Lectures on Cauchy Problem

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### Contents

4 133

1 Parabolic equations ........................................ 133
2 Energy inequality for parabolic equations ............... 136

5 145

1 Preliminaries to the study of semi-linear equations .... 145
2 Regularity of some non-linear functions ................. 159
3 An example of a semi-linear equation .................... 164
4 Existence theorems for first order systems of.......... 170
5 Existence theorems for a single semi-linear............. 178
Chapter 1

1 Preliminaries and function spaces

We will be concerned with functions and differential operators defined on the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \). The points of \( \mathbb{R}^n \) will be denoted by \( x = (x_1, \ldots, x_n) \), \( \xi = (\xi_1, \ldots, \xi_n) \), etc. and we will use the following abbreviations:

\[
|x| = \left( \sum x_j^2 \right)^{\frac{1}{2}}, \lambda x = (\lambda x_1, \ldots, \lambda x_n), x \cdot \xi = \sum_j x_j \xi_j,
\]

\( S \) will denote the sphere \( |x| = 1 \), \( dS_x \) the element of surface area on \( S \), and \( dx \) will denote the standard volume element in \( \mathbb{R}^n \). If \( \nu = (\nu_1, \ldots, \nu_n) \) is a multi-index of non-negative integers \( |
u| = \nu_1 + \cdots + \nu_n \) is called the (total) order of \( \nu \). We will also use the following standard notation:

\[
\left( \frac{\partial}{\partial x} \right)^\nu = \left( \frac{\partial}{\partial x_1} \right)^{\nu_1} \cdots \left( \frac{\partial}{\partial x_n} \right)^{\nu_n},
\]

\[
\xi^\nu = \xi_1^{\nu_1} \cdots \xi_n^{\nu_n},
\]

\[
a_\nu(x) = a_{\nu_1, \ldots, \nu_n}(x).
\]

In general \( a_\nu(x) \) will be complex valued functions on \( \mathbb{R}^n \), unless otherwise mentioned. We will also have occasion to use vectors and matrices of complex valued functions. The notation will be obvious from the context.
A general linear partial differential operator can be written in the form

\[ a \left( x, \frac{\partial}{\partial x} \right) = \sum_{\nu} a_{\nu}(x) \left( \frac{\partial}{\partial x} \right)^{\nu}. \]

The maximum \( m \) of the total orders \(|\nu|\) of multi-indices occurring in (1) for which \( a_{\nu}(x) \neq 0 \) is called the order of the operator \( a \left( x, \frac{\partial}{\partial x} \right) \). The transpose or the formal adjoint of \( a \left( x, \frac{\partial}{\partial x} \right) \) is defined by

\[ t_{a} \left( x, \frac{\partial}{\partial x} \right)[u] = \sum_{|\nu| \leq m} (-1)^{|\nu|} \left( \frac{\partial}{\partial x} \right)^{\nu} [a_{\nu}(x)u]. \]

The adjoint of \( a \left( x, \frac{\partial}{\partial x} \right) \) in \( L^2 \) is defined by

\[ a^{\ast} \left( x, \frac{\partial}{\partial x} \right)[u] = \sum_{|\nu| \leq m} (-1)^{|\nu|} \left( \frac{\partial}{\partial x} \right)^{\nu} [a_{\nu}(x)u]. \]

In most of our considerations we will be considering systems of linear differential equations of the first order. We refer to these as first order. We refer to these as first order systems. A first order system can therefore be written in the form:

\[ A \left( x, \frac{\partial}{\partial x} \right) u = \sum_{k=1}^{N} A_{jk} \left( x, \frac{\partial}{\partial x} \right) u_k, \quad j = 1, \ldots, N, \]

where \( A_{jk} \left( x, \frac{\partial}{\partial x} \right) = \sum_{\rho=1}^{n} a_{jk, \rho}(x) \frac{\partial}{\partial x_{\rho}} + b_{jk}(x) \) and \( u = (u_1, \ldots, u_N) \). The formal adjoint of \( A \left( x, \frac{\partial}{\partial x} \right) \) is defined by

\[ 'A \left( x, \frac{\partial}{\partial x} \right) v = \sum_{j} 'A_{jk} \left( x, \frac{\partial}{\partial x} \right) v_j, \quad k = 1, \ldots, N, \]
where \( A_R \left( x, \frac{\partial}{\partial x} \right) u_j = \sum_{j=1}^{n} (-1) \frac{\partial}{\partial x^\rho} (a_{jk\rho}(x) u_j) + b_{jk}(x) u_j \), and the adjoint in \( L^2 \) of \( A \left( x, \frac{\partial}{\partial x} \right) \) is defined by
\[
\left( A^* \left( x, \frac{\partial}{\partial x} \right) v \right)_k = \sum_j A^*_R \left( x, \frac{\partial}{\partial x} \right) v_j, \quad k = 1, \ldots, N
\]
where \( A^*_R \left( x, \frac{\partial}{\partial x} \right) v_j = \sum_{\rho} (-1) \frac{\partial}{\partial x^\rho} (a_{jk\rho}(x) v_j) + b_{jk}(x) v_j. \)

We shall now introduce some function spaces used in the sequel. \( U \) will denote an open set in \( \mathbb{R}^n. \) \( \mathcal{D}(U), \mathcal{E}(U), \mathcal{E}^m(U), \mathcal{D}'(U), \mathcal{E}'(U), \mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n) \) will denote the function spaces of Schwartz provided with their usual topologies. The space of \( m \) times continuously differentiable functions which are bounded together with all their derivatives up to order \( m \) in \( U \) will be denoted by \( \mathcal{B}^m(U). \) \( \mathcal{E}^m(U) \) is provided with the topology of convergence in \( L^\infty(U) \) of all the derivatives up to order \( m. \) \( \mathcal{E}^m_{L^p}(U) \) stands for the space of functions in \( L^p(U) \) whose distributional derivatives up to order \( m \) are functions in \( L^p(U). \) For \( f \in \mathcal{E}^m_{L^p}(U) \) we define
\[
\|f\|_{\mathcal{E}^m_{L^p}(U)} = \|f\|_{L^p(U)} = \left( \sum_{|\nu| \leq m} \left( \frac{\partial}{\partial x} \right)^\nu f \right)^{1/p}.
\]

\( \mathcal{E}^m_{L^p}(U) \) is a Banach space with this norm. Clearly \( \mathcal{E}^m_{L^p}(U) \subset \mathcal{E}^k_{L^p}(U) \) for \( k \leq m \) and the inclusion mapping is continuous. The space of distributions \( f \in \mathcal{D}'(U) \) which are in \( \mathcal{E}^m_{L^p}(U') \) for every relatively compact subset \( U' \) of \( U \) is denoted by \( \mathcal{E}^m_{L^p(loc)}(U). \) This space is topologized by the following sequence of semi-norms. If \( \{U_n\} \) is a sequence of relatively compact subsets of \( U, \) covering \( U, \) we define
\[
p_n(f) = \|f\|_{\mathcal{E}^m_{L^p(U_n)}} \quad \text{for} \quad f \in \mathcal{E}^m_{L^p(loc)}(U).
\]

\( \mathcal{E}^m_{L^p(loc)}(U) \) is a Frechet space with this topology. This space can also be considered as the space of distributions \( f \in \mathcal{D}'(U) \) such that \( \alpha f \in \mathcal{E}^m_{L^p}(U) \) for every \( \alpha \in \mathcal{D}(U). \) Evidently \( \mathcal{E}^m_{L^p}(U) \subset \mathcal{E}^m_{L^p(loc)}(U) \) with continuous inclusion for \( m \geq 0. \) The closure of \( \mathcal{D}(U) \) in \( \mathcal{E}^m_{L^p}(U) \) is denoted by \( \mathcal{D}^m_{L^p}(U) \) and is provided with the induced topology. As before
\( \mathcal{D}_m^m(U) \subset \mathcal{D}_k^m(U) \) for every \( k \leq m \) with continuous inclusion. In general \( \mathcal{D}_m^m(U) \neq \mathcal{D}_m^m(U) \) (for a detailed study of these spaces see Seminaire Schwartz 1954 for the case \( \rho = 2 \)). However \( \mathcal{D}_m^m(\mathbb{R}^n) = \mathcal{D}_m^m(\mathbb{R}^n) \).

When we consider spaces of vectors or matrices of functions we use the obvious notations, which, however will be clear from the context. For instance, if \( f = (f_1, \ldots, f_N) \) where \( f_j \in \mathcal{E}_m^{mL^2}(U) \) then \( \|f\|_{e^m} \) stands for \( \left( \sum_j \|f_j\|_{mL^2(U)}^2 \right)^{\frac{1}{2}} \).

When \( U = \mathbb{R}^n \) we simply write \( \mathcal{D}, \mathcal{E}, \mathcal{E}_m^{mL^2} \) etc. for \( \mathcal{D}(U), \ldots \).

We will denote the space of all continuous functions of \( t \) in an interval \([0, T]\) with values in the topological vector space \( \mathcal{E}_m \) by \( \mathcal{E}_m[0, T] \).

It is provided with the topology of uniform convergence (uniform with respect to \( t \) in \([0, T]\)) for the topology of \( \mathcal{E}_m \). Similar definitions hold for \( \mathcal{E}_m^{mL^2}[0, T], \mathcal{D}_m^{mL^2}(U), \mathcal{D}_m^{mL^2}(loc)[0, T], \mathcal{B}_m[0, T], \) etc.

We now recall, without proof, a few well-known results on the spaces \( \mathcal{E}_m^{mL^p}(U) \) and \( \mathcal{E}_m^{mL^p}(loc)(U) \).

**Proposition 1** (Rellich). Every bounded set in \( \mathcal{E}_m^{mL^p}(U) \) is relatively compact in \( \mathcal{E}_m^{mL^p}(loc)(U) \) for \( m \geq 1 \).

In other words, the proposition asserts that the inclusion mapping of \( \mathcal{E}_m^{mL^p}(U) \) into \( \mathcal{E}_m^{mL^p}(loc)(U) \) is completely continuous.

The following is a generalization due to Sobolev of a result of F. Riesz.

**Proposition 2.** Let \( g \in L^p, h \in L^q \) for \( p, q > 1 \) such that \( \frac{1}{p} + \frac{1}{q} > 1 \).

Then the following inequality holds:

\[
\left( \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{g(x)h(y)}{|x-y|^{\lambda}} \, dx \, dy \right)^{\frac{1}{p}} \leq K ||g||_{L^p} \cdot ||h||_{L^p}
\]

where \( \lambda = n \left( 2 - \frac{1}{p} - \frac{1}{q} \right) \) and \( K \) is a constant depending only on \( p, q, n \) but not on \( g \) and \( h \).
2. Cauchy Problem

**Proposition 3 (Sobolev).** If \( h \in L^p \) for \( p > 1 \) then the function

\[
f(x) = \int \frac{h(y)}{|x - y|^\lambda} dy,
\]

where \( n > \lambda > \frac{n}{p'} = n(1 - \frac{1}{p}) \), is in \( L^q \) where \( \frac{1}{q} = \frac{1}{p} + \frac{\lambda}{n} - 1 \).

**Theorem 1 (Sobolev).** Let \( U \) be an open set with smooth boundary \( \partial U \) (for instance \( \partial U \in C^2 \)). Then any function \( \varphi \in \mathcal{E}^m_{L^p}(U) \) with \( pm \leq n \) itself belongs to \( L^q(U) \) where \( q \) satisfies \( \frac{1}{q} = \frac{1}{p} - \frac{m}{n} \). Further we have an estimate

\[
\|\varphi\|_{L^q(U)} \leq C\|\varphi\|_{\mathcal{E}^m_{L^p}(U)}.
\]

The constant \( C \) depends only on \( p, q, r \) and \( n \) but not on the function \( u \).

For the study of this inequality and delicate properties of the inclusion mapping see S. Sobolev: Sur un Théorème d’analyse fonctionnelle, Mat. Sbornik, 4(46), 1938.

