Lectures On
Old And New Results On
Algebraic Curves

By

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Tata Institute Of Fundamental Research,
Bombay
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Introduction

The aim of the present course is to give a proof, due to Hans Grauert, of an analogue of Mordell's conjecture. Mordell's conjecture says that if C is a curve, of genus ≥ 2, defined over a number field K, then set C_K of K-rational points C is *finite*. This conjecture applies in particular to Fermat's curve $x^n + y^n = 1 (n \geq 4)$. 1

As a matter of notation if V is an algebraic variety defined over a field K, V_K will denote the set of K-rational points of V. If V is an affine (resp. projective) variety, we mean, by a $K-$rational point of V, a point whose affine coordinates (resp. ratios of homogeneous coordinates) all lie in K.

An analogue of Mordell's conjecture has recently been stated and proved by Ju Manin and Hans Grauert. In this, number fields are replaced by function fields.

Theorem. *Let k be an algebraically closed field of characteristic 0, and K a function field over k. If C is a curve of genus ≥ 2 defined over K such that C_K is infinite, then*

a) C is birationally equivalent, over K, to a curve C' defined over k

b) $C'_K - C'_k$ is finite (in other words, almost all points of C_K come from k).

The algebraist Manin ([3]) has given an analytical proof, in which 2
$k = \mathbb{C}$; the result, of course, remains valid for arbitrary k by the principle of Lefschetz. The analyst Grauert ([2]) gives a purely algebro-geometric proof, a large part of which is valid in characteristic $p \neq 0$. The finishing touch in characteristic p has been provided by the lecturer ([5], [6]):

Let k be an algebraically closed field of characteristic $p \neq 0$, K a function field over k and C a curve defined over K with absolute genus ≥ 2, such that C_K is infinite. Then,

(a) C is birationally equivalent(over some field) to a curve C' defined over k.

But unlike in characteristic 0, (see the example at the end of the course) this birational equivalence is in general *not* defined over K: one cannot expect $C'_K - C'_k$ to be finite; indeed, if C is birationally equivalent to a curve C' defined over a finite field $\mathbb{F}_q$, then for any $x \in C'_K - C'_k$, all the points x^{q^n} obtained from x by iterated applications of the Frobenius automorphism $x \to x^q$ (the point whose affine coordinates are the q^{th} powers of the affine coordinates of x) are again in $C'_K - C'_k$. Interestingly enough, it turns out that these are the only exceptional cases; more precisely, if C is *not* birationally equivalent to any curve defined over a finite field, then the birational equivalence $C \sim C'$ (C' defined over k) is defined over K. One can also prove that, in any case, the birational equivalence $C \sim C'$ is defined over a finite galois extension K' of K
3 and that all the points of $C'_K - C'_k$ may be obtained from a finite number among them, by applying the Frobenius process.

We assumed that the *absolute* genus of C (i.e.. the genus of C over the algebraic closure $\overline{K}$ of K) is ≥ 2; this is stronger than the assumption genus $_K C \geq 2$, since the genus of a curve may very well drop by an inseparable base-extension; a classical example is the curve

$$Y^2 = X^p - a$$

(with $p \geq 3$ and $a \in K - K^p$) whose relative genus is $\dfrac{p-1}{2}$ and absolute genus is 0. At present, nothing is known for curves of relative genus ≥ 2 and absolute genus 0 or 1.

The key to the proof is *not* the inequality $g \geq 2$ but the equivalent inequality $(2g - 2) > 0$ which means that the canonical divisor on C is *ample*.

The proof of Grauert's theorem may be divided into two parts.

1) Proving that C is birationally equivalent to a curve C' defined over k. This is the hardest and most original part.

2) Studying $C'_K - C'_k$. Here we are in the midst of nice old theorems on algebraic curves. For K may be viewed as the function field $k(D)$ of an algebraic variety D over k then the points of $C'_K - C'_k$ correspond to nonconstant rational maps of D in C' over k. The statement b) of the theorem says that these maps are finite in number in charac. 0; in charac. p. the separable ones are finite in number: We will 4
assume in our proof that D first a curve and then pass to the general case by a simple induction. We have a theorem of F. Severi ([8] ,0. 291). the separable involutions of genus ≥ 2 on a curve D (i.e., the isomorphism classes of non-constant separable rational maps of D into curves of genus ≥ 2) are finite in number. This is somewhat stronger than the finiteness of $C'_K - C'_k$ since the image curve C' is *not* fixed in Severi's theorem; but this stronger statement will be needed in clarifying the charac. p case. As a corollary we get a well known theorem of *H.A.* Schwarz and F. Klein: for a curve D of genus ≥ 2, Aut D is finite.

The lecturer has felt that it will be more germane to the spirit of Grauert's theorem (in which fields and rationality questions play a prominent part)- or may be easier for himself - to use the older algebro-geometric language of Weil and Zariski which, by now, is barely distinguishable from the language of the ancient Italian School. Many high powered classical tools of geometry will be used, (eg. the intersection theory, Chow coordinates, Zariski's Main Theorem etc.) An introductory chapter will give the necessary definitions and state (mostly without proofs) the theorems that will be used. A second chapter will give the theory of algebraic curves, with an emphasis on correspondences. The last chapter will give Grauert's proof proper.

We remark, that, for the sake of ease, we will mostly be doing our 5
geometry over "big" (universal) fields (i.e. fields that have infinite transcendence degree over the prime field).

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Chapter 1

Algebro-Geometric Background

1. Algebraic Varieties: Affine and Projective

a) We consider the affine n-space K^n over an algebraically closed field 6
K. An algebraic set H in K^n is, by definition, the set of common zeros $(x) = (x_1, \ldots, x_n) \in K^n$ of a family of polynomials $(F_\alpha), F_\alpha \in K[X_1, \ldots, X_n]$, or equivalently, of polynomials in an ideal $\mathscr{O}$ of $K[X_1, \ldots, X_n]$. A finite union of algebraic sets, and an arbitrary intersection of algebraic sets are easily seen to be themselves algebraic sets so that, in K^n, we have a topology, called the Zariski topology, whose closed sets are precisely the algebraic sets in K^n. These algebraic sets are often called the closed sets of K^n or the *affine algebraic sets.*

Given an affine algebraic set $H \subset K^n$ we define the *ideal of H* $\mathscr{J}(H)$, as

$$\mathscr{J}(H) = \{F \in K[X_1, \ldots, X_n] : F(x) = 0 \ \forall (x) \in H\}.$$

The *affine ring of H*, or the *coordinate ring of H* is, by definition, the ring $K[X_1, X_2, \ldots, X_n] \big/ \mathscr{J}(H) = K[H]$. We remark that the elements of $K[H]$ are functions on H with values in K, in a natural manner.

We say that an affine algebraic set H is *irreducible* if it is non-void and is *not* the union of two proper algebraic subsets; it is not difficult
7 to see that H is irreducible $\Leftrightarrow \mathcal{J}(H)$ is a prime ideal of $K[X_1,\ldots,X_n] \Leftrightarrow K[H]$ is a domain. An irreducible algebraic set is often called a *Variety* (We may also use the term sometimes for algebraic sets). One can show that any algebraic set H in K^n can be written, in a unique manner, as a finite irredundant union of varieties in K^n

b) Let k be a subfield of K and V an algebraic set in K^n. We say that V is defined over k, or that k is a *field of definition* for V if $\mathcal{J}(V)$ admits a set of generators in $k[X_1,\ldots,X_n] \subset K[X_1,\ldots,X_n]$, or equivalently, if $\exists$ a finite type k-algebra $k[x_1,\ldots,x_n]$ such that $K[V] \simeq k[x_1,\ldots,x_n] \otimes_k K$. As, by the basis-theorem of Hilbert, $\mathcal{J}(V)$ admits finitely many generators in $K[X_1,\ldots,X_n]$, any algebraic set in K^n admits a field of definition which is of finite type over the prime field.

Consider a variety V in K^n; then $K[V]$ is a domain. Its field of fractions is called the *function field* (or the rational function field) of V and denotes by $K(V)$. If k is any subfield of K which is a field of definition for V, then one defines the k-ideal of V by

$$\mathcal{J}(V) = \{G \in k[X_1,\ldots,X_n] : G(x) = 0 \ \forall (x) \in V\}$$

and the k-ring of functions on V by

$$k[V] = k[X_1,\ldots,X_n]/\mathcal{J}_k(V) :$$

one sees that $k[V]$ is again a domain, and further that $k[V] \otimes_k \bar{k}$, ($\bar{k}$ =
8 the algebraic closure of k) is also a domain. One can check that this last fact is equivalent to the fact that the k-function field $k(V)$ = the field of fractions of $k[V]$ is a *regular* extension of k (i.e. is separately generated over k and $\bar{k} \cap k(V) = k$) (cf. [4])

Let V be a variety in K^n and k a field of definition for V. Consider a point $(y) = (y_1,\ldots,y_n) \in V$ and the homomorphism $k[X_1,\ldots,X_n] \to k[y_1,\ldots,y_n] \subset K$ defined by $X_i \to y_i$. By the definition of $k[V]$ this homomorphism admits a factorization

$$\begin{array}{ccc} k[X_1,\ldots,X_n] & \longrightarrow & k[y_1,y_2,\ldots,y_n] \subset K \\ & \searrow \quad \nearrow_{\varphi} & \\ & k[V] & \end{array}$$

We say that the point (y) is a *generic point* for V over k if φ is an isomorphism (note that the irreducibility of V over k is necessary for the existence of such a point). One can show that a generic point for V/k exists if K has infinite transcendence degree over k.

On the other hand, let k be a subfield of K and $(x) \in K^n$ such that $k(x) = k(x_1, \ldots, x_n)$ be a regular extension of k. If one defines an ideal $\mathcal{J}$ in $k[X_1, \ldots, X_n]$ by $\mathcal{J} = \{F \in k[X_1, \ldots, X_n] : F(x) = 0\}$ then the polynomials of $\mathcal{J}$ define a variety V in K^n such that

(i) (x) is a generic point of V/K **9**

(ii) $k[V] \simeq k[X_1, \ldots, X_n]/\mathcal{J}$

(iii) $$\begin{aligned} \mathcal{J}(V) &= \mathcal{J}_k(V).k[X_1, \ldots, K_n] \\ &= \mathcal{J}.K[X_1, \ldots, X_n] \end{aligned}$$

V is called the *locus* of the point (x) in K^n.

Example. $k = \mathbb{Q}, K = \mathbb{C}, (x) = (e^2, e^3)$; then the locus of (x) is the curve $X_1^3 - X_2^2 = 0$.

c) Dimension

Let V be an (irreducible) variety in K^n and $K(V)$ the function field of V. Then $K(V)$ is of finite type over K; its transcendence degree over K, which is finite, is called the *dimension* of the variety V. Once has then the classical

Theorem. *dimension of V= the Krull dimension of* $K[V]$.

We remark that the above equality makes the definition of the dimension of V more intuitive: it says precisely that dim. V is the maximum length of strictly increasing chains of subvarieties of V.

Any point $P \in K^n$ is a variety of dimension 0; more precisely we have: a variety $V \subset K^n$ has dimension 0

$\Leftrightarrow$ V is a point

$\Leftrightarrow$ $\mathcal{J}$ (V) is a maximal ideal

(This is essentially the Hilbert-Zero-Theorem).

Proposition. *A variety V in K^n ban dimension* $(n-1)$ **10**

$$\Leftrightarrow \mathcal{J}(V) \text{ is principal}$$

Proof. $\Rightarrow$: V has dimension $(n-1)$ implies that $\mathcal{J}(V) = 0$. Thus $\exists$ a non-zero polynomial in $\mathcal{J}(V)$ and hence a non-zero irreducible polynomial F in $\mathcal{J}(V)$. Then $0 \neq (F) \subset \mathcal{J}(V)$; as (F) and $\mathcal{J}(V)$ are both prime ideals, and as $\dim K[V] =$ coheight $\mathcal{J}(V) = (n-1)$, and $\dim K[X_1, \ldots, X_n] = n$ one gets $(F) = \mathcal{J}(V)$. □

$\Leftarrow$: If $\mathcal{J}(V) = (F)$, one has $K[V] = K[x_1, \ldots, x_n]$ with $F(x_1, \ldots, x_n) = 0$; F involves at least one variable, say X_n, non-trivially. Thus, x_n is algebraic on $K(x_1, \ldots, x_{n-1})$. On the otherhand, if $\exists G \neq 0$ in $x_{11}, \ldots, x_{n-1}$ such that $G(X_1, \ldots, X_{n-1}) = 0$ then $G \in (F)$; as F involves X_n and G does not, this is impossible. Therefore $X_1, \ldots, X_{n-1}$ are algebraically independent over K. Q.E.D

A variety in K^n of dimension $(n-1)$ is called a *hypersurface* of K^n; the above proposition says merely that hypersurfaces are precisely those varieties which admit a single "equation".

d) The concept of an "abstract variety"

Let V be an algebraic set in K^n j for any $P \in V$, there is a maximal ideal $\mathcal{M}_P$ of the coordinate ring $K[V]$, namely, the ideal of functions in $K[V]$, null at P. The local ring $\mathcal{O}_P = K[V]_{\mathcal{M}_P}$ is called the *local ring $\mathcal{O}_P$ of P* (or of V at P). More generally, if W is an irreducible subvariety
11 of V, then $\exists$ a prime ideal $\mathcal{P}$ of $K[V]$, namely, the ideal of functions in $K[V]$ null on W: the local ring $K[V]_{\mathcal{P}}$ is called the *local ring $\mathcal{O}_W$ of W in V*. By the Hilbert Zero theorem, for any prime ideal $\mathcal{P}$ of $K[V]$, $\exists$ a subvariety W of V such that $\mathcal{O}_W = K[V]_{\mathcal{P}}$.

It is not difficult to see that the pairs $(P, \mathcal{O}_P)_{P\in V}$ form, in a natural way, a subsheaf of the sheaf of functions on V in to K_p: thus, we get, what is called a *ringed space*, $(V, \mathcal{O})$. A pair $(X, \mathcal{O}_X)$ where X is a topological space space and $\mathcal{O}_X$ a given a sheaf of rings on X, is called a ringed space : here, we shall be interested only in the case where all the stalks $\mathcal{O}_x, x \in X$, are local rings; thus we will assume this additional condition always satisfied). If $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ are two ringed spaces, a *morphism* $(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)$ is a pair (u, θ) where u is a continuous map of topological spaces and θ is a morphism (of sheaves of local rings over Y) $\mathscr{O}_Y \to u_*(\mathscr{O}_X)(u_*(\mathscr{O}_X)$ is the "direct image" of $\mathscr{O}_X$ under $u)\theta$; is essentially a "nice collection": $\theta = (\theta_x)_{x \in X}$, where $\theta_x : \mathscr{O}_{u(x)} \to \mathscr{O}_x$ is a homomorphism of local rings (For details see EGA. Ch. I).

Our ringed spaces can thus be made objects of a category.

Definition. *An (abstract) prevariety X over K is a ringed space $(X, \mathscr{O}_X)$ such that there is a finite open covering $(U_i)_i$ of the underlying space X with the following property:*

$\forall i$, there is an affine variety V_i over K such that the restricted ringed space $(U_i, \mathscr{O}_X \mid U_i)$ is isomorphic to the ringed space $V_i, \mathscr{O}_{Vi}$ of the affine variety V_i.

The notions of irreducibility, rational maps, morphisms, subs., prod- **12**
ucts, fields of definition and a generic point of an irreducible prevariety over a field of definition, can all be easily carried over. In particular, one defines the *diagonal map* $\Delta_X : X \to X \times X$ for a prevariety X.

Definition. *We say that a prevariety X is a variety if the diagonal map Δ_X is closed*

We remark that this condition is analogous to the Hausdorff axiom on a topological space.

In the category of varieties isomorphisms will be often called *biregular maps*. We say that a morphism f biregular at a point $x \in X$, if the stalk-map $\mathscr{O}'_{f(x)} \to \mathscr{O}_x$ defined by f is an isomorphism. One can show that biregularity at x implies biregularity in a neighbourhood of x.

e) Projective Varieties

Let K be an algebraically closed field. The *projective n-space* $\mathbb{P}_n(K)$ over K is, by definition, the quotient of $K^{n+1} - (0)$ for the action of K^* by scalar multiplication. An algebraic set in $\mathbb{P}_n(K)$ is the image of an algebraic cone in K^{n+1}. Any point in $\mathbb{P}_n(K)$ can be represented, upto a scalar multiplication, by an $(n+1)$-tuple $(x_0, \ldots, x_n)$, the $x_i \in K$, *not* all zero. Such a representation is called a system of (homogeneous) coordinates (x) for that point. An algebraic set in $\mathbb{P}_n(K)$ is then the set of points $(x) \in \mathbb{P}_n(K)$ such that $F_\alpha(X) = 0$ for a family of homogeneous polynomials $(F_\alpha), F_\alpha \in K[X_0, \ldots, X_n]$. One can again define a topology on **13**

$\mathbb{P}_n(K)$ (the Zariski topology) whose closed sets are the algebraic sets in $\mathbf{P}_n(K)$. One again defines as before the notion of a projective variety in $\mathbf{P}_n(K)$. For any algebraic set V in $\mathbb{P}_n(K)$, one defines the homogeneous ideal $\mathscr{H}\mathcal{J}(V)$ of V as the ideal generated by all homogeneous polynomials in $K[X_0, \ldots, X_n]$ vanishing on V, and the homogeneous coordinate ring or the *graded ring* of V as the quotient $K[X_0, \ldots, X_n]/\mathscr{H}\mathcal{J}(V)$.

A projective algebraic set V is a variety $\iff \mathscr{H}\mathcal{J}(V)$ is a prime ideal *not* containing the ideal $(X_0, \ldots, X_n)$. One defines as before the notion of a (homogeneous) generic point for a projective variety V, over a subfield k of K.

One has a bijective map of K^n onto the open set $X_0 \neq 0$ of $\mathbb{P}_n(K)$ given by $(y_1, \ldots, y_n) \longmapsto (1, y_1, \ldots, y_n)$. This map is a homeomorphism on to its image $X_0 \neq 0$, which we will call K_0. The hyper-plane H_0,given by $X_0 = 0$,is called the hyper-plane at infinity for the affine subspace K_0. Similarly we have hyperplanes $H_i \equiv X_i = 0$, whose complements K_i are affine,$1 \leq i \leq n$. Clearly $\mathbb{P}_n(K) = \bigcup\limits_{i=0}^{n} K_i$. If U is an affine algebraic set in K^n. given by prescribing a system of generators for $K[U], K[U] = K[X_1, \ldots, X_n]$ say, we get an imbedding of U as a subset of K_0, hence of $\mathbb{P}_n(K)$. The closure of U in $\mathbb{P}_n(K)$ is called a *projective closure* of U.

14 Let V be a projective variety in $\mathbb{P}_n(K)$ then $V = \bigcup\limits_{i-0}^{n} V_i$ where $V_i = V \cap K_i = V - H_i$, each V_i as ***** of K_i is an affine algebraic set. Further, if $K[x_0, \ldots x_n]$ with $x_0 \neq 0$ is the homogeneous coordinate ring of V and if we set $y_i = \dfrac{x_i}{x_0}, 1 \leq i \leq n$ (then the y_i are in the quotient field of $K[V]$) then $V_0 = V - H_0$ is an affine variety of affine ring $A_0 = K[y_1, \ldots, y_n]$ and $V_i, i \geq 1$, if nonvoid, is an affine variety of affine ring

$$A_i = K\left[\frac{1}{y_i}, \ldots, \frac{y_n}{y_i}\right].$$

To show that a projective to show variety V is a variety in the sense of (α), it only remains to show that the diagonal of V is closed; we leave the verifications to the reader.

Finally, we define the function field of a projective variety V as the

subfield $K(V)$ of the quotient field of $K[V]$ defined by $K(V) = \left\{\frac{a}{b}:\right.$
$\left. a, b \in K[V], b \neq 0, a, b \text{ are of the same degree in the gradation on } K[V]\right\}$

If V is *not* contained in H_i, then one can show that $K(V)$ is canonically isomorphic to the quotient field of A_i (in the above notation).

Theorem 1. *A projective variety V is complete in the following sense: Every valuation ring $\mathscr{O}$ of $K(V)$ contains some A_i.*

Proof. Let ω be the valuation defined by $\mathscr{O}$ on $K(V)$. If $\omega(y_i) \geq 0$ for all i, then $\mathscr{O} \supset A_0$; otherwise choose an i such that $\omega(y_i)$ is minimum; then $\omega(y_i) < 0$ and $\mathscr{O} \supset A_i$. □

Remark. One can define fields of definition for projective varieties as for affine varieties. If k is a field of definition for V, one can define $k[V]$ as before and V is a finite union at affine subvarieties for which again k is a field of definition. All our computations above go through and the obvious modifications of Theorem 1 is also valid. **15**

Theorem 2. *If V is a projective variety, then for any variety U the projection p of $V \times U$ on U is a closed map.*

Proof. We have to show that for any closed subset H of $V \times U$, $p(H)$ is closed in U. We may assume V, U, H all to be irreducible. Let then $(y, x) \in V \times U$ be a generic point for H over a field of definition k of V, U and H. If $x' \in \overline{p(H)}$ then x' belongs to the locus of x over k in U, and there is a homomorphism (over k) $k[x] \xrightarrow{\varphi} k[x']$,mapping x on x'. As $k[x] \subset k(V \times U)$ and as K is algebraically closed, φ extends to a valuation ring R of $k(V \times U)$ as a homomorphism $\overline{\varphi} : R \to K$. By (the remark following) Theorem 1, there is a system of affine coordinates $z_1, \ldots, z_n$ of $(y) \in V$ such that all the z_i are in R: thus there is a point $(y') \in \mathbb{P}_n(K)$ with affine coordinates $(\overline{\varphi}(Z_1), \ldots, \overline{\varphi}(Z_n))$. But as $k[x', y'] \subset k, k[x, y] \subset k(U \times V)$ and $\overline{\varphi}$ is a homomorphism mapping (x, y) on (x', y') it follows that $(y', x') \in H$ and $x' \in p(H)$. □

Finally we also draw the attention of the reader to the fact that the product of two projective spaces $\mathbb{P}_n(K), \mathbb{P}_q(K)$ is a projective variety;

this is shown by the *Segre imbedding*

$$\mathbb{P}_n(K) \times \mathbb{P}_q(K) \longrightarrow \mathbb{P}_{(n+1)(q+1)-1}(k)$$
$$((x_i), (y_j)) \longmapsto (x_i y_i).$$

16 **f) Dimension Theorems.**

If A is a noetherian ring, $x \in A$ and $\mathscr{Y}$ a minimal prime ideal of Ax then $ht\mathscr{Y} \leq i$ (Krull); if x is not a zero divisor of A, then $ht\mathscr{Y} = 1$.

1) If V is an affine variety of dimension d in K^n, and H a hyper surface in K^n, then every component of $V \cap H$ has dimension d or $(d-1)$.

   If $H \equiv (f = 0)$, $f \in k[X_1, \ldots X_n]$ then $V \cap H$ is given by $\bar{f} = 0$ in $A = K[V]$ ($\bar{f}$ image of f in $K[V]$) and its components are given by the minimal prime ideals of $A\bar{f}$ so that our assertion follows by our remark at the beginning.

   If V is irreducible and $V \not\subset H$ then the same shows that every component of $V \cap H$ had dimension $(d-1)$.

2) By an inductive argument one gets that: if V is affine, of dimension d and if $H_1, \ldots, H_q$ are hypersurfaces, then every component of $V \cap H_1 \cap \cdots \cap H_q$ has dimension $\geq (d-q)$.

3) If V and W are affine algebraic sets in K^n and C a component of $V \cap W$ then $\dim C \geq \dim V + \dim W - n$.

   In fact $V \cap W$ is isomorphic in a natural way with $(V \times W) \cap \Delta$ in $K^n \times K^n$, Δ being the diagonal in $K^n \times K^n$, thus our assertion follows from the facts that $\dim(V \times W) = \dim V + \dim W$, that Δ in $K^n \times K^n$ is the intersection of n hyperplanes and 2) above.

   We remark that 3) does *not* say that $V \cap W \neq \phi$ even if $\dim V + \dim W - n \geq 0$; $(e, .g)$ two parallel lines in K^2. However we have the following:

17 4) $\mathscr{G}V, W$ are projective varieties in $\mathbf{P}_n(K)$ such that $\dim V + \dim W - n \geq O$, then $V \cap W \neq \phi$, and each component C of $V \cap W$ has dimension $\geq \dim V + \dim W - n$.

