Lectures On
Discrete Subgroups Of Lie Groups

By

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Tata Institute Of Fundamental Research,
Bombay
1969

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Introduction

These lectures are devoted to the proof of two theorems (Theorem 8.1, the first main theorem and Theorem 9.3). Taken together these theorems provide evidence for the following conjecture:

Let Y and Y' be complete locally symmetric Riemannian spaces of non-positive curvature having finite volume and having no direct factors of dimensions 1 or 2. If Y and Y' are homeomorphic, then Y and Y' are isometric upto a constant factor (i.e., after changing the metric on Y by a constant).

The proof of the first main theorem is largely algebraic in nature, relying on a detailed study of the restricted root system of an algebraic group defined over the field $\mathbb{R}$ of real numbers. The proof of our second main theorem is largely analytic in nature, relying on the theory of quasi-conformal mappings in n-dimensions.

The second main theorem verifies the conjecture above in case Y and Y' have constant negative curvature under a rather weak supplementary hypothesis.

The central idea in our method is to study the induced homeomorphism φ of X, the simply covering space of Y and in particular to investigate the action of φ at infinity. More precisely our method hinges on the question: Does φ induce a smooth mapping $\varphi_\circ$ of the (unique) compact orbit $X_\circ$ in a Frustenberg-Stake compactification of the symmetric Riemannian space X?

There are good reasons to conjecture that not only is $\varphi_\circ$ smooth, but that $\varphi_\circ G_\circ \varphi_\circ^{-1} = G_\circ$ where $G_\circ$ denotes the group of transformations of $X_\circ$ induced by G, provided of course that X has no one or two dimensional factors. The boundary behaviour of φ thus merits further investigation.

It is a pleasure to acknowledge my gratitude to Mr. Gopal Prasad who wrote up this account of my lectures.

G.D. Mostow

Contents

Chapter 0

Preliminaries

We start with two definitions. 1

Symmetric spaces.

A Riemannian manifold X is said to be symmetric if $\forall x \in X$, there is an isometry σ_x such that $\sigma_x(x) = x$ and $\forall t \in T_x \quad \sigma_x^\circ(t) = -t$, where T_x is the tangent space at x and σ_x° denotes the differential of σ_x.

Locally symmetric spaces.

A Riemannian manifold X is locally symmetric if $\forall x \in X$, is a neighborhood N_x which is a symmetric space under induced structure.

Remark. Simply connected covering of a locally symmetric complete Riemannian space is a symmetric space (see Theorem 5.6 and Cor 5.7 pp.187-188 [8]).

Now we give an example which suggests that the last condition (that is there are no direct factors of dimensions 1 or 2 in the statement of the conjecture given in the introduction is in a sense necessary).

0.1 Example. Y, Y' Compact Riemann surfaces of the same genus > 1 and which are not conformally equivalent. By uniformization theory, the simply connected covering space of such Riemann surfaces is analytically equivalent to the interior X of the unit disc in the complex plane. Then $Y = \Gamma\backslash X, Y' = \Gamma'\backslash X$ where Γ, Γ' are fundamental groups of Y, Y' respectively. The elements of Γ, Γ' operate analytically on X.

Letting G denote the group of conformal mappings of X into itself, we have $\Gamma, \Gamma' \subset G$. It is well known that G is also the group of isometries
2 of X with respect to the hyperbolic metric $ds^2 = \frac{dz^2}{1-z^2}$ and with respect to this metric X is symmetric Riemannian space of negative curvature. Hence Y, Y' are locally symmetric spaces of negative curvature.

If Y, Y' were isometric then they would be conformally equivalent, which would be *a contradiction*.

We list some facts about linear algebraic groups, these are standard and the proofs are readily available in literature. Perhaps the use of algebraic groups is not indispensable, however we hope that this will simplify the treatment.

Let K be an algebraically closed field. For our purpose, we need only consider the case $K = \mathbb{C}$, the field of complex numbers,

Definitions.

Algebraic set: A subset A of K^n is said to be algebraic if it is the set of zeros of a set of polynomials in $K[X_1, \ldots, X_n]$.

If A is a subset of K^n, then $I(A)$ will denote the ideal of $K[X_1, X_2, \ldots, X_n]$ consisting of the polynomials which vanish at every point of A.

Zariski topology on K^n: The closed sets are algebraic sets.

Field of definition of a set: Let k be a subfield of K and A a subset of K^n. If $I(A)$ is generated over K by polynomials in $k[X_1, \ldots X_n]$ then A is said to be defined over k or k is a field of definition of A.

A subgroup of the group $GL(n, K)$ of non-singular $n \times n$ matrices over K is *algebraic* if it is the intersection with $GL(n, K)$ of a Zariski
3 closed subset of the set of all $n \times n$ matrices $M(n, K)$.

An algebraic group G is a k-group if G is defined over k, where k is a subfield of K.

Terminology.

If $k = \mathbb{R}$ or $\mathbb{C}$, we shall refer to the usual euclidean space topology as the $\mathbb{R}$-topology for $G_{\mathbb{R}}$ or for $G_{\mathbb{C}}$.

For a k-group G we write,

$$G_k = G \cap GL(n, k).$$

0.2 Theorem. *G be an algebraic group then the Zariski connected component of identity is a Zariski-closed, normal subgroup of G of finite index. ([5] Th 2, Chap. II, pp.86-88).*

0.3 Theorem (Rosenlicht)**.** *If k is a infinite perfect field, G a connected k-group then G_k is Zariski dense in G ([18] pp.25-50).*

0.4 Proposition. *If k is a perfect field, any $x \in GL(n, k)$ can be written uniquely in the form [Jordan normal form] $x = s \cdot u$ where s is semi-simple and u is unipotent; s, u commute. (use Th. 7, pp.71-72 [72])*

0.5 Theorem. *If k is a field of characteristic zero and G an algebraic k-group then there is a decomposition $G = M.U$ (semi-direct product) where U is a normal unipotent k-subgroup, M is a reductive k-subgroup. Moreover any reductive k-subgroup of G is conjugate to a subgroup of M by an element in U_k. (Th.7.1, pp.217-218, [15]).*

0.6 Proposition. *If U is a unipotent algebraic subgroup of an algebraic group defined over a field k of characteristic zero, then*

1. *U is connected ([2] §8, p.46).* **4**

2. *U is hypercentral [Engel-Kolchin] (see LA 5.7 [22]).*

3. *$U_{\mathbb{R}}$ is connected in the $\mathbb{R}$-topology if $k \subseteq R$.*

0.7 Proposition. *An abelian reductive group over algebraically closed field is diagonalizable.*

Definition. A connected abelian reductive group is called a *torus*.

0.8 Theorem. *Let G be an algebraic k-group. Then*

1. *The maximal tori are conjugate by an element of G.*

2. *Every reductive element of G_k lies in a k-torus.*

3. *A maximal k-torus is a maximal torus.*

4. *Any maximal torus is a maximal abelian subgroup if G is connected and reductive.*

Definitions. A reductive element $x \in GL(n,k)$ is called $k-split$ (or k-reductive) if $y \in GL(n,k)$ such that yxy^{-1} is diagonal, this is equivalent to saying that all the eigen-values of x are in k.

A torus T is called $k-split$ if $y \in GL(n,k)$ with yTy^{-1} diagonal, equivalently if each element of T_k is $k-split$.

Let G be a reductive group, $\mathring{G}$ its Lie algebra and T be a maximal torus.

Consider the adjoint representation

$$G \to \operatorname{Aut}\mathring{G}$$
$$x \mapsto \operatorname{Ad}_x$$

5 then $\mathring{G} = \sum_{\alpha}^{\mathring{G}} \quad \alpha \in \operatorname{Hom}(T, \mathbb{C}^*)$
where $\mathring{G}_\alpha = \left\{y | y \in \mathring{G} \quad \operatorname{Ad}_x(y) = \alpha(x)\, y \,\forall x \in T\right\}$ $\operatorname{Hom}(T, \mathbb{C}^*)$ being abelian we will use additive notation.

0.9 Theorem. *Let $\phi = \{\alpha | \alpha \in \operatorname{Hom}(T, \mathbb{C}), \mathring{G}_\alpha \neq 0, \alpha \neq 0\}$ then is called the set of roots of G on T and we have*

1. $\alpha \in \phi \Rightarrow -\alpha \in \phi$

2. $\alpha \in \Phi \Rightarrow \dim \varepsilon_\alpha = 1.$

3. $\left[\mathring{G}_\alpha, \mathring{G}_\beta\right] = \mathring{G}_{\alpha+}$ *if* $\alpha, \beta, \alpha| \in \phi$

   $\left[\mathring{G}_\alpha, \mathring{G}_\beta\right] = 0$ *if* $\alpha + \beta \notin \phi$

4. *There exists a linearly independent set $\triangle \in \phi$ such that the roots are either non-negative integral linear combination or a non positive integral linear combination of elements in $\triangle$. Such a subset is called a fundamental system of roots on T.*

Remark. A fundamental system of roots can be obtained as follows. Take any linear ordering of $\operatorname{Hom}(T, \mathbb{C}^*)$ compatible with addition. Let $\triangle$ be the set $\{\alpha \big| \alpha \in \phi, \alpha$ not a sum of two positive elements in $\phi\}$.

Notations. Let G be a group and A a subset of G, then $Z(A)$ will denote the centralizer and Norm (A) the normalizer of A.

If A and B are two subsets of G

$$A[B] = \{aba^{-1} \in A, b \in B\}.$$

Definition. Let T be a maximal torus a of a connected reductive group G. $Z(T)$ operates trivially on ϕ. The group $W = \frac{\mathrm{Norm}(T)}{Z(T)}$ is called the *Weyl group* of G. **6**

0.10 Theorem. *The Weyl group operates simply transitively on the set of fundamental systems of roots.*

Definitions. A reductive element $x \in G$ is $k-$ *regular* if $\forall y \in k$ - reductive, $\dim Z(x) \leq \dim Z(y)$.

A reductive element is called *singular* if it is not $\mathbb{R}$-regular.

Let V be a K-subspace of K^m and let k be a subfield of K, then V is a $k-subspace$ if $V = K(V \cap k^m)$i.e., $V \cap k^m$ generates the space over K.

Let G be a connected reductive k-group and ${}_kT$ a maximal k-split torus. Consider the adjoint representation of ${}_kT$ on G.

Then $\mathring{G} = \sum\limits_{\alpha} \mathring{G}_\alpha \quad \mathrm{Hom}({}_kT, \mathbb{C}^*)$.

Each $\mathring{G}_\alpha$ is a k-subspace.

The following analogue of the Theorem 0.9 is true.

0.11 Theorem. *Let* ${}_k\phi = \{\alpha \big| \mathring{G}_\alpha \neq 0, \alpha \neq 0\}$. *Then*

1. $\alpha \epsilon_k \phi \Rightarrow -\alpha \epsilon_k \phi$
2. $\left[\mathring{G}_\alpha, \mathring{G}_\beta\right] = \mathring{G}_{\alpha+\beta}$ *if* $\alpha, \beta, \alpha+\beta \epsilon_k \phi$

   $\left[\mathring{G}_\alpha, \mathring{G}_\alpha\right] = 0$ *if* $\alpha + \beta \epsilon_k \phi$
3. *There exists a linearly independent subset* ${}_k\triangle \subset {}_k\phi$ *such that the roots are either non-negative integral linear combination or non-positive integral linear combination of elements from* ${}_k\triangle \cdot {}_k\triangle$ *is called a fundamental system of restricted roots.* **7**

Let G be a connected reductive k-group, let ${}_kT$ be a maximal k-split torus in G and let T be a maximal k-torus containing ${}_kT$. Let $\triangle$ be a fundamental system of roots on T and ${}_k\triangle$ a fundamental system of restricted roots on ${}_kT$. We call $\triangle$ and ${}_k\triangle$ *Coherent* if the elements in ${}_k\triangle$ are restriction of roots in $\triangle$. If one introduces ordering of the sets ϕ and ${}_k\phi$ via lexicographic ordering with respect to $\triangle$ and ${}_k\triangle$ respectively, the resulting orders are Coherent the sense: If $\alpha \in \phi$ and $\alpha|_kT > 0$ then $\alpha > 0$.

The existence of Coherent $\triangle$ and ${}_k\triangle$ can be seen as follows. Let $X = \mathrm{Hom}(T, \mathbb{C}^*)$, the group of rational characters of T. Then X is a free abelian group. Ann ${}_kT$, the subgroup of characters which are trivial on ${}_kT$ is a direct summand of X since ${}_kT$ is connected. Therefore, one can choose a basis $\chi_1, \ldots \chi_r$ for X such that $\chi_1, \ldots, \chi_s$ is a base for Ann ${}_kT$. Now introduce lexicographic ordering on X with respect to this base. The resulting order on ϕ clearly has the property: If α and β have the same restrictions to ${}_kT$ and if $\alpha > 0$, then $\beta > 0$. Consequently, there is induced an order on ${}_k\phi$ compatible with addition. The corresponding fundamental systems $\triangle$ and ${}_k\triangle$ are Coherent.

Notations. Let $\triangle' \subset {}_k\triangle$

$$\{\triangle'\} = \mathbb{Z} - \text{linear span of } \triangle'$$
$$\alpha\triangle' = \bigcap_{\alpha\in\triangle'} \ker\alpha \subset {}_kT$$

8 Choose an ordering such that ${}_k\triangle$ consists of positive roots.

Put
$$\mathring{N}(\triangle') = \sum_{\substack{\alpha>0 \\ \alpha\notin\{\triangle'\}}} \mathring{G}_\alpha$$

$N(\triangle')$ be the complex analytic subgroup of G with Lie algebra $\mathring{N}(\triangle')$. Since $\forall x \in \sum_{\alpha>0} \mathring{G}_\alpha$ is nilpotent, $N(\triangle')$ is a unipotent group.

Let

$$\begin{aligned}
G(\triangle') &= Z({}^\perp\triangle') \\
P(\triangle') &= \text{Norm } (N(\triangle')) \\
N &= N(\phi)\ P = P(\phi)\ \phi = \text{empty set.} \\
M'_k &= \text{norm } ({}_kT)\ M_k = Z({}_kT) = G(\phi) \\
k^W &= M'_k/M_k.
\end{aligned}$$

The group $_kW$ is called *Little Weyl Group* or *relative Weyl group* and this operates transitively on the set of fundamental systems of restricted roots.

0.12 Lemma.

$$P(\triangle') = G(\triangle') \cdot N(\triangle').$$

$G(\triangle')$ is a maximal reductive subgroup of $P(\triangle')$ and $N(\triangle')$ is a maximal normal unipotent subgroup (called unipotent radical of $P(\triangle')$).

0.13 Theorem (Bruhat's decomposition)**.** *Let G be a connected reductive k-group. Then* 9

1. $G_k = N_k \cdot M'_k \cdot N_k$
2. *The natural map $M'/M \to P_k \backslash G_k / P_k$ is bijection.*
3. *Any unipotent k-subgroup of G is conjugate to a subgroup of N by an element in G_k.*
4. *Any k-subgroup containing P equals $P(\triangle')$ for some $\triangle' \subset \triangle$ (P is minimal parabolic k-subgroup). (see [4] or [21]).*

Remark. $Z(T)$ is a connected subgroup. More generally if S is any torus in G then $Z(S)$ is connected.

Now we consider for a moment the special case that k is algebraically closed.

In this case $G(\phi) = Z(T) = T$. Since T is a maximal abelian subgroup and $P = TN$ is solvable. Clearly P is connected. It follows at once from assertion 4 of the previous theorem that the connected component of the identity in $P(\triangle')$ contains P and therefore it is seen to coincide with $P(\triangle')$. In particular, every subgroup of G containing TN is connected, and TN is a maximal connected solvable subgroup.

Definition. A maximal connected solvable subgroup of an algebraic group is called a *Borel subgroup*. A subgroup containing a *Borel subgroup* is called *Parabolic*.

0.14 Theorem. *The Borel subgroups of an algebraic group are conju-* 10
gate under an inner automorphism.

0.15 Theorem. *If G is a connected reductive k-group and ${}_k\triangle$ is a fundamental system of restricted roots on a maximal k-split torus ${}_kT$, then $P(\triangle')$ is parabolic for any $\triangle' \subset {}_k\triangle$.*

Proof. Let T be a maximal k-torus containing ${}_kT$. Since all fundamental systems are conjugate under the Weyl group, it is possible to find a fundamental system $\triangle$ on T which is coherent with ${}_k\triangle$. Let ϕ^+ denote the set of positive roots on T defined by a lexicographic ordering with respect to $\triangle$. Then for any $\triangle' \subset {}_k\triangle$, the Lie algebra of $P(\triangle')$ contains $\mathring{G}_\alpha$ for all $\alpha \in \phi^+$. It follows directly that $P(\triangle')$ is parabolic. □

Remark. A subgroup Q is parabolic iff G/Q is a complete variety; equivalently, in case $k = \mathbb{C}$ if G/Q is compact in the $\mathbb{R}$-topology.

From the conjugacy of Borel subgroups and the theorem above, it is seen that any parabolic k-subgroup is conjugate to $P(\triangle')$ for some $\triangle' \subset {}_k\triangle$. Also any parabolic subgroup containing the maximal k-split torus ${}_kT$ is $\omega[P(\triangle')]\omega^{-1}$ with $\omega \in N({}_kT)$.

Since a reductive element of a connected reductive group is k-regular iff it lies in a single maximal torus, we see

0.16 Proposition. *A reductive element of a connected reductive k-group G is k regular iff it lies in at least one and at most finitely many parabolic k-subgroups of G, not equal to G.*

Chapter 1

Complexification of a real Linear Lie Group

Let G be a Lie subgroup of the Lie group of all automorphisms of a 11
real vector space V. Let $V_{\mathbb{C}}$ denote the complexification of V (i.e., $V_{\mathbb{C}} = V \otimes \mathbb{C}$) we identify the elements of $\dot{G}$, the lie algebra of G, with endomorphisms of V. We let $\dot{G}_{\mathbb{C}}$ denote the complification of the Lie algebra $\dot{G}$ and let $G^{\circ}_{\mathbb{C}}$ denote the analytic group of automorphisms of $V_{\mathbb{C}}$ that is determined by $\dot{G}_{\mathbb{C}}$. We identify the endomorphisms of V with their unique endomorphism extension to $V_{\mathbb{C}}$, so that we have $\dot{G} \subset \dot{G}_{\mathbb{C}}$ and $G^{\circ} \subset G^{\circ}_{\mathbb{C}}$. Where G° is connected component of identify in G.

Definitions. By the *complexification of a real linear Lie group G* is meant $G^{\circ}_{\mathbb{C}}$. G, it will be denoted by $G_{\mathbb{C}}$.

By a *f.c.c. group* we mean a topological group with finitely many connected components.

Suppose that G_* is a semisimple f.c.c. Lie subgroup of $GL(n, \mathbb{R})$. Then $\dot{G}_* \otimes_{\mathbb{R}} \mathbb{C} = \dot{G}_{\mathbb{C}}$ is semisimple. Hence $\dot{G}_{\mathbb{C}} = [\dot{G}_{\mathbb{C}}, \dot{G}_{\mathbb{C}}]$ is an algebraic Lie algebra [Th. 15, pp. 177-179 [5]]. Since a Zariski-connected subgroup of $GL(n, \mathbb{C})$ is topologically connected, it follows that the complex analytic analytic semisimple group $G^{\circ}_{\mathbb{C}}$is algebraic, and therefore $G_* \cdot G^{\circ}_{\mathbb{C}}$ is algebraic. Thus we have

1.1 Theorem. *The Zariski closure in $GL(n, \mathbb{C})$ of the semisimple f.c.c. Lie subgroup of $GL(n, \mathbb{R})$ is its complexification.*

Definition. A subset S of $GL(n, \mathbb{R})$ is said to be *selfadjoint* if $t_S = S$ where $t_S = \{g | t_g \in S, (t_g$ transpose of $g) \}$.

12 **1.2 Theorem.** *Let G_* be a semisimple f.c.c. Lie subgroup of $GL(n, \mathbb{R})$ then $\exists x \in GLn, \mathbb{R})$ such that xG_*x^{-1} is self adjoint. (for a proof see [12]).*

Notations. $S(n)$ will denote the set of all real $n \times n$ symmetric matrices and $P(n)$ the set of real positive definite symmetric matrices.

For any $g \in GL(n, \mathbb{R})$ $g = (g^t g)^{\frac{1}{2}} (g^t g)^{-\frac{1}{2}} \cdot g$ with $(g^t g)^{\frac{1}{2}} \in P(n)$ and $(g^t g)^{-\frac{1}{2}} g \in 0(n, \mathbb{R})$.