2 Cauchy Problem

In this section we formulate the Cauchy problem for a linear differential operator \( a \left( x, \frac{\partial}{\partial x} \right) \). To begin with we make a few formal reductions.

Let \( S \) be a hypersurface in \( \mathbb{R}^n \) defined by an equation \( \varphi(x) = 0 \) where \( \varphi \) is a sufficiently often continuously differentiable function with its gradient \( \varphi_x(x_0) = \left( \frac{\partial \varphi}{\partial x_1}(x_0), \ldots, \frac{\partial \varphi}{\partial x_n}(x_0) \right) \neq 0 \) at every point \( x_0 \) of \( S \).

Let \( n \) denote the normal at the point \( x_0 \) to \( S \) and \( \frac{\partial}{\partial n} \) denote the derivation along the normal \( n \).

Suppose \( x_0 \) is a point on \( S \); let \( u_0, \ldots, u_{m-1} \) be functions on \( S \) defined in a neighbourhood of \( x_0 \). A set \( \psi = (u_0, \ldots, u_{m-1}) \) of such functions is called a set of Cauchy data on \( S \) for any differential operator
of order $m$. The Cauchy data $\psi$ are said to be analytic (resp. of class $E^m$, resp. of class $E$) if each of the functions $u_0, u_1, \ldots, u_{m-1}$ is an analytic (resp. $m$ times continuously differentiable function resp. infinitely differentiable function) in their domain of definition.

Let there be given a function $f$ defined in a neighbourhood $U$ in $\mathbb{R}^n$ of a point $x_0$ of $S$ and Cauchy data $\psi$ in a neighbourhood $V$ of $x_0$ on $S$.

The Cauchy problem for the differential operator $a(x, \frac{\partial}{\partial x})$ with the Cauchy data $\psi$ on $S$ consists in finding a function $u$ defined in a neighbourhood $U'$ of $x_0$ in $\mathbb{R}^n$ satisfying

$$(2.1) \quad a(x, \frac{\partial}{\partial x}) u = f \text{ in } U'$$

and $u(x) = u_0(x), \frac{\partial}{\partial n} u(x) = u_1(x); \ldots; \left(\frac{\partial}{\partial n}\right)^{m-1} u(x) = u_{m-1}(x)$ for $x \in V \cap U'$. When such a $u$ exists we call it a solution of the Cauchy problem.

In the study of the Cauchy problem the following questions arise: the existence of a solution $u$ and its domain of definition, uniqueness when the solution exists, dependence of the solution on the Cauchy data and the existence of the solution in the large. The answers to these questions will largely depend on the nature of the differential operator and of the surface $S$ (supporting the Cauchy data) in relation to the differential operator besides the Cauchy data $\psi$ and $f$. In order to facilitate the formulation and the study of the above questions we first make a preliminary reduction.

By a change of variables

$$(x_1, \ldots, x_n) \rightarrow (x'_1, \ldots, x'_n)$$

with $x'_1 = x_1, \ldots, x'_{n-1} = x_{n-1}$ and $x'_n = \varphi(x)$ the equation

$$(2.1) \quad a(x, \frac{\partial}{\partial x}) u = f$$

is transformed into an equation of the form

$$h(x, \varphi_x) \left(\frac{\partial}{\partial x'_n}\right)^m u + \sum \cdots = f$$
2. Cauchy Problem

where \( \varphi = \left( \frac{\partial \varphi}{\partial x_1}, \ldots, \frac{\partial \varphi}{\partial x_n} \right) \) and

\[
h(x, \xi) = \sum_{|\nu| = m} a_{\nu}(x) \xi^\nu, \quad \xi = (\xi_1, \ldots, \xi_n).
\]

The summation above contains derivatives of \( u \) of orders < \( m \) in the \( x'_n \)-direction.

1. If \( h(x, \varphi_x(x)) \neq 0 \) in a neighbourhood of the point under consideration we can divide the above expression for the equation by the factor \( h(x, \varphi_x) \) and write

\[
\left( \frac{\partial}{\partial x'_n} \right)^m u + \sum_{|\nu| \leq m} a_{\nu}'(x') \left( \frac{\partial}{\partial x'} \right)^\nu u = \frac{f}{h(x, \varphi_x)}.
\]

This is called the normal form of the equation.

\[
a(x, \frac{\partial}{\partial x}) u = f.
\]

The Cauchy problem is now given by

\[
\left( \frac{\partial}{\partial x_n} \right)^j a(x'_1, \ldots, x'_{n-1}, 0) = u_j(x'_1, \ldots, x'_{n-1}) \quad \text{for} \quad j = 0, 1, \ldots, m-1.
\]

2. In the case in which \( h(x, \varphi_x) = 0 \) at a point \( x_0 \) of \( S \) the study of the Cauchy problem in the neighbourhood of \( x_0 \) becomes considerably more difficult. In what follows we only study the case (1) where the equation can be brought to the normal form by a suitable change of variables. This motivates the following

**Definition.** A surface \( S \) defined by an equation \( \varphi(x) = 0 \) (\( \psi \) being once continuously differentiable) in \( \mathbb{R}^n \) is said to be a characteristic variety or characteristic hypersurface of the operator \( a \left( x, \frac{\partial}{\partial x} \right) \) if \( h(x, \text{grad} \varphi(x)) = 0 \) for all the points \( x \) on \( S \).

A vector \( \xi \in \mathbb{R}^n \) is said to be a characteristic direction at \( x \) with respect to the differential operator \( a \left( x, \frac{\partial}{\partial x} \right) \) if \( h(x, \xi) = 0 \).


Clearly, if $S$ is a characteristic variety of a differential operator \( a(x, \frac{\partial}{\partial x}) \) then the vector normal to $S$ at any point on it will be a characteristic direction at that point. For any point $x \in S$ the set of vectors $\xi$ which are characteristic directions at $x$ form a cone in the $\xi$-space with vertex at the origin called the characteristic cone of the operator $a\left(x, \frac{\partial}{\partial x}\right)$ at the point $x$. In the following we restrict ourselves to the case where $S$ is not characteristic for the differential operator at any point and hence assume the operator to be in the normal form.

3 Cauchy-Kowalevsky theorem and Holmgren’s theorem

The first general result concerning the Cauchy problem (local) is the following theorem due to Cauchy and Kowalevsky. This we recall without proof. For a proof see for example Petrowsky [3].

From now on we change slightly the notation and denote a point of $\mathbb{R}^{n+1}$ by $(x, t) = (x_1, \ldots, x_n, t)$ and a point of $\mathbb{R}^n$ by $x = (x_1, \ldots, x_n)$.

Let

$$L \equiv \left( \frac{\partial}{\partial t} \right)^m + \sum_{\|\nu\| + j \leq m} a_{\nu,j}(x, t) \left( \frac{\partial}{\partial x} \right)^\nu \left( \frac{\partial}{\partial t} \right)^j$$

be a differential operator of order $m$ written in the normal form with variable coefficients.

**Theorem 1** (Cauchy-Kowalevsky). Let the coefficients $a_{\nu,j}$ of $L$ be defined and analytic in a neighbourhood $U$ of the origin in the $(x, t)$ space. Suppose that $f$ is an analytic function on $U$ and $\psi$ is an analytic Cauchy datum in a neighbourhood $V$ of the origin in the $x$-space. Then there exists a neighbourhood $W$ of the origin in the $(x, t)$-space and a unique solution $u$ of the Cauchy problem

$$Lu = f \quad \text{in} \ W$$

and

$$\left( \frac{\partial}{\partial t} \right)^j u = u_j \quad \text{on} \ W \cap \{t = 0\} \quad \text{for} \ j = 0, 1, \ldots, m - 1,$$
The domain $W$ of existence of $u$ depends on $U$, $V$ and the maximum moduli of $a_{\nu,j}$.

It is not in general, possible to assert the existence of a solution of the Cauchy problem when the Cauchy data are only of class $\mathcal{E}$. However, for a certain class of differential operators—such as Hyperbolic operators—existence (even in the large) of solutions of the Cauchy problem can be established under some conditions. This will be done in the subsequent sections.

If $u_1$ and $u_2$ are two analytic solutions of the Cauchy problem in a neighbourhood of the origin with the same analytic Cauchy data the theorem of Cauchy-Kowalevsky asserts that $u_1 \equiv u_2$. Holmgren showed that for an operator with analytic coefficients the solution is unique, if it exists, in the class $\mathcal{E}^m$ ($m$, we recall, is the order of $L$). More precisely, we have the

**Theorem 2** (Holmgren). *If the coefficients $a_{\nu,j}$ of the differential operator $L$ are analytic functions in a neighbourhood $U$ of the origin then there exists a number $\varepsilon_0 > 0$ satisfying the following: for any $0 < \varepsilon < \varepsilon_0$ if the Cauchy data $\psi$ vanish on $(t = 0) \cap D_\varepsilon$ then any solution $u \in \mathcal{E}^m$ of the Cauchy problem

\[
Lu = 0 \text{ in } D_\varepsilon \text{ and }
\left(\frac{\partial}{\partial t}\right)^j u = 0 \text{ on } (t = 0) \cap D_\varepsilon \text{ for } j = 0, 1, \ldots, m - 1,
\]

itself vanishes identically in $D_\varepsilon$, where $D_\varepsilon$ denotes the set

\[
\{(x,t) \in \mathbb{R}^{n+1} ||x|^2 + |t| < \varepsilon\}.
\]

**Proof.** By a change of variables $(x, t) \to (x', t')$ where $x'_k = x_k (k = 1, \ldots, n)$ and $t' = t + x_1^2 + \cdots + x_n^2$, the half space $t \geq 0$ is mapped into the domain

\[
\Omega = \{(x', t') \in \mathbb{R}^{n+1} | t' - |x'|^2 \geq 0\}.
\]
in the \((x', t')\) space. The transformed function \(u'(x', t')\) and its derivatives up to order \((m - 1)\) in the direction of the interior normal to the hypersurface \(t' - |x'|^2 = 0\) vanish identically on the hyper-surface. Hence extending \(u'\) by zero outside the domain \(\Omega\) we obtain a function in \(\mathcal{E}^m\), which we again denote by \(u\), with support contained in \(\Omega\). The differential operator is transformed into another differential operator of order \(m\) with analytic coefficients. \(\square\)

Thus we may assume that \(u\) is a solution of an equation

\[
(3.3) \quad Lu \equiv \left( \frac{\theta}{\partial t} \right)^m u + \sum_{|\nu| + j \leq m \atop j \leq m-1} a_{\nu,j}(x,t) \left( \frac{\partial}{\partial x} \right)^\nu \left( \frac{\partial}{\partial t} \right)^j u = 0
\]

with support contained in \(\Omega\). Let \(^tL\) be the transpose operator of \(L\) and \(V\) be a solution of \(^tL[v] = 0\) in \(\Omega_h = \Omega \cap \{0 \leq t \leq h\}\) satisfying the conditions

\[
(3.4) \quad v(x,h) = \frac{\partial}{\partial t} v(x,h) = \ldots = \left( \frac{\partial}{\partial t} \right)^{m-2} v(x,h) = 0
\]

on the hyperplane \((t = h)\). Then we have

\[
(3.5) \quad \int_{\Omega_h} (u' L[v] - v L[u]) dx \, dt = 0.
\]

On the other hand, integrating by parts with respect to the variables \(t\) and \(x\) yields

\[
\int_{\Omega_h} (u' L[v] - v L[u]) dx \, dt = \int_{t=h} (-1)^m u(x,t) \left( \frac{\partial}{\partial t} \right)^{m-1} v(x,t) dx
\]

because of the conditions \((3.4)\).

\[
(3.6) \quad \text{Hence} \quad \int_{t=h} (-1)^m u(x,t) \left( \frac{\partial}{\partial t} \right)^{m-1} v(x,t) dx = 0.
\]

Now consider the Cauchy problems

\[t_h L[v] = 0\]
4. Solvability of the Cauchy problem in the class $\mathcal{E}^m$

In this section we make a few remarks on the existence of solutions of the Cauchy problem in the class $\mathcal{E}^m$ under weaker regularity conditions on the coefficients of the differential operator. We begin with the following formal definition.

$$\left( \frac{\partial}{\partial t} \right)^j v(x,0) = 0, \; j = 1, \ldots, m, \; \left( \frac{\partial}{\partial t} \right)^{m-1} v(x,0) = P(x),$$

$P(x)$ running through polynomials. By the Cauchy Kowalevsky Theorem, there exists solutions $v(x)$, in a fixed neighbourhood $|t| \leq h$ satisfying the above Cauchy problems. Hence there is a $h > 0$ such that, for every polynomial $P(x)$ there exist $v$ in $\Omega_h$ satisfying (3.2) with $\left( \frac{\partial}{\partial t} \right)^{m-1} u(x,h) = P(x)$. Hence by (3.6) $u(x,t)$ is orthogonal to every polynomial $P(x)$ for $t \leq h$. Hence $u(x,t) \equiv 0$ for $0 \leq t \leq h$. Replacing $t$, by $-t$ we obtain $u(x,t) \equiv 0$ for $-h \leq t \leq 0$. Hence $u(x,t) \equiv 0$ in $D_E$ which finishes the prove of the theorem.