In fact V, W are images of cones V' W' in K^{n+1} has $V' \cap W' \neq \phi$ since the origin is in $V' \cap W'$ are both cones and as

$$\dim V' + \dim W' - (n+1) = \dim V + \dim W - (n-1) > 0$$

there is a ray throughout the origin in K^{n+1} completely contained in $V' \cap W'$, this implies that $V \cap W \neq \phi$ in $\mathbb{P}_n(K)$; any component C of $V \cap W$ corresponds to a component C' of $V' \cap W'$ so that by 3)

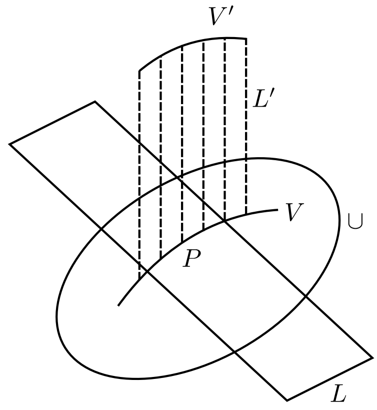
$$\begin{aligned} \dim C' &\geq \dim V' + \dim W' - (n+1) \\ &= \dim V + \dim W - n + 1 \end{aligned}$$

and $$\dim C = \dim C' - 1 \geq \dim V + \dim W - n$$

5) Let U be an algebraic variety over an algebraically closed field K. We say that a point $P \in U$ is *simple* if the local ring $\mathscr{O}_p$ of V at P is *regular.*

Proposition. *If P is a simple point on U and V and W are subvarieties of U passing through P, then any component C of VW passing through P has dimension* $\geq \dim V + \dim W - \dim U$.

Proof. The question being local we may assume that U is an affine variety imbedded in K^n and V, W are closed irreducible subvarieties of U. If $d = codim_{K^n} U$ then $\dim U = (n-d)$: and P being simple on U, the tangent space of U at P is a linear subvariety L in K^n of dimension = $(n-d)$. Let L' be a linear d-dimensional variety transverse to L and V' 18
be the cylinder parallel to L with base V (*locally*, at $P, V' \simeq V \times L'$). Then dimension of V' is equal to $(d + \dim V)$. □



It is not difficult to show that the only component of $V' \cap U$ passing through P is V. Thus, we have only to show that for any component C of $V' \cap W$ passing through $P, \dim C \geq (\dim V + \dim W - \dim U)$.

But by 3) we have:

$$\begin{aligned} \dim C &\geq \dim V' + \dim W - n \\ &= \dim V + \dim W - (n - d) \\ &= (\dim V + \dim W - \dim U) \end{aligned}$$

Q.E.D.

g) Zariski's main theorem (ZMT)**.**

A point P on an algebraic variety V is said to be *normal* of the local ring $\mathscr{O}_P$ of V at P is *integrally closed* .

An affine variety V is normal (i.e. each point $P \in V$ is normal) $\Longleftrightarrow K[V]$ is a normal domain.

19 If V is an affine variety and $K[V]$ its coordinate ring then the integral closure $K[V]'$ of $K[V]$ in $K(V)$ is a K-algebra of finite type which is a domain and therefore is the coordinate ring of at affine variety V^*, which is normal; and one has a morphism $V^* \xrightarrow{p} V$ defined in a natural way which is birational and onto.

We also remark that the same procedure can be adopted in the case of any arbitrary algebraic variety V (cf. [7]); one has only to construct pairs of the type (U_i^*, p_i) for an affine open covering (U_i) of V and patch up. We thus obtain a normal variety V^* and a birational, onto, morphism $V^* \xrightarrow{p} V$ such that the pair (V^*, p) is universal for morphisms from normal varieties V' to V. Then V^* is called the *normalisation* of V.

We shall show that if V is projective then V^* is also projective. If V is a projective variety then its homogeneous coordinate ring $K[V]$ is a graded domain and its normal closure B' in its quotient field is again a graded domain (is contained in the quotient-ring $F^{-1}A, F$ the set of homogeneous (non-zero) elements). Let $B' = \sum\limits_{n\geq 0} B'_n$ be the gradation of B' ; as above B' is a finite type K-algebra. But one cannot assert that B' is generated over K by homogeneous elements of degree 1; however, it is not difficult to see that $\exists\ d \in \mathbb{Z}^+$ s.t. $B'' = \sum\limits_{n\geq 0} B'_{nd}$ is a graded domain, and a finite type K-algebra, generated by homogeneous elements of degree 1 so that B'' defines a projective variety V^*, normal, with a birational, onto regular map $V^* \xrightarrow{p} V$ with finite fibres; so we are through. We shall now state the following theorem without proof.

Theorem. *Let A be a domain and B an over-domain which is a finite* 20
type A-algebra, $B = A[x_1, \ldots, x_n]$. Let p be a prime ideal in B which is both minimal and maximal in the set of prime ideals of B having the intersection $p \cap A$ with A. Then if A' is the integral closure of A in B, one has $B_p = A'_{p\cap A'}$.

Remark. Finiteness conditions on A are *not* needed, as shown by Grothendieck, or more elementarily, by C. Peskine: "Une generalisation du main théorème de Zariski", Bull, Soc. Math. 1966.

Corollary. *If A is integrally closed in B then*

$$B_p = A_{p\cap A}.$$

Geometrically the above theorem says that if $f : Y \to X$ is a generically surjective (i.e., dominant) morphism of irreducible varieties and if $y \in Y$ is isolated in the fibre $f^{-1}(f(y))$, then $\mathscr{O}_y$ is a ring of fractions of some finite extension of $\mathscr{O}_{f(y)}$. If, in addition, we assume that f is birational ($K(X) = K(Y)$) and $f(y)$ normal in X then $\mathscr{O}_y = \mathscr{O}_{f(y)}$ i.e. f is biregular at y.

Theorem (ZMT) . *Let X, Y be irreducible varieties and f a birational map $Y \to X$. If y is a point in Y which is isolated in the fibre $f^{-1}(f(y))$ and if $f(y)$ is normal in X, then f is biregular at y.*

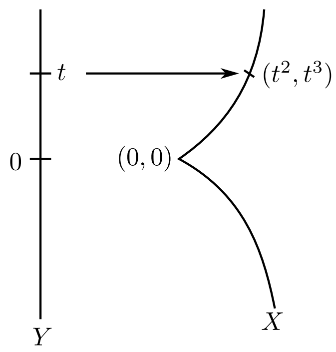
Corollary. *If X is normal, any bijective birational map $Y \to X$ is biregular.*

Remarks.

1. *A* bijective rational map need not be birational even if X is normal

   (e.g) K an algebraically closed field of charac. $p \neq 0$; $f : K \to K$ the map given by $x \mapsto x^p$.

21 2. If X is *not* normal, the above corollary in false. (e.g) Take a non-normal curve with a single cusp, for instance, $X_1^3 - X_2^2 = 0$; then the map from the normalisation (the affine line) to the curve is birational bijective but certainly not biregular at 0.



2. Divisors, Invertible Sheaves and Line Bundles

22 a) Let X be an irreducible algebraic variety over an algebraically closed field K, which is locally factorial (i.,each $\mathscr{O}_x, (x \in X)$ is factorial; this will be the case, for instance, if X is non-singular). We denote by $\mathscr{O}$ the structure sheaf of X and by $\mathscr{K}$ the constant sheaf of rational functions on X.

If W be an irreducible subvariety of X, of codimension 1, then $\mathscr{O}_W$ is a normal local ring of dimension 1 and thus is *a* discrete valuation ring; $\mathscr{O}_W$ then defines *a* discrete valuation on $K(X)$ which we denote by v_W.

Definition 1. *A* divisor *on X is an element of the free abelian group generated by irreducible subvarieties W of codimension 1 in X.*

Thus a divisor D on X can be written as $D = \sum n(W).W$ with $n(W) \in \mathbb{Z}$, almost all $n(W)$ being zero.

If f is *a* rational function on X then $v_W(f) = 0$ for almost all irreducible subvarieties of codimension 1 in X; in fact if U and U' are the domains of definition of f and f^{-1} respectively, then $v_W(f) \neq 0 \Longleftrightarrow W$ is contained in $(X - U \cap U')$. Therefore f defines *a* divisor $(f) = \sum\limits_W v_W(f)W$ on X; such divisors are called *principal divisors.*

The quotient group $\mathcal{D}/_{\mathcal{P}}$ of the group of divisors $\mathcal{D}$ on X by the group of principal divisors $\mathcal{P}$ on X is called the *Picard group Pic X* or the *group of divisor classes* on X. We say that two divisors D and D' are *linearly equivalent* $(D \sim D')$ if $(D - D')$ is principal.

Definition 2. *An* invertible sheaf *on X is a coherent sheaf of $\mathscr{O}$-modules,* **23**
which is locally free, of rank 1. equivalently it is a coherent sheaf of fractionary $\mathscr{O}$-ideals which is locally principal.

Definition 3. *A* line bundle *on X is a pair (L, π) such that*

(i) *L is an algebraic variety and $\pi : L \to X$ is a morphism or simply, a map*

(ii) *$\exists$ a (finite) open covering (U_i) of X with $L\big|\pi^{-1}(U_i) \overset{\varphi_i}{\xrightarrow{\sim}} U_i \times K$.*

In the intersections $U_i \cap U_j$, the isomorphisms φ_i and φ_j define regular maps $g_{ij} : U_i \cap U_j \to K^* = K - \{0\}$, such that $g_{ij}.g_{jk} = g_{ik}$, in $U_i \cap U_j \cap U_k$ and g_{ii} = identity.

Conversely, any line bundle is described by giving a finite open covering (U_i) of X and regular maps (g_{ij}) on the intersections $U_i \cap U_j$ to K^* with the properties: $g_{ij}.g_{jk} = g_{ik}$ on $U_i \cap U_j \cap U_k$ and g_{ii} = identity. The bundle is obtained by "recollement" of the varieties $U_i \times K$ through the biregular maps

$$(U_i \cap U_j) \times K \to (U_j \cap U_i) \times K$$
$$(x, t) \mapsto (x, g_{ij}(x)t).$$

We shall now show, by a series of constructions and reverse constructions, that the three notions we have define above are equivalent.

24 (i) **Divisor to a Bundle.**

Since we have assumed X to be locally factorial, any irreducible subvariety W of codim.1 is locally defined by a single equation (in a $u.f.d$, any prime ideal of height 1 is principal) and thus any divisor D is locally principal. We therefore get a covering (U_i) of X such that on $U_i, D = (z_i)$ for some $z_i \in K(X)$. If we define $g_{ij} = z_i/_{z_j}$ or. $U_i \cap U_j$ then all our requirement are satisfied and we get a line bundle defined by the (U_i, g_{ij}).

(ii) **Bundle to a sheaf**.

Let $L \xrightarrow{\pi} X$ be a line bundle on X. For any open set U in X a *section* of L over U is by definition a map $U \xrightarrow{s} L$ such that $\pi o s =$ identity on U; these sections form a group $\Gamma(U, L)$ which is also a $\Gamma(U, \mathscr{O})-$ module in a natural way. As the fibres $\pi^{-1}(x)$ are K-vector spaces of rank 1, these modules are locally free of rank 1 and we thus get an invertible sheaf.

(iii) **Sheaf to a Divisor**.

If $\mathscr{L}$ is a locally free $\mathscr{O}-$ module of rank 1, we choose an imbedding $\mathscr{L} \hookrightarrow \mathscr{K}$; at any points $P \in X, \mathscr{L}_P$ is a principal fractionary ideal, say of the form $\mathscr{O}_P f$, $f \in \mathscr{K}_P = K(X)$; for any irreducible subvariety W of codimension 1 in X, $\nu_W(f) = n(W)$ (say); then $n(W)$ depends only on the imbedding $\mathscr{L} \subset \mathscr{K}$ and not on the choice of f and again one see that almost all $n(W)'$are zero. One then associates to $\mathscr{L}$, the divisor $\sum\limits_{W} n(W)W$.

25 We remark that in (i) if we had started with a $D' \sim D$ oven then

we would have arrived at the same line bundle; and in (iii) if we had choosen a different imbedding $\mathscr{L} \subset \mathscr{K}$ then we would have arrived at an equivalent divisor. Thus, what we have shown is that giving an element of Pic X is equivalent to giving a line bundle or an invertible sheaf.

(iv) **Bundle to a divisor**.

In a manner similar to the one defining a line bundle on X, we may define a projective line bundle on X whose fibres are projective lines $\mathbb{P}_1(K)$. By fixing a "point at ∞" on $\mathbb{P}_1(K)$, hence giving an imbedding $K \hookrightarrow \mathbb{P}_1(K)$, we get an imbedding of a line bundle L on X in a projective line bundle $\mathbb{P}$ on X. If U is an open set in X and s is any section of L over U, different from the "O-section" S_o or the "∞-section" s_∞ of $\mathbb{P}$, take $T = \overline{s(U)} \subset \mathbb{P}$. Then the intersection cycle $T \cdot (s_o - s_\infty)$ (see §4) is of codimension 2 in $L|U$, and its projection $\pi(T \cdot (s_o - s_\infty))$ is of codimension 1 in U and therefore determines a divisor D on U.

(v) **Sheaf to a Bundle**.

Let $\mathscr{L}$ be an invertible sheaf, and P a point on X. Then $\exists$ a section z of $\mathscr{L}$ over a neighbourhood U of P such that for any $P' \in U', z_{P'} = z(P')$ generates $\mathscr{L}_{P'}$. We thus take a covering (U_i) of X and sections $z_i \in \Gamma(U_i, \mathscr{L})$ such that z_i generates $\mathscr{L}\big|U_i$; as z_i and z_j both generate $\mathscr{L}$ on $U_i \cap U_j$. $g_{ij} = z_i/_{z_j}$ defines a regular map $U_i \cap U_j \to K^*$ and we thus find a bundle.

(vi) **Divisor to a sheaf**. **26**

Let $D = \sum\limits_W n(W)\, W$ be a divisor. For any open subset U of X, we define

$$\mathscr{L}_D(U) = \{f \in K(X) : v_W(f) \geq -n(W) \text{ for all } W \text{ with } W \cap U \neq \phi\}.$$

Then $\mathscr{L}_D$ is a presheaf on X; and if U is open set such that $D = (g)$ on U for some $g \in K(X)$ then it is easily checked that $\mathscr{L}_D(U)$ is precisely $= \{f \in K(X) : fg \in \mathscr{O}_P \forall P \in U\}$ hence is free on $\Gamma(U, \mathscr{O})$ of rank 1, The sheaf $\mathscr{L}(D)$ defined by $\mathscr{L}_D$ is thus invertible.

(vii) **Certain observations:**

α) If D, D' are divisors, corresponding to line bundles L, L' and invertible sheaves $\mathscr{L}(D), \mathscr{L}(D')$ then $D + D'$ corresponds to the line bundle $L \underset{X}{\otimes} L'$ and the invertible sheaf $\mathscr{L}(D) \underset{\mathscr{O}}{\otimes} \mathscr{L}(D')$; and $-D$ corresponds to the dual bundle L^* and the dual sheaf $\mathscr{L}(D)^*$.

In fact if L is given by (U_i, g_{ij}) and L' by (U_i, g'_{ij}) then $L \underset{X}{\otimes} L'$ is given by $(U_i, g_{ij}.g'_{ij})$ and L^* by (U_i, g_{ij}^{-1}).

β) Let $u : X' \to X$ be a morphism of nonsingular (or locally factorial) irreducible varieties X', X. Let D be a divisor on X corresponding to a line bundle L and a sheaf $\mathscr{L}$; then the "pull-backs" $u^*(L)$ and $u^*(\mathscr{L})$ are both well-defined but where is no way of assuring that $u^{-1}(D)$ is always well-defined. But one can prove that $\exists$ a divisor $D_1 \sim D$ on X such that $u^{-1}(D_1)$ is (see §4) well-defined. Then $u^*(\mathscr{L})$ and $u^{-1}(D_1)$ correspond to one another.

27 γ) Suppose $L \xrightarrow{\pi} X$ is a line bundle on X, and assume it admits global sections $s_o, \ldots, s_n$ on X. Then the s_i are rational functions on X and if we suppose that they do not vanish (as functions on X) all together at any point on X, for every $P \in X$, each ratio $s_i(P)/_{s_j(P)}$ is either in K or is ∞ in $\mathbb{P}_1(K)$. Then the s_i define a morphism

$$X \to \mathbb{P}_n(K)$$
$$P \mapsto \text{the point given by the homogeneous coords.}(s_i(P)).$$

If the s_i do have common zeros, then we get only a rational map on X with values in $\mathbb{P}_n(K)$.

Definition 4. *We say that the line bundle L is* very ample *if there exist an integer n and global sections $s_o, \ldots, s_n$ of L over X giving* an isomorphism *of X on an algebraic subvariety of $\mathbb{P}_n(K)$. We say that L is* ample *if $\exists$ a $q > O$ such that $L^{\otimes q}$ is very ample. We say that a divisor D on X is very ample (resp. ample) if the line bundle L_D defined by D is so.*

For any divisor D on X take global sections $f_o, \ldots, f_n$ of the sheaf $\mathscr{L}_D$ defined by D; they are rational functions of X such that $(f_i) + D \geq 0$

and correspond to global sections of the line bundle L_D defined by D. As before we get a rational map φ_D of X with values in $\mathbb{P}_n(K)$; φ_D is a morphism if the divisors $(f_i) + D$ have no common point. If in addition, we assume that the (f_i) are such that the φ_D they define is an imbedding of X in $\mathbb{P}_n(K)$, then the divisors $(f) + D$ on X correspond on $\varphi_D(X)$ to **28**
the hyperlane sections for all $f \in \Gamma(X, \mathscr{L}_D)$.

Proposition 1. *Let D, E be divisors on X and suppose D very ample. Then $\exists n_o \in \mathbb{Z}^+$ such that for any $n \geq n_o$, the sheaf $\mathscr{L}(E + nD)$ admits a finite number of global sections, defining a morphism $\varphi(E + nD)$ on X into a projective space.*

Proof. We may assume X imbedded in a $\mathbb{P}_s$ and that D is a hyperplace section. Let $(\xi_o, \ldots, \xi_s)$ be a system of homogeneous coordinate in $\mathbb{P}_s$ and consider the affine open subset $U_i \equiv (\xi \neq 0)$ in X. The $E\big|U_i$ corresponds to a fractionary ideal of $K[U_i]$; a finite system of generators of the ideal gives a finite system (s_{ij}) of sections of the line bundle $L_E\big|U_i$, without a common zero on U_i. Then these sections extend to rational sections $(\overline{s}_{ij})$ of L_E with poles only on $\xi_i = 0$.

If $n(i)$ is the maximum of the orders of these poles then for $n \geq n_i$ the $(\overline{s}_{ij} \otimes \xi_i^n)_j$ are sections of $L_E \otimes L_{nD} = L_{E+nD}$ on X without common zeros on U_i. Then for $n \geq \sup_i n(i)$ the $(\overline{s}_{ij} \otimes \xi_i^n)_{i,j}$ are sections of L_{E+nD} without common zeros and hence define a morphism $\varphi_{(E+nD)}$ from X, into a projective space. *Q,E,P* □

Proposition 2. *If F is an ample divisor then $\exists$ a $k_o > 0$ such that for any $k \geq k_o$, there exist global sections (finitely many) of $\mathscr{L}_{kF}$ defining a morphism φ_{kF}.*

Proof. By definition some qF is very ample. Now apply Proposition 1 to $D = qF$, $E = F, 2F, \ldots, qF$. □

Proposition 3. *If D is ample then $\exists n_o > 0$ such that nD is very ample* **29**
$\forall n \geq n_o$.

Proof. Some qD is very ample; also choose k_o as in Proposition 2 and $n_o = q + k_o$. □

For $n \geq n_o, \exists$ a morphism $\varphi_{(n-q)D} : X \to \mathbb{P}_\gamma$ defined by global sections (f_i) of $\mathscr{L}_{n-q/D}$; and we have an imbedding

$\varphi_{qD} : X \to \mathbb{P}_s$ defined by global sections (g_j) of $\mathscr{L}_{qD}$. Therefore the morphism $\varphi = (\varphi_{(n-q)D}, \varphi_{qD}) : X \to \mathbb{P}_\gamma \times \mathbb{P}_s$ is an imbedding (its composite with the projection is an imbedding); composing with the Segre-imbedding $\mathbb{P}_r \times \mathbb{P}_s \to \mathbb{P}_{rs+r+s}$ we get an imbedding φ_{nD} of X in $\mathbb{P}_{rs+r+s}$ defined by global sections $(f_i g_j)$ of φ_{nD}. Q.E.D

Theorem(Grauert). *Let X be a normal irreducible algebraic variety over an algebraically closed field K and $L \xrightarrow{\pi'} X$ a line bundle on X*

Let S_o be the zero section of the dual bundle $L^* \xrightarrow{\pi} X$. Then L is ample $\Longleftrightarrow \exists$ a morphism t from L^* on an affine variety Z passing through the origin in some K^n such that

(i) $t(S_o) = 0$

(ii) $L^* - S_o \xrightarrow{\underset{\sim}{t}} Z - (0)$.

For proving the theorem we shall need the following two lemma.

Lemma 1. *Let L be a line bundle on X and $L^* \xrightarrow{\pi} X$ the dual of L. Suppose $s_o, \ldots, s_n$ are sections of L on X. One then defines a map $\tilde{s}$: $L^* \to K^{n+1}$ by $x \in L^* \mapsto \tilde{s}(x) = (\langle s_i(\pi(x)), x\rangle)_{i=0}^n$.*

30 We have also seen before that the s_i define a rational map s on X to $\mathbb{P}(K)$. We have the following:

a) $\tilde{s}(L^*) \subset$ the affine cone in K^{n+1} of $s(X)$, and if $\lambda \in K, \tilde{s}(\lambda x) = \lambda\tilde{s}(x)$

b) Any regular map $u : L^* \to K^{n+1}$ such that $u(\lambda x) = \lambda u(x), \lambda \in K, x \in L^*$ is of the form $\tilde{s}$ for a system of sections $s_0, \ldots, s_n$ of L.

c) Let S_o be the image of the zero section of L^*. Then s is a biregular imbedding of X in $\mathbb{P}_n(K) \Leftrightarrow \tilde{s}$ is biregular outside S_o on L^*.

Proof. a) The former assertion is clear, for if $\tilde{s}(x) \neq 0$, say $\tilde{s}(x)_o \neq 0$, then, for each $\dfrac{\tilde{s}(x)_i}{\tilde{s}(x)_o} = \dfrac{s_i(\pi(x))}{s_o(\pi(x))}$; the latter assertion is trivial.

b) Let $u(x) = (u_o(x), \ldots, u_n(x)), x \in L^*$. For any $a \in X$ if $x \in (L^*)_a$ then $x \to u_i(x)$ is a linear form on L^*_a and thus defines an $s_i(a) \in (L^*)^* = L_a$; and therefore we get sections $s_i \in \Gamma(X, L), 0 \leq i \leq n$ with the required property.

c) Assume $s : X \to \mathbb{P}_n$ is a biregular imbedding. Let $x \in L^* - S_o$ and $a \in X$ s. t $x \in L^*_a$; if i is such that $s_i(\pi(x)) \neq 0$ then $\tilde{s}(x)_i \neq 0$ and $\tilde{s}(x) \neq 0$. Therefore $\tilde{s}(L^* - S_0) \subset$ {cone of $s(X) - (0)$} in K^{n+1}. We will prove that this inclusion is, in fact, an equality; it is sufficient to prove our assertion when we restrict L^* to an open at $U \ni a$ in X s-t.L^* over U is isomorphic to $U \times K$. Then $L^* - S_o \simeq U \times K^*$, while, the cone of $s(U) - \{0\}$ is isomorphic to $s(U) \times K^*$.

□

As all these isomorphisms are compatible with restrictions of U to smaller open sets, we are through, we have, by hypothesis, $U \xrightarrow{\sim} s(U)$ 31

Conversely,let $s(L^* - S_o) \xrightarrow{\sim} \tilde{s}(L^*) - \{0\}$; again by choosing an open $U \subset X$ such that $L^* \simeq U \times K$ we get

$U \times K^* \xrightarrow{\sim} s(U) \times K^*$ by an isomorphism compatible with restrictions and identity on K^* i. e., $U \xrightarrow{\sim} s(U)$); and s is thus a biregular imbedding of X in $\mathbb{P}_n$.

Lemma 2. *Let E be a line bundle on X and v_q be the morphism of fibre spaces $E \xrightarrow{v_q} E^{\otimes q}$ on X, defined by*

$$x \to x \otimes \cdots \otimes x, (q \text{ times}).$$

Suppose u is any morphism $E \to K^n$ such that $u(\lambda x) = \lambda^q u(x).\lambda \in K, x \in E$. Then u admits a unique factorization

$$E \xrightarrow{v_q} E^{\otimes q} \xrightarrow{\bar{u}} K^n$$

such that $\bar{u}$ is a morphism with $\bar{u}(\lambda x) = \lambda \bar{u}(x), x \in E^{\otimes q}, \lambda \in K$.