1.3 Theorem. *Let G_* be a self adjoint Lie subgroup of $GL(N, \mathbb{R})$. If G_* is of finite index in $F_{\mathbb{R}}$, F an algebraic $\mathbb{R}$ group (equivalently $G_*^\circ = (F_{\mathbb{R}})^\circ$). Then*

1. $G_* = \{G_* \cap P(n)\} \cdot \{G_* \cap 0(n, \mathbb{R})\}$

2. $G_* \cap 0(n, \mathbb{R})$ *is a maximal compact subgroup of* G_*

3. $G_* \cap P(n) = \exp(\dot{G}_* \cap S(n))$ *(see [12]).*

1.4 Lemma. *Let G_* be a real analytic self adjoint subgroup of $GL(n, \mathbb{R})$, G its Zariski closure in $GL(n, \mathbb{C})$. Let A be a maximal connected abelian subgroup in $G_* \cap P(n)$. Let T be the Zariski closure of A in $GL(n, \mathbb{C})$, then T is a maximal $\mathbb{R}$-split torus in G and $A = (T\mathbb{R})^\circ$.*

Proof. A being a commutative group of $\mathbb{R}$-diagonalizable matrices, is $\mathbb{R}$-diagonalizable. Therefore T, its Zariski closure is $\mathbb{R}$-diagonalizable and hence an abelian subgroup of G. ◻

Since A is self adjoint, the centralizer $Z(A)$ of A in G and therefore also $G_* \cap Z(A)$ are self adjoint.

13 By the previous theorem

$$G_* \cap Z(A) = \{G_* \cap Z(A) \cap P(n)\} \circ \{G_* \cap Z(A) \cap 0(n, \mathbb{R})\}$$

By maximality of A

$$G_* \cap Z(A) \cap P(A) = A.$$

Hence
$$Z(A) \cap G_* = A \cdot \{G_* \cap Z(A) \cap 0(n, \mathbb{R})$$

Since $T \subset Z(A)$, we have

$$(T_{\mathbb{R}})^\circ = A \circ \{(T_{\mathbb{R}})^\circ \cap 0(n, \mathbb{R})\}$$

Also since $(T_{\mathbb{R}})^\circ$ is diagonalizable over $\mathbb{R}$, $(T_{\mathbb{R}})^\circ \cap 0(n, \mathbb{R})$ is finite and as $(T_{\mathbb{R}})^\circ$ is connected, this consists of identity matrix alone.

Thus $(T_{\mathbb{R}})^\circ = A$.

1.5 Lemma. *Let G_* be a semi-simple self adjoint analytic subgroup of $GL(n, \mathbb{R})$ and let G be its Zariski-closure. Let $K_{\mathbb{R}} = G \cap 0(n, \mathbb{R})$, $E = G_* \cap P(n)$ and A as above, then*

$$K_{\mathbb{R}}[A] = E.$$

Proof. Evidently $K_{\mathbb{R}}[A] \subset E$. We will prove the other inclusion. □

First we show that if e, $p \in P(n)$ and $\text{epe}^{-1} \in P(n)$ then $\text{epe}^{-1} = p$.

By the theorem 1.3 we have 14

$$Z(p) = \{Z(p) \cap P(n)\} \cdot \{Z(p) \cap 0(n, \mathbb{R})\}$$
and $$Z(p) \cap P(n) = \exp\{Z(p) \cap S(n)\}$$

where $Z(p)$ is centralizer of p.

Since

$$\text{epe}^{-1} \in P(n) \quad \text{epe}^{-1} = {}^t(\text{epe}^{-1}) = e^{-1} p\, e$$
so $$e^2 p = pe^2 \text{ i.e., } e^2 \in Z(p)$$

Since

$$e^2 = \text{Exp}(X) \text{ for some } X \in \mathring{Z}(p) \cap S(n)$$
$$e = \text{Exp}\,\frac{1}{2}X \text{ therefore } e \in Z(p) \cap P(n).$$
$$\therefore \quad ep = pe \text{ i.e. } epe^{-1} = p, \text{ as asserted.}$$

Now if $p \in E \quad p = \text{Exp}\, X$ for some $X \in \dot{G} \cap S(n)$.

The Zariski closure of one parameter group $\operatorname{Exp} tX$ is a torus which is contained in a maximal $\mathbb{R}$-split torus (say S).

By conjugacy of maximal $\mathbb{R}$-split tori, $\exists x \in G$ with $x(T_{\mathbb{R}})^{\circ} x^{-1} = (S_{\mathbb{R}})^{\circ}$ where T is the Zariski closure of A in G. By the previous lemma $(T_{\mathbb{R}})^{\circ} = A$ and hence $p \in xAx^{-1}$.

As

$$G = E \circ K_{\mathbb{R}} \qquad \text{(see Th 1.3)}$$

we have $x = ek$ with $e \in E$, $k \in K_{\mathbb{R}}$.

$$xax^{-1} = p \qquad \text{for some } a \in A.$$

Thus $ekak^{-1}e^{-1} = p$ but $kak^{-1} \in P(n)$ hence $kak^{-1} = p$.

15 **Remark.** If B is a maximal connected abelian subgroup in $K_* = (K_{\mathbb{R}})^{\circ}$ then an argument similar to the one used in the above proof yields: $K_*[B] = K_*$.

Weyl chambers. The connected components of $A - \bigcup\limits_{\alpha \in \phi} \ker \alpha$, where ϕ is a restricted root system on T, are called the Weyl chambers associated with G_* and A.

If $\triangle$ is a fundamental system of restricted roots, then $A_{\triangle} = \{a \mid a \in A, \alpha(a) > 1 \forall \alpha \in \triangle\}$ is a Weyl chamber. Observe that $(\text{Norm } T)_{\mathbb{R}}$ operates on A, for $(\text{Norm } T)_{\mathbb{R}}$ operates on $T_{\mathbb{R}}$ and hence on $(T_{\mathbb{R}})^{\circ} = A$.

If $0 \neq X_\alpha \in \dot{G}_\alpha$ then $\forall h \in T$

$$\begin{aligned} Ad\, h(X_\alpha) &= h\, X_\alpha h^{-1} = \alpha(h) X_\alpha \\ {}^t(hX_\alpha h^{-1}) &= (h^{-1})^t X_\alpha h = \alpha(h)^t X_\alpha \\ \text{i.e.} \qquad h^t X_\alpha h^{-1} &= (\alpha(h))^{-1\,t} X_\alpha \end{aligned}$$

this proves that ${}^tX_\alpha \in \dot{G}_{-\alpha}$.

Let $h_\alpha = [X_\alpha, {}^tX_\alpha]$ then h_α, X_α, ${}^tX_\alpha$ is a base for 3 dimensional split Lie algebra over $\mathbb{R}$. By taking a suitable multiple of X_α, we can assume that

$$[h_\alpha, X_\alpha] = 2X_\alpha, [h_\alpha, {}^tX_\alpha] = -2^t X_\alpha$$

then $\qquad \operatorname{Exp} \pi/2(X_\alpha - {}^tX_\alpha) \in (\text{Norm } T)$.

16 Since $X_\alpha - {}^tX_\alpha$ is skew symmetric it actually belongs to (Norm T) $\cap$ K_*.

$Ad\,\mathrm{Exp}\,\pi/2(X_\alpha - {}^tX_\alpha)$ is reflection in the Wall corresponding to α, of the Weyl chamber.

This shows that $Ad[(\text{Norm } T)_{\mathbb{R}} \cap K_*]$ contains the reflections in all the Walls of the Weyl chambers.

1.6 Theorem. $E = K_*[\overline{A_\triangle}]$.

Proof. $K_*[\overline{A_\triangle}] = K_*[(\text{Norm } A \cap K_*)[\overline{A_\triangle}]]$ ◻

Since $Ad(\text{Norm } A \cap K_*)$ contains the reflections in all the walls of Weyl chambers $(\text{Norm } A \cap K_*)[\overline{A_\triangle}] = A$.

$$\therefore \quad K_*[\overline{A_\triangle}] = K_*[A].$$

Let $X \in \dot{E}$ and let Y be an $\mathbb{R}$-regular element in A, then since K_* is compact, $\exists k \in K_*$ such that

$$d(X, k[Y]) = d(X, K_*[Y])$$

where $$d(\widetilde{X}, \widetilde{Y}) = \mathrm{Tr}\ (\widetilde{X} - \widetilde{Y})^2$$

then $$d(k[X], Y) = d(X, k[Y]) \leq d(k[X], l[Y]), \forall l \in K_*$$

therefore $\forall Z \in \dot{K}_*$ the real valued function

$$\begin{aligned} f_Z : t &\mapsto d(k[x], \mathrm{Exp}\, tZ[Y]) \\ &= \mathrm{Tr}\,[h[X] - \mathrm{Exp}(tZ)Y\,\mathrm{Exp}(-tZ)]^2 \end{aligned}$$

is minimum at $t = 0$.

$$\therefore \ \frac{\partial f_2}{\partial t}_{t=0} = 0.$$

which gives 17

$$\mathrm{Tr}\,(h[X] - Y)[Y, Z] = 0 \text{ but since } \mathrm{Tr}\ Y[Y, Z] = 0$$

we have

$$= \mathrm{Tr}\ Z[k[X], Y] = \mathrm{Tr}\ k[X][Y, Z] = 0, \forall Z \in K_*$$

hence

$$[k[X], Y] = 0.$$

$\mathbb{R}$-regularity of Y implies

$$\begin{aligned} & Z(Y) \cap \dot{G} \cap S(n) = \dot{A} \\ \therefore \quad & k[X] \in \dot{A} \end{aligned}$$

this proves that

$K_*[A] \supset E$. The other inclusion is obvious.

Definition. An algebraic k-group is said to be k-compact if it contains no k-split connected solvable subgroup, that is a connected group that can be put in triangular form over k.

Remark. If G is a reductive algebraic k-group then the following three conditions are equivalent.

18 1. G is k-compact

2. G_k has no unipotent elements

3. the elements of G_k are reductive.

Exercise. Prove the above equivalences.

[Hint (1) ⇒ (2) ⇒ (3) is obvious prove (3) ⇒ (1) by showing: not (1) ⇒ not (3).]

The following digression is included just for fun, we need it only in the case $k = \mathbb{R}$.

1.7 Theorem. *Let k be a loc. compact field of characteristic 0. Then G is k-compact iff it is compact in the k-topology.*

Proof. (⇒) Let V be the underlying vector space [i.e. G is a subgroup of $\operatorname{Aut} v$] □

Let E_d be the set of d dimensional subspaces of V. Then there is a canonical imbedding $E_d \hookrightarrow \mathbb{P}(\wedge^d(V))$ which makes E_d a closed subvariety of the projective variety $\mathbb{P}(\wedge^d(V))$. The product $\prod\limits_{d=1}^{n} E_d$ is a closed subvariety of $\prod\limits_{d=1}^{n}(\wedge^d(V))$ (which, by Segre imbedding, itself is a closed subvariety of a projective space $\mathbb{P}$ of sufficiently large dimension). Hence $\prod\limits_{d=1}^{n} E_d$ is a compact set.

The set $W = \left\{(\omega_1, \ldots \omega_n) \middle| (\omega_1, \ldots, \omega_n) \prod\limits_{d=1}^{n} E_d\right\} \omega_1 \subset \omega_2, \ldots \alpha\omega_n$ is a closed subvariety (it is called the Flag manifold) of $\prod\limits_{1}^{n} E_d$.

G operates on W. For a k-rational point $\omega \in W_k$. Let T_ω be the stabalizer of ω in G then $G/T_\omega = G.\omega$. Since T_ω is k-triangularizable **19**
(hence solvable) and as G is k-compact $(T_\omega)^\circ = \{e\}$. Hence T_ω is finite [In an algebraic group the connected component of identity is of finite index see Th. 0.2]. Therefore $G. = G/T$ has dimension equal to that of G.

Let $D \bigcap\limits_{g \in G_k} T_{g\omega}$ since $T_{g\omega} = gT_\omega g^{-1}$. D is a finite (therefore discrete) normal subgroup and so it is central.

Since T_ω is finite we can choose $g_1 \cdots g_r$ such that

$$D = \bigcap_{i=1}^{r} T_{g_i \omega}$$

let $u_i = g_i \cdot \omega$ and W^i be the Zariski closure of $G \cdot u_i$ in the projective variety.

$$G \text{ acts on } \prod_{i=1}^{r} G.u_i \left(\subset \prod_{i=1}^{r} w^i \right).$$

Since D acts trivially we get a faithful action of $G' = G/D$ on $\prod\limits_{i=1}^{r}(G.u_i)$.

Let $v = (u_1 \cdots u_r)$ and let V be the Zariski closure of the G' orbit $G'.v$ of v. Then since V is a irreducible closed set and $G'v$ is open (by Chevalley's theorem in algebraic geometry) $\dim(\widetilde{V} - G'v) \leq \dim \widetilde{V} = \dim$ (G' orbit of a k-rational element). Therefore $\widetilde{V} - G'v$ has no k-rational points.

So $\widetilde{V}_k = (G'v)_k = G'_k v$

$\therefore$ $\widetilde{V}_k$ is compact in k-topology.

20 Since the differential of the map $G'_k \to G'_k.v$ is surjective by the implicit function theorem for loc. compact fields, this map is compact.

But $G_k \to G'_k$ is open (again by implicit function th.) and so $\frac{G_k}{D} =$ Image of G_k in G'_k is open (and therefore a closed subgroup). This proves that $\frac{G_k}{D}$ is compact, but since D is finite G_k is compact in k-topology.

The converse is also true. For if G_k is compact in k-topology G cannot have a unipotent subgroup. (Any unipotent group is isomorphic as an algebraic variety to K^r and its set of k-rational point is k^r which is not compact). This proves that any element of G_k is reductive and this by the preceding remark implies that G_k is k-compact.

Chapter 2

Intrinsic characterization of K_* and E

K_* is a maximal compact subgroup of G_*, equivalently the complexification K of K_* is a maximal $\mathbb{R}$-compact subgroup of G. 21

$$E = \operatorname{Exp}(\dot{G}_* \cap S(n))$$
$$\dot{G}_* \cap S(n) = \log E.$$

$\log E$ is the orthogonal complement to $\dot{K}_*$ in $\dot{G}_*$ with respect to the Killing form (see [13]).

2.1 Theorem. *The maximal compact subgroups in a f.c.c. Lie group are conjugate by inner automorphism ([13] or Chapter XV [9]).*

For $gGL(n,\mathbb{R})$ we have a linear automorphism of $S(n)s \mapsto gs^tg$ which leaves $P(n)$ stable. This operation of $GL(n,\mathbb{R})$ on $S(n)$ is called the *canonical action*.

Now let G_* be an analytic semi-simple group with finite center and let ρ be a finite dimensional representation of G with finite kernel. By Theorem 1.2 we can assume, after conjugation that $\rho(G_*)$ is self adjoint.

We set

$$K_* = \rho^{-1}(\rho(G_*) \cap 0(n,\mathbb{R}))$$

K_* is then a maximal compact subgroup of G_*, let $\varphi : G \mapsto P(n)$ denote the map

$$g \mapsto \rho(g)^t\rho(g)$$

22 then

$$\varphi(g_1 g_2) = \rho(g_1)\varphi(g_2)^t\rho(g_1).$$

Thus under φ, left translation by g corresponds to the canonical action by $\rho(g)$ on $P(n)$. In addition

$$\begin{aligned} &\varphi(gk) = \varphi(g) &&\text{for} \quad k \in K_* \\ \text{and} \quad &\varphi(g_1) = \varphi(g_2) &&\text{iff} \quad g_1 K_* = g_2 K_* \end{aligned}$$

therefore φ induces an injection

$$\overline{\varphi} : X = G_*/K_* \to P(n).$$

Let $[S]$ denote the projective space of lines in $S(n)$ and let

$$\Pi : S(n) - 0 \to [S]$$

be the natural projection and let $\psi = \pi \circ \overline{\varphi}$ and be the composite

$$G_* \to G_*/K_* \to [S]$$

then $\overline{\psi}$ is injective because if $p_1, p_2 \in \overline{\varphi}(X)$ with $\pi p_1 = \pi p_2$, then since p_1, p_2 are positive definite matrices, $\exists\, c > 0$ such that

$$\begin{aligned} p_1 &= c p_2 \\ \text{so } |p_1| &= c^n |p_2| && \text{where } |p| = \det p. \\ \text{But since } p_1, p_2 \in \overline{\varphi}(G_*) & && |p_1| = |p_2| = +1. \end{aligned}$$

[for G_* being semi-simple, the commutator $[G_*, G_*] = G_*$ and so there
23 does not exist a non-trivial homomorphism of G_* into an abelian group. Thus $g \mapsto |\rho(g)|$ is a trivial homomorphism of G_* into $\mathbb{R}^*$].

This implies that $c = +1$ i.e., $p_1 = p_2$. The map $\overline{\psi}$ is a G-map that is $\overline{\psi}(gx) = g\overline{\psi}(x)$ for all $g \in G$, $x \in X$s thus $\overline{\psi(X)}$ is stable under G.

Definition. If ρ is irreducible over $\mathbb{R}$ then $\overline{\psi(X)}$ is called the stable compactification of X. This of course depends on ρ.

Remark. The above compactification was arrived in a measure theoretic way by Frusentenberg [7].

We shall now show that X has the structure of a symmetric Riemannian space and shall obtain a decomposition for $\overline{\psi(X)}$ in terms of symmetric Riemannian spaces.

On $P(n)$ we introduce a infinitesimal metric

$$ds^2 = \mathrm{Tr}\,(p^{-1}\dot{p})^2$$

where $p(t)$ is a differentiable curve in $P(n)$ and $\dot{p}(t) = \frac{dp}{dt}\Big|_t$.

It is easy to check that this metric is invariant under the action of $GL(n, \mathbb{R})$ on $P(n)$ and also under the map $p \mapsto p^{-1}$. This implies that $P(n)$ is a symmetric Riemannian space. (see [14]).

Let G_* be a semi-simple analytic subgroup of $GL(n, \mathbb{R})$, then by Theorem 1.3.

$$G_* = (G_* \cap P(n) \cdot (G_* \cap 0(n))).$$

Let A be a maximal connected abelian subgroup of $P(n) \cap G_*$.

Since any abelian subgroup of $P(n)$ can be (simultaneously) diagonalized, we can assume that $A \subset D(n)$ the set of real diagonal matrices. **24**

Let T be the Zariski closure of A in $GL(n, \mathbb{C})$ then by Lemma 1.4, T is a maximal $\mathbb{R}$-split tours in the Zariski closure G of G_* in $GL(n, \mathbb{C})$ and

$$A = (T_{\mathbb{R}})^\circ.$$

Let $\triangle$ be a fundamental system of restricted roots on T. There is a natural faithful representation of $GL(n, \mathbb{C})$ and therefore of G on $\mathbb{C}^n$. In this section the complex vector space $\mathbb{C}^n$ considered as a G-module under this representation will be denoted by V.

From the representation theory of semi-simple Lie algebras we have

$$V = \oplus \sum V_\mu$$

where μ's are "weights" (more precisely, restricted weights) on T. The highest weight will be denoted by $\mu_\circ$. Also we know that any other weight is of the form $\mu = \mu_\circ - \sum n_\alpha \alpha$, where each n_α is a non-negative integer.

For $h \in A_\triangle$ we have clearly

$$\psi(h) = \pi \begin{pmatrix} \ddots & & 0 \\ & (\mu(n))^2 & \\ 0 & & \ddots \end{pmatrix}$$

25 After a conjugation we can assume that the first diagonal entry is $(\mu \circ (h))^2$.

So

$$\psi(h) = \pi \begin{pmatrix} 1 \ddots & & 0 \\ & ((\mu - \mu_\circ)(h))^2 & \\ 0 & & \ddots \end{pmatrix}$$

Let $\{h_n\}$ be a sequence in $A_\triangle$ such that the sequence $\psi(h_n)$ is convergent in the projective space $[S(n)]$. If necessary by passing to a subsequence, we can assume that $\forall \alpha \in \triangle \lim\limits_{n\to\infty} \alpha(h_n)$ exists in $\mathbb{R} \cup \{\infty\}$ and is equal to ℓ_α. For a weight $\mu = \mu_\circ - \sum n_\alpha \alpha$, if we define $\operatorname{Supp}\mu = \{\alpha | h_\alpha \neq 0\}$, then clearly the diagonal entry in $\lim\limits_{n\to\infty} \psi(h_n)$, corresponding to the weight μ is zero iff $\operatorname{Supp}\mu$ contains some α with $I_\alpha = \infty$.

