Lectures on

Introduction to Algebraic Topology

By

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Notes by

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Preface

These are notes of a part of lectures which I gave at the Tata Institute of Fundamental Research in 1966. They were intented as a first introduction to algebraic Topology.

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Contents

1 Definition and general properties of the fundamental group

We call a continuous map simply a map. We denote the real line by $\mathbb{R}$ **1**
and the unit interval by I:

$$I = \{t | t \in \mathbb{R}, 0 \leq t \leq 1\}.$$

We consider maps of I into a topological space X and say that a map f_1 is equivalent to a map f_2 if there exists an automorphism φ of I fixing end points such that $f_1 = f_2 \circ \phi$. This is an equivalence relation in the set of all maps of I into X. An equivalence class under this relation is called a *path* in X. We say that a map f *defines a path* W if f belongs to the equivalence class W. Clearly any two maps defining the same path map 0 onto the same point and also 1. The images of 0 and 1 are called the *initial* and the terminal point of the path. We say that a path connects a point x to a point y if x and y are the initial and terminal points of the path.

Suppose that a map f defines a path W connecting x_0 to x_1. Then the map

$$f^{-1}(t) = f(1-t), \quad 0 \leq t \leq 1$$

defines a path W^{-1} which can be seen to be dependent only on W. This path connecting x_1 to x_0 is called the path W described in the opposite sense or the reversed path of W.

If C is a Jordan arc, i.e. a set homeomorphic to I, there are two paths defines by homeomorphism of I onto C, each reversed to the other, and corresponding to the two orientations of C.

The product of paths. **2**

Let f_1 and f_2 be two maps of I into X such that $f_1(1) = f_2(0)$. Then we can define a new map $f_1 f_2$ by setting

$$f_1 f_2(t) = \begin{cases} f_1(2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ f_2(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

If g_1 and g_2 are equivalent to f_1 and f_2 respectively and the equivalences are give by the automorphisms φ_1 and φ_2 of I;

$$f_1 = g_1 \circ \varphi_1,\ f_2 = g_2 \circ \varphi_2$$

then the automorphism

$$\varphi(t) = \begin{cases} \frac{1}{2}\varphi_1(2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ \frac{1}{2} + \frac{1}{2}\varphi_2(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

defines an equivalence between $f_1 f_2$ and $g_1 g_2$:

$$f_1 f_2 = (g_1 \circ \varphi_1)(g_2 \circ \varphi_2) = g_1 g_2 \circ \varphi.$$

Hence the path defined by $f_1 f_2$ depends only on the paths W_1 and W_2 defined by f_1 and f_2. This path is called the product of the paths W_1 and W_2 and is denoted by $W_1 W_2$. This product is associative. For suppose that W_1, W_2 and W_3 are paths defined by the maps f_1, f_2 and f_3 such that

$$f_1(1) = f_2(0) \text{ and } f_2(1) = f_3(0).$$

Then, if φ is the automorphism of I taking the points $0, \dfrac{1}{4}, \dfrac{1}{2}$ and 1 onto
3 the points $0, \dfrac{1}{2}, \dfrac{3}{4}$ and 1 respectively and which is linear on the intervals $[0, \dfrac{1}{4}]$, $[\dfrac{1}{4}, \dfrac{1}{2}]$ and $[\dfrac{1}{2}, 1]$, we have

$$f_1(f_2 f_3) \circ \varphi = (f_1 f_2) f_3$$

and hence $W_1(W_2 W_3) = (W_1 W_2) W_3$.

It is clear that each subdivision of the unit interval gives rise to a representation of a given path as a product of paths and by passing to a sufficiently fine subdivision of I we can represent that given path as the product of paths which are as small as we wish.

Homotopy of paths. Two maps f and g of I into X are *homotopic with fixed end points* if there exists a continuous family of maps $f_t : I \to X (0 \leq t \leq 1)$ such that $f_0 = f$, $f_1 = g$ and $f_t(a) = f(0)$, $f_t(1) = f(1)$. The family f_t is said to be continuous if the map $F : I \times I \to X$, defined by $F(s,t) = f_t(s)$, is continuous. We then say that f_t, or F, is a *homotopy form f to g*. This is an equivalence relation, for it is

(i) reflexive : the family $f_t = f$ defines a homotopy from f to f.

(ii) symmetric : if f_t defines homotopy from f to g then f_{1-t} defines homotopy from g to f.

(iii) transitive : if f_t and g_t define homotopies from f to g, and g to h respectively then the family

$$h_t = \begin{cases} f_{2t} & 0 \le t \le \frac{1}{2}, \\ g_{2t-1} & \frac{1}{2} \le t \le 1 \end{cases}$$

defines a homotopy from f to h. This relation is denoted by $\approx$.

Every map $\varphi : I \to I$ fixing end points is homotopic to the identity, **4**
for $\varphi_t(s) = ts+(1-t)\varphi(s)$ is such a homotopy. If $f = g\circ\varphi$, then $f_t = g\circ\varphi_t$ is a homotopy from g to f. Hence, if two maps define the same path, they are homotopic (with fixed end pints). Hence, all maps defining the same path w belong to the same class, which will be denoted by $[w]$, and we get an equivalence relation in the set of paths in X, which we still call homotopy and denote by $\approx$.

It is clear that if two paths are homotopic, then they have the same initial point and the same terminal point. If the product w_1w_2 is defined, i.e. if the terminal point of w_1 is the initial point of w_2, one sees that $[w_1w_2]$ depends only on $[w_1]$ and $[w_2]$. We then call $[w_1w_2]$ *the product of* $[w_1]$ and $[w_2]$ and we write $[w_1][w_2] = [w_1w_2]$.

We denote the set of homotopy classes of paths in X having the same point x_0 as initial and terminal point by $\pi(X, x_0)$. In this set the operation of forming products is defined for every pair of elements. Now we prove that this set form a group with respect to this operation. We have already proved that the operation is associative and if remains to prove the existence of a unit element and inverse.

Let e be the path reduced to x_0, defined by the constant map $e(s) = x_0$, and let w be any path closed at x_0, defined by the map f. Then

$$f_t(s) = \begin{cases} f(\frac{2s}{1+t}) & \text{for } 0 \le s \le \frac{1+t}{2} \\ e(s) = x_0 & \text{for} \frac{1+t}{2} \le s \le t \end{cases}$$

is a homotopy from fe to f. There is a similar homotopy from ef to f. **5**

Hence $[w][e] = [e][w] = [w]$ and $[e]$ is a unit element.

Now, for any path w with initial points x_0, we have $[ww^{-1}] = [e]$. For, if w is defined by f,

$$g_t(s) = \begin{cases} f(2st) & \text{for } 0 \leq s \leq \frac{1}{2}, \\ f((2-2s)t) & \text{for } \frac{1}{2} \leq s \leq 1 \end{cases}$$

is a homotopy from e to ff^{-1}, hence $[ww^{-1}] = [e]$. If w is closed at x_0, we have also $[w^{-1}w] = [e]$, so that $[w^{-1}][w] = [w][w^{-1}] = [e]$ and $[w^{-1}]$ is an inverse $[w]^{-1}$ to $[w]$.

The group $\pi(X, x_0)$ is called the *fundamental group of X based at* x_0.

Suppose that x_0 and x_1 are points of X connected by the path, L, and consider the map $J_L : \pi(X, x_0) \to \pi(X, x_1)$ defined by $J_L[w] = [L^{-1}wL]$. It is an homomorphism, for

$$\begin{aligned} J_L[w_1]J_L[w_2] &= [L^{-1}w_1LL^{-1}w_2L] = [L^{-1}w_1][LL^{-1}][w_2L] \\ &= [L^{-1}w_1w_2L] = J_L[w_1][w_2]. \end{aligned}$$

Further, $J_{L^{-1}} : \pi(X, x_1) \to \pi(X, x_0)$ is an inverse to J_L, so that J_L *is an isomorphism*. It is clear that it depends only on L.

If L' is another path connecting x_0 to x_1, we have $[L'] = [L'L^{-1}L] =$
6 $[CL]$ with $C = L'L^{-1}$ closed at $J_{L'}[w] = [L^{-1}C^{-1}wCL] = J_L \circ J_C[w]$ where J_C is an inner automorphism of $\pi(X, x)$. Hence $J_{L'}$ and J_L differ only by an inner automorphism.

Now suppose that the space X is arcwise connected. Then it follows that $\pi(X, x)$ for various x in X are isomorphic to each other and hence determine a group $\pi(X)$, called *the fundamental group of X*. But there are in general no canonical isomorphisms between the various $\pi(X, x)$, these isomorphisms being determined up to an inner automorphism. In case $\pi(X)$ us abelian, they are consequently determined in a canonical manner.

A space is called simply connected if it is connected and its fundamental group is trivial.

A convex set in the euclidean space is simply connected: A subset E of $\mathbb{R}^n$ is convex if the line segment connecting any two of its points is in E:

$$x \in E, y \in E \text{ implies } (1-t)x + ty \in E; 0 \leq t \leq 1.$$

Thereby it follows that any two paths connecting the same points are homotopic, and the set is simply connected.

The simplest example of a space which is not simply connected is the circle. We shall prove this fact later.

Now, suppose X and Y are two spaces and $\mu : X \to Y$ is a map. If
w is a path in X defined by f, the $\mu \circ f$ defines a path in Y, which is
called *the image of* w *by* μ, and denoted by $\mu(w)$ or μw. It is clear that 7
$\mu(w_1 w_2) = \mu(w_1)\mu(w_2)$ and if $w \approx w^1$, $\mu(w) \approx \mu(w')$. There follows that,
if $y_0 = \mu(x_0)$, we have a homomorphism

$$\mu_* : \pi(X, x_0) \to \pi(Y, y_0)$$

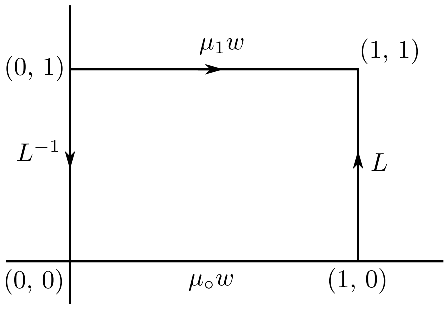
defined by $\mu_*[w] = [\mu w]$. We call μ_* the homomorphism induced by μ.

If μ_0 and μ_1 are maps of X into Y, a homotopy from μ_0 to μ_1 is a continuous family of maps $\mu_t : X \to Y (0 \leq t \leq 1)$. That means that $M(x, t) = \mu_t(x)$ is a continuous map of $X \times I$ into Y.

Theorem 1.1. *If $\mu_t : X \to Y$ is a homotopy from μ_0 to μ_1, and if L is the path connecting $y_0 = \mu_0(x_0)$ to $y_1 = \mu_1(x_0)$ in Y, defined by $\mu_t(x_0)$, then $\mu_{1*} = J_L \circ \mu_{0*}$.*

Proof. Suppose w is a path closed at x_0, define by the map $f : I \to X$. Then $\mu_0 w$ is the image of $I \times 0 = (0 \leq s \leq 1, t = 0)$ by the map $M : I \times I \to Y$, where $M(s, t) = \mu_t \circ f(s)$. The image of the other sides of the square $I \times I = (0 \leq s, t \leq 1)$ are L, $\mu_1 w$ and L^{-1}. As the square is simply connected (as a convex set), $I \times 1$ is homotopic to the path with same end points made of the three other sides, their images are also homotopic, i.e.

$$\mu_1 w \approx L^{-1}(\mu_0 w)L \quad \text{and} \quad \mu_{1*}[w] = J_L \mu_{o*}[w]$$



□

8 **Corollary 1.** *If a map $\mu : X \to X$ is homotopic to the identity map of X onto itself then μ_* is an isomorphism. Further if the homotopic leaves the base point x_0 fixed then the isomorphism coincides with the identity isomorphism since in this case L reduces to x_0.*

A map $\mu : X \to Y$ is called a *homotopy equivalence* if there exists a map $\lambda : Y \to X$ such that

$$\mu \circ \lambda \approx \mathrm{id}_Y \text{ and } \lambda \circ \mu \approx \mathrm{id}_X .$$

A subspace Y of X is called a deformation retract of X if the inclusion map of Y into X is a homotopy equivalence.

With these definitions we have the following:

Corollary 2. *If $\mu : X \to Y$ is a homotopy equivalence and X is arcwise connected then*

$$\pi(X) \simeq \pi(Y).$$

Corollary 3. *If $Y \subset X$ is a deformation retract of X then*

$$\pi(X) \simeq \pi(Y).$$

Theorem 1.2. *Suppose that X and Y are arcwise connected spaces and x_0 is in X and y_0 is in Y. Then*

$$\pi(X \times Y, (x_0, y_0)) \simeq \pi(X, x_0) \times \pi(Y, y_0).$$

in a canonical manner.

Proof. We denote the projection of $X \times Y$ onto X by p_1 and the other projection by p_2:

$$p_1(x, y) = X, \quad p_2(x, y) = y$$

and thus have homomorphisms P_{1*} and P_{2*}. Hence we have a homo- 9
morphism

$$\pi(X \times Y, (x_0, y_0)) \xrightarrow{(P_{1*}, P_{2*})} \pi(X, x_0) \times \pi(Y, y_0).$$

Suppose that w_1 and w_2 are closed paths in X and Y around x_0 and y_0 respectively defined by f and g. Then denoting the path defined by the map

$$t \longmapsto (f(t), g(t))$$

by w we have

$$(P_{1*}, P_{2*})(W) = (W_1, W_2)$$

and hence (P_{1*}, P_{2*}) is surjective.

Suppose that $W \in \pi(X \times Y, (x_0, y_0))$ is such that $P_{1*}W$ and $P_{2*}W$ are unit elements and the homotopies are given by maps F_1 and F_2 respectively. Then the map

$$(s, t) \longmapsto (F_1(s, t), F_2(s, t))$$

gives a homotopy between W and the constant path at (x_0, y_0) of $X \times Y$. Hence (P_{1*}, P_{2*}) is injective. □

2 Free products of groups and their quotients

Suppose that $\{G_i\}_{i \in I}$ is a family of groups and that the elements of G_i 10
for different i are distinct. We denote the union of the G_i by E and associated $W(E)$ to it. The elements of $W(E)$, called *words*, are finite sequences W of elements from E:

$$W = a_1 a_2 \cdots a_n; \quad a_i \in E$$

The integer n is called the *length* of the word W and is denoted by $lg(W)$. The unique word whose length is zero is called the empty word and is denoted by W_0. Each element of E gives rise to a unique word of length one and conversely. Therefore we identify E with the subset of $W(E)$ consisting of words of length one. Now if

$$W = a_1 a_2 \cdots a_m \text{ and } W' = a'_1 a'_2 \cdots a'_n$$

are two words then we define the product of W and W', denoted by WW', by the equation $WW' = a_1 a_2 \cdots a_m a'_1 a'_1 \cdots a'_n$. This product is associative and it is clear that the empty word is the unit element with respect to this operation. Hence the set $W(E)$ is a semigroup. Since $lgWW' = lgW + lgW'$ no word expect W_0 has an inverse.

Now we introduce a relation in $W(E)$. We say that *two words are equivalent* if one can be changed into the other by means of a finite number of operations of the following kind:

11 (a) Deletion from the word of an element e which is the unit element of some G_i, or introduction of such a unit element:

$$AeB \rightleftarrows AB.$$

(b) Replacement of two consecutive elements x, y belonging to the same group G_i by the element z equal to their product in G_i or replacement of z by xy:

$$AxyB \rightleftarrows AzB.$$

If the word W and W' are equivalent we write $W \sim W'$. This is an equivalence relation. We denote the equivalence class containing the empty words W_0 by W_0 itself. Then it directly follows that the operation of forming products passes down to the quotient set $W(E)/W_0$ of equivalence classes, i.e. if $W_1 \sim W'_1$ and $W_2 \sim W'_2$, then $W_1 W_2 \sim W'_1 W'_2$. Hence $W(E)/W_0$ is a semigroup. We assert that it is a group. For this we define the *reverse* of a word. If $W = a_1 a_2 \cdots a_n$, the reverse of W, denoted by W^{-1} is $a_n^{-1} a_{n-1}^{-1} \cdots a_2^{-1} a_1^{-1}$. For any words W and W' we have $(WW')^{-1} = W'^{-1} W^{-1}$, $(W^{-1})^{-1} = W$ and $W_0^{-1} = W_0$.

Now suppose that W is a words of length n. If we apply (b) and (a) successively n-times to WW^{-1}, we see that WW^{-1} is equivalent to the unit element W_0, and also that so is $W^{-1}W$. Hence $W(E)/W_0$ is a group.

The above group is called the *free product* of the family of groups G_i. It is denoted by $\underset{i}{*} G_i$ and if I is a finite set of n elements then it is also denoted by

$$G_{i_1} * G_{i_2} * \ldots * G_{i_n}.$$

The G_i are called the factor of $\underset{i}{*} G_i$. **12**

Definition. *A word $W = a_1 a_2 \cdots a_n$ is called* reduced *if*

(i) No a_i is the unit element of some $G_{i(W)}$.

(ii) No two consecutive elements a_i, a_{i+1} belongs to the same group G_i.

The simplest example of a reduced word is an element x of E different from the units of G_i.

Theorem 2.1. *Each equivalence class of $W(E)$ contains one and only one reduce word.*

Proof. For every reduced word $w = a_1 a_2 \cdots a_r$ and every $a \in E$, we get a reduced word $|aw|$ equivalent to aw by setting

$$|aw| = \begin{cases} a_1 a_2 \cdots a_r & \text{if a is a unit element} \\ a a_1 a_2 \cdots a_r & \text{if a is not a unit element} \\ & \text{and does not belong to the same} \\ & G_i \text{ as } a_i \\ b a_2 \cdots a_r & \text{if a belongs to the same } G_i \\ & \text{as } a_1 \text{ and } a a_1 = b \text{ is not} \\ & \text{a unit element} \\ a_2 \cdots a_r & \text{if } a = a_1^{-1}. \end{cases}$$

By induction on the length, one proves that every word is equivalent to at least one reduced word.