Proof. Is $x, y \in E$ are s.t. $v_q(x) = v_q(y)$ then $y = \alpha x, \alpha \in K$. Also, $v_q(y) = (y \otimes \cdots \otimes y) = \alpha^q (x \otimes \cdots \otimes x) = x \otimes \cdots x = v_q(x)$ by assumption so that α is a $q^{th}-$ root of unity. Then $u(y) = \alpha^q u(x)$ by hypothesis, and so $u(y) = u(x)$ so that the set theoretic map $\overline{u}$ on $E^{\otimes q}$ defined by

$$\overline{u}(v^q(x)) = u(x) \text{ is } \textit{well-defined}.$$

To show that $\overline{u}$ is a morphism take any open $U \subset X$ such that $E|U \simeq U \times K$ and $E^{\otimes q}|U \simeq U \times k$. If $(x) = (z, \lambda)$ is a coordinate system for $E|U$ and $(x') = (z', \mu)$ a coordinate system for $E^{\otimes q}|U$ then v_q is given by $(z, \lambda) \to (z, \lambda^q)$ and $\overline{u}$ is given by $\overline{u}(z', \mu) = \overline{u}(z', 1).\mu = u(z', 1)\mu$ Q.E.D. □

32 **Proof of Grauert's theorem**

(i) Necessity

If L is ample by Proposition 3, $\exists q > 0$ and systems of sections (s) of $L^{\otimes q}$ and (s') of $L^{\otimes(q+1)}$ defining projective imbeddings

$$s : X \to \mathbb{P}_n$$

and $$s' : X \to \mathbb{P}'_n \text{ respectively}.$$

As in Lemma 1, we obtain morphisms

$$\tilde{s} : (L^*)^{\otimes q} \to K_{n+1} \text{ and } \tilde{s} : (L^*)^{\otimes} \xrightarrow{(q+1)} K_{n'+1}$$

which are biregular imbeddings outside the respective null sections (Lemma 1, c). We define now a morphism $v : L^* \to K_{n+n'+2}$ as the composite

$$L^* \xrightarrow[v_q \oplus v_{q+1}]{} (L^*)^{\otimes q} \oplus (L^*)^{\otimes(q+1)} \xrightarrow[\tilde{s}+\tilde{s}']{} K_{n+n'+2}.$$

We shall prove the v is a biregular imbedding outside the null section S_o of L^*. It is enough to show that $v_q \oplus v_{q+1}$ is a biregular imbedding outside S_o. Again it suffices to do this over open sets $U \subset X$ such that $L^*|U \simeq U \times K$; over U the map $v_q \oplus v_{q+1}$ is given by $(x, \lambda) \mapsto (x, \lambda^q) \oplus (x, \lambda^{q+1})$; the inverse rational map form the image is given by $(x, t) \oplus (x, t') \mapsto (x, t'/_t)$; $v_q \oplus v_{q+1}$ is then certainly a biregular imbedding on $(L^* - S_O)|U \simeq U \times K^*$.

(ii) Sufficiency.

33 Let t be a morphism on L^* with the properties stated in the theorem. For any $x \in L^*$ one obtains a map $K \to K^n$ by $\lambda \mapsto t(\lambda x)$: this is defined on the whole of K and vanishes precisely for $\lambda = 0$ or $x \in S_o$. Hence there exist regular maps $t_1(x), \ldots, t_r(x)$ on L^* to K^n such that

$$t(\lambda x) = \lambda t_1(x) + \cdots + \lambda^r t_r(x).$$

Computing $t(\mu\lambda x)$ we easily see that $t_j(\mu x) = \mu^j t_j(x) \forall x \underset{\mu\in K}{\in} L^*$. The map $t(x) = t_1(x) + \cdots + t_r(x)$ is an imbedding of $(L^* - S_o)$; thus the map $u(x) = (t_1(x), \ldots, t_r(x))$ of L^* in a suitable big affine space is also an imbedding outside S_o.

Let $t_{jk}(x)$ be the components of the map $t_j(x)$; we have $t_{jk}(\mu x) = \mu^j t_{jk}(x) \forall \mu \in K, x \in L^*$. We form the graded ring $A = K\left[(t_{jk})_{j,k}\right]$, the degree of t_{jk} being j. Then A is a domain (it is the coordinate ring of the image $u(L^*)$); its integral closure A' is a graded, finite type K-algebra. Then $\exists\ q < 0$ such that the homogeneous elements in A' of degree q generate $A'(q) = \sum_{\nu\geq 0} A'_{\nu q}$ as a graded K-algebra; we may assume q to be a multiple of $1, 2, \ldots, r$. Let (w_α) be a basis over K of the K-space A'_q; (w_α) are regular functions $L^* \to K$, homogeneous of degree q. We obtain then a regular map $w : L^* \to K^N$ for an $N \in \mathbb{Z}^+$ whore components are the (w'_α)s and which therefore has the property $w(\lambda x) = \lambda^q w(x), x \in L^*, \lambda \in K$. By Lemma 2, we get a factorization

$$L* \xrightarrow{\nu_q} (L^*)^{\otimes q} \xrightarrow{\bar{w}} K^N$$

such that $\overline{w}(\mu y) = \mu\overline{w}(y), y \in (L^*)^{\otimes q}, \mu \in K$.We shall prove that $(L)^{\otimes q}$ is *very ample* by showing that $\overline{w}$ is an imbedding outside its null section (see Lemma 1,c)). 34

As $\overline{w}((L^*)^{\otimes q}) = w(L^*)$ is a *normal* subvariety of K^N (its coordinate ring is $K[(w_\alpha)]$ we shall prove our ascertain by using Zariski's main theorem (Chapter I, g)) i.e. by proving that $\overline{w}$ is bijective, birational onto its image, outside $S_\circ((L^*)^{\otimes q})$.

$\overline{w}$ is injective outside $S_o((L^*)^{\otimes q})$:

a) If $w(x) = w(y)$, then $\exists$ a q^{th} root of unity ε such that $t_{jk}(y) = \varepsilon^j t_{jk}(x) \forall j, \mathcal{R}$.

In fact, each $t_{jk}^{q/j}$ is a linear combination of the w'_α s so that $w(x) = w(y)$ implies $t_{jk}^{q/j}(x) = t_{jk}^{q/j}(y)$,

$$\text{i. e} \qquad t_{jk}(y) = \varepsilon_{jk} t_{jk}(x)$$

where ε_{jk} is a $(q/j)^{th}$root of unity. In particular, $\varepsilon = \varepsilon_{11}$ is a q^{th} root of unity. Now $t_{jk}t_{11}^{q-j}$ is again a linear combination of the w'_αs and therefore

$$t_{jk}t_{11}^{q-j}(y) = t_{jk}t_{11}^{q-j}(x)$$

i.e.,
$$\varepsilon_{jk}\varepsilon^{q-1}t_{jk}t_{11}^{q-j}(x) = t_{jk}t_{11}^{q-j}(x)$$

and for $x \notin S_o$, this means $\varepsilon_{jk} = \varepsilon^j$.

35 b) Suppose $x', y' \in (L^*)^{\otimes q}$ such that $\overline{w}(x') = \overline{w}(y')$.

Let $x' = v_q(x), y' = v_q(y)$, with $x, y \in L^*$. Then by a) we obtain $t_{jk}(y) = \varepsilon^j t_{jk}(x) = t_{jk}(\varepsilon x) \forall j, k$. This implies $t(y) = t(\varepsilon x)$ for the map $t = t_1 + \cdots + t_r$; but if $x', y' \in (L^*)^{\otimes q} - S_o((L^*)^{\otimes q})$ then $x, y \in L^* - S_o$ and t is biregular on $L^* - S_o$ by hypothesis. Therefore $y = \varepsilon x$ and $y' = v_q(y) = \varepsilon^q v_q(x) = x'$.

$\overline{w}$ is birational

We have $K(L^*) = K((t_{jk})_{j,k})$; the projection $\pi : L^* \to X$ then gives an imbedding $K(X) \hookrightarrow K((t_{jk}))$. As locally L^* is of the form $X \times K$, this means that $\exists$ a homogeneous function $\sigma \in K(L^*)$ of deg 1, $\sigma(\lambda x) = \lambda\sigma(x)$, such that $K(L^*) = K(X)(\sigma)$. Obviously, then $K((L^*)^{\otimes q}) = K(X)(\sigma^q)$.

On the other hand, $K(X)$ is generated over K by the elements in $K((t_{jk}))$ which are products of the form $\prod\limits_{j,k} t_{jk}^{\beta(j,k)}$ with $\sum\limits_{j,k}\beta(j,k) = 0$. Thus $K(X) = K((w_\alpha/_{w_{\alpha_o}})_\alpha)$ and the function field F of $\overline{w}((L^*)^{\otimes q})$ is given by

$$F = K((w_\alpha)_\alpha) = K(X)(w_{\alpha_o}).$$

Finally as $\sigma^q/w_{\alpha_o} \in K(X)$, we obtain

Q.E.D
$$F = K(X)(\sigma^q) = K((L^*)^{\otimes q)}.$$

3. Chow Coordinates

a) Let V be an algebraic set of dimension d in $\mathbb{P}_n$ (notation: V^d); we assume that V is pure of dimension d, i.e., all the components of V have the same dimension d. If H_1 is a hyper plane not containing any component of V then $H_1 \cap V$ is pure of dimension $(d-1)$. If H_2 is a hyperplane not containing any component of $H_1 \cap V$ then $H_2 \cap H_1 \cap V$ is pure of dimension $(d-2)$ and so on; and finally $H_{d+1} \cap \cdots \cap H_1 \cap V$ is empty *in general*. 36

Our purpose now is to study the systems of $(d+1)$ hyperplanes $H_o, \ldots, H_d$ such that $(H_o \cup \cdots \cap H_d) \cap V$ is *non-empty*.

Assume V to be irreducible. Any hyperplane H will be defined by an equation of the form $\sum\limits_{i=0}^{n} u_i X_i = 0$ in $\mathbb{P}_n, u_i \in K$ and is there fore determined by the point $(u_o, \ldots, u_n) \in K^{n+1}$. A system of $(d+1)$ hyperplanes will then be determined by a point in the affine space $K^{(n+1)(d+1)}$. We shall now prove that the points in $K^{(n+1)(d+1)}$ corres ponding to systems of (d+1) hyperplanes $H_P, \ldots, H_d$ in $\mathbb{P}_n$ such that $V \cap H_O \cap \cdots \cap H_d \neq \phi$ form an irreducible algebraic variety (in fact, a cone for obvious reasons) in $K^{(n+1)(d+1)}$.

Let k be a field of definition for V and $(x) = (x_1, \ldots, x_n)$ be the affine coordinates for a generic point (x) of V/k. Its homogeneous coordinates are the $(1, x_1, \ldots, x_n)$. Consider a system

$$\overline{u}_{ij}, i = 0, \ldots, d, \ j = 1, \ldots, n$$

of algebraically independent elements of K over $k(x)$ and an equation

$$\overline{u}_{io} = -\sum_{j=1}^{n} \overline{u}_{1j} x_j \qquad \text{defining } (\overline{u}_{io})_{i=0,\ldots,d}$$

Then $(\overline{u}_{ij})$ is a generic point of an irreducible variety in $K^{(n+1)(d+1)}$ (the locus of $(\overline{u}_{ij})$ over $k(x)$ which is the one we have been looking for. Its dimension is $d+n(d+1) = n(d+1)(d+1)-1$ so that it is a hypersurface in $K^{(n+1)(d+1)}$, defined by an equation: 37

$$F(u_{oo}, \ldots, u_{on}; u_{1o}, \ldots, u_{1n}; ----; u_{do,\ldots,u_{dn}}) = 0$$

in $(n+1)(d+1)$ variables; this equation is obviously multi homogeneous in the set of variables indicated, of the same degree δ, in each set of variables; if two sets of variables take the same values then the hyperplanes they define are the same and $H_o \cap \cdots \cap H_d \cap V \neq \phi$ in valid; this implies that F is containing and antisymmetric in the sets of variables indicated.

The polynomial F is called the *Chow form* (or the associated form, or the Cayley form or the Bertini form)of V.

The Chow form $F(V)$ of a variety V^d determines the variety V back uniquely. In fact for any linear variety L^{n-d} the points of intersection of L^{n-d} and V are given by the Chow form F(V) and are precisely $\delta =$ degree of $F(V)$, in number.δ is called the *degree of the variety V*.

b) Let V be any variety. By a *cycle* on V we mean an element of the free abelian group generated by irreducible subvarieties of V. If $X = \sum n_\alpha V_\alpha$ is a cycle on V. the support of X is the union $\bigcup\limits_{n_\alpha \neq = 0} V$.

We say that X is of dimension d if each V_α is of dimension d; we say that X is *positive* $(X \geq 0)$ if each n_α is ≥ 0.

38 If $X = \sum n_\alpha V_\alpha$ is a positive cycle of dimension d in $\mathbb{P}_n(K)$ and $F_\alpha(u)$ is the Chow form of each V_α the form $F(u) = \prod\limits_\alpha (F_\alpha(U))^n \alpha$ is called the *Chow form of the cycle X*. The degree of $F(u) = \sum n_\alpha \deg V_\alpha$ is called the *degree of the cycle X*. We may write Chow form $F(u)$ in the form $\sum C_\lambda(X) u_\lambda$ where the u_λ are monomials; we then call the $(C_\lambda(X))$ the *Chow coordinates* of X, and the point whose homogeneous coordinates are the $(C_\lambda(X)_\lambda)$ is called the *Chow point* of X.

Conversely, if $G(u) = d\ u$ is a given form multihomogeneous of the same degree in each set of variables, alternating and antisymmetric,one may ask "Under what condition are the (d) the Chow coordinates of a positive cycle on a subvariety $U\ n^?$".The answer is given by the following theorem which we shall not prove here. (For a proof see Chapter I,§9,5, Samuel [4]).

Theorem. *The (d_λ) are the Chow coordinates of a positive cycle X on a subvariety $U \subset \mathbb{P}_n(K)$ if and only if the (d_λ) satisfy a system of homogeneous equation with coefficients in the smallest field of definition of U.*

We call a system of cycles (X_α) an *irreducible* system if the Chow points of the X_α range over an irreducible variety.

Corollary. *Any system of positive cycles on $\mathbb{P}_n(K)$ with a given dimension d and a given degree δ is a finite union of irreducible systems.*

Example.

a) $V = \{(a_o, \ldots, a_n)\}$, *a point in* $\mathbb{P}_n$.

The Chow form $F(u)$ of V is the linear form $\sum\limits_{i=1}^{n} u_i a_i = 0$.

b) *V a hyperplane H given by* $\sum\limits_{j=0}^{n} a_j X_j = 0$. Then the Chow form of V **39**
is given by the determinant

$$\begin{vmatrix} a_o & \cdots & a_n \\ u_{oo} & \cdots & u_{on} \\ & \cdots & \\ u_{n-1}, o & \cdots & u_{n-1}, n \end{vmatrix} = 0$$

c) *V a linear subvariety of* $\mathbb{P}_n$.

In this case the Chow coordinates are essentially the Grassmann coordinates of the linear variety.

4. Results from Intersection Theorey

a) We fix, for our present consideration, an ambient variety U, over an algebraically closed field K, which we shall assume to be non singular. Let V, W be irreducible closed subvarieties of U. We say that a component C of $V \cap W$ is *proper* if $\dim C = \dim V + \dim W - \dim U$. For such a component C, the *intersection multiplicity* $i_U(C; V.W)$ is defined in the following manner:

Consider the product (nonsingular) variety $U \times U$ and let $\triangle$ be the diagonal of $U \times U$; then the component C corresponds to a component $\tilde{C}$ of $\triangle \cap (V \times W)$; $\tilde{C}$ is an irreducible subvariety of $V \times W$ and if $\mathscr{O}$ is local ring of $\tilde{C}$ on $V \times W$ then $\mathscr{O} \subset K(V \times W)$; then $\triangle \cap (CV \times W)$ is defined

by equations $(X_i - X_i')$ in $K[V \times W]$ locally and these equations generate an ideal q in $\mathscr{O}$, primary for the maximal ideal of $\mathscr{O}$. *The intersection*
40 *multiplicity of V and W on C* is defined to be the multiplicity $e(q)$ of the primary ideal q in $\mathscr{O}$.

It can be shown that $i_U(C; V.W) = 1$ means that $\mathscr{O}$ (in the above paragraph) is a regular local ring and q is the maximal ideal of $\mathscr{O}$. The first condition means that C is simple on V and W: the second condition says precisely that the tangent spaces to V and W at a generic point of C have for their intersection the tangent space to C at that point. We say then that V and W are *transversal* on C.

Remark. We observe that so far we have not made any use whatever of the fact that C is a proper component of $V\ W$. We could have defined $i_U(C; V.W)$ for any component C of $V \cap W$. But the advantage in considering a proper component is seen from the following fact (which we shall not prove):

Let C be a proper component of $V^d \cap W^{d'}$ in a (nonsingular) U^n; suppose that the primary ideal of V in $\mathscr{O}(C; U)$ (the local ring of C in U) is generated by $(n-d)$ elements $y_1, \ldots, y_{n-d}$ (this will be the case for instance if C in simple on V); then the multiplicity $i_U(C; V.W)$ is equal to the multiplicity of the ideal of $\mathscr{O}(C.W)$ generated by the classes of (y_i).

(For a proof see Chapter II, §5, no.7, Theorem 6), Samuel [4]

With the same notation,we define the "intersection" $V \cdot W$ (or $\overset{V \cdot W}{\bigcup}$ when all the components C_α of $V \cap W$ are proper, by:

$$V.W = \sum i_U(C_\alpha : V.W) C_\alpha$$

41 Then $V \cdot W$ is a *cycle* on $V \cap W$. Under obvious condition, we may also define the cycles $X \underset{U}{\cdot} Y$ for two cycle X, Y on U, by extending the above definition by linearity. When the ambient space U needs no special mention we may also write $X.Y$ for $X \underset{U}{\cdot} Y$.

Some properties

i)

$$X \underset{U}{\cdot} U = X$$

$$X \underset{U}{\cdot} Y = Y \underset{U}{\cdot} X$$

and $$(X \underset{U}{\cdot} Y) \underset{U}{\cdot} Z = X \underset{U}{\cdot} (Y \underset{U}{\cdot} Z)$$

(We remark that the associativity holds only when all the component of $U \cap V \cap W$, $(U, V, W$ being the components of X, Y, Z respectively are proper).

ii) If X, Y are cycles on U, X', Y' cycles on U' then $X \times X'$ and $Y \times Y'$ are cycles on $U \times U'$, and one has $(X \times X').(Y \times Y') = (X.Y) \times (X'.Y')$

iii) Let V be a closed, nonsingular subvariety of U, X a cycle on V, Y a cycle U then

$$X \underset{U}{\cdot} Y = X \underset{V}{\cdot} (V \underset{U}{\cdot} Y) \quad \text{(Induction formula)}.$$

iv) Let U, U' be nonsingular varieties and $p : U \times U' \to U$ the set-theoretic projection. Let V be a subvariety of $U \times U'$; we define the *algebraic projection* $pr_U(V)$ of V as follows:

α) if $\dim \overline{p(V) <} \dim V$, then $pr_U(V)$ is the "zero cycle" on U

β) if $\underline{\dim \overline{p(V)}} = \dim V$, them $K(V)$ is a $\underline{\text{finite extension}}$ of $K(\overline{p(V)})$ and $pr_U(V)$ is the cycle $\left[K(V) : K(\overline{p(V)})\right] \overline{p(V)}$.

This notion of an algebraic projection extend by linearity to cycles on $U \times U'$. 42

Proposition. *Let U' be a* complete *nonsingular variety and X be a cycle on $U \times U'$. Then for any cycle Y on U.*

$$pr_U(X \cdot (Y \times U')) = pr_U(X) \cdot Y$$

whenever both sides are defined.

b) Given a morphism $f : V \to V'$ of nonsingular varieties and a cycle X' on V', one defines the inverse image $f^{-1}(X')$ as $pr_v((V \times X') \cdot \Gamma_f)$ where Γ_f is the graph of f. Let V be a projective nonsingular variety and T an irreducible variety, also nonsingular. For any cycle X an $V \times T$, the cycle $X_t = pr_V((V \times t) \cdot X)$ on V is defined for "almost all" $t \in T$. The system of cycles (X_t) thus defined on V is said to be an *irreducible algebraic family of cycles on* V.

One can show that for an algebraic family (X_t) of cycles on V, the Chow coordinates $c(X_t)$ of the cycle X_t depend rationally on $t \in T$:thus the dimension and degree of the cycle X_t are independent of $t \in T$. Any two members of the family (X_t) are said to be *algebraically equivalent*.

Let $(X_t)_{t \in T}$ and $(Y_{t'})_{t' \in T'}$ be algebraic families of cycles on V; we may assume that the parametrizing varieties T and T' are the same by passing to $T \times T'$ in the obvious manner. Then it is checked that

$$X_t \cdot Y_t = pr_V((V \times t) \cdot (X \cdot Y))$$

where X and Y are the "defining" cycles on $V \times T$. Thus $(X_t \cdot Y_t)_{t \in T}$ is again an algebraic family on V.

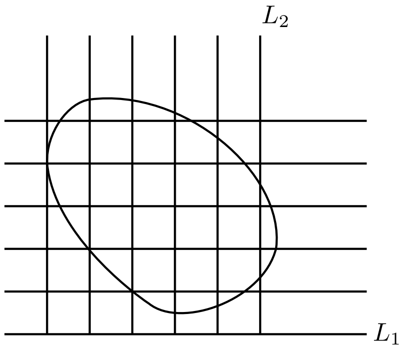
43 **Examples.** We shall be interested here only in cycle of dimension 1, i.e.linear combinations of irreducible curves.

(i) Let D and E be cycles of (dimension 1 and) degrees d and e in $\mathbb{P}_2$. The intersection number $(D.E)$ of D and E is the total number of points of intersection of D and E each point counted with the appropriate multiplicity.

We have:$(D.E) = de$ (Theorem of Bezoout). In fact , it is enough to prove this for cycles D' and E' such that D is algebraically equivalent to D' and E is algebraically equivalent to F'. One can construct an algebraic family containing D and $D' = dL$(L a line) as members and similarly for E and eL'. For the two lines L and L' our assertion is obvious.

Thus, we have a case where the intersection number is completely determine by the degrees of the cycles

ii) Consider the products of the projective lines $\mathbb{P}_1 \times \mathbb{P}_1$ imbedded in $\mathbb{P}_3$. Thus is the quadric $\xi_o\xi_1 - \xi_2\xi_3 = 0$ in $\mathbb{P}_3$.



Let L_1 and L_2 be a 'horizontal' and a 'vertical' line of this product: if $d_1 = (X \cdot L_2)$ and $d_2 = (X \cdot L_1)$ one may construct an algebraic family on $\mathbb{P}_1 \times \mathbb{P}_2$ containing X and $(d_1L_1 + d_2L_2)$ as members. Similarly for a second cycle X' in $\mathbb{P}_1 \times \mathbb{P}_1$

The intersection number of the cycles X, X' is then given by **44**

$$\begin{aligned}(X \cdot X') &= \Big((d_1L_1 + d_2L_2).(d_1'L_1 + d_2'L_2')\Big)\\ &= d_1d_2' + d_2d_1'\end{aligned}$$

while
$$\deg X = d_1 + d_2 \quad \text{and}$$
$$\deg X' = d_1' + d_2'$$

We have thus a case where the intersection number is *not* determine by the degrees of the cycles.

Chapter 2

Algebraic Curves

Let K be an algebraically closed field. An *algebraic curve* over K is a **45**
variety over K all of whose irreducible components have dimension 1. In this chapter, we shall be mainly concerned with irreducible complete, nonsingular curves.

1. The genus

a) Let C be a nonsingular curve. Then for any point $P \in C$, the local ring $\mathcal{O}_P$ is a discrete valuation ring of the function field $K(C)$. If, in addition, C is complete, then every discrete valuation ring of $K(C)$ dominates some $\mathcal{O}_P$, $P \in C$ and hence equals that $\mathcal{O}_P$. Obviously the point P is uniquely determined. Thus the structure sheaf of a nonsingular, complete, irreducible curve C is determined by $K(C)$ and any two birationally equivalent such curves are isomorphic. This fact enables us to construct a projective "model" C for a function field of one variable L ove K in the following manner:

Let $L = K(x_1, \ldots, x_n)$; consider any affine curve C' in K^n whose coordinate ring is $K[x_1, \ldots, x_n]$. We take for C the projective normalisation of a projective closure for C'. The curve is normal (therefore nonsingular) complete, irreducible, with function field L and is thus the "model" we are looking for.

But in the sequel, we need models of L with “nicer” properties in projective spaces of “small” dimension. We start with the following

46 **Definition.** *A nonsingular curve C is said to be* strange *if all its tangents have a point in common, all if it is not a straight line. It is easily seen that strange curves can occur only in characteristic $p \neq 0$.*

Theorem 1. *Any function field of one variable L over K admits:*

i) *a nonsingular model, which is* not strange, *in* $\mathbb{P}_3$.

ii) *a model in $\mathbb{P}_2$ which has for its singularities only finitely many ordinary double points.*

Proof. We shall prove i) in two stages. □

α) L admits a nonsingular, nonstrange, projective model.

Let C be a projective nonsingular model of L in $\mathbb{P}_n$, constructed as above. We may assume that C is strange; choose a system of homogeneous coordinates in $\mathbb{P}_n$, such that C does not lie entirely in the hyperplane $X_o = 0$ and that $A = (1, 0, \ldots, 0)$ is the common point of all the tangents of C. Let the homogeneous coordinate functions on C be $(1, x_1 \ldots, x_n)$.