Notations. For a non-empty subset $\triangle'$ of $\triangle$, we write

$$V(\triangle') = \sum_{\operatorname{Supp}\mu \subset \triangle'} V_\mu$$

$p_{\triangle'}$= the projection of V on $V(\triangle')$ with kernal $\sum\limits_{\operatorname{Supp}\mu \not\subset \triangle'} V_\mu$

$\pi_{\triangle'} = \pi(p_{\triangle'} h p_{\triangle'})$ for $h \in S(n)$ and let $\psi_{\triangle'}$ be the composite $G_* \to G_*/K_* \to P(n) \xrightarrow{P_{\triangle'}} [S(n)]$. ($K_* = G_* \cap 0(n, \mathbb{R})$).

26 Since $V(\triangle')$ is stable under A, we note that $p_{\triangle'} h = h p_{\triangle'} = p_{\triangle'} h p_{\triangle'}$. The preceding remarks establish

2.2 Lemma.

$$\overline{\psi(A_{\triangle'})} = \bigcup_{\triangle' \subset \triangle} \psi_\triangle(\overline{A}_\triangle)$$

Also, if

$$K_* = G_* \cap 0(n, \mathbb{R})$$

we have by theorem 1.6

$$\begin{aligned} E &= K_*[\bar{A}_\triangle] \\ G_* &= E \cdot K_* \\ &= K_*[\bar{A}_\triangle] \cdot K_* \\ \therefore \quad \overline{\psi(X)} &= \overline{\psi(G_*)} = \overline{\psi(K_*[A_\triangle] \cdot K_*)} = \overline{\psi(K_*[A_\triangle])} \\ &= \overline{\psi(K_*[A_\triangle])} = \overline{\pi\psi K_*[A_\triangle]} = \overline{\pi(K_*[\varphi A_\triangle])} \\ &= \overline{K_* \cdot (\pi\varphi A_\triangle)} = K_* \cdot \overline{\pi\varphi(A_\triangle)} \end{aligned}$$

For $h, h' \in \dot{T}$ $< h, h' >= T_r(hh')$ is a inner product on $\dot{T}$. This inner product induces an inner product on $\mathrm{Hom}(\dot{T}, \mathbb{C})$ and hence its restriction on $\mathrm{Hom}(T, \mathbb{C}^*) \hookrightarrow \mathrm{Hom}(\dot{T}, \mathbb{C})$. This restriction will again be denoted by $<, >$.

2.3 Lemma. *If $\dot{G}_\alpha V_\mu = 0$, then the following two conditions are equivalent.*

1. $\dot{G}_{-\alpha} V_\mu = 0$.

2. $< \mu, \alpha >= 0$.

Proof. We can choose $X_\alpha \in \dot{G}_\alpha$ such that $\mathrm{Tr}\,(X_\alpha {}^t X_\alpha) = 1$ set 27

$$\begin{aligned} &[X_\alpha, {}^t X_\alpha] = h'_\alpha, \text{ then for } h \in \dot{T} \text{ we have} \\ &< h, h'_\alpha > = \mathrm{Tr}\ hh'_\alpha = \mathrm{Tr}\ h[X_\alpha, {}^t X_\alpha] \\ &\qquad = \mathrm{Tr}\,[h, X_\alpha]^{tr} X_\alpha = \alpha(h) \cdot \mathrm{Tr}\ X_\alpha {}^t X_\alpha = \alpha(h) \end{aligned}$$

$\therefore \quad h'_\alpha = h_\alpha$ where h_α is the dual of α in the inner product. Therefore for any weight μ, $< \mu, \alpha >= \mu(h_\alpha)$. □

By considering the representation of 3-dimensional simple Lie algebra generated by $\{X_\alpha, h_\alpha, {}^t X_\alpha\}$ on $\sum\limits_{n \in \mathbb{Z}} V_{\mu + n\alpha}$ the result follows immediately. (see [10] or pp. IV-3 to IV-6 of [23]).

Definition. Let $E_\rho = \{\alpha \big| \alpha \in \triangle < \alpha, \mu_\circ >= 0$

A subset $\triangle'$ of $\triangle$ is said to be $\rho -$ *connected* if $\triangle' \cup \{\mu_\circ\}$ is connected in the sense of Dynkin's diagram of $\triangle'$ lies in E_ρ.

For $\triangle' \subset \triangle$ we set $\widetilde{\triangle}' = \triangle' \cup \{\alpha \big| \alpha \in E_\rho \alpha \big| \beta$ for $\forall \beta \in \triangle'\}$. The following is an easy consequence of the previous lemma.

2.4 Lemma. *A subset $\triangle'$ of $\triangle$ is ρ-connected iff there is a weight μ with support $\mu = \triangle'$.*

Proof. By induction on s = the cardinality of $\triangle'$. If $s = 1$ the result follows at once from lemma 2.3. If $s > 1$ then $\triangle'$ contains a ρ-connected subset $\triangle''$ of cardinal $s-1$, and hence there is a weight $\triangle'' = \mu_\circ - n_1\alpha_1 - \cdot n_{s-1}\alpha_{s-1}$, where $\triangle'' = \alpha_1, \ldots, \alpha_{s-1}$. ◻

28 Let $\alpha_s \in \triangle' - \triangle''$. Then $< \mu, \alpha_s >=< \mu_\circ, \alpha_s > - \sum n_k < \alpha_k, \alpha_s > \geq 0$ and is not zero since $\triangle'' \cup \{\alpha_s\}$ is ρ-connected. Hence $\mu - \alpha_s$ is a weight of support $\triangle'$.

2.5 Corollary. *$V(\triangle') = V$ (largest ρ-conn. subset in $\triangle'$) and*

$$\overline{\psi(A_\triangle)} = \bigcup_{\substack{\triangle' \subset \triangle \\ \triangle' - \rho\, conn\cdot}} \psi_{\triangle'}(\overline{A}_\triangle)$$

2.6 Lemma.

$$\overline{\pi(E)} = \bigcup_{\substack{\triangle' \subset \triangle \\ \triangle' \rho - conn.}} G_* \cdot \underset{\triangle'}{\pi}(1).$$

Proof.

$$\begin{aligned}
\overline{\pi(E)} &= \overline{\psi(E)} = \overline{\pi K_*[\overline{A}_\triangle]} \\
&= \overline{\psi K_*[A_\triangle]} = K_* \cdot \overline{\psi(A_\triangle)} \\
&= K_* \cdot \bigcup_{\substack{\triangle' \subset \triangle \\ \triangle' \rho - \text{conn}}} \psi_{\triangle'}(\overline{A}_\triangle) = \bigcup_{\substack{\triangle' \subset \triangle \\ \triangle' \rho - \text{conn.}}} K_* \cdot \psi_{\triangle'}(\overline{A}_\triangle) \\
&= \bigcup_{\triangle' \rho - \text{conn}} K_*(\overline{A}_\triangle \psi_{\triangle'}(1)) = \bigcup_{\substack{\triangle \subset \triangle \\ \triangle' \rho - \text{conn.}}} (K_* \overline{A}_\triangle)\psi_{\triangle'}(1)
\end{aligned}$$

$$= \bigcup_{\substack{\triangle' \subset \triangle \\ \triangle' \rho-\text{conn.}}} G_* \cdot \pi_{\triangle'}(1).$$

Since $\triangle$ is finite there are only finitely many subsets $\triangle' \subset \triangle$. So this lemma in particular shows that $\overline{\psi(X)} = \overline{\pi(E)}$ consists of a finite number of G_* orbits. □ **29**

2.7 Lemma. (i) *For* $h \in \perp_{\triangle'}$ *and* $v : V(\triangle')$, $hv = \mu_\circ(h)v$

(ii) $\dot{G}_\alpha V(\triangle') = 0$ *if* $\alpha > 0$ *and* $\alpha\psi\{\triangle\}$

(iii) $\dot{G}_\alpha V(\triangle') = 0$ *if* $\alpha \in \{\widetilde{\triangle'} - \triangle\}$

Proof. Parts (i) and (ii) are immediate. (iii) follows from Lemma 2.3 and (ii) of this lemma. □

For each restricted root α, set G_α the group generated by $\{\operatorname{Exp} X, X \in \dot{G}_\alpha\}$. For a subset $\triangle'$ of $\triangle$ let $G'(\triangle')$ be the group generated by G_α, $\alpha \in \{\triangle'\}$ and let $K(\triangle')$ be the subgroup generated by $\operatorname{Exp}(X - {}^tX)X \in \dot{G}_\alpha$, $\alpha \in \{\triangle'\}$ and maximum $\mathbb{R}$-compact subgroup of $Z(T)$. $G'(\triangle')$ is semisimple.

We write

$$G_*(\triangle') = G(\triangle') \cap G_*; K_*(\triangle') = K(\triangle') \cap G_*$$

$$G'_*(\triangle') = G'(\triangle') \cap G_*; K_* = K_*(\triangle) = K(\triangle) \cap G_*$$

and $$K'_*(\triangle') = K_* \cap G'(\triangle').$$

It is easy to see that $G(\triangle') = G'(\triangle') \cdot Z(T)$; $G'_*(\triangle') = (G'(\triangle')_{\mathbb{R}})^\circ$ but $G_*(\triangle')$ need not be connected. Also $K_*(\triangle')$ and $K'_*(\triangle')$ are maximal compact subgroups of $G_*(\triangle')$ and $G'_*(\triangle')$ respectively.

Remarks.

(i) Since $g \in G_\alpha$ implies $Z^{-1}gZ \in G_\alpha \forall Z \in Z(T)$ we have $G_\alpha \cdot Z(T) = Z(T) \cdot G_\alpha$.

(ii) Since $(\widetilde{\triangle} - \triangle') \perp \triangle'$, roots in $(\{\widetilde{\triangle}'\} - \{\triangle'\})$ = roots in $\{\widetilde{\triangle}' - \triangle'\}$.

(iii) The Lie algebras of $G(\triangle')$ and $P(\triangle')$ are respectively 30

$$Z(T) + \sum_{\alpha\in\{\triangle'\}} \dot{G}_\alpha \text{ and } Z(T) + \sum_{\alpha>0} G_\alpha + \sum_{\substack{\alpha<0\\ \alpha\in\{\widetilde{\triangle}'\}}} G_\alpha$$

(iv) $P(\triangle')$ is connected and for $\triangle' \supset \triangle''$ we have $P(\triangle')P(\triangle'')$. Now we prove following results, which allow us to determine the G_* orbits in $\overline{\psi(X)}$.

2.8 Lemma.

(i) *The stabalizer of* $V(\triangle')$ *is* $P(\widetilde{\triangle}')$

(ii) *The stabalizer of* $P(\triangle')\cdot\pi_{\triangle'}$ *(1) is* $P(\widetilde{\triangle}')$.

(iii) *The stabalizer of the point* $\pi_{\triangle'}$ *(1) in* G_* *is*

$$G_*(\widetilde{\triangle}' - \triangle')\cdot K_*(\triangle')N_*(\triangle')\cdot({}^\perp\triangle' \cap A).$$

Proof.

(i) It is clear that the stabalizer of $V(\triangle')$ contains $P(\triangle')$ hence is a parabolic group and therefore it is connected. $V(\triangle')$ is stable under a connected subgroup H iff it is stable under $\dot{H}$. From this it can be easily proved that the stabalizer is $P(\widetilde{\triangle}')$.

(ii) Let S be the Stabalizer of $P(\triangle')\pi_{\triangle'}$ (1) and $S_{\triangle'}$ the Stabalizer of $\pi_{\triangle'}$ (1)

□

Clearly $S \supset P(\triangle')$. If x stabalizer $P(\triangle')\pi_{\triangle'}$ (1) $x.\pi_{\triangle'}$ (1) for some $p \in P(\triangle')$, this implies that $p^{-1}x.\pi_{\triangle'}$ (1) $= \pi_{\triangle'}$ (1) i.e. $p^{-1}x \in S_{\triangle'}$.

Hence $S = P(\triangle')\cdot(S_{\triangle'} \cap S)$.

We first prove that $S_{\triangle'} \subset P(\widetilde{\triangle}')$.

31 If $g \in S_{\triangle'}$ then $gp_{\triangle'}{}^t g = cp_{\triangle'}$ for some $c \in \mathbb{R}$

i.e. $$gp_{\triangle'} = cp_{\triangle'}({}^t g)^{-1}$$

So $$gV_{\triangle'} = gp_{\triangle'}V_{\triangle'} = cp_{\triangle'}({}^t g)^{-1}V_{\triangle'} \subset V_{\triangle'}$$

$\therefore$ g the stabalizer of $V(\triangle') = P(\widetilde{\triangle}')$.

$\therefore$ $S_{\triangle'} \subset P(\widetilde{\triangle}')$

$\therefore$ $S \subset P(\triangle').P(\widetilde{\triangle}') = P(\widetilde{\triangle}')$.

From parts (ii) and (iii) of Lemma 2.7 it follows almost immediately that $N(\triangle') \subset S_{\triangle'}$ and $G_\alpha \subset S'_{\triangle}$ and $G_\alpha \subset S_{\triangle'}$, $\forall\alpha \in \{\widetilde{\triangle}' - \triangle'\}$ (& So $G'(\widetilde{\triangle}'-\triangle') \subset S_{\triangle'}$) i.e., $N(\triangle')$. $\pi_{\triangle'}$ (1)$=\pi_{\triangle'}$ (1) $= G'(\widetilde{\triangle'-\triangle'})\pi_{\triangle'}$ (1). Also from the remark (ii) after §2.7, we get $G_\alpha \subset Z(G'(\triangle'))\forall\alpha \in \{\widetilde{\triangle}' - \triangle'\}$.

Now we prove that $G'(\widetilde{\triangle}' - \triangle') \subset S$.

For $\alpha \in \{\widetilde{\triangle}' - \triangle'\}$

$$\begin{aligned}
G_\alpha \cdot P(\triangle')\pi_{\triangle'}(1)G_\alpha \cdot G(\triangle')N(\triangle').\pi_{\triangle'}(1)\\
&= G_\alpha G(\triangle')\pi_{\triangle'}(1)\\
&= G_\alpha G'(\triangle')Z(T)\pi_{\triangle'}(1)\\
&= G_\alpha G'(\triangle')G_\alpha Z(T)\pi_{\triangle'}(1)\\
&= G'(\triangle')Z(T)\cdot G_\alpha\pi_{\triangle'}(1) = G'(\triangle')Z(T)\pi_{\triangle'}(1)\\
&\subset P(\triangle')\pi_{\triangle'}(1).
\end{aligned}$$

This proves that $\forall\alpha \in \{\widetilde{\triangle}'-\triangle'\}G_\alpha \subset S$ and therefore $G'(\widetilde{\triangle}'-\triangle') \subset S$. 32

From the Lie algebra considerations it is easy to see that the group given by $G'(\widetilde{\triangle}' - \triangle')$ and $P(\triangle')$ is $P(\widetilde{\triangle}')$.

$\therefore$ $P(\widetilde{\triangle}') \subset S$. This proves that $S = P(\widetilde{\triangle}')$.

(iii) In (ii) we proved

$$S_{\triangle'} \subset P(\widetilde{\triangle}').$$

As $$P(\widetilde{\triangle}') = N(\widetilde{\triangle}')\cdot G(\widetilde{\triangle}')$$

and $$S_{\triangle'} \supset N(\triangle') \supset N(\widetilde{\triangle}')$$

we have $$S_{\triangle'} = N(\widetilde{\triangle}')\cdot(S_{\triangle'} \cap G(\widetilde{\triangle}'))$$

since $G(\widetilde{\triangle}') = G'(\widetilde{\triangle}' - \triangle')\cdot G(\triangle')$ and since

$$G'(\widetilde{\triangle}' - \triangle') \subset S_{\triangle'}$$

$$G(\widetilde{\triangle}') \cap S_{\triangle'} = G'(\widetilde{\triangle}' - \triangle')\cdot\{G(\triangle') \cap S_{\triangle'}\}$$

clearly $$S_{\triangle'} \supset K(\triangle')_{\mathbb{R}} \quad \text{and} \quad S_{\triangle'} \cap T = {}^{\perp}\triangle'$$

$$\begin{aligned}
(G(\triangle')S_{\triangle'})_{\mathbb{R}} &= (S_{\triangle'})_{\mathbb{R}} \cap (G(\triangle'))_{\mathbb{R}} = (S_{\triangle'})_{\mathbb{R}}(K(\triangle')_{\mathbb{R}} A K(\triangle')_{\mathbb{R}}) \\
&= K(\triangle')_{\mathbb{R}} \cdot ({}^{\perp}\triangle' \cap A) \\
\therefore \quad (S_{\triangle'}) \cap G_* &= N_*(\widetilde{\triangle}') \cdot (S_{\triangle'} \cap G(\triangle')) \\
&= N_*(\widetilde{\triangle}') \cdot G'_*(\widetilde{\triangle}' - \triangle') \cdot K_*(\triangle')\left({}^{\perp}\triangle' \cap A\right).
\end{aligned}$$

33

2.9 Lemma.

(1) $G_* = K_* P_*(\triangle') \quad \forall \triangle' \subset \triangle$

(2) dimension of $K_*(\triangle') \cdot N(\triangle')$ *is independent of* $\triangle'$.

Proof. (i) Since $K_* \cdot P_*(\triangle') \subset G_*$ and since for $\triangle' \subset \triangle'' P_*(\triangle') \subset P_*(\triangle'')$ it is sufficient to prove that

$$G_* = K_* \cdot P_*(\phi).$$

As a vector space
$$\dot{K_*(\triangle)} + \dot{P_*(\phi)} = \dot{G}_*$$

By implicit function theorem $K_* P_*(\phi)$ is open in G_*. Also it is closed, since K_* is compact. Connectedness of G_* implies the result.

(ii) For a positive root α, let $\{X^i_\alpha\}$, be a basis of $\dot{G}_\alpha$ then the set

$$\bigcup_{\substack{\alpha \in \{\triangle'\} \\ \alpha > 0}} \{X^i_\alpha - {}^t X^i_\alpha\} \cup \bigcup_{\substack{\alpha > 0 \\ \alpha \notin \{\triangle'\}}} \{X^i_\alpha\}$$

is a basis for the Lie algebra of $K_*(\triangle') \cdot N(\triangle')$. This shows that $\dim(K_*(\triangle') N(\triangle')) = \dim \left(\sum_{\alpha > 0} \dot{G}_\alpha\right)$.

□

34 **Definition.** If A is a right H-set and B a left H-set, where H is a group, then $A \times_H B$ denotes the set $(A \times B)_{/\mathbb{R}}$ where $\mathbb{R}$ is the equivalence relation $(ah, \underline{h^{-1}b) \sim (a, b)}, \forall h \in H$.

By the previous lemma

$$G_* = K_* P_*(\triangle') \quad \forall \triangle' \subset \triangle$$

So the G_* orbit of

$$\begin{aligned}
\pi_{\triangle'}(1) &= G_* \cdot \pi_{\triangle'}(1) \\
&= K_* \cdot P_*(\triangle')\pi_{\triangle'}(1) \\
&= K_*(P_*(\triangle')\pi_{\triangle'}(1)) \\
&\approx \frac{P_*(\triangle')}{K_*(\triangle')N_*(\triangle')({}^{\perp}\triangle' \cap A)}.K_* \text{ (by part (iii) of Lemma 2.8).} \\
&= \frac{G^*(\triangle')}{K_*(\triangle')} \times K_*.
\end{aligned}$$

Since $P_*(\triangle') = G'_*(\triangle') \cdot N_*(\triangle') \cdot (Z(T) \cap G_*)$. If we put

$$\frac{G'_*(\triangle')}{K_*(\triangle')} = X(\triangle')$$

we have the G_* orbit of $\pi_{\triangle'}(1) \approx X(\triangle') \times \underset{K_*(\triangle')}{K_*}$

This is compact iff $X(\triangle')$ is a single point set, equivalently iff $G'_*(\triangle') = K'_*(\triangle')$ i.e., iff $\triangle' = \phi$.

Then the orbit is the compact set $X_\circ = \frac{K_*}{K_*(\phi)} \cdot \pi_{\triangle'}(1)$. Also from part 35
(iii) of Lemma 2.8 it is clear that $\dim S_{\triangle'} \geq \dim S_{\triangle''}$ if $\triangle' \subset \triangle''$.

So we have proved.