To prove the unicity, following an idea of Van der Waerden, let us denote by $T(a)$, for every $a \in E$, the map of the set of all reduced words into itself which changes w into $|aw|$, and for any word $W_1 = b_1 b_2 \cdots b_s$, reduced or not, let us set $T(w_1) = T(b_1)T(b_2) \cdots T(b_r)$. It can be easily **13** verified that, if a and b belong to the same G_i and $ab = c$, then $T(c) = T(a)T(b)$, while $T(e) =$identity if e is a unit element. There follows that if the words w_1 and w_2 are equivalent, $T(w_1) = T(w_2)$. Now, a reduced word w is the image of the empty word W_0 by $T(w)$. Hence, if w_1 and w_2 are equivalent reduced words, they are the images of W_0 by the same map, $w_1 = w_2$, which proves the unicity.

If $x \in G_i$ is not the unit element of G_i, then it is a reduced word. If we map the unit element of each G_i onto the empty word of $W(E)$ and the other by inclusion into $W(E)$ we obtain an isomorphism of G_i onto a subgroup of $\underset{i}{*} G_i$; let us identify G_i with the corresponding subgroup. With this identification it is clear that

$$G_i \cup G_j = W_0 \text{ for } i \neq j.$$

□

Theorem 2.2. *Given a group G, a family G_i of groups and a family of homomorphisms*

$$h_i : G_i \to G$$

then there exists one and one homomorphism h of $\underset{i}{*} G_i$ onto G such that

$$h|G_i = h_i.$$

Proof. For the proof it is better to consider the semigroup $W(E)$. Given any map g of E into a semigroup G then we can extend it into a homomorphism g^* of $W(E)$ into G by setting

$$g^*(a_1 a_2 \cdots a_n) = g(a_1)g(a_2) \cdots g(a_n).$$

14 for every word $a_1 a_2 \cdots a_n$. Further if g is given by a family h_i of homomorphisms of G_i into G then equivalent elements of $W(E)$ go into the same element of G under the map g^* (This fact can be proved by induction on the number of operations (a) and (b) performed to change a word into its equivalent). Hence g^* gives a homomorphism of $\underset{i}{*} G_i$ onto G. The uniqueness part is clear because any homomorphism g_1^* of $\underset{i}{*} G_i$ which equal g_i on G_i has to satisfy the defining equation of g^*. Hence the result. □

The quotient of a free product by a set of relations.

Suppose we are given a family $(S_j, T_j)_{j\in J}$ of pairs of words of $W(E)$, and consider the set of relations $R = \{S_j = T_j\}_{j\in J}$. We say that two words are R-equivalent if one can be changed into the other by a finite number of operations (a), (b) and

(c) Replacement of S_j by T_j or T_j by S_j at will. If two words W, W' are R-equivalent we write

$$W \underset{R}{\sim} W'$$

It is a matter of direct verification that R-equivalence is an equivalence relation. We denote the equivalence class determined by the empty word W_0 by W_R and it follows directly that the product operation in $W(E)$ passes down to the quotient set $W(E)/W_R$ of the equivalence classes. (This fact can be proved by induction on the number of operation (a), (b), (c) used to changed a word into its equivalent). Since the operations (a) and (b) are subsumed under R-equivalence it follows that R-equivalence can be considered as an equivalence ration in $\underset{i}{*}G_i$ and that $W(E)/W_R$ can be considered as the set of equivalence classes of $\underset{i}{*}G_i$ under R. It follows that $W(E)/W_R$ is a group. This group is called *the quotient group of the free product* $\underset{i}{*}G_i$ *by the set of relations* R.

We say that two sets R and R' of relations in $W(E)$ are equivalent if R-equivalence implies and is implied by R'-equivalence, i.e. for any two words W and W' we have $W \underset{R}{\sim} W'$ if and only if $W \underset{R}{\sim} W'$. In this case we write $R \sim R'$.

Example. Consider $R = \{S_j = T_j\}$ and $R' = \{S_j T_j^{-1} = W_0\}$ then we have $R \sim R'$.

A set of relations $S_j = W_0$ is also written $S_j = 1$ and the S_j are sometimes called relaters.

Definition. *A* free group *is the free product of a family of groups each of which is infinite cyclic.*

Suppose that we are given a group G and a system $\{a_i\}_{i\in I}$ of generations of G. Suppose that Z denotes the infinite cyclic group and e a

generator of Z and that we denote the homomorphism of Z into G taking e onto a_i by h_i. Now if we take copies Z_i of Z one for each i in I and form the free product, by theorem 2.2. there exists a homomorphism h
16 of $\underset{i}{*} Z_i$ into G such that $h|G_i = h_i$. This homomorphism is not since a system $\{a_i\}_{i \in I}$ of generators of G is in the image. Hence G is isomorphic to the quotient of $\underset{i}{*} Z_i$ by the kernel of h. Hence each group can be represented as a quotient of a free group. The word in the kernel give a set of relators for G.

Theorem 2.3. *Suppose that $W \neq W_0$ is an element of finite order in $\underset{i}{*} G_i$. Then there is one and only one i in I such that W is conjugate to an element of G_i.*

Proof. Among the reduced words corresponding to the conjugates $W_1 W W_1^{-1}$ suppose that $W' = W_1 W W_1^{-1}$ is one length and that

$$W' = a_1 a_2 \cdots a_r.$$

□

To prove the result we first show that $r = 1$. Suppose that $r > 1$. Then a_1 and a_r do not belongs to the same group for if they belong to the same group then the conjugate

$$a_r^{-1} a_1 a_2 \cdots a_{r-1} = a_1' a_2 \cdots a_{r-1}$$

of W would have length $r - 1$ contradicting the minimality of r. But if a_1 and a_r are not in the same group, then $(W')^n$ is reduced for every n, therefore W^n is never the unit element W_0. Hence $r = 1$ and there is an i such that G_i contains a conjugate of W. The unicity follows from the fact that every conjugate of a_1 is represented by a reduced word of the form $Aa'A^{-1}$ with $a' \in G_i$.

17 **Theorem 2.4.** *The center of a free group $G = \underset{i}{*} G_i$ with at least two factors is trivial i.e. consists of W_0 alone.*

Proof. Suppose that $a_1 a_2 \cdots a_r$ is a reduced word in the center of the group. Suppose that a_1 is in G_i. Then since G contains at least two factors there is a group G_j different from G_i, let a be an element of G_j different from the unit. Then the word $aa_1 \cdots a_r$ is a reduced word; further the word $a_1 a_2 \ldots a_r a$ is reduced if a_r and a do not belong to the same group G_j, otherwise $a_1 a_2 \cdots a_r a$ has length r, hence $a_1 a_2 \cdots a_r a \neq a_1 a_2 \cdots a_r$; in each case $a_1 a_2 \cdots a_r a$ then $a_1 a_2 \cdots a_r a$ represent distinct equivalent classes, a contradiction. □

Theorem 2.5. *The quotient of a free product* $G = \underset{i}{*} G_i$ *of abelian group by the commutator subgroup* G' *is isomorphic to the direct sum* $\underset{i}{\times} G_i$.

Proof. The direct sum $\underset{i}{\times} G_i$ is the subgroup of the direct product of the G_i consisting of the families $(a_i)_{i \in I}$ where all a_i but for a finite number are unit elements. By theorem 2.2, there is a homomorphism

$$h : \underset{i}{*} G_i \to \underset{i}{\times} G_i$$

such that $h|G_i$ is the canonical injection of G_i into $\underset{i}{\times} G_i$. Since $\underset{i}{\times} G_i$ is abelian (G is assumed abelian), G' is in the kernel of h. Hence we have the natural maps

$$G \xrightarrow{p} G/G' \xrightarrow{\overline{h}} \underset{i}{\times} G_i$$

with $h = \overline{h} o p$. □

Now, suppose that $a = a_1 a_2 \cdots a_m$ is in the kernel of h, where 18
$a_1, a_2, \cdots, a_m$ are elements of $\bigcup_{i=1}^{n} G_i$. Let A_k be the product of the a_i which belongs to G_k, and $A = A_1 A_2 \cdots A_n$. Then $p(A) = p(a)$ and $h(A) = h(a) = 1$. Because $h(A) = (A_i)_{i \in I}$, where A_i is for $i > n$ the unit element of G_i, this implies that $A = 1$, $p(a) = 1$ and $a \in G'$. Hence G' is the kernel of h.

Theorem 2.6. *If two free products of cyclic groups are isomorphic, then their factors are the same.*

Proof. Let $G = \underset{i}{*}G_i$ and $G' = \underset{j}{*}G'_j$. be two free products of cyclic groups, which are isomorphic. For $1 < k \leq \infty$, let $N_k(resp.\ N'_k)$ be the number of the $G_i(resp.\ G'_j)$ which are of order k. We prove that $N_k = N'_k$. For $k = \infty$, N_∞ is the rank of $\underset{i}{\times}G_i$ and N'_∞ of rank of $\underset{j}{\times}G'_j$. According to theorem 2.5, $\underset{i}{\times}G_i$ and $\underset{j}{\times}G'_j$ are isomorphic, hence $N_\infty = N'_\infty$. For k finite > 1, let us consider the cyclic subgroup of G of order k which are maximal, i.e. not contained in a larger cyclic subgroup. From theorem 2.3, it follows that such maximal subgroup is conjugate to one and only one G_i. Hence N_k is the number of conjugacy classes of maximal cyclic subgroups of G of order k. Therefore, the isomorphism of G and G, implies $N_k = N'_k$.

If G is the quotient of a free group generated by $a_1, a_2, \ldots$ by the set
19 of relations $S_1 = T_1, S_2 = T_2, \ldots$, we write

$$G = \{a_1, a_2, \ldots, S_1 = T_1 S_2 = T_2 \ldots\}$$

and this is called a presentation of G.

Given a set of relations R in a free product of groups, the problem to find a procedure for deciding if any two words are R-equivalent is called the word problem. According to Novikov, in general, such a procedure does not exist. But, in some particular cases, such a procedure exists. We are now considering such a case, which is useful in Topology.

Suppose that G_i is a family of groups, H is a group and that

$$j_i : H \to H_i \subset G_i$$

is a family of isomorphisms of H onto subgroups of the G_i. Now we consider the quotient of $\underset{i}{*}G_i$ by the relation

$$J = \{j_i(a) = j_k(a), a \in H; i, k \in I\}.$$

This group denoted by $\underset{i}{*}G_i/J$ is called a *free product with an amalgamation.*

For each α in H let us identify $j_i(\alpha)$ with α for all i, and denote the union of the G_i by E. So $E = \bigcup_i G_i$ and $G_i \cap G_j = H$ for $i \neq j$.

Now we consider the semigroup $W(E)$. In this set to change a word into an equivalent one we need only perform the operations (a), (b) the other (c) being taken care of by our identifications.

Now we say that two elements a and b of G_i are H-equivalent if ab^{-1} **20**
belongs to H. This is an equivalence relation in G_i and partitions G_i into right cosets of H. Thus the whole set E is partitioned into disjoint cosets. From each of these sets, except the set H, we take one representative f and denote the collection of representatives by F.

By the definition of F it is clear that every element of E not in H can be written as hf where h is in H and f is in F. If we apply this to the product fh of a pair of elements h and h where h is in H and f is in $F \cap G_i$ it follows that there exists a pair h', f' of elements with h' in H and f' in $F \cap G_i$ such that

$$fh = h'f'.$$

To replace fh by $h'f'$ will be called commutation operation.

Definition. *A word $W = hf_1 \cdots f_2$ is called F-reduced if $h \in H$, $f_i \in F$ and no two consecutive elements f_i and f_{i+1} belongs to the same group G_k.*

Now we prove the following theorem whose statement and proof are similar to those of theorem 2.1.

Theorem 2.7. *Each J-equivalence class contains one and only one F-reduced word.*

Proof. If $W = hf_1 \cdots f_r$ is an F-reduced word and $a \in E$, we define an F-reduced word $|aw|$ which is J-equivalent to aw in the following manner. □

If $a \in H$, $ah = h'$, we set $|aw| = h'f_1 \cdots f_r$. **21**

If a belongs to the same group G_i as f_1, and if $ahf_1 \in H$, we set $|aw| = h'f_2 \cdots f_r$ with $h' = ahf_1$; if $ahf_1 \notin H$, we have $ahf_1 = h'f$ with $h' \in H$, $f \in F$ and we set $|aw| = h'ff_2 \cdots f_r$. If a does not belong to the same group G_i as f_1, we have $ah = h'f$ with $h' \in H$, $f \in F$ and we set $|aw| = h'ff_1f_2 \cdots f_r$.

Now, one proves, by induction on the length, that each word is J-equivalent to at least one F-reduced word. For the unicity the proof is exactly the same as for theorem 2.1, using the map $T(a)$ of the set of all F-reduced words into itself which changes w into $|aw|$.

By this theorem we can identity the underlying set of the group $\mathop{\ast}\limits_{i} G_i/j$ with the set of F-reduced words. Now every element of G represents an F-reduced word. Hence we identify G_i with a subgroup of $\mathop{\ast}\limits_{i} G_i/J$.

2.8 **A consequence.** Suppose that we are given a family of homomorphisms h_i of G_i into a group G such that

$$h_i|H = h_j|H$$

Then there exists one and only one homomorphism h of $\mathop{\ast}\limits_{i} G_i/J$ into G such that

$$h|G_i = h_i.$$

The proof is clear. □

22 **Remarks.** Suppose that the indexing set $I = I_1 \cup I_2$ where I_1 and I_2 are disjoint. Then there is a canonical isomorphism:

$$\mathop{\ast}\limits_{i\in I} G_i \simeq \left(\mathop{\ast}\limits_{i\in I_1} G_i\right) \ast_i \left(\mathop{\ast}\limits_{I\in I_2} G_i\right).$$

The proof is straight forward.

Suppose that a set of relations is the union of two sets, $R = R_1 \cup R_2$, where R_1 involves words from $\mathop{\ast}\limits_{i\in I_1} G_i$ alone and R_2 involves words from $\mathop{\ast}\limits_{i\in I_2} G_i$ alone. Then there is a canonical isomorphism:

$$\mathop{\ast}\limits_{i\in I} G_i \Big/_{R} \simeq \left(\mathop{\ast}\limits_{i\in I_1} G_{iR_1}\right) \ast \left(\mathop{\ast}\limits_{I\in I_2} G_{iR_2}\right).$$

Again the proof is direct.

We give an application.

Suppose that p and q are integers > 1, G_1, G_2 and H are groups each isomorphic to $\mathbb{Z}$ with generators a, b and c respectively. We define isomorphisms

$$j_1 : H \to H_1 \subset G_1 \text{ and } j_2 : H \to H_2 \subset G_2$$

by setting

$$j_1(c) = a^p \text{ and } j_2(c) = b^q.$$

We denote the resulting free product with amalgamation by $G_{p,q}$. The element $a^p = b^q$ commutes with a as well as b and hence is in the centre. Denoting the subgroup generated by a^p by Z we have

$$G_{p,q/Z} = \{a, b; a^p = 1, b^q = 1\}.$$

Now by the second remark above we have **23**

$$G_{p,q}\big/_{Z} = G_1\big/_{a^p} * G_2\big/_{b^q} = \mathbb{Z}_p * \mathbb{Z}_q.$$

By theorem (2.4) this latter group does not have centre. Hence the centre of $G_{p,q}$ is Z.

Using this fact and theorem (2.6) we prove

Theorem 2.9. *Suppose that the ordered pairs of integers greater than 1, (p, q) and (p', q') are such that $G_{p,q} \simeq G_{p',q'}$. Then either $(p, q) = (p', q')$ or $(p, q) = (q', p')$.*

Proof. We have $G_{p,q} \simeq G_{p',q'}$ and hence

$$G_{p,q/(\text{Centre})} \simeq G_{p',q'/(\text{Centre})}.$$

□

Hence $\mathbb{Z}_p * \mathbb{Z}_q$ $\mathbb{Z}_{p'} * \mathbb{Z}_{q'}$, and now by theorem (2.6) we have either $p = p'$ and $q = q'$ or $p = q'$ and $q = p'$.

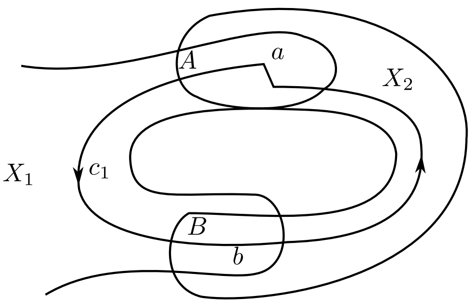
3 On calculation of fundamental groups

In this section, we have some theorems that are useful in the computation of the fundamental group. 24

Theorem 3.1. *Suppose $X = X_1 \cup X_2$ is the union of two arcwise connected open subsets X_1 and X_2. If $X_1 \cap X_2 = A \cup B$ is the union of two arcwise connected non void disjoint open sets A and B and if X_2, A and B are simply connected, then*

$$\pi(X) \simeq \pi(X_1) * \mathbb{Z}.$$

Proof. Let us fix a point a in A and a point b in B and take the fundamental groups of X and X_1 based at a, $\pi(X) = \pi(X, a)$, $\pi(X_1) = \pi(X_1, a)$. Let c_1 be a path joining a to b in X_1, c_2 a path joining b to a in X_2 and $c = c_1 c_2$.



Let us denote by γ a generator of $\mathbb{Z}$ (the infinite cyclic group written multiplicatively). By (2.2) there is a unique homomorphism $h : \pi(X_1) * \mathbb{Z} \to \pi(X)$ such that $h(j) = [c]$ and $h|\pi(X_1)$ is induced by the inclusion $X_1 \subset X$. Now, we have a more precise formulation of (3.1): □

(3.1)′. *h* is an isomorphism.

25 Consider the semi-group of words $W = W(\pi(X_1) \cup \mathbb{Z})$ and the canonical homomorphism $h_c : W \to W/W_0 = \pi(X_1) * \mathbb{Z}$ where W_0 is the equivalence class of the empty word. Let us set $\tilde{h} = h \circ h_c : W \to \pi(X)$. Then, (3.1)′ is equivalent to:

(3.1)″. $\tilde{h}$ is surjective and its kernel is W_0.

