (2 \leq j \leq 6^{t+1}) \\ &\geq \frac{1}{6}z_j + \frac{1}{2}(h(\Re_2,\alpha_{j-1}) + h(\Re_2,\alpha_j) + h(\Re_4,\alpha_{j-1}) + h(\Re_4,\alpha_j)) \end{aligned}$$

We also note that the trivial relations

$$h(\mathscr{T},\alpha_1) \geq \frac{1}{2}(h(\Re_2,\alpha_1) + h(\Re_4,\alpha_1)),$$
$$h(\mathscr{T},\alpha_{6^{t+1}}) \geq \frac{1}{2}(h(\Re_2,\alpha_{6^{t+1}}) + h(\Re_4,\alpha_{6^{t+1}})).$$

Adding all these inequalities, we obtain

$$2\sum_{j=1}^{6^{t+1}} h(\mathscr{T},\alpha_j) \geq \frac{1}{6}(6^{t+1}-1) - Z + \sum_{j=1}^{6^{t+1}} h(\Re_2,\alpha_j) + \sum_{j=1}^{6^{t+1}} h(\Re_4,\alpha_j).$$

Here in Case II,

$$\frac{1}{6}(6^{t+1}-1) - Z \geq \frac{5}{6}6^t - Z \geq \frac{5}{6}6^t - \frac{5}{7}6^t = \frac{5}{42}6^t \geq \frac{2}{120}6^{t+1}.$$

Proceeding similarly as in Case I, we finally obtain

$$\frac{1}{6^{t+1}}\sum_{j=1}^{6^{t+1}} h(\mathscr{T},\alpha_j) \geq \frac{(t+1)}{120}.$$

This completes the proof of Lemma 5D.

Proof of Theorem 5A. First suppose that $N \geq 6^4$. Pick t with $6^t \leq N <$
37 6^{t+1}. Let $\mathscr{T}$ be the interval $0 < n \leq 6^t$. Applying Lemma 5D with $s = 0$. we get

$$\frac{1}{6^t}\sum_{j=1}^{6^t} h(\mathscr{T},\beta + j6^{-t}) \geq \frac{t}{120}.$$

There exists a β with

$$h(\mathscr{T},\beta) \geq \frac{t}{120}.$$

So there exists an $n \in \mathscr{T}$ with

$$\begin{aligned} |D(n,\beta)| \geq \frac{t}{240} &\geq \frac{1}{240}(\log_6 N - 1) \\ &\geq \frac{1}{320}\log_6 N > \frac{\log N}{1000}. \end{aligned}$$

Hence

$$\triangle(n) > \frac{\log N}{1000},$$

with $1 \leq n \leq 6^t \leq N$. On the other hand, if $N < 6^4$, then $\triangle(1) \geq |D(1,\frac{1}{2})| \geq \frac{1}{2} > \frac{\log N}{1000}$. The proof of Theorem 5A is complete.

6 A question of Erdös

It was known since Ehrenfest result (see § 2), that $\triangle(n)$ is bounded. Recall that $\triangle(n) = \sup\limits_{I\subseteq U} |D(n, I)|$. P.Erdös [5] asked the following question:

Does there always exist an interval $I \subseteq U$ such that $D(n, I)$ is unbounded?

This question was answered, in the affirmative, by W. M. Schmidt [21].

A stronger result is

Theorem 6A. *There always exists an $\alpha\epsilon U$ with*

$$\limsup_{n\to\infty} \frac{|D(n,\alpha)|}{\log n} > \frac{1}{2000}.$$

(α depends on the sequence $x_1, x_2, \ldots$).

The proof of Theorem 6A depends on Lemma 5D and the following simple

Lemma 6B. *Suppose that $0 < \epsilon < n$ and $\alpha \in U$. Then there exists a closed subinterval I of U containing α such that $|I| = \epsilon/n$ and*

$$|D(n,\beta)| \geq |D(n,\alpha)| - \epsilon \tag{6.1}$$

for every $\beta\epsilon I$.

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Proof. We distinguish three cases. □

Case I. $|D(n, \alpha)| \leq \epsilon$. The lemma follows trivially, since now the right hand side of (6.1) is ≤ 0.

Case II. $D(n, \alpha) > \epsilon$. In this case we have

$$n - n\alpha \geq z(n,\alpha) - n\alpha = D(n,\alpha) > \epsilon,$$

whence

$$\alpha < 1 - \frac{\epsilon}{n}.$$

Let I be the interval

$$\alpha \leq \beta \leq \alpha + \frac{\epsilon}{n}.$$

Observe that $I \subseteq U$ with $|I| = \frac{\epsilon}{n}$, and for $\beta \in I$ we have

$$\begin{aligned}|D(n,\beta)| &\geq z(n,\beta) - n\beta \\ &\geq z(n,\alpha) - n\alpha + n(\alpha - \beta) \\ &\geq |D(n,\alpha)| - \epsilon.\end{aligned}$$

Case III. $D(n,\alpha) < -\epsilon$. We now have

$$n\alpha \geq n\alpha - z(n,\alpha) = -D(n,\alpha) > \epsilon,$$

whence

$$\alpha > \frac{\epsilon}{n}.$$

We take I to be the interval

$$\alpha - \frac{\epsilon}{n} \leq \beta \leq \alpha.$$

Then $I \subseteq U$ with $|I| = \frac{\epsilon}{n}$, and for $\beta \in I$,

$$\begin{aligned}|D(n,\beta)| &\geq n\beta - z(n,\beta) \\ &\geq n\alpha - z(n,\alpha) - n(\alpha - \beta) \\ &\geq |D(n,\alpha)| - \epsilon.\end{aligned}$$

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Proof of Theorem 6A. It is sufficient to proof the following: *There exists a nested sequence $I_1 \supseteq \ldots \supseteq I_m \supseteq \ldots$ of closed intervals and positive integers $n_1 < \ldots < n_m < \ldots$, such that for every $\beta \in I_m$,*

$$|D(n_m,\beta)| \geq \frac{1}{2000} \log n_m.$$

(Since $\bigcap_{m=I}^{\infty} I_m \neq \phi$ choose $\alpha \epsilon \bigcap_{m=I}^{\infty} I_m$. Then $|D(n_m,\alpha)| \geq \frac{\log n_m}{2000}$ for every m.

So Theorem 6A follows).

The proof is by induction on m. If $m = 1$, take $I_1 = U$, $n_1 = 1$, and the desired inequality holds trivially. Suppose that $I_1, \ldots, I_{m-1}$ and $n_1, \ldots, n_{m-1}$ are already constructed. Let I'_{m-1} by the subinterval of I_{m-1} with $|I'_{m-1}| = \frac{1}{2}|I_{m-1}|$ and with the same midpoint as I_{m-1}. Choose s so large that

$$6^{-s} \leq |I'_{m-1}|. \tag{6.2}$$

Further choose t so large that

$$t > s, t > 250 n_{m-1}. \tag{6.3}$$

Let β be the left and points of I'_{m-1}. Let $\mathscr{T}$ be the interval of integers n with $0 < n \leq 6^{s+t}$. We now apply Lemma 5D and obtain

$$\frac{1}{6^t} \sum_{j=1}^{6^t} h(\mathscr{T}, \beta + j6^{-s-t}) > \frac{t}{120}.$$

By (6.2),

$$\beta + j6^{-s-t} \in I'_{m-1} \qquad (1 \leq j \leq 6^t).$$

Hence there exists an $\alpha \in I'_{m-1}$ with **40**

$$h(\mathscr{T}, \alpha) > \frac{t}{120}.$$

There exists an $n \in \mathscr{T}$, $0 < n \leq 6^{s+t}$, with

$$|D(n, \alpha)| > \frac{t}{240}.$$

Now α lies in the interior of I_{m-1}. Hence if we choose $\epsilon > 0$ sufficiently small and apply Lemma 6B, we see the existence of a closed subinterval I_m of I_{m-1} such that

$$|D(n,\beta)| > \frac{t}{250}, \tag{6.4}$$

for every $\beta \epsilon I_m$. Since $|D(n,\beta)| \leq n$, (6.3) and (6.4) yield

$$n > \frac{t}{250} > n_{m-1}.$$

Set $n_m = n$. It follows from (6.3) and (6.4), that for every $\beta \in I_m$

$$|D(n_m, \beta)| > \frac{t}{250} > \frac{s+t}{500} \geq \frac{\log_6 n_m}{500} \geq \frac{\log n_m}{2000}.$$

This completes our inductive construction.

Theorem 6C. *For almost every α,*

$$\limsup_{n\to\infty} \frac{D(n,\alpha)}{\log\log n} > \frac{1}{2000}.$$

This is stronger than in a paper by W. M. Schmidt [21]. It is an open problem whether $\log\log n$ may be replaced by a faster increasing function, perhaps even by $\log n$.

Proof. For $k = 1, 2, \ldots$, the intervals $\frac{u}{k!} \leq x \leq \frac{(u+1)}{k!}$ with $u = 0, 1, \ldots, (k! - 1)$ will be called *interval of order k*. Given any interval I, the subinterval with the same midpoint as I and with length $\frac{1}{2}|I|$ will be denoted by I'. □

41 Let k be a fixed large integer; say $k > k_0 = 100$. Put $N = N(K) = k!$. Let I_{k-1} be an interval of order $k-1$. Choose s with $6^{s-1} \leq 2(k-1)! \leq 6^s$. Let t be such that $6^{s+t} \leq N(k) < 6^{s+t+1}$. Then $t \geq 1$ in view of $k > 100 > 72$. Let β be the left end point of I'_{k-1}. Denote by $\mathscr{T}$ the interval $0 < n \leq 6^{s+t}$. By Lemma 5D, we obtain

$$\frac{1}{6^t} \sum_{j=1}^{6^t} h(\mathscr{T}, \beta + j6^{-s-t}) \geq \frac{t}{120}.$$

Observe that $\beta + j6^{-s-t} \in I'_{k-1} (0 < j \leq 6^t)$, since $j6^{-s-t} \leq 6^{-s} \leq \frac{1}{2(k-1)!}$. Hence there is a $\beta_0 \in I'_{k-1}$ and an integer n, $0 < n \leq 6^{s+t} \leq N$, with

$$|D(n, \beta_0)| \geq \frac{t}{240}.$$

By Lemma 6B with $\epsilon = 2$, there is an interval I of length $\frac{\epsilon}{n} \geq \frac{2}{k!}$ containing β_0, such that

$$|D(n,\alpha)| \geq \frac{t}{240} - 2,$$

for every $\alpha \in I$. Since k is large, $I \leq I_{k-1}$. Now $|I| \leq \frac{2}{k!}$, and therefore I and a fortiori I_{k-1} contains a subinterval I_k of order k having

$$|D(n,\alpha)| \geq \frac{t}{240} - 2,$$

for every $\alpha \in I_k$. Now

$$6^t = 6^{s+t+1-s-1} \geq N(k)\frac{1}{6^2.2(k-1)!} = \frac{k}{72}$$

and

$$\log N(k) \leq k \log k \leq k^2.$$

Hence for every $\alpha \in I_k$,

$$|D(n,\alpha)| \geq \frac{1}{240}\log_6(\frac{k}{72}) - 2 > \frac{1}{1000}\log k \geq \frac{1}{2000}\log\log N(k) \quad (6.5)$$
$$\geq \frac{1}{2000}\log\log n.$$

For every interval I_{k-1} of order $k-1(\geq k_0-1)$, we may select a 42
subinterval I_k of order k with the property (6.5). Denote the union of the intervals I_k so obtained by $E(k)$. Let α be such that it lies in infinitely many of the sets $E(k_0), E(k_0+1), \ldots$. Then the inequality

$$|D(n,\alpha)| \geq \frac{1}{2000}\log\log n$$

holds for infinitely many n. Hence Theorem 6C is proved if we can show that almost every α lies in infinitely many of the sets $E(k_0), E(k_0+1), \ldots$.

For every natural number $K \geq k_0$, let T_K be the complement of $\bigcup_{k\geq K} E(k)$. It is sufficient to prove that $\mu(T_K) = 0$ for every K (μ denotes the Lebesgue measure). But this is so, since

$$\mu(T_K) = \prod_{k=K}^{\infty}(1-\mu(E_k)) = \prod_{k=K}^{\infty}(1-\frac{1}{k}) = 0.$$

This proves Theorem 6C.

7 The Scarcity of Intervals With Bounded Error

Recall that for $\alpha \in U$, $D(n,\alpha) = z(n,\alpha) - n\alpha$. Put

$$E(\alpha) = \sup_n |D(n,\alpha)|,$$

where the supremum is taken over all positive integers n. For every non-negative $\mathcal{K}$, let $S(\mathcal{K})$ be the set of all those $\alpha \in U$ which have $E(\alpha) \leq \mathcal{K}$.

Let $S(\infty)$ consist of α in U with $E(\alpha) < \infty$. Clearly,

$$S(\infty) = \bigcap_{\mathcal{K}=0}^{\infty} S(\mathcal{K}).$$

We shall prove that $S(\infty)$ is at most countable.

The *derivative* of a set S of real numbers is the collection of the limit points of S. It will be denoted by $S^{(1)}$. Define inductively

$$S^{(d)} = (S^{(d-1)})^{(1)} \qquad (d = 2, 3, \ldots).$$

43 For convince, we shall write $S^{(0)}$ for S.

Theorem 7A. *(W. M. Schmidt [24]) Suppose that $d > 4\mathcal{K}$. Then*

$$(S(\mathcal{K}))^{(d)} = \phi,$$

i. e., the empty set.

In view of the following lemma, Theorem 7A implies that $S(\infty)$ is at most countable.

Lemma 7B. *Suppose that S is a set of real numbers having $S(d) = \phi$ for some d. Then S is at most countable and is nowhere dense.*

Proof. We claim that for any set A of real numbers, the set theoretic difference $B = A - A^{(1)}$ is at most countable. Denote by $\mathfrak{S}$ the collection of all the open intervals N with rational end points. Note that $\mathfrak{S}$ is countable. Further observe that for x in B, there exists an interval N_x of $\mathfrak{S}$ with $N_x \cap B = x$. For distinct x_1, x_2 in B, the intervals N_{x_1}, N_{x_2} are distinct. This shows that B is countable, since $\mathfrak{S}$ is so. ◻

We have $A = (A - A^{(1)}) \cup (A \cap A^{(1)})$. Hence if $A^{(1)}$ is at most countable, then so is A. Hence if $S^{(d)} = \phi$, then S is at most countable.

Now suppose that $A^{(1)}$ is nowhere dense. Then every interval I contains a closed subinterval J_1 with $J_1 \cap A^{(1)} = \phi$. Then $J_1 \cap A$ is finite. Thus there is a subinterval J of J_1 with $J \cap A = \phi$. Thus A is nowhere dense. More generally, if $A^{(d)}$ is nowhere dense, then so is A. If $S^{(d)} = \phi$, then S is nowhere dense.

COROLLARY. For every non-negative $\mathcal{K}$, $S(\mathcal{K})$ is at most countable and is nowhere dense. The set $S(\infty)$ is at most countable.

For $I \subseteq U$, set

$$D(n, I) = z(n, I) - n|I|,$$
$$E(I) = \sup_n |D(n, I)|.$$

44

We may call $E(I)$ the *error* of I.

Theorem 7C*. *(W. M. Schmidt [26].) The lengths of all the intervals I with finite error $E(I)$ form at most a countable set*[7].

The above Theorem does not give any information about the cardinality of the set of intervals I with $E(I) < \infty$. It has the power of continuum in the following case.

Theorem 7D*. *Let α be an irrational number. Consider the sequence*

$$\{\alpha\}, \{2\alpha\}, \ldots.$$

Then $E(I)$ is finite if and only if $I = \{k\alpha\}$ for some non-zero integer k.

The "if" part is due to A. Ostrowski [17]. The "only if" part was shown by H. Kesten [11][8].

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[7]see remark on page 54.

[8]Addded June 1976. H.Furstenberg, H. Keynes and L. Shapiro (Prime flows in topological dynamics, Israel J. Math, 14(1) (1973), 26-38) and G. Halász (Remarks on the remainder in Birkhoff Ergodic Theorem (preprint)) proved this with ergodic theorey.

A proof of the "if" part is as follows. More generally, we shall consider the sequence

$$\{\alpha+\beta\}, \{2\alpha+\beta\}, \ldots.$$

Because of the arbitrary parameter β, it will be sufficient to deal with the case when I is of the type $0 \le x < \{k\alpha\}$. We may assume that $0 < \{k\alpha\} < 1$. Further we may assume that $k > 0$: For if $k < 0$, put $k = -k'$, $k' > 0$. Then the length of the complement I' of I in U is equal to $\{k'\alpha\}$, since $\{k\alpha\} + \{k'\alpha\} = 1$. We have $E(I) = E(I')$, so that $E(I)$ is
45 finite if $E(I')$ is finite, and in particular $E(I)$ is finite if the 'if' part of the theorem is true for $k' > 0$.

Consider the k sequence

$$\begin{array}{lll}
\{\alpha+\beta\}, & \{(k+1)\alpha+\beta\}, & \ldots, \\
\{2\alpha+\beta\}, & \{(k+2)\alpha+\beta\}, & \ldots, \\
\ldots\ldots\ldots & \ldots\ldots\ldots & \ldots \\
\{k\alpha+\beta\}, & \{2k\alpha+\beta\}, & \ldots.
\end{array}$$

It is sufficient to prove that $E(I)$ is finite for each of these sequences. In each of these sequences, the common differences of the arithmetic progression is $k\alpha$. Since $|I| = \{k\alpha\}$, we may replace α by $k\alpha$ and see that it will suffice to prove the assertion in the special case when $|I| = \{\alpha\}$, i. e. the special case $k = 1$. Thus the problem reduces to showing that $E(I)$, with $I : 0 \le x < \alpha$, is finite for the sequence

$$\{\alpha+\beta\}, \{2\alpha+\beta\}, \ldots.$$

Observe that

$$\begin{aligned}
z(n, I) = {} & \text{the number of } k, 1 \le k \le n, \text{ with } \{k\alpha+\beta\} < \alpha \\
= {} & \text{the number of integers } m \text{ and } k, 1 \le k \le n, \text{ satisfying the inequality}
\end{aligned}$$

$$0 \le k\alpha + \beta - m < \alpha.$$

Any m satisfying this inequality with $1 \le k \le n$ must satisfy

$$\beta < m \le n\alpha + \beta.$$

For every m, there exists a unique integer k satisfying the first inequality.

Moreover if m satisfies the second inequality, then this integer k satisfies $1 \leq k \leq n$. Hence $z(n, I)$ is equal to the number of integers m satisfying the second inequality, and $|z(n, I) - n\alpha| \leq 1$.

Therefore $|D(n.I)| \leq 1$ for every integer n, and $E(I) \leq 1$. **46**

This proves the 'if' part of Theorem 7D. The more difficult 'only if' part will not be proved here.

Denote by U^0 the open unit interval $0 < x < 1$. BY a *neighbourhood*, we shall always mean an open interval contained in U^0. For the proof of Theorem 7A, we shall need several lemmas.

Lemma 7E. *Let $\alpha \in U^0$ and $\epsilon > 0$ be given. Then there exists a neighbourhood A of α, and an integer p, such that for any $\beta \in A$ and for any interval $\mathscr{T}$ of integers with $\ell(\mathscr{T}) \geq p$,*

$$h(\mathscr{T}, \beta) \geq \frac{1}{2} - \epsilon.$$

Proof. Assume at first that $0 < \alpha \leq \frac{1}{2}$. Put

$$p = \left[\frac{1}{\alpha}\right] + 1.$$

Consider the numbers

$$\alpha, 2\alpha, \ldots, p\alpha.$$

Every number ψ with $\frac{\alpha}{2} \leq \psi \leq (p + \frac{1}{2})\alpha$ has a distance $\leq \frac{\alpha}{2}$ from (at least) one of these p numbers. Hence it has a distance $\leq \frac{1}{4}$. Thus for every ψ with $\frac{\alpha}{2} \leq \psi \leq (p + \frac{1}{2})\alpha$, there is an integer n, $1 \leq n \leq p$, such that $|n\alpha - \psi| \leq \frac{1}{4}$. ◻

Since $\frac{\alpha}{2} \leq \psi \leq (p + \frac{1}{2})\alpha$ has length $p\alpha > 1$, the translations of this interval by integers cover the real line. Therefore for any real number ψ, there exist integers n, m with $0 < n \leq p$ and $|n\alpha - m - \psi| \leq \frac{1}{4}$. Further the restriction $0 < \alpha \leq \frac{1}{2}$ can be removed, since for $\frac{1}{2} < \alpha \leq 1$, there exist integers n, m' such that $0 < n \leq p$ and $|n(1 - \alpha) - m' + \psi| \leq \frac{1}{4}$, i.e. $|n\alpha - m - \psi| \leq \frac{1}{4}$, with $m = n - m'$.

Let A consist of $\beta \in U^0$ with $p|\beta - \alpha| < \frac{\epsilon}{2}$. For every real number ψ, there are integers n, m, $0 < n \leq p$, such that $|n\beta - m - \psi| < \frac{1}{4} + \frac{\epsilon}{2}$ holds for all $\beta \in A$, since $|n\alpha - m - \psi| \leq \frac{1}{4}$ and $0 < n \leq p$, implies that
47 $|n\beta - m - \psi| < \frac{1}{4} + \frac{\epsilon}{2}$ for $\beta \in A$. Now since ψ is arbitrary, we see that for every ψ, there are integers n, m, $0 < n \leq p$, having $0 < n\beta - m\psi < \frac{1}{2} + \epsilon$ for every $\beta \in A$. *Hence for every interval $\mathscr{T}$ with $\ell(\mathscr{T}) \geq p$, every $\beta \in A$ and every ψ, there are integers n, m with $n \in \mathscr{T}$ and*

$$0 < n\beta - m - \psi < \frac{1}{2} + \epsilon.$$

For $\beta \in A$ and an interval $\mathscr{T}$ with $\ell(\mathscr{T}) \geq p$, choose integers $n \in \mathscr{T}$ and m such that

$$0 < n\beta - m + g^{-}(\mathscr{T}, \beta) < \frac{1}{2} + \epsilon. \tag{7.1}$$

Then

$$z(n, \beta) = D(n, \beta) + n\beta \geq g^{-}(n, \beta) + n\beta > m.$$

Therefore

$$z(n, \beta) \geq m + 1. \tag{7.2}$$

Combining (7.1) and (7.2), we obtain

$$\begin{aligned} h(\mathscr{T}, \beta) &= g^{+}(\mathscr{T}, \beta) - g^{-}(\mathscr{T}, \beta) \\ &\geq D(n, \beta) - g^{-}(\mathscr{T}, \beta) \\ &= z(n, \beta) - n\beta - g^{-}(\mathscr{T}, \beta) \\ &> m + 1 - m - \frac{1}{2} - \epsilon \\ &= \frac{1}{2} - \epsilon. \end{aligned}$$

This is true for every $\beta \in A$. Lemma 7E is proven.

Lemma 7F. *Suppose that $0 < \epsilon < 1$ and $q \geq 1$ is an integer. Suppose that $\alpha, \beta \in U^0$ with $0 < |\alpha - \beta| < \frac{\epsilon}{8q}$. Then there is an integer p and there are neighbourhoods A of α, B of β with the following property: If $\mathscr{T}$ is an interval with $\ell(\mathscr{T}) \leq p$ and if $\gamma \in A$, $\delta \in B$, then there exist*
48 *subintervals $\mathfrak{R}, \mathfrak{R}'$ of $\mathscr{T}$ with $\ell(\mathfrak{R}) = \ell(\mathfrak{R}') = q$ and*

$$g^{-}(\mathscr{R}, \gamma, \delta) - g^{+}(\mathscr{R}', \gamma, \delta) > 1 - \epsilon.$$

Proof. We may assume that $\alpha < \beta$. Put $p_0 = \frac{1}{\beta-\alpha} + 1$. By an argument as in the proof of Lemma 7E, one sees that for every real number ψ, there exist integers n, m, $1 \le n \le p_0$, such that

$$|n(\beta - \alpha) - m - \psi| \le \frac{1}{2}|\beta - \alpha| < \frac{\epsilon}{8}.$$

Let A consist of $\gamma \in U^0$ with

$$|\gamma - \alpha| \max(q, p_0) < \frac{\epsilon}{16},$$

and B of $\delta \in U^0$ with

$$|\delta - \beta| \max(q, p_0) < \frac{\epsilon}{16}.$$

□

Then for every $\gamma \in A$ and $\delta \in B$, and for every ψ, there are integers n, m with $0 < n \le p_0$ and

$$|n(\delta - \gamma) - m - \psi| < \frac{\epsilon}{4}.$$

Now since ψ is arbitrary, it follows that for every $\gamma \in A$, every $\delta \in B$ and every ψ, there exist integers n, m, $0 < n \le p_0$, with

$$0 < n(\delta - \gamma) - m - \psi < \frac{\epsilon}{2}.$$

Here the condition $0 < n \le p_0$ may be replaced by $n \in \mathscr{T}$, where $\mathscr{T}$ is a given interval with $\ell(\mathscr{T}) \ge p_0$.

Suppose that $\mathscr{T}_\circ$ is any interval with $\ell(\mathscr{T}_\circ) \le p_0$. Let $\gamma \in A$ and $\delta \in B$ be fixed. Choose $n_0' \in \mathscr{T}_\circ$ with

$$g^-(\mathscr{T}_\circ, \gamma, \delta) = D(n_0', \gamma, \delta).$$

Choose integers $n_0 \in \mathscr{T}_\circ$ and m such that

$$0 < n_0(\delta - \gamma) - m + g^-(\mathscr{T}_0, \gamma, \delta) < \frac{\epsilon}{2}. \tag{7.3}$$

Then

$$\begin{aligned} z(n_0,\delta) - z(n_0,\gamma) &= D(n_0,\gamma,\delta) + n_0(\delta-\gamma) \\ &\geq g^-(\mathscr{T}_\circ,\gamma,\delta) + n_0(\delta-\gamma) \\ &> m. \end{aligned}$$

49 Therefore

$$z(n_0,\delta) - z(n_0,\alpha) \geq m+1 \tag{7.4}$$

Combining (7.3) and (7.4), we obtain

$$\begin{aligned} D(n_0,\gamma,\delta) - D(n_0',\gamma,\delta) &\geq m+1-n_0(\delta-\gamma) - g^-(\mathscr{T}_\circ,\gamma,\delta) \\ &> m+1-m-\frac{\epsilon}{2} \\ &= 1-\frac{\epsilon}{2} \end{aligned} \tag{7.5}$$

Now put

$$p = p_0 + 2q.$$

Suppose that $\mathscr{T}$ is an interval with $\ell(\mathscr{T}) \geq p$. Let $\mathscr{T}_\circ$ be the interval obtained from $\mathscr{T}$ by cutting off segments of length q on either side. Then clearly $\ell(\mathscr{T}_0) \geq p_0$. Hence there exist integers n_0, $n_0' \in \mathscr{T}_\circ$ satisfying (7.5). Set

$$\mathfrak{R} : n_0 < n \leq n_0 + q \text{ and } \mathfrak{R}' : n_0' - q < n' \leq n_0'.$$

It is clear that

$$\ell(\mathfrak{R}) = \ell(\mathfrak{R}') = q \text{ and } \mathfrak{R}, \mathfrak{R}' \subseteq \mathscr{T}.$$

For every $n \in \mathfrak{R}$ we have

$$\begin{aligned} D(n,\gamma,\delta) - D(n_0,\gamma,\delta) &\geq -(n-n_0)(\delta-\gamma) \\ &\geq -q(\delta-\alpha) > -\frac{\epsilon}{4}, \end{aligned} \tag{7.6}$$

since $\gamma \in A$, $\delta \in B$ has

$$|\gamma - \delta| \leq |\gamma - \alpha| + |\alpha - \beta| + |\beta - \delta| < \frac{\epsilon}{16q} + \frac{\epsilon}{8q} + \frac{\epsilon}{16q} = \frac{\epsilon}{4q}.$$

Similarly, for every $n' \in \mathfrak{R}'$, we have

$$D(n'_0, \gamma, \delta) - D(n', \gamma, \delta) > -\frac{\epsilon}{4}. \tag{7.7}$$

Adding (7.5), (7.6) and (7.7), we get

$$D(n, \gamma, \alpha) - D(n', \gamma, \delta) > 1 - \in .$$

This holds for every $n \in \mathfrak{R}$ and every $n' \in \mathfrak{R}'$. Lemma 7F follows immediately **50**

Lemma 7G. *Suppose that $\theta_1, \ldots, \theta_t \in U^0$ and belong to the derivative $R^{(1)}$ of some set R of real numbers. Let $D_1, \ldots, D_t$ be respective neighbourhoods of $\theta_1, \ldots, \theta_t$. Let $\in > 0$ and an integer $q \geq 1$ be given. Then there exist numbers $\alpha_1, \beta_1, \ldots, \alpha_t, \beta_t$ of R such that $\alpha_1, \beta_1 \in D_1, \ldots, \alpha_t, \beta_t \in D_t$, there exist neighbourhoods $A_1, B_1, \ldots, A_t, B_t$ of α_1, $\beta_1, \ldots, \alpha_t, \beta_t$, respectively, with $A_1, B_1 \subseteq D_1, \ldots, A_t, B_t \subseteq D_t$, and there exists an integer r with the following property : If $\mathscr{T}.\mathscr{T}'$ are intervals with $\ell(\mathscr{T}) \geq r$, $\ell(\mathscr{T}') \geq r$ and if $\gamma_1, \delta_1, \ldots, \gamma_t, \delta_t$ lie in $A_1, B_1, \ldots$, A_t, B_t, respectively, then there are subintervals $\mathfrak{R} \subseteq \mathscr{T}, \mathfrak{R}' \subseteq \mathscr{T}$ with $\ell(\mathfrak{R}) = \ell(\mathfrak{R}') = q$ and*

$$h(\mathfrak{R}, \mathfrak{R}', \gamma_i, \delta_i) > 1 - \in \qquad (i = 1, \ldots, t).$$

We shall apply this lemma with $\mathscr{T} = \mathscr{T}'$, but generality is necessary to carry out the inductive proof.