Further general results on the uniqueness of the solution of the Cauchy problem were proved by Calderon [11]. We restrict ourselves to stating one of his results (3).

**Theorem 3 (Calderon).** Let $L$ be an operator of the form (3.1) with real coefficients. Assume that in a neighbourhood of the origin all the coefficients $a_{\nu,j}(x,t)$, for $|\nu| + j = m$, belong to $C^{1+\sigma}(\sigma > 0)$ and the other coefficients are bounded. Further suppose that the characteristic equation at the origin

$$(3.6) \quad P(\lambda,\xi) \equiv \lambda^m + \sum_{|\nu|+j=m} a_{\nu,j}(0,0) \xi^\nu \lambda^j = 0$$

has distinct roots for any real $\xi \neq 0$. If the solution $u$ belong to $C^m$ and has zero Cauchy data (more precisely, Cauchy data, zero in a neighbourhood of the hyperplane $t = 0$) then $u \equiv 0$ in a neighbourhood of the origin.

4 Solvability of the Cauchy problem in the class $\mathcal{E}^m$

In this section we make a few remarks on the existence of solutions of the Cauchy problem in the class $\mathcal{E}^m$ under weaker regularity conditions on the coefficients of the differential operator. We begin with the following formal definition.
Let

\[ L \equiv \left( \frac{\partial}{\partial t} \right)^m + \sum_{|\nu| + \sum_{j \leq m} |j| \leq m} a_{\nu, j}(x, t) \left( \frac{\partial}{\partial x} \right)^{\nu} \left( \frac{\partial}{\partial t} \right)^j \]

be a differential operator of order \( m \) in the normal form.

**Definition.** The Cauchy problem for \( L \) is said to be solvable at the origin in the class if for any given \( f \in E_x \) and any Cauchy datum \( \psi \) of class \( E_x \) there exists a neighbourhood \( D_\psi, f \) of the origin in the \((x, t)\) space and a solution \( u \in E_x(D_{\psi, f}) \) of the Cauchy problem for \( L \) with \( \psi \) as the Cauchy datum.

**Remark.** The Cauchy problem for a general linear differential operator \( L \) is not in general solvable in the class \( E \) as is shown by the following counter example due to Hadamard.

**Counter example (Hadamard).** Let \( L \) be the Laplacian \( \Delta \) in \( \mathbb{R}^3 \)

\[ \Delta \equiv \left( \frac{\partial}{\partial x} \right)^2 + \left( \frac{\partial}{\partial y} \right)^2 + \left( \frac{\partial}{\partial z} \right)^2 \]

and \((z = 0)\) be the hyperplane supporting the Cauchy data. Consider for the Cauchy data the conditions

\[ u(x, y, 0) = u_0(x, y) \quad \text{and} \quad \frac{\partial u}{\partial z}(x, y, 0) = 0. \]

Suppose \( u(x, y, z) \equiv u \) is a solution of \( \Delta u = 0 \) in \( z \geq 0 \) with the Cauchy data \((u_0, 0)\). Extend \( u \) to the whole of \( \mathbb{R}^3 \) by setting

\[ \tilde{u}(x, y, z) = u(x, y, z) \quad \text{for} \quad z \geq 0 \quad \text{and} \]

\[ = u(x, y, -z) \quad \text{for} \quad z \leq 0. \]

\( \tilde{u} \) satisfies the equation \( \Delta \tilde{u} = 0 \) in the sense of distributions. In fact, for any \( \varphi \in \mathcal{D}(\mathbb{R}^3) \) we have

\[ \langle \tilde{u}, \Delta \varphi \rangle = \int_{\mathbb{R}^3} \tilde{u}(x, y, z) \Delta \varphi(x, y, z) dx \, dy \, dz \]
4. Solvability of the Cauchy problem in the class $\mathcal{E}^m$

$$= \lim_{\varepsilon \to 0} \left\{ - \int_{|z| \geq \varepsilon} \frac{\partial \phi}{\partial z} \frac{\partial \tilde{u}}{\partial z} \, dx \, dy \, dz \right\}$$

and

$$\int_{|z| \geq \varepsilon} \frac{\partial \tilde{u}}{\partial z} \frac{\partial \phi}{\partial z} \, dx \, dy \, dz = \int_{|z| \leq \varepsilon} \left[ \frac{\partial \tilde{u}}{\partial z} \right] \phi \, dx \, dy - \int_{|z| \geq \varepsilon} \frac{\partial^2 \tilde{u}}{\partial z^2} \phi \, dx \, dy$$

Hence

$$\langle \tilde{u}, \Delta \phi \rangle = \lim_{\varepsilon \to 0} \left\{ \int \phi(x, y, \varepsilon) \frac{\partial \tilde{u}}{\partial z}(x, y, \varepsilon) \, dx \, dy - \int \phi(x, y, -\varepsilon) \frac{\partial \tilde{u}}{\partial z}(x, y, -\varepsilon) \, dx \, dy \right\}$$

$$= 0$$

By the regularity of solutions of elliptic equations $u$ is an analytic function of $x, y, z$ in $\mathbb{R}^3$. Since $u_0(x, y) = u(x, y, 0) = \tilde{u}(x, y, 0)$, $u_0$ is an analytic function of $(x, y)$. Thus, if $u_0$ is taken to be in $\mathcal{E}^m$, but non analytic there does not exist a solution of the Cauchy problem for $\Delta u = 0$ with the Cauchy data $(u_0, 0)$.

As far as the domain of existence of a solution of the Cauchy problem is concerned we know by the Cauchy-Kowalevsky theorem that, whenever the coefficients of $L$, $f$ and the Cauchy data $\psi$ are of analytic classes, there exists a neighbourhood of the origin and an analytic function $u$ on it satisfying $L[u] = f$ with Cauchy data $\psi$. However it is not in general possible to continue this local solution $u$ to the whole space as a solution of $L[u] = f$. This is demonstrated by the following counter example which is again due to Hadamard.

**Counter example.** Let the differential operator be

$$L \equiv \left( \frac{\partial}{\partial x} \right)^2 + \left( \frac{\partial}{\partial y} \right)^2.$$ 

A solution of $L[u] = 0$ is provided by

$$u(x, y) = \Re \frac{1}{z - a} = \frac{x - a}{(x - a)^2 + y^2} \text{ where } a > 0.$$
Clearly \( u(0, y) \) and \( \frac{\partial u}{\partial x}(0, y) \) are analytic functions of \( Y \). However this solution cannot be continued to the half plane \( x \geq a \) as can be easily seen.

For a class of differential operators the existence of solutions in the large has been established by Hadamard, Petrowsky, Leray, Garding and others. We shall prove some of these results later by using the method of singular integral operators introduced by Calderon and Zygmund.
Chapter 2

In this chapter as well as in the next chapter we will be mainly concerned with the study of the Cauchy problem for systems of differential equations of the first order, which will be referred to as first order systems.

1

If \( u(x, t) = (u_1(x, t), \ldots, u_N(x, t)) \) and \( f(x, t) = (f_1(x, t), \ldots, f_N(x, t)) \) denote vector valued functions with \( N \) components, a first order system of equations can be written in the form

\[
M[u] \equiv \frac{\partial}{\partial t} u - \sum_{k=1}^{n} A_k(x, t) \frac{\partial}{\partial x_k} u - B(x, t) u = f
\]

where \( A_k(x, t), B(x, t) \) are matrices of order \( N \) of functions whose regularity conditions will be made precise in each of the problems under consideration.

**Definition.** The Cauchy problem for a first order system \( M[u] = 0 \) is said to be locally solvable at the origin in the space \( \mathcal{E} \) (resp. \( \mathcal{B} \), resp. \( D^s_{\mathcal{L}^2} \)) if for any given \( \psi \in \mathcal{E}(U) \) (resp. \( \mathcal{B}(U) \), resp. \( D^s_{\mathcal{L}^2}(U) \)) \( U \) being an arbitrary open set in the \( x \)-space containing the origin there exists a neighbourhood \( V \) of the origin in \( \mathbb{R}^{n+1} \) and a function \( u \in \mathcal{E}(V) \) (resp. \( \mathcal{B}(V) \), resp. \( D^s_{\mathcal{L}^2}(V) \)) satisfying

\[ M[u] = 0 \text{ and } u(x, 0) = \psi(x) \]
The following proposition shows that when the system $M$ has analytic coefficients the local solvability of the Cauchy problem implies the existence of a neighbourhood $V$ independent of $ψ$ such that for any $ψ \in \mathcal{E}_x$ there exists a unique solution $u \in \mathcal{E}^1(V)$.

We define a family of open sets $D_ε$ of $\mathbb{R}^{n+1}$ by

$$(1.2) \quad D_ε = \{(x,t) \in \mathbb{R}^{n+1} \mid |t| + |x|^2 < ε \}.$$ 

Proposition 1 (P.D. Lax). Assume that the coefficients of $M$ are analytic and the Cauchy problem for $M$ is locally solvable at the origin. Then there exists a $δ > 0$ such that for any given $ψ \in \mathcal{E}_x(U)$ there exists a unique solution $u \in \mathcal{E}^1(D_δ)$ of $M[u] = 0$, $u(x,0) = ψ(x)$.

Proof. By Holmgren’s theorem there exists an $ε_0 > 0$ such that for $0 < ε \leq ε_0$ a solution, $u \in \mathcal{E}^1_{x,t}$ with $u(x,0) = ψ(x)$ on $D_ε \cap (t = 0)$ is uniquely determined in $D_ε$. Let $ε_0 > ε_1 \ldots$ be a sequence of positive numbers $ε_n \to 0$. Denote by $A_{k,m}$ the set of all $ψ \in \mathcal{E}_x(U)$ such that the solution $u$ of $M[u] = 0$ with $u(x,0) = ψ(x)$ for $x \in D_{ε_k} \cap (t = 0)$ is in $\mathcal{E}(D_{ε_k})$ and satisfies

$$||u||_{\frac{n}{2}+2} \leq m.$$ 

The sets $A_{k,m}$ are symmetric and convex. Further $\mathcal{E}(U) = \bigcup_{k,m} A_{k,m}$, by the local solvability at the origin. We shall now show that $A_{k,m}$ is closed for every $k,m$.