For (3.1)″ implies that h is an isomorphism.

Let us say that a path in X is *small*, if it is contained in X_1 or in X_2. For every point $x \in X$, let $d(x)$ be a path such that

if $x \in B$, $d(x)$ is connecting x to b in B,

if $x \in A$, $d(x)$ is connecting x to a in A,

if $x \notin A \cup B$ and $x \in X_i$, $d(x)$ is connecting x to a in $X_i (i = 1$ or $2)$. Thus, in any case, $d(x)$ is a small path. Now, let w be a path in X closed at a, so that w is an element of $\pi(X)$. If $w \subset X_1$, we denote by $[w]_1 \in \pi(X_1)$ its homotopy class in X_1. Let s be a subdivision of w into a product $w = w_1 w_2, \ldots, w_n$ of small paths w_i. Since X_1 and X_2 are open, such a subdivision always exists. We are going to define a well determined word $M(w, s)$ associated to this subdivision of w.

Let t_i be the terminal point of w_i (and also the initial point of w_{i+1}) and $d_i = d(t_i)$ the small path joining t_i to a or b. Let us set

$$w'_1 = w_1 d_1, w'_i = d_{i-1}^{-1} w_i d_i (i = 2, \ldots, n-1), w'_n = d_{n-1}^{-1} w_n;$$

so that w'_i is a small path contained in the same set X_1 or X_2 as w_i, and **26**

$$w'_1 w'_2 \cdots w'_n = w_1 d_1 d_1^{-1} w_2 d_2 \cdots d_{n-1}^{-1} w_n \approx w.$$

That will be called *the first modification* of the product.

Now, we take off those w'_i which are closed and ≈ 0 in X_1 or in X_2, in particular, as X_2 is simply connected, all w'_i which are closed in X_2. If w'_i is connecting b to a in X_2, $w'_i \approx c_2$ in X_2, $w'_i \approx c_1^{-1} c_1 c_2 = c_1^{-1} c$ in X and we replace w'_i by $c_1^{-1} c$. If w'_i is connecting a to b in X_2, $w'_i \approx c_2^{-1}$ in X_2, $w'_i \approx c_2^{-1} c_1^{-1} c_1 = c^{-1} c_1$ in X and we replace w'_i by $c^{-1} c_1$. Thus we get a product of paths, which are either contained in X_1 or equal to c or c^{-1},

$$w \cdots c^{\pm 1} w''_j \cdots w''_k c^{\pm 1} w''_1 \cdots .$$

That will be called *the second modification*.

It may happen that all w'_i have been taken off in which case $w \approx 0$ and we set $M(w, s) =$ the empty word.

If that is not the case, we replace every maximal series of consecutive paths contained in X_1 by their product: $w''_j \cdots w''_k = \bar{w}_\ell$, and we get a product

$$w \cdots c^{\pm 1} \overline{w}_\ell c^{\pm 1} \cdots$$

of paths which are either equal to c or c^{-1} or contained in X_1 and closed at a (the end points of $\overline{w}_\ell$ must be a because c is closed at (a). That is *the third and last modification*.

27 Now we define $M(w, s)$ as the word we get by replacing, in this product, every path $\overline{w}_\ell$ contained in X_1 by its homotopy class $[\overline{w}_l]_1 \in \pi(X_1)$ in X_1, and c by γ, c^{-1} by γ^{-1}:

$$M(w, s) = \cdots \gamma^{\pm 1} [\overline{w}_l]_1 \gamma^{\pm 1} \cdots .$$

We have now the following properties.

(a) $\tilde{h} M(w, s) = [w]$. For, by the definition of $\tilde{h}$, we have

$$\tilde{h}\gamma = [c], \tilde{h}\gamma^{-1} = [c^{-1}], \tilde{h}[\overline{w}_l]_1 = [\overline{w}_l].$$

There follows that *$\tilde{h}$ is surjective*.

(b) *If in the product $w_1 \cdots w_n$ we replace a small path w_j by another small path v_j, such that $v_j \approx w_j$ in X_1 or in X_2, the associated word $M(w, s)$ does not change*. For, after the first modification, v_j will be replaced by $v'_j \approx w'_j$ in X_1 or in X_2, and if $w_j \subset X_2$, the product we get after the second modification will be the same; if $w_j \subset X_1$, after the third modification one of the paths $\overline{w}_l$ will be replaced by a path $\overline{v}_l \approx \overline{w}_l$ in X_1 so that $[\overline{v}_l]_1 = [\overline{w}_l]_1$ and we get the same word

(c) *If we take off or add, in the product $w_1 \cdots , w_n$, a path contained in A and closed at a, the associated word does not change*. For if $w_i \subset A$ is closed at a, w'_i will be taken off in the second modification.

(d) *If the subdivision s' of w is obtained from s by dividing one of the small paths w_i into two paths*, $w_i = u_i v_i$, then $M(w, s') \sim M(w, s)$.

28 Here the proof, though quite easy, is a bit longer. If we replace w_i by $u_i v_i$, after the first modification, w'_i will be replaced by $u'_i v'_i \approx w'_i$ in X_1 or in X_2. If it is in X_1, the only effect after the third modification would

be to replace a path $\bar{w}_l$ by another one homotopic to it in X_1 or, if w'_i is closed and ≈ 0 in X_1 while u'_i and v'_i are not, to introduce a new path $\bar{w}_l$ homotopic to zero in X_1, so that $M(w, s') = M(w, s)$. If $u'_i v'_i \approx w'_i$ in X_2 and w'_i is connecting a to b or b to a, one of the paths u'_i or v'_i will be closed and taken off in the second modification, so that again $M(w, s') = M(w, s)$. The same holds, if $w_i \subset X_2$, u'_i and v'_i are all closed.

If $w'_i \subset X_2$ is closed at a while u'_i is connecting a to b in X_2 and v'_l b to a in X_2, w'_i will be taken off in the second modification while $u'_i v'_i$ will be replaced by $c^{-1} c_1 c_1^{-1} c$, so that if e_1 denotes the unit element of $\pi(X_1)$, we have $M(w, s) = M_1 M_2$ and $M(w, s') = M_1 \gamma^{-1} e_1 \gamma M_2$, and consequently $M(w, s') \sim M(w, s)$.

If $w'_i \subset X_2$ is closed at b while u'_i is connecting b to a and v'_i a to b, w'_i will be taken off in the second modification, while $u'_i v'_i$ will be replaced by $c_1^{-1} c c^{-1} c_1$. Since it is closed at b, it must be preceded and followed by paths in X_1, so that after the second modification we get products like

$$\ldots w''_k w''_h \ldots \quad \text{and} \quad w''_k c_1^{-1} c c^{-1} c_1 w''_h \ldots .$$

There follows that the path $\bar{w}_i$ coming from $\cdots w''_k w''_h \cdots$ after the third modification is the product of two paths $u = \cdots w''_k$ and $v = w''_h \ldots$, connecting a to b and b to a, $\bar{w}_i = uv$, and will be replaced by $\bar{w}_{l_1} c c^{-1} \bar{w}_{l_2}$ **29**
with $\bar{w}_{l_1} = u c_1^{-1}$ and $\bar{w}_{l_2} = c_1 v$. Setting $[\bar{w}_l]_1 = \xi$, $[\bar{w}_{l_1}]_1 = \xi_1$ and $[\bar{w}_{1_2}]_1 = \xi_2$, we have

$$M(w, s) = M_1 \xi M_2, M(w, s') = M_1 \xi_1 \gamma \gamma^{-1} \xi_2 M_2$$

and $M(w, s) \sim M(w, s')$ because $\xi = \xi_1 \xi_2$.

Hence (d) is proved. An immediate consequence is:

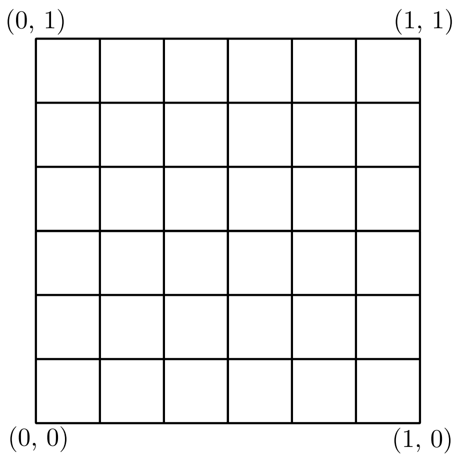
(e) *If s and s' are any two subdivisions of w into small paths,* $M(w, s) \sim M(w, s')$.

For any two subdivisions have a common refinement, and any refinement can be obtained by introducing one point at a time.

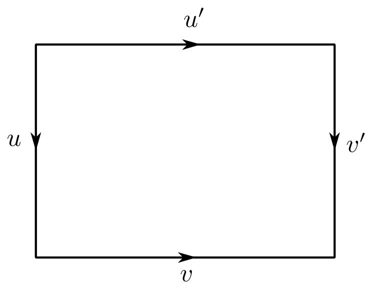
(f) *If $w \approx 0$ in X, then* $M(w, s) \sim W_o$

As $\tilde{h} M(w, s) = [w]$, this means that $\ker \tilde{h} = W_o$ and will achieve the proof of (3.1).

The homotopy $w \approx 0$ consists of a map $F : I \times I \to X$ such that $F(I \times 0) = w$, $F(0 \times I) = F(I \times 1) = F(1 \times I) = a$. Let us divide the square $I \times I = (0 \le s, t \le 1)$ into m^2 equal squares, m being great enough in order that their images by F be small in X, i.e contained in X_1 or in X_2. We number these squares in such a way that the one of centre (s, t) comes before the one of centre (s', t') if $t < t'$ or $t = t'$ and $s < s'$. Let v_j be the path, product of $2m$ edges of our chessboard, connecting $(0, 1)$ to $(1, 0)$ by passing above the first j squares and under the other. Here is a representation of v_{16} with $m = 6$.



30 $F(v_j)$ is a path closed at a and has a subdivision s_j into $2m$ small paths, the images of the edges of v_j. Let us compare the words $M_{j-1} = M(F(v_{j-1}), s_{j-1})$ and $M_j = M(F(V_j), s_j)$. Let u, v be the sides of the j^{th} square in v_{j-1} and u', v' the sides in v_j



We get the product $F(v_j)$ from $F(v_{j-1})$ by replacing $F(u)F(v)$ by $F(u')F(v')$, so that we have

$$F(v_{j-1}) = PF(u)F(v)Q, F(v_j) = PF(u')F(v')Q.$$

But $F(u)F(v) = F(uv)$ and $F(u')F(v') = F(v'v')$ are small paths,
homotopic in the same set X_1 or X_2. By (b), the words associated to 31
$PF(uv)Q$ and to $PF(u'v')Q$ are the same, and by (d) they are equivalent to M_{j-1} and M_j, so that $M_{j-1} \sim M_j$. Hence $M_o \sim M_{m^2}$. But, by (c) and (d), $M_o \sim M(w,s)$, and as $F(v_{m^2}) = a$, m_{m^2} = the empty w so that $M(w,s) \in W_o$.

Theorem 3.2. *Suppose $X = X_1 \cup X_2$ is the union of two arcwise connected open sets X_1 and X_2, whose intersection $X_o = X_1 \cap X_2$, is non void and arcwise connected. The groups $\pi(X)$ and $\pi(X_i)$ being based at the same point $a \in X_o$, let $j_i : \pi(X_o) \to \pi(X_i)$ $(i = 1,2)$ be the homomorphism induced by the inclusion $X_o \subset X_i$, and J the set of relations $J = \{j_1(\alpha) = j_2(\alpha), \alpha \in \pi(X_o)\}$. Then*

$$\pi(X) \simeq \pi(X_1) * \pi(X_2)/J.$$

By (2.2), there is a unique homomorphism $h : \pi(X_1) * \pi(X_2) \to \pi(X)$ such that $h|\pi(X_i)$ is induced by the inclusion $X_i \subset X (i = 1,2)$. Now, a precise formulation of (3.2) is: (3.2)′ *h is surjective and has the same kernel as the canonical homomorphism of $\pi(X_1) * \pi(X_2)$ onto $\pi(X_1) * \pi(X_2)/J$.*

Consider the semi-group of words $W = W(\pi(X_1) \cup \pi(X_2))$, let W_o be the equivalence class of the empty word and $W_J \supset W_0$ its J-equivalence class. Then $W/_{W_J} = \dfrac{\pi(X_1) * \pi(X_2)}{J}$. Let h_c be the canonical homomorphism $h_c : W \to W_{W_o} = \pi(X_1) * \pi(X_2)$, and $\tilde{h} = hoh_c$. Then (3.2)′ is equivalent to (3.2)″*$\tilde{h}$ is surjective and its kernel is W_J.*

The proof of similar to that of (3.1)″, but easier. We say, again, that 32
a path is small, if it is contained in X_1 or in X_2, and for every $x \in X$, we choose a path $d(x)$ connecting x to a, such that $d(x) \subset X_i$ if $x \in X_i (i = 0,1,2)$. For any path u closed at a, we denote by $[u] \in \pi(X)$ its homotopy class in X and if $u \subset X_i$, by $[u]_i \in \pi(X_i)$ its homotopy class in X_i.

Let w be a path closed at a in X and s a subdivision of w into a product $w = w_1 w_2 \cdots w_n$ of small paths. We are going to define two words $M^1(w, s)$ and $M^2(w, s)$. We set again $d_i = d(t_i)$ with t_i = terminal point of w_i, $w'_i = d^{-1}_{i-1} w_i d_i$ $(i = 2, \dots, n-1)$, $w'_n = d^{-1}_{n-1} w_n$, and we get a new product

$$w'_1 w'_2 \cdots w'_n \approx w$$

where w'_i is a small path contained in the same set X_1 or X_2 as w_i, and closed at a. Now, replacing, in this product, w'_i by $[w'_i]_1 \in \pi(X_1)$ if $w'_i \subset X_1$ and by $[w'_i]_2 \in \pi(X_2)$ if $w'_i \not\subset X_1$, we get the word $M^1(w, s)$. Similarly, replacing w'_i by $[w'_i]_2 \in \pi(X_2)$ if $w'_i \subset X_2$ and by $[w'_i]_1 \in \pi(X_1)$ if $w'_i \not\subset X_2$, we get the word $M^2(w, s)$. These two words differ only if some w'_i are contained in $X_1 \cap X_2 = X_o$, in which case we have $[w'_i]_1 = j_1[w'_i]_o$ in $M^1(w, s)$ and $[w'_i]_2 = j_2[w'_i]_o$ in $M^2(w, s)$. As $\tilde{h}[w'_i]_j = [w'_i] \in \pi(X)$, by the definition of $\tilde{h}$, we have,

33 (a′) *The two words* $M^1(w, s)$ *and* $M^2(w, s)$ *are J-equivalent and* $\tilde{h}M^1(w, s) = \tilde{h}M^2(w, s) = [w]$,

Hence $\tilde{h}$ is *surjective*. We have further the following properties.

(b′) *If, in the product* $w = w_1 \cdots w_n$ *we replace* w_i *by another small path* v_i *homotopic to* w_i *in* X_j $(j = 1$ *or* $2)$, *the word* $M^j(w, s)$ *does not change*.

That is immediate because we will have $v'_i \approx w'_i$ in X_j. The following is also immediate.

(c′) If we take off or add in the product $w_1 \cdots w_n$ *a small path closed at* a *and* ≈ 0 *in* X_o, *the only effect will be to take off or to add in* $M^j(w, s)$ *a unit element*.

(d′) *If the subdivision* s' *of* w *is obtained from* s *by dividing one of the small paths* w_i *into two parts,* $w_i = u_i v_i$, *and if* $w_i \subset X_j$ $(j = 1$ *or* $2)$, *then* $M^j(w, s')$ *is J-equivalent to* $M^j(w, s)$.

The proof is similar to that of (d) in the former case, but easier. For w'_i will be replaced by $u'_i v'_i$, $w'_i = d^{-1}_{i-1} w_i d_i = d^{-1}_{i-1}(u_i v_i) d_i$ and $u_i v'_i = d^{-1}_{i-1} u_i d d^{-1} v'_i d_i$, where $d = d$ (terminal point of u_i). As all these paths are in X_j, $w'_i \approx u'_i v'_i$ in X_j and $[w'_i]_j = [u'_i]_j [v'_i]_j$, we have $M^j(w, s')$ is J-equivalent to $M^j(w, s)$.

(e′) *If s and s' are any two subdivisions of w into small paths, $M^j(w, s')$ and $M^j(w, s)$ are J-equivalent.*

(f′) *If $w \approx 0$ in X, then $M^j(w, s) \in W_j$.* The proof is the same as that 34
of (c) and (f) in the former case, so that (3.2)″ is established.

An important corollary is the following.

Theorem 3.3. *The notations and the hypotheses being the same as in (3.2), let $i_1 : \pi(X_o) \to \pi(X_1)$ and $i_2 : \pi(X_o) \to \pi(X_2)$ be the homomorphisms induced by the inclusions $X_o \subset X_1$ and $X_o \subset X_2$:*

$$\begin{array}{ccccc} & & \pi(X_1) & & \\ & \overset{i_1}{\nearrow} & & \overset{j_1}{\searrow} & \\ \pi(X_0) & & & & \pi(X) \\ & \underset{i_2}{\searrow} & & \underset{j_2}{\nearrow} & \\ & & \pi(X_2) & & \end{array}$$

If i_1 and i_2 are injective, then j_1 and j_2 are also injective.

That follows from the fact that if i_1 and i_2 are injective, $\pi(X) \simeq \pi(X_1) * \pi(X_2)/J$ is a free product with amalgamation.

If in the statements (3.1) and (3.2) one replaces the condition that X_1 and X_2 be open by that they be closed, without changing the other hypothesis, the conclusion do not remain valid in general, as can be shown by counter examples. Nevertheless, for many applications, the following remark is useful.

Remark 3.4. If $X = X_1 \cup X_2$ is the union of two closed sets X_1 and X_2, satisfying all other hypotheses of theorem (3.1) (resp. (3.2) and (3.3)
and if there are open sets Y_1 and Y_2, such that $Y_1 \supset X_1$, $Y_2 \supset X_2$, X_i is a 35
deformation restract of Y_i ($i = 1, 2$) and $X_1 \cap X_2$ is a deformation retract of $Y_1 \cap Y_2$, then the conclusion of (3.1)-(resp. (3.2) and (3.3)-remain valid.