Proof. Suppose at first that $t = 1$. Then $\theta_1 \in R^{(1)}$, a neighbourhood D_1 of θ_1, $\in > 0$ and an integer $q \geq 1$ are given. We may suppose that $0 < \in < 1$. Since $\theta_1 \in R^{(1)}$, there exist elements $\alpha_1, \beta_1 \in R \cap D_1$ with $0 < |\alpha_1 - \beta_1| < \frac{\epsilon}{8q}$. We now apply Lemma 7F. There exist neighbourhoods A_1 of α_1, B_1 of β_1, and an integer p as follows: If $\mathscr{T}$ is an interval with

$\ell(\mathscr{T}) \geq p$ and if $\gamma_1 \in A_1$, $\delta_1 \in B_1$, then there exist intervals $\Re_1, \Re_2 \subseteq \mathscr{T}$ such that $\ell(\Re_1) = \ell(\Re_2) = q$ and

$$g^-(\Re_1, \gamma_1, \delta_1) - g^+(\Re_2, \gamma_1, \delta_1) > 1 - \epsilon \tag{7.8}$$

We may choose A_1, B_1 such that $A_1, B_1 \subseteq D_1$. $\square$

51 Let $\mathscr{T}'$ be another interval with $l(\mathscr{T}') \geq p$. By applying Lemma 7F again, we conclude that these exist $\Re_1', \Re_2' \subseteq \mathscr{T}'$ with $\ell(\Re_1') = \ell(\Re_2') = q$ and

$$g^-(\Re_1', \gamma_1, \delta_1) - g^+(\Re_2', \gamma_1, \delta_1) > 1 - \epsilon. \tag{7.9}$$

From (7.8) and (7.9) it follows that either

$$g^-(\Re_1, \gamma_1, \delta_1) - g^+(\Re_2', \gamma_1, \delta_1) > 1 - \epsilon$$

or

$$g^-(\Re_1', \gamma_1, \delta_1) - g^+(\Re_2, \gamma_1, \delta_1) > 1 - \epsilon.$$

Therefore

$$\text{either } h(\Re_1, \Re_2', \gamma_1, \delta_1) > 1 - \epsilon \text{ or } h(\Re_2, \Re_1', \gamma, \delta) > 1 - \epsilon.$$

In the first case put $\Re = \Re_1, \Re' = \Re_2'$, and in the second case out $\Re = \Re_2, \Re' = \Re_1'$.

Hence

$$h(\Re, \Re', \gamma_1, \delta_1) > 1 - \epsilon.$$

Thus the lemma is proved for $t = 1$ with $r^{(1)} = p$.

We now shall do the inductive step from $t - 1$ to t. Suppose we are given $\theta_1, \ldots, \theta_t \in R^{(1)}$ with respective neighbourhoods $D_1, \ldots, D_t$, $\epsilon > 0$ and an integer $q \geq 1$. By induction hypothesis there exist $\alpha_1, \beta_1, \ldots, \alpha_{t-1}, \beta_{t-1}$ and an integer $r^{(t-1)}$ satisfying the conclusions of the lemma. Therefore if $\mathscr{T}$ and $\mathscr{T}'$ are intervals with $\ell(\mathscr{T})$, $\ell(\mathscr{T}') \geq r^{(t-1)}$ and if $\gamma_1 \in A_1, \delta_1 \in B_1, \ldots, \gamma_{t-1} \in A_{t-1}, \delta_{t-1} \in B_{t-1}$, then there exist $\Re \subseteq \mathscr{T}$, $\Re' \subseteq \mathscr{T}'$ with $\ell(\Re) = \ell(\Re') = q$ and

$$h(\Re, \Re', \gamma_i, \delta_i) > 1 - \in \qquad (i = 1, \ldots, t-1). \tag{7.10}$$

52 Apply the case $t = 1$ of the lemma to θ_t, D_t and to $r^{(t-1)}$ in place of q.

There exist α_t, β_t of R with respective neighbourhoods $A_t, B_t \subseteq D_t$ and an integer $\overline{r}^{(t)}$ with the following property: If $\mathscr{T}$ and $\mathscr{T}'$ are intervals with $\ell(\mathscr{T}), \ell(\mathscr{T}') \geq r^{-(1)}$ and if $\gamma_t \in A_t$, $\delta_t \in B_t$, then there exist $\mathscr{T}_0 \subseteq \mathscr{T} . \mathscr{T}_0' \subseteq \mathscr{T}'$ with $\ell(\mathscr{T}_0) = \ell(\mathscr{T}_0') = r^{t-1}$, such that

$$h(\mathscr{T}_0, \mathscr{T}_0', \gamma_t, \delta_t) > 1 - \epsilon. \tag{7.11}$$

Since $\ell(\mathscr{T}_0) = \ell(\mathscr{T}_0') = r^{(t-1)}$, and by the case $t-1$ discussed above, there exist subintervals $\mathfrak{R} \subseteq \mathscr{T}_0$, $\mathfrak{R}' \subseteq \mathscr{T}_0'$ with $\ell(\mathfrak{R}) = \ell(\mathfrak{R}') = q$, satisfying (7.10). We also have $h(\mathfrak{R}, \mathfrak{R}', \gamma_t, \delta_t) > 1- \in$ by (7.11). The proof of Lemmas 7G is complete.

Lemma 7H. *Let $\in > 0$ and an integer $d \geq 0$ be given. Let R be a set of real numbers such that $R^{(d)}$ has non - empty intersection with U^0. Then there exist $w = 2^d$ elements $\lambda_1, \ldots, \lambda_w$ of R contained in U^0, with respective neighbourhoods $L_1, \ldots, L_w$, and there exists an integer p, such that if $\mathscr{T}$ is any interval with $\ell(\mathscr{R}) \geq p$ and if $\mu_1 \in L_1, \ldots, \mu_w \in L_w$, then*

$$\frac{1}{w} \sum_{j=1}^{w} h(\mathscr{T}, \mu_j) > \frac{1}{2}(d+1) - \epsilon.$$

Proof. The proof is by induction on d. When $d = 1$, our lemma reduces to Lemma 7E. (Put $w = 1$, $\lambda_1 = \alpha \in R$, $L_1 = A$). Assume the truth of the lemma for $d-1$; we shall proceed to prove it for d. ◻

Put $t = 2^{d-1}$. Apply the case $d-1$ of the lemma to $R^{(1)}$. Therefore are t elements $\theta_1, \ldots, \theta_t$ of $R^{(1)}$ contained in U^0, with respective neighbourhoods $D_1, \ldots, D_t$. and there is an integer $p^{(d-1)}$, so that if $\mathscr{T}_\circ$ is an interval with $\ell(\mathscr{T}_\circ) \geq p^{(d-1)}$ and if $\eta_1 \in D_1, \ldots, \eta_t \in D_t$, then

$$\frac{1}{t} \sum_{j=1}^{t} h(\mathscr{T}_0, \eta_j) > \frac{1}{2} d - \frac{\epsilon}{2}. \tag{7.12}$$

Apply Lemma 7G to $\theta_1, \ldots, \theta_t$ and R, to $q = p^{(d-1)}$ and to $D_1, \ldots,$ **53**
D_t. There exist elements $\alpha_1, \beta_1, \ldots, \alpha_t, \beta_t$ of R with respective neighbourhoods $A_1, B_1 \subseteq D_1, \ldots, A_t, B_t \subseteq D_t$ and an integer r, so that if $\mathscr{T}$ is an interval with $\ell(\mathscr{T}) \geq r$, then there exist intervals $\mathfrak{R}, \mathfrak{R}' \subseteq \mathfrak{T}$ with

$\ell(\Re) = \ell(\Re') = p^{d-1}$, such that for any $\gamma_1 \in A_1, \delta_1 \in B_1, \ldots, \gamma_t \in A_t, \delta_t \in B_t$, we have

$$h(\Re, \Re', \gamma_i, \delta_i) > 1 - \frac{\epsilon}{2} \qquad (i = 1, \ldots, t).$$

An application of Lemma C yields

$$\begin{aligned} &h(\mathscr{T}.\gamma_i) + h(\mathscr{T}, \delta_i) \\ &> 1 - \frac{\epsilon}{2} + \frac{1}{2}(h(\Re, \gamma_i) + h(\Re, \delta_i) + h(\Re', \gamma_i) + h(\Re', \delta_i)) \quad (i = 1, \ldots, t). \end{aligned}$$

Taking the sum of these inequalities for $1 \leq i \leq t$ and dividing by $2t$, we get

$$\begin{aligned} &\frac{1}{2t}\left(\sum_{i=1}^{t} h(\mathscr{T}, \gamma_i) + \sum_{i=1}^{t} h(\mathscr{T}, \delta_i)\right) \\ &> \frac{1}{2} - \frac{\epsilon}{4} + \frac{4}{4t}(t(\frac{1}{2}d - \frac{1}{2}\epsilon)) > \frac{1}{2}(d+1) - \epsilon, \end{aligned}$$

since $\ell(\Re) = \ell(\Re') = p^{(d-1)}$, $\gamma_i \subseteq A_i \subseteq D_i$, $\delta_i \in B_i \in D_i (1 \leq i \leq t)$, and so (7.12) is valid for each of the four sums

$$\sum_{i=1}^{t} h(\Re, \gamma_i), \sum_{i=1}^{t} h(\Re, \delta_i), \sum_{i=1}^{t} h(\Re', \gamma_i), \sum_{i=1}^{t} h(\Re', \delta_i).$$

Hence Lemma 7H is true with $p = r$ and

$$\begin{aligned} &\lambda_1 = \alpha_1, \ldots, \lambda_t = \alpha_t, \lambda_{t+1} = \beta_1, \ldots, \lambda_{w=2t} = \beta_t, \\ &L_1 = A_1, \ldots, L_t = A_t, B_{t+1} = B_1, \ldots, L_{w=2t} = B_t. \end{aligned}$$

Proof of Theorem 7A. The proof is by contradiction. Suppose that $S^{(d)}(\mathcal{K}) \neq \phi$ where $d > 4\mathcal{K}$. Put $d_1 = d - 1$. Then $S^{(d_1)}(\mathcal{K})$ has a non-empty intersection with U^0. Put $\in = \frac{1}{2}d - 2\mathcal{K} > 0$. By Lemma 7H
54 there is an element $\lambda \in S(\mathcal{K})$ and an interval $\mathscr{T}$ with

$$h(\mathscr{T}, \lambda) > \frac{1}{2}(d_1 + 1) - \epsilon.$$

So there exists an integer $n \in \mathscr{T}$ with

$$|D(n, \lambda)| > \frac{1}{4}(d_1 + 1) - \frac{\epsilon}{2} = \mathcal{K}.$$

Thus

$$E(\lambda) > \mathcal{K}.$$

which contradicts the fact that $\lambda \in S(\mathcal{K})$. This completes the proof of Theorem 7A.

The restriction $d > 4\mathcal{K}$ can be replaced somewhat relaxed, but it can-not be replaced by $d \geq \mathcal{K} - 1$. We shall now show that for the Van der Corput sequence (see § 1), $S^{(d)}(d + 1) \neq \phi$.

For every non-negative integer d, define a set R_d as follows:

$$R_d = \begin{cases} 0, & \text{if } d = 0 \\ 0 & \text{and the collection of all the numbers } 2^{-g_1} + \ldots + 2^{-g_t}, \\ & \text{with } 0 < t \leq d \text{ and } 0 < g_1 \ldots < g_t, \text{ if } d \neq 0. \end{cases}$$

Observe that R_d is precisely the collection of the numbers of $[0, 1)$ which have at most d "ones" in their dyadic expansion. Let $\alpha \in R_d$ with $d \neq 0$. Consider the interval $0 \leq x < \alpha$. It is the disjoint union of at most d elementary intervals, in the sense defined in § 1.

It was shown in § 1 that $|D(n, I)| \leq 1 (n = 1, 2, \ldots)$, i.e., that $E(I) \leq 1$ for elementary intervals I. The interval $0 \leq x < \alpha$ with $\alpha \in R_d$, being the union of at most d elementary intervals, has error $\leq d$, and since the elements of the Van der Corput sequence are distinct, the closed interval $0 \leq x \leq \alpha$ has error $\leq d + 1$. Thus

$$|D(n, \alpha)| \leq d + 1 \qquad (n = 1, 2, \ldots).$$

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This is true for every $\alpha \in R_f$, so that

$$R_d \subseteq S(d + 1).$$

It is easy to check that

$$R_d^{(1)} = R_{d-1}, (d = 1, 2, \ldots),$$

whence

$$R_d^{(d)} = R_0 = \{0\} \neq \phi.$$

Since $R_d^{(d)} \subseteq S^{(d)}(d+1)$, we obtain

$$S^{(d)}(d+1) \neq \phi.$$

We finally mention that Theorem 7C* can be generalized to k dimensions: The volumes of the boxes B with finite error $E(B)$ from at most a countable set. This result is complemented by

Theorem 7I*. *Suppose $k > 1$. There is a sequence $\underline{\underline{x}}_1, \underline{\underline{x}}_2, \ldots$ in U^k, such that for every μ in $0 \leq \mu \leq 1$, there is a convex set S in U^k with measure $\mu(S) = \mu$ and with $E(S) \leq \frac{1}{2}$.*

This last result was generalized by *H*. Niederreiter [15].

Chapter 2

The Method of Integral Equations

1 A Theorem on Balls

Let $\underline{\underline{p}}_1, \ldots, \underline{\underline{p}}_N$, be points in U^k, all of whose coordinates are less than 1. A point $\underline{\underline{\ell}}$ in k-dimensional space R^k will be called an *integer point* if all its coordinates are integers. Integer points will be denoted by $\underline{\underline{\ell}}, \underline{\underline{\ell}}_1, \underline{\underline{\ell}}_2, \ldots$. Denote by $\mathscr{P}$ the set of points $\underline{\underline{p}}_i + \underline{\underline{\ell}}$, $1 \leq i \leq n$, where $\underline{\underline{\ell}}$ runs over all the integer points. Thus $\mathscr{P}$ is a "periodic" set. Let A be a bonded set with volume $\mu(A)$. Write 56

$z(A)$ for the number of points of $\mathscr{P}$ in A,

and

$$D(A) = z(A) - N\mu(A).$$

Call $D(A)$ the 'error'. Let $K(r, \underline{\underline{c}})$ be the ball with radius r and centre $\underline{\underline{c}}$, i. e. the set of all the points $\underline{\underline{x}}$ with $|\underline{\underline{x}} - \underline{\underline{c}}| \leq r$. Put

$$z(r, \underline{\underline{c}}) = z(K(r, \underline{\underline{c}})), D(r, \underline{\underline{c}}) = D(K(r, \underline{\underline{c}})), \mu(r) = \mu(K(r, \underline{\underline{c}})).$$

Set

$$E(r, s) = \int_{U^k} D(r, \underline{\underline{c}}) D(s, \underline{\underline{c}}) d\underline{\underline{c}}.$$

Theorem 1A (W. M. Schmidt [22]). *Let $k > 0$, $\epsilon > 0$. Suppose $\delta > 0$ satisfies $N\delta^k > N^\epsilon$. Then*

$$\int_0^\delta r^{-1}E(r,r)dr \gg (n\delta^k)^{1-\frac{1}{k}-\epsilon},$$

where the constant implied by $\gg$ depends on k and ϵ, but is independent of N and δ.

Theorem 1B. *Let $k, \in, \delta$ be as in Theorem 1A. Then there exists a ball $K(r, \underline{\underline{c}})$ with $r \leq \delta$ and*

$$|D(r, \underline{\underline{c}})| \gg (n\delta^k)^{\frac{1}{2}-\frac{1}{2k}-\epsilon},$$

57 *where the constant implied by $\gg$ depends on k and ϵ, but is independent of N and δ.*

According to Theorem 1B, there exists a ball with 'error' very large as compared with that of boxes with sides parallel to the axes. We recall: Roth proved (see Chapter I, § 2), that there exists a box B with sides parallel to the axes contained in U^k with

$$|D(B)| > c(\log N)^{\frac{k-1}{2}}.$$

On the other hand, there are distributions of N points such that $|D(B)| < c'(\log N)^{k-1}$ (see Chapter I, § 2).

Remarks.

(i) If δ and k are fixed, the hypotheses of Theorem 1A and Theorem 1B are satisfied for large N. On the other hand, we can allow δ to be small, namely as small as $\delta^k = N^{-1+2\epsilon}$, and still we get a lower bound which tends to infinity with N.

(ii) The ball $K(r, \underline{\underline{c}})$ of Theorem 1B is not necessarily contained in U^k.

(iii) One would like to ask if there exists a ball with positive $D(r, \underline{\underline{c}})$ and with

$$D(r, \underline{\underline{c}}) \gg (N\delta^k)^{\frac{1}{2}-\frac{1}{2k}-\epsilon},$$

or a ball with negative $D(r, \underline{c})$ and with

$$-D(r, \underline{c}) \gg (N\delta^k)^{\frac{1}{2}-\frac{1}{2k}-\epsilon}.$$

(iv) We do not know what would be the best possible exponent in the estimate of Theorem 1B, Using probabilistic methods, one sees the existence of points $\underline{p}_1, \ldots, \underline{p}_N$ with

$$|D(r, \underline{c})| \ll N^{\frac{1}{2}+\epsilon}, \text{ if } r \leq 1.$$

(See, e.g., W. Philipp [18].) **58**

We shall prove that Theorem 1A implies Theorem 1B. For this we require the following:

Lemma 1C.

$$|E(r, s)| \ll N^2 \max(r^k s^k, \min(r^k, s^k)).$$

Remark. Here and later, the constant in $\ll$ depends on k and ϵ, but is independent of n, δ.

Proof. Note that

$$|E(r, s)| \leq \int_{U^k} (z(r, \underline{c}) + N\mu(r))(z(s, \underline{c}) + N\mu(s))d\underline{c}.$$

□

Observe that

$$z(s, \underline{c}) \ll N \max(1, s^k),$$
$$\int_{U^k} z(s, \underline{c})d\underline{c} = N\mu(s) \ll Ns^k. \tag{1.1}$$

Using these relations, we obtain

$$|E(r, s)| \ll N \max(1, r^k) \int_{U^k} (z(s, \underline{c}) + N\mu(s))d\underline{c}$$

$$\begin{aligned} &\ll N \max(1, r^k) N s^k \\ &= N^2 \max(s^k, r^k s^k). \end{aligned}$$

Since $|E(r,s)|$ is symmetric in r and s, we also obtain

$$|E(r,s)| \ll N^2 \max(r^k, r^k s^k).$$

Hence

$$|E(r,s)| \ll N^2 \max(r^k s^k, \min(r^k, s^k)).$$

This proves Lemma 1C.

59 The proof that Theorem 1A implies Theorem 1B runs as follows. If $N\delta^k \ll 1$, take a ball with centre $\underline{\underline{p}}_i$ and very small radius r. Then

$$\begin{aligned} |D(r, \underline{\underline{c}})| &\geq |z(r, \underline{\underline{c}})| - |N\mu(r)| \\ &\geq 1 - \frac{1}{2} = \frac{1}{2} \gg (N\delta^k)^{\frac{1}{2} - \frac{1}{2k} - \epsilon}. \end{aligned}$$

So we may assume that $N\delta^k > c'$, where c' is a large positive constant depending only on k and ϵ. Put

$$\eta = N^{-2/k}$$

Then $\eta < \delta$, since $N\eta^k = N^{-1} \leq N^{\epsilon} < N\delta^k$. By Lemma 1C,

$$\int_0^{\eta} r^{-1} E(r,r) dr \ll N^2 \int_0^{\eta} r^{k-1} dr \gg N^2 \eta^k = 1. \tag{1.2}$$

From Theorem 1A and (1.2), we obtain

$$\int_{\eta}^{\delta} r^{-1} E(r,r) dr \gg (N\delta^k)^{1 - \frac{1}{k} - \epsilon}, \tag{1.3}$$

if c' is large enough. Notice that

$$\left| \int_{\eta}^{\delta} r^{-1} E(r,r) dr \right| \leq (\max_{\eta \leq r \leq \delta} |E(r,r)|) \int_{\eta}^{\delta} r^{-1} dr. \tag{1.4}$$

But

$$\int_{\eta}^{\delta} r^{-1}dr = \log(\frac{\delta}{\eta}) = \log(\delta N^{2/k})$$
$$= \frac{1}{k}\log(\delta^k N, N)$$
$$\ll \log(\delta^k N) \ll (\delta^k N)^{\epsilon} \tag{1.5}$$

since $\delta^k N > N^{\epsilon}$ and since $N\delta^k$ is large. Combining (1.3), (1.4), (1.5), we conclude that there is an r with $\eta \leq r \leq \delta$ and

$$|E(r,r)| \gg (N\delta^k)^{1-\frac{1}{k}-\epsilon}.$$

Hence there exists a $\underline{\underline{c}} \in U^k$ such that

$$|D(r,\underline{\underline{c}})| \gg (N\delta^k)^{\frac{1}{2}-\frac{1}{2k}-\frac{\epsilon}{2}}.$$

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Thus Theorem 1B is true.

2 Setting up an Integral Equation

Let k be an integer > 1. Put

$$\nu = \begin{cases} 1, & \text{if } k \text{ is odd },\\ 0, & \text{if } k \text{ is even.}\end{cases}$$

Let $0 < \alpha < 1$, and let β be such that $\alpha + \beta = 1 + \nu$. Assume that $\delta > 0$. Put

$$A = \underset{r+s\leq 1}{\int_0^1 \int_0^1} E(\delta r, \delta s)|r-s|^{-\alpha}|r+s|^{-\beta}dr \quad ds.$$

We shall show that

$$|A| \ll \int_0^{\delta} E(r,r)r^{-1}dr, \tag{2.1}$$

where the constant implied by $\ll$ depends only on α and k. Note that

$$\begin{aligned}|2E(r,s)| &\leq \int_{U^k} 2|D(r,\underline{\underline{c}})D(s,\underline{\underline{c}})|d\underline{\underline{c}}\\ &\leq \int_{U^k}(D^2(r,\underline{\underline{c}})+D^2(s,\underline{\underline{c}}))d\underline{\underline{c}}\\ &= E(r,r)+E(s,s).\end{aligned}$$

Therefore

$$|A| \leq \frac{1}{2}\underset{r+s\leq 1}{\int_0^1\int_0^1}(E(\delta r,\delta r)+E(\delta s,\delta s))|r-s|^{-\alpha}|r+s|^{-\beta}dr \quad ds.$$

Since $|r-s|^{-\alpha}|r+s|^{-\beta}$ is symmetric in r and s, we have

$$\begin{aligned}|A| &\leq \underset{r+s\leq 1}{\int_0^1\int_0^1} E(\delta r,\delta r)|r-s|^{-\alpha}|r+s|^{-\beta}dr \quad ds\\ &= \int_0^1 E(\delta r,\delta r)r^{-\nu}dr\int_0^{\frac{1}{r}-1}|1-t|^{-\alpha}|1+t|^{-\beta}dt,\end{aligned}$$

61 by introducing a new variable t given by $s=rt$. For $0<r\leq 1$, observe that the inner integral

$$\int_0^{\frac{1}{r}-1}|1-t|^{-\alpha}|1+t|^{-\beta}dt \ll \begin{cases}1, & \text{if } \alpha+\beta=1+\nu=2,\\ 1, & \text{if } 1+\nu<2 \text{ and } \frac{1}{r}\leq 10,\\ \log\frac{1}{r}, & \text{if } 1+\nu=1 \text{ and } \frac{1}{r}>10.\end{cases}$$

So in general the above integral is

$$\ll (1+\log\frac{1}{r})^{1-\nu} \ll r^{\nu-1} \qquad (0<r\leq 1).$$

Hence

$$|A| \ll \int_0^1 E(\delta r,\delta r)r^{-1}dr = \int_0^{\delta} E(r,r)r^{-1}dr,$$

and (2.1) is proved.

Put

$$f(r, \underline{\underline{c}}; \underline{\underline{x}}) = \begin{cases} 1, & \text{if } \underline{\underline{x}} \epsilon K(r, \underline{\underline{c}}), \\ 0 & \text{otherwise} \end{cases},$$

so that $f(r, \underline{\underline{c}}; \underline{\underline{x}})$ for fixed $r, \underline{\underline{c}}$ is the characteristic function of $K(r, \underline{\underline{c}})$.