Let D be a nontrivial derivation of L over K. For any point $P \in C$, the parametric equations of the tangent T_p to C at P are given by

(1) $$X_i = \alpha x_i(P) + (Dx_i)(P), \alpha \in K, i = 0, \ldots, n.$$

The parameter α_A of A on T_P is a rational function of P, say $u \in L$. Thus,

(2) $$0 = ux_i + Dx_i \text{ for} i = 1, 2, \ldots, n$$

and the system of homogeneous coordinates of A, given by (1) is
47 $(u, 0, \ldots, 0)$ whence $u \neq 0$.

We now claim that $\exists y \in L$ such that $uy + Dy$ is not proportional to u, i.e., such that $y + u^{-1}$ Dy $\in K$: in fact, as we are in characteristic $p \neq 0$, we may take any $y \in \mathrm{L}^p - K$ (since $Dy = 0$). Thus any model of L for which the system of homogeneous coordinates is of the form

$$(3) \qquad (1, x_1 \ldots, x_n, y, z_1 \ldots, z_o), \quad y, z_j \in L$$

is *nonstrange* . It remains to prove that there exists such a nonsingular model.

Let $\mathcal{G}$ be a positive divisor on C such that $(y) \geq -\mathcal{G}$. Then $1,y$ $L(\mathcal{G})$ and C being projective, $L(\mathcal{G})$ is finite dimensional with a basis $(1, y_1 \ldots, y_r)$ with $y_1 = y$. These then define a morphism $\varphi_{\mathcal{G}} : C \to \mathbb{P}_r$ (Chapter I, §2) and as in Chapter I, §2. Proposition 3, we get an imbedding $\psi : C \xrightarrow{(1,\varphi\mathcal{G})} \mathbb{P}_n \times \mathbb{P}_r \xrightarrow{\sigma} \mathbb{P}_{\mathrm{nr+r+n}}$ of C in $\mathbb{P}_{\mathrm{nr+r+n}}$. The homogeneous coordinate functions on $\psi(C)$ are the functions $(x_i y_j)$ with $x_o = y_o = 1$ and among then we have $1, x_1, \ldots, x_n, y; \psi(C)$ is nonsingular and we have a model of type (3).

β) L admits a nonsingular, nonstrange model in $\mathbb{P}_3$.

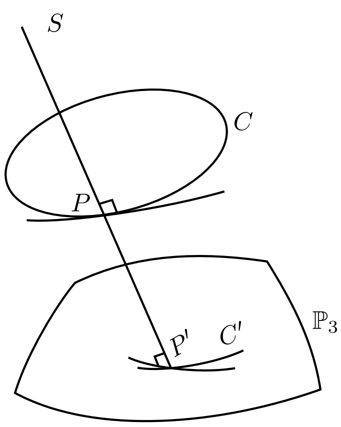
Let C be a nonsingular, nonstrange model in some $\mathbb{P}_n$ (by α)); let $n \geq 4$. The tangents T,T' at two generic points of C do *not* meet: otherwise any two tangents meet. We recall here that an irreducible system of lines in $\mathbb{P}_n$ such that any two lines in the system meet, either is a system of coplanar lines or has the property that all the lines of system pass through a point. This then implies that C is either planar or **48**
strange. Therefore the union $\underset{\lambda}{V}$ of lines in $\mathbb{P}_n$ which meet T and T' has dimension ≤ 3 in $\mathbb{P}_n$.

We now consider the set of all chords pp' in $\mathbb{P}_n$, $P, P,' \in C$, P *not* necessarily distinct from P' (thus, the tangents to C are also included); we claim that the union of these chords forms an irreducible algebraic variety W of dimension ≤ 3 in $\mathbb{P}_n$. In fact, let k be a field of definition for C, (x) a generic point of C/k, (x') a generic point of $C/_{k(x)}$. Then the "generic" chord has the parametric equations

$$y_i = tx_i + (1-t)x'_i.$$

Taking t transcendental over $k(x, x')$, the point (y) is a generic point over k for the "chord variety", and our assertion is proved.

We now take any point S in $\mathbb{P}_n$ *not* in $V \cup W$ and project $\mathbb{P}_n$ into $\mathbb{P}_3$ with S as the vertex of projection. This projection π is defined and injective on C; also as C is nonsingular the "geometric tangent space" is the same as the "Zariski tangent space" for C so that the mapping induced by π on the Zariski tangent space is also injective, which means that the maximal ideal at $P \in C$ is generated by the maximal ideal at $\pi(P) = P' \in \pi(C) = C'$. As π is an injection on C and on its Zariski tangent spaces, π is *birational* which proves that C is the normalisation of $\pi(C) = C'$.



Now (by using the injectivity of π again) one easily proves that $\mathscr{O}_p$is a finite $\mathscr{O}_{P'}$-module. As $\mathscr{O}_{P'} \subset \mathscr{O}_P$ and as both have the same residue field, viz, K, and as $\mu_{P'}, \mathscr{O}_P = \mu_P$, by Nakayama's lemma we get $\mathscr{O}_{P'} = \mathscr{O}_P$. That is, π is biregular. Therefore, C' is nonsingular; in addition it is *non-strange*: in fact, as $S \notin V$, the projections $\pi(T)$ and $\pi(T)'$ are tangents to C' which do not meet.

49 (ii) L **admits a (birational) model in $\mathbb{P}_2$(K) with only modes as singular points.**

Take a nonsingular nonstrange model C of L in $\mathbb{P}_3$ (by (i)). The tangents to C form a variety of dimension 2 in $\mathbb{P}_3$; and as C is nonstrange the union of chords PP' of C such that tangents to C at P,P' meet form a variety of dimension ≤ 2. Finally, the possibility that any chord PP' is a trisecant (i.e.meets C at a third point) is ruled out as in that case any two tangents to C will have to meet; thus the union of trisecants to C forms a variety of dimension ≤ 2 in $\mathbb{P}_3(K)$. Therefore, choose a point $S \in \mathbb{P}_3(K)$ avoiding:

α) the surface of tangents to C

β) the trisecants to C

γ) the chords at whose "extremities" the tangents are coplanar.

Project $\mathbb{P}_3(K)$ into $\mathbb{P}_2(K)$ with S as vertex. γ) and α) ensure that the 50
projection is a birational map C onto its image C'.

$\beta), \gamma$) ensure that the only singularities on C' and ordi-nary double points. C' is then the require models in $\mathbb{P}_2(K)$ Q.E.D

b) The Riemann -Roch formula

A divisor D on a nonsingular projective curve C is a finite linear combination, over $\mathbb{Z}$, of points on $C, D = \sum\limits_{P \in C} n(P).P$, $n(P) = 0$ for almost all $P \in C$. The *degree* of $D = \sum n(P).P$ is by definition $d(D) = \sum n(P)$. Any $x \in K(C)$ defines a divisor $(x) = \sum\limits_{P \in C} V_p(x), P, V_p$ being the valuation of $K(C)$ defined by the local ring $\mathscr{O}_P$; we write $(x) = (x)_\circ - (x)_\infty$ where $(x)_\circ$ and $(x)_\infty$ are both positive and are disjoint, the former defined by the zeros of x and the latter by the poles. Elementary valuation theory proves that $(x)_\circ, (x)_\infty$ have degrees equal to $[K(C) : K(x)]$. One deduces that $\deg(x) = 0 \forall x \in K(C)$; thus one may talk of the degree of a divisor class of C.

For any divisor D on C, the K-space
$L(D) = \{x \in K(C) : (x) \geq -D\} = \Gamma(C, \mathscr{L}_D)$ is finite dimensional (as C is projective), say of dimension $l(D)$. Then we have the following

Theorem 1. (Riemann-Roch) (Preliminary form). *There exists an integer $g \geq 0$ and a divisor Ω on C such that for any divisor D on C the* 51
following equality holds:

$$l(D) = d(D) - g + 1 + l(\Omega - D)$$

We shall assume the theorem in this (a proof can be found in Serre [7] Ch. The, 1.4, or in Chevalley [1] Ch. II. §5, Th. 3).

Consequences of the theorem

(i) The integer g is uniquely determined by the above property.

In fact, let Ω', g' be another pair for which the above assertion holds. Take a divisor D on C such that $d(D) > d(\Omega)$ and $d(D) > d(\Omega')$; then for this D, $l(\Omega - D) = 0 = l(\Omega' - D)$ and one obtains

$$g = g' = d(D) - l(D) + 1.$$

(ii) $g = l(\Omega)$, and $2g-2 = d(\Omega)$ as one sees by writing $D = 0$ and $D = \Omega$ in the equality.

Proposition. *Upto a linear equivalence the divisor Ω is uniquely determined by the properties: $l(\Omega) \geq g$ and*

$$d(\Omega) = (2g-2).$$

In fact, suppose Ω' is a divisor on C such that $d(\Omega') = 2g-2$ and $l(\Omega') \geq g$; then

$$g \leq (2g-2) - g + 1 + l(\Omega - \Omega')$$

i. e.,
$$l(\Omega - \Omega') \geq 1;$$

52 on the other hand, $d(\Omega - \Omega') = 0$.

The former implies that $\exists$ an $x \neq 0$ in $K(C)$ such that $(x) \geq (\Omega' - \Omega)$; but we know that $d((x)), = 0$; it follows that $(x) = \Omega' - \Omega$ and $\Omega' \sim \Omega$.

Remark. The integer g occurring in the Riemann-Roch Theorem is called the *genus* of the curve C; it can be defined for *any* irreducible curve and in fact for *any* function L of one variable over K; by definition it is the genus of any projective nonsingular model C for L.

Theorem 2. *Let C be any irreducible curve of degree d in $\mathbb{P}_2(K)$. Then the genus g of C is given by*

$$g = \frac{(d-1)(d-2)}{2} - \sum_{P \in C} \dim_K(\mathscr{O}'_P / \mathscr{O}_p)$$

where $\mathscr{O}'_P$ is the integral closure of $\mathscr{O}_P$.

Proof. Choose a system of affine coordinates x, y for C such that C has d distinct points of intersection with the line at ∞ and the affine equation for C is of the form $F(x,y) = 0, F$ a polynomial of degree d. The d distinct points of C at ∞ define a positive divisor D on C, of degree d. If A is the affine coordinate ring of C and A' its integral closure, one has the equality

$$A' = \bigcup_{n \geq 0} L(nD);$$

(Recall that for $u \in K(C), P \in C, P$ is *not* a pole of $u \iff u$ is integral on $\mathscr{O}_p$). □

Denote $L(nD)$ by A'_n; and $A'_n \cap A$ by A_n. As A' is the integral closure of $A, \exists\, z \neq 0$in A such that $zA' \subset A$. 53

Therefore $\dim_K(A'/A) = \dim_K\left(\frac{zA'}{zA}\right) \leq \dim\left(\frac{A}{zA}\right) < \infty$.

(A has Krull-dimension 1 and therefore $A/_{ZA}$ Krull dimension 0, this means that $A/\sqrt{zA}$is a finite direct product of copies of K.) It follows that $(\dim_K \mathscr{O}'_P/\mathscr{O}_P) < \infty$ for all finite P; and this is trivial for P at infinity so that $\sum\limits_{P\in C} \dim_K(\mathscr{O}'_P/\mathscr{O}_P)$is a well defined integer, and is equal to $\dim_K(A'/A)$ by the properties of localization of the integral closure.

On the other hand $A'_{n/A_n} = A'_{n/A'_n} \cap A$ is contained in $A'/_A$; as, from above, $\dim_K(A'/_A) < \infty$ and as (A'_n) is an increasing family it follows that, for large n one must have $A'_{n/A_n} = A'/A$ and hence the equality

$$\dim_K A'_n = \dim_K A_n + \dim_K(A'/_A).$$

Now A_n consists of the classes modulo F of polynomials $f \in K[X, Y]$ such that $d \circ f \leq n$; thus

$$\begin{aligned}\dim_K A_n &= \frac{1}{2}(n+1)(n+2) - \frac{1}{2}(n-d+1)(n-d+2)\\ &= nd + 1 - \frac{1}{2}(d-1)(d-2).\end{aligned}$$

Now applying the Riemann-Roch formula to the divisor nD, for large n we obtain $nd+1-\frac{1}{2}(d-1)(d-2)+\dim_K(A'/_A) = nd-g+1+0$ 54

i.e.,
$$\begin{aligned}g &= \frac{1}{2}(d-1)(d-2) - \dim_K(A'/A)\\ &= \frac{1}{2}(d-1)(d-2) - \sum_{P\in C} \dim_K(\mathscr{O}'_{P/\mathscr{O}_P})\end{aligned}$$

Q.E.D

Examples.

(i) If C is nonsingular of degree d in $\mathbb{P}_2(K)$, then the genus of $C = \dfrac{(d-1)(d-2)}{2}$.

(ii) Let P be *a* node on C.

Choose a system of affine coordinates (x, y) on C_λ such that P is the point $(0, 0)$. Then C is given at P by an equation of the form $0 = (F \equiv$

$(ax^2 + bxy + cy^2)$ + (terms of degree ≥ 3); by hypothesis the square terms do not form a perfect square and the tangents at P are given by $2ax + by = 0, bx + 2cy = 0$; these are linearly independent and the maximal ideal at P is given by

$$\mu = (xy) = (F'_x, F'_y).$$

One can show easily that the integral closure $\mathscr{O}'$ of $\mathscr{O} = \mathscr{O}_p$ satisfies the relation $F'_x\mathscr{O}' \subset \mathscr{O}, F'_y\mathscr{O}' \subset \mathscr{O}$, hence $\mu\mathscr{O}' \subset \mathscr{O}$ and $x\mathscr{O}' \subset \mathscr{O}$; we then have $\mathscr{O} \supset x\mathscr{O}' \supset x\mathscr{O}$; as $\dim_K(\mathscr{O}/_{x\mathscr{O}}) = 2$ and $\dim_K(\mathscr{O}/x\mathscr{O}') \geq 1$, it follows that $\dim_K\left(\frac{\mathscr{O}'}{\mathscr{O}}\right) = \dim_K\left(\dfrac{x\mathscr{O}'}{x\mathscr{O}}\right) = 1$. Thus the contribution to the sum $\sum\limits_{P\in C} \dim_K(\mathscr{O}'_P/\mathscr{O}_P)$ from a node 1. In particular,if C is an an irreducible plane curve, wish r nodes on it as the only singularities, and of degree d, then the genus g of C is $\dfrac{(d-1)(d-2)}{2} - r$.

2. Differentials on an curve

55 Let A be a K-algebra. Then there is an A-module $\Omega_{A/K}$ which is universal for K-derivations of A in an A-module. It is called the A- module of K-differentials of A. If A is of finite type as a K-algebra,then $\Omega_{A/K}$ is a finite A-module; if $d; A \to \Omega_{A/K}$ is the structural derivation then $\Omega_{A/K}$ is generated over A by elements $dx_\alpha, (x_\alpha)$ being a system of K- generators of A.

If C is an algebraic curve over K then $\Omega_{K(C)/K}$ is a $K(C)$ module; if $x \in K(C)$ is a separating base of $K(C)/K$ then $\Omega_{K(C)/K}$ has a base (dx).

One may also define a sheaf Ω of differentials on the curve C (more generally, on any variety); this is defined by the presheaf $U \longmapsto \Omega_{Au/K}$ on affine open sets U with affine algebra A_U; the stalk Ω_P of Ω at a point $P \in C$ is given by $\Omega_{\mathscr{O}_{P/K}}$. The sheaf Ω is locally free of rank 1 on $\mathscr{O}$ and hence is a *line bundle* denoted by $T^*(C)$, and called the *cotangent bundle* on C; it is the dual of the tangent bundle on C as is shown by the duality between derivations and differentials. An $\omega \in \Omega_{K(C)/K}$is called a *rational differential* on C.

To any such differential ω on C we associate a divisor (ω) on C in the following manner: at any point. $P \in C, \omega$ is of the form x. dt, $x \in$

$K(C)$ and t a uniformising parameter P; since $t - t(Q)$ is a uniformising parameter at all Q close to $P, \omega = x\, dt$ in a neighbourhood of P and we 56
set $(\omega) = \sum v_p(x).P$ in this neighbourhood. It can be shown that this expression is independent of the choice of t so that we obtain a divisor (ω) on $C, (\omega) = \sum\limits_{P} v_P(\omega).P$.

Proposition 1. *Let C' be a model in $\mathbb{P}_2(K)$ for C, of degree d, such that the only singularities of C' are nodes. Choose a system of affine coordinates (x, y) in $2^{(K)}$ such that C' has affine equation $F(x, y) = 0$ and all the nodes are at finite distance; assume also that the points $x = 0, y = 0$ on the line at ∞ are not on C'. Let T be a polynomial in two variables such that the curve $T(x, y) = 0$ passes through all the nodes on C'. Then the differential*

$$\omega = \frac{T(x, y)dx}{F'_y} = \frac{T(x, y)dy}{F'_x}$$

on C' (i.e. of $K(C')$) defines a differential ω on C with the property that $(\omega) \geq 0$ at all points P at finite distance on C; if, in addition, $d^\circ T \leq (d - 3)$ then $(\omega) \geq 0$ on C (we may say in this case that ω is regular or is of the 1^{st} kind on C).

Proof.

Case (i). Let P be a simply point on C'. Then $F'_x \neq 0$ or $F'_y \neq 0$ at P and correspondingly y or x will serve as a uniformiser at P.Then $v_p(\omega) = v_p(T) \geq 0$.

Case (ii). By hypothesis, $T \in (F'_x, F'_y)$ (§1, Ch II) this proves that 57
$v_P(\omega) \geq 0$ for every branch of a node; moreover if $T = 0$ passes through the nodes in a "nice" manner (i,e transversal to both the branches of C' at any node) T/F'_x and T/F'_y will be with invertible on each at any node, and $v_P(\omega)$ (which , in any case will be ≥ 0) becomes zero.

Case (iii). If P is at ∞ on C', (by our choice of coordinates) we may take $z = \dfrac{1}{x}$ to be a uniformiser at P ; then ω takes the form $\dfrac{-T}{F'_y}\dfrac{dz}{z^2}$ and

$$\begin{aligned} v_P(\omega) &= (-\deg .T - 2 + (d-1)) \\ &= d - 3 - d^\circ T; \end{aligned}$$

if $d^\circ T \leq (d-3)$, this means that $v_P(\omega) \geq 0$ for P at ∞. *Q.E.D.* □

Consequences of Proposition 1.

Corollary 1. *The differentials ω of the first kind on C form a K- space of dimension $\geq g$, the genus of C.*

By Proposition 1, case (iii), the space of regular differentials on C is of dimension $\geq \dfrac{(d-3+1)(d-3+2)}{2} - r$, where d is the degree of the plane model C' of C and r its number of nodes; the number on the right hand side is precisely the genus g of C', hence of C.

58 **Corollary 2.** *For any differential ω in C, $d((\omega)) = 2g-2$. As $\omega_{K(0)/K}$ is one dimensional ove $K(C)$ and as $d((f)) = 0 \forall f \in K(C)$, we may take ω to be of the form $\dfrac{T\,dx}{F_y}$ as in the proposition 1 with T such that $T(x,y) = 0$ passes through the nodes on the plane model C' of C in a "nice" manner and does* not *pass through any point on C' at ∞.*

Thus we have, on the one hand

$$d((\omega)) = \sum_{\substack{P \text{ finite} \\ \text{simple on} C'}} v_P(T) + d(d - 3 - d^o T)$$

and on te other hand

$$\begin{aligned} (T \cdot C') &= d \cdot (d^o T) \\ &= \sum_{\substack{P \text{ finite} \\ \text{simple on} C'}} v_P(T) + 2r \end{aligned}$$

$$\begin{aligned} \text{Thus ,} \quad d((\omega)) &= d \cdot (d^o T) - 2r + d(d - 3 - d^o T) \\ &= -2r + d(d-3) \qquad \text{Q.E.D} \\ &= 2g - 2. \end{aligned}$$

Let C be a nonsingular, complete irreducible curve and ω a differential on C. Then the divisor (ω) defined by has the properties $l((\omega)) \geq g$ (by Cor. 1 to Proposition 1: $(f) \geq -(\omega) \Longleftrightarrow (f\omega) \geq 0$), and $d((\omega)) = (2g-2)(Cor.2)$. Thus, as we have seen before (niemann-Roch, prelimi- **59**
nary form) the divisor class Ω of (ω) satisfies

Theorem (Riemann Roch-Final form). *For any divisor D on C, one has the equality*

$$l(D) = d(D) - g + 1 + l(\Omega - D).$$

As $\Omega_{K(C)/K}$ is one -dimensional over $K(C)$, it follows that Ω is the class of all differentials on C; we call it the canonical class on C.

3. Projective Imbeddings of a curve

Let C be a complete, nonsingular irreducible curve and D a divisor on C. Let $(f_o, \ldots, f_n)$ be a basis for $L(D)$; as before, they define a morphism $\varphi = \varphi_D$ from C to $\mathbb{P}_n(\mathrm{K})$.

(i) Assume that P is a point on C with $L(D) \underset{\neq}{\supset} L(D-P) \underset{\neq}{\supset} L(D-2P)$.

Then $\mathscr{O}_{\varphi(P)}$ contains a uniformising parameter for $\mathscr{O}_p$.

In fact, if $x \in L(D) - L(D-P)$, and $y \in L(D-P) - L(D-2P)$ and $z = y/x$ then $z \in \mathscr{O}_{\varphi(P)}$ and $v_P(z) = v_P(y) - v_P(x)$

$$= v_P(-D) + 1 - v_P(-D) = 1.$$

(ii) Assume that $P, P' \in C$ such that

$$L(D) \underset{\neq}{\supset} L(D-P-P) \underset{\neq}{\supset} L(D-P-P').$$

Then $\varphi(P) \neq \varphi(P')$.

Let $\lambda \in L(D) - L(D-P)$ and $y \in L(D-P) - L(D-P-P')$, and **60**
$u = y/x$; then u defines a function $\tilde{u}$ on the image in a natural way. And $\tilde{u}(\varphi(P')) = u(P') = 0$ while $\tilde{u}(\varphi(P')) = u(P') \neq$ so that $\varphi(P) \neq (P')$.

Theorem. *For a divisor D on C with deg. $D \geq 2g+1$ φ_D is an imbedding.*

Proof. For $P, P' \in C$ one has $d(D-P) \geq 2g$ and $d(D-2P) = d(D-P-P') \geq 2g-1 > (2g-2)$ so that by the Riemann-Roch formula

$$\begin{aligned} l(D) &= d(D) - g + 1 \geq g + 2 \\ l(D-P) &= l(D) - 1 \geq g + 1 \\ l(D-P-P') &= l(D) - 2 = 1(D-2P) \geq g \end{aligned}$$

and one obtains:

$$\begin{aligned} L(D) \underset{\neq}{\supset} L(D-P) \underset{\neq}{\supset} L(D-2P) \\ L(D) \underset{\neq}{\supset} L(D-P) \underset{\neq}{\supset} L(D-P-P'). \end{aligned}$$

□

By (i) preceding the theorem, φ_D is an unramified morphism and by (ii) it is injective. As C is nonsingular, one proves that φ_D is an imbedding as in §1. Q.E.D

Examples.

(i) $g = 0, 1$: classical line, cubic curve.

(ii) $g \geq 2$: the divisor $D = 3\Omega$ satisfies $d(D) \geq 2g+1$ and thus defines an imbedding φ_D *(the Tricanonical imbedding)*.

(iii) on (any nonsingular) C, *any positive divisor is ample*.

4. Morphisms of algebraic curves

61 (a) Let C be a nonsingular, irreducible complete curve and Δ the diagonal in $C \times C$. Then "self-inter section" $\Delta \cdot \Delta$ is a divisor class on Δ defined in the following manner; let φ be a function on $C \times C$ such v_Δ $(\varphi) = 1$; then $\Delta.(\Delta-(\varphi'))$ is well-defined on Δ ; if φ' is any other function

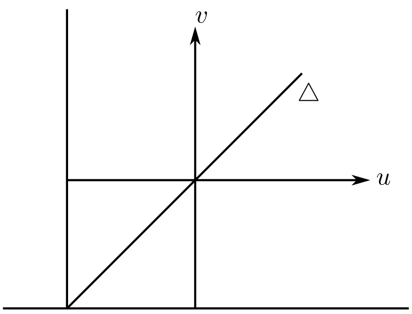
on $C \times C$ such that $v_\Delta\,(\varphi') = 1$ then $\varphi'/_\varphi$ can be restricted to a regular function θ on Δ and one has

$$\Delta \cdot (\Delta - (\varphi)) - \Delta \cdot (\Delta - (\varphi')) = \Delta \cdot (\frac{\varphi'}{\varphi}) = (\theta)$$

so that the class of $\Delta \cdot (\Delta - (\varphi))$ is independent of the choice of φ; call it Δ, Δ. We have the

Theorem. $\Delta \cdot \Delta = -\Omega_\Delta$, *the class on* Δ *defined by the canonical class* Ω *on* C*; therefore, the intersection number* ($\Delta \cdot \Delta$) *is* $(2 - 2g)$.