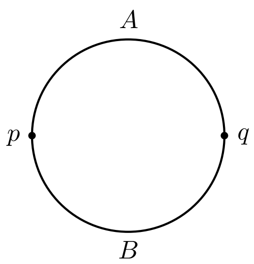
For we can apply the theorem (3.1) to Y_1 and Y_2 instead of X_1 and X_2, and as $\pi(Y_i) = \pi(X_i)$ ($i = 1, 2$ or 0) we get the conclusion.

Note. The theorem (3.2) is usually named van *Kampen's* theorem. In the case where X is a simplicial complex and X_1 and X_2 are subcomplexes of X, it was first given by Seifert. Van Kampen gave a more general theorem which contains both (3.1) and (3.2).

4 Examples

36 **1. The circle S^1.**

Suppose that p and q are two points on the circle S^1. Then $S^1 - p$ and $S^1 - q$ are both simply connected open subsets of S^1 and their intersection consists of two arcwise connected simply connected sets. Hence by theorem (3.1) we have



$$(S^1) \simeq 1 * \mathbb{Z} = \mathbb{Z}.$$

2. The sphere $S^n = \{x = (x_1, \dots, x_{n+1}), \sum_1^{n+1} x_i^2 = 1\}, n \geq 2$.

It follows from (3.2) that the union of two simply connected open sets, whose intersection is arcwise connected, is simply connected. Here, if p, q are two distinct points of S^n, as $S^n = (S^n - p) \cup (S^n - q)$ and $(S^n - p) \cap (S^n - q)$ is arcwise connected for $n \geq 2$, $\pi(S^n)$ is trivial.

3. The wedge of n circles. We consider a set of n circles $C_1, C_2, \dots, C_n$ every two of which meet at a point p. We denote the union of $C_1, \dots, C_n$ by B_n.

Let q be a point of C_n, $q \neq p$. B_{n-1} is a deformation retract of $B_n - q$, hence (corollary 1.3) $\pi(B_n - q) = \pi(B_{n-1})$. Now $B_n = (B_n - q) \cup (C_n - p)$, while $(B_n - q) \cap (C_n - p) = C_n - p - q$ is the union of two disjoint simply connected open sets. Hence, by (3.1), $\pi(B_n) = \pi(B_{n-1}) * \mathbb{Z}$, and by induction $\pi(B_n) = \mathbb{Z} * \cdots * \mathbb{Z}$ (n copies).

Now suppose that we take the circles C_i in the (x, y) plane as 37

$$C_i = \{(x, y) : (x - i)^2 + y^2 = i^2\}.$$

We provide B_∞ with the topology induced from the plane. In this topology every compact set in B is contained in some B_n. Hence if F gives a homotopy between two paths in B_∞ it does so in some B_n. Now using the previous result we obtain

$$\pi(B\infty) = \mathop{*}_{i \in I} \mathbb{Z}_i$$

where I is a countable indexing set.

If we take the circle $(x - \frac{1}{n})^2 + y^2 = \frac{1}{n^2}$ for C_n then the group $\pi(B_\infty)$ is much more complicated since, in this case, a path need not be contained in a finite number of the C_i.

4. Attaching an *n*-cell to a space.

We denote the coordinate in euclidean n-space by $x_1, x_2, \ldots, x_n$. We denote the n-disc by D^n, its boundary the $(n-1)$-sphere by S^{n-1} and the n-cell by e^n:

$$\begin{aligned} D^n &= \left\{(x) | x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1\right\} \\ S^{n-1} &= \left\{(x) | x_1^2 + x_2^2 + \cdots + x_n^2 = 1\right\} \\ e^n &= \left\{(x) | x_1^2 + x_2^2 + \cdots + x_n^2 < 1\right\} \end{aligned}$$

Suppose that X is a topological space and that f is a map of S^{n-1} into X. Then we consider the disjoint union $X \cup D^n$ of X and D^n and 38 introduce the following relation:

$$x = y \text{ if } x = y \text{ or } x = f(y) \text{ or } y = f(x) \text{ or } f(x) = f(y).$$

It is clear that this is an equivalence relation and we denote the quotient space by $X \cup_f D^n$. We note that the point set underlying this space can be identified with $X \cup e^n$ and that with this identification $X \cap e^n = \phi$. This process of constructing a new space is called attaching an n-cell. Now we study the effect, on the fundamental group, of attaching an n-cell.

We choose an interior point p of D^n. We denote the set $X \cup_f D^n - p$ by X_1 and e^n by X_2. Then since the radial projection of $D^n - p$ gives a deformation retraction of the set onto S^{n-1} we have that X is a deformation retract of X_1. Hence

$$\pi(X_1) \simeq \pi(X).$$

It is clear that S^{n-1} is a deformation retract of $e^n - p$ and hence we have

$$\pi(e^n - p) \simeq \pi(S^{n-1}).$$

Applying our theorem (3.2) to X_1, X_2 we obtain in view of (2)

$$\pi(X \cup_f D^n) \simeq \pi(X) \text{ if } n \geq 3.$$

$\boldsymbol{n = 1}$. In this case $e^1 - p$ consists of two connected components. We know that e^1 is simply connected. Hence applying theorem (3.1) with $X_1 = X \cup_f D^1 - p$, $X_2 = e^1$ we obtain

$$\pi(X \cup_f D^1) \simeq \pi(X) * \mathbb{Z}.$$

39 $\boldsymbol{n = 2}$. In this case $e^2 - p$ is connected and $\pi(e^2 - p)$ is generated by a single element say a. We know that $\pi(e^2) = 1$ and hence applying theorem (3.2) to $X_1 = X \cup_f D^2 - p$ and $X_2 = e^2$ we obtain

$$\pi(X \cup_f D^2) \simeq \frac{\pi(X) * 1}{\overline{a} = 1} = \pi(X)/_{\overline{a}=1}$$

where $\overline{a}$ is the image of a in $\pi(X_1)$. Since the boundary of D^2 is a deformation retract of $D^2 - p$ and a is in $D^2 - p$ it follows that $\overline{a}$ is the class given by the path $f|S^1$.

5. Closed surfaces of genus *p*.

Now we give some constructions.

We take a wedge X of $2p$ circles $a_1, a_2, \ldots, a_p$, $b_1, b_2, \ldots, b_p$ and a 2-disc D^2. We subdivide the circles S^1, boundary of D^2, into $4p$ segments C_i. We homeomorphically map $C_{4i}, C_{4i+1}, C_{4i+2}, C_{4i+3}$ onto $a_{i+1}, b_{i+1}, a_{i+1}^{-1}, b_{i+1}^{-1}$ in that order. We denote the resulting map of the S^1 by f. Then attaching D^2 to X by f we obtain an oriented closed surface of genus p. We denote it by S_p. If we denote the generator of $\pi(X)$ corresponding to the circle a_i by a_i itself by (4) above we obtain

$$\pi(S_p) \simeq \{a_1, b_1, a_2, b_2, \ldots, a_n, b_n; a_1, b_1 a_1^{-1} b_1^{-1} \cdots a_n^{-1} b_n^{-1} = 1\}.$$

Now we take a wedge X of p-circles $a_1, \ldots, a_p$. We subdivide S^1 into $2p$ segments C_i. We map C_{2i}, C_{2i+1} homeomorphically onto a_i, a_i, and denote the resulting map of S^1 into X by f. Then attaching D^2 to X by f we obtain a non-oriented closed surface of genus p. We denote it **40**
by S'_p. If we denote the generator of $\pi(X)$ corresponding to the circle a_i by a_i itself by (4) we obtain

$$\pi(S'_p) \simeq \{a_1, a_2, \ldots, a_p; a_1 a_1 a_2 a_2 \ldots a_p a_p = 1\}.$$

Now for any space X we define $H_1(X)$, the first homology group of X, to be the quotient of $\pi(X)$ by the commutator subgroup. It is clear that the the image of $a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_n b_n a_n^{-1} b_n^{-1}$ in $H_1(S_p)$ reduces to the unit element and hence $H_1(S_p)$ is a free abelian group with $2p$ generators. Similarly it follows that $H_1(S'_p)$ is the direct product of a free abelian group with $(p-1)$ generators and a group of order 2. Hence it follows that $\pi(S_p)$ and $\pi(S'_q)$ for various p and q are all distinct. Hence the surfaces we obtain are pairwise distinct.

The theorem on the classification of surfaces shows that we thus obtain all closed surfaces.

6. Torus knots.

A knot is a simple closed curve in $\mathbb{R}^3$. The fundamental group of the complement of a knot is called the group of the knot.

We say that two knots are of the same type if there exists an automorphism of $\mathbb{R}^3$ taking one onto the other. In this case the corresponding complements in $\mathbb{R}^3$ are homeomorphic and hence two knots of the same type have the same group. A knot which is of the type as a circle is called a trivial knot.

41 A knot lying over a torus is called a torus knot.

We take a Cartesian coordinate system (x, y, z) in $\mathbb{R}^3$. Let C' be the circle

$$(y-10)^2 + z^2 = 1, \quad x = 0;$$

or parametrically

$$y - 10 = \cos\varphi, \quad z = \sin\varphi$$

and T be the torus generated by the rotation of C' around the z axis. The parametric equations of T are

$$\begin{aligned} x &= (10 + \cos\varphi)\sin\theta \\ y &= (10 + \cos\varphi)\cos\theta \\ z &= \sin\varphi \end{aligned}$$

with angular parameters φ and θ. We denote the interior of T by T^i and its exterior by T^e. Now let p, q be two positive coprime integers and $C_{p,q}$ the closed curve

$$\theta = pt, \varphi = qt, \quad 0 \leq t \leq 2\pi.$$

Since p, q are coprime it follows that $C_{p,q}$ is a simple closed curve. We denote the circle trace by the centre of C' under the rotation around 0_z (the z axis) by C. We prove that the group of this knot is $G_{p,q} = (a, b; a^p = b^q)$.

Let $X_1 = \overline{T^i} - C_{p,q}$, $X_2 = \overline{T^e} - C_{p,q}$. Then $X = X_1 \cup X_2 = \mathbb{R}^3 - C_{p,q}$ and $X_1 \cap X_2 = T - C_{p,q}$. Here X_1 and X_2 are closed in X. Nevertheless, according to the remark (3.4), we can apply (3.2), as there are open sets
42 $Y_1 \supset X_1$ and $Y_2 \supset X_2$ such that X_1, X_2, $X_1 \cap X_2$ are deformation retracts of Y_1, Y_2 and $Y_1 \cap Y_2$ respectively.

$\pi(X_1) \simeq \mathbb{Z}$, has as generator the homotopy class of C, and $\pi(X_2) \simeq \mathbb{Z}$ has as generator the homotopy class of a circle C'' concentric with C' with radius 2. Now $T - C_{p,q}$, the torus cut along a closed curve, is homeomorphic to an annulus and $\pi(T - C_{p,q}) = \mathbb{Z}$, with generator the homotopy class of a curve equidistant of $C_{p,q}$ along the torus. This curve is homotopic to C^p in X_1 and to C''^q in X_2. Setting $a = [C]$ and $b = [C'']$, we have by (3.2) $\pi C_{p,q} = (a, b; a^p = b^q) G_{p,q}$.

It follows, by (2.8), that if p, q, p', q' are integers > 1 and $(p, q) \neq (p', q') \neq (q, p)$, $C_{p,q}$ and $C_{p',q'}$ are not of the same type.

This result was proved by Antoine by a different and highly geometric method.

Now we prove that the knots $C_{p,q}$ and $C_{q,p}$ are of the same type. For this we compactify $\mathbb{R}^3$ and obtain the 3-sphere

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1.$$

Now it is easy to see that by means of the transformation

$$(x_1, x_2, x_3, x_4) \mapsto (x_3, x_4, x_1, x_2)$$

we can obtain $C_{q,p}$ from $C_{p,q}$.

5 The group of a tame link given by a good plane projection

43

A *link* is a collection of a finite number of pairwise disjoint knots. We say that two links are of the same type if there exists an automorphism of $\mathbb{R}^3$ carrying one onto the other. The fundamental group of the complement of a link is called the group of the link.

We say that a link is *tame* if it is of the same type as a link of polygonal knots, i.e. of knots defined by piecewise linear simple curves.

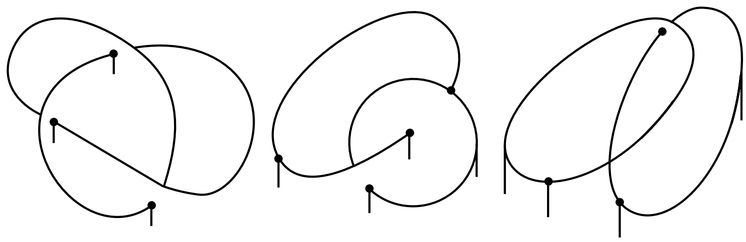
We say that a projection of a link into a plane if *good*, if the image is a curve with a continuous tangent and a continuous curvature, whose only singularities are double points with distinct tangents. The preimage of a double point consists of two points, while the preimage of any other point of the projection consists of only one point.

It can be shown that a link having a good projection is tame and that any tame link is of the same type as a knot having a good projection (we are not going to prove this here). We shall now describe a presentation of the group of a link C given by a good projection.

Suppose the link lies between the planes $z = -z_0$, $z = z_0$ in the interior of the cylinder $x^2 + y^2 = r^2$, and has a good projection into the plane $z = 0$. We choose an orientation on each of its curves, and take the point $(0, 0, z_0)$ as the base point for the fundamental group.

44 We denote by S the surface generated by the open half ray in the direction of negative z-axis with end point moving along C

$$S = \{(x, y, z) | \exists\, z' > z \text{ with } (x, y, z') \in C\}$$



The curves of the link intersect the surface S at points which divide C into a certain number of arcs. It may happen that a curve of C does not intersect S; in that case we mark a point on it. Now C is divided by all these points $P_1, \ldots, P_p$ into p arcs $a_1, \ldots a_p$, a_i starting from P_i. We call them principal arcs and denote the surface

$$\{(x, y, z) | \exists\, z' > z \text{ with } (x, y, z') \in a_i\}$$

by F_i'. If a P_j belongs to F_i' then we omit all points of F_i' lying below P_j including P_j from F_i' and obtain a surface F_i which is simply connected. We denote the open half ray through P_i in the direction of the negative z-axis by d_i.

To compute the group of C we note that $\mathbb{R}^3 - C$ is obtained by successive addition of $F_1, \ldots, F_p$ and $d_1, \ldots, S_p$ to $\mathbb{R}^3 - S$.

45 We set

$$D_0 = \mathbb{R}^3 - S, D_i = D_{i-1} \cup F_i \text{ for } i > 1$$

and

$$E_0 = \mathbb{R}^3 - (C \cup D), E_i = E_{i-1} \cup d_i, i > 1.$$

First let us compare $\pi(D_i)$ and $\pi(D_{i-1})$. Since C has a good projection it is easy to construct a simply connected set F_i'' such that $D_i \supset$

$F_i'' \supset F_i$ and such that $F_i'' \cap D_{i-1}$ consists of two simply connected sets. Hence by (3.1) we have

$$\pi(D_i) = \pi(D_{i-1}) * \mathbb{Z}.$$

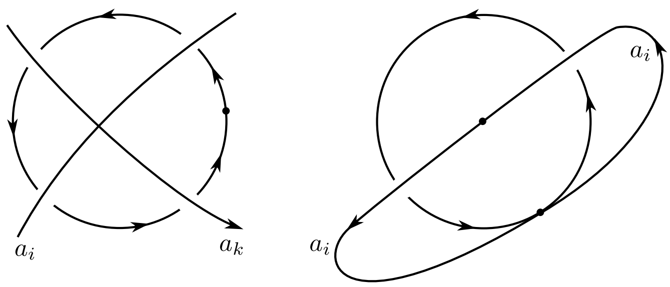
We denote the element of $\pi(D_i)$ corresponding to the generator of $\mathbb{Z}$ and represented by a path crossing S only at one point from the left to the right by a_i itself. Now we note that $\mathbb{R}^3 - S$ is simply connected. Hence

$$\pi(E_i) = \mathbb{Z} * \ldots * \mathbb{Z} \ (p \text{ times}).$$

Now let us compare $\pi(E_i)$ and $\pi(E_{i-1})$. Since C has a good projection we can construct an open simply connected neighbourhood U_i of d_i contained in E_i such that $\pi(U_i - d_i)$ is infinite cyclic and $E_{i-1} \cap U_i = U_i - d_i$ is connected.

In the case where the projection of d_i is a double point, crossed by **46**
the projection of a_k and separating the projection of a_j and a_i a generator of $\pi(U_i - d_i)$ in E_{i-1} is represented by $a_j a_k a_i^{-1} a_k^{-1}$ and the relation given is $a_j a_k = a_k a_i$ (see figure).

In the case where the projection of d_i is not a double point and corresponds to a point marked on a curve which does not cross S, a generator of $\pi(U_i - d_i)$ is represented by $a_i a_i^{-1}$ (see figure).



Now by (3.2) we have

$$\begin{aligned}\pi(E_i) &= \pi(E_{i-1}) / \left\{ a_i a_k a_j^{-1} a_k^{-1} = 1 \right\} \\ &= \pi(E_{i-1}) / (a_i a_k = a_k a_j)\end{aligned}$$

in the first case and

$$\pi(E_i) = \pi(E_{i-1})$$

it the second.

Now, we show that $\pi(E_{p-1}) \simeq \pi(E_p)$.