Write

$$g(r, \underline{\underline{c}}; \underline{\underline{x}}) = \sum_{\underline{\underline{\ell}}} f(r, \underline{\underline{c}}; \underline{\underline{x}} + \ell),$$

where the sum is taken over all the integer points ℓ. Notice that $f(r, \underline{\underline{c}}; \underline{\underline{x}} + \underline{\underline{\ell}}) = 0$ except for finitely many $\underline{\underline{\ell}}$. Further observe that

$$z(r, \underline{\underline{c}}) = \sum_{i=1}^{N} \sum_{\underline{\underline{\ell}}} f(r, \underline{\underline{c}}; \underline{\underline{p}}_i + \underline{\underline{\ell}}) = \sum_{i=1}^{N} g(r, \underline{\underline{c}}; \underline{\underline{p}}_i). \tag{2.2}$$

Put

$$\omega(\underline{\underline{x}}, \underline{\underline{y}}) = \min_{\underline{\underline{\ell}}} |\underline{\underline{x}} - \underline{\underline{y}} - \underline{\underline{\ell}}|.$$

where the minimum is taken over all the integer points. We could interpret ω as the "distance modulo 1" of $\underline{\underline{x}}$ and $\underline{\underline{y}}$. Denote by $K(r, s, \omega)$ the volume of the intersection of two balls with radii r and s, whose centres **62**
have distance ω. *We shall show that if r and s are positive real numbers with $r + s < \frac{1}{2}$, then*

$$E(r, s) = \sum_{i=1}^{N} \sum_{j=1}^{N} (K(r, s, \omega)(\underline{\underline{p}}_i, \underline{\underline{p}}_j)) - \int_{U^k} \int_{U^k} K(r, s, \omega(\underline{\underline{x}}, \underline{\underline{y}}) d\underline{\underline{x}} \quad d\underline{\underline{y}}). \tag{2.3}$$

We start by noting that

$$\begin{aligned} E(r, s) &= \int_{U^k} D(r, \underline{\underline{c}}) D(s, \underline{\underline{c}}) d\underline{\underline{c}} \\ &= \int_{U^k} (z(r, \underline{\underline{c}}) - N\mu(r))(z(s, \underline{\underline{c}}) - N\mu(s)) d\underline{\underline{c}} \\ &= \int_{U^k} (z(r, \underline{\underline{c}}) z(s, \underline{\underline{c}})) d\underline{\underline{c}} - N^2 \mu(r) \mu(s) \end{aligned} \tag{2.4}$$

since

$$\int_{U^k} z(r,\underline{\underline{c}})d\underline{\underline{c}} = \sum_{i=1}^{N}\sum_{\underline{\underline{\ell}}}\int_{U^k} f(r,\underline{\underline{c}}-\underline{\underline{\ell}};\underline{\underline{p}}_i)d\underline{\underline{c}} = \sum_{i=1}^{N}\int_{R^k} f(r,\underline{\underline{c}};\underline{\underline{p}}_i)d\underline{\underline{c}} = N\mu(r).$$

From (2.2) and (2.4),

$$E(r,s) = \sum_{i=1}^{N}\sum_{j=1}^{N}\left(\int_{U^k} g(r,\underline{\underline{c}};\underline{\underline{p}}_i)g(s,\underline{\underline{c}};\underline{\underline{p}}_i)d\underline{\underline{c}} - \mu(r)\mu(s)\right) \tag{2.5}$$

The integral

$$\int_{U^k} g(r,\underline{\underline{c}}'\underline{\underline{x}})g(s,\underline{\underline{c}};\underline{\underline{y}})d\underline{\underline{c}}$$

is periodic in $\underline{\underline{x}}$ and $\underline{\underline{y}}$. (That is, the integral remains unchanged if $\underline{\underline{x}}$ and $\underline{\underline{y}}$ are replaced by $\underline{\underline{x}}+\underline{\underline{\ell}}_1$ and $\underline{\underline{y}}+\underline{\underline{\ell}}_2$, respectively). Therefore to study the integral we may assume that $\underline{\underline{x}} = (x_1,\ldots,x_k)$, $\underline{\underline{y}} = (y_1,\ldots,y_k)$ satisfy $|x_i - y_i| \leq \dfrac{1}{2}(1 \leq i \leq k)$. Observe that

$$\begin{aligned}\int_{U^k} g(r,\underline{\underline{c}};\underline{\underline{x}})g(s,\underline{\underline{c}};\underline{\underline{y}})d\underline{\underline{c}} &= \int_{U^k}\sum_{\underline{\underline{\ell}}_1}\sum_{\underline{\underline{\ell}}_2} f(r,\underline{\underline{c}}-\underline{\underline{\ell}}_1;\underline{\underline{x}})f(s,\underline{\underline{c}}-\underline{\underline{\ell}}_2;\underline{\underline{y}})d\underline{\underline{c}}\\ &= \sum_{\underline{\underline{\ell}}}\int_{U^k} f(r,\underline{\underline{c}}-\underline{\underline{\ell}};\underline{\underline{x}})f(s,\underline{\underline{c}}-\underline{\underline{\ell}};\underline{\underline{y}})d\underline{\underline{c}},\end{aligned}$$

63 since if $\underline{\underline{\ell}}_1 = (\ell_{11},\ldots,\ell_{1k})$, $\underline{\underline{\ell}}_2 = (\ell_{21},\ldots,\ell_{2k})$, $\underline{\underline{c}} = (c_1,\ldots,c_k)$ and $f(r,\underline{\underline{c}}-\underline{\underline{\ell}}_1;\underline{\underline{x}}) = f(s,\underline{\underline{c}}-\underline{\underline{\ell}}_2;\underline{\underline{y}}) = 1$, then

$$|c_i - \ell_{1i} - x_i| \leq r \text{ and } |c_i - \ell_{2i} - y_i| \leq s \qquad (1 \leq i \leq k).$$

So $|\ell_{1i} - \ell_{2i}| \leq r + s + |x_i - y_i| < \dfrac{1}{2} + \dfrac{1}{2} = 1 \quad (1 \leq i \leq k)$. which implies that $\ell_{1i} = \ell_{2i}$ for $1 \leq i \leq k$, i.e. that $\underline{\underline{\ell}}_1 = \underline{\underline{\ell}}_2$. Therefore

$$\int_{U^k} g(r,\underline{\underline{c}};\underline{\underline{x}})g(s,\underline{\underline{c}};\underline{\underline{y}})d\underline{\underline{c}} = \int_{R^k} f(r,\underline{\underline{c}};\underline{\underline{x}})f(s,\underline{\underline{c}};\underline{\underline{y}})d\underline{\underline{c}}$$

$$= K(r, s, |\underline{\underline{x}} - \underline{\underline{y}}|).$$

This holds if $|x_i - y_i| \leq \dfrac{1}{2}(1 \leq i \leq k)$. In general we have

$$\int_{U^k} g(r, \underline{\underline{c}}; \underline{\underline{x}}) g(s, \underline{\underline{c}}; \underline{\underline{y}}) d\underline{\underline{c}} = K(r, s, \omega(\underline{\underline{x}}, \underline{\underline{y}})) \tag{2.6}$$

The last equation yields

$$\begin{aligned} &\int_{U^k} \int_{U^k} K(r, s, \omega(\underline{\underline{x}}, \underline{\underline{y}})) d\underline{\underline{x}} \quad d\underline{\underline{y}} \\ &\qquad = \int_{U^k} \int_{U^k} \int_{U^k} g(r, \underline{\underline{c}}; \underline{\underline{x}}) g(s, \underline{\underline{c}}; \underline{\underline{y}}) d\underline{\underline{c}} \quad d\underline{\underline{x}} \quad d\underline{\underline{y}} \\ &\qquad = \int_{U^k} (\int_{U^k} g(r, \underline{\underline{c}}; \underline{\underline{x}})(d\underline{\underline{x}})(\int_{U^k} g(s, \underline{\underline{c}}; \underline{\underline{y}}) d\underline{\underline{y}}) d\underline{\underline{c}} \\ &\qquad = \mu(r)\mu(s) \end{aligned} \tag{2.7}$$

since $\int_{U^k} g(r, \underline{\underline{c}}; \underline{\underline{x}}) d\underline{\underline{x}} = \sum_{\underline{\underline{\ell}}} \int_{U^k} f(r, \underline{\underline{c}}; \underline{\underline{x}} + \underline{\underline{\ell}}) dx = \int_{R^k} f(r, \underline{\underline{c}}; \underline{\underline{x}}) = \mu(r)$.

Combining (2.5), (2.6) and (2.7), we obtain (2.3).

Lemma 2A (Fundamental Lemma). *Suppose that* $0 < \delta < \dfrac{1}{2}$. *Suppose* $f(r)$ *is continuous in* $0 < r \leq \dfrac{1}{2}$ *with*

$$|f(r)| \ll r^{1-k-\alpha} \text{ as } r \to 0,$$

and satisfies the integral equation

$$\int_0^{\frac{1}{2}} K(r\delta, r\delta, \omega) f(r) dr = \underset{r+s\leq 1}{\int_0^1 \int_0^1} \frac{K(r\delta, s\delta, \omega)}{|r-s|^{\alpha} |r+s|^{\beta}} dr \quad ds \quad (\omega \leq 0). \tag{2.8}$$

Then 64

$$\int_0^{\frac{1}{2}} E(\delta r, \delta r) f(r) dr = \int_0^1 \int_0^1 \frac{E(\delta r, \delta s)}{|r-s|^{\alpha} |r+s|^{\beta}} . dr \quad ds. \tag{2.9}$$

Proof. The lemma follows immediately from (2.3). All the functions occurring in the integrals are summable. □

Remark. *If (2.8) is true for some $\delta > 0$, then (2.8) and (2.9) are true for every $\delta > 0$.* This is seen as follows. For a positive integer m, denote by $\mathscr{P}^*$ the set $\dfrac{1}{m}\mathscr{P}$. Define $E^*(r, s)$ with reference to $\mathscr{P}^*$. Notice that for every $r, s > 0$,

$$E^*\left(\frac{r}{m}, \frac{s}{m}\right) = E(r, s), \tag{2.10}$$

since the set $\mathscr{P}$ is periodic and $D^*(\dfrac{r}{m}, \dfrac{\underline{\underline{c}}}{m}) = D(r, \underline{\underline{c}}), D^*(\dfrac{s}{m}, \dfrac{\underline{\underline{c}}}{m}) = D(s, \underline{\underline{c}})$.

Suppose that (2.8) is true for some $\delta > 0$. Since for any $c > 0$, $K(cr, cs, c\omega) = c^k K(r, s, \omega)$, (2.8) is true for $c\delta$ and hence for every $\delta > 0$. Now given $\delta > 0$, choose an integer in such that $\delta/m < \dfrac{1}{2}$. Then by Lemma 2A,

$$\int_0^{\frac{1}{2}} E^*\left(\frac{\delta}{m}r, \frac{\delta}{m}r\right) f(r)dr = \underset{r+s\leq 1}{\int_0^1 \int_0^1} \frac{E^*\left(\frac{\delta}{m}r, \frac{\delta}{m}s\right)}{|r-s|^\alpha |r+s|^\beta} dr \quad ds,$$

which along with (2.10) gives (2.9).

3 Differentiating the Integral Equation

In view of the Remarks at the end of § 2, we may restrict ourselves to the equation (2.8) with $\delta = 1$, i. e. to

$$\int_0^{\frac{1}{2}} f(r)K(r, r, \omega)dr = \underset{r+s\leq 1}{\int_0^1 \int_0^1} \frac{K(r, s, \omega)}{|r-s|^\alpha |r+s|^\beta} dr \quad ds \quad (\omega \geq 0). \tag{3.1}$$

We shall determine a solution $f(r)$ of this integral equation. It is sufficient to consider (3.1) for $0 \leq \omega < 1$, since both sides of (3.1) are zero when $\omega \geq 1$.

65 Further we may assume that $0 < \omega < 1$, since both sides of (3.1)

are continuous functions of ω. In fact they satisfy a Lipschitz condition, since

$$|K(r,s,\omega) - K(r,s,\omega')| \ll |\omega - \omega'| \min(r^{k-1}, s^{k-1}), \tag{3.2}$$

since

$$\int_0^{\frac{1}{2}} |f(r)| r^{k-1} dr \ll 1 \tag{3.3}$$

by virtue of $f(r) \ll r^{1-k-\alpha}$, and since

$$\underset{r+s\leq 1}{\int_0^1 \int_0^1} \frac{\min(r^{k-1}, s^{k-1})}{|r-s|^{\alpha} |r+s|^{\beta}} dr \quad ds \ll 1. \tag{3.4}$$

Now suppose that $|r-s| < \omega < r+s$. Let $K(r, \underline{\underline{c}})$, $K(s, \underline{\underline{d}})$ be balls with $|\underline{\underline{c}} - \underline{\underline{d}}| = \omega$. The boundaries of $K(r, \underline{\underline{c}})$, $K(s, \underline{\underline{d}})$ intersect in a sphere S^{k-2}. Denote the radius of S^{k-2} by e, the distance from the centre of S^{k-2} to $\underline{\underline{c}}$, $\underline{\underline{d}}$ by a, b, respectively. (The reader might want to draw a sketch). We have

$$a + b = \omega, a^2 + e^2 = r^2 \text{ and } b^2 + e^2 = s^2.$$

Eliminating a and b, we obtain

$$e^2 = \frac{1}{4}\left(1 - \frac{(r-s)^2}{\omega^2}\right)((r+s)^2 - \omega^2).$$

We have

$$K(r,s,\omega) = c_1 \int_0^e z^{k-2} \left(\sqrt{r^2 - z^2} + \sqrt{s^2 - z^2} - \omega\right) dz.$$

(The constant c_1 and subsequent constants $c_2, c_3, \ldots$ are positive and depend only on k and on α). Since the integrand vanishes for $z = e$, we obtain

$$\frac{\partial}{\partial\omega} K(r,s,\omega) = -c_1 \int_0^e z^{k-2} dz = -c_2 e^{k-1}.$$

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All this holds for $|r-s| < \omega < |r+s|$. For $\omega \leq |r-s|$, $K(r,s,\omega)$ is independent of ω, and for $\omega \geq r+s$, we have $K(r,s,\omega) = 0$. From all this, it follows that

$$\frac{\partial}{\partial\omega}K(r,s,\omega) = \begin{cases} -c_2 e^{k-1} & \text{if } |r-s| < \omega < r+s, \\ 0 & \text{otherwise .} \end{cases}$$

Now we differentiate both sides of (3.1) with respect to ω. By (3.2), (3.3) and Lebesgue's Theorem on dominated convergence, (see e.g., § 19 of [14]), the left hand side of (3.1) can be differentiated inside the integral.

By (3.2), (3.3) and the dominated convergence Theorem, the same can be done for the right hand side of (3.1). The derivative of the left hand side of (3.1) is

$$-c_2 \int_{\omega/2}^{\frac{1}{2}} f(r)(r^2 - \frac{\omega^2}{4})^{(k-1)/2} dr. \tag{3.5}$$

The derivative of the right hand side of (3.1) is

$$-c_2 \underset{|r-s|\leq\omega\leq r+s\leq 1}{\int_0^1\int_0^1} \frac{\left(\frac{1}{4}\left(1-\frac{(r-s)^2}{\omega^2}\right)((r+s)^2-\omega^2)\right)^{(k-1)/2}}{|r-s|^\alpha |r+s|^\beta} dr \quad ds.$$

Put $x = r+s$ and $y = |r-s|$. The above integral becomes

$$\begin{aligned} &- c_3 \left(\int_\omega^1 (x^2-\omega^2)^{(k-1)/2} x^{-\beta} dx\right)\left(\int_0^\omega (1-\frac{y^2}{\omega^2})^{(k-1)/2} y^{-\alpha} dy\right) \\ &= -c_4 \omega^{1-\alpha} \int_\omega^1 x^{-\beta}(x^2-\omega^2)^{(k-1)/2} dx, \end{aligned}$$

since substituting $y = t\omega$ we get

$$\int_0^\omega \left(1-\frac{y^2}{\omega^2}\right)^{(k-1)/2} y^{-\alpha} dy = \omega^{1-\alpha}\int_0^1 (1-t^2)^{(k-1)/2} t^{-\alpha} dt.$$

Hence after differentiation, the equation (3.1) becomes

$$\int_{\omega/2}^{\frac{1}{2}} f(r)(r^2 - \frac{\omega^2}{4})^{(k-1)/2} dr$$

$$= c_5\omega^{1-\alpha}\int_{\omega}^{1} x^{-\beta}(x^2-\omega^2)^{(k-1)/2}dx \quad (o < \omega < 1). \quad (3.6)$$

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Any solution of this integral equation is also a solution of (3.1), since both sides of (3.1) are continuous and are zero for $\omega = 1$. Putting $\omega^2 = t$ in (3.6), we get

$$\int_{\frac{\sqrt{t}}{2}}^{\frac{1}{2}} f(r)\left(r^2 - \frac{t}{4}\right)^{(k-1)/2} dr$$
$$= c_5 t^{(1-\alpha)/2}\int_{\sqrt{t}}^{1} x^{-\beta}(x^2-t)^{(k-1)/2}dx\,(0 < t < 1). \qquad (3.7)$$

We now differentiate (3.7)

$$m = \left[\frac{k}{2}\right] = \frac{k-\nu}{2}$$

times with respect to t. The left hand side becomes

$$c_6(-1)^m\int_{\frac{\sqrt{t}}{2}}^{\frac{1}{2}} f(r)\left(r^2-\frac{t}{4}\right)^{(\nu-1)/2} dr.$$

If on the right hand side we differentiate $t^{(1-\alpha)/2}$ precisely i times, and the integral $\int_{\sqrt{t}}^{1} x^{-\beta}(x^2-t)^{(k-1)/2}dx$ $(m-i)$ times, we obtain

$$(-1)^m c_6^{(0)} t^{(1-\alpha)/2}\int_{\sqrt{t}}^{1} x^{-\beta}(x^2-t)^{(\nu-1)/2}dx$$
$$= (-1)^m c_6^{(0)}\int_{1}^{1/\sqrt{t}} y^{-\beta}(y^2-1)^{(\nu-1)/2}dy \text{ if } i = 0$$

and

$$(-1)^m c_6^{(i)} t^{(1-\alpha-2i)/2}\int_{\sqrt{t}}^{1} x^{-\beta}(x^2-t)^{(k-1-2m)/2}dx$$
$$= (-1)^m c_6^{(i)}\int_{1}^{1/\sqrt{t}} y^{-\beta}(y^2-1)^{(\nu-1+2i)/2}dy \text{ if } 1 \le i \le m.$$

Here all the constants $c_6^{(i)}$ are positive $(0 \leq i \leq m)$. Hence (3.7) becomes

$$\begin{aligned}
&\int_{\sqrt{t}/2}^{\frac{1}{2}} f(r)(r^2 - \frac{t}{4})^{(\nu-1)/2} dr \\
&= c_7^{(0)} \int_1^{1/\sqrt{t}} y^{-\beta}(y^2-1)^{(\nu-1)/2} dy \\
&- \sum_{i=1}^{m} c_7^{(i)} \int_1^{1/\sqrt{t}} y^{-\beta}(y^2-1)^{(\nu-1+2i)/2} dy \; (0 < t < 1). \qquad (3.8)
\end{aligned}$$

68 Now the left hand side and the right hand side of (3.7) and their first $m-1$ derivatives are continuous in t and vanish for $t = 1$. From this fact it follows that every solution $f(r)$ of (3.8) is also a solution of (3.7). We now write $\sqrt{t} = \omega$ and rewrite (3.8) as

$$\begin{aligned}
&\int_{\omega/2}^{\frac{1}{2}} f(r)(r^2 - \frac{\omega^2}{4})^{(\nu-1)/2} dr \\
&= c_7^{(0)} \int_1^{1/\omega} y^{-\beta}(y^2-1)^{(\nu-1)/2} dy - \sum_{i=1}^{m} c_7^{(i)} \int_1^{1/\omega} y^{-\beta}(y^2-1)^{(\nu-1+2i)/2} dy \\
&= \ell_0(\omega) - \sum_{i=1}^{m} \ell_i(\omega), \qquad (3.9)
\end{aligned}$$

say. This equation is to hold for $0 < \omega < 1$. For every i with $0 \leq i \leq m$, we shall find a solution $f_i(r)$ of the integral equation

$$\int_{\omega/2}^{1/2} f_i(r)(r^2 - \frac{\omega^2}{4})^{(\nu-1)/2} dr = \ell_i(\omega) \qquad (0 < \omega < 1). \qquad (3.10)$$

Then

$$f(r) = f_0(r) - \sum_{i=1}^{m} f_i(r)$$

will be a solution of (3.9).

4 Solving the Integral Equation

Case I. k is odd, i.e. $\nu = 1$.

Differentiate (3.10) with respect to ω. We obtain

$$-\frac{1}{2} f_i\left(\frac{\omega}{2}\right) = -c_7^{(i)} \omega^{\beta-\nu-2i-1}(1-\omega^2)^{(\nu-1+2i)/2}. \tag{4.1}$$

Hence

$$f_i\left(\frac{\omega}{2}\right) = 2c_7^{(i)} \omega^{-2+\beta-2i}(1-\omega^2)^i,$$

or

$$f_i(r) = c_8^{(i)} r^{-\alpha-2}(1-4r^2)^i \qquad (0 < r \leq \frac{1}{2}).$$

It is clear that $f_i(r)$ is in fact a solution of the integral equation (3.10). **69**
Put

$$f_*(r) = \sum_{i=1}^{m} f_i(r) \text{ and } f(r) = f_0(r) - f_*(r).$$

Observe that the functions $f_i(r)$ $(0 \leq i \leq m)$ are continuous in $0 < r \leq \frac{1}{2}$. When $r \to 0$,

$$f_0(r) \gg\ll r^{-\alpha},\text{[1]}$$

$$f_*(r) \gg\ll \sum_{i=1}^{(k-1)/2} r^{-\alpha-2i} \gg\ll r^{1-k-\alpha}. \tag{4.2}$$

Therefore $f(r)$ is continuous in $0 < r \leq \frac{1}{2}$, and

$$|f(r)| \ll r^{1-k-\alpha} \text{ as } r \to 0.$$

Hence by Lemma A and the remark below it,

$$\int_0^{\frac{1}{2}} E(\delta r, \delta r) f(r) dr = \underset{r+s\leq 1}{\int_0^1 \int_0^1} \frac{E(\delta r, \delta s)}{|r-s|^\alpha |r+s|^\beta} dr \quad ds = A.$$

---

[1] The notation $f \gg\ll g$ means that both $f \ll g$ and $g \ll f$.

Put differently,

$$\int_0^{\frac{1}{2}} E(\delta r, \delta r) f_0(r) dr - A = \int_0^{\frac{1}{2}} E(\delta r, \delta r) f_*(r) dr. \tag{4.3}$$

Now $|A| \ll \int_0^{\delta} E(r, r) r^{-1} dr$ by (2.1), and further more

$$\int_0^{\frac{1}{2}} E(\delta r, \delta r) f_0(r) dr \ll \int_0^1 E(\delta r, \delta r) r^{-1} dr = \int_0^{\delta} E(r, r) r^{-1} dr,$$

since $f_0(r) \ll r^{-\alpha}$ as $r \to 0$ and since $0 < \alpha < 1$. Therefore the left hand side of (4.3) is

$$\ll \int_0^{\delta} E(r, r) r^{-1} dr. \tag{4.4}$$

Now we shall show that the right hand side of (4.3) is large. Notice that

$$|D(r, \underline{\underline{c}})| \geq \|N\mu(r)\|,$$

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where $\|\zeta\|$ denotes the distance from ζ to the nearest integer. It follows that

$$E(r, r) \geq \|N\mu(r)\|^2.$$

In view of this and of (4.2), the right hand side of (4.3) exceeds

$$\int_0^{c_9} \|N\mu(\delta r)\|^2 c_{10} r^{1-k-\alpha} dr. \tag{4.5}$$

Let the interval I consist of $r > 0$ with

$$\frac{1}{4} \leq N\mu(\delta r) = c_{11} r^k \delta^k N \leq \frac{3}{4}.$$

Then every $r \in I$ satisfies

$$(4c_{11})^{-1/k} (N\delta^k)^{-1/k} \leq r \leq 3^{1/k} (4c_{11})^{-1/k} (N\delta^k)^{-1/k}$$

and

$$r^{1-k-\alpha} \gg (N\delta^k)^{-(1-k-\alpha)/k}.$$

Observe that $I \subseteq [0, c_9]$ if N is large enough and that

$$|I| \gg (N\delta^k)^{-1/k}.$$

Therefore

$$\int_0^{c_9} \|N\mu(\delta r)\|^2 c_{10} r^{1-k-\alpha} dr \geq \int_I \|N\mu(\delta r)\|^2 c_{10} r^{1-k-\alpha} dr \gg (N\delta^k)^{1+\frac{\alpha}{k}-\frac{2}{k}}. \tag{4.6}$$

Combining (4.3), (4.4), (4.5) and (4.6), we get

$$\int_0^{\delta} E(r, r) r^{-1} dr \gg (N\delta^k)^{1+\frac{\alpha}{k}-\frac{2}{k}}.$$

This is true for every α with $0 < \alpha < 1$. Putting $\alpha = 1 - \epsilon$, we obtain

$$\int_0^{\delta} E(r, r) r^{-1} dr \gg (N\delta^k)^{1-\frac{1}{k}-\epsilon}. \tag{4.7}$$

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Thus Theorem 1A is true when k is odd.

Case II. k is even, i. e. $\nu = 0$.

Putting $\nu = 0$ in (3.10), we have

$$\int_{\omega/2}^{\frac{1}{2}} f_i(r)\left(r^2 - \frac{\omega^2}{4}\right)^{-\frac{1}{2}} dr = \ell_i(\omega) \qquad (0 < \omega < 1) \tag{4.8}$$

This is an Abel integral equation. We check that

$$f_i(r) = -\frac{4}{\pi} \int_{2r}^{1} r(t^2 - 4r^2)^{-\frac{1}{2}} \ell_i'(t) dt \qquad \left(0 < r < \frac{1}{2}\right) \tag{4.9}$$

is a solution. Namely, substituting (4.9) into left hand side of (4.8), we get

$$\begin{aligned} &-\frac{4}{\pi} \int_{\omega/2}^{\frac{1}{2}} \left(r^2 - \frac{\omega^2}{4}\right)^{-\frac{1}{2}} dr \int_{2r}^{1} r(t^2 - 4r^2)^{-\frac{1}{2}} \ell_i'(t) dt \\ = &-\frac{4}{\pi} \int_{\omega/2}^{\frac{1}{2}} \ell_i'(t) dt \int_{\omega/2}^{t/2} r\left(r^2 - \frac{\omega^2}{4}\right)^{-\frac{1}{2}} (t^2 - 4r^2)^{-\frac{1}{2}} dr. \end{aligned}$$

The inner integral equals

$$\int_0^{\frac{1}{2}\sqrt{t^2-\omega^2}} (t^2-4u^2-\omega^2)^{-\frac{1}{2}}du = \frac{1}{2}\int_0^1 (1-u^2)^{-\frac{1}{2}}du = \frac{\pi}{4}.$$

So the left hand side of (4.8) becomes

$$\ell_i(\omega) = -\int_\omega^1 \ell_i'(t)dt = \ell_i(\omega) - \ell_i(1) = \ell_i(\omega).$$

Notice that $f_i(r)$ $(0 \le i \le m)$ is continuous in $0 < r < \frac{1}{2}$ and can be extended continuously in $0 < r \le \dfrac{1}{2}$. Further $f_i(r) \ge 0$, since $\ell_i'(t) < 0$ in $0 < t \le 1$ (see (4.1)). Using (4.9) and (4.1), we have

$$f_i(r) = c_{12}\int_{2r}^1 r(t^2-4r^2)^{-\frac{1}{2}}t^{-\alpha-2i}(1-t^2)^{(2i-1)/2}dt \qquad (0 \le i \le m).$$

72 Suppose that r is small, say $0 < r < \dfrac{1}{3}$. Write

$$f_i(r) = c_{12}r\int_{2r}^{\frac{1}{2}}\ldots + c_{12}r\int_{\frac{1}{2}}^1 \ldots .$$

The second integral is $\ll r < r^{1-\alpha-2i}$. The first integral is

$$\begin{aligned}
&\gg\ll r\int_{2r}^{\frac{1}{2}}(t^2-4r^2)^{-\frac{1}{2}}t^{-\alpha-2i}dt\\
&\gg\ll r^{1-\alpha-2i}\int_1^{1/(4r)}\frac{du}{u^{\alpha+2i}(u^2-1)^{1/2}}\\
&\gg\ll r^{1-\alpha-2i}.
\end{aligned}$$

It follows that for $r \to 0$,

$$f_i(r) \gg\ll r^{1-\alpha-2i} \quad (0 \le i \le m).$$

We now many proceed exactly as in Case I and conclude that (4.7) holds. This completes the proof of Theorem 2A.