Theorem (Satake). *$\overline{\psi(X)}$ consists of a finite number of G_* orbits. Among these there is a unique compact orbit $X_\circ$, also characterized as the orbit of minimum dimension.*

Chapter 3

$\mathbb{R}$-regular elements

When $k = \mathbb{R}$ and G is a semi-simple algebraic $\mathbb{R}$-group we can give another description of *reductive* $\mathbb{R}$-regular elements. **36**

Let G be a semi-simple $\mathbb{R}$-group without loss of generality we can (and we will) assume that G is self adjoint (cf. [13]). Let T be a maximal $\mathbb{R}$-split torus in G. Let $A = (T_{\mathbb{R}})^{\circ}$.

We can assume that $A \subset P(n)$ [see Lemma 1.4]. Let $\triangle$ be a fundamental system of restricted roots on T, let

$$A^t = \{x \big| x \in A \quad \alpha(x) > t \quad \forall \alpha \in \triangle\}$$

then
$$A^1 = A_{\triangle}.$$

$$Z(T)_{\mathbb{R}} = Z(A)_{\mathbb{R}} = A.(Z(A) \cap 0(n, \mathbb{R}))$$

we put
$$Z(A) \cap 0(n, \mathbb{R}) = L.$$

Then L is the unique maximum compact-subgroup of $Z(A)_{\mathbb{R}}$. The only $\mathbb{R}$-regular elements in A are those in (Norm A) $[A_{\triangle}]$. More generally the $\mathbb{R}$-regular elements in $Z(T)_{\mathbb{R}}$ are of the form m.a with $m \in L$ and $a \in$ (Norm A)$[A_{\triangle}]$. For given such an element it lies in $P(\triangle')$ for any $\triangle' \subset \triangle$. Moreover if P is parabolic and $m.a \in P$ then $Z(m.a) \subset P$

$$\therefore \quad T \subset P \& P = P(\triangle') \text{ for some } \triangle' \subset \triangle$$

This implies that $m.a$ is $\mathbb{R}$-regular.

Since all the max $\mathbb{R}$-split tori are conjugate by an element from $G_{\mathbb{R}}$ **37**
it follows that the set of reductive $\mathbb{R}$-regular elements in G is $G_{\mathbb{R}}[L.A_{\triangle}]$.

3.1 Lemma (Polar decomposition)**.** *If x is a reductive element of $GL(n, \mathbb{R})$ then x can be written uniquely in the form $x = p.k$ with $p, k \in GL(n, \mathbb{R})$, the eigenvalues of p are positive, the eigenvalues of k are of absolute value 1 and $pk = kp$.*

Proof. Let V be the underlying complex vector space. □

$V = \oplus \sum_{\lambda} V_\lambda$, whee λ varies over the eigenvalues of x.

Let

$$p : v \mapsto |\lambda| V \quad \text{for } v \in V_\lambda$$

and

$$k : v \mapsto \frac{\lambda}{|\lambda|} v. \text{ for } v \in V_\lambda.$$

Then p, k satisfy the requirements of the lemma.

Definition. p is called the *polar part* of x.

The polar decomposition provides the following characterization of $\mathbb{R}$-regular elements.

Proposition. *A reductive element is $\mathbb{R}$−regular iff its polar part is $\mathbb{R}$-regular.*

The rest of this section will be devoted to the proof of the

3.2 Theorem. *Let G be a semi-simple $\mathbb{R}$−group and y be an $\mathbb{R}$-regular reductive element in $G_{\mathbb{R}}$. Then there is an algebraic subset S_y, not containing 1, such that for all large n, xy^n is $\mathbb{R}$-regular, provided $x \in G_{\mathbb{R}} - S_y$.*

38 We introduce the following new notations:

N^+ = The unipotent analytic subgroup with Lie algebra $\sum_{\alpha>0} \dot{G}_\alpha$

N^- = The unipotent analytic subgroup with Lie algebra $\sum_{\alpha<0} \dot{G}_\alpha$

$L_{\mathbb{C}}$ = The Zariski closure of L= maximal $\mathbb{R}$-compact subgroup of $Z(T)$.

$F \;= N^-_{\mathbb{R}} \cdot N^+_{\mathbb{R}}$

we have Bruhat's decomposition

$$G = N^{-}(\text{Norm } T)N^{+}$$
$$G_{\mathbb{R}} = N^{-}_{\mathbb{R}}N^{+}_{\mathbb{R}}.$$

we need following Lemmas.

3.3 Lemma. *Let V be a finite dimensional vector space and let $v_i \in V$ and $d_i \in GL(V) i = 1, 2, \ldots$*

Assume

(i) $\lim_{i\to\infty} v_i = v = \lim_{i\to\infty} d_i v_i$ *and*

(ii) $(d_i - 1)^{-1}$ *are bounded uniformly in i then $v = 0$.*

Proof. Set $w_i = (d_i - 1)v_i$ then $w_i \to 0$ as $i \to \infty$

$$\therefore \quad v_i = (d_i - 1)^{-1} w_i \to 0 \quad \text{i.e.,} v = 0.$$

□

3.4 Lemma. *Let K be a compact subset of $G_{\mathbb{R}}$, let W and U_A be neighbourhoods of 1 in $G_{\mathbb{R}}$ and A respectively. Let $t > 1$, then there is a nbd.* **39**
U of 1 in G such that

$$(kW)[LaU_A] \supset k[La].U \ \forall k \in K, a \in A^t.$$

Proof. Since the rank of the map $(g, b) \to g(b)$ of $(W \cap F) \times LA^1$ into $G_{\mathbb{R}}$ at $(1, b)$ equals the dimension of $[\dot{G}_{\mathbb{R}}, \dot{L} + \dot{A}] + \dot{L} + \dot{A} = \dot{G}$ the map is open in a nbd. of $(1, b)$. By taking a open subset of U_A we can assume that $\forall a' \in U_A \ t^{-1} < \alpha(a') < t \ \forall \alpha \in \triangle$ and $\overline{U}_A$ is compact. Then $\forall a \in A^t$ $aU_A \subset A^1$. If necessary, by passing to a open subset, we can assume that the above map has maximal rank on $(W \cap F) \times LA^1$, $\overline{W}$ is compact and $\overline{W} \cap$ Norm $A \subset Z(A)_\circ$. Then the set $kW[LaU_A]$ is a nbd. of identity. It remains only to show that

$$\bigcap_{\substack{a\in A^t, k\in K \\ m\in L}} (k[ma])^{-1}(kW)[LaU_A] \text{ is a nbd. of identity.}$$

□

Since for any nbd. U of 1, $\cap_{k\in K} k[U]$, for K compact, is a nbd. of 1, it is sufficient to show that

$$(*) \qquad \bigcap_{a\in A^t, m\in L} (ma)^{-1} W[LaU_A] \text{ is a nbd. of 1.}$$

Let

$$\pi : G_{\mathbb{R}} \to G_{\mathbb{R}/A}$$

set
$$\widetilde{W} = \pi(W)$$
define
$$f_{m,a} : \widetilde{W} \times U_A \times L \to G$$
$$f_{m,a} : (WA, a', m') \mapsto (ma)^{-1}(w(m'aa'))$$

40 then (*) is equivalent to

$$(**) \qquad \bigcap_{a\in A^t, m\in L} \text{Image } f_{m,a} \text{ is a nbd. of 1.}$$

It is easy to see that the condition (**) fails iff there is a sequence of points $x_i \in W \times U_A \times L$ and a sequence $(m_i, a_i) \in L \times A^t$ such that $x_i \mapsto$ boundary of $= \widetilde{W} \times U_A \times L$ in $G_{\mathbb{R}/A} \times A \times L$ and $\lim_{i\to\infty} f_{m_i,a_i}(x_i) = 1$. Hence to prove (**) it suffices to show that if $\lim_{i\to\infty} f_{m_i,a_i}(w_iA, a_i', m_i') = 1$ with $a_i' \in U_A, m_i' \in L$ and if $\lim(w_i, a_i', m_i, m_i') = (w, a', m, m')$ then $w = 1$ and $a' = 1$. For then it will follow that

$$(w_iA, a_i', m_i') \to (A, 1, m)$$

which is not a boundary point of $\widetilde{W} \times U_A \times L$.

The previous statement is equivalent to

$$(***) \qquad \begin{cases} \text{If } (m_ia_i)^{-1}(w_i[m_i'a_ia_i']) \mapsto 1 \text{ and if} \\ (w_i, a_i', m_i, m_1') \to (w, a', m, m') \in (\overline{W} \cap F) \times \overline{U}_A \times L \times L \\ \text{then } w = 1 \text{ and } a' = 1. \end{cases}$$

We prove (***)

From the uniqueness of Bruhat's decomposition it follows that $N_{\mathbb{R}}^-LAN_{\mathbb{R}}^+$, being the image of $N_{\mathbb{R}}^- \times LA \times N_{\mathbb{R}}^+$ under a homeomorphism, is open (invariance of domain).

41 Let $b \in A$ be close enough to the identity 1, so that

$$w[b] \in N_{\mathbb{R}}^{-} LAN_{\mathbb{R}}^{+} \quad \forall w \in \overline{W}.$$

Then

$$w_i[b] = p_i c_i q_i \quad p_i \in N_{\mathbb{R}}^{-}, c_i \in LA, q_i \in N_{\mathbb{R}}^{+}$$

and $$w[b] = pcq \quad p \in N_{\mathbb{R}}^{-}\ c \in LA, q \in N_{\mathbb{R}}^{+}.$$

Set $$b_i = m_i a_i \quad \text{then } b_i^{-1} = v_i(w_i[(m_i' a_i a_i')^{-1}])$$

where $$v_i = b_i^{-1}(w_i[m_i' a_i a_i'])$$

$$\begin{aligned} \therefore \quad (b_i^{-1} w_i)[b] &= v_i(w_i[(m_i' a_i a_i')^{-1}]) w_i b w_i^{-1} (w_i[m_i' a_i a_i']) v_i^{-1}. \\ &= v_i w_i[b]. \end{aligned}$$

Since $v_i \to 1$ and $w_i \to w$ we have

$$\lim_{i\to\infty} (b_i^{-1} w_i)[b] = w[b]$$

i.e., $$\lim_{i\to\infty} b_i^{-1} w_i[b] = \lim_{i\to\infty} w_i[b]$$

but $$b_i^{-1}[w_i[b]] = b_i^{-1}[p_i] \cdot c_i \cdot b_i^{-1}[q_i]$$

so

$$\begin{aligned} \lim b_i^{-1}[w_i[b]] &= \lim b_i^{-1}[p_i] \cdot c_i \cdot b_i^{-1}[q_i] \\ &= \lim p_i \cdot c_i \cdot q_i \end{aligned}$$

$$\therefore \qquad \lim b_i^{-1}[p_i] = p = \lim p_i$$

and $$\lim b_i^{-1}[q_i] = q = \lim q_i.$$

Since N^+, N^- are nilpotent. By induction on the lengths of the descending central series of N^-, N^+ and using the previous lemma we get 42

$$\lim p_i = p = 1 = q = \lim q_i.$$

$$\therefore \qquad w[b] = c \in LA$$

since A is connected any nbd. of 1 in A generates A we have

$$w[A] \subset LA$$

$$\therefore \qquad w[A] = A$$

$$w \in \overline{W} \cap \text{ Norm } A \subset Z(A)$$

$$\therefore \qquad w \in \overline{W} \cap LA \cap F.$$

But from Bruhat's decomposition $\overline{W} \cap LA \cap F = \{1\}$.

$\therefore \quad w = 1.$

From (***)

$$b_i^{-1}(w_i[m'_i a_i a'_i]) \to 1.$$

Since $w_i \to 1$ we have

$$b_i^{-1} m'_1 a_i a'_i \to 1$$

$$\therefore \qquad \underbrace{m_i^{-1} m'_i}_{\in L} a'_i \to 1$$

$$\therefore \qquad a'_i \to 1 \qquad \text{since } L \cap A = \{1\}.$$

$$\text{i.e.,} \qquad a' = 1.$$

43 This proves the Lemma.

3.5 Lemma. *Let C be a compact subset of $N_{\mathbb{R}}^-$ and let $t > 1$. Then there exists a compact subset $K \subset N_{\mathbb{R}}^-$ such that*

$$Cb \subset K[b] \quad b \in LA^t.$$

Proof. (By induction on the length of the derived series of $N_{\mathbb{R}}^-$).

Set

$$N_\circ = N_{\mathbb{R}}^- \quad N_{i+1} = [N_i, N_i]$$

Suppose $N_\circ$ is abelian. □

Then

$$ub = v[b] \text{ iff } ub = vbv^{-1}b^{-1}b$$

$$\text{i.e.,} \quad u = vb\, v^{-1}\, b^{-1}$$

$$
\begin{aligned}
&= v - ad\ b\ v \quad \text{(written in additive from)}\\
&= (1 - adb)v.
\end{aligned}
$$

44 Iff $v = (1 - adb)^{-1}v$ where $adb(x) = bxb^{-1}$.
Since $(1 - adb)^{-1}$ is uniformly bounded for

$$b \in LA^t, \quad \bigcup_{b \in LA^t} (1 - adb)^{-1}C$$

is a subset of a compact set K. This proves the statement for the case when $N_\circ$ is abelian.

In general, given a fixed $b \in LA^t$, by applying the above argument to $N_\circ/N_1$, we can find elements $v \in N_\circ$ and $n \in N_1$ such that $nub = v[b]$; moreover since $N_\circ = N^-_{\mathbb{R}}$, as u varies over compact set C, n and v vary over compact sets.

$ub = n^{-1}v[b] = v[n_1 b]$ where $n_1 \in N_1$ and varies over a compact set K_1.

By induction $n_1 b = v_1[b]$ and v_1 varies over a compact set as n_1 varies over compact set K_1 and b over LA^t.

We have

$$ub = v[n_1 b] = v[v_1[b]] = vv_1[b]$$

as both v, v_1 vary over compact sets $v \cdot v_1$ varies over a compact set proving the Lemma.

Now we prove Theorem 3.2.

We can assume (see pp. 36-37) that $y \in LA^t$, $t > 1$. Let $S_y = G - N^- Z(A) N^+$. Then since $S_y \Big| Z(A)N^+$ is union of N^- orbits of lower dimensions in $GZ(A)N$, S_y is Zariski closed in G. Let

$$x \in G_{\mathbb{R}} - S_y = N^-_{\mathbb{R}} Z(Z)_{\mathbb{R}} N^+_{\mathbb{R}}$$

then 45

$$
\begin{aligned}
x &= u^- b u^+ \text{ with } b \in LA, u^- \in N^-_{\mathbb{R}} \quad \& u^+ \in N^+_{\mathbb{R}}\\
xy^n &= u^- b u^+ y^n = (u^- b y^n)(y^{-n} u^+ y^n)\\
&= v[by^n](y^{-n} u^+ y^n) \text{ for some } v \in K \text{ (by Lemma 3.5).}
\end{aligned}
$$

Since y is $\mathbb{R}$-regular reductive element, given a nbd. U of 1, $\exists n_\circ(U)$ such that $(y^{-n}u^+y^n) \in U$ for $n > n_\circ(U)$. Hence by Lemma 3.4, $\exists n_\circ(y, x)$ such that $xy^n \in G[LA^t]$ if $n > n_\circ(y, x)$.

3.6 Lemma. *S_y contains no conjugacy class of G.*

Proof. Suppose $E \subset S_y$ with $G[E] = E$. Since S_y is Zariski closed we can also assume that $E = E^*$. Let $0 = G - E$, then for $g \in N^+$

$$
\begin{aligned}
& g[N^- Z(A) N^+] = g[G - S_y] \subset g[G - E] \subset G - E = 0 \\
\therefore \quad & N^+[N^- Z(A) N^+] = N^+ N^- Z(A) N^+ \subset 0.
\end{aligned}
$$

□

But

$$
\begin{aligned}
N^+N^-Z(A)N^+ &= N^+N^-Z(A)Z(A)N^+ = N^+Z(A)N^-Z(A)N^+ \\
&= \underline{N^+Z(A)N^+}\ \underline{N^-Z(A)N^+} = JJ^{-1} \subset 0 \\
&\quad \text{where } J = N^+Z(A)N^+
\end{aligned}
$$

Since J is Zariski open in G, for any $g \in G$, $gJ \cap J$, being intersection of two Zariski open (hence dense) sets, is nonempty.

Therefore

$$
\begin{aligned}
& g \in JJ^{-1} \subset 0. \\
\therefore \quad & 0 = G \\
\therefore \quad & E = G = 0 = \phi.
\end{aligned}
$$

46 This proves the assertion.

Chapter 4

Discrete Subgroups

In this and the following sections we will use the following notations. **47**

G will denote a semi-simple (complex analytic) algebraic $\mathbb{R}-$group. $G_{\mathbb{R}} = G \cap GL(n, \mathbb{R})$ and $G_* = G^{\circ}_{\mathbb{R}}$. For any subset S of G, S^* and $\bar{S}$ are respectively the Zariski closure and the closure in $\mathbb{R}$-topology of S in G.

We state the following useful Theorem, for a proof the reader is referred to [3] or [16].

4.1 Theorem. *Let G be a connected algebraic $\mathbb{R}-$group with no $\mathbb{R}$-compact factors and let Γ be a $\mathbb{R}$-closed subgroup of $G_{\mathbb{R}}$. If $G_{\mathbb{R}/\Gamma}$ has an $G_{\mathbb{R}}$-invariant finite measure, then Γ is Zariski dense in G.*

Here after we assume that Γ is a closed subgroup of $G_{\mathbb{R}}$ such that $G_{\mathbb{R}/\Gamma}$ has an $G_{\mathbb{R}}$ invariant finite measure and G has no $\mathbb{R}$-compact factors.

Now we prove a few "density" results.

4.2 Lemma. *If $\Gamma_{\circ}$ is the set of reductive $\mathbb{R}$-regular elements in Γ then $\Gamma^*_{\circ} = G$.*

Proof. We first show that $\Gamma_{\circ}$ is non-empty. ☐

Fix an element x of $LA^t, t > 1$, let U be a symmetric nbd. of 1 in G. Then since the sett $x^n U\Gamma$. **** have same non zero measure and since the total measure is finite, at least two of them intersect **48**

let $\qquad x^m U\Gamma \cap x^k U\Gamma \neq \phi$ for $k > m$

then $\qquad \Gamma \cap U x^{k-m} U \neq \phi$

i.e. for some $n \geq 1$

$$\Gamma \cap Ux^n U \neq \phi.$$

but $$Ux^n U \subset U[x^n] \cdot U^2.$$

If U is sufficiently small by Lemma 3.4 $U[x^n] \cdot U^2 \subset G[LA^t]$. This implies that

$$\Gamma \cap G[LA^t] \neq \phi$$

$\therefore$ $\Gamma_\circ \supset \Gamma \cap G[LA^t]$ is non-empty.

Let $\gamma_\circ \in \Gamma_\circ$, if $\gamma \in \Gamma - S_{y_\circ}$ then by Theorem 3.2,

$$\gamma\gamma_\circ^n \in G[LA^t] \text{ for all } n > n_\circ(\gamma).$$

Set

$$B_\circ = \left\{\gamma_\circ^n; n > n_\circ(\gamma)\right\}$$

then $$B_\circ \cdot B_\circ \subset B_\circ \text{ hence } B_\circ^* \cdot B_\circ^* = B_\circ^*.$$

Since the ideal of polynomials vanishing on $B_\circ$ is stable under translation by $x \in B_\circ^*$ and therefore under translation by x^{-1} for $x \in B_\circ^*$ (see Lemma 1 on p. 80 [50]), we have

$$(B_\circ^*)^{-1} \subset B_\circ^*$$

49 Therefore

$$(B_\circ^*)^{-1}(B_\circ^*) \subset B_\circ^*.$$

$$\therefore \qquad 1 \in B_\circ^*.$$

Also since

$$\gamma B_\circ \in \Gamma_\circ$$
$$\gamma B_\circ^* \subset \Gamma_\circ^*$$
$$\therefore \qquad \gamma^1 = \gamma \in \Gamma_\circ^*.$$

This proves that $\Gamma - S_{y_\circ} \subset \Gamma_\circ^*$. But since $S_{y_\circ}$ is a Zariski closed proper subset of G, $\Gamma - S_{y_\circ}$ is Zariski dense and therefore $\Gamma_\circ^* = G$.

4.3 Lemma. *Let* $\gamma_1 \in \Gamma$, *set*

$$\Gamma_1 = \{\gamma | \gamma \in \Gamma, \gamma, \gamma^n \in \Gamma_\circ \textit{ for } n > n_\circ(\gamma)\}$$

and $$\Gamma_2 = \{\gamma^n | \gamma \in \Gamma_1 \quad n \geq n_\circ(\gamma)\}$$

then $$\Gamma_\circ^* = \Gamma_\circ^* = G.$$

Proof. Since the Zariski closure of $\{x^n, n \geq n_\circ\}$ for $x \in G$ is a group (see the proof of previous lemma) $x : \{x^n, n \geq n_\circ^*\}$. □

This shows that

$$\Gamma_1 \subset \Gamma_2^*.$$

Hence it is sufficient to prove that Γ_1 is Zariski dense in G. Given
$y \in \Gamma_\circ$, since by Lemma 3.6, S_y does not contain any conjugacy classes, 50
$\exists \gamma$ such that $\gamma[\gamma_1] \notin S_y$.