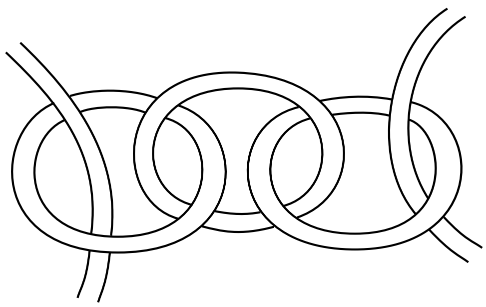
47 For we have $E_{p-1} = \mathbb{R}^3 - (CUd_p)$ and $E_p = \mathbb{R}^3 - C$, and it is easy to see that a small path around d_p is ≈ 0 in E_{p-1}. Hence

$$\pi(E_{p-1}) \simeq \pi(E_p).$$

Theorem. *Let C be a link with a good projection, To every principal arc a_i of C is associated an element of $\pi(\mathbb{R}^3 - C)$ denoted by the same letter a_i and represented by a path which crosses the cylinder S in only one point, under a_i, from the left to the right. To every double point of the projection is associated a relation described in the figure. The elements a_i generate $\pi(\mathbb{R}^3 - C)$ and these relations form a complete set of relations, moreover, any one of them is a consequence of all the others.*

6 Antoine's Necklace

48 We consider a solid torus T and $2p$ solid torus $T_1, \ldots, T_{2p}$, situated in the interior of T as in the figure.



T_i is linked with T_{i+1}, all T_i are congruent to T_1 and similar to T, and a rotation through an angle of $2\pi/p$ around the axis of T carries T_i onto

T_{i+2}. (We take $i(\text{mod}2p)$). The union $N_1 = \bigcup_{i=1}^{2p} T_i$ forms an elementary necklace.

Let h_j be a similarity of $\mathbb{R}^3$, i.e. a contraction followed by an isometry, taking T onto T_j. Then by means of the h_j we get $(2p)^2$ tori $h_j T_i$ of second order and, more generally, we get $(2p)^n$ tori

$$h_{ji} \circ \ldots \circ h_{j_n} T$$

which we call the tori of order n. Their union

$$N_n = \bigcup_{j_1 \cdots j_n}^{1 \ldots 2p} h_{j_1} \circ \ldots \circ h_{j_n} T$$

will be called a necklace of order n. The intersection **49**

$$A = \bigcap_{1}^{\infty} N_n$$

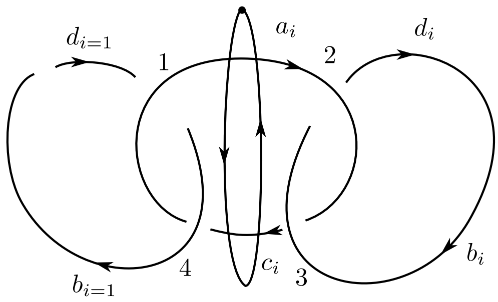
is a perfect totally discontinuous set, discovered and studied by Antoine, which is called Antoine's Necklace.

We will show that $\pi(\mathbb{R}^3 - A)$ *is not trivial, even not finitely generated.*

First, *we prove that a meridian of T represents an element of infinite order in $\pi(\mathbb{R}^3 - N_1)$.*

If C is the link consisting of the $2p$ circles c_i $(i = 1, \ldots, 2p)$ where c_i is the locus of the centres of the meridians of T_i, $\mathbb{R}^3 - N_1$ is a deformation retract of $\mathbb{R}^3 - C$, we have $\pi(\mathbb{R}^3 - N_1) = \pi(\mathbb{R}^3 - C)$. Now, $\pi(\mathbb{R}^3 - C)$ has 4p generators $a_i, b_i, c_i, d_i(i(\text{mod}p))$, corresponding to the principal arcs of C as in the figure, with the relations

$$(1)\ \ a_i b_{i-1} = d_{i-1} a_i, \quad (2)\ \ a_i b_i = d_i a_i,$$
$$(3)\ \ c_i b_i = b_i a_i \qquad (4)\ \ c_i b_{i-1} = b_{i-1} a_i.$$



50 Now a meridian X of T represents the element $X = a_i c_i^{-1}$. There is a homomorphism of $\pi(\mathbb{R}^3 - C)$ onto the free group of rank two with generators a and b, under which the images of a_i, b_i, c_i, d_i are respectively a, b, bab^{-1} and aba^{-1} for every i. The image of X being $aba^{-1}b^{-1}$, an element of infinite order, X is also of infinite order. As X represents a generator of the infinite cyclic group $\pi(\mathbb{R}^3 - \overset{o}{T})$, there follows that *the homomorphism* $\pi(\mathbb{R}^3 - \overset{o}{T}) \to \pi(\mathbb{R}^3 - N_1)$ *induced by inclusion is injective.*

Now, if we write $D = T = \overset{o}{N}_1$, we have

$$T - A = \bigcup_{n=0}^{\infty} \bigcup_{j_1 \dots j_n}^{1 \dots 2p} h_{j_1} \circ \dots \circ h_{j_n} D.$$

Let us number the series $(j_1 \dots j_n)$, taking in succession those for which $n = 0, 1, 2, \dots$, and if $m = m(j_1, \dots, j_n)$ is the number of $(j_1, \dots, j_n)$ let us set $h_{j_1} \circ \dots \circ h_{j_n} D = D_m$, (where $D_0 = D$, $D_1 = h_1 D = T_1 - h_1 N_1, \dots$). Then we have

$$T - A = \bigcup_{k=0}^{\infty} D_k$$

and we set

$$E_m = \bigcup_{k=0}^{m} D_k \cup (\mathbb{R}^3 - T).$$

We have $D_m \cap E_{m-1} = h_{j_1} \circ, \dots \circ h_{j_n} (T - \overset{o}{T})$, and E_{m-1} is a deformation retract of a set open in E_m.

Now, we will prove that the inclusion homomorphisms$\pi(E_{m-1}) \to \pi(E_m)$ and $\pi(D_m) \to \pi(E_m)$ are injective. If we set $X_1 = E_{m-1}$, $X_2 = D_m$, 51
we have $X_1 \cap X_2 = X_0 = h_{j_1} \circ \ldots \circ h_{j_n}(T - \overset{\circ}{T})$ and $X_1 \cap X_2 = E_m$. From theorem (3.3) and remark (3.4), we have only to show that the inclusion homomorphisms

$$i_1 : \pi(X_0) \to \pi(X_1) \text{ and } i_2 : \pi(X_0) \to \pi(X_2)$$

are injective.

X_0 is the surface of the torus $T' = h_{j_1} \circ \ldots \circ h_{j_n} T$ and $\pi(X_0)$ is a free abelian group with two generators represented by a meridian M and a parallel P of that surface. If $M^a P^b \approx 0$ in E_{m-1}, then $M^a P^b \approx M^a \approx 0$ in $\mathbb{R}^3 - \overset{\circ}{T}'$ because $P \approx 0$ in $\mathbb{R}^3 - \overset{\circ}{T}'$ and $E_{m-1} \subset \mathbb{R}^3 - \overset{\circ}{T}'$. This implies $a = 0$. Then $M^a P^b = P^b \approx 0$ in E_{m-1} implies $b = 0$, because P is linked with a following torus or the elementary necklace contained in it, in the same way as X was linked with T or N_1. Hence i_1 is injective. Similarly one sees that i_2 is injective.

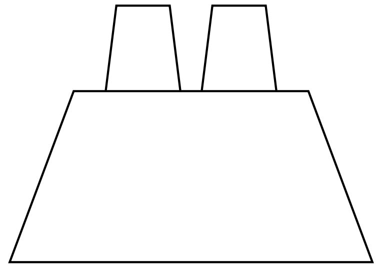
Thus the following inclusion homomorphisms

$$\pi(\mathbb{R}^3 - T^0) \to \pi(E_1) \ldots \pi(E_{m-1}) \to \pi(E_m) \ldots$$

are injective, consequently $\pi(\mathbb{R}^3 - T) \to \pi(\mathbb{R}^3 - A)$ is also injective, because if a path in $\mathbb{R}^3 - A$ is ≈ 0 in $\mathbb{R}^3 - A$, it is contained and ≈ 0 in E_m for m great enough. A meridian of T is not ≈ 0 in $\mathbb{R}^3 - A$, and the same is true for a meridian of any torus of any order q. If q is great enough, such a meridian is not homotopic to a path contained in E_m, and it follows that $\pi(\mathbb{R}^3 - A)$ is not finitely generated.

Antoine's horned sphere.

We take a solid truncated cone H_0 with base in the plane $z = 0$ and 52
the top in $z = 1$. We erect on its top $2p$ congruent disjoint truncated cones $H_i (i = 1, \ldots, 2p)$ with tops in the plan $z = 2$. We assume the union $H_0 \cup H_1 \cup \ldots \cup H_{2p} = H$ is a solid bounded by a disk D in the plane $z = 0$, the base of H_0, and a surface $z = f(x, y)$, which contains $2p$ discs D'_i $(i = 1 \ldots 2p)$ in the plane $z = 2$, the top of the H_i.



Now, let h_i by a similarity consisting of a contraction followed by a translation which takes the base of H_0 onto the top of H_i. Now let us set

$$B = \bigcup_{n=0}^{\infty} \bigcup_{i_1 \dots i_n}^{1 \dots 2p} h_{i_1} \circ \dots \circ h_{i_n} H.$$

The closure $\overline{B}$ of B is a 3-disk, bounded by the base of H and a surface $z = g(x, y)$, where $g(x, y)$ is continuous in the base of H and zero on its boundary. Moreover, $0 \leq g(x, y) \leq \dfrac{2}{1-r}$, where r is the ratio of the similarities h_i. The set $\overline{B} \cap \{z = \dfrac{2}{1-r}\}$ is a perfect totality discontinuous set A' on the boundary of $\overline{B}$.

Now, let us consider a 2-disk D on the boundary of T and its image $h_i D = D_i$ on the boundary of T_i $(i = 1, \dots, 2p)$. One can find a homeomorphism f of H into $T - N_1^\circ$, such that $fD' = D$, $fD_i' = D_i$ and $f \circ h_i'(x) = h_i \circ f(x)$ for any $x \in D'$ This homeomorphism has a unique
53 extension from H to $\overline{B}$ satisfying $f \circ h_i'(x) = h_i f(x)$ for any $x \in \overline{B}$. Then $f(\overline{B})$ is a 3-disk contained in T, whose boundary contains A. For A is the image under f of the set $\overline{B} \cap \{z = 2/1 - r\}$, the top of $\overline{B}$. Hence, a closed path linked with T in $R^3 - T$ is not ≈ 0 in $R^3 - f(\overline{B})$, and we see that *the complement* $R^3 - f(\overline{B})$ *of the 3-disc* $f(\overline{B})$*, is not simply connected.*

Antoine proved, more generally, that for any compact totally discontinuous set A in $\mathbb{R}^n$, there is an imbedding $f : D^n \to \mathbb{R}^n$ such that $f(S^{n-1}) \supset A$.

7 Elementary ideals-Alexander polynomials

In the last section we have seen how to get a presentation of the group of a link. It is often quite difficult to decide, from presentations, whether two groups are distinct. In this section, we give some invariants of a group that can be calculated easily from a presentation. 54

Suppose that A is a commutative ring with a unit element 1 and that M is a finitely generated A-module. Then we associate to M a finite sequence of ideals of A called elementary ideals. If M_1 is an A-module and M_2 is a submodule of M_1 then the values $m(M_2)$ taken over M_2 of linear forms m on M_1 generate an ideal which we denote by $\mathfrak{a}(M_1, M_2)$. Now we take a finite system $s = [\alpha_1, \ldots, \alpha_n]$ of generators of M and consider the free module $U = A^n$ with the natural basis $u_1, \ldots, u_n$. We denote the homomorphism of U onto M taking u_1 onto α_i by h and denote the kernel of h by V. We define the ideals $\mathfrak{a}_p(s)$ by setting

$$\mathfrak{a}_p(S) = \begin{cases} \mathfrak{a}(\overset{P}{\Lambda} U, \overset{P}{\Lambda} V) & p \geq 1, \\ (1) = A, & p \leq 0. \end{cases}$$

We have $\mathfrak{a}_p(s) \supset \mathfrak{a}_{p+1}(s)$. Now we proceed to show that this sequence of ideals essentially depends only on M. For this we examine $\mathfrak{a}(U, V)$ more closely. The linear forms on U are generated by $u_i^*(i = 1, \ldots, n)$ where u_i^* is the linear form defined by

$$u_i^*(u_j) = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j. \end{cases}$$

$u_i^*(m)$ is precisely the i-th component of m. Hence $\mathfrak{a}(U, V)$ is the ideal generated by the components of elements in V. To study the relation between $a_p(s)$ and $a_p(s')$ where s' is another system of generators of M we start with $s' = [\alpha_1, \ldots, \alpha_n, \alpha_{n+1}]$. Since s generates M we have 55

$$\alpha_{n+1} = \sum_{j=1}^{n} C_j \alpha_j$$

for some elements C_j of the ring A. We take the natural basis $u_1, u_2, \ldots, u_n, u_{n+1}$ of $A^{n+1} = U'$ and set

$$h'(u_i) = \begin{cases} \alpha_i \text{ for } i \leq n, \\ \alpha_{n+1} \text{ for } i = n+1. \end{cases}$$

We denote the kernel of h' by V'. To compute $\mathfrak{a}(U', V')$ we change the basis of U' so that the elements of V' can be obtained easily from those of V. The new basis consists of $u_1, u_2, \ldots, u_n$ and $u = u_{n+1} - \sum_{i=1}^{n} c_i u_i$. Then $h'(u) = \alpha_{n+1} - \sum_{i=1}^{n} c_i \alpha_i = 0$. And hence an element $\sum_{i=1}^{n} b_i u_i + bu$ of U' belongs to V' if and only if $\sum_{i=1}^{n} b_i u_i$ belongs to V. So $V' = V \oplus A_u$ and

$$\mathfrak{a}_{p+1}(s') = \mathfrak{a}_p(s).$$

Now, let s_1 and s_2 be any two systems of n_1 and n_2 generators. Then $s = s_1 \cup s_2$ is a system of $n_1 + n_2$ generators and it follows from the above that $\mathfrak{a}_{r+1}(s) = \mathfrak{a}_r(s_1)$, $\mathfrak{a}_{r+n_1}(s) = \mathfrak{a}_r(s_2)$.

56 Consequently

$$\mathfrak{a}_{r+n_1}(s_2) = \mathfrak{a}_{r+n_2}(s_1).$$

Let us denote the last non-zero ideal $\mathfrak{a}_p(s)$ in the sequence $\mathfrak{a}_i(s)$, where s is any system of generators, by $\mathfrak{a}_p(s) = A_1(M)$ and set

$$A_i(M) = \mathfrak{a}_{p-i+1}(s) \ (i = 1, 2, \ldots.)$$

Form the above relation it follows that this sequence $A_i(M)$ depends only on M. These ideals are called the *elementary ideals of* M. We have $A_i(M) \subset A_{i+1}(M)$ and $A_i(M) = A$ for i great enough.

Suppose that V is generated by the elements

$$v_i = \sum_{j=1}^{n} a_{ij} u_j \quad (i = 1, 2, \ldots, m).$$

Then, for the definition, it follows that $\mathfrak{a}(\wedge U, \wedge V)$ is the ideal generated by the minors of order p in the matrix (a_{ij}) and is zero if $p > n$ or $p > m$.

Now let us take a knot group $G = (a_1, \ldots, a_p; R_1, \ldots, R_q)$. Let us denote $[G, G]$ by G' and $[G', G']$ by G'' and the images of a_i in G/G' by $\overline{a}_i$. Since it is a knot group, G/G' is the infinite cyclic group. We denote its group ring by A and denote a generator of G/G' by x, the other one being x^{-1}. We have

$$A = \mathbb{Z}[x, x^{-1}],$$

the polynomial ring of two variables X, Y quotiented by the ideal $(XY - 1)$.

Let us show now that G'/G'' is an A-module, and is finitely generated. If $t \in G$ has x as canonical image in G/G', $x = tG'$, and if $\alpha \in G'/G''$, the automorphism of G'/G'' defined by $\alpha \to t\alpha t^{-1}$ depends only on x. Hence to x is associated a well determined automorphism of G'/G'' and each element of A represents a well-determined endomorphism of G'/G'', so that G'/G'' is an A-module. **57**

Suppose further that $a_1, \ldots, a_n$ generate G. Let x^{n_i} be the canonical image of a_i in G/G'; then $a_i t^{-n_i} \in G'$ and the elements t, $a_1^{t^{-n1}}, \ldots, a_n^{t^{-nn}}$ generate G. Now, the elements $t^k(a_i t^{-n_i})\ t^{-k}(i = 1, \ldots, n;\ k = 0, \pm 1, \pm 2, \ldots)$ generate G'. Let α_i be the canonical image of $a_i t^{-n_i}$ in G'/G''. There follows that $\alpha_1, \ldots, \alpha_n$ generate G'/G'' as an A-module. Moreover, the relations of the presentation of G give a set of relations defining this A-module.

Every element $\neq 0$ of $\mathbb{Z}[x, x^{-1}]$ can be written uniquely in the form

$$\pm x^k P(x),$$

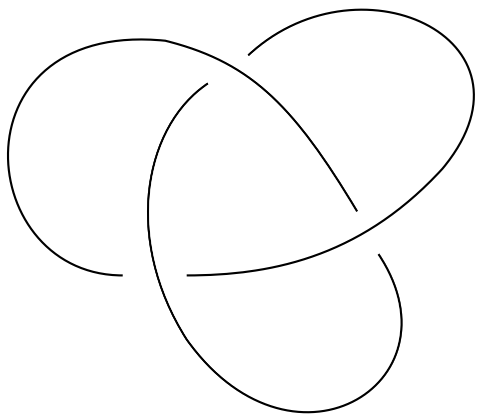
where $P(x) \in \mathbb{Z}[x]$ is such that $P(0) > 0$. We say that $P(x)$ is a normalized polynomial. The units of $\mathbb{Z}[x, x^{-1}]$ are the elements $\pm x^k$ $(k \in \mathbb{Z})$. As $\mathbb{Z}[x]$ is a unique factorization domain, there follows that any set of elements of A (containing at least one element $\neq 0$) has a greatest common divisor, (i.e. an element which divides all these elements and is divided by each of their common divisors,) represented by a well determined normalized polynomial.