Proof. Take a separating function $f \in K(C)$; then $df \neq 0$. Define φ on $C \times C$ as $\varphi(P, Q) = f(P) - f(Q)$. One has clearly $v_\Delta(\varphi) = 1$.



Also, if $f = \sum_{n \geq n_o} a_n u^n$ is the Taylor expansion of f at a point P on C, then φ is locally given by $\varphi = \sum a_n(u^n - v^n)$ on $C \times C$; the local equation of $(\varphi) - \Delta$ is thus $\sum_{n \geq n_\circ} a_n \frac{u^n - v^n}{u - v}$; from this one deduces by an easy computation that the local education of

$pr(\Delta \cdot ((\varphi) - \Delta))$ is $\sum_{n \geq n_o} n a_n u^{n-1} = \dfrac{df}{du}$ and the coefficient of (P, P) in 62
$\Delta \cdot ((\varphi) - \Delta)$ is therefore $v_p(df)$. Q.E.D. □

Corollary. *Let* $f \in K(C)$*; if* $\varphi \in K(C \times C)$ *is given by* $\varphi(P, Q) = f(P) - f(Q)$*, then the divisor* $\Delta.(\Delta - (\varphi))$ *on* Δ *in* $C \times C$ *is given by* $-(df)_\Delta$.

(b) Let C, C' be nonsingular, irreducible, complete curves and $\pi : C \to C'$ a morphism; let $\pi^* : K(C') \to K(C)$ be the cohomomorphism of π. For any $P' \in C'$, the discrete valuation $v_{P'}$ with centre P' of $K(C')$ extends to discrete valuations $v_{P_1}, \ldots, v_{P_r}$ of $K(C)$ with centres $P_1, \ldots, P_r$; we thus define a divisor $\pi^*(P') = \sum_{i=1}^{r} e_i P_i$ where e_i is the ramification index of the extension v_{P_i} over $v_{P'}$. The degree of $\pi^*(P')$ is then equal to $\sum_{i=1}^{r} e_i = n = [K(C) : K(C')]$. The definition is extended by linearity to all divisors on C'. We say that π is *unramified* if all the e_i are 1. If (f'), $f' \in K(C')$, is a principal divisor on C', $\pi^*((f'))$ is principal on C defined by $\pi^* f' \in K(C)$. If π is separable (i.e. if $K(C)/K(C')$ is separable) and $\omega' = x'dt'$ is a differential on C', then $\pi^*(\omega') = \pi^* x'.d(\pi^* t')$ is a differential on C; and for any $z' \in K(C'), \pi^*(z'\omega') = \pi^* z'.\pi^*(\omega')$. We shall now prove a theorem which shows how $\pi^*((\omega'))$ and $(\pi^*(\omega'))$ are related on C.

63 **Theorem (Hurwitz-Zenthem).** *If π is a separable morphism $C \to C'$, then for any differential ω' on C'*

$$(\pi^*(\omega')) - \pi^*(\omega') = \underline{d}$$

where $\underline{d}$ is the different of $K(C)/K(C')$.

Proof. Let $P \in C$ and $P' = \pi(P) \in C', z$ a uniformising parameter at P', t a uniformising parameter at P. We have $\mathscr{O} = \mathscr{O}_p \supset \mathscr{O}_{P'} = \mathscr{O}'$ and $K[[t]] = \hat{\mathscr{O}} \supset \hat{\mathscr{O}}' = K[[Z]]$ and finally $\widehat{\mathscr{O}} = \widehat{\mathscr{O}'}[t]$. One has an exact sequence of $\hat{\mathscr{O}}$-modules

$$\Omega_{\hat{\mathscr{O}}'/K} \otimes \hat{\mathscr{O}} \to \Omega_{\hat{\mathscr{O}}/K} \to \Omega_{\hat{\mathscr{O}}/\hat{\mathscr{O}}'} \to 0.$$ □

Thus $\Omega_{\hat{\mathscr{O}}/\hat{\mathscr{O}}}$ is identified to the quotient of $\Omega_{\hat{\mathscr{O}}/K}$ by dF where F is the minimal polynomial of t over $\mathscr{O}'$. Identifying $\Omega_{\hat{\mathscr{O}}/K}$ with $\hat{\mathscr{O}}$, thanks to the base dt, one finds that $\Omega_{\hat{\mathscr{O}}/\hat{\mathscr{O}}}$ is identified with $\hat{\mathscr{O}}/_{(F'(t))}$. The different of $\hat{\mathscr{O}}/_{\hat{\mathscr{O}}'}$ is thus generated by $(F'(t))$.

Now, we first observe that $(\pi^*(\omega')) - \pi^*((\omega'))$ is independent of ω'; thus, set $\omega' = dz$. Then $v_{P'}(\Omega') = 0$; and therefore if $dz = ydt$ one obtains

$$v_P(dz) = v_{P'}(\pi^*(\omega')) = v_P(y) = v_P(F'(t))$$

and $$v_P((\pi^*(\omega')) - \pi^*((\omega'))) = v_P(F'(t)) - v_{P'}(\omega') = v_P(d) - 0.$$

64 One deduces then that

$$(\pi^*(\omega')) - \pi^*((\omega')) = d..$$

Q.E.D.

Corollaries. Let g and g' be the genera of C and C'.

(i) By the above theorem we obtain

$$2g - 2 - n(2g' - 2) = d^{\circ}(d) \geq 0.$$

(ii) *Luroth's theorem.*

If $g = 0$ *then* $g' = 0$.

From (i), we obtain for $g = 0$,

$$-2 - n(2g' - 2) \geq 0$$
$$n(2g' - 2) \leq -2$$

which implies that $g' = 0$.

(iii) $g' = 0, \pi$ unramified $\Rightarrow n = 1, g = 0$.

Note that π unramified $\Leftrightarrow d = 0$ so that by (i),

$2 - n(-2) = 0$ and $n = 1 - g$ which implies $g = 0$, $n = 1$.

(iv) $g = 1 \Rightarrow g' = 0, 1$.

By (i), $-n(2g' - 2) \geq 0$, i.e. $g' = 0$ or 1

(v) $g = 1, g' = 1 \Rightarrow \pi$ unramified

(vi) $g' = 1, \pi$ unramified $\Rightarrow g = 1$.

(c) Let $\pi : C \to C'$ be a separable morphism as in (b). Then the graph $T = (\pi \times \pi)^{-1}(\Delta')(\Delta' =$ diagonal in $C' \times C'$) of the equivalence relation defined by π on C is a cycle on $C \times C$ in a natural way. If d is **65**
the different of $K(C)/_{K(C')}$ then one has

Theorem.

$$\Delta.(T - \Delta) = (d)_{\Delta}$$

Proof. Write $T = \Delta + S$. Then the set of all points P'_i in $\pi(pr_C(\Delta \cap S))$ is finite and one can find a function f' on C' which is a uniformiser at all $P'_i s$. We define a function u' on $C' \times C'$ by $u'(P', Q') = f'(P') - f'(Q')$. Then $v_{\Delta'}(u') = 1$ and $\Delta'.((u') - \Delta') = (df')_\Delta$ by the corollary to the theorem of (a). We set $f = \pi^* f'$ and $u = \pi^* u'$ so that $u(P, Q) = f(P) - f(Q)$ on $C \times C$. Also, if $(u') = \Delta' + X'$ then $(u) = T + X$ with $\pi^{-1} = (X') = X$ and thus $(u) = \Delta + S + X$. One then deduces that □

$$
\begin{aligned}
(\Delta + S).X &= \pi^*(\Delta' \cdot X') = \pi^*(df')_\Delta \\
&= (df)_\Delta - d_\Delta (\text{by Hurwitz's theorem}).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
(\Delta + S).X &= \Delta.((u) - \Delta - S) + S.X \\
&= (df)_\Delta - \Delta.S + S.X.
\end{aligned}
$$

Therefore,

$$\Delta \cdot S - \underline{d}_\Delta = S.X.$$

But if $P \in S.X$ then $\pi(P) = P' \in S'.X'$ so that

$$P \in (S.X) \cap \Delta \Rightarrow P' \in (S'.X') \cap \Delta$$

and $(df')(P') = 0$. But we have chosen f' to be a uniformising param-
66 eter at $P' \in \pi(pr_C(\Delta \cap S))$ and hence $(df')(P')$ cannot be zero. This proves that $S.X$ as a cycle on $C \times C$, disjoint from Δ, and we deduce then from the last equality that

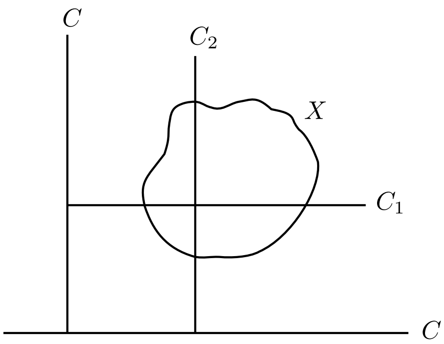
$$S.X = 0$$

and $$\Delta.S = \underline{d}_\Delta$$

i.e. $$d_\Delta = \Delta.(T - \Delta)$$ Q.E.D.

Theorem (Schwarz-Klein). *A curve C of genus ≥ 2 admits only finitely many automorphisms.*

Proof. Let X be any cycle on the product $C \times C$. If C_1 and C_2 are the "horizontal" and "vertical" in $C \times C$, one defines the *indices* of X as the intersection numbers $(X.C_1) = d_1$ and $(X.C_2) = d_2$. If $C \times C \hookrightarrow \mathbb{P}_n$ is a projective imbedding of $C \times C$ and X is a cycle of dim 1 on $C \times C$, then the projective degree of X in $\mathbb{P}_n$ is completely determined by its indices d_1 and d_2 in $C \times C$ and the degree of $C \times C$ in $\mathbb{P}_n$. More precisely, consider an imbedding $C \hookrightarrow \mathbb{P}_r$; this defines an imbedding $C \times C \hookrightarrow \mathbb{P}_r \times \mathbb{P}_r \hookrightarrow \mathbb{P}_{r^2+2_r}$ and the homogeneous coordinates in the image are of the form $(z_{ij}) = (x_i y_j)$.



Consider a hyperplane $H \equiv \sum\limits_{i,j} a_j b_j z_{ij} = 0$ in $\mathbb{P}_{r^2+2r}$, which is the product of two hyper planes $H_1 \equiv \sum a_i x_i = 0$ and $H_2 \equiv \sum_{b_j x_j} = 0$ in $\mathbb{P}_r$. If the degree of C in $\mathbb{P}_r$ is d and d if X is a cycle of inlices d_1 and d_2 in $C \times C$ then the intersection $H(C \times C)$ in $\mathbb{P}_{r^2+2r}$ is given by

$$H(C \times C) = (H_1.C) \times C + C \times (H_2.C)$$

so that the intersection number $(H.X)$ is equal to $d(d_1 + d_2)$. Thus, the **67**
positive cycles of dim 1 on $C \times C$ with given indices d_1 and d_2 form a finite union of irreducible algebraic families. (of Theory of Chow coordinates). Now a positive cycle T of dimension 1 in $C \times C$ is the graph of an automorphism σ of $C \Leftrightarrow T$ has indices $1, 1$; it follows in particular that the graphs of automorphisms of C form a finite union of irreducible algebraic families. We will be through, therefore, if we prove that every irreducible system of graphs of automorphisms of C is of dimension 0; let $(T_\sigma)_\sigma$ be such a system; if its dimension ≥ 1, fixing an automorphism $\sigma_\circ$ of C, the system $(T_\sigma T_{\sigma_\circ}^{-1})_\sigma$ is irreducible, contains Δ and a $T \neq \Delta$ by assumption. We then have $(T.\Delta) = (\Delta.\Delta)$. On the

other hand, $(T.\Delta) \geq 0$ while $(\Delta.\Delta) = 2 - 2g < 0$(note that $g \leq 2$). This contradiction proves our theorem. □

Corollary. *Let K be an algebraically closed field and K' any extension of K. Let L_1 and L_2 be function fields of one variable over K, of genus ≥ 2 and linearly disjoint from K' over K.Then any K'-isomorphism $K'(L_1) \to K'(L_2)$ is the extension of a K-isomorphism $L_1 \to L_2$.*

68 *Proof.* Choose a "big" algebraically closed $\Omega \supset K'$ models C_1 and C_2 for $K'(L_1)$ and $K'(L_2)$ over Ω. We have bijections

$$\begin{aligned} &\left\{ K'\text{−isomorphisms of } K' \ (L_1) \text{ on } K'(L_2)\right\} \\ \leftrightarrow &\left\{ K'\text{−isomorphisms of } C_2 \text{ on } C_1\right\} \\ \leftrightarrow &\left\{ K'\text{−automorphisms of } C_1\right\}. \end{aligned}$$

By hypothesis of linear disjointness, the genus of C_1 over the algebraic closure $\overline{K'}$ of K' is ≥ 2 and by the theorem of Schwarz-Klein the number of K'-automorphisms of C_1 is finite.

Consider any K'-isomorphisms $K'(L_1) \to K'(L_2)$. If $L_1 = K(x_{1,\dots,}x_p)$ and $L_2 = K(y_{,\dots,}y_q)$, this defines rational functions R_i and S_j over K such that

$$\begin{aligned} \varphi(x_i) &= R_i(y_{1,\dots,}y_{q,}\lambda_1, \dots, \lambda_s) \\ \varphi^{-1}(y_j) &= S_j(x_{1,\dots,}x_p, \lambda_{1,\dots,}\lambda_s) \end{aligned}$$

with the $\lambda's$ in K'. The locus of $(\lambda_1, \dots, \lambda_s) \in \Omega^s$ over K is zero dimensional as each specialisation gives a K'-isomorphism. As K is algebraically closed, one concludes that $(\lambda_1, \dots, \lambda_s) \in K^s$. The corollary is proved. □

Remark. This corollary shows that, if C is a curve of genus ≥ 2 defined over an algebraically closed field K, then every automorphism of C is defined over K.

Theorem (Severi). *Let K be an algebraically closed field and L a function field of one variable over K. Then the intermediary extensions*
69 $K \underset{\neq}{\subset} L' \subset L$ *such that genus $L' \geq 2$ and $L/_{L'}$ is separable, are finitely many in number.*

Proof. Let C be a model for L over K. any L' with the given property, take any model C' of L' over K; the inclusion $L' \hookrightarrow L$ defines a morphism $\pi : C \to C'$; if genus $C = g$ and genus $C' = g'$; we have the equality $2g-2 = n(2g'-2)+d^\circ(\underline{d})$ where $n = [K(C) : K(C')] = [L : L']$ and $\underline{d}$ is the different of L over L'. As $d^\circ(d) \geq 0$ and g is given, the number of choices for $g' \geq 2$ and n is *finite. Thus we may assume that n* and g' *are also given.* □

Take then a curve C' such that $K(C')$ is of genus g' and $K(C)$ is separable of degree n over $K(C')$. Consider the graph T in $C \times C$ of the equivalence relation defined by the morphism $\pi : C \to C'$. Then T is a cycle of dimension 1 of the form $T = \Delta + S$, symmetric about Δ.

T can be considered as a correspondence of C in C, i.e. as a divisor on $C \times C$. Thus one can form the composite correspondence ToT; more generally, if A is a correspondence of C_1 in C_2 and B is a correspondence of C_2 in C_3, the composite correspondence AoB of C_1 in C_3 is defined by

$$AoB = pr_{13}((A_{12} \times C_3).(C_1 \times B_{23}))$$

where pr_{13} is the *algebraic projection* $C_1 \times C_2 \times C_3 \to C_1 \times C_3$. Also, if $P \in C_1$ and the "value" of A at P is defined as $A(P) = pr_2(A(\{p\} \times C_2))$, one has the following equality

$$A(B(P)) = (AoB)(P).$$

Therefore, in the present case, if $[K(C) : K(C')] = n$, it follows that 70

(I) $$\boxed{T \circ T = nT}$$

From which one obtains:

$$(\Delta + S)o(\Delta + S) = n\Delta + nS$$
$$i.e. \Delta + 2S + SoS = n\Delta + nS \quad \text{and thus}$$

(II) $$\boxed{S \circ S = (n-1)\Delta + (n+2)S}$$

One concludes then that the correspondences T on $C \times C$, for a fixed n, form a union of irreducible, algebraic system (cf. Theory of Chow Coordinates). Our aim now is to show that each irreducible system $(T_\alpha)_\alpha$ of correspondences in $C \times C$, defined as above by morphisms π of C into an arbitrary curve C' such that genus $C' = g$ (fixed) ≥ 2, and $[K(C) : K(C')] = n$ (fixed) is zerodimensional. First note that if A and B are correspondence of C in C then

(III) $$\boxed{(A \cdot B) = ((AoB) \cdot \Delta)}$$

In fact, consider the 0-dimensional cycle $Z = (A_{12} \times C_3) \cdot (C_1 \times B_{23}) \cdot (\Delta_{13} \times C_2)$ in $C_1 \times C_2 \times C_3$ (each $C_i = C$).

One has

$$pr_{13}(Z) = (A \circ B) \cdot \Delta_{13} \qquad \textit{and}$$
$$pr_{12}(Z) = A.B.$$

71 As a 0-dimensional cycle has the same degree as its projections, our assertion is proved.

Write now $T_\alpha = \Delta + S_\alpha, S_\alpha$ symmetric for every T_α in the irreducible family (T_α). Assume that the dimension of the family is ≥ 1. Then the S_α are distinct from Δ and one has $(\Delta . S_\alpha) \geq 0$.

Case (i). All the components of S_α are "moving", i.e., each $S_\alpha . S_\beta, \alpha \neq \beta$ is defined. In this case, one has $(S_\alpha . S_\alpha) \geq 0$ while by (III) and (II) we get

$$\begin{aligned}(S_\alpha . S_\alpha) &= (n-1)(2-2g) + (n-2).d^o(\underline{d}) \\ &= (n-1)(2-2g) + (n-2)(2g-2-n(2g'-2)) \\ &< 0 as g, g' \geq 2.\end{aligned}$$

This contradiction establishes the result in this case.

Case (ii). (General Case) In general, one may write $S_\alpha = F + m_\alpha$, where F is "fixed" and M_α "moving", both symmetric. One has then

$$\begin{aligned}S_\alpha o S_\alpha &= FoF + FoM_\alpha + M_\alpha oF + M_\alpha oM_\alpha \\ &= (n-1)\Delta + (n-2)F + (n-2)M_\alpha.\end{aligned}$$

As $(n-2)M_\alpha$ is not fixed while FoF is fixed one obtains

(IV) $$\boxed{F \circ F \leq (n-1)\Delta + (n-2)F}$$

Now consider the symmetric correspondence **72**

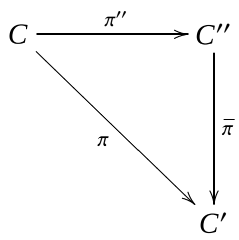
$$G = \Delta + F \text{ in } G \times C.$$

From IV it follows easily that

$$G \circ G \subset G \text{ set theoretically.}$$

Thus G can be identified with the graph (set theoretically) of the equivalence relation defined by a morphism $C \xrightarrow{\pi} C''$ of C in another curve C''. (In fact, the set theoretic map $P \to G(P)$) gives a morphism of C in a suitable symmetric power of C)

The a priori set theoretic map $C'' \xrightarrow{\overline{\pi}} C'$ which makes the diagram



commutative can be easily checked to be a morphism. To prove the zero dimensionality for graphs of equivalence relations defined by morphisms $\pi : C \to C'$ it is enough to prove the same for the morphisms $C'' \xrightarrow{\overline{\pi}} C'$. But the graphs of the equivalence relations defined by the $\overline{\pi}'s$ in $C'' \times C''$ are $\pi''(T_\alpha) = \pi''(G + M_\alpha) = \Delta'' + \pi''(M_\alpha)$ and by hypothesis the components $\pi''(M_\alpha)$ are all "moving" and we are back to case (i). Q.E.D

Corollary 1. *Let k be an arbitrary field and L a function fields of one* **73**
variable over k, which is regular extension of k. Then the intermediary fields $k \subset L_1 \subset L_\lambda$ such that

(i) L_1 is a function of one variable over k, of *genus (absolute)* ≥ 2.

(ii) L is separable over a L_1

are finite in number.

Proof. By making a base change $k \to$ the algebraic closure $\bar{k}$ and using the fact that L and $\bar{k}$ are linearly disjoint over k it follows that

$$L_1 = \bar{k}(L_1) \cap L$$

$$\begin{array}{ccc}
L & \text{———} & \bar{k}(L) \\
| & & | \\
L_1 & \text{———} & \bar{k}(L_1) \\
| & & | \\
k & \text{———} & \bar{k}
\end{array}$$

for any intermediary L_1. As, by hypothesis, the genus of $\bar{k}(L_1)$ over $\bar{k}$ is ≥ 2, it follows that the number of $\bar{k}(L_1)$ is finite. Our assertion follows. □

Corollary 2. *Let k be an arbitrary field and L an algebraic function field over k, regular over k. Then the number of intermediary fields $k \subset L_1 \subset L$ such that (i) L_1 is a function field of one variable over k of absolute genus ≥ 2 (ii) L is separable over L_1, is finite.*

Proof. As in Corollary 1, we may assume *k algebraically closed*

We apply an induction on the transcendence degree d of L/k. For $d = 1$, we are through by Corollary 1. □

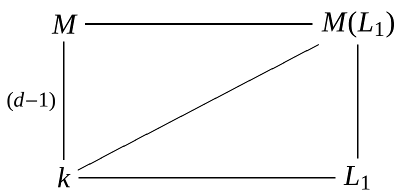
74 Let now $u_1, \ldots, u_d$ be a separating base for L/K. We set M to be the algebraic closure of $k(u_1, \ldots, u_{d-1})$ in L.

Case (i). The number of L_1 with the required properties, contained in M, is finite by induction hypothesis.

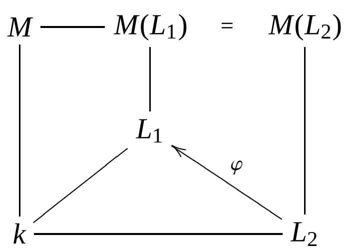
Case (ii). Consider an L_1 of the required type, $L_1 \not\subset M$. Then $M(L_1)$ is transcendental over M and L is hence separably algebraic over $M(L_1)$,

$$M \subset M(L_1) \subset L.$$

Thus M and L_1 are linearly disjoint and it follows that the absolute genus of $M(L_1)$ over M is ≥ 2 . As the transcendence degree of L over M is 1, it follows by Corollary 1 that the number of $M(L_1), L_1$ being of the required type over $k, L, \not\subset M$, is *finite*.



If L_1, L_2 are two extensions of the required type, with $L_1 \not\subset M, L_2 \not\subset M$, and if in addition $M(L_1) = M(L_2)$, then by the corollary to Schwarz-Klein it follows that there is a k-isomorphism



$$\varphi : L_2 \to L_1$$

extending to identity on $M(L_1) = M(L_2)$. Thus φ has to be the identity map. Case (ii) and thus corollary 2 is proved.

Corollary 3 (de Franchis)**.** *Let V be an algebraic variety and C an* 75
algebraic curve of absolute genus ≥ 2, over a field k. Then almost all rational maps $V \to C$ are either constant or separate.

Proof. Any nonconstant rational map $V \longrightarrow V$ defines an inclusion $k(C) \hookrightarrow k(V)$ and if the map is separable the field $k(V)$ is a separable extension of $k(C)$, whose absolute genus is ≥ 2. By corollary 2, such inclusion are finitely many An application of Schwarz-Klein concludes the proof. □

Q.E.D.

APPENDIX TO CHAPTER II
Nonsingular strange curves

For proving the existence of a plane model of a function field with **76**
only nodes (§1, Ch II), we had avoid the "strange" curves of characteristic p, i.e, the curves C in projective space all the tangents of which have a fixed point in common. A posterior (i.e. by using facts about divisors of differentials) one can prove that we were fighting against a phantom ; more precisely:

Theorem. *The only nonsingular projective strange curves are the lines, and in characteristic* 2*, also the plane conics.*

That a plane conic $ayz + bzx + cxy + dx^2 + d'y^2 + d''z^2 = 0$ is strange in characteristic 2 is well known and easily proved. The equation of the tangent at (x, y, z) is

$$XF'_x + YF'_y + ZF'_z = X(bz + cy) + Y(cx + az) + Z(ay + bx) = 0$$

and is satisfied by the point (a, b, c) (we have$(a, b, c) \neq (0, 0, 0)$, otherwise our conic is a double line).

Conversely let C be a strange nonsingular curve in $\mathbb{P}_n$, defined over an algebraically closed field K of characteristic $p \neq 0$. By a suitable choice of coordinates, we may assume that the point A common to all tangents to C has homogeneous coordinates $(1, 0, \ldots, 0, 0, 0)$ and that (except perhaps for A) C does not contain point for which two coordinates vanish. Let L be the function field $K(C)$ of C, and $(x, x_2, \ldots, x_n)$
$(x, x_i \in L)$ be the affine coordinate functions on C outside of the hyper- **77**
plane H (last coordinate = 0). By hypothesis all point of $C \cap H$ lie in the affine piece with coordinates $\left(\frac{1}{x}, \frac{x_2}{x}, \ldots, \frac{x_n}{x}\right)$.