5 A Good Distribution of Points

We shall show that Theorem 1*A* is in a sense best possible.

Theorem 5A (K. B. Stolarsky, [28a]). *For every k and N, there exists a distribution of N points in U^k, such that*

$$E(r,s) \le c_1(k)N^{1-(1/k)} \min(r^{k-1}, s^{k-1}) \text{ for } r+s < \frac{1}{2},$$

whence that

$$\int_0^\delta r^{-1}E(r,r)dr \le c_2(k)(N\delta^k)^{1-(1/k)} \text{ for } 0 < \delta < \frac{1}{4}.$$

For the *proof*, we note that there is a partition of U^k into N disjoint subsets $S_1, \dots, S_N$, each with measure $1/N$ and with diameter $\le c_3(k)N^{-1/k}$.

Let $E(r,s;\underline{\underline{p}}_1,\dots,\underline{\underline{p}}_N)$ be $E(r,s)$ with respect to the N points 73
$\underline{\underline{p}}_1,\dots,\underline{\underline{p}}_N$ in U^k. Theorem 5A is an immediate consequence of

Lemma 5B. *For any r, s with $r+s<\frac{1}{2}$, we have*

$$N^N \int_{S_1} d\underline{\underline{p}}_1 \dots \int_{S_N} d\underline{\underline{p}}_N\, E(r,s;\underline{\underline{p}}_1,\dots,\underline{\underline{p}}_N) \ll N^{1-(1/k)} \min(r^{k-1}, s^{k-1}).$$

Proof. In view of (2.3), we have to show that

$$F^* - F \ll N^{1-(1/k)} \min(r^{k-1}, s^{k-1}), \tag{5.1}$$

where

$$F^* = N^N \int_{S_1} d\underline{\underline{p}} \dots \int_{S_N} d\underline{\underline{p}}_N \sum_{i=1}^{N}\sum_{j=1}^{N} K(r,s,\omega(\underline{\underline{p}}_i, \underline{\underline{p}}_j)),$$

$$F = N^2 \int_{U^k}\int_{U^k} K(r,s,\omega(\underline{\underline{x}},\underline{\underline{y}}))d\underline{\underline{x}} \quad d\underline{\underline{y}}.$$

□

Now

$$F^* = N^2 \sum_{\substack{i=1 \\ i\neq j}}^{N} \sum_{j=1}^{N} \int_{S_i} d\underline{\underline{p}}_i \int_{S_j} d\underline{\underline{p}}_j K(r, s, \omega(\underline{\underline{p}}_i, \underline{\underline{p}}_j))$$

$$+ N \sum_{i=1}^{N} \int_{S_i} d\underline{\underline{p}}_i K(r, s, \omega(\underline{\underline{p}}_i, \underline{\underline{p}}_i))$$

$$= F + N^2 \sum_{i=1}^{N} \int_{S_i} d\underline{\underline{x}} \int_{S_i} d\underline{\underline{y}} (K(r, s, \omega(\underline{\underline{x}}, \underline{\underline{x}})) - K(r, s, \omega(\underline{\underline{x}}, \underline{\underline{y}}))).$$

Now since S_i has diameter $\ll N^{-1/k}$, we have

$$\omega(\underline{\underline{x}}, \underline{\underline{y}}) - \omega(\underline{\underline{x}}, \underline{\underline{x}}) = \omega(\underline{\underline{x}}, \underline{\underline{y}}) \ll N^{-1/k},$$

whence by (3.2),

$$\left| K(r, s, \omega(\underline{\underline{x}}, \underline{\underline{x}})) - K(r, s, \omega(\underline{\underline{x}}, \underline{\underline{y}})) \right| \ll N^{-1/k} \min(r^{k-1}, s^{k-1}),$$

if both $\underline{\underline{x}}, \underline{\underline{y}} \in S_i$. Thus

$$F^* - F \ll N^{-1/k} \min(r^{k-1}, s^{k-1}) N^2 \sum_{i=1}^{N} \int_{S_i} d\underline{\underline{x}} \int_{S_i} d\underline{\underline{y}},$$

and (5.1) follows.

6 Balls Contained in the Unit Cube

74 Let $\underline{\underline{p}}_1, \ldots, \underline{\underline{p}}_N$ be points in the unit cube U^k. If the ball $K(r, \underline{\underline{c}})$ is contained in U^k, write $z(r, \underline{\underline{c}})$ for the number of points $\underline{\underline{p}}_i$ in $K(r, \underline{\underline{c}})$. Also write

$$D(r, \underline{\underline{c}}) = z(r, \underline{\underline{c}}) - N\mu(r),$$

$$D(r, \underline{\underline{c}}, M) = z(r, \underline{\underline{c}}) - M\mu(r).$$

Put

$$F(r, M) = \int\limits_{K(r,\underline{\underline{c}})\subseteq U^k} D(r, \underline{\underline{c}}, M)^2 d\underline{\underline{c}}.$$

The domain of integration consists of all $\underline{\underline{c}}$ for which $K(r, c)$ is contained in the unit cube.

Theorem 6A. *Suppose that $k > 1$. Let δ, M and ϵ be positive real numbers satisfying*

$$0 < \in < \frac{1}{2(k+2)} \textit{ and } 1 < M^{\epsilon} < M\delta^k \leq M^{1/(k+2)-\epsilon}.$$

Then

$$\int_0^{\delta} r^{-1} F(r, M) dr \gg (M\delta^k)^{1-(1/k)-\epsilon}.$$

Theorem 6B. *Let $k, \in, \delta, M$ be as in Theorem 6A. Then there exists a ball $K(r, c) \subseteq U^k$ with $r \leq \delta$, having*

$$|D(r, c, M)| \gg (M\delta^k)^{\frac{1}{2}-(1/2k)-\epsilon}.$$

Now suppose $\eta > 0$. Put $\epsilon = \eta/2$ and

$$\delta = M^{-\frac{k+1}{k(k+2)}-\frac{\epsilon}{k}}.$$

For sufficiently small η the conditions of Theorem 6B are satisfied and we obtain the

COROLLARY. Suppose $M > 1$, $\eta > 0$. Then there exists a ball **75**
$K(r, \underline{\underline{c}}) \subseteq U^k$ with $r \leq M^{-\frac{(k+1)}{k(k+2)}}$ and

$$|D(r, \underline{\underline{c}}, M)| \gg M^{\frac{k-1}{2k(k+2)}} - \eta.$$

(The constant in $\gg$ depends only on k, η). Clearly, the interesting case for these results is when $M = N$. The general case will be needed in Theorem 6C below.

We shall now show that Theorem 6A implies Theorem 6B. The hypotheses imply that $0 < \delta < 1$. Since $(\delta/8)^k \gg \delta^k$, we may assume that

$0 < \delta < \dfrac{1}{8}$. Let U^{k*} be the cube of side $\dfrac{1}{2}$ whose centre is the same as that of U^k. This cube has the property that if $\underline{\underline{c}} \in U^{k*}$, then $K(\delta, \underline{\underline{c}}) \subseteq U^k$.

Earlier we derived Theorem 1B from Theorem 1A (see Chapter II, § 1). We used Lemma 1C of Chapter II. In fact, we used

$$|E(r,r)| \ll N^2 r^k \text{ if } 0 < r < 1.$$

We can similarly show that

$$|F(r,M)| \ll M^2 r^k,$$

provided that $N \leq 2^{k+1}M$ and $0 < r < 1$. In this case the proof that Theorem 6A implies Theorem 6B is similar to the proof that Theorem 1A implies Theorem 1B.

Let N^* be the number of points $\underline{\underline{p}}_i$ in U^{k*}.

Suppose at first that $N^* \leq 2M$, and put $M' = 2^{-k}M$, so that $N^* \leq 2^{k+1}M'$. Blow up U^{k^*} and N^* points in it by the factor 2. By what we just said, there is a ball $K(r, \underline{\underline{c}})$ in this blown up cube with $r \leq \delta$ and

$$\left| z'(r, \underline{\underline{c}}) - M'\mu(r) \right| \gg (M'\delta^k)^{\frac{1}{2}}$$

where $z'(r, \underline{\underline{c}})$ is counting function in this blown up cube. Hence in U^{k^*} itself, there is a ball $K\left(\dfrac{r}{2}, \underline{\underline{d}}\right)$ with

$$\begin{aligned} |z\left(\frac{r}{2}, \underline{\underline{d}}\right) - M\mu\left(\frac{r}{2}\right)| &= |z'(r, \underline{\underline{c}}) - M'\mu(r)| \\ \gg (M'\delta^k)^{\frac{1}{2}-(1/2k)-\epsilon} &\gg (M\delta^k)^{\frac{1}{2}-(1/2k)-\epsilon}. \end{aligned}$$

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It remains to dispose of the case when $N^* > 2M$. If $\underline{\underline{p}}_1, \ldots, \underline{\underline{p}}_{N^*}$ are the points in U^{k^*}, then

$$\int_{K(\delta,\underline{\underline{c}}) \subseteq U^k} z(\delta, \underline{\underline{c}}) d\underline{\underline{c}} \geq \int_{K(\delta,\underline{\underline{c}}) \subseteq U^k} \left(\sum_{\substack{1 \leq i \leq N^* \\ |\underline{\underline{c}} - \underline{\underline{p}}_i| \leq \delta}} 1 \right) d\underline{\underline{c}}$$

$$= \sum_{i=1}^{N^*} \int\limits_{\substack{K(\delta,\underline{\underline{c}})\subseteq U^k \\ |\underline{\underline{c}}-\underline{\underline{p}}_i|\leq\delta}} d\underline{\underline{c}} = \sum_{i=1}^{N^*} \int_{|\underline{\underline{c}}-\underline{\underline{p}}^*_i|\leq\delta} d\underline{\underline{c}} = N^*\mu(\delta).$$

Hence $\int_{K(\delta,\underline{\underline{c}})\subseteq U^k} D(\delta,\underline{\underline{c}},M)d\underline{\underline{c}}$

$$\begin{aligned} &= \int_{K(\delta,\underline{\underline{c}})\subseteq U^k} z(\delta,\underline{\underline{c}})d\underline{\underline{c}} - m\mu(\delta)\int_{K(\delta,\underline{\underline{c}})\subseteq U^k} d\underline{\underline{c}} \\ &\geq \mu(\delta)N^* - \mu(\delta)M > M\mu(\delta). \end{aligned}$$

Therefore there exists a $\underline{\underline{c}}$ with $K(\delta,\underline{\underline{c}})\subseteq U^k$ and

$$|D(\delta,\underline{\underline{c}},M)| \geq M\mu(\delta) \gg M\delta^k > (M\delta^k)^{\frac{1}{2}-(1/2k)-\epsilon}.$$

This completes the proof that Theorem 6B follows from Theorem 6A. We postpone the proof of Theorem 6A until after Theorem 6C below.

Let $\mathfrak{S}$ be a completely arbitrary set of points in k-dimensional space. Put

$$z^*(r,\underline{\underline{c}}) \text{ for the number of points of } \mathfrak{S} \text{ in } K(r,\underline{\underline{c}}),$$

and

$$D^*(r,\underline{\underline{c}}) = z^*(r,\underline{\underline{c}}) - \mu(r).$$

Theorem 6C. *Suppose that $k > 1$, $\in > 0$ and $R > 1$. Then there is a ball* **77**
$K(r,\underline{\underline{c}})$ with $r \leq R$, $|\underline{\underline{c}}| \leq R^{k+2}$ and

$$|D^*(r,\underline{\underline{c}})| \gg R^{\frac{k-1}{2}-\epsilon},$$

where the constants implied by $\gg$ depends on ϵ and K, but is independent of R.

This theorem was conjectured by P. Erdós [5].

Remark. If $k = 2$ and $\mathfrak{S}$ is the set of integer points in R^2, then the above theorem has some relation with the circle problem.

DEDUCTION OF THEOREM 6C. we may assume that $R \geq k$. Put

$$L = \left[R^{k+2} k^{-1/2}\right] \text{ and } M = L^k.$$

Construct the set $\mathscr{P} = \left(\frac{1}{L}\mathfrak{S}\right) \cap U^k$. We can assume that $\mathscr{P}$ is finite, for if $\mathscr{P}$ is infinite, it has a limit point and the theorem follows immediately. Now we apply the Corollary to Theorem 6B with reference to the set $\mathscr{P}$. There is a ball $K(s, \underline{\underline{d}})$ contained in the unit cube with

$$s \leq M^{-\frac{k+1}{k(k+2)}} = L^{-\frac{k+1}{k+2}}.$$

and with

$$|D(s, \underline{\underline{d}}, M)| \gg M^{\frac{k-1-\epsilon}{2k(k+2)}} = L^{\frac{k-1-\epsilon}{2(k+2)}}.$$

Put

$$r = Ls \text{ and } \underline{\underline{c}} = L\underline{\underline{d}}.$$

We shall show that $K(r, \underline{\underline{c}})$ satisfies the conclusions of the theorem. Observe that

$$z^*(r, \underline{\underline{c}}) = z(s, \underline{\underline{d}}),$$

78 since $K(s, \underline{\underline{d}}) \subseteq U^k$. Now

$$\begin{aligned}
D^*(r, \underline{\underline{c}}) &= z^*(r, \underline{\underline{c}}) - \mu(r)\\
&= z(s, \underline{\underline{d}}) - L^k \mu(s)\\
&= z(s, \underline{\underline{d}}) - M\mu(s)\\
&= D(s, \underline{\underline{d}}, M).
\end{aligned}$$

Hence

$$\begin{aligned}
&|D^*(r, \underline{\underline{c}})| \gg L^{\frac{k-1-\epsilon}{2(k+2)}} \geq R^{\frac{k-1}{2}-\epsilon},\\
&r = Ls \leq LL^{-(k+1)/(k+2)} = L^{1/(k+2)} \leq R,
\end{aligned}$$

and

$$|\underline{\underline{c}}| = L|\underline{\underline{d}}| \leq Lk^{\frac{1}{2}} \leq R^{k+2}.$$

This completes the proof of Theorem 6C.

We finally come to the proof of Theorem 6A. It will be advantageous to use to the cube V^k of points $\underline{\underline{x}}$ with $|x_i| \leq \frac{1}{2}(i = 1, \ldots, k)$, rather that U^k. If $\lambda > 0$, let $V(\lambda)$ be the cube $2\lambda V^k$, and if $0 < \lambda' < \lambda$, let $V(\lambda', \lambda)$ consist of the complement of $V(\lambda')$ in $V(\lambda)$.

Again we may assume that $0 < \delta < \frac{1}{8}$. We may further assume that δ is such that there is no point $\underline{\underline{p}}_i$ on the boundary of $V\left(\frac{1}{2} - 2\delta\right)$. Note that

$$\underline{\underline{c}} \in V\left(\frac{1}{2} - \delta\right) \text{ and } r \leq \delta$$

imply that $K(r, \underline{\underline{c}}) \subseteq V\left(\frac{1}{2}\right) = V$.

Suppose N_0 of the points of $\mathscr{P}$ lie in $V\left(\frac{1}{2} - 2\delta\right)$; let these points be $\underline{\underline{q}}_1, \ldots, \underline{\underline{q}}_{N_0}$. Let $\mathscr{P}_1$ be the set of points

$$\underline{\underline{q}}_i + (1 - 4\delta)\underline{\underline{\ell}}$$

with $1 \leq i \leq N_0$ and $\underline{\underline{\ell}}$ an integer point. Then $\mathscr{P}_1$ is a "periodic" set of **79**
points with period $1 - 4\delta$. Write $z_1(r, \underline{\underline{c}})$ for the number of points of $\mathscr{P}_1$ in $K(r, \underline{\underline{c}})$.

Lemma 6D.

$$\int_0^\delta dr \int_{V(\frac{1}{2} - 2\delta)} d\underline{\underline{c}}\; r^{-1}(z_1(r, \underline{\underline{c}}) - M\mu(r))^2 \gg (M\delta^k)^{1-(1/k)-\epsilon}. \tag{6.1}$$

Proof. It is more convenient to work with $\mathscr{P}_2 = (1 - 4\delta)^{-1}\mathscr{P}_1$, i. e. the set points

$$(1 - 4\delta)^{-1}\underline{\underline{q}}_i + \underline{\underline{\ell}}.$$

□

Then $\mathscr{P}_2$ has period 1, i.e. it is invariant under translation by integer points, and has N_0 points in V^k. Define $z_2(r, \underline{\underline{c}})$, $D_2(r, \underline{\underline{c}})$ and $E_2(r, s)$ with respect to $\mathscr{P}_2$, in the obvious way. Now theorem 1A shows us that

$$\int_0^\delta r^{-1} E_2(r, r) dr \gg (N_0 \delta^k)^{1-(1/k)-\epsilon},$$

provided

$$N_0 \delta^k > N_0^{\epsilon/2} \tag{6.2}$$

Putting $M' = M(1 - 4\delta)^k$, we have

$$\begin{aligned}\int_{V^k} (z_2(r, \underline{\underline{c}}) - M'\mu(r))^2 d\underline{\underline{c}} &= \int_{V^k} (z_2(r, \underline{\underline{c}}) - N_0 \mu(r))^2 d\underline{\underline{c}} + (M' - N_0)^2 \mu(r)^2\\ &= E_2(r, r) + (M' - N_0)^2 \mu(r)^2,\end{aligned}$$

whence

$$\begin{aligned}\int_0^\delta r^{-1} dr \int_{V^k} d\underline{\underline{c}}(z_2(r, \underline{\underline{c}}) - M'\mu(r))^2 &\\ &\gg (N_0 \delta^k)^{1-(1/k)-\epsilon} + (M' - N_0)^2 \int_0^\delta r^{2k-1} dr\\ &\gg (N_0 \delta^k)^{1-(1/k)-\epsilon} + (M' - N_0)^2 \delta^{2k}\\ &\gg (M' \delta^k)^{1-(1/k)-\epsilon} \qquad (6.3)\\ &\gg (M \delta^k)^{1-(1/k)-\epsilon}.\end{aligned}$$

80 All this holds if (6.2) is true. But if (6.2) is not true, then $N_0 \delta^k \leq N_0^{\epsilon/2}$, and together with $M\delta^k > M^\epsilon$ and $0 < \delta < 1/8$, we obtain

$$(4^k M')^{1-\epsilon} > M^{1-\epsilon} > \delta^{-k} > N_0^{1-(\epsilon/2)},$$

whence

$$N_0 < (4^k M')^{(1-\epsilon)/(1-\epsilon/2)} < (4^k M')^{1-(\epsilon/2)} < \frac{1}{2} M'$$

if M is sufficiently large. The left hand side of (6.3) is therefore again

$$\gg (M' - N_0)^2 \delta^{2k} \gg (M' \delta^k)^2 \gg (m \delta^k)^{1-(1/k)-\epsilon}.$$

The proof of Lemma 6D is completed by noting that

$$\begin{aligned}
&\int_0^{\delta} dr \int_{V(\frac{1}{2}-2\delta)} d\underline{\underline{c}} r^{-1}(z_1(r,\underline{\underline{c}}) - M\mu(r))^2 \\
&\quad = (1-4\delta)^k \int_0^{\delta(1-4\delta)^{-1}} dr' \int_{V^k} d\underline{\underline{c}}' r'^{-1}(z_2(r',\underline{\underline{c}}') - M'\mu(r'))^2 \\
&\quad \gg (M\delta^k)^{1-(1/k)-\epsilon}.
\end{aligned}$$

Now write A for the contribution to the left hand side of (6.1) by the points $\underline{\underline{c}}$ which are not in $V\left(\frac{1}{2}-2\delta\right)$. i.e. which are in $V\left(\frac{1}{2}-3\delta, \dfrac{1}{2}-2\delta\right)$. If A is small as compared with the right hand side of (6.1), then

$$\begin{aligned}
\int_0^{\delta} r^{-1}F(r,M)dr &\geq \int_0^{\delta} dr \int_{V(\frac{1}{2}-3\delta)} d\underline{\underline{c}}\, r^{-1}(z(r,\underline{\underline{c}}) - M\mu(r))^2 \\
&= \int_0^{\delta} dr \int_{V(\frac{1}{2}-3\delta)} d\underline{\underline{c}}\, r^{-1}(z_1(r,\underline{\underline{c}}) - M\mu(r))^2 \\
&\gg (M\delta^k)^{1-(1/k)-\epsilon},
\end{aligned}$$

and Theorem 6A is true. We therefore shall assume henceforth that

$$A \gg (M\delta^k)^{1-(1/k)-\epsilon}. \tag{6.4}$$

Put

$$\begin{aligned}
B &= \int_0^{\delta} dr \int_{V(\frac{1}{2}-3\delta,\frac{1}{2}-2\delta)} d\underline{\underline{c}}\, r^{-1} z_1(r,\underline{\underline{c}})^2, \\
B' &= \int_0^{\delta} dr \int_{V(\frac{1}{2}-3\delta,\frac{1}{2}-\delta)} d\underline{\underline{c}}\, r^{-1} z(r,\underline{\underline{c}})^2, \\
C &== \int_0^{\delta} dr \int_{V(\frac{1}{2}-3\delta,\frac{1}{2}-2\delta)} d\underline{\underline{c}}\, r^{-1} M^2\mu(r)^2, \\
C' &= \int_0^{\delta} dr \int_{V(\frac{1}{2}-3\delta,\frac{1}{2}-\delta)} d\underline{\underline{c}}\, r^{-1} M^2\mu(r)^2.
\end{aligned}$$

Then

$$
\begin{aligned}
C \le C' \ll \delta \int_0^\delta r^{-1} M^2 \mu(r)^2 dr &\ll M^2 \delta^{2k+1} \\
&= (M\delta^k)^{1-(1/k)-2\epsilon} (M\delta^k)^{1+(2/k)+2\epsilon} M^{-1/k} \\
&< (M\delta^k)^{1-(1/k)-2\epsilon} M^{(\frac{k+2}{k}+2)(\frac{1}{k+2}-\epsilon-(1/k))} \qquad (6.5)\\
&< (M\delta^k)^{1-(1/k)-2\epsilon}
\end{aligned}
$$

by virtue of $M\delta^k \le M^{(1/(k+2))-\epsilon}$, which is a hypothesis in Theorem 6A.

Lemma 6E. $B \le 2^k B'$.

Proof. Suppose $\underline{\underline{c}} \in V(\frac{1}{2} - 3\delta, \frac{1}{2} - 2\delta)$ and $r \le \delta$, Observe that $z_1(r, \underline{\underline{c}})$ is the number of points $\underline{\underline{q}}_i + (1-4\delta)\underline{\underline{\ell}}$ in $K(r, \underline{\underline{c}})$. For such points, $|\underline{\underline{q}}_i + (1-4\delta)\underline{\underline{\ell}} - \underline{\underline{c}}| \le r \le \delta$, whence $\underline{\underline{c}} - (1-4\delta)\underline{\underline{\ell}} \in V(\frac{1}{2} - 2\delta + \delta) = V(\frac{1}{2} - \delta)$, since $\underline{\underline{q}}_i \in V(\frac{1}{2} - 2\delta)$. As $\underline{\underline{\ell}}$ runs through the integer points, $\underline{\underline{c}} - (1-4\delta)\underline{\underline{\ell}}$ lies in $V\left(\frac{1}{2} - 2\delta\right)$ only for $\underline{\underline{\ell}} = 0$, and hence all these points lie outside $V\left(\frac{1}{2} - 3\delta\right)$. □

Thus we have

$$
\underline{\underline{c}} - (1-4\delta)\underline{\underline{\ell}} \in V(\frac{1}{2} - 3\delta, \frac{1}{2} - \delta). \qquad (6.6)
$$

For given $\underline{\underline{c}}$, there are at most 2^k such integer points $\underline{\underline{\ell}}$.
We have

$$
z_1(r, \underline{\underline{c}}) \le \sum_{\underline{\underline{\ell}} \text{ satisfies } (6.6)} z(r, \underline{\underline{c}} - (1-4\delta)\underline{\underline{\ell}}).
$$

82 and by Cauchy's inequality

$$
z_1^2(r, \underline{\underline{c}}) \le 2^k \sum_{\underline{\underline{\ell}} \text{ satisfies } (6.6)} z^2(r, \underline{\underline{c}} - (1-4\delta)\underline{\underline{\ell}}).
$$

This yields

$$
\int_{V(\frac{1}{2}-3\delta, \frac{1}{2}-2\delta)} z_1^2(r, \underline{\underline{c}}) d\underline{\underline{c}} \le 2^k \int_{V(\frac{1}{2}-3\delta, \frac{1}{2}-\delta)} z^2(r, \underline{\underline{c}}) d\underline{\underline{c}},
$$

and Lemma 6E follows.

The proof of Theorem 6A is now completed as follows. By the general inequality $(a-b)^2 \leq 2a^2 + 2b^2$, we have

$$A \leq 2B + 2C \tag{6.7}$$

Now set

$$A' = \int_0^\delta dr \int_{V(\frac{1}{2}-3\delta, \frac{1}{2}-\delta)} d\underline{\underline{c}}\, r^{-1}(z(r,\underline{\underline{c}}) - M\mu(r))^2.$$

By the general inequality $(a-b)^2 \geq \frac{1}{2}a^2 - b^2$, we have

$$A' \geq \frac{1}{2}B' - C'.$$

Hence by Lemma 6E, by (6.4), (6.5) and (6.7), we obtain

$$A' \geq 2^{-k-1}B - C' \geq 2^{-k-2}A - C - C' \gg (M\delta^k)^{k-(1/k)-\epsilon}.$$

Theorem 6A follows.

7 Rectangles in Arbitrary Position

For any $\underline{\underline{u}} = (u_1, u_2)$, denote by $R(\underline{\underline{u}})$ the rectangle of points $\underline{\underline{x}} = (x_1, x_2)$ with $0 \leq x_1 \leq |u_1|$, $0 \leq x_2 \leq |u_2|$. Denote by τ_φ the rotation by the angle φ about $\underline{\underline{0}}$ (i. e. if $\underline{\underline{x}} = (x_1, x_2)$), then $\tau_\varphi \underline{\underline{x}} = (x_1 \cos\varphi - x_2 \sin\varphi, x_1 \sin\varphi + x_2 \cos\varphi)$. Write $R(\underline{\underline{u}}, \underline{\underline{v}}, \varphi)$ for the rectangle of points $\tau_\varphi \underline{\underline{x}} + \underline{\underline{v}}$ with $\underline{\underline{x}} \in R(\underline{\underline{u}})$. It is easy to see that all the rectangles of diameter δ are of the form **83**

$$R(\delta \underline{\underline{w}}(\psi), \underline{\underline{v}}, \varphi), \quad (0 \leq \varphi, \psi \leq 2\pi)$$

where $\underline{\underline{w}}(\psi) = (\cos\psi, \sin\psi)$.

Let $\underline{\underline{p}}_1 = (x_1, y_1), \ldots, \underline{\underline{p}}_N = (x_N, y_N)$ be N points in the plane with $0 \leq x_i, y_i < 1 (1 \leq i \leq N)$. Denote by $\mathscr{P}$ the set of points $\underline{\underline{p}}_i + \underline{\ell}$, $1 \leq i \leq N$, where $\underline{\underline{\ell}}$ runs through all the integer points. Define the set function $D(A)$ as in § 1.