In particular, the elements of the ideal $A_i(G'/G'')$ have as a greatest common divisor a normalized polynomial $A_i(x)$. These are called the *Alexander polynomials* of the knot. $A_i(x)$ is divisible by $A_{i+1}(x)$, and $A_i(x) = 1$ for i great enough. The most interesting is the first $A_1(x)$, also **58** denoted by $A(x)$. Form our definition, there follows that these polynomials are well determined by the group G of the knot and the choice of a generator x of G/G'. By taking the other generator x^{-1} instead of x, one would get a second series of polynomials. But it can be shown that the elementary ideals are invariant by the conjugation $x \mapsto x^{-1}$ in $\mathbb{Z}[x, x^{-1}]$ and that the Alexander polynomials are reciprocal. Hence the two series are identical (we shall not prove it here. But this will be verified in the examples), and *if two knots are of the same type, they have the same chain of elementary ideals and the same Alexander polynomials.*

Now we compute the Alexander polynomial and elementary ideals of some knots.

First we remark that if a group G is given by a presentation $\{a_i : R_j\}$ then $G/[G, G]$ is given by $\{a_i; R_j, a_k a_\ell = a_\ell a_k\}$.

1) **Trefoil knot.**



The group G is given by the presentation

$$G = \{a, b, c; ba = ac = cb\}.$$

59 If we set $\alpha = ac^{-1}, \beta = bc^{-1}$ then we obtain

$$G = \left\{c, \alpha, \beta; \beta c\alpha c = \alpha c^2 = c\beta c\right\}$$

or $$c, \ , \ ; \beta c\alpha c^{-1} = \alpha = c\beta c^{-1}$$

Denoting the generator of G/G' corresponding to c by x and passing to the additive notation and replacing $c\alpha c^{-1}$ by $x\alpha$, $c\beta c^{-1}$ by $x\beta$, we have the following relations:

$$(x-1)\alpha + \beta = 0 \ (\text{mod} G'')$$
$$-\alpha + \beta = 0 \ (\text{mod} G'').$$

Hence

$$A(x) = x^2 - x + 1.$$

and the second elementary ideal is (1).

2) **Torus knot of type (p, q).**

We have $G = (a, b; a^p = b^q)$, $(p, q) = 1$. We suppose that $p > 1$, $q > 1$, $p < q$, and that p', q' are such that $pq' - p'q = 1$, $p' > 0$, $q' > 0$. Now set $t = a^{-p'}b^{q'}$. Then we have

$$\begin{aligned} a = a^{pq'-p'q} &= (a^{-p'})^q (b^{q'})^q. \\ &\equiv (a^{-p'}b^{q'})^q (\text{mod} G') \\ &\equiv t^q \ (\text{mod} G') \\ \textit{and} \qquad &\equiv t^p (\text{mod} G'). \end{aligned}$$

Now let elements α, β be defined by the equations

$$a = \alpha t^q, b = \beta t^p.$$

From the paragraphs above, it follows that α, β together with their 60
conjugates with respect to t and its powers generate G'. We have $\alpha = at^{-q}, \beta = bt^{-p}$ and the relations $a^p = b^q$ and $a^{p'}t = b^q$. The first gives

$$\begin{aligned} &\alpha t^q \alpha t^q \cdots \alpha t^q = \beta t^p \cdots \beta t^p \\ \text{i.e.} \qquad &t^q t^{-q} t^{2q} t^{-2q} t^{3q} = t^p t^{-p} t^{2p} \ldots . \end{aligned}$$

Hence, passing to the additive notation, replacing $t^n \alpha t^{-n}$ by $x^n \alpha$ and $t^n \beta t^{-n}$ by $x^n \beta$, we have

$$(1 + x^q + x^{2q} + \cdots + x^{(p-1)}q)\alpha = (1 + x^p + x^{2p} + \cdots + x^{(q-1)}p)\,\beta \ (\text{mod } G'')$$

$$\text{i.e.} \qquad \frac{(x^{pq}-1)}{(x^{q}-1)}\alpha - \frac{(x^{pq}-1)}{(x^{p}-1)}\beta = 0 \pmod{G''}.$$

Similarly from the other we obtain

$$\frac{(x^{p'q}-1)}{(x^{q}-1)}\alpha - \frac{(x^{q'p}-1)}{(x^{p}-1)}\beta = 0 \pmod{G''}.$$

Hence

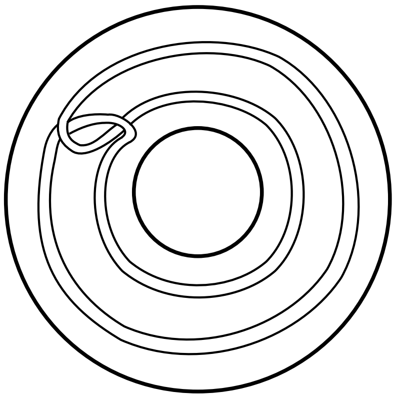
$$A(x) = \frac{(x^{pq}-1)(x^{p'q}-x^{q'p})}{(x^{p}-1)(x^{q}-1)} \text{ and the second ideal is (1) .}$$

For many other examples, see *Crowel and Fox: Introduction to the Knet Theory* (Ginn and Company, 1963).

8 Construction of 3-manifolds

61 **(a) Whitehead Manifold.**

Suppose that T is a solid torus and solid torus T' is situated in the interior of T as shown in the figure.



There exists a homeomorphism h of $\mathbb{R}^3 + (\infty) = S^3$ which fixes points outside a compact set of $\mathbb{R}^3$ and which takes T onto T'. We set

$$T^{(n)} = h(T^{(n-1)}) = h^n(T)$$

$$T^{\infty} = \bigcap_{n=0}^{\infty} \text{ and } W = S^3 - T^{\infty}.$$

We note that $T^{(n)} \subset T^{(n-1)}$ and W is a manifold. We know that $S^3 - \mathring{T}$ is a torus T_0. We set

$$T_n = h^n(T_0).$$

Then we have **62**

$$T_{n-1} \subset T_n \text{ and } W = \bigcup_{n=0}^{\infty} T_n.$$

Hence W is the union of an increasing sequence of solid tori. It has the following properties:

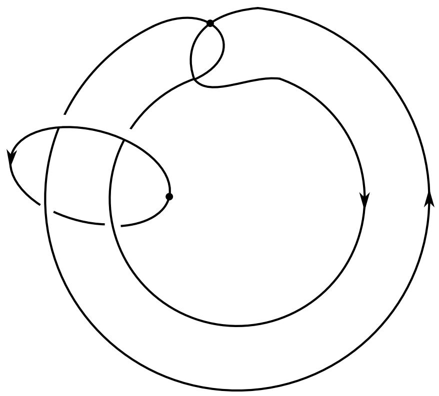
(1) W is simply connected.

(2) W is not homeomorphic to $\mathbb{R}^3$

(3) $W \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^4$.

We prove the first two of these and only sketch a proof of the third.

(1) Suppose that m is the meridian of T. Then m is homotopic to the null path in $T_1 = S^3 = \mathring{T}'$. First it is clear that $S^3 = \mathring{T}'$ is a deformation retract of $S^3 - C_1$ where C_1 is the locus of the center of D^2 in a representation of T' as $S^1 \times D^2$.



With the notation of the figure we have

$$\pi(T_1) = \{a, b; a^2 = ba\}$$
$$[m] = a^{-1}b$$

and hence $m \approx 1$.

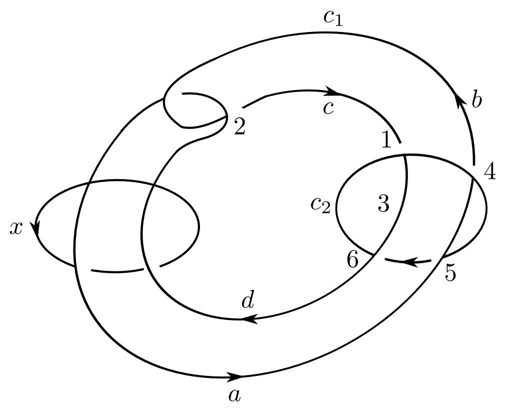
It is clear that the homotopy class of every closed path in $S^3 - \mathring{T}$ is a power of $[m]$ and hence it follows that every closed path in T_0 is homotopic to the null path in T_1. By induction it follows that every path in T_n is homotopic to the null path in T_{n+1}. Since every path in W has to be contained in some T_n it follows that W is simply connected.

63 (2) In $\mathbb{R}^3$ every compact set is contained in a compact set whose complement is simply connected. We prove that the compact set T_0 of W is not contained in a compact set of W whose complement is simply connected. If not there exists a compact set $K \subset W$ and hence $K \subset T_n$ for some n such that $W \supset T_o$ and $W - K$ is simply connected. Then the meridian of $T^{(n+1)}$ is homotopic to the null path in $W - K$ and hence in $W - T_0 \subset T - T^\infty$. We prove that this is not the case.

Let C_1 be the image of the circle $S^1 \times 0$ by a homeomorphism of $S^1 \times D^2$ onto T' and C_2 a circle in $\mathbb{R}^3 - T$ bounding a 2-disk which cuts T along a meridian. Then it is clear that $T - \mathring{T}'$ is a deformation retract of $S^3 - C_1 - C_2$ and hence

$$G = \pi(T - T') \simeq \pi(S^3 - C_1 - C_2) \simeq \pi(\mathbb{R}^3 - C_1 - C_2)$$

is isomorphic to the group of the link $C_1 \cup C_2$, given as in the figure.



Then the homotopy class of a meridian of T is given by X. With the notation of the figure the group G is given by generators a, b, c, d, e, f and the following relations: 64

(1) $db = ba$, (2) $bd = dc$, (3) $ed = ce$
(4) $ea = be$ (5) $ae = fa$, (6) $fd = de$.

Since one relation is a consequence of the others we can drop (6) and eliminate f by means of (5). We are thus left with generators a, b, c, d, e and relations 1,2,3,4. From (4) we have $a = e^{-1}be$, and from (1) $d = bab^{-1}$. Eliminating c from (2) and (3) we obtain

$$d^{-1}bd = c = ede^{-1},$$
$$\text{i.e.} \qquad be^{-1}b^{-1}ede^{-1}beb^{-1} = ede^{-1}beb^{-1}e^{-1}.$$

Hence the group is given by the presentation

$$G = \{e, b; eb^{-1}e^{-1}be^{-1}b^{-1}ebe^{-1}beb^{-1}ebe^{-1}b^{-1} = 1\}.$$

In this X is represented by the element

$$a^{-1}d = e^{-1}b^{-1}ebe^{-1}beb^{-1}.$$

Now we prove that this element is of infinite order in G. For this consider the map of b and e into $\mathbb{Z}_2 * \mathbb{Z}_2 = (\alpha, \beta, \alpha^2 = \beta^2 = 1)$ defined by

$$b \mapsto \beta, e \mapsto \beta\alpha.$$

Substituting these values in the relator we obtain the word

$$\beta\alpha\beta\alpha\beta\beta\alpha\beta\beta\beta\alpha\beta\alpha\beta\beta\beta\alpha\beta\beta\alpha\beta\alpha\beta\beta$$

Since $\alpha^2 = \beta^2 = 1$ this becomes the empty word. Hence the above map can be extended into a homomorphism of G into $\mathbb{Z}_2 * \mathbb{Z}_2$.

65 Under this $[X]$ goes into

$$\alpha\beta\beta\beta\alpha\beta\alpha\beta\beta\beta\alpha\beta = (\alpha\beta)^4$$

The word $\alpha\beta$ is reduced and it follows that $\alpha\beta$ and hence $[X]$ is of infinite order.

Now we can prove that the inclusion homomorphism $\pi(S) \to \pi(T - \mathring{T}')$, where $S = T - \mathring{T}$, is injective. Denoting the parallel of T by Y, if $X^a Y^b \approx 0$ in $T - \mathring{T}'$, then we have $X^a Y^b \approx 0$ in T and hence $Y^b \approx 0$ in T. This implies that $b = 0$. Then the result above gives $a = 0$.

Denoting the boundary of T' by S' one can prove, in a similar manner, that the inclusion homomorphism

$$\pi(S') \to \pi(T - \mathring{T}')$$

is injective. But we prefer to indicate a way of obtaining an involution of $T - \mathring{T}'$ permuting the boundaries S' and S. The following sequence of figures gives an involution of $S^3 - C_1 - C_2$ exchanging C_1 and C_2. Since $T - \mathring{T}'$ is a deformation retract of $S^3 - C_1 - C_2$ we get the result.

Thus the inclusion homomorphism

$$\pi(S') \to \pi(T - \mathring{T}')$$

is injective.

Using (3.3) and induction on n we can show that the conclusion homomorphism 66

$$\pi(h^n(S)) \to \pi(T - \overset{\circ}{T}^{(n)})$$

is injective. From this fact and the fact that the inclusion homomorphism

$$\pi(h^n(S)) \to \pi(T^{(n)} - \overset{\circ}{T}^{(n+1)})$$

is injective by (3.3) it follows that the inclusion homomorphism

$$\pi(T - \overset{\circ}{T}^{(n)}) \to \pi(T - \overset{\circ}{T}^{(n+1)})$$

is injective.

Hence the meridian of $T^{(n)}$ cannot be homotopic to the null path in $T^{(n)} - T^{\infty}$ and (2) is proved.

(3) We have seen that

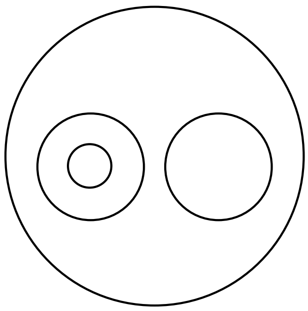
$$W = \bigcup_n T_n, \;\; T_n \subset T_{n+1}$$

where each T_n is a torus. Hence

$$W \times \mathbb{R} = \bigcup_n T_n \times I_n$$

where I_n is the interval $[-n, n]$.

Now let g be a homothety which changes the t hours T'' into a torus $g(T'')$, such that there is a sphere in $g(T'')$ containing T''.

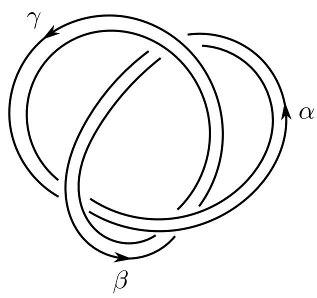


Let $T''_n = g(T'')$. It is clear that $\bigcup\limits_0^\infty T''_n = \mathbb{R}^4$. 67

Now, it can be shown that there exists homeomorphisms $\varphi_n : T_n \times I_n \to T''_n \times I_n$ such that φ_{n+1} is an extension of φ_n, because one can unknot $T_n \times I_n$ in $T_{n+1} \times I_{n+1}$. [See T. Glimm, Bull. Soc. Math France, 1962, p]

(b) Homology 3-spheres of Dehn.

Now we proceed to give a method, due to Dehn, of constructing homology 3-spheres (i.e. a compact closed manifold M such that $H_*(M) = H_*(S^3)$.), with a fundamental group G identical to its commutator subgroup G'. For this we take a tubular neighbourhood of a trefoil knot. Then we remove its interior from S^3 and attach a torus to this with a twist.



Let T be a tubular neighbourhood of a trefoil knot. Let the notation be as in the figure and let 0 be the base point. We have

$$\pi(S^3 - \overset{\circ}{T}) = (a, b, c; ad = bc = ca);$$

68 where a, b, c are the generators corresponding to the principal arcs of the trefoil. Setting $ab = bc = ca = \sigma$ we have

$$b = a^{-1}\sigma, c = \sigma a^{-1}$$
$$\sigma = bc = a^{-1}\sigma^2 a, \text{ i.e. } \sigma^2 = a\sigma a.$$

Hence $\pi(S^3 - \overset{\circ}{T}) = (a, \sigma; \sigma^2 = a\sigma a)$.

We have $\sigma a\sigma a = \sigma\sigma^2 = \sigma^2\sigma = a\sigma a\sigma$,

$$\sigma^3 a = a\sigma a\sigma a = a\sigma^3,$$

and hence σ^3 is in the centre.

T is homeomorphic to a solid torus and the fundamental group of the surface $S = T - \overset{circ}{T}$, based at the same point 0, is generated by the homotopy classes $[m]$ and [1] of a meridian m and a "parallel" ℓ, whose homotopy classes in $S^3 - \mathring{T}$ are a and $c^{-1}a^{-1}b^{-1}$. It is of course also generated by $[m]$ and $[lm^k]$, for any integer k. The homotopy class of lm^3 in $S^3 - \mathring{T}$ is $\lambda = c^{-1}a^{-1}b^{-1}a^3$. As $c^{-1}a^{-1}b^{-1} = a\sigma^{-1}a^{-1}\sigma^{-1}a$, we have

$$\lambda = a\sigma^{-1}a^{-1}\sigma^{-1}a^4 = a(\sigma^{-1}a^{-1}\sigma^{-1}a^{-1})a^5 = a\sigma^{-3}a^5 = \sigma^{-3}a^6.$$

Let T' be a solid torus and let h be a homeomorphism of $S' = T' - \mathring{T}'$ onto S, under which a parallel and a meridian of S' go into curves of S of homotopy classes λ and $a\lambda^{-k}$, where k any integer. We set

$$V_k = (S^3 - \mathring{T}) \bigcup_h T'.$$

(Using (3.2), since the meridian of S' is homotopic to the null path in 69
T' and $\pi(T')$ is generated by the homotopy class of a parallel of S', we see that $\pi(V_k)$ is the quotient of $\pi(S^3 - \mathring{T})$ by the relation $a\lambda^{-k} = 1$ or $a\sigma^{3k}a^{-6k} = 1$ or $\sigma^{3k} = a^{6k-1}$, and we get the presentation

$$\pi(V_k) = (a, \sigma; \sigma^2 = a\sigma a, \sigma^{3k} = a^{6k-1}).$$

It is clear that the quotient of this group by its commutator subgroup is trivial. Hence it follows that $H_1(V_k) = 0$ and that V_k is a homology 3-sphere. When $k = 0$ we have the 3-sphere, and when $k \neq 0$ we show that $G_k = \pi(V_k)$ is not trivial. In fact G_k and $G_{k'}$ are not isomorphic if $k \neq k'$.