Since all tangents to C pass through A, we have $Dx_2 = \cdots = Dx_n = 0$ for any K-derivation D of L, i.e.

$$x_2, \ldots, x_n \in L^p. \tag{1}$$

We are going to compute the divisor (dx) . At a point $P \in C - (C \cap H), C$ is transversal to the hyperplane $X_1 = 0$, whence $x - x(P)$ is uniformizing at P. Thus

$$v_P(dx) = 0. \tag{2}$$

For points $P \in C \cap H$, we set $y = \frac{1}{x}, y_1 = \frac{x_i}{x}(i = 2, \ldots, n)$; thus $y \in L^p Y_i$ for $i = 2, \ldots, n$. Suppose, first, that $P \neq A$. We have $y(P) = 0, y_i(P) \neq 0$ for $i = 2, \ldots, n$. Since the maximal ideal of the local ring $\mathscr{O}_p$ (i.e. of the valuation ring of v_P) is generated by $y, y_2 - y_2(P), \ldots, y_n - y_n(P)$, there exists an index i such that $y_i - y_i(P)$ is a uniformizing variable t at P. Since $y \in L^p y_i$ and since $v_P(y) > 0$, the power series equation of y with respect to t is

$$y = (y_i(P) + t)(\alpha_0 t^{pj_P} + \alpha_1 t^{p(jP+1)} + \cdots),\ (\alpha_0 \neq 0, j_P > 0);$$

it contains terms of degree pj_p and $pj_p + 1$ with nonzero coefficients. Hence $v_P(y) = pj_P, v_P\left(\frac{dy}{dt}\right) = pj_P$. Since $dx = -\frac{dy}{y^2}$, we have

$$v_P(dx) = -pj_p, (j_P > 0). \tag{3}$$

78 Now, if $A \in C$ we have $y(A) = y_2(A) = \cdots = y_n(A) = 0$. As above one of the y_i is a uniformizing variable t at A. From $y \in L^p y_i$ and $v_A(y) > 0$, we get the power series expansion

$$y = t(\alpha_0 t^{pj_A} + \alpha_1 t^{p(j_A+1)} + \cdots),\ (\alpha_0 \neq 0, j_A \geq 0).$$

Hence $v_A(y) = pj_A + 1, v_P\left(\frac{dy}{dt}\right) = pj_A$. From $dx = -\frac{dy}{y^2}$, we get now

$$v_A(dx) = -pj_A - 2, (j_A \geq 0). \tag{4}$$

From (2), (3) and (4), and from the fact that $C \cap H \neq \phi$ we see that the degree of the divisor (dx) is < 0. Since it is $2g - 2$ (g denoting the genu of C), it is necessarily 2 and we have $g = 0$. Looking at (3) and (4), we see that only two cases may happen:

a) $C \cap H$ consists on only one point $P \neq A$. Then $v_P(dx) = -2, p = 2, j_P = 1, v_P(y) = 2$; this last relation shows that $C.H = 2P$ Whence C has degree 2; we get a conic in characteristic 2.

b) $C \cap H$ contains only A. Then $v_A(dx) = -2, j_A = 0, \ v_A(y) = 1 C.H = A$; thus C has degree 1 and is a straight line.

Remark. There exist, of course many, *singular* strange curves in characteristic p: take a function field L of transcendence degree 1 over K, functions, $z_2, \ldots z_n \in L$ which generate L^p over K and $z \in L - L^p$ then $L = K(z, z_2, \ldots, z_n)$; the affine curve D with coordinates function $(z, z_2, \ldots, z_n)$ is a model of L; take its projective closure $\bar{D}$; It i easily seen that all tangent to $\bar{D}$ pass through the point $(1, 0 \ldots)$,

Chapter 3

The Theorem of Grauert (Mordell's conjecture for function fields)

1. Description of the method

In this section we shall describe, often rather loosely, the method of attack in the proof of Grauert. (Paragraphs which do make a precise mathematical sense will be starred). 79

(∗) Let k be an algebraically closed field and K, a function field over k; let C be a curve defined over K, with its absolute genus $g(\geq 2)$ equal to its relative genus over K. We shall do the geometry over a "big" universal domain Ω. Following Grauert ([2]) we are going to analyse the cases when the set C_K of K- rational point of C (i.e. the points of C having coordinates) in K is *infinite*. The complete results (at least in characteristic 0) have been stated at the beginning of these notes. Our first aim will be to prove the following

(∗)

Theorem 1. *If C_K is infinite then C is birationally equivalent (over some extension of K) to a curve C' defined over k.*

If this is done, the theorems of Severi and be Franchis proved in chapter II will enable us to study easily the rational points of C' over K (or some extension of K) and to obtain complete result about the structure of C_K.

80 $(*)$ We shall first prove theorem 1 under the additional hypothesis that the transcendence degree of K/k is 1. Then an easy induction d on will give theorem 1. It should be noted that this is not really essential (Grauert studies case of arbitrary d).

However, this makes the proof simpler and more understandable.

Gruert's method is the inverse of the so called "Picard's Method" whereas Picard liked to consider a surface as "curve over a function field" here the curve C will be interpreted as a surface over k. Roughly speaking let $F(x, y) = 0$ be a plane model of C the coefficients of F are elements of K, i.e. rational functions of parameters $t = (t_1, \ldots, t_n)$ such that $K = k(t)$. Then (upto a factor in $k(t)$) we may write $F(x, y)$ as a polynomial $G(t, x, y)$ over k. The surface will be $G(t, x, y) = 0$.

$(*)$ Let us be more precise. The field K is k-isomorphic to the function field $k(R)$ of an affine curve R, which we may assume to be nonsingular; thus we may write $K = k(r)$ where r is a generic point of R over k. We may also assume that C is a nonsingular curve in some $\mathbb{P}_n(\Omega)$; let x be a generic point of C over K. Then our surface X will be the *locus of the point* $(r, x) \in R \times \mathbb{P}_n$ *over* k. The projection on the first factor gives a fibration $\pi : X \to R$; for $t \in R$, we set $X_t = \pi^{-1}(t)$ (a priori, this is a cycle on X). The generate fibre X_r is essentially to curve C.

81 **Lemma 1.** *Almost all fibres $X_t, t \in R$, are irreducible nonsingular curves, having the same genus g as C.*

Irreducibility is proved elements by taking a plane model C_1 of C, the corresponding surface $X_1 \subset R \times \mathbb{P}_2$ and by noticing that the non-absolutely - irreducible homogeneous polynomials in 3 variables of a given degree d form a closed subset of the space of all homogeneous polynomials of degree d. As to nonsingularity, the singular set X' of X does not intersect the generic fibre X_r(X_r is nonsingular); Since $\pi : X \to R$ is proper, it mean that X' is in a finite union of fibres say $\bigcup_{i=1}^{q} X_{t_i}$; then for $t \neq t_i$, X_t is nonsingular. Finally all the nonsingular fibres X_t have the same genus according to a theorem of Igusa:

Sketch of the proof : Consider the fibre product $X \underset{R}{x} X$ and the iagonal Δ_t in $X_t \times X_t \subset X \underset{R}{x} X$; the genus g_t of X_t is given by $2-2g_t = (\Delta_t.\Delta_t)$ (Chapter II); this intersection number is constant by the "principle of conservation of the intersection number"(of. The Theory of Chow coordinates).

(∗) A rational point (x') of C over $K = k(r)$ corresponds to a rational section $s : R \to X$ of $\pi : X \to R$: the coordinates of (x'_i) are rational function $x'_i(r)$ (over k); thus the section is given by $\mapsto (x'_i(t))$. Since R is nonsingular and X is complete s is even a *morphism* $R \to X$.

Suppose that we have prove that C is birationally equivalent to a curve defined over k. Then X is birationally equivalent to $R \times C'$ and the section $R \to X$ correspond to maps $R \to C'$. By the theorem of de Franchis (Chapter II) almost all these maps are constant or inseparable, i.e have a derivative equal to 0: in other words, their graphs are tangent to the "*horizontal directions field*" or $R \times C'$. Coming back to X we see that we shall have on X,a field E of tangent directions, everywhere transversal to the fibres and such that almost all sections $s : R \to X$ are tangent to E. **82**

(∗) More precisely, we replace R by an open $R_o \subset R$ such that every fibre of $X \mid R_0$ is irreducible and nonsingular. We denote R_o by R and $X \mid R_0$ by X, so that X is nonsingular. At each point $(t, x) \in X$ the tangent directions to X form a projective line, with a marked point namely the tangent to the fibre X_t at (t, x). We thus get a *projective line bundle* $\theta : \hat{F} \to X$ (bundle of tangent direction) with a section F^{∞} (the tangent to the fibres); we set $F = \hat{F} - F^{\infty}$; this is an affine *Line bundle* on X. For $t \in R$. We set $\hat{F}_t = \hat{F} \mid X_t, F_t = F \mid X_t$ and $F^{\infty}_t = F^{\infty} \mid X_t$.

Sine an affine space "carries functionally in the its structure" its vector space of "translations" F_t admits a *vector-line-bundle* of translation S_t.

Lemma 2. *The translation bundle $S_t \to X_t$ of $F_t \to X_t$ is isomorphic to the tangent bundle $T(X)$.* **83**

Proof. Let $T(X), T(R)$ be the tangent bundles to X, R and $\tilde{T}(R)$ be the pull back $\pi^* T(R)$ of $T(R)$ to X. We have the exact sequence

$$(1) \qquad 0 \longrightarrow T(X_t) \longrightarrow T(X) \mid X_t \longrightarrow \tilde{T}(R) \mid X_t \longrightarrow 0.$$

Local sections α, β of $F_t \to X_t$ give in each fibre $T(X) \mid X_t$ supplementary subspaces of the corresponding fibres of $T(X_t)$. Hence they be viewed as local spitting $\alpha, \beta : \tilde{T}(R) \mid X_t \to T(X) \mid X_t$ of the exact sequence (1). Thus their *difference* $\alpha - \beta$ is a local section of Hom $(\tilde{T}(R) \mid X_t, T(X_t))$. Now, for a fixed $t \in R$, the bundle $\tilde{T}(R) \mid X_t$ is trivial whence Hom $(\tilde{T}(R) \mid X_t, T(X_t)) \simeq T(X_t)$.
Q.E.D. □

If the "horizontal direction field" E exists, it is a section of $F \to X$. For fixed $t \in R, E_t = E \mid X_t$ is a section of $F_t \to X_t$ which then identifies F_t with $T(X_t)$(Lemma 2) (fixing a point in an affine space makes it a vector space). Now since $(2g-2) > 0, T(X_t)$ admits only the zero section; thus, if E exists, it is *unique*. Furthermore, we have, by Grauert's criterion, (Chapter I) a morphism of $T(X_t)$ into an affine space which contracts the zero section to a point and which is biregular elsewhere. Thus we should look for a nice morphism φ_t of F_t into an affine space the curves on the surface F_t which are contracted to points by φ_t.

84 In order to be "coherent" with respect to t" we will look for a nice morphism φ of F into an affine spaces. We shall successively:

a) construct the morphism φ

b) study the "blowing down set E" of φ (i.e. the set of all $y \in F$ such that $\dim_y \varphi^{-1}(\varphi(y) > 0)$ and prove that it is a section of $F \to X$

c) prove that almost all sections s:$R \to X$ are tangent to E

d) prove that the existence of a direction field E or X enables us to "descend" to a smaller field of definition for the generic fibre X_r namely $k(r^p)$ (p = charac. k, so that $k(r^p) = k$ if $p = 0$)

e) in case p= charac $k \neq 0$ lower the field of definition of X_r successively to $k(r^{p^s})$, $s \geq 1$; then use a lemma (a construction analogous to the one used by Mumford in his theory of module) to prove that X_r is birationally equivalent to a curve C' defined over $\bigcap\limits_{s \geq 1} k(r^{p^s}) = k$.

2. The Proof

(A) Construction of a morphism.

For $t \in R$, the dual $T(X_t)^*$ of the tangent bundle $T(X_t) \xrightarrow{\mu} X_t$ corresponds to the canonical k_{-t} on X_t. As $(2g-2) > 0$ this bundle is ample and it follows that $\exists$ a should section s of $T(X_t)^*$ such that the divisor class of (s) is k_t on X_t (class are denoted by the same symbol as the divisors) on the bundle $L = T(X_t)$ one may then define a homogeneous linear map by.

$$\tilde{s}(y) = \langle s(\mu(y)) \cdot y \rangle, y \in L.$$

Completing L to a projective line bundle $\hat{L}$ and extending $\tilde{s}$ to $\hat{L}$, we 85
complete immediately the divisor of $\tilde{s}$ or $\hat{L}$:

$$(\tilde{s}) = L^\circ - L^\infty + \mu^{-1}(\underline{k}_t)$$

where L°, L^∞ are the null and infinite sections of $\hat{L}$ and $\mu : \hat{L} \to X_t$ the canonical projection.

Take now any rational section σ of $\hat{F}_t$ over X_t, as X_t is nonsingular and $\hat{F}_t$ complete σ is in fact a section. If $x \in F_t$, $x - \sigma(\theta(x))$ is in the affine space L for almost all x and the rational function $x \longmapsto \tilde{s}(x - \sigma(\theta(x))$ on F_t extends to a rational function α on $\hat{F}_t$:

$$\alpha(x) = \tilde{s}(x - \sigma(\theta(x)), x \in F_t.$$

The divisor class of (α) is then

$$(\alpha) = M - F_t^\infty + \theta^{-1}(\underline{k}_t) - \theta^{-1}(M.F_t^\infty)$$

where $M = Im.\sigma, \theta : \hat{F}_t \longrightarrow X_t$ the canonical projection, with an identification $F_t^\infty \underset{\to}{\backsim} X_t$. Under this identification the self-intersection of the divisor $D = F_t^\infty$ on the ruled - surface $\hat{F}_t$ corresponds to the canonical class $\underline{k}_t$ on X_t; in fact, this class is given by

$$F_t^\infty.(F_t^\infty + (\alpha)) = M.F_t^\infty + \underline{k}_t - MF_t^\infty = \underline{k}_t$$

Consider the divisor $nD = nF_t^\infty$ on $\hat{F}_t$ for large values of n;the corresponding line bundle L_{nD} on $\hat{F}_t$ induces, as is seen above, the line bundle 86

$L_n\underline{k}_t$ on F_t^∞. On $\hat{F}_t$ we have the exact sequence of coherent sheaves

$$0 \longrightarrow \mathscr{L}((n-1)D) \longrightarrow \mathscr{L}(nD) \longrightarrow \mathscr{L}(n\underline{k}_t) \longrightarrow 0$$

which gives a cohomology exact sequence:

$$0 \longrightarrow H^\circ(\hat{F}_t, (n-1)D) \longrightarrow H^\circ(\hat{F}_t, nD) \xrightarrow{\gamma} H^\circ(F_t^\infty, \underline{k}_t \longrightarrow$$
$$\xrightarrow{\beta} H'(\hat{F}_t, (n-1)D) \xrightarrow{\alpha} H'(\hat{F}_t, nD) \longrightarrow H'(F_t^\infty, (1-n)\underline{k}_t).$$

On F_t^∞, one has $H^1(F_t^\infty, n\underline{k}_t) \simeq H^\circ(F_t^\infty, (1-n)\underline{k}_t$ (cf. Serre [7], ch II) and if n is large enough $H^\circ(F_t^\infty, (1-n)\underline{k}_t) = 0$ so that α is surjective. Now the $H'(\hat{F}_t, nD)$ are finite dimensional vector spaces over k and from the surjectivity of α_λ it follows that $h^1(nD) = \dim_k H^1(\hat{F}_t, nD)$ is a decreasing positive integral valued function of n and has therefore to remain constant for n large. It follows that α is an isomorphism for n large and therefore that γ is surjective. Fix such an n.

The surjectivity of γ means that the liner system $| nD | = \{(f) + nD : (f) \geq -nD\}$ induces on $D = F_t^\infty$ the complete linear system $| n\underset{-t}{k} |$; since $|n\underset{-t}{k}|$ has no base point on F_t^∞ and as nD as a member of $|nD|$, it follows that $|nD|$ has no base point. This means that the rational map φ_{nD} of $\hat{F}_t$ into a projective space, defined by a basis for $L(nD)$ is a *morphism* (Chapter I). On the other hand $n\underline{k}_t$ is very ample for n large (Chapter
87 II)and the surjectivity of γ means that $\varphi_{n\underline{k}_t}$ is induced on F_t^∞ by φ_{nD}. Let $1 = u_\circ, u_1, \ldots, u_r$ be a basis of $L(nD)$. We set $g_t = \varphi_{nD}$; we have thus proved a major part of

Lemma 1. (i) *$g_t(F_t^\infty) \subset H$, the hyperplane at ∞ in $\mathbb{P}_r$, for the affine coordinates $(u_1(x), \ldots, u_r(x))$; $g_t \mid F_t^\infty$ is an imbedding.*

(ii) *$g_t(F_t) \subset \mathbb{P}_r - H$.*

Proof. If u_1 is the function with a pole of maximum order along F_t^∞ among the u_i, it is clear that, for

$$x \in F_t^\infty, g_t(x) = \left(\frac{1}{u_1(x)}, \ldots, \frac{u_r(x)}{u_1(x)}\right) \in H.$$

Also, by choice, $g_t \mid F_t^\infty$ is a multicanonical imbedding. This proves (*i*) To prove (*ii*) it is enough to observe that the polar varieties of (u_i) lie in F_t^∞. Q.E.D □

Fix a generic point $t_\circ$ of R/K. The varieties $F_{t_\circ}^\infty, \hat{F}_{t_\circ}, F_{t_\circ}$ are all defined over $k(t_\circ) = k(R) = K$ and $D = F_t^\infty$ is a K-rational divisor on $F_{t_\circ}$. Thus, by the "last theorem of Weils Foundations" the u_i can be assumed to be functions in $k(t_\circ)(\hat{F}_{t_\circ}) = k(\hat{F})$ and defined over $K(t_\circ) = K$. As the homogeneous system (u_i) of functions on $\hat{F}$ does not have common zeros on the generic fibre $\hat{F}_{t_\circ}$, by restricting t to an open subset of R, we may assume that the system (u_i) does not have common zeros on $\hat{F}$. Then we and define a homomorphism $\hat{F} \to \mathbb{P}_r$ by $x \longmapsto (1 = u_\circ(x), \ldots, u_r(x))$.

The following two lemmas are then easy deductions from lemma 1.

Lemma 2. (i) *$g(F^\infty) \subset H$, the hyperplane at ∞ in $\mathbb{P}_r$ for the affine coordinates $(u_1(x), \ldots, u_r(x))$; also, g restricted to each F_t^∞ is an imbedding.* 88

(ii) *$g(F) \subset \mathbb{P}_r - H$.*

Thus, if we define a morphism $\hat{\varphi} = \pi\theta \times g : \hat{F} \to R \times \mathbb{P}_r$, then:

Lemma 3. (i) *$\hat{\varphi} \mid F^\infty$ is biregular into $R \times H$*

(ii) *$\hat{\varphi}(F) \subset R \times (\mathbb{P}_r - H)$.*

(B) The contraction set of the morphism $\hat{\varphi}$.

Definition. *Let $Y \xrightarrow{g} z$ be a dominant morphism of varieties; the* contraction *set $E(g)$ of g is, by definition, the set*

$$E(g) = \left\{ y \in Y : \dim_y(g^{-1}g(y)) > 0 \right\}.$$

Our aim in this section will be to study the contraction set $E(\hat{\varphi})$ of the morphism $\hat{(\varphi)}$ we have constructed above.

(i) *$E(\hat{\varphi})$ is a closed subset of $\hat{F}$.*

Infact, we will prove that the contraction set $E(g)$ of an arbitrary dominant morphism $Y \xrightarrow{g} Z$ is closed; by an obvious reduction one may assume successively that Y is normal, Z is normal, and then by replacing Z by its normalisation in $k(Y)$ that g is birational (note that the case $\dim Z = \dim Y$ is the only non-trivial one). But then, by ZMT g is a local isomorphism an all points $y \notin E(g)$. Our assertion follows.

89 (ii) If $\varphi = \hat{\varphi} \mid F$, then $E(\varphi) = E(\hat{\varphi})$.

This follows from (i) of Lemma 3, (A). In particular. $E = E(\varphi) \subset F$.

(iii) $E_t = E \mid F_t = E \cap F_t = E(\varphi \mid F_t)$.

Follows from the fact that φ separates fibres.

(iv) E_t is complete.

Follows from (iii) and (i).

(v) *If $E_t \neq \phi$ then $\theta \mid E_t : E_t \longrightarrow X_t$ is a bijection.*

Let $\tilde{E}_t = \left\{ e - e' \mid e, e' \in E_t \text{ such that } \theta(e) = \theta(e') \right\}$.

Then $\tilde{E}_t \subset T(X_t); \tilde{E}_t$ is one-dimensional, since E_t is one-dimensional: Furthermore the map $E_t \underset{\times_t}{\times} E_t \longrightarrow T(X_t)$ given by $(e, e') \longmapsto (e - e')$ has E_t for its image which is therefore complete.

Now, as the cotangent bundle $T(X_t)^*$ on X_t is ample $((2g - 2) > 0)$, it follows, by Grauert's criterion of amplitude (Chapter I, §2), that $\exists$ a morphism $\tau : T(X_t) \to U, U$ affine, such that τ contracts the null section of $T(X_t)$ to a point and is biregular outside it; as $\tilde{E}_t \subset T(X_t)$ is complete, its image under τ, which is affine, reduces to a finite subset of U. From the one-dimensionality of $\tilde{E}_t$ and the biregularity of τ outside the null section of $T(X_t)$, one deduces that $\tilde{E}_t$ is contained in the null section of $T(X_t)$, in other words, that $e = e'$ if $\theta(e) = \theta(e'), e, e' \in E$.

90 Consider the morphism $\varphi = \hat{\varphi} \mid F$; we have seen that $\varphi(F) \subset R \times$

$(\mathbb{P}_r - H)$ which is affine; replacing the affine closure $\overline{\varphi(F)}$ in $R \times (\mathbb{P}_r - H)$ by its normalisation in $k(F)$ we may assume that φ is a morphism of F into an affine space A, which is *birational* onto $\varphi(F)$, and therefore (by ZMT) *biregular outside the contraction set* E.

(C) Finiteness of sections of X/R not tangent to E.

Let $s : R \to X$ be any section of X/R. For $t \in R$, the tangent to $s(R)$ at $s(t)$ is then well defined and not "vertical" and is thus in F_t; one can therefore define a section $\tilde{s} : R \to F$ so that $s = \theta.\tilde{s}$. To say that "s is tangent to E'' means precisely that $\tilde{s}(R) \subset E$. Denote by $\sum$ the set of all sections s of X/R and by $\sum'$ the subset of all s such that $\tilde{s}(R) \not\subset E$.

Proposition 1. $\sum'$ is a finite subset of $\sum$.

Proof. For any section s of $X|R$ the composite $\varphi \circ \tilde{s}$ belongs to Mor $(R, A) = V$. ◻

a) The map $s \longmapsto \varphi o \tilde{s}$ is a map of $\sum$ into V and is *injective* on $\sum'$.

The first assertion is trivial; to prove the second, take $s, s' \in \sum$, $s \neq s'$, such that $\tilde{s}(R) \not\subset E$ and $\tilde{s}'(R) \not\subset E$; then $\exists . t \in R$ such that $\tilde{s}(t) \neq \tilde{s}'(t)$ and $\tilde{s}(t), \tilde{s}'(t) \not\subset E$. By the biregularity of φ outside E, one gets $\varphi o \tilde{s}(t) \neq \varphi \circ \tilde{s}'(t)$.

b) The elements $\varphi o \tilde{s}$, $s \in \sum$, belong to a *finite dimensional* vector subspace V_1 of V.

For a large q, one has F imbedded in $R \times \mathbb{P}_q$; let $\overline{R}$ be a nonsingular 91
projective closure of R and $\overline{F}$ be the closure of F in $\overline{R} \times \mathbb{P}_q$. Any section $s : \tilde{R} \longrightarrow$ extends to a rational section $\overline{s} : \overline{R} \longrightarrow \overline{F}$ in a natural way. Also, because $\overline{R}$ is non singular and $\overline{F}$ complete, $\overline{s}$ is a section; $\overline{A}$ is a projective closure of A, the morphism $\varphi : F \to A$ extends to a rational map $\overline{\varphi} : \overline{F} \to \overline{A}$ and the composite $\overline{\varphi} o \overline{s} : \overline{R} \longrightarrow \overline{A}$ is a morphism.