Theorem 7A. *Let* $\in > 0$*,* $\delta > 0$ *with* $N\delta^k > N^\epsilon$*. Then there exists a rectangle* R *with diameter* δ *and with*

$$|D(R)| \gg (N\delta^2)^{\frac{1}{4}-\epsilon}.$$

Theorem 7A follows from

Theorem 7B. *Let* $\in$*,* δ *be as in Theorem 7A. Then*

$$\frac{1}{4\pi^2}\int_0^{2\pi} d\psi \int_0^{2\pi} d\varphi \int_{U^2} d\underline{\underline{v}} D(\delta \underline{\underline{w}}(\psi), \underline{\underline{v}}, \varphi)^2 \gg (N\delta^2)^{\frac{1}{2}-\epsilon},$$

where $D(\underline{\underline{u}}, \underline{\underline{v}}, \varphi) = D(R(\underline{\underline{u}}, \underline{\underline{v}}, \varphi))$*.*

Weaker results were proved in [21 a].

Put

$$f_R(\underline{\underline{u}}, \underline{\underline{v}}, \varphi; \underline{\underline{x}}) = \begin{cases} 1 & \text{if } \underline{\underline{x}} \in R(\underline{\underline{u}}, \underline{\underline{v}}, \varphi), \\ 0 & \text{otherwise ,} \end{cases}$$

so that $f_R(\underline{\underline{u}}, \underline{\underline{v}}, \varphi; \underline{\underline{x}})$ for fixed $\underline{\underline{u}}, \underline{\underline{v}}, \varphi$ is the characteristic function of $R(\underline{\underline{u}}, \underline{\underline{v}}, \varphi)$, and $f_R(\underline{\underline{u}}, \underline{\underline{v}} + \underline{\underline{c}}, \varphi; \underline{\underline{x}} + \underline{\underline{c}}) = f_R(\underline{\underline{u}}, \underline{\underline{v}}, \varphi; \underline{\underline{x}})$ for any $\underline{\underline{c}}$. Next put

$$g_R(\underline{\underline{u}}, \underline{\underline{v}}, \varphi; \underline{\underline{x}}) = \sum_{\underline{\underline{\ell}}} f_R(\underline{\underline{u}}, \underline{\underline{v}}, \varphi; \underline{\underline{x}} + \underline{\underline{\ell}}),$$

where the sum is taken over all the integer points $\underline{\underline{\ell}}$. Notice that only finitely many summands are non-zero. Further observe that $g_R(\underline{\underline{u}}, \underline{\underline{v}}, \varphi; \underline{\underline{x}})$ is periodic in $\underline{\underline{x}}$.

Lemma 7C. *For any* δ *with* $0 < \delta < \dfrac{1}{2}$*,*

$$\begin{aligned} &\frac{1}{4\pi^2}\int_0^{2\pi} d\psi \int_0^{2\pi} d\varphi \int_{U^2} d\underline{\underline{v}} D(\delta \underline{\underline{w}}(\psi), \underline{\underline{v}}, \varphi)^2 \\ &= \sum_{i,j=1}^{N} (\ell_R(\delta, \omega)(\underline{\underline{p}}_i, \underline{\underline{p}}_j)) - \int_{U^2}\int_{U^2} \ell_R(\delta, \omega(\underline{\underline{x}}, \underline{\underline{y}})) d\underline{\underline{x}} \quad d\underline{\underline{y}}, \end{aligned} \tag{7.1}$$

84 where $\omega(\underline{\underline{x}}, y)$ is as in § 2, and where

$$\ell_R(\delta, \omega) = \frac{1}{2\pi} \int_0^{2\pi} d\psi K_R(\delta \underline{\underline{w}}(\psi), \omega),$$

with

$$K_R(\underline{\underline{u}}, \omega) = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \int_{R^2} d\underline{\underline{v}} f_R(\underline{\underline{u}}, \underline{\underline{0}}, 0; \underline{\underline{v}}) f_R(\underline{\underline{u}}, \underline{\underline{0}}, 0; \underline{\underline{v}} + \omega \underline{\underline{w}}(\varphi)).$$

Let $\underline{\underline{u}} \in U^2$ with $0 < |\underline{\underline{u}}| < \dfrac{1}{2}$. Let $\underline{\underline{x}} = (x_1, x_2)$ and $\underline{\underline{y}} = (y_1, y_2)$ be in the plane, and assume, for the moment, the $|x_i - y_i| < \dfrac{1}{2} (i = 1, 2)$. Put $h_R(\underline{\underline{u}}, \underline{\underline{x}}, \underline{\underline{y}})$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\varphi \int_{U^2} d\underline{\underline{v}} g_R(\underline{\underline{u}}, \underline{\underline{v}}, \varphi; \underline{\underline{x}}) g_R(\underline{\underline{u}}, \underline{\underline{v}}, \varphi; \underline{\underline{y}})$$

$$\sum_{\underline{\underline{\ell}}_1} \sum_{\underline{\underline{\ell}}_2} \frac{1}{2\pi} \int_0^{2\pi} d\varphi \int_{U^2} d\underline{\underline{v}} f_R(\underline{\underline{u}}, \underline{\underline{0}}, \varphi; \underline{\underline{x}} - \underline{\underline{v}} + \underline{\underline{\ell}}_1) f_R(\underline{\underline{u}}, \underline{\underline{0}}, \varphi; \underline{\underline{y}} - \underline{\underline{v}} + \underline{\underline{\ell}}_2).$$

The new variable $\underline{\underline{v}}' = \underline{\underline{x}} - \underline{\underline{v}} + \underline{\underline{\ell}}_1$ ranges over the whole plane R^2. Hence the above sum equals

$$\sum_{\underline{\underline{\ell}}_2} \frac{1}{2\pi} \int_0^{2\pi} d\varphi \int_{R^2} d\underline{\underline{v}}' f_R(\underline{\underline{u}}, \underline{\underline{0}}, \varphi; \underline{\underline{v}}') f_R(\underline{\underline{u}}, \underline{\underline{0}}, \varphi; \underline{\underline{v}}' + \underline{\underline{y}} - \underline{\underline{x}} + \underline{\underline{\ell}}_2)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\varphi \int_{R^2} d\underline{\underline{v}} f_R(\underline{\underline{u}}, \underline{\underline{0}}, \varphi; \underline{\underline{v}}) f_R(\underline{\underline{u}}, \underline{\underline{0}}, \varphi; \underline{\underline{v}} + \underline{\underline{y}} - \underline{\underline{x}}), \qquad (7.2)$$

since if $f_R(\underline{\underline{u}}, \underline{\underline{0}}, \varphi; \underline{\underline{v}}') f_R(\underline{\underline{u}}, \underline{\underline{0}}, \varphi; \underline{\underline{v}}' + \underline{\underline{y}} - \underline{\underline{x}} + \underline{\underline{\ell}}_2) = 1$, then both $\underline{\underline{u}}'$ and $\underline{\underline{v}}' + \underline{\underline{y}} - \underline{\underline{x}} + \underline{\underline{\ell}}_2$ belong to $R(\underline{\underline{u}})$. Therefore $|\underline{\underline{y}} - \underline{\underline{x}} + \ell_2| < \frac{1}{2}$. If, say, $\underline{\underline{\ell}}_2 = (\ell_{2,1}, \ell_{2,2})$, then $|\ell_{2,1}| < \dfrac{1}{2} + \dfrac{1}{2} = 1$, since $|y_i - x_i| < \dfrac{1}{2}$. This implies 85
that $\underline{\underline{\ell}}_2 = (0, 0)$.

Note that $f_R(\underline{\underline{u}}, \underline{\underline{0}}, \varphi; \underline{\underline{v}}) = 1$ if and only if $\underline{\underline{v}} = \tau_\varphi \underline{\underline{z}}$ with $\underline{\underline{z}} \in R(\underline{\underline{u}})$, i. e. if $\tau_\varphi^{-1}\underline{\underline{v}} \in R(\underline{\underline{u}})$. Therefore (7.2) equals

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} d\varphi \int_{R^2} d\underline{\underline{v}}\, f_R(\underline{\underline{u}}, \underline{\underline{o}}, o_j \tau_\varphi^{-1}\underline{\underline{v}})\, f_R(\underline{\underline{u}}, \underline{\underline{0}}, 0; \tau_\varphi^{-1}\underline{\underline{v}} + \tau_\varphi^{-1}(\underline{\underline{y}} - \underline{\underline{x}}))\\
&= \frac{1}{2\pi}\int_0^{2\pi} d\varphi \int_{R^2} d\underline{\underline{v}}\, f_R(\underline{\underline{u}}, \underline{\underline{0}}, 0; \underline{\underline{v}})\, f_R(\underline{\underline{u}}, \underline{\underline{0}}, 0 : \underline{\underline{v}} + \omega(\underline{\underline{x}}, \underline{\underline{y}})\underline{\underline{\omega}}(\varphi)),\\
&= K_R(\underline{\underline{u}}, \omega(\underline{\underline{x}}, \underline{\underline{y}})).
\end{aligned}$$

for as φ ranges from 0 to 2π, $\tau_\varphi^{-1}(\underline{\underline{y}} - \underline{\underline{x}})$ ranges over the circle

$$|\underline{\underline{y}} - \underline{\underline{x}}|\underline{\underline{w}}(\varphi) = \omega(\underline{\underline{x}}, \underline{\underline{y}})\underline{\underline{w}}(\varphi) \text{ with } 0 \le \varphi \le 2\pi.$$

Hence for every $\underline{\underline{u}}$ with $0 < |\underline{\underline{u}}| < \frac{1}{2}$ and every $\underline{\underline{x}} = (x_1, x_2)$, $\underline{\underline{y}} = (y_1, y_2)$ with $|x_i - y_i| < \frac{1}{2}$, $(i = 1, 2)$, we have

$$h_R(\underline{\underline{u}}, \underline{\underline{x}}, \underline{\underline{y}}) = K_R(\underline{\underline{u}}, \omega(\underline{\underline{x}}, \underline{\underline{y}})). \tag{7.3}$$

Since both sides of (7.3) are periodic in $\underline{\underline{x}}$ and $\underline{\underline{y}}$, this equation holds in fact arbitrary $\underline{\underline{x}}$ and $\underline{\underline{y}}$. We have

$$\begin{aligned}
&\int_{U^2}\int_{U^2} K_R(\underline{\underline{u}}, \omega(\underline{\underline{x}}, \underline{\underline{y}}))d\underline{\underline{x}} \quad d\underline{\underline{y}}\\
&= \int_{U^2}\int_{U^2} h_R(\underline{\underline{u}}, \underline{\underline{x}}, \underline{\underline{y}})d\underline{\underline{x}} \quad d\underline{\underline{y}}\\
&= \frac{1}{2\pi}\int_0^{2\pi} d\varphi \int_{U^2} d\underline{\underline{v}} \int_{R^2}\int_{R^2} f_R(\underline{\underline{u}}, \underline{\underline{v}}, \varphi; \underline{\underline{x}})\, f_R(\underline{\underline{u}}, \underline{\underline{v}}, \varphi; \underline{\underline{y}})d\underline{\underline{x}} \quad d\underline{\underline{y}}\\
&= \mu(R(\underline{\underline{u}}))^2.
\end{aligned} \tag{7.4}$$

Proceeding similarly as in § 2 and using (7.3) and (7.4), we obtain

$$\frac{1}{2\pi} d\varphi \int_{U^2} d\underline{\underline{v}} D(\underline{\underline{u}}, \underline{\underline{v}}, \varphi)^2$$

$$= \sum_{i,j=1}^{N} (K_R(\underline{\underline{u}}, \omega(\underline{\underline{p}}_i, \underline{\underline{p}}_j)) - \int_{U^2} \int_{U^2} K_R(\underline{\underline{u}}, \omega(\underline{\underline{x}}, \underline{\underline{y}})) d\underline{\underline{x}} d\underline{\underline{y}}). \qquad (7.5)$$

provided that $0 < |\underline{\underline{u}}| < \dfrac{1}{2}$ **86**

We now consider rectangles with diameter δ. If $0 < \delta < \dfrac{1}{2}$, then the relation (7.5) is valid for all $\underline{\underline{u}} = \delta \underline{\underline{w}}(\psi)$. Substitute $\underline{\underline{u}} = \delta \underline{\underline{w}}(\psi)$ in (7.5) and integrate both the sides with respect to ψ from 0 to 2π. We obtain (7.1), and Lemma 7C is proved.

We now recall formula (2.3) the special case $k = 2$, $r = s$:

$$E(r,r) = \sum_{i=1}^{N} \sum_{j=1}^{N} (K(r, r, \omega(\underline{\underline{p}}_i, \underline{\underline{p}}_j))$$
$$- \int_{U^2} \int_{U^2} K(r, r, \omega(\underline{\underline{x}}, \underline{\underline{y}})) d\underline{\underline{x}}\, d\underline{\underline{y}}) \left(0 < r < \frac{1}{4}\right) \qquad (7.6)$$

Lemma 7D ("Fundamental Lemma"). *Suppose that* $0 < \delta < \dfrac{1}{2}$. *Let* $f(r)$ *be non - negative and continuous in* $0 < r < \dfrac{1}{2}$. *Further assume that* $f(r)$ *satisfies the integral equation*

$$\int_0^{\frac{1}{2}} K(\delta r, \delta r, \omega) f(r) dr = \ell_R(\delta, \omega) \qquad (\omega \geq 0). \qquad (7.7)$$

Then

$$\int_0^{\frac{1}{2}} E(\delta r, \delta r) f(r) dr$$
$$= \frac{1}{4\pi^2} \int_0^{2\pi} d\psi \int_0^{2\pi} d\varphi \int_{U^2} d\underline{\underline{v}} D(R(\delta \underline{\underline{w}}(\psi), \underline{\underline{v}}, \varphi))^2 \qquad (7.8)$$

Proof. The Lemma follows immediately from (7.1) and (7.6). □

8 Solving the Integral Equation for Rectangles

We shall require the following :

Lemma 8A. *Write $\ell(\omega)$ for $\ell_R(1,\omega)$. Then*

$$\ell(\omega) = \frac{2}{\pi^2}((1+2\omega^2)arc\cos\omega - 3\omega(1-\omega^2)^{1/2}) \qquad (0 \leq \omega \leq 1),$$
$$\ell'(\omega) = \frac{8}{\pi^2}(\omega arc\cos\omega - (1-\omega^2)^{1/2}) \qquad (0 \leq \omega \leq 1),$$
$$\ell''(\omega) = \frac{8}{\pi^2} arc\cos\omega \qquad (0 < \omega < 1),$$
$$\ell'''(\omega) = -\frac{8}{\pi^2}(1-\omega^2)^{-1/2} \qquad (0 < \omega < 1).$$

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Here arc $\cos\omega \in [0, \pi/2]$ for every ω with $0 \leq \omega \leq 1$.

We postpone the proof of this lemma until page 106.

We shall determine a solution $f(r)$ of the integral equation (7.7) such that $f(r)$ is continuous in $0 < r \leq \dfrac{1}{2}$, and

$$|f(r)| \ll r^{-1} \text{ as } r \to 0, \tag{8.1}$$

hence such that

$$\int_0^{\frac{1}{2}} |f(r)| r dr \ll 1 \tag{8.2}$$

By the argument of § 2 (see page 64), we see that if (7.7) is true for some $\delta > 0$, then (7.7) and (7.8) are true for every $\delta > 0$. So it will suffice to consider the integral equation (7.7) with $\delta = 1$, i. e.

$$\int_0^{\frac{1}{2}} K(r,r,\omega) f(r) dr = \ell_R(1,\omega) = \ell(\omega) \qquad (\omega \geq 0). \tag{8.3}$$

Observe that the left hand side vanishes for $\omega \geq 1$. Further $\ell(1) = 0$ by Lemma 8A, and $\ell(\omega) = 0$ for $\omega > 1$ by the definition of $\ell(\omega)$. Hence it will be enough to consider (8.3) for $0 \leq \omega < 1$. Further we may assume that $0 < \omega < 1$, since both sides of (8.3) for $0 \leq \omega < 1$. Further we may assume that $0 < \omega < 1$, since both sides of (8.3) are continuous in $0 \leq \omega < 1$. (The right hand side is continuous by Lemma 8A. That the left hand side is continuous in ω follows from (3.2) with $k = 2$ and from (8.2)).

We now differentiate both sides of (8.3) with respect to ω. By (3.2)
88 (8.2) and by Lebesgue's Theorem on dominated convergence, the left hand side of (8.3) can be differentiated under the integral sign. Proceeding similarly as in § 3, we see the derivative of the left hand side of (8.3) equals

$$-c_1 \int_{\omega/2}^{\frac{1}{2}} f(r)(r^2 - \frac{\omega^2}{4})^{1/2} dr.$$

(See (3.5) with $k = 2$). Here c_1 is a positive constant[2]. Hence by differentiating the integral equation (8.3), we get

$$-c - 1 \int_{\omega/2}^{\frac{1}{2}} f(r)(r^2 - \frac{\omega^2}{4})^{\frac{1}{2}} dr = \ell'(\omega) \qquad (0 < \omega < 1). \tag{8.4}$$

Any solution of this integral equation is also a solution of (8.3), since both sides of (8.3) are continuous in ω and are zero for $\omega = 1$. Differentiating again with respect to ω, we obtain

$$\frac{1}{4} c_1 \int_{\omega/2}^{\frac{1}{2}} f(r)(r^2 - \frac{\omega^2}{4})^{-1/2} \omega dr = \ell''(\omega) \qquad (0 < \omega < 1),$$

i. e.

$$\int_{\omega/2}^{\frac{1}{2}} f(r)(r^2 - \frac{\omega^2}{4})^{-1/2} dr = m(\omega) \qquad (0 < \omega < 1) \tag{8.5}$$

where

$$m(\omega) = \frac{4}{c_1} \frac{\ell''(\omega)}{\omega}.$$

Now the left hand side and the right hand side of (8.4) are continuous in ω and vanish for $\omega = 1$. So it follows that every solution of (8.5) is also a solution of (8.4).

Using the argument of § 3, we see that

$$f(r) = -\frac{4}{\pi} \int_{2r}^{1} r(t^2 - 4r^2)^{-1/2} m'(t) dt \qquad \left(0 < r < \frac{1}{2}\right)$$

is a solution of (8.5). By Lemma 8A, 89

[2]The numbering of constants is begin a new in each section.

$$m'(t) = -c_2(\frac{arc\cos t}{t^2} + \frac{1}{t(1-t^2)^{\frac{1}{2}}}), \quad (0 < t < 1).$$

Therefore

$$\lim_{r\to\frac{1}{2}} f(r) = \frac{4c_2}{\pi} \lim_{r\to\frac{1}{2}} \int_{2r}^{1} r(t^2-4r^2)^{-\frac{1}{2}} \left(\frac{arc\cos t}{t^2} + \frac{1}{t(1-t^2)^{\frac{1}{2}}}\right) dt.$$

Observe that

$$\int_{2r}^{1} (t^2-4r^2)^{-\frac{1}{2}} (arc\cos t) t^2 dt \ll r^{-2} \int_{1}^{1/2r} (u^2-1)^{-\frac{1}{2}} du \to 0,$$

as $r \to \frac{1}{2}$. Further

$$\begin{aligned}\lim_{r\to\frac{1}{2}} \int_{2r}^{1} r(t^2-4r^2)^{-\frac{1}{2}} \frac{dt}{t(1-t^2)^{\frac{1}{2}}}& \\ &= \lim_{r\to\frac{1}{2}} r \int_{0}^{\sqrt{1-4r^2}} \frac{dt}{(t^2+4r^2)^{\frac{1}{2}}(1-4r^2-t^2)^{\frac{1}{2}}} \\ &= \lim_{r\to\frac{1}{2}} r \int_{0}^{1} \frac{dt}{(1-t^2)^{\frac{1}{2}}((1-4r^2)t^2+4r^2)^{\frac{1}{2}}} \\ &= \frac{1}{2}\frac{\pi}{2} = \frac{\pi}{4}.\end{aligned}$$

Hence $f(r)$ can be extended continuously in $0 < r <\leq \frac{1}{2}$, and is positive through out. We next check the behaviour of $f(r)$ as $r \to 0$. We have

$$\begin{aligned}f(r) &\ll \int_{2r}^{1} r(t^2-4r^2)^{-frac12} \left(\frac{1}{t^2} + \frac{1}{t(1-t^2)^{\frac{1}{2}}}\right) dt \\ &= \int_{2r}^{\frac{1}{2}} \ldots + \int_{\frac{1}{2}}^{1} \ldots.\end{aligned}$$

The second integral is clearly bounded as $r \to 0$. The first integral is

$$\ll \int_{2r}^{1} rt^{-2}(t^2-4r^2)^{-\frac{1}{2}} dt = \frac{1}{4r} \int_{1}^{1/2r} \frac{du}{(u-1)^{\frac{1}{2}}} \ll \frac{1}{r}, \text{ as } r \to 0.$$

90 Now if t is small, then

$$|m'(t)| \gg t^{-2}.$$

Therefore if r is small, then

$$\begin{aligned} f(r) &\gg \int_{2r}^{4r} r(t^2 - 4r^2)^{-\frac{1}{2}} t^{-2} dt \\ &= \frac{1}{4r} \int_1^2 \frac{dt}{t^2(t^2-1)^{\frac{1}{2}}} \gg \frac{1}{r}, \text{ as } r \to 0. \end{aligned}$$

Hence

$$f(r) \gg\ll r^{-1}.$$

So (8.1) is satisfied. We conclude that there exists a function $f(r)$ satisfying the integral equation (8.3), which is continuous in $0 < r \leq \frac{1}{2}$. Hence by Lemma 7D, we have

$$\frac{1}{4\pi^2} \int_0^{2\pi} d\psi \int_0^{2\pi} d\varphi \int_{U^2} d\underline{\underline{v}} D(\delta \underline{\underline{w}}(\psi), \underline{\underline{v}}, \varphi)^2 = \int_0^{\frac{1}{2}} f(r) E(\delta r, \delta r). \quad (8.6)$$

In view of an earlier remark, the above relation is true for all $\delta > 0$. The right hand side of (8.6) exceeds

$$c_3 \int_0^{\frac{1}{2}} r^{-1} E(\delta r, \delta r) dr = c_3 \int_0^{\frac{1}{2}\delta} r^{-1} E(r, r). \quad (8.7)$$

By Theorem 1A,

$$c_3 \int_0^{\frac{1}{2}\delta} f^{-1} E(r, r) dr \gg (N\delta^2)^{1-\frac{1}{2}-\epsilon} = (N\delta^2)^{\frac{1}{2}-\epsilon} \quad (8.8)$$

Combining (8.6), (8.7) and (8.8), we obtain Theorem 7B.

Proof of Lemma 8A. Assume that $0 \leq \omega \leq 1$. Put

$$< \alpha >= \begin{cases} \alpha & \text{if } \alpha \geq 0, \\ 0 & \text{otherwise }. \end{cases}$$

Observe that

$$\begin{aligned}\ell(\omega) &= \frac{1}{2\pi}\int_0^{2\pi} d\psi\, \ell_R(\underline{\underline{w}}(\psi), \omega)\\ &= \frac{1}{4\pi^2}\int_0^{2\pi} d\psi \int_0^{2\pi} d\varphi \int_{R^2} d\underline{\underline{v}}\, f_R(\underline{\underline{w}}(\psi), \underline{\underline{v}}, 0; \omega\underline{\underline{w}}(\varphi))\, f_R(\underline{\underline{w}}(\psi), \underline{\underline{v}}, 0; \underline{\underline{0}}).\end{aligned}$$

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For fixed $\underline{\underline{u}} = (u_1, u_2)$ and $\underline{\underline{x}} = (x_1, x_2)$, consider the integral

$$\int_{R^2} d\underline{\underline{v}}\, f_R(\underline{\underline{u}}, \underline{\underline{v}}, 0; \underline{\underline{x}})\, f_R(\underline{\underline{u}}, \underline{\underline{v}}, 0; \underline{\underline{0}}). \tag{8.9}$$

Note that

$$f_R(\underline{\underline{u}}, \underline{\underline{v}}, 0; \underline{\underline{x}})\, f_R(\underline{\underline{u}}, \underline{\underline{v}}, 0; \underline{\underline{0}}) = \begin{cases} 1, & \text{if } |x_1| \le |u_1|, |x_2| \le |u_2| \text{ and if } \underline{\underline{v}} \\ & \text{lies in a certain rectangle of area} \\ & (|u_1| - |x_1|)(|u_2| - |x_2|), \\ 0, & \text{otherwise .} \end{cases}$$

Therefore the integral (8.9) above is equal to

$$\langle |u_1 - |x_1|| \rangle \langle |u_2| - |x_2| \rangle .$$

Hence

$$\ell(\omega) = \frac{1}{4\pi^2}\int_0^{2\pi} d\psi \int_0^{2\pi} d\varphi\, \langle |\cos\psi| - \omega|\cos\varphi| \rangle \langle |\sin\psi| - \omega|\sin\varphi| \rangle .$$

Since $\langle |\cos\psi| - \omega|\cos\varphi| \rangle \langle |\sin\psi| - \omega|\sin\varphi| \rangle$ is periodic in ψ and in φ with period π, we may reduce everything to the domain

$$-\frac{\pi}{2} \le \varphi, \psi \le \frac{\pi}{2}.$$

We may further restrict ourselves to the domain

$$0 \le \varphi, \psi \le \frac{\pi}{2},$$

since the integrand is an even function of φ and of ψ. Hence

$$\begin{aligned}\ell(\omega) &= \frac{4}{\pi^2}\int_0^{\pi/2} d\psi \int_0^{\pi/2} d\varphi \,\langle\cos\psi - \omega\cos\varphi\rangle\,\langle\sin\psi - \omega\sin\varphi\rangle\\ &= \frac{4}{\pi^2}\int_0^{\pi/2} d\varphi \int_{S(\omega,\varphi)}^{C(\omega,\varphi)} d\psi(\cos\psi - \omega\cos\varphi)(\sin\psi - \omega\sin\varphi),\end{aligned}$$

where $C(\omega,\varphi) = arc\cos(\omega\cos\varphi)$ and $S(\omega,\varphi) = arc\sin(\omega\sin\varphi)$. The 92
above integral is of the type

$$\frac{4}{\pi^2}\int_0^{\pi/2} T(\omega,\varphi)d\varphi \tag{8.10}$$

and after a lengthy computation we get

$$\begin{aligned}T(\omega,\varphi) = \frac{1}{2}(1+\omega^2) - \omega(\cos\varphi\sqrt{1-\omega^2\sin^2\varphi} + \sin\varphi\sqrt{1-\omega^2\cos^2\varphi})\\ +w^2\sin\varphi\cos\varphi(C(\omega,\varphi) - S(\omega,\varphi)).\end{aligned}$$

Substituting this into (8.10), we get

$$\begin{aligned}\ell(\omega) &= \frac{4}{\pi^2}\int_0^{\pi/2} T(\omega,\varphi)d\varphi\\ &= \frac{2}{\pi^2}((1+2\omega^2)arc\cos\omega - 3\omega(1-\omega^2)^{\frac{1}{2}}) \qquad (0 \leq \omega \leq 1).\end{aligned}$$

The lemma now follows by obvious differentiation. The proof of Theorem 7A is complete.