For this we examine the representations of G_k into the group of motions of the non-euclidean plane. In this latter group every element different from the identity belongs to a unique one-parameter subgroup

which is its normaliser and which consists of all the motions with the same fixed points. If the representation is non-trivial then the images A and S of a and σ are such that $A \neq 1$, $S^3 = 1$. For $A = 1$ implies that $S = 1$. Further if $S^3 \neq 1$ then $AS^3 = S^3 A$ gives that A and S^3 have the same fixed points. Hence A, S belong to the same one parameter
70 subgroup and hence $AS = SA$ which implies A and hence S is 1. This is a contradiction. Hence

$$S^3 = (AS)^2 = A^{6k-1} = 1.$$

and every representations of G_k comes from a representation of $G'_k = (A, S; S^3 = (AS)^2 = A^{6k-1} = 1)$.

Now we will show that there exists a faithful representation of G'_k into the group of motions of the non euclidean plane (except if $k = 1$, in which case G'_1 is the ikosaeder group A_5), and that $6k-1$ is the maximum order of the elements of finite order. This will imply the nonisomorphy of G_k and $G_{k'}$ for $k \neq k'$.

More generally, let us consider the group

$$G(l, m, n) = \{A, B; A^l = B^m = (AB)^n = 1\} \tag{1}$$

where l, m, n are positive integers. Or, what is the same,

$$G(l, m, n) = \{A, B, C; A^l = B^m = C^n = ABC = 1\}.$$

Clearly, $G'_k = G(1, 2, 3)$ with $l = 6k - 1$ if $k > 0$ and $l = 1 - 6k$ if $k < 0$.

Let $s = \dfrac{1}{l} + \dfrac{1}{m} + \dfrac{1}{n}$. We suppose $s < 1$ and we will construct a
71 faithful representation of $G(l, m, n)$ into the group of the motions of take non euclidean plane (for $s = 1$ or $s > 1$, one would have to take the euclidean plane or the sphere and there is a similar construction).

Let Δ be a triangle in the non euclidean plane with vertices A', B', C' and angles $\alpha = \dfrac{\pi}{l}$, $\beta = \dfrac{\pi}{m}$, $\gamma = \dfrac{\pi}{n}$. Let us denote by a, b, c the symmetries of the (non euclidean) plane with respect to the sides $B'C'$, $C'A'$, $A'B'$ of Δ in that order. Then $A = bc$, $B = ca$ and $C = ab$ are rotations around A', B' and C' of angles 2α, 2β and 2γ respectively, and we have

$$A^l = B^m = C^n = ABC = 1.$$

The group G generated by A, B and C gives a representation of $G(l, m, n)$.

We will show that this representation is faithful. For this, it will be sufficient to show that if a product $X_1X_2\ldots X_n$, where X_i is an element of the set $E = \{A, B, A^{-1}, B^{-1}\}$, is the identity element of G, then the word $X_1X_2\ldots X_n$ is equivalent to the empty word under the relations given in (1).

The quadrilateral $Q = \Delta \cup c(\Delta)$, union of Δ and its reflection $c(\Delta)$, has as vertices A', B', C' and $c(C')$. It is adjacent to its four images $XQ(X, \in E)$ i. e. images by A, A^{-1}, B, B^{-1}. Take the *disjoint* union M of its images $g(Q)$ by all $g \in G$. There is a natural projection of M into the plane. Then, for any $g \in G$ and $X \in E$, let us identify the points 72
of $g(Q)$ and $gX(Q)$ which have the same projection (they are the points of the side along which $g(Q)$ and $gX(Q)$ are adjacent). After all these identifications, we get from M a surface $\tilde{M}$ and a natural projection p of $\tilde{M}$ into the plane. It is easy to see that p is a covering map, and since the plane is simply connected and M is connected, that $\tilde{M}$ is the same as the plane. Without reference to the theory of covering maps, it means that one has only to show that if $w(t)$ $(0 \leq t \leq 1)$ is a path in the plane and $x_o \in \tilde{M}$ such that $p(x_o) = w(0)$, there is a unique path $\tilde{w}(t)$ in $\tilde{M}$ such that $\tilde{w}(0) = x_0$ and $p\tilde{w}(t) = w(t)$, and if $w_s(t)$ $(0 \leq s \leq 1)$ is a homotopy in the plane, with $w_o(t) = w(t)$, there is a unique homotopy $\tilde{w}_s(t)$ in $\tilde{M}$ such that $\tilde{w}_0(t) = w(t)$ and $p\tilde{w}_s(t) = w_s(t)$. This means that *the quadrilaterals $g(Q)$ cover the plane and do not overlap*.

Now, we consider a path w closed at a point $q \in \mathring{Q}$, which does not pass through any vertex of any quadrilateral $g(Q)$ and meets the sides of the $g(Q)$ only in a finite number of simple crossings. Such a path w goes through a series of quadrilaterals

$$Q, X_1(Q), X_1X_2(Q), \ldots, X_1X_2\ldots X_{n-1}(Q), X_1, X_2\ldots X_n(Q) = Q,$$

where $X_i \in E$ and $X_1X_2\ldots X_n = 1$. We will say that the path w and the word $X_1X_2\ldots X_n$ are associated. Clearly, for each such word, representing the unit element of G, one can find such that a path associated to it.

If w is contained in the star A' i.e. the star with centre A', i.e. in 73
the union $\bigcup_{i=0}^{l-1} A^k(Q)$ of the $g(Q)$ such that $A' \in g(Q)$, then $X_i = A$ or A^{-1}

for each i and it is easy to see that the associated word $X_1X_2 \dots X_n$ is equivalent to the empty word on account of the relation $A^l = 1$. There is a similar result if w is contained in the star $\bigcup\limits_{k=0}^{m-1} B^k(Q)$ with centre B' or in the star $\bigcup\limits_{k=0}^{n-1}(AB)^k\ (Q \cup A(Q))$ with centre C'.

Now, let us take any word $T = X_1X_2 \dots X_n$ representing the unit element of G, and let w be an associated path. As w is homotopic to zero, by the method used in §3, one can find a series of paths $w_i(i = 0, 1, 2, \dots, N)$ associated to words T_i, such that $w_o = w$, $w_N \subset \overset{\circ}{Q}$ and w_i differs from w_{i-1} only in the image by some $g \in G$ of a star S of centre A', B' or C'. That means that w_i and w_{i-1} are products like $w_i = u_1u_2u_3$, $w_{i-1} = u_1u'_2u_3$, where u_2 and u'_2 are contained in the image of the same star S. There follows that T_i and T_{i-1} are products of the form $T_i = U_1U_2U_3$, $T_{i-1} = U_1U'_2U_3$, and $U'_2U_2^{-1}$ is associated to the path $U_1^{-1}\left((u'_2u_2^{-1})\right)$; the image by U_1^{-1} of the path $u'_2u_2^{-1}$, which is contained in the star S. By the remark above, $U'_2U_2^{-1}$ is equivalent to the empty word (on account of the relations (1)), there follows that U'_2 is equivalent to U_2, T_{i-1} is equivalent to T_i and $T = T_0$ is equivalent to T_N which is the empty word because $w_N \subset \overset{\circ}{Q}$.

By this, we have proved that G is a faithful representation of $G(l, m, n)$.

74 Now, let us consider an element of finite order of G. In the plane (non - euclidean like euclidean), any motion of finite order is a rotation and leaves a point fixed. For an element of G, this point is necessarily a vertex of some quadrilateral $g(Q)$ and there follows that it is conjugate to a power of A, B or C. Hence its order is a divisor of l, m, n.

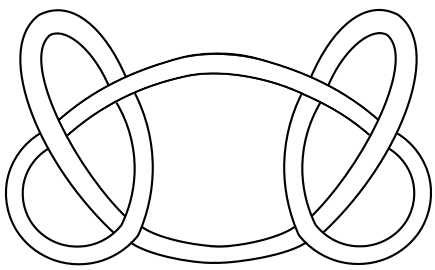
That is all what we needed to prove the non isomorphy of $G_{k'}$ for $k \neq k'$.

9 Involutions of S^4. Four dimensional contractible manifolds whose boundaries are not simply connected. One parameter group of homeomorphisms of S^5

75

Let us fix some notations. $\mathbb{R}^{n-1}$ shall be considered as a subspace of $\mathbb{R}^n$ defined by $x_n = 0$ and we shall denote the half spaces $x_n \geq 0$ and $x_n \leq 0$ bounded by $\mathbb{R}^{n-1}$ in $\mathbb{R}^n$ by $\mathbb{R}^n_+$ and $\mathbb{R}^n_-$. We shall always suppose that $\mathbb{R}^n$ is compactified with a point at infinity which allows us to identify it with S^n.

Let us consider a tubular neighbourhood T_+ of a simple arc joining two points of $\mathbb{R}^2$ in $\mathbb{R}^3$ and knotted as a trefoil knot (Figure 1). We shall suppose that T_+ is bounded by two circular discs in $\mathbb{R}^2$ and a differentiable surface lying orthogonally on the circles which bound the above discs in such a manner that the union of T_+ with its reflection T_- with respect to $\mathbb{R}^2$ is a solid torus $T = T_+ \cup T_-$ homeomorphic, even diffeomorphic to the product $D^2 \times S^1$ (Figure 1)



By taking the unit disc $|z| \leq 1$ in the complex number plane for D^2 **76**
and by denoting the angular variable on S^1 by θ, the points of T are described, thanks to the above homeomorphism of T and $D^2 \times S^1$, by (z, θ). We can suppose that $-\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}$ in T_+ and the points (z, θ) and $(z, \pm\pi - \theta)$ are symmetric with respect to $\mathbb{R}^2$. Let us call the arc described by (z, θ), where θ varies from $-\dfrac{\pi}{2}$ to $\dfrac{\pi}{2}$ as z is kept fixed, *fibre of* T_+. The union of this arc with its reflection with respect to $\mathbb{R}^2$ is a simple, closed, differentiable curve $z \times S^1$ which we *shall call a fibre of* T.

By rotating T_+ around $\mathbb{R}^2$ in $\mathbb{R}^4$ we generate a four dimensional manifold A. Each fibre of T_+ generates an S^2 which we shall call a fibre of A. Let (z, θ, ψ) be a point of a A which is obtained from the point (z, θ) of T_+ by a rotation through an angle ψ around $\mathbb{R}^2$ and $s = s(\theta, \psi)$ be the point on the standard sphere S^2 with latitude θ and longitude ψ. To each pair (z, s), $z \in D^2$, $s \in S^2$ corresponds a point of A and conversely. Thus we may write $A = D^2 \times S^2$.

The symmetry with respect to $\mathbb{R}^3$ changes the point (z, s) of A into the point (z, s') where $s' = s(\theta, \psi)$ and $s' = s(\theta, -\psi)$ are symmetric on S^2 relative to the plane containing the meridian $\psi = 0$ and $\psi = \pm\pi$. These two meridians together form the great circle S^1 of S^2 and one has

$$A \cap \mathbb{R}^3 = T_+ \cup T_- = T = D^2 \times S^1.$$

77 Each fibre of A intersects $\mathbb{R}^3$ in a fibre of T. Denoting the boundary ∂D^2 of D^2 by S^1_0 we have

$$\partial A = S^1_0 \times S^2, \partial A \cap \mathbb{R}^3 = \partial T = S^1_0 \times S^1.$$

Now let us consider the two varieties A and $B = \mathbb{R}^4 - \mathring{A}$, whose union id $\mathbb{R}^4$ and whose intersection is their boundary $\partial A = \partial B$, as two disjoint manifolds and let $h : \partial A \to \partial B$ be a homeomorphism compatible with the symmetry relative to $\mathbb{R}^3$, i.e. which changes two points of ∂A symmetric relative to $\mathbb{R}^3$ into symmetric points. By attaching A to B with the homeomorphism h we obtain a four dimensional manifold $V(h) = A \bigcup_h B$ which admits an involution J which restricted to A as well as to B reduces to the symmetry relative to $\mathbb{R}^3$. The set of fixed points of this involution is a three dimensional manifold $M(h)$ which we obtain by attaching $A \cap \mathbb{R}^3 = T$ to $B \cap \mathbb{R}^3 = \mathbb{R}^3 - \mathring{T}$ by the homeomorphism h restricted to $\partial A \cap \mathbb{R}^3 = \partial T$. If h is a diffeomorphism one can define a differentiable structure on $V(h)$ and $M(h)$ in a natural way and the involution J will then ve a diffeomorphism.

Let r_φ be the rotation of S^2 through an angle φ around the diameter perpendicular to the plane containing the great circle S^1. If s and s' are symmetric relative to this plane $r_\varphi(s)$ and $r_\varphi(s')$ are also symmetric.

Consequently the homeomorphism

$$g : \partial A \to \partial A$$

defined by setting

$$g(e^{i\varphi}, s) = (e^{i\varphi}, r_\varphi(s))$$

is compatible with the symmetry relative to $\mathbb{R}^3$. Now we have the fol- **78**
lowing proposition.

Proposition 1. *Whatever be to the homeomorphism $h : \partial A \to \partial B$ compatible with the symmetry relative to $\mathbb{R}^3$, the manifolds $V(h \circ g^2)$ and $V(h)$ are homeomorphic.*

Proof. One knows that $S0(3)$, the group of rotations of S^2 is homeomorphic to the real projective space of three dimensions and that its fundamental group is cyclic of order 2. It follows that the map of S^1_0 into $S0(3)$ which sends $e^{i\varphi}$ to $r_{2\varphi}$ is homotopic to a constant map (This fact, however, can be easily verified directly). Consequently one can find a continuous, even differentiable map of $S^1_0 \times I$ into $S0(3)$, $(e^{i\varphi}, t) \to r_{2\varphi,t}$ ($t \in I = [0,1]$) such that

$$r_{2\varphi,1} = r_{2\varphi} \text{ and } r_{2\varphi,t} = \text{ identity for } t \leq \frac{1}{2}.$$

Then by setting

$$f(te^{i\varphi}, s) = (te^{i\varphi}, r_{2\varphi,t}(s))$$

we can define a homeomorphism, even diffeomorphism

$$f : A \to A$$

which extends g^2 to the interior of A. Finally we define a homeomorphism

$$F : V(h \circ g^2) \to V(h)$$

by $F|A = f$ and $F|B =$ identity. Since the pair of points of $a \in \partial A$,
$b = h \circ g^2(a) \in \partial B$ which are identified in $V(h \circ g^2)$ is changed by F into **79**

the pair of points $f(a) = g^2(a)$ and $b = h \circ g^2(a)$ which are identified in $V(h)$ the map F is well defined.

Let us remark that this method does not prove that $V(h \circ g)$ and $V(h)$ are homeomorphic since the map of $S'_\circ$ into $S0(3)$ which takes $e^{i\varphi}$ to r_φ is not homotopic to a constant map (see Addendum).

It is evident that if one takes for h the identity then $V(h) = \mathbb{R}^4$ and $M(h) = \mathbb{R}^3$, J then reducing to symmetry relative to $\mathbb{R}^3$. By composing the identity with a power g^k of g one obtains manifolds $V_k = V(g^k)$ and $M_k = M(g^k)$. Then from proposition 1 it follows that V_{2k} *is homeomorphic to* $\mathbb{R}^4$ *(i.e.* S^4*) and even diffeomorphic to it and that the involution* J *provides a differentiable involution* J_{2k} *of* S^4 *whose fixed point set is* M_{2k}.

The interest in this arises from the following proposition whose proof shall be given later. □

Proposition 2. *The fundamental groups* $G_k = \pi(M_k)$ *of the manifolds* M_k *for* $k = 0, 1, 2, \ldots$ *are pairwise non-isomorphic.*

Consequently the manifolds M_k are pairwise non-homeomorphic and thus we have infinitely many differentiable involutions J_{2k} of S^4 which are distinct.

It follows from the proposition 1 that V_{2k+1} is homeomorphic to S^4 (see Addendum).

80 The manifold $V_{2k} = S^4$ is partitioned by M_{2k} into two manifolds V^+_{2k} and V^-_{2k} which are permuted by the involution J_{2k}. We obtain V^+_{2k} by attaching $A_+ = A \cap \mathbb{R}^4_+$ to $B_+ = B \cap \mathbb{R}^4_+$ with the homeomorphism g^{2k} restricted to $\partial A \cap \mathbb{R}^4_+$. The sphere S^2 is partitioned by S^1 into two half-spheres S^2_+ and S^2_- and each fibre $z \times S^2$ of A is partitioned by the $z \times S^1$ of T into two half-spheres $z \times S^2_+ \subset A_+$ and $z \times S^2_- \subset A_-$.

By rotating $\mathbb{R}^4_+$ around $\mathbb{R}^3$, $\mathbb{R}^5$ is generated and is partitioned by $\mathbb{R}^4$ into $\mathbb{R}^5_+$ and $\mathbb{R}^5_-$. Every half sphere $z \times S^2_+$ generates a three dimensional sphere $z \times S^3$ which is partitioned by $z \times S^2$ into two half- spheres $z \times S^3_+$ and $z \times S^3_-$ and A_+ generates a five dimensional manifold A fibred by the spheres $z \times S^3$ and partitioned by A into $\tilde{A}_+$ and $\tilde{A}_-$. We see that $\tilde{A}_+$ is obtained, in a way, from A by replacing the fibers of A, which are two dimensional spheres $z \times S^2$ by the three dimensional half spheres $z \times S^3_+$.