$\therefore$ $T_y = \{\gamma, \gamma|\gamma_1| \in S_y\}$ is a proper algebraic subset of G.

For any $\gamma \in \Gamma - T_y$, $\gamma[y_1] \notin S_y$ so

$$\gamma[\gamma_1]y^n \in \Gamma_\circ \text{ for all } n > m_\circ(\gamma)$$

$$\gamma\gamma_1\gamma^{-1}y^n \in \gamma_\circ \ \therefore\ \gamma_1\gamma^{-1}y^n\gamma \in \gamma^{-1}\Gamma_\circ\gamma = \Gamma_\circ$$

i.e. $\gamma_1(\gamma^{-1}[y])^n \in \Gamma_\circ$ for $n > m_\circ(\gamma))$

i.e. $\gamma^{-1}[y] \in \Gamma_1$ if $\gamma \notin T_y$

$\therefore$ $\gamma \notin T_y^{-1}$ implies $\gamma[y] \in \gamma_1$

$\therefore$ $\Gamma_1^* \supset (\Gamma - T_y^{-1})^*[y]$.

Bur $\Gamma - T_y^{-1}$ is dense in G, so

$$y \in G^*[y] \subset \Gamma_1^*$$

$$\therefore \quad \Gamma_\circ \subset \Gamma_1^*.$$

By the previous lemma we have $\Gamma_1^* = \Gamma_\circ^* = G$.

Following is a refinement of the above result

4.4 Lemma. *Let S be a proper algebraic subset of G, let n be a positive integer and* $\gamma_1 \in \Gamma$. *Then* $\exists \gamma_\circ \subset \Gamma_\circ - S$ *such that* $\gamma_\circ, \gamma_\circ^2, \ldots, \gamma_\circ^n$ *and* $\gamma_1\gamma_\circ, \gamma_1\gamma_\circ^2, \ldots, \gamma_1\gamma_4^n \in \Gamma_\circ - S$.

Proof. $\forall m, S_m = \{x | x \in G, \gamma_1 x^m \in S\} \cup \{x | x \in G, x^m \in S\}$ is a proper algebraic subset of G. □ **51**

Hence $S_1 \cup S_2 \cup \ldots \cup S_n$ is a proper algebraic subset. Since Γ_2 is Zariski dense, we can find a $\gamma_\circ$ in $\Gamma_2 - S_1 \cup S_2 \cdots \cup S_n$. Obviously such a $\gamma_\circ$ satisfies the requirements of the lemma.

4.5 Lemma. *Let G be a semi-simple $\mathbb{R}$-group and let ${}_{\mathbb{R}}T$ be a maximal $\mathbb{R}$–split torus. Let T be a maximal $\mathbb{R}$–torus containing ${}_{\mathbb{R}}T$. Set $A = ({}_{\mathbb{R}}T)^\circ\mathbb{R}$, $H = (T_{\mathbb{R}})^\circ$. Then*

$$(G_*[\Gamma] \cap H)^* = T.$$

Proof. $Z(A)_{\mathbb{R}} = Z({}_{\mathbb{R}}T)_{\mathbb{R}} = L.A.$ □

Since

$$\begin{aligned} H &= (H \cap L) \cdot A \\ L^\circ[H] &= L^\circ[H \cap L] = L^\circ.A \\ \text{and} \quad G_*[G_*[\Gamma] \cap H] &= G_*[\Gamma \cap G_*[H]] \\ &= G_*[\Gamma \cap G_*[L^\circ.A]] \\ &\supset G_*[\Gamma_\circ] \supset \Gamma_\circ. \end{aligned}$$

By taking Zariski closure we get, since $G_*^* = G$

$$G = \Gamma_\circ^* = (G_*[G_*[\Gamma] \cap H])^* = G[(g_*[\Gamma] \cap H^*)].$$

Therefore

$$\dim G = \dim G[(G_*[\Gamma] \cap H)^*] = \dim G/Z(x) + \dim(G_*[\Gamma] \cap H)^* \quad \text{for some } x \in H.$$

52 Since $\dim Z(x) \geq \dim T$, we find $\dim(G_*(\Gamma) \cap H)^* = \dim T$ and thus $(G_*[\Gamma] \cap H)^* = T$.

Chapter 5

Some Ergodic Properties of Discrete Subgroups

5.1 Lemma (Mautner)**.** *Given a group $B{\cdot}A$, where B is an additive group of reals or complex numbers and A is an infinite cyclic subgroup of the multiplicative group of complex numbers a with $|a| < 1$ and assume that* 53

$$a \circ b \circ a^{-1} = a.b \quad \begin{array}{l}\textit{0 is group operation is}\\ \textit{ordinary multiplication in } \mathbb{C}\end{array}$$

Let V be a Hilbert space and let ρ be a unitary representation of $B \cdot A$ on V, then any element $v \in V$ whose line is fixed under A is fixed under B.

Proof. Since ρ is unitary

$$\begin{aligned}
\rho(a)v &= \alpha v \text{ with } |a| = 1 \text{ for } b \in B\\
< \rho(b)v, v > &=< \rho(a)\rho(b)v, \rho(a)v >\\
&=< \rho(a)\rho(b)\rho(a^{-1})\rho(a)v, \rho(a)v >\\
&=< \rho(a \circ b \circ a^{-1})\alpha v, \alpha v >=< \rho(a \circ b \circ a^{-1})v, v >
\end{aligned}$$

So for $\forall n$ positive

$$\begin{aligned}
< \rho(b)v, v > &=< \rho(a^n \circ b \circ a^{-n})v, v >\\
&=< \rho(a^n.b)v, v >
\end{aligned}$$

$$\text{as } n \to \infty$$
$$< \rho(b)v, v > =< v, v >$$
$$\therefore \quad \rho(b)v = v.\text{(use Schwarz's inequality).}$$

54 This proves the assertion. □

5.2 Lemma. *Let G be an analytic semis-simple group having no compact factors. Let ρ be a unitary representation of G on a Hilbert space V, let x be a reductive $\mathbb{R}$-regular element in G, if for some element $v \in V$, $\rho(x)v = \alpha v$ then $\rho(G)v = v$.*

Proof. Take the decomposition of G with respect to x. Let A be the group generated by x and B a root space. The previous Lemma applies. □

Remark. The above result holds for any x not contained in a compact subgroup (see [11]).

5.3 Theorem. *Let x be a reductive $\mathbb{R}$−regular element of G. Then x operates ergodically on $G_*|\Gamma$, i.e. any measurable subset of $G_*|\Gamma$ stable under left translation by x is either of measure zero or its complement has measure zero.*

Proof. Let $V = \mathscr{L}^2(G_*/\Gamma)$. □

Since the measure on G_*/Γ is G_*-invariant, the canonical action of G_* on V is unitary.

Let $Z \subset G_*/\Gamma$ with $xZ \subset Z$ and let v be the characteristic function of Z. Then since measure of $x^{-1}Z - Z$ is zero

$$x.v = v.$$

55 Therefore by the previous lemma

$$G_* \cdot v = v.$$
$$\therefore \quad v = 1 \text{ almost every where}$$
$$\text{or} \quad v = 0 \text{ almost every where.}$$

This implies that either Z or $G/\Gamma - Z$ has zero measure.

Remark. Let M be a separable topological measure space [i.e. the open sets are measurable and have positive measure] and let $f : M \to M$ be a measurable transformation. Let $A^+ = \{f^n, n = 1, 2, \ldots\}$. Then if f is ergodic, for almost all $p \in M$, A^+p is dense in M.

[*Proof*: Let $\{U_i\}$ be a denumerable base of open sets. Let $\{W_i = p|p \in M, A^+p \cap U_i = \phi\}$ then W_i is measurable. Also $p \in W_i \Rightarrow fp \in W_i$ therefore $fW_i \subset W_i$. Since f is ergodic and $U_i \subset M - W_i, W_i$ is of measure zero.

$$\therefore \quad E = \bigcup_{i=1}^{\infty} W_i \quad \text{has measure 0}$$

$p \notin E$ implies $A^+p \cap U_i \neq \phi \;\forall i$ and this proves that for almost all $p \in M$, A^+p is dense].

5.4 Theorem. *Let G_* be a semi-simple analytic linear group. Let Γ be a subgroup such that G_*/Γ has a finite invariant measure. Let P be a $\mathbb{R}$-parabolic subgroup of G_*^* (= the complexification of G_*). Set $P_* = P \cap G_*$ then $\overline{\Gamma P_*} = G_*$.*

Proof. Let T be a maximal $\mathbb{R}-$split torus in P. Let $x \in T^\circ_{\mathbb{R}}$ such that **56**
for any restricted root α on T with $G_\alpha \subset U^+$ the unipotent radical of $P, \alpha(x) > 1$. ◻

Let U^- be the opposite (i.e. $\dot{U}^- = \sum \dot{G}_{-\alpha}$ where $\dot{U}^+ = \sum \dot{G}_\alpha$) of U^+ and K_* a maximal compact subgroup of G_*. Also let W^- be a nbd. of 1 in $U^- \cap G_*$ whose logarithm is a convex set. Since W^-P_* is a nbd. of 1 in G_*, and K_* is compact $\exists$ a nbd. W of 1 in G_* with $W \subset W^-P_*$ and $K_*[W] = W$.

$Ux^nW\Gamma$ is stable under x and contains an non-empty open set, hence by Theorem 5.3 it differs from $G|\Gamma$ in a set of measure zero. Therefore $\exists\; n = n(k)$ such that

$$W^{-1}k^{-1}\Gamma \cap x^nW\Gamma \neq \phi$$

$$\therefore \quad \Gamma \cap kWx^nW \neq \phi,$$

$$\text{but} \qquad x^nW \subset x^nW^-P_* = x^nW^-x^{-n} \cdot P_* \subset W^-P_*$$

(using the convexity of logarithm of W^-)

$$\therefore \quad \Gamma \cap kWW^- P_* \neq \phi$$
$$\therefore \quad \Gamma P_* \text{ meets } kWW^-.$$

This proves that $K_* \subset \overline{\Gamma P_*}$. $\quad \therefore K_* P_* \subset \overline{\Gamma P_*} \subset G_*$.

But we know that $K_* \cdot P_* = G_*$.

Therefore $G_* = \overline{\Gamma P_*}$.

Chapter 6

Real Forms of Semi-simple Algebraic Groups

In this and the following sections, G will denote a semi-simple $\mathbb{R}$-groups 57
${}_{\mathbb{R}}T$ a maximal $\mathbb{R}$-split Torus, T a maximal $\mathbb{R}-$torus containing ${}_{\mathbb{R}}T$; ${}_{\mathbb{R}}\triangle$ a fundamental system of restricted roots on ${}_{\mathbb{R}}T$, a fundamental system of roots on T whose restriction to ${}_{\mathbb{R}}T$ consists of ${}_{\mathbb{R}}\triangle \cup \{0\}$ (such a $\triangle$ can always be found). Φ, Φ^{+} will denote respectively the set of roots and the set of positive roots and Φ^{*} the set of positive roots whose restriction to ${}_{\mathbb{R}}T$ is non-zero. $\triangle_{\circ}$ will denote the subset of $\triangle$ consisting of those roots which are constant on ${}_{\mathbb{R}}T$.

Given $\alpha \in \Phi$ we define $\alpha' \in \Phi$ by the formula $\alpha'(\bar{x}) = \overline{\alpha(x)} \forall x \in T$, $\bar{x}$ is complex conjugate of x. Then for any $\alpha \in \triangle_{\circ}$ $\alpha' = -\alpha$ on $\triangle - \triangle_{\circ}$. We can define a permutation σ by

$$\alpha' = \sigma(\alpha) + \sum_{\beta \in \triangle_{\circ}} n_{\beta}\beta \quad n_{\beta} \quad \text{non-negative integers.}$$

Satake's Diagrams of semi-simple $\mathbb{R}-$groups.

In Dynkin's diagram every root in $\triangle_{\circ}$ is denoted by a back circle $\bullet$ and every root of $\triangle - \triangle_{\circ}$ by a white circle $\circ$. If $\alpha \in \triangle - \triangle_{\circ}$ then the white circles corresponding to α and $\sigma(x)$ are joined by a arrow .

Definition. $G_{\mathbb{R}}$ is said to be $\mathbb{R}$-simple if $(G_{\mathbb{R}})^{\circ}$ has no proper normal subgroups of positive dimension.

If $G_{\mathbb{R}}$ is $\mathbb{R}$-simple, but G is not simple then $\dot{G}$ = restriction $\dot{H}$ = **58**
$\dot{H} \otimes_{\mathbb{R}} \mathbb{C}$, where $\dot{H}$ is a simple Lea algebra over $\mathbb{C}/\mathbb{R}$

Thus $\dot{G} = \dot{H} \oplus \dot{H}$, and the diagram of $\dot{G}$ consists of two copies of Dynkin's diagram of $\dot{H}$, with vertices corresponding under complex conjugation joined by arrows

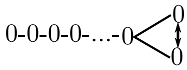
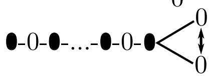
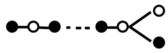
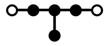
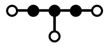
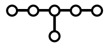
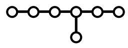
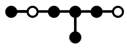
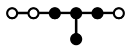
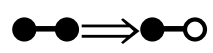
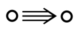
$$\begin{array}{ccccc} 0- & 0- & 0- & \cdots & -0 \\ \updownarrow & \updownarrow & \updownarrow & & \updownarrow \\ 0- & 0- & 0- & \cdots & -0 \end{array}$$

Real forms of semi-simple Lie groups have been determined by F-Gautmacher (cf. Matsbornik (47) V. 5 (1939) pp. 217-249).

The following is a complete list of $\mathbb{C}$-simple $\mathbb{R}$-groups (cf. [1], [20] & [24]).

$$1 = \sharp\,\triangle \qquad p = \sharp_{\mathbb{R}}\triangle$$

	Group	$\triangle$	Type of ${}_{\mathbb{R}}\triangle$	p
A I	$SL(l+1,\mathbb{R})$	0-0-0-0-…-0	A_1	$p = 1$
A II	$SU^*(l+1)$	$\bullet$-0-$\bullet$-0-$\bullet$-0----$\bullet$	A_p	$p = \frac{l-1}{2}$
A III	$SU(p,q)$			
	$(p < q, p+q = l+1)$	$\begin{array}{ccccc} 0- & 0- & 0- & ----- & 0 \\ \updownarrow & \updownarrow & \updownarrow & & \updownarrow\bullet\bullet \\ 0- & 0- & 0- & ----- & 0 \end{array}$	B_p	$p \leq 1/2$
	$SU\left(\frac{l+1}{2},\frac{l+1}{2}\right)$	$\begin{array}{cccccc} 0- & 0- & 0- & \cdots & -0 & \\ \updownarrow & \updownarrow & \updownarrow & & \updownarrow & >0 \\ 0- & 0- & 0- & \cdots & -0 & \end{array}$	C_p	$p = \frac{l+1}{2}$
B I	$S0(p,2l+1-p)$	0-0-0-…-$\bullet$-$\bullet$-$\bullet$-…$\bullet\Rightarrow\bullet$	B_p	$p \leq 1$
C I	$S_p(n,\mathbb{R})$	0-0-0-----0$\Longleftarrow$0	C_l	
C II	$S_p(p,l-p)$	$\bullet$-0-$\bullet$-0----0-$\bullet$-0-$\bullet\Longleftarrow\bullet$	B_p	$p \leq \frac{l-1}{2}$, l odd
	$S_p(l/2,l/2)$	$\bullet$-0-$\bullet$-…-$\bullet\Longleftarrow$0	C_p	$p = l/2$, l even
D I	$SO(p,2l-p)$	0-0-0-0…-$\bullet$-$\bullet$-…$\bullet\!<\!\begin{smallmatrix}\bullet\\ \bullet\end{smallmatrix}$	B_p	$p \leq l-2$
	$SO(l.l)$	0-0-0-…-0$<\!\begin{smallmatrix}0\\ 0\end{smallmatrix}$	D_1	

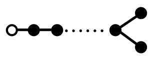
	$SO(l-1, l+1)$		B_{l-1}	
D III	$SO^*(21)$		B_p	$p = \frac{l-1}{2}$
	$SO^*(21)$		C_p	$p = l/2$
	E_6		A_2	
			B_2	
			E_6	
	E_7		C_3	
	E_7		F_4	
	E_8		F_4	
	F_4		F_4	
			$\vee_1$	
	G_2		G_2	

59

Definition. $\mathbb{R}$-rank of an algebraic group is the dimension of a maximal $\mathbb{R}$-split torus. **60**

From the diagrams above, we can excerpt the diagrams of groups of $\mathbb{R}$-rank 1 and we list the dimension of the restricted root spaces.

Let $\triangle - \triangle_\circ = \{\alpha\}$

Associated symmetric space	diagram	$\dim \dot{G}_2$	$\dim \dot{G}$
Hyperbolic	0	0	1
		0	$2l-1$
		0	$2l-2$
Hermitian hyperbolic		1	$2l-2$
Quaternianic hyperbolic		3	$4l-8$
Cayley hyperbolic		7	8

Now we collect some results whose proof require examining these diagrams. **61**

6.1 Lemma. *If N is the set of automorophisms of G stablizing T and ${}^{T}_{\mathbb{R}}$, then any automorphism τ of restricted root system ${}_{\mathbb{R}}\Phi$ is induced by an element of N.*

Proof. The result is true for inner automorphisms (i.e. elements in little Weyl group). For given any element $\sigma \in N({}_{\mathbb{R}}T)$, both $\sigma T\sigma^{-1}$ and T are contained in $Z({}_{\mathbb{R}}T)$ and therefore are maximal $\mathbb{R}$-tori of the connected algebraic group $Z({}_{\mathbb{R}}T)$. By the conjugacy of maximal tori, $\exists z \in Z({}_{\mathbb{R}}T)$ such that

$$z\sigma T\sigma^{-1}z^{-1} = T \quad \text{i.e.,} \quad z\sigma \in N(T)$$

The inner automorphism given by $z\sigma$ is the desired element of N. □

In case τ, is an outer automorphism we can without loss of generality assume that $\tau({}_{\mathbb{R}}\triangle) = {}_{\mathbb{R}}\triangle$.

From the usual root diagrams it is clear that only the restricted root systems of type A, E and D_1 admit an outer automorphism.

In case A and E the automorphism is an inner automorphism composed with the "opposition" map $\alpha \mapsto -\alpha$; since each of these extend to Φ, then so does the outer automorphism.

If the restricted diagram is of type D_l then the Satake diagram shows that $\triangle = {}_{\mathbb{R}}\triangle$; that is the group splits over $\mathbb{R}$ and hence the conclusion is hypothesis.

62 **6.2 Lemma.** *If $[Z({}_{\mathbb{R}}T), Z({}_{\mathbb{R}}T)] = J$ then $\dot{J} = \dot{J}_1 + \dot{J}_2$ is simple (possibly zero) and $\dot{J}_1$ is sum of compact Lie algebras of rank 1.*

Proof. We remark first that $\triangle_\circ$ is a fundamental system of roots for J. Now simply observe that the diagram of $\triangle_\circ$ satisfies the condition required by the conclusion. □

Note. $\dot{J}_2$ is of rank 1 only if the group is $Sp(1, 2)$.

6.3 Lemma. *Let W^* be the subgroup of the Weyl group of T, which stabalizer ${}_{\mathbb{R}}T$. Then W^* is irreducible on $\dot{J}_1 \cap \dot{T}$ and $\dot{J}_2 \cap \dot{T}$.*

Proof. $\dot{J}_2 \cap \dot{T}$ is a cartan subalgebra of the simple Lie algebra $\dot{J}_2$ and, as is well known, the Weyl group of a simple Lie algebra operates irreducibly of its associated Cartan subalgebra. It remains only to prove that W^* is irreducible on $\dot{J}_1 \cap \dot{T}$. □

Inspection of the position of $\triangle_\circ$ in $\triangle$, as pictured in the diagrams shows that $\dot{J}_1$ is contained in a subalgebra of type A_{2p-1} with the diagram ●–○–●– – – – – –○–●.