Let $(\varphi_1, \ldots, \varphi_d)$ be the coordinate functions of φ; each φ_i is finite on F but may have poles on $\overline{F} - F \subset (\overline{R} - R) \times \mathbb{P}_q$. Chose a rational function u on $\overline{R}$ such that each $u\varphi_i$ is finite on $(\overline{R} - R) \times \mathbb{P}_q$; then $u\overline{\varphi}_i$ is finite on $(\overline{R} - R) \times \mathbb{P}_q$. Therefore the composite $u(\overline{\varphi} o \overline{s}) : \overline{R} \longrightarrow$

$\overline{A}$ is finite on $\overline{R} - R$; if $(t_1, \ldots, t_l)$ are the coordinate functions of $\varphi \circ \tilde{s}$, it follows that the t_j which are rational functions on $\overline{R}$ have the property:

$$(t_i) \geq -h \text{ on } \overline{R} - R$$

where h is the polar divisor of u on $\overline{R} - R$; in other words, we have $t_i \in L(h)$ on $\overline{R}$ and this is a finite dimensional vector space. Our assertion follows.

c) The elements $\varphi o \tilde{s}$, $s \in \sum'$, form a closed subset of the linear variety V_1.

In fact, if $v \in V_1 (\subset \text{ Mor } (R, A))$ is of the form $\varphi \circ \tilde{s}$, $s \in \sum$, then it has the following properties:

92 (1) $v(R) \subset \varphi(F)$ (More precisely $\forall t \in R, v(t) \in \varphi(F_t)$).

(2) if $v(R) \not\subset \varphi(E)$ then $\varphi^{-1}.v$ is defined and is a section $R \to F$; also $\theta.\varphi^{-1} \cdot v$ is a section $R \to X$; furthermore the "direction function" involved in $\varphi^{-1}v$ must be the "derivative" (i.e must give the tangent direction) of the "point function" $\theta \cdot \varphi^{-1} \cdot v$ with the above notation we can write as $\theta\widetilde{\varphi^{-1}}v = \varphi^{-1}v$.

Conversely, if $v \in V_1$ satisfies (1), we have either $v(t) = \varphi(E_t) \forall t$, so that v comes from a section tangent to E or (2) holds so that $v = \varphi o \underset{\sim}{s}$ with $s = \theta\varphi^{-1}v$. Now notice that (1) and (2) are algebraic conditions on $v \in V_1$ (taking into account the fact that there is at most one $v_o \in V_1$ such that $v_1(t) = \varphi(E_t) \forall t \in R$).

Therefore $Im \sum$, $Im \sum'$ are closed subs.of the linear variety V_1.

As the map $\sum \to V_1$ is injective on $\sum'$ (by a)) the algebraic structure on $Im \sum'(c))$ can be pulled to $\sum'$. To prove Proposition 1, it is then enough to prove that:

d) $\dim . \sum' = 0$

We shall show that any morphism $Q \xrightarrow{\psi} \sum'$ of on irreducible curve Q into $\sum'$ is constant.

We first remark that Q may be assumed to be complete. Indeed for every fixed t ∈ R the morphism

$$Q \to X_t$$
$$q \longmapsto \psi(q)(t)$$

admits as extension a morphism $\bar{Q} \xrightarrow{\psi} X_t$ ($\bar{Q}$ a projective closure of Q) so that $\psi_q = \psi(q)$ is a section in $\sum'$, for all $q \in \bar{Q}$.

Now consider, for a fixed $t \in R$. the morphism 93

$$\bar{Q} \longrightarrow A$$
$$q \longmapsto \varphi \circ \tilde{\psi}_q(t).$$

The image, being affine, is point ; this means that $\varphi o \tilde{\psi}_q(t) = \varphi o \tilde{\psi}_{q'}(t)$ for all $q, q' \in \bar{Q}$. This is true for all $t \in R$ and as φ is biregular outside E while $\tilde{\psi}_q(t) \notin E$ for almost all t it follows that $\tilde{\psi}_q(t) = \tilde{\psi}_q(t)$ for almost all $t \in R$. One then concludes that $\psi_q = \psi_{q'}$. Q.E.D.

(D) Case of an infinity of sections.

Proposition 2. *If $X \xrightarrow{\pi} R$ admits an infinity of sections then*

a) E is an irreducible surface on F

b) $\theta|E : E \to X$ is biregular.

In view of Proposition 1, it follows that there exists at least one section $s : R \to X$ which is tangent to E i.e., $s(t) \in E_t$ for all t; this means that $E_t \neq \phi$ for any $t \in R$. Assertion a) is proved.

Regarding b), as we have already seen that $\theta \mid E_\alpha$ bijective we will be through, by ZMT, if we prove the birationality of $\theta|E$.

Characteristic 0 offers no trouble. *We assume therefore that charac $k = p$, $p \neq 0$, to prove the birationality of $\theta|E$.*

We have a chain 94

$$k \subset k(R) \subset k(X).$$

Let $k(X) = k(x_1, x_2, x_3)$ and assume that x_1, x_2 form a separating bases of $k(X)|k$. We consider the minimal equation

1). $G(x_1, x_2, x_3) = 0$ with an irreducible $G \in k[X_1, X_2, X_3]$ with $G'_{x_3}(x_1, x_2, x_3) \neq 0$.

A tangent direction at $(x) \in X$ is defined by homogeneous coordinates (y_1, y_2, y_3) satisfying

2). $\sum\limits_{i=1}^{3} y_i G'_{x_i}(x) = 0.$

As $G'_{x_3}(x) \neq 0$ this direction is defined completely by $y = y_2/y_1$.

Now since $\theta|E$ is bijective, $\theta|E$ is purely inseparable so that

3). the direction E is defined by an equation

$$y^{p^n} = H(x_1, x_2, x_3);$$

we assume that H is *not* a p^{th} power in $k(X)$.

Any section s of $X \to R$ defined by $s_1, s_2, s_3 \in k(R)$ such that $G(s_1, s_2, s_3) = 0$. Then the tangent to $s(R)$ the parameters Ds_1, Ds_2, Ds_3 (where D is any non-trivial derivation of $k(R)/k$). To say that $\tilde{s}(R) \subset E$ is then equivalent to

4). $\left(\dfrac{Ds_2}{Ds_1}\right)^{p^n} = H(s_1, s_2, s_3).$

95 If $p^n = 1$, then trivially $\theta \mid E$ will be birational; assume then that $p^n \neq 1$.

Now deriving 4 we obtain

5). $$0 = \sum_{i=1}^{3} H'_{x_i}(s) Ds_i.$$

Consider now the locus Y of the tangent direction $\left((x_1, x_2, x_3), \dfrac{G'_{x_1} H'_{x_3} - H'_{x_1} G'_{x_3}}{G'_{x_2} H'_{x_3} + H'_{x_2} G'_{x_3}}\right)$ at (x_1, x_2, x_3),over k. Then y is an irreducible variety whose elements satisfy

6). $$\sum_{i=1}^{3} H'_{x_i}(x) y_i = 0.$$

We claim that Y is not the whole of F; in fact, if it were, for any k-derivation Δ of $K(X) = k(x_1, x_2, x_3)$, the tangent direction $(\Delta x_1, \Delta x_2, \Delta x_3)$ will be in Y so that $\sum H'_{x_i}(x)\Delta x_i = \Delta(H(x)) = 0$. This means that H is a p^{th} power. Thus Y is a surface.

By the very definition of $Y, \theta \mid Y$ is birational from Y onto X. from 5), it follows that for any $s \notin \sum'$, $\tilde{s}(R) \subset Y$. By hypothesis, there are an infinity of $s \notin \sum'$ (Proposition 1); thus, the two irreducible surfaces E and Y have an infinity of common curves, and therefore must coincide: $E = Y$. In particular, it follows that $\theta \mid E$ is birational and $p^n = 1$ in 3). Q.E.D

(E) Conclusion in charac. $p \neq 0$.

Proposition 3. *With the same notation as before, let C be a curve defined over $K = k(R)$ such that* 96

$$\text{genus}\ {}_K{}^C = \text{ absolute genus } C \geq 2.$$

If C_K is infinite, then C is birationally equivalent, over K, to a curve C_1 defined over K^P.

Proof. Proceeding as in (D), we may first assume that $x_1 \in k(R)$. Let D be the nontrivial derivation of $k(R)/k$ such that $Dx_1 = 1$. We extend this to a derivation D of $k(X)$ such that $Dx_2 = H(x)$. (see 3) of the proof of proposition 2, (D). Then one has

$$D^P k(R) = 0 \qquad \text{and}$$
$$G'_{x_1} + G'_{x_2} H + G'_{x_3} D x_3 = 0.$$

□

Take any section $s_\alpha = (s^\alpha_1, s^\alpha_2, s^\alpha_3)$ of X/R; then $s^\alpha_1 = x_1$ and

$$G(s^\alpha_1, s^\alpha_2, s^\alpha_3) = 0. \text{ Also },$$
$$\tilde{s}\alpha(R) \subset E \text{ means } Ds^\alpha_2 = H(s^\alpha_1, s^\alpha_2, s^\alpha_3).$$

Now $s^\alpha(R)$ is a curve on X thus defines a discrete valuation ring $\mathscr{O}_\alpha$ in $k(X)$; $s^\alpha(R)$ is k(R)-rational means that, for the canonical homomorphism

$$\sigma_\alpha : \mathscr{O}_\alpha \to k(R) = K$$

of $\mathscr{O}_\alpha$ onto its residue field, we have **97**

$$\sigma_\alpha(x_i) = s_i^\alpha.$$

The equations that we have obtained above mean that $\sigma_\alpha D = D\sigma_\alpha$ or, more precisely, that if $D' = D \mid k(R)$ then

$$\sigma_\alpha D = D'\sigma_\alpha.$$

Iteration gives: $\sigma_\alpha D^p = D'^p\sigma_\alpha = 0$. As this is true for an infinity of α' s we conclude that $D^p = 0$.(if $y \in k(X), \sigma_\alpha(D^p y) = 0$ means that $D^p y$, which is a rational function, has $s^\alpha(R)$ as part of its polar variety; recall that a rational function cannot have an infinite number of polar varieties).

Consider now the diagram of field extensions:

$$\begin{array}{ccccc} L^p & \text{———} & \operatorname{Ker} D & \text{———} & k(X) = L \\ & \diagdown & \Big| & & \Big| \\ k & \text{———} & K^p & \overset{p}{\text{———}} & K = k(R). \end{array}$$

As K and L are function fields of one and two variables respectively over the algebraically closed k, it follows that $[K : K^P] = P$ and $[L : L^p] = p^2$. We also have $\dim_{L_p}(\ker D^p) \leq p.\dim_{L_p}(\ker D)$. (This can be proved easily as follows: if $u, v \in \operatorname{End} V$, V a vector space, then the sequence

$$0 \longrightarrow \operatorname{Ker} v \longrightarrow \operatorname{Ker} uv \xrightarrow{v} \operatorname{Ker} u$$

is exact so that $\quad \dim(\ker\ uv) \leq \dim(ker\ u) + \dim(\ker\ v)$.

98 Recalling that $\ker D^p = L$, one gets

$$p^2 \leq p.\dim_{L^p}(\ker\ D)$$

i.e., $[\ker D : L^p] \leq p$; as $\ker D \neq L$, it follows that $[\ker D : L^p] = [L : \ker D] = p$. Consequently, $K(= k(R))$ and $\ker D$ are K^P-linearly disjoint; thus if C_1 is a (nonsingular) curve over K^P such that $K^P(C_1) = \ker D$, then $K(C_1) = K(\ker D) = L = k(X) = K(C)$. Q.E.D.

Corollary 1. $(C_1)_{k^p}$ is infinite. *With notations as in the proof of the above proposition, let* $\mathscr{O}'_\alpha = \mathscr{O}_\alpha \cap K^p(C_1)$; *then the* $\mathscr{O}'_\alpha$ *are discrete valuation rings in* $K^p(C_1)$ *and are infinitely many in number (since* $L/K^p(C_1)$) *is purely inseparable,* $\mathscr{O}_\alpha \neq \mathscr{O}_{\alpha'}, \Rightarrow \mathscr{O}'_{\alpha'} \neq \mathscr{O}'_{\alpha'}$*.To prove that the* $\mathscr{O}'_\alpha$ *define* K^p*-rational sections of* C_1*, one has to prove that* $\sigma_\alpha(\mathscr{O}'_\alpha) \subset K^p$*.*

In fact, if $\sigma_\alpha(y) = \bar{y}, y \in K^p(C_1) = \ker D$, then $D\bar{y} = D\sigma_\alpha y = \sigma_\alpha D y = 0$, whence $\bar{y} \in K^p$.

Corollary 2.

$$\begin{aligned} \text{genus}_{K^p} C^1 &= \text{absolute genus of } C_1 \\ &= \text{absolute genus of } C. \end{aligned}$$

We shall prove the corollary by showing that $\exists$ a projective model D_1 of C_1 over K^p, such that D_1 is *absolutely normal*. (We recall here that the genus drop of a curve for extension of base fields comes from nonabsolutely normal points on it.)

By corollary 1, on C_1 there exist an infinity of K^p-rational points; **99**
we form a divisor O_1 on C_1, with such points, and with such a large degree that O_1 is very ample; in view of proposition 3, this induces a very ample divisor O on C. By the last theorem of Weil's Foundations, one obtains an isomorphism over K

$$L_K(O, C) \xleftarrow{\sim} \underset{K^p}{L}(O, C_1) \underset{K^p}{\otimes} K.$$

Thus, we may assume that there exist rational functions in $K(C_1) = K(C)$, which give a projective imbeddings $C \xrightarrow{\eta} C'$, $C_1 \xrightarrow{\eta} C'_1$ in the same $\mathbb{P}_n$. We may assume in addition that $\exists$ points $P_{1,0,}P_{1,1,\ldots,}P_{1,n,}P_{1,n+1}$ in C_1, corresponding to some of the above defined valuation rings O_α such that

$$\eta_1(P_{1,i}) = (0, 0 \ldots, \overset{i}{1}, \ldots, 0) \in \mathbb{P}_n, 0 \leq i \leq n,$$

and $\eta_1(P_{1,n+1}) = (1, 1, \ldots, 1) \in \mathbb{P}_n$. After a projective transformation, we may also assume that the corresponding points $P_{0,\ldots,}P_{n+1} \in C$ are such that

$$\eta(P_i) = (0, 0, \ldots \overset{i}{1}, \ldots, 0) \in P_n, 0 \leq i \leq n,$$

$$\eta(P_{n+1}) = (1, 1, \ldots, 1) \in P_n.$$

This follows from: $D\sigma_\alpha = \sigma_\alpha D$. But as the national functions defining η and η_1 generate the same vector space there is a projective transformation u of $\mathbb{P}_n$ such that.

$$\begin{array}{ccc} C & \xrightarrow{\eta} & C' \\ \text{birational}\big\downarrow & & \big\downarrow u/C' \\ C_1 & \xrightarrow{\eta_1} & C'_1 \end{array}$$

100 is commutative. But as u fixes the base points $(0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0)$ and the unit point $(1, 1, \ldots, 1)$ in $\mathbb{P}_n, u$ has be the identity in $\mathbb{P}_n$; so that $C' = C'_1$. By hypothesis C is absolutely normal and thus $C'_1 = C'$ is the absolutely normal model for C_1 that we were looking for. Q.E.D.

This done, one may now suppose that curve C_1 again satisfies the conditions in the enunciation of proposition 3 (Corollaries 1, 2). Thus, one obtains, by iteration, a sequence $(C_i)_{i \geq 0,} C_\circ = C$, of absolutely normal curves defined respectively over fields K_i/k and such that $C_i \underset{K_i}{\sim} C_{i+1}$. The proof of the Grauert-Manin theorem will thus be completed, in charac. $p \neq 0$, by the following two lemmas.

Lemma 4. *If K is an algebraic function field over an algebraically closed K, charac. $k = P \neq 0$, then $\bigcap\limits_{n \geq 0} K^{p^n} = k$.*

Proof. $F = \bigcap\limits_{n \geq 0} K^{p^n}$ is clearly perfect; secondly, as F is a subfield of a finite type extension of k, F is also of finite type over k. We claim that F is purely algebraic over k; in fact, for any finite type extension $L/k, [L : L^p]$ equals p^d means $d = t\,\mathrm{tr}\deg_k^L$ as $F = F^P$ our lemma follows. □

101 **Lemma 5.** *Let $(C_n)_{n \geq 0}$ be a sequence of also normal curves, each C_n*

defined over a K_n, such that $k = \bigcap_{n \geq 0} K_n$ is algebraically closed. Assume that, for each n, $genus_{K_n} C_n$(=absolute genus of C_n) ≥ 2 and that $C_n \sim C_{n+1}$ over some field. Then each C_n is birationally equivalent to a curve D defined over K.

Proof. Let $g = genus\ C_n (\forall n)$. As $g \geq 2$ by hypothesis, each C_n is tricanonically imbedded in $\mathbb{P}_{5g-6}$, as a positive one dimensional cycle of degree $(6g-6)$. Let $G = \mathbb{P}GL(5g-6)$. Then G acts, in a natural way, on the Chow variety Z of positive one dimensional cycles of degree $(6g-6)$ in $\mathbb{P}_{5g-6}$ (we recall that this variety is defined over a "small" subfield of k and in particular over k itself.) This action is given by a morphism $G \times Z \longrightarrow Z$. As C_n and C_{n+1} are birationally equivalent, the Chow points x_n of C_n all lie in the same orbit $V = Gx_n$ for this action of G. □

Let $\bar{V}$ be the closure of this orbit V in Z. As x_n is K_n-rational, $\bar{V}$ is defined over K_n; thus the smallest field of definition for $\bar{V}$ is contained in K_n for each n and hence in k.

By the Hilbert-Zero -Theorem it follows that $\exists$ an $x \in V$, rational over k; then x is the Chow point of a curve D in $\mathbb{P}_{5g-6}$ defined over k; $x \in V$ means that $C_n \sim D$. Q.E.D.

Remark. We have made same construction above as Mumford has done in construction the moduli variety for curves. But when as Mumford naturally had to consider the entire orbit space we had to deal only with a single orbit; hence our result is quite elementary.

(F) Conclusion in charac. 0. **102**

Proposition 4. *Let C be e curve defined over function field of one variable K/k of genus ≥ 2. If C_K is infinite, then $C \sim D$. a curve defined over k.*

Proof. With the same notations and procedure as above, we obtain a tower $k \subset k(R) = K \subset k(X) = L$. A contrivial derivation D' of K/k is extended to D on L; corresponding to an infinity of sections of $X \mid R$, we obtain, as before, valuation rings $\mathscr{O}_\alpha$ of L/K such that the residue field of each $\mathscr{O}_\alpha$ is K and such that if $\sigma_\alpha : \mathscr{O}_\alpha \longrightarrow K$ is the canonical homomorphism, then $\sigma_\alpha D = D'\sigma_\alpha$. □

Now observe that for any valuation ring $\mathscr{O}$ of $L/K, D\mathscr{O} \subset \mathscr{O}$ (geometrically if f is a function on X, which is finite along the divisor defined by $\mathscr{O}$, then Df is also finite along that divisor; this is due to the fact that the direction field E is a morphism $X \longrightarrow F$, not merely a rational map). Consider now that formal power series fields $K((T)) \subset L((T))$. We define an automorphism S of $L((T))($ that restricts to an automorphism of $K((T))$ as follows:

$$S(T) = T$$
$$S(y) = y + TDy + \cdots + T^n \frac{D^n y}{n!} + \cdots, y \in L.$$

(One can divide by $n!$ in charac. 0 !) The Leibnitz' formula readily shows that $S(yy') = S(y)S(y')\forall y, y' \in L$. Also, as $\sigma_\alpha D = D'\sigma_\alpha$ one obtains

$$\sigma_\alpha S = S\sigma_\alpha.$$

103 Now consider the curve C as a over $K((T))$; its function field is then $K((T))(x, y)$ where $L = K(x, y)$ is the function field of C/K. The automorphism C of $K((T))$ defines a curve C^S, the S-conjugate of C (replace the coefficients in the defining equations of $C/K((T))$ by their S-conjugates). Then

$$\begin{aligned}\text{genus}_{K((T))^C} &= \text{genus (absolute) of C}\\ &= \text{absolute genus of } C^S\\ &= \text{genus}_{K((T))} C^S \geq 2.\end{aligned}$$

Also the $K((T))$-function field of C^S is $K((T))(Sx, Sy)$; we claim that C and C^S are birationally equivalent over $K((T))$. In fact, in view of the above relation on their genera and in view of Hurwitz-Zenthen (Chapter II) it is enough to prove that

$$K((T))(Sx, Sy) \subset K((T))(x, y).$$

To start with we know already that $D\mathscr{O}_\alpha \subset \mathscr{O}_\alpha$ for all α; in addition if μ_α is the maximal ideal of $\mathscr{O}_\alpha$ then for $x' \in \mu_\alpha$ one has $\sigma_\alpha(Dx') =$

$D(\sigma_\alpha x') = 0$, i.e. $Dx' \in \mu_\alpha$ so that $D\mu_\alpha \subset \mu_\alpha$. Also we claim that the valuation v_α on L/K defined by $\mathscr{O}_\alpha$ has the property

$$v_\alpha(Dz) \geq v_\alpha(z) \quad \forall z \in L.$$

In fact, if t is a uniformiser for $\mathscr{O}_\alpha$ then $Dt = at$, $a \in \mathscr{O}_\alpha$ from above, and thus if $z = ur^n, u \in \vartheta_\alpha, n \in Z$,

$$\begin{aligned} DZ &= t^n Du + nut^{n-1} Dt \\ &= t^n (Dn + nua) \end{aligned}$$

whence $v_\alpha(Dz) \geq v_\alpha(Z)$,(since $(Du + nua) \in \mathscr{O}_\alpha$).

Now consider the divisor P_α on X defined of some $\mathscr{O}_\alpha$; for large 104
$q \in \mathbb{Z}, \mathcal{O}_\circ = qp_\alpha$ is a divisor on X such that $L(\mathcal{O}_\circ)$ contains an x transcendental over K. Then we may write $L = K(x, y)$ with y *integral* on $K[X]$ $\begin{pmatrix} \text{charac.} K = 0 \\ \text{Thus } L/K \text{ separable} \end{pmatrix}$ If we write $\mathcal{O} = \max(-v_\alpha(x), -v_\alpha(y)).P$, then $x, y \in L(\mathcal{O})$.

From the fact that $\vartheta_\alpha(Dz) \geq v_\alpha(z)$ it follows that $Dx, Dy \in L(\mathcal{O})$ and by iteration $D^n x, D^n y \in L(\mathcal{O})$.

Therefore,

$$\begin{aligned} S(x) &= \sum_{n \geq 0} T^n \frac{D^n x}{n!} \\ &\in L(\mathcal{O})[[T]] \\ &= K[[T]][L] \\ &\subset K((T))(x, y). \end{aligned}$$

Similarly for $S(y)$. Our assertion is proved.

Now we take, as before, projective imbeddings of C and C^S over $K((T))$; we fix, as before, base and unit points $P_{\alpha_1}, \ldots, P_{\alpha_{n+2}} \in C$ corresponding to some of the valuation rings $\mathcal{O}_\alpha$ defined above; if $u : C \to C^S$ is the birational correspondence over $K((T))$ constructed above, then the image $u(P_\alpha)$ of P_α is P^s_α (defined by $\mathcal{O}^S_\alpha$)U follows easily from the equality $S\sigma_\alpha = \sigma_\alpha S$. One proves as be for that $C = C^S$.

We project C and C^S now into the place, from a centre of projection 105 (linear variety of dimension $(n-3)$) which is rational/k. If the images C_1 and C_1^S are defined by polynomial equations $\varphi(X, Y) = 0$ (coefficients a_{ij}) and $\varphi^S(X, Y) = 0$ (coefficients $S(a_{ij})$) then the equality of C and C^S (hence of C_1 and C_1^S) gives that $S(a_{ij})$ are proportional to a_{ij}; but since we may assume that some $a_{ij} = 1$ we obtain $S(a_{ij}) = a_{ij} \forall i, j$. This means the $a_{ij} \in \ker D = k$.

The projection onto the plane is a birational map (defined over k) from C onto C_1 ; as C_1 is defined over k the proposition is proved.

Q.E.D.

3. Definite Results

We still have to remove the extra hypothesis on K that we made in II namely, tr. $\deg_k K = 1$. We are now going to do it and later we shall see over what fields C and C' are isomorphic and then analyse C_K.