Thus

$$\tilde{A} = D^2 \times S^3 \qquad , \tilde{A}_+ = D^2 \times S^3_+$$
$$\partial\tilde{A} = S^1_o \times S^3 \qquad , \partial\tilde{A}_+ = (S^1_0 \times S^3_+) \cap A.$$

Likewise B_+ generates the manifold $\tilde{B}$ which is partitioned by B into $\tilde{B}_+$ and $\tilde{B}_-$. Every rotation r of S^2 extends in a natural way into a rotation $\tilde{r}$ of S^3 which preserves S^3_+ and S^3_-. Consequently the homeomorphism $g : \partial A \to \partial A$ where $\partial A = S^1_0 \times S^2$ and $g(e^{i\varphi}, s) = (e^{i\varphi}, r_\varphi(s))$ extends to a homeomorphism $\tilde{g} : S^1_0 \times S^3 \to S^1_0 \times S^3$. By attaching $\tilde{A}$ to $\tilde{B}$ with the **81**
homeomorphism $\tilde{g}^{2k}$ we obtain a manifold $\tilde{V}_{2k}$, which is partitioned by V_{2k} into two manifolds $\tilde{V}^+_{2k}$ and $\tilde{V}^-_{2k}$ the first coming from $\tilde{A}_+$ and $\tilde{B}_+$ and the second from $\tilde{A}_-$ and $\tilde{B}_-$. But the homeomorphism $f : A \to A$, which is an extension of g^2 to the interior of A, extends, in the same way, into a homeomorphism $\tilde{f} : \tilde{A} \to \tilde{A}$ which is an extension of $\tilde{g}^2$ to the interior of $\tilde{A}$ preserving $\tilde{A}_+$ and $\tilde{A}_-$. Then we obtain a homeomorphism

$$\tilde{F}_k : \tilde{V}_{2k} \to \mathbb{R}^5$$

by setting

$$\tilde{F}_k|\tilde{A} = \tilde{f}^k \text{ and } \tilde{F}_k|B = \text{ identity}.$$

Since the two points $a \in \partial\tilde{A}$ and $b = \tilde{g}^{2k}(a) \in \partial\tilde{B}$ which are identified in $\tilde{V}_{2k}$ are changed by $\tilde{F}_k$ into the two points $\tilde{f}^k(a) = \tilde{g}^{2k}(a)$ and $b = \tilde{g}^{2k}(a)$ which are identified in $\mathbb{R}^5$ the above map is well defined.

Thus $\tilde{V}_{2k}$ and $\tilde{V}^4_{2k}$ are respectively homeomorphic (even diffeomorphic) to $\mathbb{R}^5$ (i.e. S^5) and $\mathbb{R}^5_+$ (i.e. S^5_+ or D^5).

Now let us further show that $\tilde{V}_{2k}$ (therefore S^5) admits a one parameter group automorphisms whose fixed point set is M_{2k}.

In $\mathbb{R}^5$ the rotations around $\mathbb{R}^3$ from a one parameter group which changes each of the manifolds $\tilde{A}$ and $\tilde{B}$ into itself. Let R_u be the rotation through an angle u. Then there exists an automorphism Φ_u of $\tilde{V}_{2k}$ which coincides with R_u on $\tilde{B}$ and with $\tilde{F}_k^{-1} \circ R_u \circ \tilde{F}_k$ on $\tilde{A}$. In fact, the Φ_u, defined as above, is compatible with the attaching homeomorphism $\tilde{g}^{2k}$: **82**
$\partial\tilde{A} \to \partial\tilde{B}$ since two points $a \in \partial A$ and $b = g^{2k}(a) \in \partial B$ which are identified in $\tilde{V}_{2k}$ are changed into two points $\Phi_u(a) = \tilde{F}_k^{-1} \circ R_u \circ \tilde{F}_k(a) = \tilde{g}^{-2k} \circ R_u \circ \tilde{g}^{2k}(a)$ and $\phi_u(b) = R_u(b) = R_u \circ \tilde{g}^{2k}(a) = \tilde{g}^{2k}\Phi_u(a)$ which

too are identified in $\tilde{V}_{2k}$. Thus the Φ_u form a one parameter group of automorphisms of $\tilde{V}_{2k} = S^5$ whose fixed point set is M_{2k}.

As u varies from 0 to π, $\Phi_u(V^+_{2k})$ generates $\tilde{V}^+_{2k}$. By setting $\Phi(x, u) = \Phi_u(x)$ for $x \in V^+_{2k}$ and $u \in [0, \pi]$ we obtain a map

$$\Phi : \tilde{V}^+_{2k} \times [0, \pi] \to \tilde{V}^+_{2k}$$

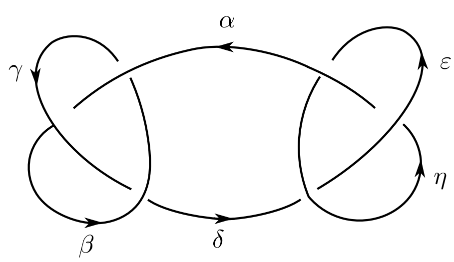
whose restriction to $(V^+_{2k} - M_{2k}) \times [0, \pi]$ is a homeomorphism onto $\tilde{V}^+_{2k} - M_{2k}$.

Let U be an open tabular neighbourhood of M_{2k} in $\tilde{V}^+_{2k}$ (homeomorphic to the product of M_{2k} by a semiopen interval) such that $\tilde{V}^+_{2k} - U$ is homeomorphic to $\tilde{V}^+_{2k}$. As u varies from 0 to π, $\Phi_u(U)$ generates an open tubular neighbourhood of M_{2k} in $\tilde{V}^+_{2k}$ and $\tilde{V}^+_{2k} - \tilde{U}$ is homeomorphic to $\tilde{V}^+_{2k}$. But $(\tilde{V}^+_{2k} - U) \times [0, \pi]$ is homeomorphic, by Φ, to $\tilde{V}^+_{2k} - \tilde{U}$. Consequently $\tilde{V}^+_{2k} \times [0, \pi]$ is homeomorphic to $\tilde{V}^+_{2k}$, i.e. to D^5.

Thus, in view of proposition 2 which shall be proved presently, we have *an infinity of compact four dimensional manifolds $\tilde{V}^+_{2k}$ with boundary $\partial\tilde{V}^+_{2k} = M_{2k}$ which is not simply connected but whose product with an interval is homeomorphic to D^5*. This fact gives immediately that manifolds $\tilde{V}^+_{2k}$ are contractible.

83 **Proof of the Proposition 2.** *To determine the group $G_k = \pi(M_k) = \pi(TU_{gk}(\mathbb{R}^3 - \overset{\circ}{T}))$ we utilise the theorem (3.2).*

$(\mathbb{R}^3 - \overset{\circ}{T})$ is nothing but the group of the knot in $\mathbb{R}^3$ formed by a fibre of T which is the union of a fibre of T_+ with its reflection in T_- as in the figure 2.



For the definition of $\pi(T)$ and $\pi(\mathbb{R}^3 - \overset{\circ}{T})$ let us choose a base point in $\mathbb{R}^2$ on the boundary of T in the portion above the arc α. The method of §4 gives the presentation of $\pi(\mathbb{R}^3 - \mathring{T})$;

$$\pi(R^3 - \mathring{T}) = (\alpha, \beta, \gamma, \varepsilon, \eta; \beta\alpha = \alpha\gamma = \gamma\beta = \beta\delta, \eta\delta = \varepsilon\eta = \alpha\varepsilon = \eta\alpha)$$

Setting $\beta\alpha = \sigma$ and $\eta\delta = \tau$ we get

$$\beta = \sigma\alpha^{-1}, \gamma = \alpha^{-1}\sigma, \delta = \alpha, \eta = \tau\alpha^{-1}, \varepsilon = \alpha^{-1}\tau$$
$$\alpha^{-1}\sigma^2\alpha^{-1} = \sigma, \alpha^{-1}\tau^2\alpha^{-1} = \tau.$$

Then one can take α, σ, τ as generators and the relations reduce to

$$\sigma^2 = \alpha\sigma\alpha, \tau^2 = \alpha\tau\alpha$$

giving **84**

$$\pi(R^3 - \mathring{T}) = (\alpha, \sigma, \tau; \sigma^2 = \alpha\sigma\alpha; \tau^2 = \alpha\tau\alpha).$$

From these relations it follows that

$$\sigma^3\alpha = \alpha\sigma^3, \tau^3\alpha = \alpha\tau^3$$
$$\gamma\alpha\beta = \alpha^{-1}\sigma\alpha\sigma\alpha^{-1} = \alpha^{-2}\alpha\sigma\alpha\sigma\alpha^{-1} = \alpha^{-2}\sigma^3\alpha^{-1} = \sigma^3\alpha^{-3}$$
$$\varepsilon\alpha\beta = \alpha^{-1}\tau\alpha\tau\alpha^{-1} = \alpha^{-2}\alpha\tau\alpha\tau\alpha^{-1} = \alpha^{-2}\tau^3\alpha^{-1} = \tau^3\alpha^{-3}.$$

The group $\pi(\partial T)$ is a free abelian group of two generators for which we take the homotopy class of the meridian m i.e. the boundary of the disc bounding T_+ containing the base point and that of l the longitude formed by the fibre of T passing through the base point. The group $\pi(T)$ is infinite cyclic and in the homomorphism $\pi(\partial T) \to \pi(T)$ induced by the inclusion l goes to a generator and m goes to the identity element, the meridian of ∂T being homotopic to zero in T. On the other hand, considering ∂T as the boundary of $R^3 - \mathring{T}$, the inclusion homomorphism $\pi(\partial T) \to \pi(R^3 - \mathring{T})$ takes m onto α and l onto the homotopy class of a fibre of ∂T which is

$$\chi = \gamma\alpha\beta\eta^{-1}\alpha^{-1}\varepsilon^{-1} = \gamma\alpha\beta(\varepsilon\alpha\eta)^{-1} = \sigma^3\tau^{-3}$$

as one deduces from Figure 2 and the above relations.

When the point $(e^{i\varphi}, s)$ describes a meridian of T, φ varying 0 to 2π and s remaining fixed in the meridian $\psi = 0$ of S^2, $r_{k\varphi}(s)$ describes the great circle of S^2 formed by the meridians $\psi = 0$ and $\psi = \pm\pi$, which corresponds to a fibre of T, k-times. Consequently $g^k(e^{i\varphi}, s) =$ **85**
$(e^{i\varphi}, r_{k\varphi}(s))$ describes a curve whose homotopy class is ml^k and therefore the homomorphism induced by g^k takes m onto ml^k which in its turn goes onto $\alpha\alpha^k = \alpha(\sigma^3\tau^{-3})^k$ by the inclusion homomorphism.

Then it results from (3.2) that G_k is obtained from $\pi(R^3 - \overset{\circ}{T})$ by adding the relation

$$\alpha\chi^k = 1$$

and thus

$$G_k = \pi(M_k) = (\alpha, \sigma, \tau; \sigma^2 = \alpha\sigma\alpha, \tau^2 = \alpha\tau\alpha, \alpha = (\tau^3\sigma^{-3})^k).$$

For $k = 0$ we have $\alpha = 1$ and consequently $\sigma = \tau = 1$ and G_o reduces to the unit element agreeing with the fact that $M_0 = \mathbb{R}^3$. Also let us remark that G_k and G_{-k} are isomorphic: since $(\sigma^3\tau^{-3})^{-k} = (\tau^3\sigma^{-3})^k$ the permutation of σ and τ realises the isomorphism.

Now it remains to prove that for $k = 0, 1, 2, \ldots$ the groups G_k are mutually non- isomorphic. This is an immediate consequence of the following proposition which we are going to prove.

Proposition. *There are exactly two minimal invariant subgroups N and N' of G_k such that G_k/N and G_k/N' admit faithful representations into the group G of non-euclidean motions or (in the exceptional case $k = 1$) into $S0(3)$. The maximum of the orders of elements of finite order in G_k/N and G_k/N' is $6k - 1$ for one and $6k + 1$ for the other.*

86 *Proof.* Let $\alpha \to A$, $\sigma \to S$, $\tau \to T$ be a non-trivial representation of G_k into G or into $S0(3)$. We have three relations

$$S^2 = ASA, T^2 = ATA, A = (S^3T^{-3})^k$$

and

$$S^3A = AS^3, T^3A = AT^3$$

□

Under these circumstances one cannot have $A = 1$ for then the representations would be trivial. In virtue of the properties of G and of $S0(3)$, if $S^3 \neq 1$ the relation $S^3A = AS^3$ gives $AS = SA$ which in view of the relation $S^2 = ASA$ gives $S = A^2$. Likewise if $T^3 \neq 1$ we have $T = A^2$. One can have neither $S^3 \neq 1$ and $T^3 \neq 1$ nor $S^3 = T^3 = 1$ since in these cases $S^3 = T^3$ and then $A = 1$ and the representation would be trivial.

If $S^3 = 1$ and $T^3 \neq 1$ one has $T = A^2$ and one is reduced to the representation of the quotient of G_k by the relation $\sigma^3 = 1$ and $\tau = \alpha^2$, i.e. N being the corresponding invariant subgroup, to the representation of

$$G_k/N = (\alpha, \sigma; \sigma^3 = (\alpha\tau)^2 = \alpha^{6k-1} = 1)$$

If $S^3 \neq 1$ and $T^3 = 1$ one is reduced to the representation of $G_k/N' = (\alpha, \tau; \tau^3 = (\alpha\tau)^2 = \alpha^{6k+1} = 1)$.

These are the groups $(2, 3, 6k-1)$ and $(2, 3, 6k+1)$ which, as we have already seen, admit a faithful representation into G with the exception of (2, 3, 5) which is the alternating group of 5 letters and which admits a faithful representation onto the icosahedral group in $SO(3)$.

Thus $6k + 1$ is the maximum of orders of elements of finite order for **87**
homomorphic images of G_k in G. This is sufficient to ensure that the groups $G_k (k \geq 0)$ are all distinct.

References. Other constructions of contractible manifolds analogous to the V_{2k} have been Mazur and Poénaru.

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Addendum

88 As Mr. G.A. Swarup has kindly pointed out to me (letter March 1968) is follows from the work of H. Gluck "The embedding of two-spheres in the four sphere" (Trans. of the Am. Math. Soc. Vol. 104, 1962, pp 308 - 333) that V_1 and consequently V_{2k+1} is also homeomorphic to $\mathbb{R}^4$. The proof is sketched here.

Given a continuous map $\gamma : S^1_0 \to SO(3)$ let us say that *the homeomorphism* $h : \partial A \to \partial A = \partial B$ defined by $h(e^{i\varphi}, s) = (e^{i\varphi}, r(s))$ where $r = \gamma(e^{i\varphi}) \in SO(3)$ is the rotation of S^2, the image of $e^{i\varphi} \in S^1_0$ by γ *is associated to* γ. Then one can state the following generalisation of the proposition 1.

If $h_1, h_2 : \partial A \to \partial B$ *are homeomorphisms associated to two homotopic maps* $\gamma_1, \gamma_2 : S^1_O \to SO(3)$ *then* $V(h_1)$ *and* $V(h_2)$ *are homeomorphic.*

If fact, let $r_1 = \gamma_1(e^{i\varphi})$, $r_2 = \gamma_2(e^{i\varphi})$; it results from the hypothesis that the map γ given by $r_1^{-1} \circ r_2$ is homotopic to zero. The homeomorphism associated to γ is $f = h_1^{-1} \circ h_2$ so that $h_2 = h_1 \circ f$ and the rest of the proof is as in proposition 1, the role of g^2 being played here by f.

Now let us denote by ρ_α the rotation of S^2 through an angle α around the axis of the poles which changes $S(\theta, \psi)$ into $S(\theta, \gamma + \alpha)$. The two
89 maps γ_1 and γ_2 defined by $\gamma_1(e^{i\varphi}) = r_\varphi$ and $\gamma_2(e^{i\varphi}) = \rho_\varphi$ are homotopic. Let $\alpha = \alpha(\varphi)$ be a continuous function monotonically increasing from 0 to 2π in the interval $- \in \leq \varphi \leq \in$, $\in$ being a positive number less than π. The map $\gamma : S^1_o \to SO(3)$ defined by setting $\gamma(e^{i\varphi}) = \rho_\alpha$ if $|\varphi| \leq \varepsilon$ and $\gamma(e^{i\psi}) =$ identity if $\in \leq |\varphi| \leq \pi$ is homotopic to γ_2 and hence to γ_1. If h is the homeomorphism associated to γ, as g is associated to γ_1, it follows from the above proposition that $V_1 = V(g)$ is homeomorphic to $V(h)$. Thus one is reduced to prove that $V(h)$ is homeomorphic to $\mathbb{R}^4$. For this *it is enough to prove that* $h : \theta B \to \partial B$ *can be extended into a homeomorphism* $\tilde{h} : B \to B$ since one will then obtain a homeomorphism of $\mathbb{R}^4$ onto $V(h)$ by taking identity in A and $\tilde{h}$ in B.

To make this extension let us remark, to start with, that the point $h(e^{i\varphi}, s)$ is obtained from the point $(e^{i\varphi}, s)$ by a rotation through an angle $\alpha = \alpha(\varphi)$ around $\mathbb{R}^2$. In fact, if $s = s(\theta, \varphi)$, then $h(e^{i\varphi}, s) = (e^{i\varphi}, s^1)$ with $s' = \rho_\alpha(s) = s(\theta, \psi + \alpha)$. Now let us consider the knot formed in $\mathbb{R}^3$

by a fibre $e^{i\varphi}\alpha S^1$ of δT. By the method of Seifert (Uber des geschlecht von knoten, Math. arn. vol. 10, pp 571-592) one can construct a surface F_φ which is compact, orientable, differentiable and contained in $R^3 - \overset{\circ}{T}$ and is bounded by this fibre $e^{i\varphi} \times S^1$. One can construct it in such a way that it is symmetric relative to $\mathbb{R}^2$. Moreover, for sufficintly small $\varepsilon > 0$, one can make this construction for every φ with $|\varphi| \leq \varepsilon$ in such a way that the set of surfaces F_φ from a fibration of a three dimensional **90**
manifold which is contained in $R^3 - \overset{\circ}{T}$ and is bounded by F_ε, $F_{-\varepsilon}$ and the portion of $\partial T = S^1_\circ \times S^1$ corresponding to the arcs $|\varphi| \leq \varepsilon$ of $S^1_\circ$. By rotation around $\mathbb{R}^2$, F_φ generates a three dimensional manifold $W_\varphi \subset B$. Let B' be the union of $W_\varphi, |\varphi| \leq \varepsilon$. This is a manifold fibred by W_φ. The homeomorphism $\tilde{h} : B \to B$ which reduces to the identity outside B' and coincides on $W_\varphi \subset B'$ with the rotation through the angle $\alpha = \alpha(\varphi)$ around $\mathbb{R}^2$ gives a desired extension of h to B.