As is known, the Weyl group of $A_{2p-1} \approx SL(2p)$ is the group of permutations of the standard basis vectors $e_1, \ldots, e_{2p}$ in $\mathbb{C}^{2p}$. The roots $\triangle_{\circ;1}$ of $\dot{J}_1$ become identified with $\{\alpha_{2i-1} - \alpha_{2i}, i = 1, \ldots p\}$ where α_i denotes the i^{th} matrix coefficient. Clearly the stabalizer of $\dot{J}_1$ contains the conjugation by the matrix sending each. $e_{2i-1} \to e_{2\pi(i)-1}$ and $e_{2i} \to e)2\pi(i)$, for any permutation π of $\{1, \ldots, p\}$. Since these automorphisms of $\dot{J}_1$ induce the full symmetric group on the elements of the set $\triangle_{\circ,1}$, we conclude that W^* is irreducible on $\dot{J}_1 \cap \dot{T}$.

6.4 Lemma. *Let G_1, G_2 be two $\mathbb{C}$-simple $\mathbb{R}$-groups. Assume $\tau : T_1 \to$* **63**
T_2 is an isomorphism sending ${}_{\mathbb{R}}T_1 \to {}_{\mathbb{R}}T_2$ and $\Phi_1^* \to \Phi_2^*$. Then $\dot{G}_1 \approx \dot{G}_2$ and ${}^{\tau}|_{\mathbb{R}}\dot{T}_1$ can be induced by an isomorphism of $\dot{G}_1$ and $\dot{G}_2$.

Proof. Suppose first that p, the $\mathbb{R}$-rank of G_1 and G_2 is one. Then the τ-corresponding restricted root spaces must have the same dimension. The listed values in our table for $\dim G_\alpha$ and $\dim G_{2\alpha}$ show that these determine the group of $\mathbb{R}$-rank 1. Thus $\dot{G}_1 \approx \dot{G}_2$ in the rank 1 case. Moreover, the isomorphism θ of $\dot{G}_1$ to $\dot{G}_2$ can be taken so as to map the restricted root spaces of $\dot{G}_{1,\alpha}$ of $\dot{G}_1$ to the restricted root space $\dot{G}_{2,\tau(\alpha)}$ of $\dot{G}_2$. It follows at once that θ and τ induce the same map on ${}_{\mathbb{R}}\dot{T}$ and thus the lemma is proved for $p = 1$. □

Suppose now that $\sharp_{\mathbb{R}}\triangle > 1$. We need only consider the case that the groups are not split over $\mathbb{R}$ (i.e., $\triangle \neq {}_{\mathbb{R}}\triangle$), otherwise the result is a well-known theorem of Weyl. Assuming therefore that $\triangle \neq {}_{\mathbb{R}}\triangle$ we find that the restricted root diagrams are of type A, B, C, F_4. In neither of these cases does the Dynkin diagram of ${}_{\mathbb{R}}\triangle$ have a branch point. Therefore given a Satake diagram $\triangle$ of a non $\mathbb{R}$-split group, one can form a sequence of the subdiagrams $\triangle^{(1)} \subset \triangle^{(2)} \ldots \subset \triangle^{(p)} = \triangle$ such that

(a) $\sharp_{\mathbb{R}}\triangle^{(k)} = k$

(b) $\triangle^{(k+1)} = \triangle^{(k)} \cap D^{k+1}$ where $\sharp_{\mathbb{R}}D^{k+1} = 1$.

(c) $D^k \cap D^{k+1} = D^{k+1} \cap \triangle^{(k)}$.

64 Given now the two groups G_1 and G_2 and the map τ, we decompose the Satake diagram $\triangle_i$ of G_i as above, getting $\triangle_i = \triangle_i^{(p-1)} \cup D_i^p$. By induction there are isomorphisms $\theta^{(p-1)} : \triangle_1^{(p-1)} \to \triangle_2^{(p-1)}$ and $\theta_p : D_1^p \to D_2^p$ induced by isomorphisms of the corresponding Lie algebras $\dot{G}_i^{(p-1)}$ and F_i^p. Let φ denote the restriction of $\theta_p^{-1} \cdot \theta^{p-1}$ to $\dot{G}_1^{(p-1)} \cap \dot{F}_i^p$.

Set $\triangle_{i,\circ}^p = \triangle_i^{(p-1)} \cap D_i^p$. Then $\triangle_{i,\circ}^p$ is connected since the root diagram $\triangle$ has no loops. In fact by property (c) $\triangle_{i,\circ}^p = D_i^{p-1} \cap D_i^p$ and is in fact a connected component in $\triangle_{i,\circ}$, the diagram of the $\mathbb{R}$-compact part of $Z(T_i)$. An additional inspection of the diagrams shows that no connected component of the diagram of $Z(T_1)$ admits an (outer) automorphisms. Hence φ is an inner automorphisms of $\dot{G}_1^{(p-1)} \cap \dot{F}_1^p$ and thus extends to an automorphisms χ of $\dot{G}_1$. Replacing θ_p by $\theta_p \cdot \chi$, we obtain the derived isomorphisms of $\dot{G}_1$ onto $\dot{G}_2$.

The following is an easy consequence of previous lemma.

6.5 Theorem. *Let G be a semisimple $\mathbb{R}$-group having no compact factors. Let $\tau : T \to T$ be an isomorphism which stabilizes ${}_{\mathbb{R}}T$ and Φ^*. Then there exists an automorphism θ of G such that $\theta \cdot \tau$ stabilizes T and on ${}_{\mathbb{R}}T$ it is identity.*

6.6 Lemma. *Let G be a semisimple $\mathbb{R}$-group with no $\mathbb{R}$-compact factors. We also assume that $G_{\mathbb{R}}$ is simple of $\mathbb{R}$-rank 1. If τ is an automorphisms*
65 *of T stabilizing ${}_{\mathbb{R}}T$ and Φ^*, then it stabilizes $\dot{J} \cap \dot{T}$, $\dot{J}_1 \cap \dot{T}$ and $\dot{J}_2 \cap \dot{T}$.*

Proof. Let
$$B^* = \sum_{\alpha \in \Phi^*} \alpha^2$$
Then τ preserves B^*. Let $B, B_\circ$ denote the killing forms of G and $Z(T)$ respectively then $B = B_\circ + 2B^*$. So any two subspaces of $\dot{T}$ orthogonal with respect to both B and $B_\circ$ are orthogonal with respect to B^*. ◻

Let $X, X' \in \dot{J}_1$ and $Y \in \dot{J}_2$, then
$$B([X, X'], Y) = -B(X', [X, Y]) = 0$$
and since $[\dot{J}_2, \dot{J}_2] = -\dot{J}_2$, we have $B(\dot{J}_1, \dot{J}_2) = 0$ similarly $B_\circ(\dot{J}_1, \dot{J}_2) = 0$
$$\therefore \quad B^*(\dot{J}_1 \cap \dot{T}, \dot{J}_2 \cap \dot{T}) = 0.$$

Composing τ with an automorphisms of $\dot{G}$, we can assume by the Theorem 6.5 that τ induces identity on ${}_{\mathbb{R}}T$. Thus τ stabilizes the set of all roots having a non-trivial restriction on ${}_{\mathbb{R}}T$ and we can assume accordingly that ${}_{\mathbb{R}}\triangle$ consists of a single element, and that the set of positive roots in ${}_{\mathbb{R}}\Phi$ is either $\{\overline{\alpha}\}$ or $\{\overline{\alpha}, 2\overline{\alpha}\}$. Let S denote the set of roots restricting to $\overline{\alpha} \cdot \tau$ stabilizes Φ^* and therefore also the set $S - S$ of differences of roots in S. These differences clearly lie in linear span of $\{\triangle_\circ\}$, conversely, given any root $\beta \in \{\triangle_\circ\}$ we shall show that β occurs in $S - S$. We can assume $\beta > 0$.

The hypothesis that G contains no $\mathbb{R}$- compact factors is tantamount **66**
to the hypothesis that $\{G_\alpha, \pm\alpha \in S\}$ generates G. Hence $< \beta, S > \neq 0$. Let α be the least root in S for which $< \beta, \alpha > \neq 0$. Then $\sigma_\beta(\alpha) = \alpha + q(\alpha,\beta)\beta$ is a root where $q(\alpha,\beta) = \frac{-2<\alpha,\beta>}{<\beta,\beta>}$ is a positive integer. Thus $\alpha + q(\alpha,\beta)\beta \in S$ and $\beta \in S - S$. Hence $\{S - S\} = \{\triangle_\circ\}$ as asserted.

Therefore τ stabilizes the intersection of the kernels of the linear functions in $\triangle_\circ$ e.g. τ stabilizes $Z(J) \cap \dot{T}$. Now $\dot{J} \cap \dot{T}$ is the orthogonal complement of $Z(J) \cap \dot{T}$ with respect to both killing forms B and $B_\circ$ and therefore with respect to $B^* = B - 2B_\circ$. Since τ stabilizes B^*, it stabilizes $\dot{J} \cap \dot{T}$.

Having assumed that G has $\mathbb{R}$-rank 1, we see that τ is simple in all cases except $G = C_l(l = 3)$ or $G = D_3$. In the second case $\dot{J} = \dot{J}_1, \dot{J}_2 = 0$ and the Lemma is established. In case $G = C_l(l = 3)$ the diagram is

●—○—●- - - - -●⇐●

and the roots in Φ^* having the same restriction to ${}_{\mathbb{R}}T$ as $2\alpha_2$ are $2\alpha_2 + \ldots + 2\alpha_{l-1} + \alpha_l$; $\alpha_1 + 2\alpha_2 + \ldots 2\alpha_{l-1} + \alpha_1$, $2\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{l-1} + \alpha_l$. Since τ permutes this set, it permutes the differences and therefore $\tau\alpha_1 = \pm\alpha_1$. Hence τ stabilizes $\dot{T}_2 = \ker \alpha_1 \cap \dot{J}$, and therefore stabilizes $\dot{T}_1$ which is the orthogonal complement of $\dot{T}_2$ in $\dot{T} \cap \dot{J}$ with respect to B^*. The proof of the Lemma is now complete.

Chapter 7

Automorphisms of Φ^*

7.1 Lemma. *Let G be an $\mathbb{R}$-group with no compact factors. Let $\tau : \dot{T} \to$* **67**
$\dot{T}$ be a automorphism stabilizing ${}_{\mathbb{R}}T$ and Φ^* then τ preserves the Killing form.

Proof.
$$\dot{T} = (\dot{J}_1 \cap \dot{T}) + (\dot{J}_2 \cap \dot{T}) + (Z(\dot{J}) \cap \dot{T})$$
we know that (i) τ preserves $B^* = \sum_{\alpha\in\Phi^*} \alpha^2$ (ii) the three subspaces $\dot{J}_1 \cap \dot{T}$, $\dot{J}_2 \cap \dot{T}$ and $Z(\dot{J}) \cap \dot{T}$ are stable under τ and (iii) if W^* is the subgroup of Weyl group stabilizing ${}_{\mathbb{R}}T$ then $\dot{J}_1 \cap \dot{T}$ and $\dot{J}_2 \cap \dot{T}$ are irreducible under W^*. □

Since B^* and B are preserved by W^*, $B_i = C_i B_i^*$ $i = 1, 2$. $c_i \neq 0$. Here $B_i, B_i^*(i = 1, 2)$ are restrictions to $\dot{J}_i \cap \dot{T}$ of B, B^* respectively. Let B_3, B_3^* are restrictions to $Z(\dot{J})_i \cap \dot{T}$ of B, B^* respectively, then $B_3 = B_3^*$.

Hence on T, $B = B_3^* + C_1 B_1^* + C_2 B_2^*$. As τ preserves $B_1^*, B_2^* \& B_3^*$ it also preserves B.

7.2 Lemma. *Let G be an R-group without compact factors and let $\tau : \dot{T} \to \dot{T}$ be an automorphism stabilizing ${}_{\mathbb{R}}\dot{T}$ and Φ^* then τ is restriction to $\dot{T}$ of an automorphism of $\dot{G}$.*

Proof. Let W' be the subgroup of Weyl group of T generated by $\{\sigma_\alpha$, $\alpha \in \Phi^*\}$. We shall first prove that $W = W'$. Given $B \in \{\triangle_\circ\}$, $< \beta, \Phi^* > \neq 0$ since G has no $\mathbb{R}$-compact factors and hence $\{G_{\pm\alpha}, \alpha \in \Phi^*\}$ generates G. □

We can find $\alpha \in \Phi^*$ with $< \beta, \alpha >< 0$.

68 As

$$\sigma_\alpha(\beta) = \beta + q(\beta, \alpha)\alpha \text{ where } q(\beta, \alpha) = \frac{-2 < \beta, \alpha >}{< \alpha, \alpha >} > 0$$

$$\sigma_{\sigma_\alpha}(\beta) \in \Phi^*$$

$$\therefore \quad \sigma_{\sigma_\alpha(\beta)} \in W' \text{ but } \sigma_{\sigma_\alpha(\beta)} = \sigma_\alpha \sigma_\beta \sigma_\alpha^{-1}$$

$$\therefore \quad \sigma_\beta = \sigma_\alpha^{-1} \sigma_{\sigma_\alpha(\beta)} \sigma_\alpha \in W' \text{ for all } \beta \in \{\triangle_z\}$$

$$\therefore \quad W' = W.$$

$\tau\sigma_\alpha\tau^{-1} = \sigma_{\tau(\alpha)}$ is a reflection since for $\alpha \in \Phi^* \tau(\alpha) \in \Phi^*$

$$\tau W' \tau^{-1} = W'$$

$$\therefore \quad \tau W \tau^{-1} = W.$$

Thus τ permutes reflections in W, i.e. τ permutes the set $\{\sigma_\alpha, \alpha \in \Phi\}$.

$$\therefore \quad \tau\Phi = \Phi$$

$\therefore \quad \tau$ extends to an automorphisms of $\dot{G}$.

Chapter 8

The First Main Theorem

This section is devoted to the proof of **69**

8.1 Theorem. *Let G_* be a semi-simple analytic group with no compact factors and no center. K be a maximal compact subgroup. Let $X = G/K$ and let Γ, Γ' be two discrete subgroups of G_*, isomorphic under an isomorphism $\theta : \Gamma \to \Gamma'$. We assume that $G_*\big|\Gamma, G_*\big|\Gamma'$ have finite Haar measure. Let $X_\circ$ be the unique compact G_* orbit in some Satake-compactification of X. Let $\varphi : X \to X$ be a homeomorphism such that (i) $\varphi(\gamma x) = \theta(\gamma)\varphi(x)$ $\forall \gamma \in \Gamma$, $x \in X$: (ii) φ extends to a homeomorphism of $X \cup X_\circ$ whose restriction to $X_\circ$ is a diffeomorphism of $X_\circ$, then θ extends to an automorphisms of G_*.*

[Conjecture. Condition (ii) is superfluous if G has no factors isomorphic to $PSL(2, \mathbb{R})$.]

For the proof of the theorem we need following lemmas.

8.2 Lemma. *Let G be a connected reductive linear algebraic group. Let k^T be a maximal k-split torus and T be a maximal k-torus containing k^T. Let t_1, t_2 be elements in k^T conjugate in G. Then t_1, t_2 are conjugate by an element in Norm $({}_K T) \cap$ Norm T.*

Proof. From Bruhat's decomposition

$$G = N^+(\text{Norm }{}_k T)N^+.$$

Suppose

$$xt_1x^{-1} = t_2 \text{ with } x \in G$$

and $$x = u\,w\,v \quad u, v \in N^+, w \in \text{ Norm } k^T$$

70 then

$$uwvt_1 = t_2uwv$$
$$\therefore \quad u\underline{wt_1}t_1^{-1}[v] = t_2[u]\underline{t_2w}v.$$

□

By the uniqueness of Bruhat's decomposition $wt_1 = t_2w$. Thus t_1, t_2 are conjugate by w (Norm k^T). Since T and wTw^{-1} are contained in $Z({}_kT)$, by the conjugacy of maximal tori, $\exists\lambda \in Z({}_kT)$ such that $\lambda T\lambda^{-1} = wTw^{-1}$ i.e., $\lambda^{-1}wTew^{-1}\lambda = T$ i.e., $\lambda^{-1}w \in$ Norm $(k^T) \cap$ Norm T.

It is clear that t_1 and t_2 are conjugate by $\lambda^{-1}w$.

8.3 Lemma. *Let G be the Zariski closure of a real linear algebraic group G_*, let ${}_{\mathbb{R}}T$ be a maximal $\mathbb{R}$-split torus in G and T be a maximal $\mathbb{R}$-torus containing ${}_{\mathbb{R}}T$. W^A = Norm ${}_{\mathbb{R}}T$. P_* be the stabalizer in G_* of a point in $X_\circ$, P the Zariski closure of P_*, U the unipotent radical of P. We assume that $P \supset T$. $\mathscr{R}_U$ the set of roots occurring in U, $\mathscr{R}_{N^+}$ the set of roots occurring in N^+, then*

$$W^A(\mathscr{R}_U) = \pm\mathscr{R}_{N^+}.$$

Proof. From our description of Satake compactfication in § 2, we know that $P = P(\triangle')$ for some $\triangle' \subset {}_{\mathbb{R}}\triangle$. Indeed in the notation of § 2, $\triangle' = E_\rho$ where ρ is an $\mathbb{R}$-irreducible representation with finite kernel, and thus $P(\triangle')$ contains no normal subgroup of positive dimension, equivalently, the subset $\triangle'$ contains no connected component of the fundamental system of restricted roots ${}_{\mathbb{R}}\triangle$. □

71 We have $P(\triangle') = G(\triangle')$. $N(\triangle')$, $U = N(\triangle')$ and $N^+ = N(\phi)$. It is easy to see that if ${}_{\mathbb{R}}\triangle$ is connected, then $\mathscr{R}_U$ contains a root whose restriction to ${}_{\mathbb{R}}T$ has length equal to the length of any restricted root in ${}_{\mathbb{R}}\triangle$. We recall that the Weyl group of a connected root system permutes transitively all roots having the same length. Applying this observation to each connected component of ${}_{\mathbb{R}}\triangle$, we find W^A (restriction of $\mathbb{R}_U$ to ${}_{\mathbb{R}}T$)= all restricted roots. Hence

$$W^A(\mathbb{R}_U) = \pm\mathbb{R}_{U^+}.$$

8.4 Lemma. *Let $A = ({}_{\mathbb{R}}T_{\mathbb{R}})^{\circ}$ and let $b \in L.A = Z({}_{\mathbb{R}}T)_{\mathbb{R}}$. Then b is $\mathbb{R}$-regular iff b keeps fixed exactly $m/m_{\circ}$ points [here $m = \sharp W^{A}$ and $m_{\circ}$= order of the Weyl group of $G(\triangle')$] and on the tangent space at these points, the eigenvalues are different from 1 in absolute value.*

Proof. Let U^{-} denote the opposite of U. Suppose b is $\mathbb{R}$-regular then the eigenvalues of b on tangent spaces at the points fixed under b are the values of $W^{A}(\mathbb{R}_{U^{-}})$ on b. Conversely if $b \in LA$ has only finitely many fixed points on $X_{\circ}$, then b is $\mathbb{R}$-regular. The Lemma is now clear. □

8.5 Lemma. *Let G_{*}, Γ, Γ' be as in the hypothesis of the Theorem 8.1. Let γ be a reductive $\mathbb{R}$-regular element of Γ. Then $\theta(\gamma)$ is also reductive $\mathbb{R}$-regular.*

Proof. Let $p_{\circ} \in X_{\circ}$ and let P_{*} be the stabilizer of $p_{\circ}$ in G_{*}. For $g \in P_{*}$ **72**
we denote by $\hat{g}$ the operation of g on $\dot{G}_{*}/\dot{P}_{*}$. An element $g \in P_{*}$ is reductive $\mathbb{R}$-regular iff $Ad_{N} + g$ has eigenvalues different from 1 in absolute value; this will be true if g keeps fixed $m/m_{\circ}$ points of $X_{\circ} = G_{*/P_{*}}$ and on each of the tangent spaces at the fixed points, takes eigenvalues $\neq 1$ in absolute value. □

Thus $g \in G$ is reductive R-regular iff it keeps fixed $m/m_{\circ}$ points in $X_{\circ}$ and on the tangent space each point has eigenvalues $\neq 1$ in absolute value. From this it will follow that if γ is $\mathbb{R}$-regular then $\theta(\gamma)$ is also $\mathbb{R}$-regular.