Let $ψ_j$ be a sequence in $A_{k,m}$ converging to $ψ_0$ in $\mathcal{E}(U)$. The corresponding sequence of solutions $u_j$ is a bounded set in $\mathcal{E}^1_{L^2}(D_{ε_k})$ and hence has a subsequence $u_{j_p}(x,t)$ weakly convergent in $\mathcal{E}(D_{ε_k})$. In view of the Prop. [1] of Chap. II §1 we can, if necessary by choosing a subsequence, assume that $u_{j_p}(x,t)$ converges in $\mathcal{E}^1_{L^2(\text{loc})}(D_{ε_k})$. Let this limit be $u_0$. Since $u_{j_p} \to u_0$ weakly in $\mathcal{E}^1_{L^2}(D_{ε_k})$ we have $||u_0||_{\frac{n}{2}+2} \leq m$. By prop ?? of Chap II §1 (Sobolev’s lemma) $u_0 \in \mathcal{E}^1(D_{ε_k})$ and further $M[u_0] = 0$. Again $u_{j_p} \to u_0$ in $\mathcal{E}^1_{L^2(\text{loc})}(D_{ε_k})$ implies that this conver-
gence is uniform on every compact subset of $D_{\varepsilon_k}$ and hence $u_0(x, 0) = \psi_0(x)$. Thus $A_{k,m}$ is a closed subset of $E_x(U)$.

Now by Baire’s category theorem one of the $A_{k,m}$, let us say $A_{k_0,m_0}$ contains an open set of $E_x(U)$. $A_{k_0,m_0}$, being symmetric and convex contains therefore a neighbourhood of 0 in $E_x(U)$. Since any $\psi \in E_x(U)$ has a homothetic image $\lambda \psi$ in this neighbourhood, there is a unique solution $u \in E_2[0, 2] L^2(D_{\varepsilon_{k_0}})$, a fortiori, in $E_1(D_{\varepsilon_{k_0}})$ of $M[u] = 0$ with $u(x, 0) = \psi(x)$. $\varepsilon_{k_0}$ can be taken to be the required $\delta$. □

**Theorem 1.** Let the coefficients $A_k(x, t), B(x, t)$ of $M$ be analytic. If the Cauchy problem is locally solvable at the origin in the space $E$ then the linear mapping $\psi(x) \rightarrow u(x, t)$ is continuous from $E(U)$ in to $E_1(D_{\varepsilon_0})$.

**Proof.** The graph of the mapping $\psi \rightarrow u$ is closed in $E(U) \times E_1(D_{\varepsilon_0})$ because of the uniqueness of the solution of $M[u] = 0$, with $u(x, 0) = \psi(x)$ in $D_{\varepsilon_0}$. Hence by the closed graph theorem of Banach the mapping is continuous. □

This leads us to the notion of well-posedness of the Cauchy problem in the sense of Hadamard. This we consider in the following section.

## 2 Well-posedness and uniform-well posedness of the Cauchy problem

By a $k$-times differentiable function on a closed interval $[0, h]$ we mean the restriction to $[0, h]$ of a $k$-times continuously differentiable function on an open interval containing $[0, h]$.

The space of continuous functions of $t$ in $[0, h]$ with values in the space $E_x^m$ is denoted by $E_t^m[0, h]$. It is provided with the topology of uniform convergence in the topology of $E_x^m$ (uniform with respect to $t$ in $[0, h]$). In other words, a sequence $\varphi_n \in E_t^m[0, h]$ converges to 0 in the topology of $E_t^m[0, h]$ if $\varphi_n(t) = \varphi_n(x, t) \rightarrow 0$ in $E_x^m$ uniformly with respect to $t$ in $[0, h]$. A vector valued function $u = (u_1, \ldots, u_N)$ is said to belong to $E_t^m[0, h]$ if each of its components $u_j$ belong to $E_t^m[0, h]$. Similarly one can define the spaces $E_t^m[0, h] \cdot D_t^1[0, h], L^2[0, h] = D_t^0[0, h]$ etc. These will be the spaces which we shall be using in our
discussions hereafter. We also write \( \mathcal{B}[0, h], \mathcal{E}[0, h], D_{L^2}[0, h] \) instead of \( \mathcal{B}^\infty[0, h], \mathcal{E}^\infty[0, h], D_{L^2}^\infty[0, h] \). Following Petrowsky \[2\] we give the

**Definition.** The forward Cauchy problem for a first order system \( M \) is said to be well posed in the space \( \mathcal{E} \) in an interval \( [0, h] \) if

1. for any given function \( f \) belonging to \( \mathcal{E}[0, h] \) and any Cauchy data \( \psi \in \mathcal{E}_x \) there exists a unique solution \( u \) belonging to \( \mathcal{E}[0, h] \) and once continuously differentiable with respect to \( t \) in \( [0, h] \) (with its first derivative w.r.t. \( t \) having its values in \( \mathcal{E}_x \)) of \( M[u] = f \) with \( u(x, 0) = \psi(x) \); and

2. the mapping \( (f, \psi) \rightarrow u \) is continuous from \( \mathcal{E}[0, h] \times \mathcal{E}_x \) into \( \mathcal{E}[0, h] \).

**Definition.** The forward Cauchy problem for a first order system \( M \) is said to be uniformly well posed in the space \( \mathcal{E} \) if for every \( t_0 \in [0, h] \) the following condition is satisfied:

1. for any given function \( f \) belonging to \( \mathcal{E}[0, h] \) and any Cauchy data \( \psi \in \mathcal{E}_x \) there exists a unique solution \( u = u(x, t, t_0) \) belonging to \( \mathcal{E}[t_0, h] \) and once continuously differentiable with respect to \( t \) in \( [t_0, h] \) (the first derivative having its values in \( \mathcal{E}_x \)) of \( M[u] = f \) with \( u(x, t_0, t_0) = \psi(x) \); and

2. the mapping \( (f, \psi) \rightarrow u \) is uniformly continuous from \( \mathcal{E}[0, h], \mathcal{E}_x \) into \( \mathcal{E}[t_0, h] \).

The condition of uniform continuity can also be analytically described as follows: given an integer \( l \) and a compact set \( K \) of \( \mathbb{R}^n \) there exists an integer \( t' \), a compact set \( K' \) of \( \mathbb{R}^n \) and a constant \( C \) (all independent of \( t_0 \) in \( [0, h] \)) such that

\[
\sup_{0 \leq t \leq h} |u(x, t, t_0)|_{\mathcal{E}_x^l} \leq C |\psi(x)|_{\mathcal{E}_x^{l'}} + \sup_{0 \leq t \leq h} |f(x, t)|_{\mathcal{E}_x^{l'}}
\]

where \( |g(x)|_{\mathcal{E}_x^l} = \sup_{x \in K} \left| \frac{\partial}{\partial x} \right|^l g(x) \).

Similar statements hold also for the spaces \( \mathcal{B} \) and \( D_{L^2}^\infty \).
We shall now give some criteria for the well posedness of the forward Cauchy problem for first order systems $M$. For this purpose we introduce the notions of characteristic equation and of the characteristic roots of a first order system $M$.

The polynomal equation

$$(2.2) \quad \det \left( \lambda I - i \sum_{k} A_k(x,t) \xi_k - B(x,t) \right) = 0$$

is called the characteristic equation of $M$ and the roots $\lambda_1(x,t,\xi), \ldots, \lambda_N(x,t,\xi)$ of this equation are called the characteristic roots of $M$.

It will be useful for our future considerations to introduce the notions of characteristic equation and of characteristic roots for a single equation of order $m$ of the form

$$(2.3) \quad L = \left( \frac{\partial}{\partial t} \right)^m + \sum_{\nu+j=m} a_{\nu,j}(x,t) \left( \frac{\partial}{\partial x} \right)^\nu \left( \frac{\partial}{\partial t} \right)^j.$$

Consider the principal part of $L$ and write it in the form

$$(2.4) \quad \left( \frac{\partial}{\partial t} \right)^m + \sum_{j=0}^{m-1} a_j(x,t,\xi) \frac{\partial^j}{\partial t^j}$$

where $a_j(x,t,\xi) = \sum_{\nu+j=m-j} a_{\nu,j}(x,t) \xi^\nu$ is a homogeneous polynomial in $\xi$ of degree $m-j$. The characteristic equation of $L$ is defined to be

$$(2.5) \quad \lambda^m + \sum_{j=0}^{m-1} a_j(x,t,\xi) \lambda^j = 0$$

and its roots are called the characteristic roots of $L$.

We remark here that if we take \( \left( u, \frac{\partial u}{\partial t}, \ldots, \left( \frac{\partial}{\partial t} \right)^{m-1} u \right) \) as a system of unknown functions, say \( (u_1, u_2, \ldots, u_m) \), we have

$$(2.6) \quad \frac{\partial}{\partial t} \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 - a_1 & -a_2 & \ldots & -a_{m-1} \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} = H \begin{pmatrix} x, t, \frac{\partial}{\partial x} \\ \vdots \\ \frac{\partial}{\partial t} \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}.$$
and \( \det(\lambda I - H(x, t, \xi)) = \lambda^n + \sum_{j=0}^{n-1} a_j(x, t, \xi) \lambda^j \). Thus the characteristic roots of \( L \) are the same as those of the system \( (2.6) \).

We now obtain necessary and sufficient condition for the well posedness of the Cauchy problem for first order systems in the case where the coefficients depend only on \( t \):

\[
\frac{\partial u}{\partial t} = \sum A_k(t) \frac{\partial u}{\partial x_k} + B(t)u. \tag{2.7}
\]

These conditions depend on the nature of the roots of its characteristic equation

\[
\det(\lambda I - \sum A_k(t) \xi_k - B(t)) = 0 \tag{2.8}
\]

In the case where \( A_k \) and \( B \) are constant matrices, we have the following proposition.

**Proposition 1** (Hadamard). *Let the coefficients \( A_k \) and \( B \) of \( M \) be constants. A necessary condition in order that the forward Cauchy problem for \( M \) be well posed in the space \( \mathcal{B} \) is that there exist constants \( c \) and \( p \) such that*

\[
\text{Re} \lambda_j(\xi) \leq p \log(1 + |\xi|) + c \quad (j = 1, \ldots, N). \tag{2.9}
\]

**Proof.** Assume that the forward Cauchy problem for \( M \) is well posed but the condition \( (2.9) \) is not satisfied. First of all we observe that, if \( \lambda(\xi) \) is any characteristic root of \( M \) there exists a non-zero vector \( P(\xi) \in \mathbb{C}^N \) with \( |P(\xi)| = 1 \) such that

\[
\left( \lambda(\xi) I - \sum A_k(t) \xi_k - B(t) \right) P(\xi) = 0.
\]

Then \( u(x, t) = \exp(\lambda(\xi)t + ix \cdot \xi) \). \( P(\xi) \) is a solution of \( M[u] = 0 \). By assumption for any \( p > 0 \) there exists a vector \( \xi \), \( |\xi| \geq 2 \), and a characteristic root \( \lambda(\xi) \) such that,

\[
\text{Re} \lambda(\xi) \geq p \log(1 + |\xi|).
\]

For this \( \lambda(\xi) \) we have
2. Well-posedness and uniform-well posedness.....

(i) \( M[u] \equiv M[\exp(\lambda(\xi)t + ix\cdot\xi) \cdot P(\xi)] = 0; \)

(ii) \(|u(x, t)| = \exp(\text{Re} \lambda(\xi)t) \cdot |P(\xi)| \geq (1 + |\xi|)^p t > 0; \) and

(iii) \( \sum_{|\nu| \leq l} \left( \frac{\partial}{\partial x} \right)^{\nu} u(x, 0) \leq C(1)(1 + |\xi|)^l. \)

The inequalities (ii) and (iii) show that the forward Cauchy problem is not well posed which contradicts the assumption. Hence Proposition 1 is proved. \( \square \)

For a smooth function \( u \) (for instance a function in \( L^2 \) or \( \mathcal{S} \)) the Fourier transform \( \hat{u} \) with respect to \( x \) is defined by

\[
\hat{u}(\xi, t) = \int u(x, t) \exp(-2\pi i x \cdot \xi) \, dx.
\]

More precisely if \( u \) belongs to \( \mathcal{S}' \) then its Fourier image is denoted by \( \hat{u} \) and \( \hat{u} \) belongs to \( \mathcal{S}' \).