Let the dimension k be arbitrary, For any subset B of R^k, define the diameter as the supermum of the distance $|\underline{\underline{x}} - \underline{\underline{y}}|$ for all pairs of points $\underline{\underline{x}}, \underline{\underline{y}} \in B$. By a box in arbitrary position, we mean a box obtained from a box parallel to axes (as studied in Chapter I) by a rotation. An analogue of Theorem 7B for $k > 2$ is as follows :

Theorem 8B*. *Let N and $\mathscr{P}$ be as usual. Suppose that $k \geq 3$, $\in > 0$ and $\delta > 0$, with $N\delta^k > N^\epsilon$. Then there exists a box B (in arbitrary position) with diameter δ and with*

$$|D(B)| \gg (N\delta^k)^{(1/3)-\epsilon}. \tag{8.11}$$

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Remark. It may be seen that it is sufficient to prove Theorem 8B* in case $k = 3$. In this case, the Theorem can be obtained by the method used for Theorem 7A. For arbitrary k the computations become awkward. One would like to be able to replace the right hand side of (8.11) by $(N\delta^k)^{\frac{1}{2}-(1/2k)-\epsilon}$.

9 Triangles

Theorem 9A. *Suppose that* $k = 2$ *and* $\in > 0$. *Let* N *and* $\mathscr{P}$ *be as usual. Then there exists a closed right angle triangle* $T \subseteq U^2$ *with sides containing the right angle parallel to co-ordinate axes, and with*

$$|D(T)| \gg N^{(1/4)-\in},$$

where the constant implied by $\gg$ *depends only on* ϵ. *There exists such a triangle with no points of* $\mathscr{P}$ *on its hypotenuse.*

This theorem gives rise to a rather paradoxical phenomenon : Let T be a triangle as given by Theorem 9A. There is a unique right triangle T' such that $T \cup T'$ is a rectangle R with sides parallel to the coordinate axes.

Since no points of $\mathscr{P}$ lies on the hypotenuse of T, we have

$$D(R) = D(T) + D(T').$$

We know that there exist sets (see § 1 of Chapter I) with

$$|D(R)| \ll \log N$$

for every rectangle R with sides parallel to the axes. We have thus

$$|D(T)| \gg N^{(1/4)-\epsilon}, |D(T')| \gg N^{(1/4)-\epsilon},$$
$$\Big||D(T)| - |D(T')|\Big| \ll \log N.$$

So there exist sets $\mathscr{P}$ which are quite irregularly distributed with re-
94 spect to triangles such as T and T', but mush more regularly with respect
to rectangles R. We may say that $D(T)$ and $D(T')$ "almost cancel".

Proof of Theorem 9A.

Case I. Suppose that there exists a line segment of length 1, *parallel to one of the axes, with* $\geq N^{(1/4)-\epsilon}$ *points of* $\mathscr{P}$ *on it.*

In this case there exists a line segment S of length ≤ 1 contained in U^2 with $\geq \dfrac{1}{2}N^{(1/4)-\epsilon}$ points on it. Construct a triangle T of very small area with S as a base. This triangle will have $D(T) \gg N^{(1/4)-\epsilon}$.

Case II. Suppose that case I does not hole, but that there exists a line segment of length 1 *with* $N^{(1/4)-(\epsilon/2)}$ *points of* $\mathscr{P}$ *on it.*

Then there exists a line segment of length ≤ 1 contained in U^2 with $\geq \dfrac{1}{3}N^{(1/4)-(\epsilon/2)}$ points of $\mathscr{P}$ on it Since case I is ruled out, there are $< 4N^{(1/4)-\epsilon}$ points of $\mathscr{P}$ on the boundary of U^2. Hence these is a line segment S in the interior of U^2 with $\gg N^{(1/4)-(\epsilon/2)}$ points on it. Construct the right angled triangle T with S as hypotenuse. Let T_+ be a right angled triangle which is slightly larger than T and contains T. Let T_- be a right angled triangle which is slightly smaller than T and is contained in T. Since the line segment S is contained in the interior of U^2, it is possible to choose triangles T_+ and T_- as described above such that they are contained in U^2. Further they can be chosen in such a way that neither the hypotenuse of T_+ nor of T_- contains any point of $\mathscr{P}$ on it. It is clear that

$$D(T_+) - D(T_-) \gg N^{(1/4)-(\epsilon/2)} - \in \geq N^{(1/4)-\epsilon},$$

if the area of $T_+ - T_-$ is small. Hence either $|D(T_+)| \gg N^{(1/4)-\epsilon}$ or $|D(T_-)| \gg N^{(1/4)-\epsilon}$.

Case III. Suppose that the Cases I and II do not hold. Then on every line segment of length 1, there lie $< N^{(1/4)-\epsilon}$ points of $\mathscr{P}$. Now we apply Theorem 7A with $\epsilon/3$ in place of ϵ and with $\delta = 1$. There exists a rectangle R of diameter 1 with **95**

$$|D(R)| \gg N^{(1/4)-(\epsilon/3)}.$$

Now if R has sides parallel to the coordinate axes, then it is the union of two right triangles T_1, T_2 with sides parallel to the axes, and this union is disjoint except for line segments. Since there are less that $N^{(1/4)-\epsilon}$ points of $\mathscr{P}$ on such line segments, we have either

$$|D(T_1)| \gg N^{(1/4)-(\epsilon/3)} \text{ or } |D(T_2)| \gg N^{(1/4)-(\epsilon/3)}.$$

If the sides of R are not parallel to the coordinate axes, then there exist right triangles T_1, T_2, T_3, T_4 with sides containing the right angle parallel to the coordinate axes, such that $R \cup T_1 \cup T_2 \cup T_3 \cup T_4$ is a disjoint union except for line segments and constitutes a rectangle R' with sides parallel to the axes. Again $R' = T_5 \cup T_6$ with right angled triangles T_5, T_6, and this union is again disjoint except for a line segment. Hence for some T_i, $1 \leq i \leq 6$, we have

$$|D(T_i)| \gg N^{(1/4)-(\epsilon/3)}.$$

So in either case there is a triangle of diameter at most 1 with large $|D(T)|$. This triangle is a union of a bounded number of right triangles which are contained in triangles of U^2 by integer points. One finds eventually the existence of a right angled triangle in the interior of U^2 with large $D(T)$. By slightly enlarging T we may assure that there is no point of $\mathscr{P}$ on the hypotenuse of this triangle.

10 Points on a sphere

96 Denote by $S = S^k$ the sphere consisting of points $\underline{\underline{x}} = (x_1, \ldots, x_{k+1})$ of R^{k+1} with $|\underline{\underline{x}}| = 1$. Let $\underline{\underline{p}}_1, \ldots, \underline{\underline{p}}_N$ be points on S. Denote by $d_S\,\underline{\underline{x}}$ the volume of S and assume that it is normalised so that

$$\int_S d_S\,\underline{\underline{x}} = 1.$$

If $A \subset S$, then set

$$\mu(A) = \int_A d_S\,\underline{\underline{x}},$$

if it exists. Write

$$z(A) \text{ for the number of points } \underline{\underline{p}}_i \text{ in } A.$$

$$D(A) = z(A) - N\mu(A).$$

Write $\omega(\underline{\underline{x}}, \underline{\underline{y}})$ for the spherical distance of points $\underline{\underline{x}}, \underline{\underline{y}}$ on S. For $\underline{\underline{c}} \in S$ and r with $0 < r \le \dfrac{\pi}{2}$, denote by $C(r, \underline{\underline{c}})$ the *spherical cap* with centre , $\underline{\underline{c}}$ and radius r consisting of the points $\underline{\underline{x}} \in S$ with $\omega(\underline{\underline{x}}, \underline{\underline{c}}) \le r$. Put $\mu(r) = \mu(C(r, \underline{\underline{c}}))$, $z(r, \underline{\underline{c}}) = z(C(r, \underline{\underline{c}}))$, $D(r, \underline{\underline{c}}) = D(C(r, \underline{\underline{c}}))$,

$$E(r, s) = \int_S D(r, \underline{\underline{c}}) D(s, \underline{\underline{c}}) d_S \underline{\underline{c}}.$$

Theorem 10A. *Suppose that* $k > 1$ *and* $\epsilon > 0$. *Let* δ *satisfy* $0 < \delta \le \dfrac{\pi}{2}$ *and* $N\delta^k > N^{\epsilon}$. *Then*

$$\int_0^{\delta} E(r, r) r^{-1} dr \gg (N\delta^k)^{1-(1/k)-\epsilon}. \tag{10.1}$$

It can easily be shown that an analogue of Theorem 5A holds, and hence that the exponent on the right hand side of (10.1) is essentially best possible.

Theorem 10B. *Suppose that* k, ϵ *and* δ *are as in Theorem 10A. Then there exists a spherical cap* $C(r, \underline{\underline{c}})$ *with* $0 < r \le \delta$ *and with*

$$|D(r, \underline{\underline{c}})| \gg (N\delta^k)^{\frac{1}{2}-(1/2k)-\epsilon}.$$

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Lemma 10C.

$$E(r, s) \ll N^2 \min(r^k, s^k).$$

Proof.

$$\begin{aligned}
E(r, s) &= \int_S D(r, \underline{\underline{c}}) D(s, \underline{\underline{c}}) d_S \underline{\underline{c}} \\
&\le \int_S (z(r, \underline{\underline{c}}) + N\mu(r))(z(s, \underline{\underline{c}}) + N\mu(s)) d_S \underline{\underline{c}} \\
&\le 2N \int_S (z(s, \underline{\underline{c}}) + N\mu(s)) d_S \underline{\underline{c}}
\end{aligned}$$

$$= 4N^2\mu(s) \ll N^2 s^k.$$

□

The lemma follows by symmetry.

Using Lemma 10C, one can easily derive Theorem 10B from Theorem 10A. The procedure is the same as was followed in § 1 to derive Theorem 2B from Theorem 2A.

A *half sphere* is a spherical cap of radius $\frac{\pi}{2}$. A *slice* is an intersection of two half spheres.

Theorem 10D ***.** *Suppose that* $k > 1$ *and* $\in > 0$*. Then there exists a slice* L *with*

$$|D(L)| \gg N^{\frac{1}{2}-(1/2k)-\epsilon}.$$

For $k = 2$, there exists a spherical triangle T with 2 right angles having

$$|D(T)| \gg N^{(1/4)-\epsilon}.$$

Theorem 10D* may be proved by combining the argument of [21b] with Theorem 10A.

11 The Integral Equation for Spherical Caps

98 Set

$$\nu = \begin{cases} 0 & \text{if } k \text{ is even ,} \\ 1 & \text{if } k \text{ is odd.} \end{cases}$$

Let $0 < \alpha < 1$, and let β be the number with

$$\alpha + \beta = 1 + \nu.$$

In what follows, the constant implied by $\gg$ will depend on k and on α.

For $0 < \delta \leq \frac{\pi}{2}$, set

$$A = \delta^{\nu-1} \underset{r+s\leq\delta}{\int_0^\delta \int_0^\delta} E(r,s) \cos\frac{r-s}{2} \cos\frac{r+s}{2} \left(\sin\frac{|r-s|}{2}\right)^{-\alpha} \left(\sin\frac{r+s}{2}\right)^{-\beta} dr\, ds.$$

Note that

$$|2E(r,s)| \leq E(r,r) + E(s,s).$$

Therefore

$$\begin{aligned} |A| &\leq \delta^{\nu-1} \underset{r+s\leq\delta}{\int_0^\delta \int_0^\delta} E(r,r) \left(\sin\frac{|r+s|}{2}\right)^{-\beta} dr\, ds \\ &\ll \delta^{\nu-1} \underset{r+s\leq\delta}{\int_0^\delta \int_0^\delta} E(r,r)|r-s|^{-\alpha}(r+s)^{-\beta} dr\, ds, \end{aligned}$$

since $\sin \gg\ll x$, whenever $0 \leq x \leq \dfrac{\pi}{2}$. The above integral is equal to

$$\underset{r+s\leq\delta}{\int_0^\delta \int_0^\delta} E(\delta r, \delta r)|r-s|^{-\alpha}(r+s)^{-\beta} dr\, ds.$$

Using the argument of § 2, we may conclude that

$$|A| \ll \int_0^\delta E(r,r) r^{-1} dr. \tag{11.1}$$

For $\underline{\underline{c}}, \underline{\underline{x}}$ on S, put

$$f(r, \underline{\underline{c}}; \underline{\underline{x}}) = \begin{cases} 1 & \text{if } \underline{\underline{x}} \in C(r, \underline{\underline{c}}), \text{ i. e. if } \omega(\underline{\underline{c}}, \underline{\underline{x}}) \leq r, \\ 0 & \text{otherwise,} \end{cases}$$

so that $f(r, \underline{\underline{c}}; \underline{\underline{x}})$ for fixed $r, \underline{\underline{c}}$ is the characteristic function of $C(r, \underline{\underline{c}})$. **99**

Notice that $f(r, \underline{\underline{c}}; \underline{\underline{x}})$ is symmetric in $\underline{\underline{x}}$ and $\underline{\underline{c}}$. Put

$$h(r, s; \underline{\underline{x}}, \underline{\underline{y}}) = \int_S f(r, \underline{\underline{c}}; \underline{\underline{x}}) f(s, \underline{\underline{c}}; \underline{\underline{y}}) d_S \underline{\underline{c}}.$$

The integrand

$$f(r, \underline{\underline{c}}; \underline{\underline{x}}) f(s, \underline{\underline{c}}; \underline{\underline{y}}) = \begin{cases} 1 & \text{if } \underline{\underline{c}} \in C(r, \underline{\underline{x}}) \cap C(s, \underline{\underline{y}}), \\ 0 & \text{otherwise .} \end{cases}$$

So

$$\begin{aligned} h(r, s; \underline{\underline{x}}, \underline{\underline{y}}) &= \mu(C(r, \underline{\underline{x}}) \cap C(s, \underline{\underline{y}})) \\ &= K(r, s; \omega(\underline{\underline{x}}, \underline{\underline{y}})), \end{aligned}$$

where $K(r, s; \omega)$ denotes the volume of the intersection of two spherical caps with radius r and s whose centres have spherical distance ω. Proceeding similarly as in § 2, one obtains

$$E(r, s) = \sum_{i=1}^{N} \sum_{j=1}^{N} (K(r, s; \omega(\underline{\underline{p}}_i, \underline{\underline{p}}_j)) - \int_S \int_S K(r, s; \omega(\underline{\underline{x}}, \underline{\underline{y}})) d_S \underline{\underline{x}} d_S \underline{\underline{y}}). \tag{11.2}$$

Lemma 11A (Fundamental Lemma). *Suppose that* $0 < \delta \leq \frac{\pi}{2}$.

Suppose that $f(r) = f_\delta(r)$ *is continuous in* $0 < r \leq \dfrac{1}{2}$, *has*

$$|f(r)| \ll r^{1-k-\alpha}, \text{ as } r \to 0,$$

and satisfies the integral equation

$$\int_0^{\frac{1}{2}} K(\delta r, \delta r; \omega) f(r) dr = \delta^{\nu-1} \underset{r+s \leq \delta}{\int_0^\delta \int_0^\delta} K(r, s; \omega) \cos \frac{r-s}{2} \cos \frac{r+s}{2} \left(\sin \frac{|r-s|}{2}\right)^{-\alpha} \underset{(0 \leq \omega \leq \pi)}{\left(\sin \tfrac{r+s}{2}\right)^{\beta}} dr\, ds \tag{11.3}$$

100 Then

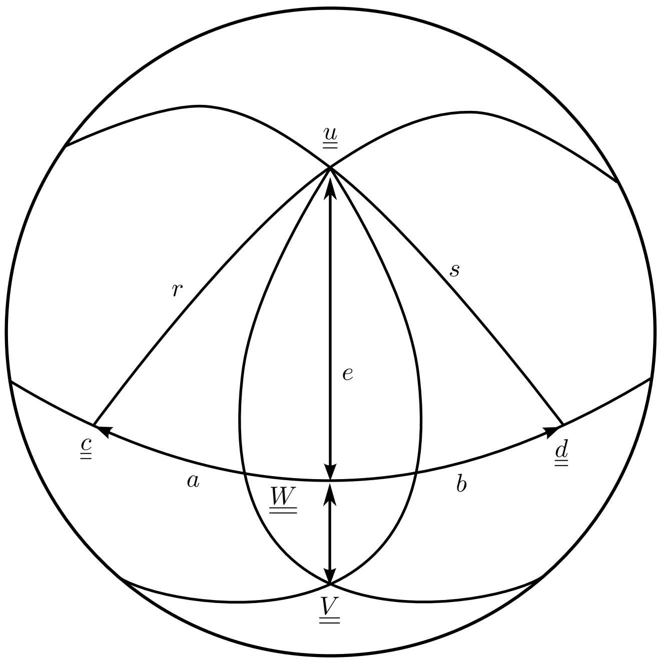
$$\int_0^{\frac{1}{2}} E(\delta r, \delta r) f(r) dr = \delta^{\nu-1} \underset{r+s \leq \delta}{\int_0^1 \int_0^1} E(r, s) \cos \frac{r-s}{2} \cos \frac{r+s}{2} \left(\sin \frac{|r-s|}{2}\right)^{-\alpha} \left(\sin \frac{r+s}{2}\right)^{-\beta} dr\, ds.$$

Proof. The lemma follows immediately from (11.2). In contrast to the situation in § 2, a solution $f(r) = f_\delta(r)$ of the integral equation will now depend on δ. ◻

We now proceed to determine a solution $f(r)$ of our integral equation. Now $K(r, s, \omega) = 0$ if $\omega \geq r+s$, and hence both sides of the integral equation (11.3) are zero if $\omega \geq \delta$. So it is sufficient to consider the interval $0 \leq \omega < \delta$. Since both sides of the equation are continuous in ω, we may in fact restrict ourselves to $0 > \omega < \delta$.

Assume, for a moment, that $k = 2$ and that

$$|r - s| < \omega < r + s.$$



Let $\underline{\underline{c}}, \underline{\underline{d}}$ be points with $\omega(\underline{\underline{c}}, \underline{\underline{d}}) = \omega$. The boundaries of $C(r, \underline{\underline{c}})$ and **101**
$C(s, \underline{\underline{d}})$ will intersect in two points $\underline{\underline{u}}$ and $\underline{\underline{v}}$. The big circle through $\underline{\underline{u}}, \underline{\underline{v}}$

will intersect the big circle through $\underline{\underline{c}}, \underline{\underline{d}}$ in two antipodal points $\underline{\underline{w}}, \underline{\underline{w}}'$. Of these two points, let $\underline{\underline{w}}$ be the one with $\omega(\underline{\underline{w}}, \underline{\underline{d}}) = \omega$. Put

$$\omega(\underline{\underline{w}}, \underline{\underline{u}}) = \omega(\underline{\underline{w}}, \underline{\underline{v}}) = e,$$
$$\omega(\underline{\underline{w}}, \underline{\underline{c}}) = a, (\underline{\underline{w}}, \underline{\underline{d}}) = b$$

Then

$$a + b = \omega,$$

and using the right spherical triangles $\underline{\underline{w}}, \underline{\underline{u}}, \underline{\underline{c}}$ and $\underline{\underline{w}}, \underline{\underline{u}}, \underline{\underline{d}}$ we obtain

$$\cos a \cos e = \cos r, \cos b \cos e = \cos s.$$

Combining these relations with the identities

$$\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$$

and

$$\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2},$$

we get

$$\cos \frac{a+b}{2} \cos \frac{a-b}{2} \cos e = \cos \frac{r+s}{2} \cos \frac{r-s}{2},$$
$$\sin \frac{a+b}{2} \sin \frac{a-b}{2} \cos e = \sin \frac{r+s}{2} \sin \frac{r-s}{2}.$$

We now multiply the first of these two equations by $\sin \frac{\omega}{2}$, the second by $\cos \frac{\omega}{2}$, then square both and add. Since $\omega = a + b$, we obtain

$$\begin{aligned} &\sin^2 \frac{\omega}{2} \cos^2 \frac{\omega}{2} \cos^2 e \\ &= \sin^2 \frac{\omega}{2} \cos^2 \frac{r+s}{2} \cos^2 \frac{r-s}{2} + \cos^2 \frac{\omega}{2} \sin^2 \frac{r+s}{2} \sin^2 \frac{r-s}{2}. \end{aligned}$$

which gives

$$\sin^2 e = \left(\sin^2\frac{\omega}{2} - \sin^2\frac{r-s}{2}\right)\left(\sin^2\frac{r+s}{2} - \sin^2\frac{\omega}{2}\right)^{-2}\left(\cos\frac{\omega}{2}\right)^{-2}. \tag{11.4}$$

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We have seen that for $k = 2$, the boundaries of $C(r, \underline{\underline{c}})$, $C(s, \underline{\underline{d}})$ intersect two points $\underline{\underline{u}}, \underline{\underline{v}}$, and we defined numbers a, b, e in terms of ω, r, s. In general, we may assume a rotation that $\underline{\underline{c}}, \underline{\underline{d}}$ lie on the x_1, x_2- plane. Construct $\underline{\underline{w}}$ so that $\omega(\underline{\underline{c}}, \underline{\underline{w}}) = a$, $\omega(\underline{\underline{d}}, \underline{\underline{w}}) = b$; then $\underline{\underline{w}}$ also lies in the (x_1, x_2)-plane. By a further rotation, we may assume that $\underline{\underline{w}} = (1, 0, \ldots, 0)$. Now let $\underline{\underline{z}}$ be an arbitrary point on the intersection of the boundaries of $C(r, \underline{\underline{c}}), C(s, \underline{\underline{d}})$. There exists a rotation ρ leaving points on the (x_1, x_2) - plane invariant such that $\rho(\underline{\underline{z}})$ lies in the (x_1, x_2, x_3) - coordinate subspace. Then $\rho(\underline{\underline{z}})$ again lies in the intersection of the boundaries of the two spherical caps, and therefore $\rho(\underline{\underline{z}}) = \underline{\underline{u}}$ or $\underline{\underline{v}}$. Thus

$$\rho(\underline{\underline{z}}) = (\cos e, 0, \pm\sin e, 0, \ldots, 0),$$

and $\underline{\underline{z}}$ itself is

$$\underline{\underline{z}} = (\cos e, 0, y_3 \sin e, \ldots, y_{k+1} \sin e),$$

where

$$y_3^2 + \ldots + y_{k+1}^2 = 1.$$

It follows that the intersection of the boundaries of the spherical caps $C(r, \underline{\underline{c}})$, $C(s, \underline{\underline{d}})$ consists of the points on the hyperplane $x_2 = 0$ whose spherical distance from $\underline{\underline{w}}$ is e.

The intersection $C(r, \underline{\underline{c}}) \cap C(s, \underline{\underline{d}})$ itself is contained in the set $W(e)$ of points whose spherical distance from the circle $x_1^2 + x_2^2 = 1$, $x_2 = \ldots x_{k+1} = 0$ is $\leq e$. Put differently, $W(e)$ consists of points $(y_1 \cos\varphi, y_2 \cos\varphi, y_3 \sin\varphi, \ldots, y_{k+1} \sin\varphi)$ with $y_1^2 + y_2^2 = 1$, $y_3^2 + \ldots + y_{k+1}^2 = 1$ and $0 \leq \varphi \leq e$. We note that **103**

$$\mu(w(e)) = c_1 \int_0^e \cos\varphi |\sin\varphi|^{k-2} d\varphi = c_2 (\sin e)^{k-1}.$$

Still under the assumption that $|r-s| < \omega < r+s$, a simple geometric argument shows that as $\omega' \to \omega$,

$$K(r,s,\omega) = K(r,s;\omega') = \frac{(\omega'-\omega)}{2\pi}\mu(W(e)) + o(\omega-\omega').^{3}$$

This implies that

$$\frac{\partial}{\partial\omega}(K(r,s;\omega)) = -\frac{1}{2\pi}\mu(w(e)) = -c_3(\sin e)^{k-1}.$$

In general we have

$$\frac{\partial}{\partial\omega}(K(r,s;\omega)) = \begin{cases} -c_3(\sin e)^{k-1}, & \text{if } |r-s| < \omega < r+s, \\ 0, & \text{otherwise.} \end{cases}$$

The derivative of the left hand side of the integral equation (11.3) with respect to ω is

$$\begin{aligned} c_3 &\int_{\omega/2\delta}^{\frac{1}{2}} (\sin e(\delta r, \delta r, \omega))^{k-1} f(r) dr \\ &= -c_3\delta^{-1} \int_{\omega/2}^{\delta/2} f(r/\delta)(\sin e(r,r,\omega))^{k-1} dr. \end{aligned}$$

In view of (11.4), this integral is equal to

$$-c_3\delta^{-1}(\cos\omega)^{1-k}\int_{\omega/2}^{\delta/2} f(r/\delta)(\sin^2 r - \sin^2\frac{\omega}{2})^{(k-1)/2}dr. \tag{11.5}$$

Put $\Omega = \sin\dfrac{\omega}{2} / \sin\dfrac{\delta}{2}$, $R = \sin r/2 \sin\dfrac{\delta}{2}$ and

$$F(R) = f(r/\delta)\cos r.$$

104 Now if ω is in $0 \le \omega \le \delta$, then Ω is in $0 \le \Omega \le 1$, and if $0 \le r \le \delta/2$, then $0 \le R \le 1/2$. With these substitutions, (11.5) becomes

$$-c_4\delta^{-1}(\sin\frac{\delta}{2})^k(\cos\frac{\omega}{2})^{1-k}\int_{\Omega/2}^{\frac{1}{2}} F(R)(R^2 - \frac{\Omega^2}{4})^{(k-1)/2}dR.$$

[3]The notation $f(t) = o(t)$ means that $f(t)/t$ tends to zero as $t > 0$ tends to zero.

The right hand side of the integral equation after differentiation with respect to ω becomes

$$-c_5\delta^{\nu-1}\underset{|r-s|\le\omega\le r+s\le\delta}{\int_0^\delta\int_0^\delta}(\sin e(r,s,\omega))^{k-1}\cos\frac{r-s}{2}\cos\frac{r+s}{2}$$
$$\left(\sin\frac{|r-s|}{2}\right)^{-\alpha}\left(\sin\frac{r+s}{2}\right)^{-\beta}dr\,ds.$$

Substituting the value of $\sin e$ as given by (11.4), we obtain

$$-c_5\delta^{\nu-1}\left(\sin\frac{\omega}{2}\right)^{-(k-1)}\left(\cos\frac{\omega}{2}\right)^{-(k-1)}\times$$
$$\underset{|r-s|\le\omega\le r+s\le\delta}{\int_0^\delta\int_0^\delta}\left(\sin^2\frac{\omega}{2}-\sin^2\frac{r-s}{2}\right)^{(k-1)/2}\left(\sin^2\frac{r+s}{2}-\sin^2\frac{\omega}{2}\right)^{\left(\frac{k-1}{2}\right)}\times$$
$$\cos\frac{r-s}{2}\cos\frac{r+s}{2}\left(\sin\frac{|r-s|}{2}\right)^{-\alpha}\left(\sin\frac{r+s}{2}\right)^{-\beta}dr\,ds.$$

Put

$$\Omega=\sin\frac{\omega}{2}/\sin\frac{\delta}{2},x=\sin\frac{r+s}{2}/\sin\frac{\delta}{2},y=\sin\frac{|r-s|}{2}/\sin\frac{\delta}{2}.$$

The above integral then becomes

$$-c_6\delta^{\nu-1}(\sin\frac{\delta}{2})^{k-\alpha-\beta+1}(\cos\frac{\omega}{2})^{1-k}\times$$
$$\int_\Omega^1(x^2-\Omega^2)^{(k-1)/2}x^{-\beta}dx\int_0^\Omega(1-\frac{y^2}{\Omega^2})^{(k-1)/2}y^{-\alpha}dy$$
$$=-c_7\delta^{\nu-1}(\sin\frac{\delta}{2})^{k-\nu}(\cos\frac{\omega}{2})^{1-k}\Omega^{1-\alpha}\int_\Omega^1(x^2-\Omega^2)^{(k-1)/2}x^{-\beta}dx.$$

Hence we obtain the new integral equation

$$\int_{\Omega/2}^{\frac{1}{2}}F(R)(R^2-\frac{\Omega^2}{4})^{(k-1)/2}dR$$

$$= c_8 \left(\frac{\delta}{\sin \frac{\delta}{2}} \right)^{\gamma} \Omega^{1-\alpha} \int_{\Omega}^{1} (x^2 - \Omega^2)^{(k-1)/2} x^{-\beta} dx \quad (0 < \Omega < 1). \quad (11.6)$$

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If $F(R)$ is a solution of this integral equation, then $f(r) = F(r\delta) \cos r\delta$ is a solution of the given integral equation (11.3).