Remark. If G is a reductive algebraic group over any field k, then it follows immediately from definitions that an element of G is k-regular iff it keeps only a finite number of points in G/p fixed for $\forall P = P(\triangle')$. $\triangle' \subset {}_{k}\triangle$. It can be proved that the element is reductive iff the number of fixed points is $\frac{\text{order of the Weyl group of } G}{\text{order of the Weyl group of P(i.e. of } C(\triangle'))}$; it is unipotent iff the number of fixed points is precisely 1.

8.6 Lemma. *If $H = T_{\mathbb{R}}^{\circ}$, there exists an automorphism τ of H and a Zariski dense subset H_{τ} of $\mathbb{R}$-regular elements in H such that $\forall h \in H$, h and $\tau(h)$ operate equivalently on $X_{\circ}$, i.e., there exists a diffeomorphism $\Phi_{\circ}$ of $X_{\circ}$ such that $h = \Phi_{\circ}^{-1}\tau(h)\Phi_{\circ}$.*

Proof. Let $A^1 = \{a; a \in A\ \alpha(a) > 1 \forall \alpha \in {}_{\mathbb{R}}\triangle\}$. Let K be a maximal compact subgroup of G_*. Recall $Z({}_{\mathbb{R}}T)_{\mathbb{R}} = L.A$. We can assume tht $K \supset L$. Let (1) be the projection of 1 in $X = G_*/K$. ◻

73 Let $p_\circ = \lim\limits_{n\to\infty} a^n(1)$, $a \in A^1$ and let P_* be the stabilizer of $p_\circ$.

$$P = P(\triangle') = P(\Phi).$$

Set V = tangent space to $X_\circ$ at $p_\circ$, then $V \approx \dot{G}_*/\dot{P}_*$. Let $g \in P_*$ and let $\hat{g}$ denote the operation of g on V. If $H \subset P_*$, $\hat{H} \subset C$; where C is a Cartan subgroup of $GL(V)$.

Let $W = \frac{N(C)}{Z(C)}$ be the Weyl group of C. For any element $\gamma \in \Gamma$ set $\gamma' = \theta(\gamma)$.

Given a reductive $\mathbb{R}$-regular element γ of Γ, there exists a $g \in G_*$ such that $g[\gamma]$ belongs to $H \cap LA'$. The element $\theta(\gamma)$ is also reductive $\mathbb{R}$-regular. Therefore $\exists g' \in G_*$ such that $g'[\gamma'] \in H \cap LA^1$.

Since

$$\begin{aligned}
&\varphi(\gamma p) = \theta(\gamma)\varphi(p)m, \text{ we can write}\\
&\theta(\gamma) = \varphi\gamma\varphi^{-1}(= \varphi[\gamma])\\
&g'[\gamma'] = g'[\varphi[\gamma]] = g'[\varphi[g^{-1}g[\gamma]]]\\
\therefore\quad &g'(\gamma') = g'\varphi g^{-1}[g[\gamma]]\\
&g'\hat{[}\gamma'] = \sigma^y g[\gamma\hat{]}(\sigma^y)^{-1}
\end{aligned}$$

where σ^γ is the differential of $g'\varphi g^{-1}$ at $p_\circ$.

Therefore there is an element τ^γ in W, the Weyl group of C such that

$$g'\widehat{[\gamma']} = \widehat{\tau\gamma(g[\gamma])}.$$

74 For any element $w \in W$, let H_w denote the subset of $H \cap LA' \cap G_*[\Gamma]$ on which the map $\gamma \to \tau^\gamma$ is constant. Since $H \cap LA' \cap G_*[\Gamma]$ is Zariski dense in H and W is finite, there exists a $\tau \in W$ such that H_τ is Zariski dense in H. Denoting Zariski closure by superscript $*$, we can write

$$H_\tau^* = H^*$$

since

$$\tau(H_\tau) \subset H$$
$$\therefore \quad \tau(H^*) = H^* \text{ and therefore } \tau(H) = H.$$

Thus τ induces an automorphism of H, and by definition, h and $\tau(h)$ operate equivalently on $x_\circ$ for all $h \in H_\tau$.

Proof of the Theorem 8.1.

Let $S_1 = \bigcup_{\substack{w \in W \\ H_w \text{ not Zariski dense}}} H_w$ and let $S = S_1^* \cup$ non $\mathbb{R}$-regular elements in H.

Then clearly
$$S^* \neq H^*.$$

Let τ be an automorphism of H given by the previous lemma. Then τ permutes the roots Φ^*, that is

$$\{\alpha(h); h \in H_\tau, \alpha \in \Phi^*\} = \{\alpha(\tau(h)); \alpha \in \Phi^*, h \in H_\tau\}$$

By the lemma 8.3 τ permutes Φ. Hence Tr $Adg[\gamma]$ = Tr $Ad(g'[\gamma'])$ **75**
that is, Tr $Ad\gamma$ = Tr $Ad\gamma'$ $\forall g \in \Gamma \cap G_*[H_\tau]$. It follows that Tr $Ad\gamma$ = Tr $Ad\gamma'$ for all $\gamma \in \Gamma \cap G_*[H - S_1]$.

Since G is without center we can identify it with AdG.

Given $\gamma \in \Gamma$ and $S \subset H$ with $S^* \neq H^*$ and n any positive integer, by Lemma 4.4. $\exists \gamma_\circ \in \Gamma \cap G[H - S]$ such that $\gamma_\circ, \gamma_\circ^2, \ldots \gamma_\circ^n$, $\gamma\gamma_\circ$, $\gamma\gamma_\circ^2, \ldots \gamma\gamma_\circ^n \in G[H - S]$.

Let $n = \dim G$. Then $\mathrm{Tr}(\gamma\gamma_\circ^m) = \mathrm{Tr}\ \theta(\gamma\gamma_\circ^m) = \mathrm{Tr}\ \theta(\gamma)\theta(\gamma_\circ)^m$ for $m = 1, \ldots n$. We can write $1 = c_1\gamma_\circ + c_2\gamma_\circ^2 + \cdots + c_n\gamma_\circ^n = f(\gamma_\circ)$ by setting the characteristic polynomial of $\gamma_\circ$ equal to zero.

Then

$$\begin{aligned}\mathrm{Tr}\ \gamma = \mathrm{Tr}\ \gamma f(\gamma_\circ) &= \sum_{m=0}^{n} \mathrm{Tr}(c_m \gamma\gamma_\circ^n) \\ &= \sum_{m=0}^{n} c_m\, \mathrm{Tr}\ \theta(\gamma)\theta(\gamma_\circ)^m\end{aligned}$$

$$= \text{Tr } \theta(\gamma) f(\theta(\gamma_\circ)).$$

But Tr $\gamma_\circ^m = \text{Tr } \theta(\gamma_\circ)^m$ for $m = 1, \ldots, n$ and thus $\gamma_\circ$ and $\theta(\gamma_\circ)$ have the same characteristic polynomial, by Newton's formulae.

Hence $f(\theta(\gamma_\circ)) = 1$ and Tr $\gamma = \text{Tr } \theta(\gamma)$ for all $\gamma \in \Gamma$.

Suppose $\sum\limits_{\gamma\in\Gamma} \mathbb{C}\gamma\gamma = 0)$. Then $0 = \text{Tr} \left(\sum\limits_{\gamma\in\Gamma} \mathbb{C}_\gamma \gamma \sum d_{\gamma^*}\gamma^* \right) \forall d_{\gamma*} \in \mathbb{C}$ $\forall \gamma^* \in \Gamma$. This will imply

$$\text{Tr} \left(\sum_{\gamma\in\Gamma} \mathbb{C}_\gamma \theta(\gamma) \cdot \sum_{\gamma^*\in\Gamma} d_{\gamma*}\theta(\gamma^*) \right) = 0$$

76 Let $\mathcal{E}$ denote the $\mathbb{C}$ linear span of Γ. Clearly $\mathcal{E}$ is an associative matrix algebra. By the density theorem (that the Zariski closure of Γ is G), the linear span of Γ s linear span of G_*. Thus Tr $\sum\limits_{\gamma\in\Gamma} \mathbb{C}_\gamma \theta(\gamma) e = 0$ for all $e \in \mathcal{E}$.

We can (and we will) assume that AdG is self adjoint. The we can assert

$$\text{Tr} \left(\sum C_\gamma \theta(\gamma)\right)^t \sum C_\gamma \theta(\gamma) = 0$$

This implies that ${}^t \sum C_\gamma \theta(\gamma) = 0$. Therefore θ induces a linear isomorphism of $\mathcal{E}$ onto $\mathcal{E}$, since $\sum C_\gamma \gamma = 0$ implies $\sum C_\gamma \theta(\gamma) = 0$. Clearly θ is an $\mathbb{R}$-algebra automorphism $\theta(\Gamma^*) \cap \mathcal{E}_\mathbb{R} = (\theta(\Gamma))^* \cap \mathcal{E}_\mathbb{R}$ implies that $G_* = G_\mathbb{R}^\circ = (G \cap \mathcal{E}_\mathbb{R})^\circ = \theta((G \cap \mathcal{E}_\mathbb{R})^\circ) = \theta(G_*)$, since Γ and $\theta(\Gamma)$ are Zariski dense in G.

Thus we have proved that θ extends to an automorphism of G_*.

Chapter 9

The Main Conjectures and the Main Theorem

Let G be a real analytic semi-simple group with no center and no compact factors, and let K be a maximal compact subgroup. Let $X = G/K$ and let Γ, Γ' be two discrete subgroups of G, isomorphic under an isomorphism $\theta : \Gamma \to \Gamma'$. We assume that G/Γ, G/Γ' have finite Haar measure. Let $\varphi : X \to X$ be a homeomorphism such that $\varphi(\gamma x) = \theta(\gamma)\varphi(x)$ to $\forall \gamma \in \Gamma$ and $x \in X$. Then 77

Conjecture 1. θ extends to an analytic automorphism of G provided G contains no factor locally isomorphic to $SL(2, \mathbb{R})$.

Conjecture 2. Let $X_\circ$ be the unique compact G-orbit of a Satake compactification of X. Then φ extends to a homeomorphism of $X \cup X_\circ$. Let $\varphi_\circ$ be the restriction to $X_\circ$ of the extension, then $\varphi_\circ G \varphi_\circ^{-1} = G$ as transformation of $X_\circ$, provided G has no factor locally isomorphic to $SL(2, \mathbb{R})$.

It is not difficult to see that Conjecture 2 implies Conjecture 1. Indeed we remark first that G operates faithfully on $X_\circ$, since G has no compact factors and no center. Since X is topologically dense in $X \cup X_\circ$ we have $\varphi_\circ(\gamma x) = \theta(\gamma)\varphi_\circ(x)$ for all $x \in X_\circ$ and all $\gamma \in \Gamma$; that is, $\theta(\gamma) = \varphi_\circ \gamma \varphi_\circ^{-1}$ as transformations of $X_\circ$. If $\varphi_\circ G \varphi_\circ^{-1} = G$, then $g \mapsto \varphi_\circ g \varphi_\circ^{-1}$ is a continuous automorphism of G with respect to the compact open topology of G as a transformation group of $X_\circ$. As is well-known; this implies that $g \mapsto \varphi_\circ g \varphi_\circ^{-1}$ is a continuous automorphism of the analytic group G and hence an analytic automorphism.

78 The following example shows that $SL(2, \mathbb{R})/\pm 1$ violates the conjecture.

9.1 Example. Let $G = SL(2, \mathbb{R})/\pm 1$, $K = SO(2, \mathbb{R})/\pm 1$. Then X is the upper half plane with G operating as linear fraction transformations $z \to \frac{az+b}{cz+d}$. Alternatively, we may identify X with the interior of the unit ball in the plane.

Let S and S' be two compact Riemann surfaces of genus > 1 which are diffeomorphic but not conformally equivalent. Let $\Gamma = \pi_1(S)$ and $\Gamma' = \pi_1(S')$ be the fundamental groups of S and S'. Let $\psi : S \to S'$ be a diffeomorphism, let $\theta : \Gamma \to \Gamma'$ be the induced isomorphism of fundamental groups, and let $\varphi : X \to X$ be the lift of ψ to the simply connected covering spaces of S and S'; by uniformization theory, the latter may be identified with X. Then $\varphi(\gamma x) = \theta(\gamma)\varphi(x)$ for all $\gamma \in \Gamma$, $x \in X$. As transformation groups on X we can therefore write $\Gamma' = \varphi\gamma\varphi^{-1}$. However $G \neq \varphi G \varphi^{-1}$ unless φ is a Mobius transformation of X.

Pursuing the example further, the map φ is a so-called quasiconformal map (cf. next sections for definitions and properties) and therefore induces a homeomorphism $\varphi_\circ$ of the boundary $X_\circ$ of the unit ball. Then $\varphi_\circ \Gamma' \varphi_\circ^{-1} = \Gamma'$ as transformations of $X_\circ$ since X is dense in $X \cup X_\circ$. However $G \neq \varphi_\circ G \varphi_\circ^{-1}$ unless $\varphi_\circ$ is a Moebius transformation of the circle $X_\circ$.

79 The following trivial example serves to illustrate that once θ is given $\varphi_\circ$ is uniquely determined by contrast with φ which is not unique; and that $\varphi G \varphi^{-1} = G$ is not necessary even when $\varphi_\circ G \varphi_\circ^{-1} = G$.

9.2 Example. Let $\Gamma = \Gamma'$, $\theta =$ Identity, ψ a homeomorphism which is the identity map except on some small neighbourhood of X/Γ. Then $\varphi G \varphi^{-1} \neq G$ since otherwise φ would have to be the identity map. However $\varphi_\circ$ is the identity map and in particular $\varphi_\circ G \varphi_\circ^{-1} = G$.

In these lectures we prove a slightly modified form of conjecture 2 for the group $G = 0(1, n)/\pm 1$ where $n > 2$. More precisely

9.3 Theorem. *Let $G = 0(1, n)/\pm 1$, $n > 2$, and let X be the associated Riemannian space. Let Γ, Γ' be discrete subgroups such that G/Γ and G/Γ' have finite Haar measure. Let $\varphi : X \to X$ be a homeomorphism and $\theta : \Gamma \to \Gamma'$ an isomorphism such that $\varphi(\gamma x) = \theta(\gamma)\varphi(x)$ for all $\gamma \in \Gamma$, $x \in X$. Assume that φ is quasi-conformal (cf. below for definition) then φ induces a diffeomorphism $\varphi_\circ$ of the boundary component $X_\circ$ of the Satake compactification of X and moreover $\varphi_\circ G \varphi_\circ^{-1} = G$.*

Note. The condition that φ be quasi-conformal is automatically fulfilled if G/Γ and G/Γ' are compact and φ is diffeomorphism.

The proof of this theorem is based on the theory of quasi conformal mappings cf. [17]. In the following section we present a summary of our proof.

Chapter 10

Quasi-conformal Mappings

Definition. Möbius n-space is the one point compactification of eucli- 80
dean n-space $\mathbb{R}^n$, it will be denoted by $\mathbb{R}^n \cup \{\infty\}$.

$GM(n)$ the Möbius group of Möbius n-space is the group of transformations generated by "inversion" in the sphere S^n

$$\eta_1^2 + \eta_2^2 + \cdots + \eta_{n+1}^2 = 1.$$

If we set $\eta_i = \frac{y_i}{y_i}(i = 1, \ldots, n+1)$ then S^n is realized as the projective variety $y_\circ^2 - y_1^2 - - - - - - y_{n+1}^2 = 0$, and one can prove that $GM(n)$ becomes identified with $0(1, n+1)/\pm 1([17]p.57)$.

10.1 Theorem. *The subgroup G' of $GM(n)$ which stabilizes the hemisphere $S_-(\eta_{i+1} < 0)$ is isomorphic to $GM(n-1)$ under the restriction homomorphism into its action on the equatorial $n-1$ sphere $\eta_{n+1} = 0$. Moreover G' operates transitively on S_- and keeps invariant a positive definite quadratic differential form dS^2. Under stereographic projection from $(0, 0, \ldots, 0, 1)$, S_- maps onto the unit ball $x_1^2 + \cdots + x_n^2 < 1$ and its invariant metric dS^2 upto a constant factor becomes $\frac{dx^2}{1-|x|^2}$, where dx is usual euclidean metric.* *(loc. cit. pp. 58-59)*

The unit ball $|x| < 1$ with metric $\frac{dx^2}{1-|x|^2}$ where dx is euclidean metric, is a Riemannian space called the hyperbolic n-space, the isotropy subgroup at a point is $0(n)$ (In this realization of hyperbolic space, the isometries of hyperbolic metric preserve euclidean angles).

Hence the spaces have constant curvature. **81**

We introduce following notations:

Let V, W be two Riemannian spaces and let $\varphi : V \to W$ be a homeomorphism.

Let

$$\begin{aligned} L_\varphi(p,r) &= \sup_{d(p,q)=r} d(\varphi(p),\varphi(q)) \\ l_\varphi(p,r) &= \inf_{d(p,q)=r} d(\varphi(p),\varphi(q)) \\ H_\varphi(p) &= \overline{\lim_{r\to 0}} \frac{L_\varphi(p,r)}{l_\varphi(p,r)} \\ I_\varphi(p) &= \overline{\lim_{r\to 0}} \frac{L_\varphi(p,r)}{r}. \\ J_\varphi(p) &= \overline{\lim_{r\to 0}} \frac{m(\varphi(T_r(p)))}{(mT_r(p))} \end{aligned}$$

m is the Hausdorff measure. *** and *** where for any subset E of V, $T_r(E)$ denotes the tubulor neighbourhood of E of radius r.

$$T_rE = \{v; v \in V \quad d(v,E) \leq r\}$$

Definitions. *** is said to be *quasi-conformal* iff there exists a constant B with $H\varphi(p) \leq B\ \forall p \in V$.

A quasi-conformal is said to be *k-quasi-conformal* iff $H_\varphi(p) \leq k$ for almost all $p \in V$.

The foregoing definition is not well-suited for proving some of the basic theorems concerning quasi-conformal mappings. The development below leads to an alternative definition of quasi-conformal mapping in terms of the modulus of a shell.

82 **Definitions.** *A shell* D in Möbius n-space $\mathbb{R}^n \cup \{\infty\}$ is an open connected set whose complement consists of two connected components $C_\circ$ and C_1. *A* shell not containing the point ∞ is called a *shell in* $\mathbb{R}^n$. The component C_1 of its complement which contains ∞ is the un *bounded* component and the other component $C_\circ$ will be referred to as the *bounded* component.

For a shell D in Möbius-n space, we define its *conformal capacity*

$$C(D) = \inf_u \int_D |\nabla u|^n dD$$

where u varies over C^1-functions with $u(c_\circ) = 0 u(c_1) = 1$ $C_\circ, C_1$ being connected components of the complement of D. We will call such a function u a *smooth admissible function*. It is easy to see that $C(D)$ is invariant under conformal mapping, since the integral $\int_D |\nabla u|^n dD$ is invariant.

Let C_{n-1} denote the area of the surface of the unit n-ball. Then we define

$$\text{mod } D = \left(\frac{C_{n-1}}{C(D)}\right)^{\frac{1}{n-1}}.$$

10.2 Example. If $D_{a,b} = m\{x, x \in \mathbb{R}^n a < |x| > b\}$ then

$$C(D_{a,b}) = C_{n-1}\left(\log\frac{b}{a}\right)^{-(n-1)} \quad \text{and} \quad \text{mod } D_{a,b} = \log\frac{b}{a}.$$

Proof. Let u be a smooth admissible function for $D_{a,b}$ then **83**

$$1 \leq \int \int_a^b |\nabla u| dr = \int_a^b |\nabla u| r^{\frac{n-1}{n}} r^{-\frac{n-1}{n}} dr.$$

□

By Hölder's inequality

$$1 \leq \int_a^b |\nabla u| dr < \left(\int_a^b |\nabla u|^n r^{n-1} dr\right)^{1/n} \left(\int_a^b r^{-1} dr\right)^{\frac{n-1}{n}}$$

raising to the power n, and integrating over all rays

$$c_{n-1} \leq \left(\int_D |\nabla u|^n dD\right)\left(\log\frac{b}{a}\right)^{(n-1)}$$

$$\therefore \quad C(D_{a,b}) \geq c_{n-1}\left(\log\frac{b}{a}\right)^{-(n-1)}.$$

On the otherhand by taking smooth admissible approximations of the function

$$u = \begin{cases} 0 & |x| \leq a \\ \frac{\log x - \log a}{\log b - \log a} & a \leq |x| \leq b \\ 1 & b \leq |x| \end{cases}$$

we get

$$\begin{aligned} C(D_{a,b}) \leq \int |\nabla u|^n dD &= C_{n-1}\left(\log\frac{b}{a}\right)^{-n}\int_a^b \left(\frac{1}{r}\right)^n r^{n-1}dr \\ &= C_{n-1}\left(\log\frac{b}{a}\right)^{-(n-1)} \\ \therefore \quad C(D_{a,b}) &= C_{n-1}\left(\log\frac{b}{a}\right)^{-(n-1)}. \end{aligned}$$

84 Therefore

$$\text{mod } (D_{a,b}) = \log\frac{b}{a}.$$

Definition. Let D, D' be two shells with $C'_\circ \supset C_\circ$ and $C'_1 \supset C_1$ then we say "D' *separates the boundary of* D". Clearly in this case $C(D') \geq C(D)$ and mod $D' \leq$ mod D.