Let us now assume that the coefficients \( A_k \) and \( B \) of \( M \) are continuous functions of \( t \) in \([0, h]\) but do not depend on \( x \). Consider the system of ordinary differential equations

\[
\frac{d}{dt} \hat{u}(\xi, t) = \left( 2\pi i \sum_k A_k(t)\xi_k + B(t) \right) \hat{u}(\xi, t).
\]

If \( v^j_0 \) denotes the vector in \( \mathbb{R}^N \) whose \( j \)th component is 1 and the other components are 0, let \( v^j(\xi, t, t_0) \) be the fundamental system of solutions of the system (2.11) (defined in \([t_0, h]\)) with the initial conditions \( v^j(\xi, t_0, t_0) = v^j_0 \). Then we have the

**Proposition 2** (Petrowsky). Let the coefficients \( A_k \) and \( B \) of \( M \) be continuous functions of \( t \) in \([0, h]\). A necessary condition in order that the forward Cauchy problem for \( M \) be uniformly well posed in the spaces \( \mathcal{B} \) and \( \mathcal{D}_L^\infty \) is that there exist constant \( c \) and \( p \), both independent of \( t_0 \) in \([0, h]\), such that

\[
|v^j(\xi, t, t_0)| \leq c(1 + |\xi|)^p.
\]
Proof. Necessity in the space $\mathcal{B}$. Assume that the forward Cauchy problem is uniformly well posed in the space $\mathcal{B}$ but the condition (2.12) is not fulfilled. Then for any $p$, one can find $\xi^*$, $t^*$, $t_0^*$ and $k$ such that we have the inequality

$$|V_k^h(\xi^*, t^*, t_0^*)| \geq p(1 + |\xi^*|)^p.$$  

The function $u(x, t, t_0^*) = (u_1(x, t, t_0^*), \ldots, u_N(x, t, t_0^*))$ with

$$(2.13) \quad u(x, t, t_0^*) = \exp(\text{i}x \cdot \xi^*) \cdot v_k^h(\xi, t, t_0^*)$$

is a solution of $M[u] = 0$ and satisfies the inequalities

(i) $|u(x, t^*, t_0^*)| \geq p(1 + |\xi^*|)^p$ where $t_0^* \leq t^* \leq h$, and

(ii) $\sum_{|\nu| = 1} \left| \frac{\partial^{|\nu|}}{\partial x^{|\nu|}} u(x, t_0^*, t_0^*) \right| \leq c(l)(1 + |\xi^*|)^l,$

c(l) being a constant depending only on $l$ which again show that the forward Cauchy problem is not uniformly well posed, thus arriving at a contradiction to the assumption. □

Necessity in the space $\mathcal{D}^{\infty}_{L^2}$. Again assume that the forward Cauchy problem is uniformly well posed in $\mathcal{D}^{\infty}_{L^2}$ but the condition (2.12) does not hold. We can therefore assume that for any $p$, there exist $\xi^*$, $t^*$, $t_0^*$ and $k$ such that we have the inequality

$$|V_k^h(\xi, t^*, t_0^*)| \geq p(1 + |\xi|)^p, t^* \geq t_0^*.$$ 

holds for all $\xi$ in a neighbourhood $U$ of $\xi_0^*$ in $\mathbb{R}^n$. Let $f \in L^2$ with its support contained in $U$ and $\|f\| = 1$. Then the function $u(x, t, t_0^*) = (u_1(x, t, t_0^*), \ldots, u_N(x, t, t_0^*))$, with

$$(2.14) \quad u(x, t, t_0^*) = \int \exp(\text{i}x \cdot \xi) v_k^h(\xi, t, t_0^*) f(\xi) d\xi \text{ for } t \geq t_0^*,$$

is a solution of $M[u] = 0$. By Plancheral’s theorem we have

$$\|u\| = (2\pi)^{n/2} \left( \int |v_k^h(\xi, t, t_0^*)|^2 |f(\xi)|^2 d\xi \right)^{1/2}$$
2. Well-posedness and uniform-well posedness.....

\[ (2\pi)^{n/2} p (1 + \| \mathcal{E}^* \|)^p \]

where \( \| \mathcal{E}^* \| = \text{dist} (0, \text{supp} \, f) \). On the other hand again by applying Plancheral’s theorem we have, for any \( \ell \), that

\[
\sum_{|\ell| \leq \ell} \left\| \left( \frac{\partial}{\partial x} \right)^\nu u(x, t_0^+, t_0^+) \right\| = \sum_{|\ell| \leq \ell} \left( 2\pi \right)^{n/2} \left( \int |\xi^{\nu} v^k(\xi, t_0^+, t_0^+)|^2 |f(\xi)|^2 \, d\xi \right)^{1/2}
\]

(2.16)

where \( c(\ell) \) is a constant depending only on \( \ell \). The two inequalities (2.15), (2.16) together show that the forward Cauchy problem is not uniformly well posed leading to a contradiction to the assumption.

**Proposition 3 (Petrowsky).** Let the coefficients \( A_k \) and \( B \) of \( M \) be continuous functions of \( t \). Then the condition (2.12) is sufficient in order that the forward Cauchy problem be uniformly well posed in the spaces \( \mathcal{D}_{L^2}^\infty, \mathcal{B} \) and \( \mathcal{C} \).

**Proof.** **Sufficiency in the space** \( \mathcal{D}_{L^2}^\infty \). The inequality (2.12)

\[ |v^j(\xi, t, t_0)| \leq c (1 + |\xi|)^p \]

shows that there exists a \( \sigma \) such that \( (1 + |\xi|)^\sigma v^j(\xi, t, t_0) \in \mathcal{B}_{L^2}^0 \) and this depends continuously on \( (t, t_0) \). In fact, \( v^j(\xi, t, t_0) \) satisfies (2.11)

\[
\frac{d}{dt} v^j(\xi, t, t_0) = (i \wedge \xi + B) v^j(\xi, t, t_0) A_\xi = \sum A_k \xi_k
\]

consider

\[
V^j(\xi, t, t_0) - v^j(\xi, t_0, t_0) = \int_{t_0}^t (iA(s) \cdot \xi + B(s)) v^j(\xi, s, t_0) \, ds.
\]

This implies that \( (1 + |\xi|)^{-p-1} v^j(\xi, t, t_0) \) is continuous in \( (t, t_0) \) in the space \( \mathcal{B}_{L^2}^0 \). Hence the inverse Fourier image \( R^j(t, t_0) \) of \( V^j(\xi, t, t_0) \) with respect to \( \xi \) belongs to \( \mathcal{S}' \) and the operator \( R^j(t, t_0)_{(\xi)} \) has the following properties:
(1) for any $\varphi \in D_{L^2}^s \mathcal{L}_2 \varphi \in D_{L^2}^{s+\sigma} [t_0, h]$ and

(2) for any $f \in D_{L^2}^s [0, h]$, the integral

$$\int_{t_0}^t R^j_\delta (t, \tau) \ast_{(x)} f (x, \tau) d\tau$$

belongs to $D_{L^2}^{s+\sigma} [t_0, h]$. Further the linear mappings

$$\varphi \rightarrow R^j_\delta (t, t_0) \ast_{(x)} \varphi, \quad f \rightarrow \int_{t_0}^t R^j_\delta (t, \tau) \ast_{(x)} f (x, \tau) d\tau$$

are continuous. Now given $\psi = (\varphi_1, \ldots, \varphi_N)$ with $\varphi_j \in D_{L^2}^s$ and $f = (f_1, \ldots, f_N)$ with $f_j \in D_{L^2} [0, h]$ define $u(x, t, t_0) = (u_1(x, t, t_0), \ldots, u_N(x, t, t_0))$ by

$$u(x, t, t_0) = \sum_j R^j_\delta (t, t_0) \ast_{(x)} \varphi_j (x) + \int_{t_0}^t R^j_\delta (t, \tau) \ast_{(x)} f_j (x, \tau) d\tau.$$

Then $u(x, t, t_0)$ is a solution of $M[u] = f$ with the Cauchy data $u(x, t_0, t_0) = \psi (x)$. In view of (2.18) we conclude that the forward Cauchy problem is uniformly well posed in the space $D_{L^2}^\infty$.

**Sufficiency in the space $\mathcal{B}$.** We recall that $(v^j(\xi, t, t_0))$ is a fundamental system of solutions of the system (2.11)

$$\frac{d}{dt} v = (2\pi i \sum A_k(t) \xi_k + B(t)) V.$$

Hence each $v^j(\xi, t, t_0)$ is an entire function of exponential type for complex $\xi \in \mathbb{C}^n$. In fact, if $|v(J, t, t_0)|^2$ stands for $\sum v^j(\xi, t, t_0)^2$, we have since $A_k(t)$ and $B(t)$ are bounded

$$|2\pi i \sum A_k(t) \xi_k + B(t)| v(\xi, t, t_0) | \leq c (1 + |\xi|) |v(\xi, t, t_0)|$$

(2.19)
with a constant $c$ independent of $\zeta$ and $v$. Further

$$\frac{d}{dt} |v(\zeta, t, t_0)|^2 = \sum_j \left( \frac{dv^j}{dt}(\zeta, t, t_0)\overline{v^j(\zeta, t, t_0)} + v^j(\zeta, t, t_0) \frac{dv^j}{dt}(\zeta, t, t_0) \right)$$

$$\leq 2 \left| \frac{d}{dt}v(\zeta, t, t_0) \right| |v(\zeta, t, t_0)|$$

$$= 2 |\sum A_k(t) \zeta + B(t)|v(\zeta, t, t_0)||v(\zeta, t, t_0)|$$

$$\leq 2c' |v(\zeta, t, t_0)|^2 (1 + |\zeta|).$$

Hence $|v(\zeta, t, t_0)| \leq c'' e^{c' |\zeta| - t_0}$ and consequently for large $\zeta$, we have, for each $j = 1, \ldots, N$ the inequality

$$|v^j(\zeta, t, t_0)| \leq c_1 e^{c_2 |\zeta| - t_0}.$$

Hence by Paley-Wiener’s theorem $R^j_{\zeta}(t, t_0)$ is a distribution with compact support contained in $\{|x| < \sqrt{2} \}$ and depends continuously on $(t, t_0)$. By the structure of distribution with compact supports we can write

$$R^j_{\zeta}(t, t_0) = \sum_{|\nu| \leq s_j} \frac{\partial}{\partial x} \left[ g^j_{\nu}(x, t, t_0) \right] (j = 1, \ldots, N),$$

where $g^j_{\nu}(x, t, t_0) \in \mathcal{B}^0[0, t]$ with support contained in $\{|x| < \sqrt{2} \}$ and the derivatives are taken in the sense of distributions. This implies that

1. for any $\varphi \in \mathcal{B}$ we have $R^j_{\zeta}(t, t_0) \ast \varphi \in \mathcal{B}[0, h]$,

2. for any $f \in \mathcal{B}[0, h]$ the integral

$$\int_{t_0}^{t} R^j_{\zeta}(t, \tau) \ast \varphi(x, \tau) d\tau \in \mathcal{B}[0, h].$$

Further the linear maps

$$\varphi \rightarrow R^j_{\zeta}(t, t_0) \ast \varphi, f \rightarrow \int_{t_0}^{t} R^j_{\zeta}(t, \tau) \ast \varphi(x, \tau) d\tau$$
are continuous. Now the same argument as in the first part of the proposition shows that the Cauchy problem is uniformly well posed in the base $\mathcal{B}$.

**Sufficiency in the space $\mathcal{E}$**. In the above proof we observe that, since $R^j_{1}(t, t_0)$ is a distribution with compact support, we have

1. for any $\varphi \in \mathcal{E}^{}, R^j_{1}(t, t_0) \ast (\cdot) \varphi \in \mathcal{E}[t_0, h]$,

2. for any $f \in \mathcal{E}[O, h]$ the integral

$\int_{t_0}^{h} R^j_{1}(t, \tau) \ast (\cdot) f(x, \tau) d\tau$

belongs to $\mathcal{E}[t_0, h]$. Again the linear maps

$\varphi \rightarrow R^j_{1}(t, t_0) \ast (\cdot) \varphi, f \rightarrow \int_{t_0}^{h} R^j_{1}(t, \tau) \ast (\cdot) f(x, \tau) d\tau$

are continuous and an argument similar to the one used earlier shows that the forward Cauchy problem is uniformly well posed in the space $\mathcal{E}$.