Except for the factor $(\delta/\sin(\delta/2))^{\nu}$ and except for the notation, this integral equation is the same as (3.6). Hence there is a solution of (11.6) of the type

$$F(R) = F_0(R) - F_*(R),$$

where $F_0(R)$ and $F_*(R)$ are positive and continuous in $0 < R \leq 1/2$, and have

$$F_0(R) \gg\ll R^{1-\nu-\alpha} \text{ and } F_*(R) \gg\ll R^{1-k-\alpha}, \quad (11.7)$$

as $R \to 0$. These functions depend on δ; they are $(\delta/\sin(\delta/2))^{\nu}$ times functions which are independent of δ. Since

$$\delta/\sin(\delta/2) \gg\ll 1, \qquad (0 < \delta < \pi/2)$$

the estimates (11.7) hold uniformly in δ.

By what we said above the function

$$f(r) = f_0(r) - f_*(r)$$

with $f_0(r) = F_0(\delta r) \cos \delta r$, $f_*(r) = F_*(\delta r) \cos \delta r$, is a solution of the original equation (11.3). The functions $f_0(r)$ and $f_*(r)$ are positive and continuous in $0 < r \leq 1/2$, and $r \to 0$ they satisfy

$$f_0(r) \gg\ll r^{1-\nu-\alpha} \text{ and } f_*(r) \gg\ll r^{1-k-\alpha},$$

uniformly in δ.

106 In view of the Fundamental Lemma and by the definition of A,

$$\int_0^{\frac{1}{2}} E(\delta r, \delta r) f(r) dr = A.$$

and

$$\int_0^{\frac{1}{2}} E(\delta r, \delta r) f_0(r) dr - A = \int_0^{\frac{1}{2}} E(\delta r, \delta r) f_*(r) dr. \quad (11.8)$$

In view of (11.1) and since $f_0(r) \ll r^{1-\nu-\alpha} \ll r^{-1}$, the left hand side of this equation is

$$\ll \int_0^\delta E(r,r)r^{-1}dr.$$

On the other hand $|D(r,\underline{\underline{c}})| \geq \|N\mu(s)\|$, whence $E(r,r) \geq \|N\mu(r)\|^2$, and the right hand side of (11.8) is

$$\gg \int_0^{c_9} r^{1-k-\alpha}E(\delta r,\delta r)dr \geq \int_0^{c_9} r^{1-k-\alpha}\|N\mu(\delta r)\|^2 dr.$$

A suitable adaptation of the argument below (4.5) shows that this is $\gg (N\delta^k)^{1+(\alpha/k)-(2/k)}$, and hence also

$$\int_0^\delta E(r,r)r^{-1}dr \gg (N\delta^k)^{1+(\alpha/k)-(2/k)}.$$

Since α was arbitrary in $0 < \alpha < 1$, Theorem 10A is proved.

12 Point with Weights

Many results of these lectures may be generalized to distributions of points with weights. As a sample, we will now mention a partial generalization of Theorem 10A.

Let $\underline{\underline{p}}_1, \ldots, \underline{\underline{p}}_N$ be points on the sphere $S = S^k$. Suppose that non-negative weights $w_1, \ldots, w_N$ are attached to these points. Now given a measurable subset A of S, write

$$z(A) = \sum_{\underline{\underline{p}}_i \in A} w_i,$$

$$D(A) = z(A) - \mu(A)z(S).$$

Theorem 12A*. (W. M. Schmidt [23]). *Suppose that* $\in > 0$. *There exists* 107
a spherical cap $C(r,\underline{\underline{c}})$ *with*

$$|D(C(r,\underline{\underline{c}}))| \gg (w_1^2 + \ldots + w_N^2)^{1/2} N^{-(1/2k)-\epsilon}.$$

Results about countable distributions of points with a finite total weight may also be obtained.

13 Convex Sets

Theorem 13A(W. M. Schmidt [27]). *Suppose* $0 < \delta < 1$. *There exists a convex set* S *in* U^k *with diameter* $\leq \delta$ *and with*

$$|D(S)| \gg (N\delta^k)^{1-(2/(k+1))}.$$

This is stronger than an earliest estimate due to S. K. Zaremba [30].

By a *quadrant set* we shall understand a subset Q of U^k with the property that if $\underline{\underline{y}} = (y_1, \ldots, y_k) \in Q$, then the box consisting of $\underline{\underline{x}} = (x_1, \ldots, x_k)$ with $0 \leq x_i \leq y_i (i = 1, \ldots, k)$ is contained in Q.

Theorem 13B. *Suppose* $0 < \delta < 1$. *There is a quadrant set* Q *of diameter* $\leq \delta$ *with*

$$|D(Q)| \gg (N\delta^k)^{1-(1/k)}$$

Proof of Theorem 13A. We may suppose that $k > 1$, and that $N\delta^k$ is large. Let B be a ball of diameter δ contained in U^k, and let S be the surface of B. Let C be a closed spherical cap on S with spherical radius ρ. (With the radius normalized such that a half sphere has radius $\frac{\pi}{2}$). The convex hull $\overline{C}$ of C is a solid spherical cap. For $0 < \rho < \frac{\pi}{2}$, $\mu(\overline{C})$ is a continuous function of ρ with

$$\rho^{k+1}\delta^k \ll \mu(\overline{C}) \ll \rho^{k+1}\delta^k. \tag{13.1}$$

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If $N\delta^k$ is sufficiently large, there is a number ρ_0 such that a cap C of spherical radius ρ_0 has

$$\mu(C) = \frac{1}{2N}.$$

In view of (13.1), $0 < \rho_0 \ll (N\delta^k)^{-1/(k+1)}$. We now pick as many pairwise disjoint caps with radius ρ_0 as possible; say $C_1, \ldots, C_M$. For large $N\delta^k$ and hence small ρ_0 we have $M \gg \rho_0^{-(k-1)}$, whence

$$M \gg (N\delta^k)^{(k-1)/(k+1)}. \tag{13.2}$$

Given a sequence of numbers $\sigma_1, \ldots, \sigma_M$, with each σ_i either $+1$ or -1, let $B(\sigma_1, \ldots, \sigma_M)$ consist of all $\underline{\underline{x}} \in B$ which do not lie in a cap

$\overline{C}_i$ with $\sigma_i = -1$. In other words, $B(\sigma_1, \ldots, \sigma_M)$ is obtained from B by removing the solid caps $\overline{C}_i$ for which $\sigma_i = -1$.

Now the function $D(A)$ is *additive*, i. e. it satisfies

$$D(A \cup A') = D(A) + D(A')$$

if $A \cap A' = \phi$. It follows easily that

$$D(B(\sigma_1, \ldots, \sigma_M)) - D(B(-\sigma_1, \ldots, -\sigma_M)) = \sum_{i=1}^{M} \sigma_i D(\overline{C}_i).$$

We have

$$D(\overline{C}_i) = z(\overline{C}_i) - N\mu(\bar{(C)}_i) = z(\overline{C}_i) - \frac{1}{2}.$$

Hence for every i, either $D(\overline{C}_i) \geq \frac{1}{2}$ or $D(\overline{C}_i) \leq -\frac{1}{2}$. Choose σ_i such that $\sigma_i D(\overline{C}_i) \geq \frac{1}{2}(1 \leq i \leq M)$. Then

$$D(B(\sigma_1, \ldots, \sigma_M)) - D(B(-\sigma_1, \ldots, -\sigma_M)) \geq \frac{1}{2}M,$$

and either $S = B(\sigma_1, \ldots, \sigma_M)$ or $S = B(-\sigma_1, \ldots, -\sigma_M)$ has $|D(S)| \geq$ **109**
$\frac{1}{2}M$. Thus by (13.2),

$$|D(S)| \gg (N\delta^k)^{(k-1)/(k+1)}.$$

Theorem 13A is proven.

Proof of Theorem 13B. We may suppose that $k > 1$, and that $N\delta^k$ is large. Let G consist of points in U^k with $x_1 + \ldots, +x_k \leq \delta/k$, and H of points with $x_1 + \ldots + x_k = \delta/k$. Let $\in > 0$ be small, and let $\underline{\underline{x}} = (x_1, \ldots, x_k)$ be a points on H with $(k-1)\delta \in < x_i (i = 1, \ldots, k)$. Let $\overline{P}(\underline{\underline{x}})$ consist of points $\underline{\underline{y}} = (y_1, \ldots, y_k)$ with

$$y_1 + \ldots + y_k. > \delta/k \text{ and } y_i \leq x_i + \delta \in (i = 1, \ldots, k).$$

Then $P(\underline{x})$ lies in U^k and has volume $\mu(P(\underline{x})) = (k\delta\epsilon)^k/k!$. If $N\delta^k$ is sufficiently large, we may choose ϵ such that this volume equals $1/(2N)$. Then $\epsilon \ll (N\delta^k)^{-1/k}$.

Pick as many pairwise disjoint sets $P(\underline{\underline{x}})$ as possible ; say $P_1, \ldots, P_M$. Clearly $M \gg \in^{-(k-1)}$, whence

$$M \gg (n\delta^k)^{(k-1)/k}.$$

For any sequence $\sigma_1, \ldots, \sigma_M$ of $+1$ and -1 signs, let $Q(\sigma_1, \ldots, \sigma_M)$ be the union of G with the "blisters" P_i for which $\sigma_i = 1$. Th set $Q(\sigma_1, \ldots, \sigma_M)$ is a quadrant set of diameter $\leq \delta$. We have

$$D(Q(\sigma_1, \ldots, \sigma_M)) - D(Q(-\sigma_1, \ldots, -\sigma_M)) = \sum_{i=1}^{M} \sigma_i D(Q_i).$$

By an argument used in the proof of Theorem 13A, we obtain a quadrant set Q of diameter $\leq \delta$ with

$$|D(Q)| \geq \frac{1}{4} M \gg (N\delta^k)^{(k-1)/k}.$$

14 Comparison of different discrepancies

110 If $\mathfrak{a}$ is anon-empty class of measurable sets in U^k, write

$$D(\mathfrak{a}) = \sup |D(A)|,$$

where the supermum is over all $A \in \mathfrak{a}$. Further put

$$\triangle(\mathfrak{a}) = D(\mathfrak{a})/N.$$

It is clear that $0 \leq D(\mathfrak{a}) \leq N$ and $o \leq \triangle(\mathfrak{a}) \leq 1$. One could call $D(\mathfrak{a})$ the *discrepancy with respect to* $\mathfrak{a}$ of the given N points; but some authors prefer to call $\triangle(\mathfrak{a})$ the discrepancy.

Let $\mathscr{J}$ be the class of boxes in U^k of the type $a_1 \leq x_1 \leq b_1, \ldots, a_k \leq x_k \leq b_k$, let $\mathfrak{M}$ be the class of closed cubes in U^k with sides parallel to the coordinate axes, $\mathscr{L}$ the class of closed balls in U^k, $\mathfrak{S}$ the class of convex subsets of U^k and $\mathfrak{q}$ the class of quadrant sets.

We have already seen that

$$\triangle(\mathfrak{J}) \gg N^{-1}(\log N)^{(k-1)/2} \qquad \text{(Ch. I, Theorem 2B)},$$

$$\triangle(\mathfrak{J}) \gg N^{-1}(\log N) \text{ if } k = 2 \qquad \text{(Ch. I, Theorem 5B)},$$
$$\triangle\mathscr{L} \gg N^{(k-1)/(2k(k+2))-1-\epsilon} \qquad \text{(Corollary to Theorem 6A)}$$
$$\triangle\mathfrak{S} \gg N^{-2/(k+1)} \qquad \text{(Theorem 13A)}$$

and that

$$\triangle(\mathfrak{q}) \gg N^{-1/k} \qquad \text{(Theorem 13B)}.$$

Now $\mathfrak{a} \leq \mathfrak{a}'$ implies $\triangle(\mathfrak{a}) \leq \triangle(\mathfrak{a}')$, so that

$$\triangle(\mathfrak{M}) \leq \triangle(\mathfrak{J}) \leq \triangle(\mathfrak{S}),$$
$$\triangle(\mathscr{L}) \leq \triangle(\mathfrak{S}).$$

Theorem 14A (W. M. Schmidt [27]).

$$\triangle(\mathfrak{S}) \ll \triangle(\mathfrak{M})^{1/k}$$
$$\triangle(\mathfrak{q}) \ll \triangle(\mathfrak{M})^{1/k}$$
.

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Earlier, E. Hlawka [10] (see also [9]) had shown that $\triangle(\mathfrak{S}) \ll \triangle(\mathfrak{m})^{1/(k+1)}$ and $\triangle(\mathfrak{S}) \ll \triangle(\mathfrak{J})^{1/k}$.

Write $\exp x = e^x$.

Theorem 14B. *Both* $\triangle(\mathfrak{S})$ *and* $\triangle(\mathfrak{q})$ *are*

$$\ll \triangle(\mathscr{L})^{1/k} exp(2(\log 2)^{1/2} k^{-1} |\log \triangle(\mathscr{L})|^{1/2}).$$

In particular, it follows that for $\in > 0$, both $\triangle(\mathfrak{S})$ and $\triangle(\mathfrak{q})$ are

$$\ll \triangle(\mathscr{L})^{(1/k)-\epsilon}.$$

This is stronger than estimates of Hlawka [10]. J. W. S. Cassels (unpublished) and C. J. Smyth [28]. But on the other hand Smyth (thesis, Cambridge, Engaland) obtained

$$\triangle(\mathfrak{J}) \ll \triangle(\mathscr{L})^{1/k}(1 + \log|\triangle(\mathscr{L})|^{c(k)}).$$

which will not be proved here.

Let $B(r, \underline{\underline{c}})$ be the closed ball with radius r and center $\underline{\underline{c}}$. Given a subset S of U^k, let $S(r)$ consist of points $\underline{\underline{x}}$ for which $B(r, \underline{\underline{x}}) \subseteq S$. Let S' be the complement of S in U^k.

For each $\sigma > 0$. let $\mathfrak{S}(\sigma)$ be the class of subsets S of U^k having

$$\mu(S(r)) \geq \mu(S) - \sigma r, \mu(S'(r)) \geq \mu(S') - \sigma r \tag{14.1}$$

for every $r > 0$.

Lemma 14C. *There are constants $c_1(k)$, $c_2(k)$ such that*

$$\mathfrak{S} \subseteq \mathfrak{S}(c_1).\mathfrak{q} \subseteq (c_2).$$

112 The proof of this lemma is easy and will not be given here. For a rather more thorough discussion of $\mu(S(r))$ for convex sets S, see H. G. Eggleston [4].

Theorem 14A, 14B are respective consequences of

Theorem 14C.

$$\triangle(\mathfrak{S}(\sigma)) \leq c_3(k, \sigma)\triangle(\mathfrak{M})^{1/k}$$

Theorem 14D.

$$\triangle(\mathfrak{S}(\sigma)) \leq c_4(k, \sigma)\triangle(\mathscr{L})^{1/k} exp(2(\log 2)^{1/2} k^{-1} |\log \triangle(\mathscr{L})|^{1/2}).$$

Proof of Theorem 14C. Let $S[r]$ consist of points $\underline{\underline{x}} \in U^k$ which have a distance $< r$ from the boundary of S. Every $\underline{\underline{x}} \in S[r]$ is either in S but not in $S(r)$, or is in S' but not in $S'(r)$. Hence for $S \in \mathfrak{S}(\sigma)$.

$$\mu(S[r]) \leq 2\sigma r.$$

Now if $k = 1$ and if $z_1, \ldots, z_M$ are on the boundary of S and in the interior of U, then for small r, $S[r]$ contains the M open intervals with centres $z_1, \ldots, z_M$ and of length $2r$. Hence for small $r, \mu(S[r]) \geq 2rM$, and we get $M \leq \sigma$. Thus S has at most $\sigma + 2$ boundary points, and is therefore the union of a bounded numbers of points and intervals. Hence Theorem 14C is true for $k = 1$.

We may henceforth assume that $k > 1$. Pick a point $\underline{\underline{a}} = (a_1, \ldots, a_k)$ such that for each of the given points $\underline{\underline{p}}_i$ $(i = 1, \ldots, N)$, each coordinate of $\underline{\underline{p}}_i - \underline{\underline{a}}$ irrational. For a positive integer n, let $\mathfrak{M}(n)$ be the class of cubes

$$a_i + \frac{u_i}{n} \leq x_i \leq a_i + \frac{u_i + 1}{n} (i = 1, \ldots, k)$$

with integers $u_1, \ldots, u_k$. Let $\mathfrak{W}(n)$ be the set of cubes of $\mathfrak{M}(n)$ which are contained in S. Since a cube of $\mathfrak{M}(n)$ has diameter $k^{\frac{1}{2}}/n$, it follows that the cubes of $\mathfrak{W}(n)$ cover $S(k^{\frac{1}{2}}/n)$, and their number $\nu(n)$ satisfies

$$n^k \mu(S) \geq \nu(n) \geq n^k \mu(S(k^{\frac{1}{2}}/n)) \geq n^k \mu(S) - n^{k-1} \sigma k^{\frac{1}{2}}. \tag{14.2}$$

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For each positive integer i, the union of the cubes of $\mathfrak{W}(2^i)$ contains the union of the cubes of $\mathfrak{W}(2^{i-1})$. Put $\mathfrak{W}_1 = \mathfrak{W}(2^1)$, and for $i \geq 2$. let $\mathfrak{W}i$ consist of the cubes of $\mathfrak{W}(2^i)$ which are not contained in a cube of $\mathfrak{W}(2^{i-1})$. If ν_i is the number of cubes in $\mathfrak{W}_i$, then $\nu_1 = \nu(2)$, and for $i \geq 2$ we have

$$2^{-ik} \nu_i + 2^{-(i-1)k} \nu(2^{i-1}) \leq \mu(S),$$

whence by (14.2),

$$\nu_i \leq 2^{ik} \nu(S) - 2^k \nu(2^{i-1}) \leq \sigma k^{\frac{1}{2}} 2^{i(k-1)+1}.$$

Since any two distinct cubes in any of the sets $\mathfrak{W}_1, \mathfrak{W}_2, \ldots$ are disjoint except possibly for their boundaries, and since by our choice of $\underline{\underline{a}}$ none of the given N points lie on such a boundary, we have for every positive integer M,

$$\begin{aligned} z(S) &\geq \sum_{i=1}^{M} \sum_{W \epsilon \mathfrak{W}_i} z(W) \\ &\geq \sum_{i=1}^{M} \sum_{W \epsilon \mathfrak{W}_i} (N\mu(W) - N\triangle(\mathfrak{M})) \\ &= N\left(\left(\sum_{W \epsilon \mathfrak{W}(2^M)}\right) \mu(W)) - \triangle(\mathfrak{M}) \sum_{i=1}^{M} \nu_i\right) \end{aligned}$$

$$\geq N\left(\mu\left(S\left(k^{\frac{1}{2}}2^{-M}\right)\right) - \triangle(\mathfrak{M})\left(2^k + \sigma k^{\frac{1}{2}}\sum_{i=2}^{M} 2^{i(k-1)+1}\right)\right)$$
$$\geq N\mu(S) - Nc_5(k,\sigma)(2^{-M} + \triangle(\mathfrak{M})2^{M(k-1)}),$$

since $k > 1$. Now if we choose M such that $2^{M-1} \leq \triangle(\mathfrak{M})^{-1/k} < 2^M$, then

$$z(S) - N\mu(S) \geq -Nc_5(k,\sigma)\triangle(\mathfrak{M})^{1/k}(1 + 2^{k-1}) = -Nc_3(k,\sigma)\triangle(\mathfrak{M})^{1/k}.$$

This inequality remains true if we replace S by S'. Hence

$$|z(S) - N\mu(S)| \leq Nc_3(k,\sigma)\triangle(\mathfrak{W})^{1/k},$$

114 and Theorem 14C follows.

15 Proof of Theorem 14D by "successive sweeping"

Given a set A,let $rA + \underline{\underline{y}}$ be the set of points $r\underline{\underline{a}} + \underline{\underline{y}}$ with $\underline{\underline{a}} \in A$. Let $\mathfrak{a}(A)$ be the class of sets $rA + \underline{\underline{y}}$ with $r > 0$ which are contained in U^k. Thus if B is any closed ball, then $\mathfrak{a}(B) = \mathscr{L}$. Hence Theorem 14D follows from

Theorem 15A. *Suppose $\mu(A) > 0$ and $A \in \mathfrak{S}(\tau)$ for some τ. Then*

$$\triangle(\mathfrak{S}(\sigma)) \leq c_1(A,\sigma)\triangle(\mathfrak{a}(A))^{1/k} exp(2(\log 2)^{\frac{1}{2}} k^{-1}|\log \triangle(\mathfrak{a}(A))|^{\frac{1}{2}}).$$

The proof of this theorem will require a series of lemmas. Denote the distance of points x, y by

$$|\underline{\underline{x}} - \underline{\underline{y}}|.$$

Now let $r_1, r_2, \ldots$ be positive reals with

$$r_{i+1} \leq \frac{1}{2} r_i \quad i = 1, 2, \ldots. \tag{15.1}$$

and set $s_i = k^{\frac{1}{2}} r_i$. The set $r_i A$ has diameter $\leq s_i$.

For a set T, let $\mathcal{X}(T|\underline{\underline{x}})$ be the characteristic function of T. Let S be a set belonging to $\mathfrak{S}(\sigma)$.

We are going to construct functions $f_\nu(\underline{\underline{x}})$, $g_\nu(\underline{\underline{x}})$, $h_\nu(\underline{\underline{x}})$ $(\nu = 0, 1, 2, \ldots)$.

We begin by setting

$$f_0(\underline{\underline{x}}) = 0.$$

If a continuous function $f_\nu(\underline{\underline{x}})$ is given, write

$$g_\nu(\underline{\underline{x}}) = \mathcal{X}(S|\underline{\underline{x}}) - f_\nu(\underline{\underline{x}}),$$

$$h_\nu(\underline{\underline{x}}) = \min_{|\underline{\underline{y}}-\underline{\underline{x}}|\le s_{\nu+1}} g_\nu(\underline{\underline{y}}),$$

$$f_{\nu+1}(\underline{\underline{x}}) = f_\nu(\underline{\underline{x}}) + (\nu(r_{\nu+1}A))^{-1} \int \mathcal{X}(r_{\nu+1}A + \underline{\underline{y}}|\underline{\underline{x}}) h_\nu(\underline{\underline{y}}) d\underline{\underline{y}}.$$

Lemma 15B. *We have* **115**

(i) $0 \le f_{\nu-1}(\underline{\underline{x}}) \le f_\nu(\underline{\underline{x}}) \le \mathcal{X}(S|\underline{\underline{x}}) \qquad (\nu = 1, 2, \ldots),$

(iia) $|f_\nu(\underline{\underline{x}}) - f_\nu(\underline{\underline{x}}')| \le c_2(A) r_\nu^{-1} |\underline{\underline{x}} - \underline{\underline{x}}'| \qquad (\nu = 1, 2, \ldots)$, *and in particular* $f_\nu(\underline{\underline{x}})$ *is continuous.*

(iib) $|f_\nu(\underline{\underline{x}}) - f_\nu(\underline{\underline{x}}')| \le 2^{\nu-i} c_2(A) r_i^{-1} |\underline{\underline{x}} - \underline{\underline{x}}'|$ *if* $i \le i \le \nu - 1$ *and if* $|\underline{\underline{x}} - \underline{\underline{x}}'| \le s_\nu$ *and* $\underline{\underline{x}}, \underline{\underline{x}}' \in S(3(s_{i+1} + \ldots + s_\nu))$.

(iiia) $f_\nu(\underline{\underline{x}}) = 1$ *if* $\underline{\underline{x}} \in S(2s_1) \qquad (\nu = 1, 2, \ldots),$

(iiib) $f_\nu(\underline{\underline{x}}) \ge 1 - 2^{\nu-i} c_3(A)(s_\nu/s_i)$ *if* $1 \le i \le \nu - 1$ *and* $\underline{\underline{x}} \in S(6s_{i+1})$.

Our construction may be interpreted as follows. We first sweep S with a broom of the size and shape of r_1A. We can sweep the middle of S, more precisely $S(2s_1)$, very well. But we cannot sweep the border areas of S very well. We then take a smaller broom of the size and shape of r_2A. And so on. We obtain a better and better sweeping of S which is expressed by (*i*) and (*iiib*). But it would have been inefficient to sweep right away with a very small broom of the size and shape of $r_\nu A$.