10.3 Lemma. *Let $S_r = \{x | x \in \mathbb{R}^n, |x| = r\}$ and let u be a C^1 function on S_r then there exists a constant A depending only on n such that*

$$(CSC_{S_r} u)^n \leq A.r \int_{S_r} |\nabla u|^n dS_r$$

(For a proof see p.69 [17]).

10.4 Lemma (Loewner)**.** *Let D be a shell in Möbius n-space and let $C_\circ$, C_1 denote the connected components of the complement of D, then $C(D) > 0$ if neither $C_\circ$ nor C_1 consists of a single point.*

Proof. Choose a point p in $\mathbb{R}^n$ such that S_r, the sphere with center at p and radius r meets $C_\circ$ and C_1 for all r with $0 < r_1 < r < r_2$ then

$$\int_D |\nabla u|^n dD = \int_n |\nabla u|^n dx \geq \int_{D_{r_1,r_2}} |\nabla u|^n dx$$

$$= \int_{r_1}^{r_2} \int_{S_r} |\nabla u|^n d\sigma dr \text{ where } d\sigma \text{ is } n-1 \text{ measure on } S_r.$$

$\square$

By the previous lemma 85

$$\int_{S_r} |\nabla u|^n d\sigma \geq A^{-1} r^{-1} (CSC_{S_r} u)^n = A^{-1} r^{-1}.$$

Thus $\int_D |\nabla u|^n dD \ A^{-1} \int_{r_1}^{r_2} \frac{dr}{r} = A^{-}1 \log \frac{r_2}{r_1}$ for all smooth admissible functions u. Hence $C(D) \geq A^{-1} \log \frac{r_2}{r_1} > 0$.

Definition. *A* continuous function f on the interval $0 \leq x \leq b$ is called *absolutely continuous* if its derivative $\frac{df}{dx}$ exists almost everywhere and is integrable and $\int_{x_\circ}^{x_1} \frac{df}{dx} dx = f(x_1) - f(x_\circ)$ for all $a \leq x_\circ, x_1 \leq b$.

A function u on an open subset D of $\mathbb{R}^n$ is called *ACL in D*, if in any closed ball lying in D it is absolutely continuous on almost all lines in the ball parallel to the coordinate axes.

Notations.

$$E_+ = \{x; x \in \mathbb{R}^n \quad x_n > 0\}$$
$$S_r^+ = S_r \cap E_+.$$

10.5 Lemma. *If u is an ACL function on E^+ then*

$$\int_a^b \left(\underset{S_r^+}{OSC} u\right)^n \frac{dr}{r} \leq 2A \int_{E_+} |\nabla u|^n dx.$$

This is a slight generalization of Lemma 10.3, for a proof see pp. 72-73 [17].

86

10.6 Lemma.

$$I^n_\varphi(p) \leq (H_\varphi(p))^n J_\varphi(p)$$

and
$$I^n_{\varphi^{-1}}(\varphi(p)) \quad (H_\varphi(p))^n J_{\varphi^{-1}}(\varphi(p)).$$

Proof. The first inequality comes from

$$\left(\frac{L_\varphi(p,r)}{r}\right)^n = \left(\frac{L_\varphi(p,r)}{l_\varphi(p,r)}\right)^n \left(\frac{l_\varphi(p,r)}{r}\right)^n .$$

The proof of the second inequality is similar. □

Remark. It can be proved that if φ is differentiable at p then

$$I^n_\varphi(p) \leq (H_\varphi(p))^{n-1} J_\varphi(p).$$

10.7 Lemma. *Let φ be a quasi-conformal mapping then φ exists almost everywhere.*

Proof. By the previous lemma

$$I^n_\varphi(p) \leq (H_\varphi(p))^n J_\varphi(p).$$

By hypothesis $H_\varphi(p) < B\ \forall p$. By Lebesgue's theorem (Saks [19] p. 115)

$$\begin{aligned} & J_\varphi(p) < \infty \text{ a.e.} \\ \therefore\quad & I_\varphi(p) < \infty \text{ a.e.} \\ \text{i.e.}\quad & \overline{\lim_{q\to p}} \frac{\varphi(q)-\varphi(p)}{|q-p|} < \infty \text{ a.e.,} \end{aligned}$$

By the Radamacher-Stepnoff theorem ([19] pp. 310-312) $\dot{\varphi}$ exists a.e., □

87 **10.8 Lemma.** *Let D, D' be open in $\mathbb{R}^n$ and let $\varphi : D \to D'$ be homeomorphism of D into D'. Let p be a hyperplane in $\mathbb{R}^n$, if $H_\varphi(p) < k$ for $p \in D - p$, then φ in ACL on D and φ^{-1} is ACL on $\varphi^{-1}(D)$. (See [17] for a proof.)*

Definition. Given a shell D; a continuous function u on $\bar{D}$, ACL in D is said to be *admissible* if $u(L_\circ \cap \bar{D}) = 0$ and $u(C_1 \cap \bar{D}) = 1$, $C_\circ, C_1$ being connected components of the complement of D.

10.9 Lemma.

$$C(D) = \inf_{u \text{ admissible}} \int_D |\nabla u|^n dD$$

(See [17] pp. 64 for a proof).

10.10 Lemma. *Let $\varphi : D \to D'$ be a homeomorphism of shells, if is ACL and $I^n\varphi \leq k^{n-1} J_\varphi$ almost everywhere, then* $\mod \varphi(D) \leq k \mod D$.

Proof. Given u an admissible function on D, set $u' = u \circ \varphi^{-1}$ then $u \leftrightarrow u'$ is bijective correspondence between admissible functions on D and D'. ◻

$$\begin{aligned}
\nabla(u)(p) &= \overline{\lim_{q\to p}} \frac{|u(q) - u(p)|}{|q-p|} \\
&= \frac{u(q) - u(p)}{|\varphi(q) - \varphi(p)|} \cdot \frac{|\varphi(q) - \varphi(p)|}{|q-p|} \\
&= |\nabla(u')\varphi(p)| I_\varphi(p).
\end{aligned}$$

$$\begin{aligned}
\therefore \quad C(D) &\int |(\nabla(u'))(\varphi(p))|^n k^{n-1} J_\varphi(p). \\
&= k^{n-1} \int_{D'} |\nabla u'|^n dD \\
\therefore \quad & C(D) \leq k^{n-1} C(D') \\
\therefore \quad & \mod D' \leq k \mod D.
\end{aligned}$$

We now define the *spherical symmetrization* of a shell for the purpose of obtaining a rough quantitative estimate for the modulus of a shell. **88**

Let L denote the ray $\{(t, 0, \ldots 0) - \infty < t \leq 0\}$ in $\mathbb{R}^n$, and let E be a set, open or closed, in $\mathbb{R}^n$. For each sphere $S_r = \{x, x \in \mathbb{R}^n, |x| = r\}$ place along S_ra spherical cap (of dimension $n-1$) with center at $S_r \cap E$. Take the cap open if E is open closed if E is closed, and equal to S_r if $S_r \subset E$. The resulting set is denoted by E^*. Clearly E^* is open resp. closed, resp. connected) if E is open (resp. closed, resp. connected).

Definition. Let D be a shell in $\mathbb{R}^n$. The spherical symmetrization of D is the set $D^\circ = (D \cup C_\circ)^* - C_\circ^*$.

where $C_\circ$ is the bounded component of D. It is clear that $D_\circ$ is a shell.

10.11 Theorem. $C(D) \geq C(D^\circ)$

The proof of this theorem makes use of the isoperimetric inequalities for both euclidean and spherical space (cf. Mostow, loc. cit, p.87). Intuitively the result is plausible because the spherical symmetrization
89 of D is a "smoothing" of D and hence admissible function for D° need to be "twist less", accordingly $C(D^\circ) \leq C(D)$.

In the proof of next lemma, we will estimate the modules of a shell by comparing it with a special shell which generalizes a special slit plane domain considered by Teichmuller.

Definition. The *Teichmuller shell* $D_+(b)$ is the shell in $\mathbb{R}^n$ whose complementary components consist of the segment $-1 \leq x_1 \leq 0$, $x_2 = \cdots = x_n = 0$ and the ray $b \leq x_1 < \infty$, $x_2 = x_3 = \cdots - x_n = 0$ where $b > 0$.

10.12 Lemma. *$\varphi : R \to R'$ be a homeomorphisms of domains in $\mathbb{R}^n$, assume* $\mod \varphi(D) \leq k \mod D$*, then $H_\varphi < C^k$, where C depends only on n.*

Proof. For $p \in R$, we consider the spherical shell $D_{l_\varphi(p,r), L_\varphi(p,r)}$ centered at $\varphi(p)$. Let $D = \varphi^{-1}(D_{l_\varphi(b,r), L_\varphi(p,r)})$ then $\log \frac{L_\varphi(p,r)}{l_\varphi(p,r)} = \mod D_{l(p,r), L(p,r)} \leq k \mod D \leq k \mod D^\circ$ (by 10.11) [D° is spherical symmetrization of D]

$$\leq k \mod D_\tau(1).$$

Since D° *separates the boundaries of* $D_\tau(1)$. $\square$

Set $C = \mod D_\tau(1)$.
Then $\frac{L_\varphi(p,r)}{l_\varphi(p,r)} = C^k$

$$\therefore \quad H_\varphi(p) \le C^k \forall p \in R.$$

Note. The idea of comparing $\mod D$ with $\mod D_\tau(1)$ is due to A. Mori cf. his posthumous paper in the Transaction of the AKS V. 84 (1957) pp. 56-77.

Putting together Lemmas 10.8, 10.10 and 10.12, we can now assert **90**

10.13 Theorem. *Let $\varphi : E \to E'$ be a homeomorphism of domains in $\mathbb{R}^n$ ^{n}. Then φ is quasi-conformal iff*

(1) φ is ACL in E.

(2) For all shells $D \subset E$, $k^{-1} \mod D \le \mod \varphi(D) \le k \mod D$, for some constant k.

We now prove two theorems that are of central importance for our main theorem.

10.14 Theorem. *Let φ be a quasi-conformal mapping of an open ball in $\mathbb{R}^n$ onto itself. Then φ extends to a homeomorphism of the closed ball.*

Proof. Mapping the domain of φ onto the upper half space $X = \{(x_1 \ldots x_n), x_n > 0\}$ via a Möbius transformation, the theorem is seen to be equivalent to the assertion a quasi conformal mapping $\varphi : X \to Y = \{y : |y| < 1\}$ extends to a continuous mapping at any point x of the boundary of X. for convenience, we take $x = 0$. □

The proof is by contradiction. If $\lim\limits_{p \to 0} \varphi(p)$ $(p \in X)$ does not exist, we can find two sequences $\{p_k\}$ and $\{q_k\}$ in X approaching 0 with $\lim\limits_{k\to\infty} \varphi(p_k) = p'$, $\lim\limits_{k\to\infty} \varphi(q_k) = q'$ and $|q' - p'| = a > 0$. Denoting by $\overline{pq}$ the line segment joining two points p and q, we select points $p'_\circ$ and $q'_\circ$ in Y such that $d(\overline{p'_\circ p'_k}, \overline{q'_\circ q_k}) > a$ for all large k, where $p'_k = (p_k), q'_k = (q_k)$. Set $p_\circ = \varphi^{-1}(p'_\circ)$, $q_\circ = \varphi^{-1}(q'_\circ)$. Then for $\sup(|p_k|, |q_k|) < r < \inf(p_\circ, q_\circ)$, the hemisphere $S^+_r = \{x; |x| = r, x_n > 0$ meets the curves **91**

$\varphi^{-1}\overline{(p'_\circ p'_k)}$ and $\varphi^{-1}\overline{(q'_\circ q'_k)}$. For each such r at least one of the coordinate functions of $\varphi(x) = (\varphi_1(x), \ldots \varphi_n(x))$ satisfies

$$\underset{S_r^+}{OSC}\varphi_i > a/\sqrt{n}.$$

Hence

$$\sum_i \int_\circ^\infty \left(\underset{S_r^+}{OSC} + \varphi_i\right)^n \frac{dr}{r} = \infty.$$

By Lemma (10.8), φ_i is ACL in X. Applying Lemma 10.5, we get for each $i = 1, \ldots, n$

$$\begin{aligned}\int_\circ^\infty \left(\underset{S_r^+}{OSC}\varphi_i\right)^n \frac{dr}{r} &\leq 2A \int_X |\nabla\varphi_i|^n dx \leq 2A \int_X I_\varphi^n dx \\ &\leq 2A \int_X K^{n-1} J_\varphi dx \leq 2AK^{n-1} \int_Y dy.\end{aligned}$$

This yields a contradiction.

10.15 Theorem. *Let φ be a k-quasi conformal mapping of an open ball B^n in $\mathbb{R}^n$, onto itself, $n \geq 2$, and let $\varphi_\circ$ denote the boundary homeomorphism induced by φ. Then $\varphi_\circ$ is C^k-quasi conformal where $c =$* mod $D_\tau(1)$ *depends only on n.*

Proof. By mapping B^n onto upper half space E_+ via Möbius transformation we can replace B^n by E_+ in the theorem. By previous theorem φ
92 extends to the boundary. Let φ also denote its extension by symmetry to $\mathbb{R}^n$. φ is k-quasi conformal in $\mathbb{R}^n$-hyperplane $x_n = 0$. Hence φ is ACL in $\mathbb{R}^n$ by Lemma 10.8. ◻

We have $H_\varphi(p) \leq k$ a.e. in $\mathbb{R}^n$.

By Lemma 10.6 and the remark following it

$$(I_\varphi(p))^n \leq k^{n-1} J_\varphi(p) \quad a.e.$$

Therefore for any shell D in $\mathbb{R}^n$ $\bmod \varphi(D) \leq k \bmod D$ by Lemma 10.10. Applying Lemma 10.12, we get that $\varphi_\circ$ is C^k-quasi conformal.

The following two Lemmas round out prerequisites for our main theorem

10.16 Lemma. *Let $\varphi : S^n \to S^n$ be a 1-quasi conformal map then φ is a Möbius transformation if $n > 2$ (See [17] pp. 101-102.)*

10.17 Lemma. *Let φ be a quasi conformal mapping of a domain of $\mathbb{R}^n$ into $\mathbb{R}^n$, $n > 1$. Then $m(\varphi(E)) = \int_E J_\varphi dx$ for any measurable set E in the domain of φ. (cf. loc. cit. p. 94).*

Now we prove theorem 9.3.

Theorem. *Let $G = 0(1, n)/\pm 1$, $n > 2$ and let X be the associated symmetric Riemannian space. Let Γ, Γ' be discrete subgroups such that G/Γ and G/Γ' have finite Haar measure. Let $\varphi : X \to X$ be a homeomorphism and $\theta : \Gamma \to \Gamma'$ an isomorphism such that $\varphi(\gamma x) = \theta(\gamma)\varphi(x)$ for all $\gamma \in \Gamma$, $x \in X$. Assume that φ is quasi-conformal. Then φ induces* 93
a diffeomorphism $\varphi_\circ$ of the boundary component $X_\circ$ of the Satake compactification of X and moreover $\varphi_\circ G \varphi_\circ^{-1} = G$ as transformations of $X_\circ$.

Proof. The symmetric space X is the hyperbolic n-space which we identify with the open unit ball $B^n : |x| < 1$ in $\mathbb{R}^n$ with metric $ds_H^2 = \frac{|dx|^2}{1-|x|^2}$. the Satake compactification of X then can be identified with the closed unit ball and $X_\circ$ is its bounding sphere S^{n-1}. □

Quasi-conformality of φ with respect to dS_H implies that φ is quasi-conformal with respect to $|dx|$. so in view of Theorem 10.14, φ extends to a homeomorphism of the closed ball. Let $\varphi_\circ$ be the restriction of this extension to the boundary $X_\circ = S^{n-1}$. By Theorem 10.15 and Lemma 10.7, $\varphi_\circ$ is almost everywhere differentiable. Furthermore since X is dense in $X \cup X_\circ$, $\varphi_\circ(\gamma x) = \theta(\gamma)\varphi_\circ(x), \forall \gamma \in \Gamma$ and $x \in X_\circ$. Also note that G which is the full group of isometriese of X acts canonically on $X_\circ$ and conversely from the identification of $GM(n-1)$ with G (cf. [17] p. 57 and p. 98), it follows that each Möbius transformation of S^{n-1} extends to a unique isometry of X. We replace $X_\circ$ by $\mathbb{R}^{n-1} \cup \{\infty\}$ via stereographic projection. Let ψ denote the homeomorphism of $\mathbb{R}^{n-1} \cup \{\infty\}$ onto itself induced by $\varphi_\circ$. Let A be the 1-parameter subgroup of G corresponding to the 1-parameter subgroup of Möbius transformations of S^{n-1} obtained from the homotheties $x \mapsto \lambda x$ (with $\lambda \in \mathbb{R}^+$, $x \in \mathbb{R}^{n-1}$) and $\infty \mapsto \infty$ of $\mathbb{R}^{n-1} \cup \{\infty\}$.

Let p be a point at which the differential ψ exists; we can assume
94 that $p = 0$ for convenience. We identify the tangent space to $\mathbb{R}^{n-1}$ at 0 with $\mathbb{R}^{n-1}$ in the usual way.

Define

$$\begin{aligned} &f : G \to \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}) \text{ by} \\ &f(g) = (\psi g)(0) \end{aligned}$$

Set $F(g) = (t_f(g) f(g))(\det {}^t f(g) f(g))^{-1/m}$ whose $m = n-1$. Since a linear map L is conformal iff

$$\frac{< L(x), L(y) >}{\|L(x)\| \cdot \|L(y)\|} = \frac{< x, y >}{\|x\| \cdot \|y\|}$$

For any two orthogonal unit vectors x, y we deduce from $< x+y, x-y >= 0$ that $0 =< L(x-y), L(x+y) >= \|L(x)\|^2 - \|L(y)\|^2$. Thus L maps the unit ball into a ball and

$${}^tLL = (\det t_{LL})^{1/m} \cdot Id \text{ where } m = \text{dimension of the vector space.}$$

Thus if L is conformal we have $({}^tLL)(\det {}^tLL)^{-1/m} = Id$. Moreover L is K-quasi-conformal iff the ratio of largest to the smallest eigenvalue of ${}^tLL$ is K^2.

From the above it follows that $f(g)$ is a conformal mapping of the tangent space at 0 iff $F(g) =$ identity. One can check that $F(ga) = F(g)$, $\forall a \in A$ and $F(\gamma g) = F(g)$ for $\gamma \in \Gamma$. Moreover F is a measurable mapping of G into $\mathrm{Hom}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1})$. It has bounded (by K^2 for some K) entries almost everywhere since ψ is k-quasi-conformal for some K. Therefore F gives rise to an element of $\mathscr{L}^2(G/\Gamma, \mathrm{Hom}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}))$; which we again denote by F. For an element $\wedge \in \mathscr{L}^2(G/\Gamma, \mathrm{Hom}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}))$ let norm $\| \wedge \|^2 = \int_{G/\Gamma} \mathrm{Tr}\,({}^t \wedge (g) \cdot \wedge(g)) d\mu$.

95 G operates on $\mathscr{L}^2(G/\Gamma, \mathrm{Hom}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}))$ via $(Z.f)(g) = f(gz)$ unitarily and we have $A.F = F$.

Hence by Lemma 5.2,

$G.F = F$ i.e. F is constant almost everywhere. In particular

$$F(gk) = F(g), \forall k \in 0(n-1)$$

i.e., the group of rotations about 0. ${}^t k = k^{-1}$ implies by the special choice of F that

$$F(g) = F(gk) = k^{-1}G(g)k.$$

Since $F(g)$ commutes with $0(n-1)$, we conclude

$$F(g) = \text{const. } Id.$$

Since the matrix $F(g)$ is positive definite and of determinant 1, the constant must equal 1.

Therefore ψ is 1-quasi-conformal and therefore ψ is Möbius transformation by Lemma 10.16.

In particular $\varphi_\circ G \varphi_\circ^{-1} = G$.

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