This completes the proof of the proposition. \qed

### 3 Cauchy problem for a single equation of order $m$

By an argument similar to those used in the previous section we shall presently prove a necessary and sufficient condition in order that the forward Cauchy problem for a single equation of order $m$ be uniformly well posed in the space $\mathcal{E}$. Let

$$L \equiv \left( \frac{\partial}{\partial t} \right)^m + \sum_{|\nu| + j \leq m} a_{\nu,j}(t) \left( \frac{\partial}{\partial x} \right)^{\nu} \left( \frac{\partial}{\partial t} \right)^{j}$$

be a linear differential operators of order $m$ whose coefficients $a_{\nu,j}(t)$ are
(m − 1) times continuously differentiable functions of t in an interval [0, h]. By Fourier transforms in the x-space we are lead to the following ordinary differential equation of order m with (m − 1)-times continuously differentiable coefficients in t:

\[ \tilde{L}[V] = \left( \frac{d}{dt} \right)^m v(\xi, t) + \sum_{|\nu| + j \leq m} a_{\nu,j}(t)(i\xi)^\nu \left( \frac{d}{dt} \right)^j v(\xi, t) = 0. \]

Let \( v(\xi, t, t_0) \) be a solution of \( \tilde{L}[v] = 0 \) satisfying the initial conditions on \( (t = t_0) \).

\[ v(\xi, t_0, t_0) = 0, \ldots, (\frac{d}{dt})^{m-2} v(\xi, t_0, t_0) = 0, (\frac{d}{dt})^{m-1} v(\xi, t_0, t_0) = 1. \]

Then we have the

**Proposition 1.** If the coefficients \( a_{\nu,j} \) of \( L \) are \( m-1 \) times continuously differentiable functions of t in an interval [0, h] the forward Cauchy problem for \( L \) is uniformly well posed in the space \( \mathcal{E} \) if and only if there exist constants \( c \) and \( p \) both independent of \( t_0 \) such that

\[ |v(\xi, t, t_0)| \leq c(1 + |\xi|)^p. \]

**Proof.** Suppose the Cauchy problem for \( L \) is uniformly well posed for the future in the space \( \mathcal{E} \) but the condition (3.3) does not hold. Then for any given \( p > 0 \) there exist \( \xi^*, t_0^* \) and \( t, t \geq t_0^* \), such that we have the inequality

\[ |v(\xi^*, t, t_0^*)| \geq p(1 + |\xi^*|)^p. \]

Then The function \( u(x, t, t_0^*) = \exp(i x \cdot \xi^*) v(\xi^*, t, t_0^*) \) is a solution of \( Lu = 0 \) and has the properties.

- (i) \( u(x, t, t_0^*) \in \mathcal{E}[t_0^*, h] \) and once continuously differentiable in t with values in \( \mathcal{E}_x \),

- (ii) \( |u(x, t, t_0^*)| = |v(\xi^*, t, t_0^*)| \geq p(1 + |\xi^*|)^p \), and

- (iii) \( \sum_{|\nu| \leq 1} \left| \left( \frac{\partial}{\partial x} \right)^\nu u(x, t_0^*, t_0^*) \right| = \sum_{|\nu| \leq 1} \left| (i \xi^*)^\nu v(\xi^*, t_0^*, t_0^*) \right| \leq c(l)(1 + |\xi^*|)^l \)
The last two inequalities together show that forward Cauchy problem is not uniformly well posed in the space $E$ which contradiction the assumption.

Conversely, assume that the condition (3.3) is satisfied. The forward Cauchy problem is uniformly well posed in the space $E$. First of all we prove that the condition (3.3) implise that $v(\xi, t, t_0)$ and all its derivatives upto order $(m-1)$ with respect to $t$ are uniformly majorized in $[t_0, h]$ by polynomials in $\xi$. For this purpose we rewrite the equation $\tilde{L}[V] = 0$ in the form

$$
(3.4) \quad \left(\frac{d}{dt}\right)^m v(\xi, t, t_0) + \sum_{j=0}^{m-1} a_j(t, \xi) \left(\frac{d}{dt}\right)^j v(\xi, t, t_0) = 0
$$

where $a_j(t, \xi) = \sum_{|\nu|=m-j} a_{\nu,j}(t)(i\xi)^\nu$ for $j = 0, 1, \ldots, (m-1)$ $a_j(t, \xi)$ are hence polynomials of degree at most $(m-j)$ in $\xi$ with coefficients which are $(m-1)$-times continuously differentiable functions of $t$ in the interval $[0, h]$. Hence we may assume that there exists a constant $c$ such that

$$
(3.5) \quad |a_j(t, \xi)| \leq c(1 + |\xi|)^{m-j}, \quad j = 0, 1, \ldots, (m-1) \quad \text{for} \quad t \in [0, h]
$$

Integrating (3.4) once with respect to $t$ over the interval $[t_0, h]$ we obtain, after using the initial conditions at $t = t_0$,

$$
\left(\frac{d}{dt}\right)^{m-1} v(\xi, t, t_0) - 1 = - \sum_{j=0}^{m-1} \int_{t_0}^t a_j(\tau, \xi) \left(\frac{d}{d\tau}\right)^j v(\xi, \tau, t_0) d\tau.
$$

Integrating by parts the terms in the right hand side in view of the initial conditions satisfied by $v(\xi, t, t_0)$ we obtain

$$
\left(\frac{d}{dt}\right)^{m-1} v(\xi, t, t_0) - 1 = - \sum_{j=0}^{m-1} \left\{ \sum_{p=0}^{j-1} (-1)^p \left(\frac{d}{dt}\right)^p (a_j(t, \xi)) \left(\frac{d}{dt}\right)^{j-1-p} v(\xi, t, t_0) \right. \right.
$$

$$
\left. \left. + (-1)^j \int_{t_0}^t \left(\frac{d}{d\tau}\right)^j (a_j(\tau, \xi)) v(\xi, \tau, t_0) d\tau \right\}.
$$
3. Cauchy problem for a single equation of order $m$

By successive integration with respect to $t$ over the interval $[t_0, h]$ $(m-1)$-times, using the initial conditions and the inequality (3.5) we show that $\frac{d}{dt}v(\xi, t, t_0), \ldots, \left(\frac{d}{dt}\right)^{m-1}v(\xi, t, t_0)$ are all majorized by polynomials of the form $c_j(1+|\xi|)^p (j = 1, 2, \ldots, m)$, $C_j, p_j$ being independent of $t_0$.

Thus it follows that there exist $\sigma_0, \ldots, \sigma_m$ such that $(1+|\xi|)^\sigma_j \left(\frac{d}{dt}\right)^j v(\xi, t, t_0) \in B_0([t_0, h])$ for $j = 0, 1, \ldots, (m-1)$. Let $R_j^i(t, t_0)$ denote the inverse Fourier image of $\left(\frac{d}{dt}\right)^j v(\xi, t, t_0)$ in the $\xi$-space.

We shall show that each $R_j^i(t, t_0)$ has compact support in the $x$-space.

In view of the theorem of Paley-Wiener we have only to show that each $\left(\frac{d}{dt}\right)^j v(\zeta, t, t_0)$ are of exponential type for complex $\zeta \in \mathbb{C}^n$.

Denoting $(1+|\zeta|)$ for $\zeta \in \mathbb{C}^n$ by $K$ we have $|a_j(t, \zeta)| \leq cK^{m-j}$ for all $j = 0, 1, \ldots, m-1$. The equation (3.4) can now be written in the form

$$\left(\frac{d}{dt}\right)^m v(\zeta, t, t_0) + a_{m-1}(t, \zeta) \left(\frac{d}{dt}\right)^{m-1} v(\zeta, t, t_0) + \frac{a_{m-2}}{K} \left(\frac{d}{dt}\right)^{m-2} v(\zeta, t, t_0) + \ldots + \frac{a_0(t, \zeta)}{K^{m-1}} K^{m-1} v(\zeta, t, t_0) = 0$$

Taking for the new set of function $w = (w_0, w_1, \ldots, w_{m-1})$ where

$$w_0(\zeta, t, t_0) = K^{m-1} v(\zeta, t, t_0),$$
$$w_1(\zeta, t, t_0) = K^{m-2} \frac{dv}{dt}(\zeta, t, t_0),$$
$$w_{m-2}(\zeta, t, t_0) = K \left(\frac{d}{dt}\right)^{m-2} v(\zeta, t, t_0),$$
$$w_{m-1}(\zeta, t, t_0) = \left(\frac{d}{dt}\right)^{m-1} v(\zeta, t, t_0).$$

the above equation can be written as a system of ordinary differential
equations in the following way:

\[
\frac{d}{dt} \begin{pmatrix}
  w_0 \\
  w_1 \\
  \vdots \\
  w_{m-1}
\end{pmatrix} = K \begin{pmatrix}
  0 & 1 & 0 & \ldots & 0 & 0 \\
  0 & 0 & 1 & \ldots & \cdot & \cdot \\
  \cdot & \cdot & \cdot & \ddots & \cdot & \cdot \\
  -\frac{a_0}{K} & -\frac{a_1}{K^{m-1}} & -\frac{a_2}{K^{m-2}} & \ldots & -\frac{a_{m-2}}{K^2} & -\frac{a_{m-1}}{K}
\end{pmatrix} \begin{pmatrix}
  w_0 \\
  w_1 \\
  \vdots \\
  w_{m-1}
\end{pmatrix}
\]

Denote the matrix of the system (3.6) by \( H(t, \zeta) \). Since \(|a_j(t, \zeta)| \leq cK^{m-j} \) the elements of the matrix \( H(t, \zeta) \) are bounded in modulus by a constant \( C_1 \) independent of \( \zeta \) in \( C^m \) and hence \( H(t, \zeta) \) as a linear transformation in an \( m \)-dimensional vector space is bounded in norm by a constant \( C_2 \) which depends only on \( m \) but not on \( \zeta \) in \( C^m \). Denoting by \(|w(\zeta, t, t_0)|^2\) the sum \( \sum_j |w_j(\zeta, t, t_0)|^2 \) and by \( w(\zeta, t, t_0) \cdot w'(\zeta, t, t_0) \) the sum \( \sum_j w_j(\zeta, t, t_0) \cdot w_j'(\zeta, t, t_0) \) we have

\[
\frac{d}{dt} |w(\zeta, t, t_0)|^2 = \frac{d}{dt} w(\zeta, t, t_0) \cdot w(\zeta, t, t_0) + w(\zeta, t, t_0) \frac{d}{dt} w(\zeta, t, t_0) = K(H(t, \zeta) + \overline{H(t, \zeta)})|w(\zeta, t, t_0)|^2
\]

on account of the system of equation (3.6) satisfied by \( w(\zeta, t, t_0) \). Hence

\[
\left( \frac{d}{dt} \right) |w(\zeta, t, t_0)|^2 \leq 2C_2K|w(\zeta, t, t_0)|^2
\]

which, by integration with respect to \( t \) over the interval \([t_0, t]\) implies that

\[
|w(\zeta, t, t_0)|^2 \leq \exp(2C_2K|t - t_0|) = \exp 2C_2(1 + |\zeta|)|t - t_0|
\]

since \(|w(\zeta, t, t_0)| = 1 \) consequently we have, since \( k \geq 1 \),

\[
\left| \frac{d}{dt} ^j \nu(\zeta, t, t_0) \right| \leq \exp[C_2(1 + |\zeta|)|t - t_0|].
\]

Hence, by the theorem of Paley-Wiener it follows that \( R_j^0(t, t_0) \) are distributions with compact support in the \( x \)-space and depend continuously on \((t, t_0)\).
Let $\Psi = (\varphi_0, \ldots, \varphi_{m-1})$ with $\varphi_j \in \mathcal{E}_x$ and $f \in \mathcal{E}[0, h]$ be given.