Proof. We proceed by induction on ν. Assume that either $\nu = 0$ or that the lemma is true for a particular value of $\nu > 0$. We have $0 \leq f_\nu(\underline{\underline{x}}) \leq \mathcal{X}(S\,|\underline{\underline{x}})$, whence $0 \leq g_\nu(\underline{\underline{x}}) \leq \mathcal{X}(S\,|\underline{\underline{x}})$. Now if $\underline{\underline{y}} \in r_{\nu+1}A + \underline{\underline{x}}$, then $\underline{\underline{y}} - \underline{\underline{x}} \in r_{\nu+1}A$, whence $|\underline{\underline{y}} - \underline{\underline{x}}| \leq s_{\nu+1}$, whence $h_\nu(\underline{\underline{y}}) \leq g_\nu(\underline{\underline{x}})$. We obtain

$$
\begin{aligned}
0 \leq \int \mathcal{X}(r_{\nu+1}A + \underline{\underline{y}}\backslash\underline{\underline{x}})h_\nu(\underline{\underline{y}})d\underline{\underline{y}} &\leq g_\nu(\underline{\underline{x}}) \int \mathcal{X}(r_{\nu+1}A + \underline{\underline{y}}\backslash\underline{\underline{x}})d\underline{\underline{y}} \\
&= g_\nu(\underline{\underline{x}})\mu(r_{\nu+1}A).
\end{aligned}
$$

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$$f_\nu(\underline{\underline{x}}) \leq f_{\nu+1}(\underline{\underline{x}}) \leq f_\nu(\underline{\underline{x}}) + g_\nu(\underline{\underline{x}}) = \mathcal{X}(S\,\backslash\underline{\underline{x}}).$$

Hence (i) is true for $\nu + 1$. □

Now it is clear that $\ell_{\nu+1}(\underline{\underline{x}}) = f_{\nu+1}(\underline{\underline{x}}) - f_\nu(\underline{\underline{x}})$ has

$$
\begin{aligned}
\ell_{\nu+1}(\underline{\underline{x}}) - \ell_{\nu+1}(\underline{\underline{x}}') = {} & (\mu(r_{\nu+1}A))^{-1} \\
& \int \Big(\mathcal{X}(r_{\nu+1}A + \underline{\underline{y}}\backslash\underline{\underline{x}}) - \mathcal{X}(r_{\nu+1}A + \underline{\underline{y}}\backslash\underline{\underline{x}}')\Big) h_\nu(\underline{\underline{y}})d\underline{\underline{y}}.
\end{aligned}
$$

Since $0 \leq h_\nu(\underline{\underline{y}}) \leq 1$, we obtain

$$\ell_{\nu+1}(\underline{\underline{x}}) - \ell_{\nu+1}(\underline{\underline{x}}') \leq (\mu(r_{\nu+1}A))^{-1}\mu(C_1),$$

where C_1 consists of $\underline{\underline{y}}$ for which $-\underline{\underline{y}}$ lies in $r_{\nu+1}A - \underline{\underline{x}}$ but not in $r_{\nu+1}A - \underline{\underline{x}}'$. Now

$$\mu(C_1) = r^k_{\nu+1}\mu(C_2),$$

where C_2 consists of $\underline{\underline{y}}$ which are in $A - r^{-1}_{\nu+1}(\underline{\underline{x}} - \underline{\underline{x}}')$ but not in A. Now if $\underline{\underline{y}} \in C_2$ lies in U^k, then $\underline{\underline{y}} \in A'$ and $\underline{\underline{y}} \notin A'(r^{-1}_{\nu+1}|\underline{\underline{x}} - \underline{\underline{x}}'|)$. Hence by virtue of $A \in \mathfrak{S}(\tau)$, the intersection $C_2 \cap U^k$ has volume $\leq \tau r^{-1}_{\nu+1}|\underline{\underline{x}} - \underline{\underline{x}}'|$. On the other hand if $\underline{\underline{y}} \in C_2$ lies putside of U^k, then, it has distance $\leq r^{-1}_{\nu+1}|\underline{\underline{x}} - \underline{\underline{x}}'|$ from U^k, and if $r^{-1}_{\nu+1}|\underline{\underline{x}} - \underline{\underline{x}}'| \leq 1$, then the part of C_2

outside U^k has volume $\leq 6^k r_{\nu+1}^{-1}|\underline{\underline{x}} - \underline{\underline{x}}'|$. Thus if $|\underline{\underline{x}} - \underline{\underline{x}}'|$ is small, then $\mu(C_2) \leq (\tau + 6^k) r_{\nu+1}^{-1}|\underline{\underline{x}} - \underline{\underline{x}}'|$, and therefore

$$\ell_{\nu+1}(\underline{\underline{x}}) - \ell_{\nu+1}(\underline{\underline{x}}') \leq \frac{1}{2} c_2(A) r_{\nu+1}^{-1}|\underline{\underline{x}} - \underline{\underline{x}}'|.$$

with $c_2(A) = 2\mu(A)^{-1}(\tau + 6^k)$. It follows that for *every* $\underline{\underline{x}}$, $\underline{\underline{x}}'$,

$$|\ell_{\nu+1}(\underline{\underline{x}}) - \ell_{\nu+1}(\underline{\underline{x}}')| \leq \frac{1}{2} c_2(A) r_{\nu+1}^{-1}|\underline{\underline{x}} - \underline{\underline{x}}'|. \tag{15.2}$$

Now if $\nu = 0$, we have $f_{\nu+1}(\underline{\underline{x}}) = f_1(\underline{\underline{x}}) = \ell_1(\underline{\underline{x}})$, and the case $\nu = 1$ of (iia) follows. If $\nu > 0$, we use our inductive assumption, (15.1), (15.2) and the relation $f_{\nu+1}(\underline{\underline{x}}) = f_\nu(\underline{\underline{x}}) + \ell_{\nu+1}(\underline{\underline{x}})$ to obtain

$$|f_{\nu+1}(\underline{\underline{x}}) - f_{\nu+1}(\underline{\underline{x}}')| \leq (c_2(A) r_\nu^{-1} + \frac{1}{2} c_2(A) r_{\nu+1}^{-1})|\underline{\underline{x}} - \underline{\underline{x}}'| \leq c_2(A) r_{\nu+1}^{-1}|\underline{\underline{x}} - \underline{\underline{x}}'|.$$

Thus (iia) is true for $\nu + 1$. **117**

Before taking up the proof of (iib) we observe the following.

Suppose that either

$$i = \nu \text{ and } \underline{\underline{z}}, \underline{\underline{z}}' \in S(s_{\nu+1}), \tag{15.3}$$

or that

$$1 \leq i \leq \nu - 1, |\underline{\underline{z}} - \underline{\underline{z}}'| \text{ and } \underline{\underline{z}}, \underline{\underline{z}}' \in S(3(s_{i+1} + \ldots + s_\nu) + s_{\nu+1}) \tag{15.4}$$

Now $h_\nu(\underline{\underline{z}})$ equals $g_\nu(\underline{\underline{w}})$ for some $\underline{\underline{w}}$ with $|\underline{\underline{w}} - \underline{\underline{z}}| \leq s_{\nu+1}$. Since $h_\nu(\underline{\underline{z}}')$ is defined as the minimum of $g_\nu(\underline{\underline{u}})$ for $|\underline{\underline{u}} - \underline{\underline{z}}'| \leq s_{\nu+1}$, and since $\underline{\underline{w}}' = \underline{\underline{w}} + \underline{\underline{z}}' - \underline{\underline{z}}$ has $|\underline{\underline{w}}' - \underline{\underline{z}}'| \leq s_{\nu+1}$, we get $h_\nu(\underline{\underline{z}}') \leq g_\nu(\underline{\underline{w}}')$, whence

$$h_\nu(\underline{\underline{z}}') - h_\nu(\underline{\underline{z}}) \leq g_\nu(\underline{\underline{w}}') - g_\nu(\underline{\underline{w}}). \tag{15.5}$$

Our hypotheses on $\underline{\underline{z}}$, $\underline{\underline{z}}'$ imply that $\underline{\underline{w}}$, $\underline{\underline{w}}' \in S$, whence $(S \setminus \underline{\underline{w}}) = \mathcal{X}(S \setminus \underline{\underline{w}}') = 1$ and

$$g_\nu(\underline{\underline{w}}') - g_\nu(\underline{\underline{w}}) = f_\nu(\underline{\underline{w}}) - f_\nu(\underline{\underline{w}}'). \tag{15.6}$$

Now if (15.3) holds, apply (*iia*) to $\underline{\underline{w}}, \underline{\underline{w}}'$. On the other hand, if (15.4) holds, then $|\underline{\underline{w}} - \underline{\underline{w}}'| = |\underline{\underline{z}} - \underline{\underline{z}}'| \leq s_\nu$ and $\underline{\underline{w}}, \underline{\underline{w}}' \in S(3(s_{i+1} + \ldots + s_\nu))$. In this case we apply (*iib*) to $\underline{\underline{w}}, \underline{\underline{w}}'$. We may do so, since (*iib*) is true for our particular value of ν by induction. In either case, we get

$$|f_\nu(\underline{\underline{w}}) - f_\nu(\underline{\underline{w}}')| \leq 2^{\nu-1} c_2(A) r_i^{-1} |\underline{\underline{w}} - \underline{\underline{w}}'| = 2^{\nu-1} c_2(A) r_i^{-1} |\underline{\underline{z}} - \underline{\underline{z}}'|.$$

Combining this with (15.5) and (15.6), we may conclude that both (15.4) or (15.4) implies

$$|h_\nu(\underline{\underline{z}}') - h_\nu(\underline{\underline{z}})| \leq 2^{\nu-i} c_2(A) r_i^{-1} |\underline{\underline{z}} - \underline{\underline{z}}'|.$$

118 Now suppose that $1 \leq i \leq \nu$, that $|\underline{\underline{x}}, \underline{\underline{x}}'| \leq s_{\nu+1}$ and that $\underline{\underline{x}}, \underline{\underline{x}}' \in S(3(s_{i+1} + \ldots + s_{\nu+1}))$. We have

$$\ell_{\nu+1}(\underline{\underline{x}}) - \ell_{\nu+1}(\underline{\underline{x}}') = (\mu(r_{\nu+1}A))^{-1} \int \mathcal{X}(r_{\nu+1}A + \underline{\underline{y}} \backslash \underline{\underline{x}}')(h_\nu(\underline{\underline{y}} + \underline{\underline{x}} - \underline{\underline{x}}') - h(\underline{\underline{y}}))d\underline{\underline{y}}.$$

The integrand is zero unless $|\underline{\underline{y}} - \underline{\underline{x}}'| \leq s_{\nu+1}$, hence is zero unless $\underline{\underline{y}} \in S(3(s_{i+1} + \ldots + s_\nu)$ [4] $+2s_{\nu+1})$. But then $\underline{\underline{y}} + \underline{\underline{x}} - \underline{\underline{x}}' \in S(3(s_{i+1} + \ldots + s_\nu) + s_{\nu+1})$. We apply the remark made above to $\underline{\underline{z}} = \underline{\underline{y}}$, $\underline{\underline{z}}' = \underline{\underline{y}} + \underline{\underline{x}} - \underline{\underline{x}}'$, and we obtain $|h_\nu(\underline{\underline{y}} + \underline{\underline{x}} - \underline{\underline{x}}') - h_\nu(\underline{\underline{y}})| \leq 2^{\nu-i} c_2(A) r_i^{-1} |\underline{\underline{x}} - \underline{\underline{x}}'|$. Hence

$$|\ell_{\nu+1}(\underline{\underline{x}}) - \ell_{\nu+1}(\underline{\underline{x}}')| \leq 2^{\nu-i} c_2(A) r_i^{-1} |\underline{\underline{x}} - \underline{\underline{x}}'|.$$

Since $f_{\nu+1}(\underline{\underline{x}}) = f_\nu(\underline{\underline{x}}) + \ell_{\nu+1}(\underline{\underline{x}})$ and since $|f_\nu(\underline{\underline{x}}) - f_\nu(\underline{\underline{x}}')| \leq 2^{\nu-i} c_2(A)$ $r_i^{-1} |\underline{\underline{x}} - \underline{\underline{x}}'|$ by induction, we obtain

$$|f_{\nu+1}(\underline{\underline{x}}) - f_{\nu+1}(\underline{\underline{x}}')| \leq 2^{\nu+1-i} c_2(A) r_i^{-1} |\underline{\underline{x}} - \underline{\underline{x}}'|.$$

Thus (*iib*) is true foe $\nu + 1$.

We have

$$f_1(\underline{\underline{x}}) = \mu(r_1 A)^{-1} \int \mathcal{X}(r_1 A + \underline{\underline{y}} \backslash \underline{\underline{x}}) h_0(\underline{\underline{y}}) d\underline{\underline{y}}.$$

[4]The empty sum occurring when $i = \nu$ is to be interpreted as zer0.

If $\underline{\underline{x}} \in S(2s_1)$ and if $\underline{\underline{x}} \in r_1 A + \underline{\underline{y}}$, then $|\underline{\underline{y}} - \underline{\underline{x}}| \le s_1$ and $\underline{\underline{y}} \in S(s_1)$. Since g_0 is the characteristic function of S, the definition of $h_0(\underline{\underline{y}})$ implies that $h_0(\underline{\underline{y}}) = 1$ for $\underline{\underline{y}} \in S(s_1)$. Therefore $\underline{\underline{x}} \in S(2s_1)$ implies that $f_1(\underline{\underline{x}}) = 1$. Since $f_1(\underline{\underline{x}}) \le f_\nu(\underline{\underline{x}}) \le 1$ by (i), we obtain ($iiia$).

There remains (iiib). Suppose $1 \le i \le \nu$ and $\underline{\underline{x}} \in S(3(s_{i+1} + \ldots + s_{\nu+1}))$.

We have

$$f_{\nu+1}(\underline{\underline{x}}) = \mu(r_{\nu+1}A)^{-1} \int \mathcal{X}(r_{\nu+1}A + \underline{\underline{y}} \backslash \underline{\underline{x}})(f_\nu(\underline{\underline{x}}) + h_\nu(\underline{\underline{y}}))d\underline{\underline{y}}. \tag{15.7}$$

Here $b_\nu(\underline{\underline{y}}) = g_\nu(\underline{\underline{w}})$ for some $\underline{\underline{w}}$ with $|\underline{\underline{w}} - \underline{\underline{y}}| \le s_{\nu+1}$. In particular, **119**
if $\underline{\underline{x}} \in r_{\nu+1}A + \underline{\underline{y}}$, we have $|\underline{\underline{y}} - \underline{\underline{x}}| \le s_{\nu+1}$, whence $|\underline{\underline{w}} - \underline{\underline{x}}| \le 2s_{\nu+1}$. In particular $\underline{\underline{w}} \in \overline{S}$, so that $g_\nu(\underline{\underline{w}}) = 1 - f_\nu(\underline{\underline{w}})$ and

$$f_\nu(\underline{\underline{x}}) + h_\nu(\underline{\underline{y}}) = 1 + f_\nu(\underline{\underline{x}}) - f_\nu(\underline{\underline{w}}).$$

Now either $i = \nu$; then we estimate $f_\nu(\underline{\underline{x}}) - f_\nu(\underline{\underline{w}})$ by (iia). Or $i \le \nu - 1$, $|\underline{\underline{w}} - \underline{\underline{x}}| \in 2s_{\nu+1} \le s_\nu$, and both $\underline{\underline{x}}, \underline{\underline{w}} \in \overline{S}(3(s_{i+1} + \ldots + s_\nu))$. Then we estimate $f_\nu(\underline{\underline{x}}) - f_\nu(\underline{\underline{w}}) - f(\underline{\underline{w}})$ by (iib). In either case we get

$$\begin{aligned} |f_\nu(\underline{\underline{x}}) - f_\nu(\underline{\underline{w}})| &\le 2^{\nu-i} c_2(A) r_i^{-1} |\underline{\underline{x}} - \underline{\underline{w}}| \le 2^{\nu-1} c_2(A)(2s_{\nu+1}/r_i) \\ &= 2^{\nu-i} c_3(A)(s_{\nu+1}/s_i), \end{aligned}$$

say¿ Thus every $\underline{\underline{y}}$ with $\underline{\underline{x}} \in r_{\nu+1}A + \underline{\underline{y}}$ has

$$f_\nu(\underline{\underline{x}}) + h_\nu(\underline{\underline{y}}) \ge 1 - 2^{\nu-1} c_3(A)(s_{\nu+1}/s_i),$$

and (15.7) yields

$$f_{\nu+1}(\underline{\underline{x}}) \ge 1 - 2^{\nu-i} c_3(A)(s_{\nu+1}/s_i).$$

Since $S(6s_{i+1}) \subseteq S(3(s_{i+1} + \ldots + s_{\nu+1}))$ by (15.1), the lemma is proved.

Let $r_1, r_2, \ldots$ and $s_1, s_2, \ldots$ be as above, and let M be an integer greater than 1.

The space $\Omega = \mathfrak{a}(A)$ of sets $rA+\underline{\underline{y}}$ in U^k may be parametrized by the pair $(r,\underline{\underline{y}})$. We introduce a measure ω on Ω by the formula

$$\int_\Omega a(r,\underline{\underline{y}})d\omega = \sum_{\nu=0}^{M-1}(\mu(r_{\nu+1}A))^{-1}\int a(r_{\nu+1},\underline{\underline{y}})h_\nu(\underline{\underline{y}})d\underline{\underline{y}}.$$

This formula is valid for functions $a(r,\underline{\underline{y}})$ on Ω for which the integrals on the right are defined.

Lemma 15C. *We have*

(i) $\int_\Omega (rA+\underline{\underline{y}}\backslash\underline{\underline{x}})d\omega \leq \mathcal{X}(S\backslash\underline{\underline{x}})$,

(ii) $\int_\Omega d\omega \leq c_4(A,\sigma)(r_1^{-k}+2r_2^{-k}r_1+\ldots+2^{M-1}r_M^{-k}r_{M-1})$,

(iii) $\int_\Omega \mu(rA)d\omega \leq \mu(S)-2^M c_5(A,\sigma)r_M$.

120 *Proof.* We begin by observing that

$$\begin{aligned}\int_\Omega \mathcal{X}(rA+\underline{\underline{y}}\backslash\underline{\underline{x}})d\omega &= \sum_{\nu=0}^{M-1}(\mu(r_{\nu+1}A))^{-1}\int \mathcal{X}(r_{\nu+1}A+\underline{\underline{y}}\backslash\underline{\underline{x}})h_\nu(\underline{\underline{y}})d\underline{\underline{y}}\\ &= \sum_{\nu=0}^{M-1}\ell_\nu(\underline{\underline{x}})\\ &= f_{M-1}(\underline{\underline{x}}) \leq \mathcal{X}(S\backslash\underline{\underline{x}}).\end{aligned}$$

□

Next,

$$\int_\Omega d\omega = \sum_{\nu=0}^{M-1}(\nu(r_{\nu+1}A))^{-1}\int h_\nu(\underline{\underline{y}})d\underline{\underline{y}} \leq \sum_{\nu=0}^{M-1}(\mu(r_{\nu+1}A))^{-1}\int g_\nu(\underline{\underline{y}})d\underline{\underline{y}}. \tag{15.8}$$

We have

$$\int g_0(\underline{\underline{y}})d\underline{\underline{y}} = \int \mathcal{X}(S,\underline{\underline{y}})d\underline{\underline{y}} = \mu(S) \tag{15.9}$$

For $\nu \geq 1$ we write

$$\int g_\nu(\underline{\underline{y}})d\underline{\underline{y}} = \int_{S_1} + \int_{S_2} + \ldots + \int_{S_\nu} + \int_{S^*_\nu},$$

where $S_1 = S(6s_1)$, where S_j is the complement of $S(6s_{j-i})$ in $S(6s_j)$ ($j = 2, 3, \ldots$), and where S^*_ν is the complement of $S(6s_\nu)$ in S. By (iiia) of Lemma 15B, $g_\nu(\underline{\underline{y}}) = 0$ for $\underline{\underline{x}} \in S_1$. so that the integral over S_1 is zero. By (iiib) of Lemma 15B, we have

$$g_\nu(\underline{\underline{y}}) \leq 2^{\nu-(j-1)} c_3(A)(s_{\nu/s_{j-1}})$$

if $\underline{\underline{y}} \in S_j$ with $2 \leq j \leq \nu$. On the other hand we have $\mu(S_j) \leq 6s_{j-1}\sigma$, because $S \in \mathfrak{S}(\sigma)$. Thus for $2 \leq j \leq \nu$. **121**

$$\int_{S_j} g_\nu(\underline{\underline{y}})d\underline{\underline{y}} \leq 6c_3(A)\sigma s_\nu 2^{\nu-j+1}.$$

On S^*_ν we have $g_\nu(\underline{\underline{y}}) \leq 1$, and since $\mu(S^*_\nu) \leq 6s_\nu\sigma$, the integral over S^*_ν is $\leq 6\sigma s_\nu$. Combining our estimates, we obtain

$$\int g_\nu(\underline{\underline{y}})d\underline{\underline{y}} \leq 6\sigma(1+c(A))s_\nu(2^{\nu-1}+2^{\nu-2}+\ldots+1), c_5(A,\sigma)2^\nu r_\nu. \quad (15.10)$$

In view of (15.8) and (15.9) we obtain part (ii) of the lemma.
Finally,

$$\int h_\nu(\underline{\underline{y}})d\underline{\underline{y}} = (\mu(r_{\nu+1}A))^{-1} \int\int \mathcal{X}(r_{\nu+1}A + \underline{\underline{y}} \backslash \underline{\underline{x}})h_\nu(\underline{\underline{y}})d\underline{\underline{x}}d\underline{\underline{y}} = \int \ell_{\nu+1}(\underline{\underline{x}})d\underline{\underline{x}}.$$

Thus

$$\begin{aligned}
\int_\Omega \mu(rA)d\omega &= \sum_{\nu=0}^{M-1} \int h_\nu(\underline{\underline{y}})d\underline{\underline{y}} = \sum_{\nu=1}^{M} \int \ell_\nu(\underline{\underline{x}})d\underline{\underline{x}} \\
&= \int f_M(\underline{\underline{x}})d\underline{\underline{x}} = \mu(S) - \int g_M(\underline{\underline{x}})d\underline{\underline{x}} \\
&\geq \mu(S) - 2^M c_5(A,\sigma)r_M
\end{aligned}$$

by (15.10).

The *proof of Theorem 15A* is now finished as follows. We may assume that $\triangle = \triangle(\mathfrak{a}(A))$ is so small that

$$|\log\triangle|/(\log 2) \geq 9k^2. \tag{15.11}$$

Repeated application of Lemma 15C yields

$$\begin{aligned}
z(S) = \sum_{N}^{i=1} \mathcal{X}(S\setminus \underline{\underline{p}}_i) &\geq \int_\Omega \sum_{i=1}^{N} \mathcal{X}(rA + \underline{\underline{y}}\setminus \underline{\underline{p}}_i) d\omega \\
&= \int_\Omega z(rA + \underline{\underline{y}}) d\omega \\
&\geq \int_\Omega (N\mu(rA) - N\triangle(\mathfrak{a}(A))) d\omega \\
&= N(\int_\Omega \mu(rA) d\omega - \triangle \int_\Omega d\omega) \\
&\geq N(\mu(S) - 2^M c_5(A,\sigma) r_M - \triangle c_4(A,\sigma) R_M)
\end{aligned} \tag{15.12}$$

122 with

$$R_M = r_1^{-k} + 2r_2^{-k} r_1 + \ldots + 2^{M-1} r_M^{-k} r_{M-1}.$$

Choose the integer M with

$$M - 1 \leq |\log\triangle|^{\frac{1}{2}} (\log 2)^{-\frac{1}{2}} k^{-1} < M \tag{15.13}$$

Then $M \geq 3$ by (15.11). Let d be the number with

$$\log d = |\log\triangle|/(Mk+1).$$

Now by (15.11), (15.13)

$$\begin{aligned}
|\log\triangle|/(Mk+1) &\geq |\log\triangle|/(2|\log\triangle|^{\frac{1}{2}}(\log 2)^{-\frac{1}{2}} + 1) \\
&\geq \frac{1}{3}|\log\triangle|^{\frac{1}{2}}(\log 2)^{\frac{1}{2}} \geq \log 2,
\end{aligned}$$

so that $d \geq 2$.

Put $r_i = d^{-i}(i = 1, 2, \ldots)$. Then

$$R_M = d^k + 2d^{2k-1} + \ldots + 2^{M-1}d^{Mk-(M-1)} \geq 2^M d^{M(k-1)+1},$$

so that

$$2^M r_M + \triangle R_M \leq (2/d)^M (1 + \triangle d^{Mk+1}) = 2(2/d)^M, \tag{15.14}$$

by our choice of d, We have

$$\begin{aligned} M(\log d - \log 2) &= (M/(Mk+1))|\log\triangle| - M\log 2 \\ &\geq |\log\triangle|((1/k) - (1/k^2 M)) - M\log 2 \\ &\geq (1/k)|\log\triangle| - (2/k)|\log\triangle|^{\frac{1}{2}}(\log 2)^{\frac{1}{2}} - \log 2 \end{aligned}$$

by (15.13), so that by (15.14), **123**

$$2^M r_M + \triangle r_M \leq 4\triangle^{1/k} exp(2(\log 2)^{1/2} k^{-1} |\log\triangle|^{1/2}).$$

This in conjunction with (15.12) gives

$$z(S) \geq N(\mu(S)) - c_1(A,\sigma)\triangle^{1/k} exp(\ldots)).$$

The same inequality holds with S replaced by S'. Both inequalities together yield

$$|z(S) - N\mu(S)| \leq N(c_1(A,\sigma)\triangle^{1/k} exp(2(\log 2)^{1/2} k^{-1}|\log\triangle|^{1/2}).$$

This holds for every $S \in \mathfrak{S}(\sigma)$, and Theorem 15A is established.

16 Open Problems

We noted that Theorem 5B of Chapter *I* about rectangles with sides parallel to the axes is best possible (except for the value of the constant). Also Theorem 13B of Chapter II is best possible, since it is easy to construct distributions of N points in U^k with $|D(Q)| \ll (N\delta^k)^{1-(1/k)}$ for quadrant sets Q of diameter $\leq \delta$. But all the other known estimates of this type are almost certainly not best possible. It appears to be a difficult problem to improve Roth's Theorem 2B of Chapter I, and Theorems 1B, 6B, 6C, 7A, 8B, 9A, 10B, 10D and 13A of chapter II.

Almost certainly there is a generalization of Davenport's Theorem 4A (Chapter I). Namely, that for any k and large N, there exists a distribution of N points in U^k with

$$\int_{U^k} \cdots \int (Z(x_1, \ldots x_k) - N x_1 \ldots x_k)^2 dx_1 \ldots dx_k \ll (\log N)^{k-1}.$$

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As was already mentioned in Chapter I, Theorem 6C is probably capable of improvement.

Let $f(x)$ be a periodic function of period 1, and L^1-integrable in $0 \leq x \leq 1$. Write $f_t(x)$ for the translated function $f(x-t)$. Given a sequence $x_1, x_2, \ldots$, put

$$z(n, f) = \sum_{i=1}^{n} f(x_i),$$
$$D(n, f) = z(n, f) - n \int_0^1 f(x) dx,$$
$$\triangle(n, f) = \sup_t |D(n, f_t)|.$$

Are there functions f such that $\triangle(n, f) \to \infty$, no matter what the given sequence? Which functions f have this property?

Perhaps related is a question of Erdós [5]. Let $\underline{\underline{x}}_1, \underline{\underline{x}}_2, \ldots$ be a sequence of points on the unit circle $|\underline{\underline{x}}| = 1$. For any $\underline{\underline{p}}$ on this circle, put

$$\Pi(n, \underline{\underline{p}}) = \pi_{i=1}^{n} |\underline{\underline{x}}_i - \underline{\underline{p}}|.$$

Let $\Pi(n)$ be the supermum of $\Pi(n, \underline{\underline{p}})$ over all $\underline{\underline{p}}$ on the unit circle. Is it true that $\Pi(n) \to \infty$?

In Theorem 1B of Chapter II we noted the existence of balls B with large $|\triangle(B)|$. It is an open question whether there are balls with $\triangle(B) > 0$ and large, or whether there are balls B with $\triangle(B) < 0$ and $|\triangle(B)|$ large. There is a similar question with regard to Theorem 10B, i. e. with regard to spherical caps.

Now suppose we have N points in a circular disc of area 1. Define $D(A)$ for measurable subsets A of this disc in the obvious way. K. F. Roth asked whether there is a segment S, i. e. an intersection of the disc 125
with a half plane, having large $|D(S)|$.

Let $\mathscr{J}$ be the class of rectangles $a_1 \leq x_1 \leq b_1$, $a_2 \leq x_2 \leq b_2$ in U^2, let $\mathscr{J}'$ be the subclass of rectangles $0 \leq x_1 \leq b_1$, $0 \leq x_2 \leq b_2$, and let $\mathscr{L}$ be the class of closed discs in U^2. In the notation of § 14,

$$\triangle(\mathscr{J}') \leq \triangle(\mathscr{J}) \leq 2^2 \triangle(\mathscr{J}). \tag{16.1}$$

We know that the estimate $\triangle(\mathscr{J}) \ll N^{-1} \log N$ of Theorem 5A (Chapter I) is best possible, but for $\triangle(\mathscr{L})$ there is the much better estimate $\triangle(\mathscr{L}) \ll N^{-(7/8)-\epsilon}$ (Corollary to Theorem 3A, Chapter II). Why is this? Probably, because $\mathscr{L}$ has more "essential" parameters than $\mathscr{J}$. Namely, $\mathscr{L}$ is a 3-parameter family. On the other hand, $\mathscr{J}$ is a 4-parameter family, but in view of (16.1) and the close connection between $\mathscr{J}$ and the 2-parameter family $\mathscr{J}'$, we may argue that $\mathscr{J}$ has only two "essential" parameters. It would be desirable to have a theory of classes of sets depending on a given number of essential parameters, and to give estimates which depend on the number of parameters and on the dimension of the space (U^k, S^k, or more general spaces) in which these sets lie.

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