Proposition 1. *Let K be any field L a regular finite type extension of K. Let C be a curve defined over K such that $\text{genus}_K C$ = absolute genus of $C \geq 2$. Then either $C_L - C_K$ is finite or $C \sim D$ a curve defined over a finite field.*

Proof. Each $x_\alpha \in (C_L - C_K)$ satisfies $K(x_\alpha) \simeq K(C)$ so that

$$K \underbrace{\subset}_{\text{tr . deg .1}} K(x_\alpha) \subset K(X) = L.$$

106 By Severi $\exists$ only finitely many $K(x_\alpha)$ such that L is separable over $K(x_\alpha)$; and by Sobwarz-Klein, $\exists$ only finitely many x_α such that $K(x_\alpha)$ has a given value. *Therefore $C_L - C_K$ infinite rules out the possibility*: *charac*. $K = 0$. Let $p \neq 0$ be charac. K. For each x_α , let $q(\alpha)$ be the largest power of p such that $K(x_\alpha) \subset K(L^{q(\alpha)})$. ◻

Then

$$K\left(x_\alpha^{\frac{1}{q(\alpha)}}\right) \subset L$$

$$\not\subset K(L^p)$$

and $L/K\left(x_\alpha^{\frac{1}{q(\alpha)}}\right)$ is separable. By Severi, therefore, the number of $K\left(x_\alpha^{\frac{1}{q(\alpha)}}\right)$ is finite; moreover, we have

$$K\left(x_\alpha^{\frac{1}{q(\alpha)}}\right) \underset{sep.}{\subset} L \Longleftrightarrow K(x_\alpha) \underset{sep.}{\subset} K(L^{q(\alpha)})$$

so that for a given $q(\alpha)$, the number of x_β such that $K(x_\beta) \underset{sep.}{\subset} K(L^{q(\alpha)})$ is finite (Severi). If follows that $\exists q(\alpha), q(\beta), q(\alpha) \neq q(\beta)$ such that $K\left(x_\alpha^{\frac{1}{q(\alpha)}}\right)$ $= K\left(x_\alpha^{\frac{1}{q(\alpha)}}\right)$. If we write say $q(\alpha) = p^n.q(\beta), n \geq 1$, then it follows that

$$K(x_\alpha) = K\left(x_\beta^{p^n}\right) = K\left(x_\beta^q\right)(\text{ say }).$$

(We remark that since L and K are linearly disjoint, to prove the above equalities, we could have assumed that K is algebraically closed). This means that the curve C^q conjugate to C under the isomorphism $x \mapsto x^q$ of K, is K-birationally equivalent to C.

Thus, to prove proposition 1, we shall now prove the **107**

Proposition 2. *Let C be a curve defined over K, charac* $K = p \neq 0$. *Let* $q = p^n, n \geq 1$. *Then* $C \sim C^q \Rightarrow C \sim D$ *a curve defined over a finite field.*

Proof. For C we choose an absolutely normal model, defined over $\overline{F}_p(z_1, \ldots, z_r)$ say (this is a finite type extension of $\overline{F}_p$); then C^q is absolutely normal, defined over $\overline{F}_p(z^q)$ and so on; we also know from our computations preceding the proposition that $C \sim C^q$ an so on. Thus, by lemma 5 II, it follows that $C \sim$ D′ defined over $\overline{F}_p$; obviously, we can consider D as defined over a finite field. □

Q.E.D.

Theorem 1. *Let k be an algebraically closed field, K/k any function field, and C be an algebraic curve defined over K such that K-genus of C = absolute genus C* ≥ 2. *If* C_K *is infinite then*

1) $C \sim C'$ defined over k

2) An isomorphism $C \to C'$ is defined over K if C is not birationally equivalent to any curve defined over a finite field

Proof. 1) We shall prove 1) by induction on the transcendence degree d of K/k. The case $d = 1$ has already been tackled (part II). If, at any stage of the inductive proof, we obtain $C \sim D, D$ defined over a finite field, we will be through; therefore we will rule out this possibility in the entire proof (so that we may Proposition 1 effectively).

108 Let $\operatorname{tr.deg}_k K = d$; choose a $K_1, k \subset K_1 \subset K$ such that $\operatorname{tr.deg}_{K_1} K = 1$. By part II, $C \sim C_1, C_1$ defined over $\overline{K}_1$; we may assume that C_1 is defined over a finite extension L_1 of K_1 ; the birational correspondence $C \sim C_1$ is then defined over a finite extension K' of $L_1(K)$; replacing L_1 by a finite extension L_2 of L_1, we may assume that K' is separable over L_1; again replacing L_1 by its algebraic closure in K' we may assume that L_1 is algebraically closed in K'. Thus K' is a regular extension of L_1 and by hypothesis $(C_1)_{K'}$ is infinite; by the remark is made at the beginning and by Proposition 1 we obtain: $(C_1)_{K'} - (C_1)_{K'}$ is finite so that $(C_1)_{K'}$ is infinite. The inductive assumption now completes the proof of 1).

2) Let $u : C \to C'$ be an isomorphism, defined over a finite extension K' of K. We have $C'_{K'}$ infinite; then by hypothesis and by proposition 1, $(C'_{K'} - C'_K) \subset (C'_{K'} - C'_K)$ is finite. □

This means that there are an infinity of $x_i \in C_K$ such that $u(x_i) \in C'_K$. We now take tricanonical models D and D' in P_r of C and C' respectively (these imbeddings are defined over K, by hypothesis). We may choose on D base points and unit point $x_1 \ldots, x_{r+2}$ among the (x_i) all rational over K: and on D' choose the base points and unit point as the K-rational points $u(x_i)$. The isomorphism $u : C \to C'$ then defined a projective transformation $u : D \to D'$ which is necessarily defined over K as the base points an unit point are K-rational. Q.E.D.

109 **Remark.** If $v : C' \to C$ is an isomorphism defined over K, then v defines a bijection $C'_k \to C_K$; but $C'_K - C'_k$ is finite by Proposition 1, which means that almost all points in C_K are in $v(C'_k)$.

We had left out some "exceptional cases" in Theorem 1, 2. Our aim now is to study the situation in this "exceptional case". (In the following we shall use the term "isomorphism" for a "birational map").

Theorem 2. *Let k be algebraically closed, K any function field over k; C is a curve (as usual projective nonsingular) defined over K such that K-genus of C = absolute genus of $C \geq 2$ and C_K infinite. Assume that $C \sim C', C'$ defined over a finite field F_p such that all the members of Aut C' (by Schwarz-Klein these are all defined over F_p) are defined over F_q. Let f be the automorphism $x \mapsto x^q$ of $F_q(C')$ giving an automorphism $f : C' \mapsto C'$. Then*

(1) *$\exists$ a finite galois extension K' of K, a K'- isomorphism $u : C \to C'$ and a monomorphism $\sigma \mapsto h_\sigma$ of $G = G(K'/K)$ into Aut C' such that*
$$h_\sigma = u^\sigma \circ n^{-1}$$

(2) *$\exists$ a finite family (z_i) of transcendental points of $C'_{K'}$ such that $z_i^\sigma = h_\sigma(z_i) \forall \sigma \in G$. Also every $x \in C_K$ is either some $u^{-1}(f^n(z_i))$ or some $u^{-1}(z)$ with $z \in C'_K$ and*
$$z^\sigma = h_\sigma(z) \quad \forall \sigma \in G.$$

Proof. (1) Let $w : C \to C'$ be the birational correspondence given; we **110**
may assume that w is defined over a finite extension K'' of K. Let (x_i) be the infinite family of points of C_K ; for each i, one has $k(w(x_i)) \subset K''$ and by Severi almost all of the $k(w(x_i))$ are contained in K''^p by iteration of this procedure, we may assume that w is defined over a finite separable extension, whence also over a finite galois extension K''/K. If $\sigma \in G(K''/K)$ then $g_\sigma = w^\sigma.w^{-1} \in \text{Aut } C'$; we claim that $\sigma \mapsto g_\sigma$ is a homomorphism: in fact, that g_σ is a cocycle follows from
$$\begin{aligned} g_{\sigma\tau} = w^{\sigma\tau}.w^{-1} &= (w^\sigma)^\tau w^{-\tau} w^\tau w^{-1} \\ &= (g_\sigma)^\tau g_\tau \end{aligned}$$

and, as g_σ is defined over K, we are through. □

Now consider the kernel of this homomorphism $\sigma \mapsto g_\sigma$; it is the galois group of K'' over a galois extension K' of K. For $\sigma \in G(K''/K')$ then one has $w^\sigma = w$ and thus w is defined over K', denote it by u. For $\sigma \in G(K'/K)$ if we set $h_\sigma = u^\sigma.u^{-1}$ then $\sigma \mapsto h_\sigma$ is a monomorphism. Hence (1).

(2) Suppose $z \in C'$ is of the form $u(x), x \in C$. Then

$$x \in C_K \Longleftrightarrow \{z \in C'_{K'} \text{ and } z^\sigma = h_\sigma(z), \forall\sigma\}$$

In fact $\Rightarrow$ is trivial ; on the other hand observing that for every $\sigma \in G(K'/K)$

$$\begin{aligned} z^\sigma = h_\sigma(z) \Leftrightarrow z^\sigma &= u^\sigma u^{-1}(z) \\ &\Longleftrightarrow u^{-1}(z) = u^{-\sigma}(z^\sigma) \\ &\Longleftrightarrow u^{-1}(z) = (u^{-1}(z))^\sigma \end{aligned}$$

111 and thus $z \in C'_{K'}, h_\sigma(z) = z^\sigma, \forall\sigma$, imply that $x = u^{-1}(z) \in C_K$.

Now by an easy iteration of Severi we prove that there are only finitely many (transcendental) points $y_1, \ldots, y_r \in C'_{K'}$ such that $k(y_i) \not\subset K'^q$. If $z \in (C'_{K'} - C'_{k'})$ then it follows that for some n,

$$k \underset{\neq}{\subset} k(z^{q^{\frac{1}{n}}}) \subset K'$$
$$\not\subset K'^q$$

i. e. $z^{q^{\frac{1}{n}}} = y_i$ for some i

i. e. $z = f^n(y_i)$ for some i.

Now if $z \in C'_{K'} - C'_k$ and if $z^\sigma = h_\sigma(z)\forall\sigma$, then the y_j for which $z = f^n(y_j)$ has the property $y_j^\sigma = h_\sigma(y_j)\forall\sigma$: this follows from the equality $h_\sigma f^n = f^n h_\sigma$ (recall that h_σ is defined over F_q).

The proof of the theorem is complete. Q.E.D.

We shall end up by giving an example which will show that the part (2) of Theorem 2 cannot be strengthened.

Example. Let k be an algebraically closed field of characteristic $p \neq$
112 2. Let C' be the plane curve defined over k whose affine equation is

$x^4 + y^4 + 1 = 0; C'$ is nonsingular: indeed, the derivatives $4x^3, 4y^3, 4z^3$ of $x^4 + y^4 + z^4$ cannot all vanish at any point on $x^4 + y^4 + z^4 = 0$(in P_2), thus genus $C' = 3$, and the imbedding $C' \hookrightarrow P_2$ is canonical.

Let $K' = k(r, s), r$ transcendental over k and s such that at $r^4+s^4+1 = 0$. Let σ be the automorphism of K'/k such that $\sigma(r) = -r, \sigma(s) = -s$; then the fixed field K of σ is $k(r^2, s^2, rs)$, and K'/K is a galois extension of degree 2 whose galois group is $G = \{1, \sigma\}$.

Let C be the curve $X^4 + Y^4 + r^4 = 0$ defined over K.

a) *C_K is infinite.*

In fact, the infinity of points (r^{1+p^n}, rs^{p^n}) are K - rational (if $p > 2, p^n + 1$ is even; also $rs^{p^n} = rs.s^{p^n-1}$ and $p^n - 1$ is even)

b) $C \underset{K'}{\sim} C'$

In fact, the homothety $u : C' \to C$ given by $X = rx, Y = ry$ is a projective transformation in P_2, defined over K'.

The automorphism h_σ of C' is then the automorphism $(x, y) \to (-x, -y)$

c) *C is not K* - isomorphic to any curve defined over k(in particular $C \underset{K}{\not\sim} C'$). Indeed, if $C \underset{K}{\sim} D$, D defined over k then $D_{\tilde{K'}}C'$ from b); as D and C' are both defined over k, k algebraically closed, it follows that $D \underset{k}{\sim} C'$; thus $C \underset{K}{\sim} C'$, say through $w : C \to C'$ defined over K; then $wu \in$ Aut C' is defined over k so that $u = w^{-1}(wu)$ is defined over K. This is clearly false since $r \notin K$.

d) *However, C is K - isomorphic to C^{p^n} for all $n \geq 1$.* **113**

Write

$$X' = r^{p^n-1}.X$$
$$Y' = r^{p^n-1}.Y.$$

Then this is a projective transformation defined over $K(p^n - 1)$ is even) and transforms $C \equiv X^4 + Y^4 + r^4 = 0$ into

$$(X')^4 + (Y')^4 + (r^4)^{p^n} = 0$$

Which is the curve $(C')^{p^n}$.

e) *Rational points P' of $C'_{K'}$ and P of C_K.*

By theorem 2, part (2), the transcendental points of C'_K are obtained by applying the "iterated Frobenius maps" to points (x_1, x_2) such that $k(x_1, x_2)$ is of transcendence degree 1 over k and $k(x_1, x_2) \underset{\text{(separably)}}{\subset} k(r, s) = K'$. As these fields have the same genus 3, it follows by Hurwitz′ theorem that $k(x_1, x_2) = k(r, s) = K'$. Thus to find the K' - rational points on C' one has to find and k-automorphisms of C'. Some of them, for instance, are given by (see appendix 3 for a complete determination of Aut (C'))

i) $(r, s) \mapsto (r, -s)$

ii) $(r, s) \mapsto (-r, -s)$

iii) $(r, s) \mapsto (\alpha r, \beta s), \alpha^4 = \beta^4 = 1$.

iv) $(r, s) \mapsto \left(\dfrac{1}{s}, \dfrac{1}{r}\right)$.

As we have seen in the proof of theorem 2, the rational points P of
114 $C_K - C_k$ are given by rational points P' of $C'_{K'} - C'_K$ which have the property $P'^{\sigma} = h_\sigma(P'), \forall \sigma \in G$. For instance, from among the above four automorphisms it is clear that (i), (ii), (iii) satisfy this requirement but the fourth does not.

Finally, to get the k-rational points P on C, we take k-rational points P' on C' which have the property $h_\sigma(P') = P'$ These are the points at infinity of C', and give the points at infinity on C.

APPENDICES TO CHAPTER III

Appendix 1. For a purely aesthetic reason, we shall prove here a stronger form of Proposition 2, III of Chapter III. **115**

Proposition $\tilde{2}$. *Let C be a curve of absolute genus ≥ 2 in characteristic $p \neq 0$. Suppose C is birationally equivalent to C^q with $q = p^n, n > 0$. Then $C \sim D$ a curve defined over $\mathbb{F}_q$ (the finite field with q elements).*

Proof. In view of Proposition 2, III, we may now assume that C is defined over an $\mathbb{F}_{q^n}$; by choosing n large, we may also assume that the elements of Aut C, which are finitely many, are all defined over $\mathbb{F}_{q^n}$. We shall set $\mathbb{F}_q = F, \ldots, \mathbb{F}_{q^r} = F_r, \ldots$ and $G_r =$ the group of all F-automorphisms of $F_r(C)(r$ large). We now define a homomorphism

$$\begin{aligned} G_r &\longrightarrow G(F_r/F) \\ \sigma &\longmapsto \sigma|F_r. \end{aligned}$$

□

If r is large, (if $n \mid r$) the kernel of this homomorphism is the group of F_r -automorphisms of $F_r(C)$ i. e. is Aut C. We claim that this homomorphism is onto. In fact, if φ denotes the Frobenius automorphism $x \mapsto x^q$ of F_r, φ extends to a $\varphi : F_r(C) \longrightarrow F_r(C^q)$; on the other hand, the hypothesis $C \sim C^q$ (we may assume that this birational correspondence is defined over $\mathbb{F}_{q^n} = F_n$) gives an isomorphism $\omega : F_r(C^q) \to F_r(C)$ so that $\omega\varphi \in G$; if $n \mid r$, we have

$$\omega\varphi|F_r = \varphi|F_r = \varphi.$$

Thus, we get an exact sequence (for $n \mid r$) **116**

(1) $1 \to \text{Aut}\, C \to G_r \to G(^F r/F) \to 1.$

(2) We now assert that

$$\begin{gathered} C \underset{F_\gamma}{\sim} D \text{ defined over } F \\ (\Longleftrightarrow \text{ the above sequence ``splits''}) \\ \Longleftrightarrow \exists\ r\text{-cyclic subgroup } G'_r \text{ of } G_r \end{gathered}$$

such that $G'_r \cap \operatorname{Aut} c = (1)$.

i.e. $\Longleftrightarrow \exists$ a $\sigma \in G_r$

with $\sigma^\gamma = 1$ and $\sigma \mid F_r = \varphi$.

$\Rightarrow$: is quasi obvious D, whence for C.

$\Leftarrow$: Suppose $\exists$ such a $\sigma \in G_r$

Let L be the fixed field of σ. By galois theory $[F_r(C) : L] = r$ so that F_r and L are F-linearly disjoint. If $L = F(D)$, D a curve defined over F then $F_r(C) = F_r(D)$ Q.E.D.

$$\begin{array}{ccc} L & \overset{r}{\text{———}} & F_r(C) \\ | & & | \\ F & \underset{r}{\text{———}} & F_r \end{array}$$

Our aim therefore will obviously be to make (1) "split" for large multiples r of n. For large r, r' with $n \mid r$ and $n \mid r'$ and $r' \mid r$, we have an obvious commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Aut\ C & \longrightarrow & G_r & \longrightarrow & G(F_{r/F}) & \longrightarrow & 1 \\ & & \| & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & Aut\ C & \longrightarrow & G_{r'} & \longrightarrow & G(F_{r'/F}) & \longrightarrow & 1 \end{array}$$

117 so that we have an inverse system of exact sequences; as the so-called Mittag-Leffler condition is trivially verified, in the limit we get an exact sequence

$$1 \to \operatorname{Aut} C \to G \to G(\bar{F}/_F) \to 1.$$

The group $G(\bar{F}/F)$ is the limit of the inverse system $(\mathbb{Z}/_{r\mathbb{Z}})_r$ and is thus the " *universal pro-cyclic group* $\hat{\mathbb{Z}}$. (It is the completion of $\mathbb{Z}$ for the topology where a fundamental system of neighbourhoods of U is $(r\mathbb{Z})_{r \neq o}$). It is topologically one-generated (with the limit topology it is compact, Hausdorff and totally disconnected) viz , by the Frobenius automorphism φ of $\bar{F}/F$. We take a $\sigma \in G$ which maps onto $\varphi \in \hat{\mathbb{Z}}$. If G' is the closed topological subgroup of G generated by $\sigma \in G$, then

G' maps onto $\hat{\mathbb{Z}}$. But by a well-known property of $\hat{\mathbb{Z}}$ (namely that for compact, totally disconnected Hausdorff group H and any $h \in H, \exists$ a continuous homomorphism $f_h : \hat{\mathbb{Z}} \longrightarrow H$, such that $\varphi \longmapsto h$, (see Corps Locaux, Sence Chapter XIII)) it follows that $G' \to \hat{\mathbb{Z}}$ is injective.

Therefore, Aut $C \cap G' = (1)$

Let $\theta_r : G \to G_r$ be the canonical homomorphisms; set $G'_r = \theta - r(G')$ = subgroup generated by $\theta_r(\sigma)$. Obviously, one has

$$G'_r \cap \text{ Aut } C \supset G'_{r'} \cap \text{ Aut } C \quad \forall r'|r;$$

since all these groups are finite, the decreasing chain $(G'_r \cap \text{ Aut } C)_r$ is stationary for large for r. We will through if we prove that, for large $\varphi G'_r \cap \text{Aut} C = (1)$. For this we have merely to show that $\bigcap_r (G'_r \text{ Aut } C) =$ (1); in fact, if $\alpha \in \bigcap_r (G'_r \cap \text{ Aut } C)$ then considering α as element of **118**
$G, \forall r, \theta_r(\alpha) \in G'_r = \theta_r(G')$ so that $\varphi \in$ Aut $C \cap G' = (1)$. Q.E.D.

Appendix 2. Our aim in this appendix is to remove from hypotheses on our curve of investigation C the condition (in charac $p \neq 0$) : $genus_K C$ = absolute genus $C \geq 2$ we shall prove now the

Theorem. *Let k be an algebraically closed field of charac. $p \neq 0, K$ a function field over k, and C a curve defined over K, with absolute genus $C \geq 2$. If C_K is infinite, then C admits an absolutely normal model defined on K (so that $genus_K C$ = absolute genus of C). (More precisely, the normalisation of C is absolutely normal).*

Proof. We may assume C normal. The normalisation C' of C in $K^{p-\infty}$ is absolutely normal so that we may assume that $\exists$ a finite, purely inseparable, extension K'/K over which the normal model C' of C is *absolutely normal*. By hypothesis, genus $C' \geq 2$ and we may apply our results in section III. Let $u : C \to C'$ be the (natural) birational correspondence (defined over K'). By Theorem 2 III , of Chapter III, $\exists$ a curve C'' defined over k and a birational correspondence $C' \xrightarrow{v} C''$ defined over a finite galois extension k''/k' such that for almost all points $x \in C_K, v.u(x)$ is in K''^{p^n} (n that large whence in the separable closure L of K in K''. One may again choose imbeddings and argue as before **119**
with base points and unit point to prove that $v \cdot u$ is defined over L :

as L is separable over K and as $\text{genus}_K C = \text{genus}_L C = \text{genus}_k C'' = \text{genus}_{K'} c'$ = abs. genus of C it follows that C, which is normal/L, is already absolutely normal. □

Q.E.D.

APPENDIX 3

Automorphisms of the curve $x^4 + y^4 + z^4 = 0$

We have seen that, in characteristic $\neq 2$, the plane curve $C : x^4+y^4+$ **120**
$z^4 = 0$ has genus 3 and that imbedding $C \hookrightarrow P_2$ is the canonical one. Thus automorphisms of C are induced by projective transformations. Among those we immediately see:

a) The permutations of the variables x, y, z; these form a group G_1 of order 6;

b) The multiplications of x, y, z by arbitrary fourth roots of unity; these form a group G_2 of order 16.

Clearly G_1 and G_2 are permutable subgroups of $PGL(2)$ such that $G_1 \cap G_2 = \{1\}$. Thus they generate a subgroup $G \subset PGL(2)$ of order 96. We claim that:

The group G is the group of automorphisms of C.

To determine Aut (C) we may look for projective peculiarities of C. Let us call a point P of C a *superflex* if the tangent to C at P intersects C with multiplicity 4 at P (and therefore has no other common point with C). Clearly the points of C on the coordinate axis ($e.g.(1, \alpha, 0)$ with $\alpha^4 = -1$) are superflexes (the tangent at $(1, \alpha, 0)$ being $y - \alpha x = 0$). We are going to find all superflexes of C. Disregarding the points at infinity ($z = 0$), and the points at which the tangent passes through $(0, 1, 0)$ (i.e. the points on $y = 0$), we may take affine coordinates, consider a point $(a, b) \in C(a^4 + b^4 + 1 = 0)$, and express that a line

$$x = a + \lambda y = b + t\lambda (b \neq 0)$$

121 has a quadruple intersection with C at (a, b); in other words $\lambda = 0$ must be a quadruple root of

$$(a + \lambda)^4 + (b + t\lambda)^4 + 1 = 0.$$

This means

$$a^4 + b^4 + 1 = 0, 4(a^3 + tb^3) = 0, 0(a^2 + b^2t^2) = 0, 4(a + bt^3) = 0.$$

We deduce $t = -\dfrac{a^3}{b^3}$ whence $a - b\dfrac{a^9}{b^9} = \dfrac{a}{b^8}(x - a^8) = 0$. We have solutions $a = 0, b^4 = -1, t = 0$ on the axis $x = 0$, thus we may assume $a \neq 0$. In characteristic $\neq 3$, the relations $a^2 + b^2t^2 = 0, t = -a^3/b^3$ give $a^4 + b^4 = 0$ impossible. In characteristic 3 the third relation in (1) disappears. From $b^8 - a^8 = 0$ and $a^4 + b^4 = -1$ we deduce $a^4 - b^4 = 0$, whence $2a^4 = -1, a^4 = 1$, and a, b are 4^{th} roots of unity. Thus the superflexes of C are:

a) The 12 points of C on the coordinate axis (for any characteristic $\neq 2$)

b) In addition the 16 points $(a, b, 1)$ such that $a^4 = b^4 = 1$, in characteristic 3.

Notice that, in characteristic 3, the 23 tangents to C at the superflexes are the famous 28 bitangents to C (replaced with the 27 Lines on a cubic surface).

Through each base point pase C lines in the general case (resp. 6 lines in characteristic 3) such that each line contains 4 superflexes. The base points are the only points with this property: this is clear in the general case; a simple computation has to be made in characteristic 3 (here the 28 superflexes are rational over $\mathbb{F}_q$).

122 Hence the base points are characterized by an invariant projective property of C. Therefore any automorphism of C is induced by a projective transformation u which permutes the base points. Then uv^{-1}, with some $v \in G_1$, leaves fixed the base points, i.e., $uv^{-1}(x) = \lambda x, uv^{-1}(y) = \mu y, uv^{-1}(z) = \nu z$. Since C is globally invariant by uv^{-1}, this implies that λ, μ, γ, are proportional to 4^{th} roots of unity, i.e., $uv^{-1} \in G_2$. Therefore $u \in G$. Q.E.D.

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