The above argument can be modified a little in order to get convolution operators $\tilde{R}_j^j(x, t_0)$ similar to $R_j^j(x, t_0)$. This we do as follows:

Let $v_j(\xi, t, t_0)$ be the solution of $\tilde{L}[v_j] = 0$ with the initial values given by

$$\left. \frac{\partial^j}{\partial t^j} v_j(\xi, t, t_0) \right|_{t=t_0} = \delta_j^j.$$ 

($\delta_j^j$ are Kronecker's symbols). We see that $v_j(\xi, t, t_0)$ is connected with the solution $v(\xi, t, t_0)$ in the following way.

Let $w_j(\xi, t, t_0) = v_j(\xi, t, t_0) - \frac{(t-t_0)^j}{j!}$, $t \geq t_0$. Then $w_j$ vanishes at $t = t_0$ together with derivatives up to order $(m - 1)$. Now $w_j$ satisfies the equation

$$\tilde{L} \left[ w_j + \frac{(t-t_0)^j}{j!} \right] = 0 \text{ or } \tilde{L}[w_j] = -\frac{1}{j!} \tilde{L}[(t-t_0)^j] = \mu_j(\xi, t, t_0).$$

$\mu_j(\xi, t, t_0)$ are obviously polynomials in $\xi$ and we have

$$|\mu_j(\xi, t, t_0)| \leq c_3 (1 + |\xi|)^m \text{ for } 0 \leq t_0 \leq t \leq h,$$

here $c_3$ is a constant. Hence

$$w_j(\xi, t, t_0) = \int_{t_0}^t v(\xi, \tau, t) \mu_j(\xi, \tau, t_0) d\tau.$$

This implies that

$$|w_j(\xi, t, t_0)| \leq \int_{t_0}^t |v(\xi, \tau, t)||\mu_j(\xi, \tau, t_0)| d\tau \leq c_3 (t-t_0)(1 + |\xi|)^m \exp[c_4(1 + |\xi|)(t-t_0)].$$

Hence the inverse Fourier image $\tilde{R}_j^j(t, t_0)$ of $v_j(\xi, t, t_0) = w_j(\xi, t, t_0) + \frac{(t-t_0)^j}{j!}$ has its support in $|x| \leq c'_4(t-t_0)$.
Then the function

\[ u(x, t, t_0) = \sum_{j=0}^{m-1} \tilde{R}_j(t, t_0) * \varphi_j + \int_{t_0}^{t} R_x(t, \tau) * f(x, \tau) d\tau \]

is a solution of \( L[u] = f \) with Cauchy data \( \Psi \) on \( t = t_0 \). (Here \( R_x(t, t_0) \) stand for the inverse Fourier image of \( v(\xi, t, t_0) \).) The linear mappings

\[ \varphi_j \rightarrow R_j(t, t_0) * \varphi_j, \quad f \rightarrow \int_{t_0}^{t} R_x(t, \tau) * f(s, \tau) d\tau \]

being continuous the forward Cauchy problem is uniformly well posed in the space \( \mathcal{E} \). This completes the proof of the proposition. \( \square \)

4

**Proposition 1.** Let the coefficients \( A_k \) and \( B \) of a first order system of differential operators \( M \) be continuous functions of \( t \) in an interval \([0, h]\). If the forward Cauchy problem is well posed in the space \( \mathcal{E} \) then it is uniformly well posed in \( \mathcal{E} \).

**Proof.** In view of Prop. of §2 it is sufficient to prove that if \( v^j(\xi, t, t_0) \) is the fundamental system of solutions of the system of ordinary differential equations

\[ \frac{d}{dt}v(\xi, t, t_0) = (iA(t)\xi + B(t))v(\xi, t, t_0), \quad A(t) \cdot \xi = \sum A_k(t)\xi_k \]

with intial conditions \( v^j(\xi, t_0, t_0) = v^j_0 \) then \( v^j(\xi, t, t_0) \) are majorized by polynomials in \( |\xi| \). (We recall that \( v^j_0 \) denotes the vector in \( \mathbb{R}^N \) having 1 for the \( j \)th component and 0 for the others). If \( \xi^0 = \frac{\xi}{|\xi|} \) we can write the above system as

\[ \frac{d}{dt}v(\xi, t, t_0) = (i|\xi|A(t) \cdot \xi^0 + B(t))v(\xi, t, t_0). \]
The element $a_{kl}(t,\xi^0)$ of the matrix $A(t) \cdot \xi^0$ are homogeneous functions of $\xi^0$ of degree one having for coefficients continuous functions of $t$ in $[0,h]$. We remark that $v^j(\xi,t,0)$ define the columns of the Wronskian $W(t,\xi)$ of the above system of differential equations. From the theory of linear ordinary differential equations we know that

\begin{equation}
4.2 \quad w(t,\xi) = W(0,\xi) \exp \left\{ i\xi \sum_j \int_0^t a_{jj}(\tau,\xi^0) d\tau + \sum_j \int_0^t b_{jj}(\tau) d\tau \right\}.
\end{equation}

The forward Cauchy problem being well posed we can assume that $\sum \int_0^t a_{jj}(\tau,\xi^0) d\tau$ is real for every $(t,\xi^0)$, $\xi^0$ real. For otherwise we may assume, if necessary by changing $\xi^0$ to $-\xi^0$ that

$$\text{Re} \left( \sum_j \int_0^t a_{jj}(\tau,\xi^0) d\tau \right) > 0.$$ 

By the assumption of the well posedness of the forward Cauchy problem it follows that

\begin{equation}
4.3 \quad |v^j(\xi,t,0)| \leq c(1 + |\xi|)^p
\end{equation}

for suitable constants $c$ and $p$, and so $W(t,\xi)$ is majorized by a polynomial in $|\xi|$. On the other hand, as $\rho \to +\infty$,

$$|w(t,\xi)| \sim |W(0,\xi)| \exp \left\{ \rho|\xi^0| \sum_j \text{Re} \left( \int_0^t a_{jj}(\tau,\xi^0) d\tau \right) \right\}, \quad \xi = \rho\xi^0.$$ 

Thus $W(t,\xi)$ tends to $+\infty$ exponentially as $\rho \to +\infty$ contradicting the inequality (4.3). Hence it follows that $\sum \int_0^t a_{jj}(\tau,\xi^0) d\tau$ is real for every $(t,\xi^0)$ with real $\xi^0$. We now have

$$|W(t,\xi)| = |W(0,\xi)| \exp \left\{ \sum_j \text{Re} \left( \int_0^t b_{jj}(\tau) d\tau \right) \right\}.$$
and hence

\[ |W(t, \xi)| \geq |W(0, \xi)| \exp \left\{ - \sum_j \int_0^t |b_{jj}(\tau)| d\Gamma \right\} \geq \delta > 0 \quad \text{for all } (t, \xi). \]

\( \xi \) real. Further we observe that, as \( v^i(\xi, t, 0) \) form a basis for the solutions of the system of ordinary differential equations

\[ v^i(\xi, t, t_0) = \sum c^i_j(\xi)v^j(\xi, t, 0). \]

Putting \( t = t_0 \) solving for \( c^i_j(\xi) \) we see that, since \( \det(v^i(\xi, t_0, 0)) \) is the Wronskian \( W(t_0, \xi) \) which is minorized by a polynomial in \( |\xi| \) and since \( v^i(\xi, t, 0) \) are majorized by polynomials in \( |\xi| \), \( c^i_j(\xi) \) are themselves majorized by polynomials. Hence \( v^i(\xi, t, t_0) \) are majorized by polynomials in \( |\xi| \) independently of \( t \) and \( t_0 \) which implies that the forward Cauchy problem is uniformly well posed for \( M \). Hence proposition \( \Box \) is proved.

Correspondingly we have the following result for a single differential equation of order \( m \). Let

\( (4.4) \)

\[ L \equiv \left( \frac{\partial}{\partial t} \right)^m + \sum_{|\nu|+j \leq m \atop j \leq m-1} a_{\nu,j}(t) \left( \frac{\partial}{\partial x} \right)^\nu \left( \frac{\partial}{\partial t} \right)^j \]

be a linear differential operator of order \( m \) with the coefficients depending only on \( t \) in the interval \([0, h]\).

**Proposition 2.** Let the coefficients \( a_{\nu,j} \) of \( L \) be \( (m-1) \) times continuously differentiable functions of \( t \) in an interval \([0, h]\). If the forward Cauchy problem for \( L \) is well posed then it is uniformly well posed for the future for \( L \).

**Proof.** Writing the operator \( L \) in the form

\( (4.5) \)

\[ \left( \frac{\partial}{\partial t} \right)^m + \sum_{j=0}^{m-1} a_j \left( t, \frac{\partial}{\partial x} \right)^j \left( \frac{\partial}{\partial t} \right)^i \]
where \( a_j(t, \xi) = \sum_{|\nu| = m-j} a_{\nu,j}(i\xi)^\nu \) (\( j = 0, 1, \ldots, m - 1 \)), we are lead to the following ordinary differential equation of order \( m \):

\[
(4.6) \quad \left( \frac{d}{dt} \right)^m v(\xi, t) + \sum_{j=0}^{m-1} a_j(t, \xi) \left( \frac{d}{dt} \right)^j v(\xi, t) = 0.
\]

Denoting the Wronskian of the equation (4.6) by \( w(t, \xi) \) we have from the theory of ordinary differential equations

\[
(4.7) \quad W(t, \xi) = W(0, \xi) \exp \left\{ -\int_0^t a_{m-1}(\tau, \xi) d\tau \right\}.
\]

Write \( a_{m-1}(\tau, \xi) = a_{m-1}^{(1)}(\tau, \xi) + b(\tau) \) where \( a_{m-1}^{(1)}(\tau, \xi) \) is homogeneous in \( \xi \) of degree one with coefficients continuous functions of \( t \) in \([0, h]\). Then

\[
\begin{align*}
W(t, \xi) &= W(0, \xi) \exp \left\{ -|\xi| \int_0^t a_{m-1}^{(1)}(\tau, \xi^0) d\tau - \int_0^t b(\tau) d\tau \right\}.
\end{align*}
\]

Now arguing as in the proof of the proposition one can show that the Cauchy problem is uniformly well posed using again the prop. 36 of §2. Finally we shall show that for first order systems with constant coefficients the condition of Hadamard implies the condition of Petrowsky. This will prove that for first order systems with constant coefficients these two conditions are equivalent. For this we need the

**Lemma 1** (Petrowsky). Let a system of differential equations with constant coefficients

\[
(4.8) \quad \frac{d}{dt} v(t) = Av(t)
\]
where $A = (a_{jk})$ and $v(t) = \begin{pmatrix} v_1(t) \\ \vdots \\ v_N(t) \end{pmatrix}$ with $|a_{jk}| \leq K$ be given. Then, given any positive number $\varepsilon$ such that $\varepsilon \leq (N - 1)!2^NK$ we can find a non-singular matrix $C$ such that

\[(4.9) \quad CA = DC \quad \text{where} \quad D = \begin{pmatrix} a_{11}^* & 0 \\ a_{22}^* & \ddots \\ \vdots & \ddots & \ddots \\ a_{jk}^* & \cdots & a_{nn}^* \end{pmatrix}\]

where all $a_{jk}^*$, $k < j$ satisfy $|a_{jk}^*| < \varepsilon$. Moreover

\[(4.10) \quad |\det C| = \left[ \frac{(N - 1)!2^NK}{\varepsilon} \right]^{(N-1)}
\]

and the elements $c_{jk}$ of $C$ satisfy

\[(4.10') \quad |c_{jk}| \leq \left[ \frac{(N - 1)!2^NK}{\varepsilon} \right]^{(N-1)}.
\]

For a proof see Petrowsky [2].

**Proposition 3.** Let the coefficients $A_k$ and $B$ of $M$ be constants. Then the condition 9 of § 2 of Hadamard implies the condition 12 of § 2.

**Proof.** Consider the system of ordinary differential equations

\[(4.11) \quad \frac{d}{dt}v(\xi, t) = (iA_\xi + B)v(\xi, t).
\]

Let us fix $\xi^0$. Taking $(iA_\xi + B)$ as the given matrix in the lemma [1] there exist constants $c_0$, $c_1$ such that

\[(4.12) \quad |a_{jk}(\xi^0) + b_{jk}| < c_0|\xi^0| + c_1
\]

($c_0$, $c_1$ are independent of $\xi^0$). We take $K = c_0|\xi^0| + c_1$ and $\xi = (N - 1)!2^NK = (N - 1)!2^N(c_0|\xi^0| + c_1)$. Then, by the lemma [1] we can find
5. Hyperbolic and strongly hyperbolic systems

A matrix $C(\xi^0)$ such that $|\det C(\xi^0)| = 1$ and its elements $c_{jk}(\xi^0)$ satisfy $|c_{jk}(\xi^0)| \leq 1$. So denoting $c(\xi^0)v$ by $w$ we have

\vspace{1em}
(4.13) \quad \frac{d}{dt} w(\xi^0, t) = \begin{pmatrix} \lambda_1(\xi^0) & \lambda_2(\xi^0) & \cdots & \lambda_N(\xi^0) \\ a_{1k}(\xi^0) & \lambda_1(\xi^0) & \cdots & \lambda_{N-1}(\xi^0) \\ \vdots & & \ddots & \vdots \\ a_{Nk}(\xi^0) & \cdots & \cdots & \lambda_1(\xi^0) \end{pmatrix} w(\xi^0, t)

where $\lambda_1(\xi^0), \ldots, \lambda_N(\xi^0)$ are the roots of the equation

\vspace{1em}
(4.14) \quad \det(\lambda I - iA_0 - B) = 0

and $|a_{jk}(\xi)| \leq (N - 1)!2^N(c_0|\xi^0| + c_1)$. By Hadamard’s condition we have

\vspace{1em}
$$\Re \lambda_j(\xi^0) < p\log(1 + |\xi^0| + logc).$$

Now since $w(\xi^0, t, t_0)$ is a solution of the above system it follows that

\vspace{1em}
$$|w(\xi^0, t, t_0)| \leq c'(1 + |\xi^0|)^{p_0 h} \text{ for } 0 \leq t_0 \leq t \leq h$$

with the constants $c'$, $p_0$ independent of $t, t_0, \xi^0$. Finally since $v(\xi^0, t, t_0) = c(\xi^0)^{-1}w(\xi^0, t, t_0)$ we have desired property. \hfill \Box

5 Hyperbolic and strongly hyperbolic systems

The notion of well posedness of the Cauchy problem is closely related to the nature of the given system of differential equations. In this section we introduce hyperbolic and strongly hyperbolic systems of differential equations. We give criteria, in order that a given system of differential operators be of this type, in terms of the characteristic roots of the system.

$A_k \equiv A_k(x, t)$, $B \equiv B(x, t)$ will be matrices of order $N$ of functions on $\mathbb{R}^n \times [0, h]$ the regularity conditions of which will be prescribed later in each case. Consider the first order system of differential operators

\vspace{1em}
(5.1) \quad M \equiv \frac{\partial}{\partial t} - \sum_k A_k(x, t) \frac{\partial}{\partial x_k}$
Definition. A system of differential operators \( M \) is said to be hyperbolic if the forward and backward Cauchy problems are well posed.

Definition. A first order system of differential operators \( M \) is said to be strongly hyperbolic if for any choice of the matrix \( B(x, t) \) the Cauchy problem (forward as well as backward) is well posed for the system

\[
\frac{\partial}{\partial t} - \sum_k A_k(x, t) \frac{\partial}{\partial x_k} - B(x, t)
\]

Let \( \lambda_1(x, \xi, t), \ldots, \lambda_N(x, \xi, t) \) be the roots of the equation

\[
\det(\lambda I - A(x, t) \cdot \xi) = 0
\]

where \( A(x, t) \cdot \xi \) denotes the matrix \( \sum_k A_k(x, t) \cdot \xi_k \).

Proposition 1. If the coefficient matrices \( A_k \) of \( M \) are constant matrices then a necessary condition in order that \( M \) be strongly hyperbolic is that

1. \( \lambda_j(\xi) \) is real for all real \( \xi \neq 0 \) \((j = 1, \ldots, N)\)
2. the matrix \( A \cdot \xi \) is diagonalizable for all \( \xi \).

We shall actually prove a slightly stronger result: If one of the \( \lambda_j(\xi) \) is not real for some real \( \xi \neq 0 \), then for any choice of \( B \) (a constant matrix) the Cauchy problem for

\[
\frac{\partial}{\partial t} - \sum_k A_k \frac{\partial}{\partial x_k} - B
\]

is not well posed.

Proof. If the condition (1) is not satisfied for same real \( \xi^* \neq 0 \), there exists a root, say \( \lambda_1(\xi^*) \), with non vanishing imaginary part of the equation \( \det(\lambda I - A \cdot \xi) = 0 \). For \( \xi = \tau \xi^* \), \( \lambda = \tau \lambda' \) we can write

\[
\det(\lambda I - i A \cdot \xi - B) = \tau^N \det(\lambda' I - i A \cdot \xi^* - B) \left( \frac{B}{\tau} \right)
\]
5. Hyperbolic and strongly hyperbolic systems

for any matrix $B$. Denoting $\det(\lambda' I - iA \cdot \xi^*)$ by $P(\lambda')$ we have

\begin{equation}
\det(\lambda I - iA \cdot \xi - B) = \tau^N \left\{ P(\lambda') + \frac{1}{\tau} Q(\lambda', \tau) \right\}
\end{equation}

where $Q(\lambda', \tau)$ is a polynomial in $\lambda'$ of degree at most $N - 1$ having for coefficients polynomials in $\tau^{-1}$. Since $\lambda_1(\xi^*)$ is not real we may, without loss of generality, assume that $\text{Im} \lambda_1(\xi^*) < 0$ (if necessary after changing $\xi^*$ by $-\xi^*$ in the equation). Then $i\lambda_1(\xi^*)$ is a root of $P(\lambda') = 0$.

By continuity of the roots there exists a root of $P(\lambda') + \frac{1}{\tau} Q(\lambda', \tau) = 0$ in a neighbourhood of $i\lambda_1(\xi^*)$ in the complex plane. More precisely there exists a root $\lambda'_1(\tau)$ for large $\tau$ of the equation $P(\lambda') + \frac{1}{\tau} Q(\lambda', \tau) = 0$ such that $\lambda'_1(\tau) = i\lambda_1(\xi^*) + \frac{1}{\tau} \in (\frac{1}{\tau})$ where $\in (\tau) \to 0$ as $\tau \to +\infty$. Hence $\Re \lambda'_1(\tau) \geq \frac{1}{2}(-\text{Im} \lambda_1(\xi^*))$ for large $\tau$. In other words there exists a root $\lambda_1(\tau)$ of the equation

\begin{equation}
\det(\lambda I - iA \cdot \xi - B) = 0
\end{equation}

such that $\Re \lambda_1(\tau) \leq c\tau$ (with a positive constant $c$), which tends to $+\infty$ as $\tau \to \infty$. Hence the forward Cauchy problem is not well posed for the system $M - B$ by prop. 2 of §2.

(2) Assume again that the system $M$ is strongly hyperbolic, but that for a certain $\xi^*$ the matrix $A \cdot \xi^*$ is not diagonalizable. There exists a non-singular matrix $N_0$ such that $N_0(A \cdot \xi^*)N_0^{-1}$ has the Jordan canonical form

\begin{equation}
\begin{pmatrix}
\lambda_1 & 0 & \cdots & 0 \\
1 & \lambda_1 & \cdots & 0 \\
& & \ddots & \ddots \\
& & & \lambda_1
\end{pmatrix}
\end{equation}

Consider for $B$ a matrix determined by

\[
N_0BN_0^{-1} = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
& \cdots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}
\]
We shall show that the Cauchy problem is not well posed for the system of differential operators

$$\frac{\partial}{\partial t} - \sum A_k \frac{\partial}{\partial x_k} - B.$$ 

Consider the characteristic equation of this system, namely

$$\det(\lambda I - iA\xi - B) = 0.$$ 

Taking for $\xi$ the vector $\tau\xi^\ast$ ($\tau$ a real parameter $\rightarrow \infty$) this equation becomes

$$\det(\lambda I - i\tau A\xi^\ast - B) = \det((\lambda I - i\tau N_0(A\xi^\ast)N^{-1}_0 - N_0BN^{-1}_0)$$

$$= \begin{vmatrix} \lambda - i\tau\lambda_1 & -1 & 0 & \ldots & 0 \\
-\tau\lambda_1 & \lambda - i\tau\lambda_1 & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

Hence $(\lambda - i\tau\lambda_1)^2 - i\tau = 0$, the roots of which are $\lambda(\tau) = i\tau\lambda_1 \pm \sqrt{i\tau}$ whose real part $\Re \lambda(\tau) \rightarrow \infty$ along with $\tau$. Hence the Cauchy problem for the system $M - B$ is not well posed by prop 2 of §2, which contradicts the assumption.  \(\square\)

**Proposition 2.** A sufficient condition in order that the system $M$ be strongly hyperbolic is that one of the following two conditions is satisfied:

(i) the characteristic roots $\lambda_i(\xi)$ are real and distinct for all real $\xi \neq 0$;

(ii) $A_k$ are Hermitian.

*Proof.* Supposing the condition (i) is satisfied. We shall show that this implies that the Cauchy problem is well posed for the system $M - B$ for any choice of $B$. Consider the equation $\det(\lambda I - iA\xi - B) = 0$. Denoting the projection $\xi_i$ of $\xi$ on the unit sphere by $\xi^0$ and $\frac{\lambda}{|\xi|}$ by $\lambda'(\xi)$ we can write this equation in the form

$$\det(\lambda' I - iA\xi^0 - \frac{B}{|\xi|}) = 0.$$
If \( \lambda_1(\xi^0), \ldots, \lambda_N(\xi^0) \) are the roots of the equation \((\det \lambda I - A_\xi^0) = 0\) we can write

\[
(5.6) \quad \det(\lambda' I - iA_\xi^0 - \frac{B}{|\xi|}) = \prod_{j=1}^{N}(\lambda' - i\lambda_j(\xi^0)) + \frac{Q(\lambda', \xi^0)}{|\xi|} = 0,
\]

where \(Q(\lambda', \xi)\) is a polynomial \(a_0(|\xi|)\lambda' N - 1 + \cdots + a_{N-1}(|\xi|)\) with coefficients bounded for \(|\xi| \geq 1\). If \(\Omega_0\) is the projection of \(\Omega\) on the unit sphere we have

\[
\inf_{\xi \in \Omega_0} |\lambda_j(\xi^0) - \lambda_k(\xi^0)| \geq d > 0
\]

since \(\lambda_1(\xi^0) \ldots \lambda_N(\xi^0)\) are all distinct.

Let \(K = \sup_{\xi \in \Omega_0} |\lambda_j(\xi)|\) and \(m = \sup_{|\xi| \geq 1} |Q(\lambda', \xi)|\),

be circles in the complex plane of radius \(C(d/2)^{N-1} \geq 2m\) and \(\Gamma_1, \ldots, \Gamma_N\) respectively. On \(\Gamma_K\) we have

\[
\left| \prod_{j}(\lambda' - i\lambda_j(\xi^0)) \right| \geq \frac{C}{|\xi|} \left(\frac{d}{2}\right)^{N-1} \geq \frac{2m}{|\xi|} \quad \text{and} \quad \frac{|Q(\lambda', \xi)|}{|\xi|} \leq \frac{m}{|\xi|}.
\]

Hence by Rouche’s theorem there exists a unique root of

\[
\prod_{j}(\lambda' - i\lambda_j(\xi^0)) + \frac{Q(\lambda', \xi)}{|\xi|} = 0
\]

in the disc enclosed by \(\Gamma_K\). More precisely there exists a root \(\lambda_j'(\xi)\) of \(\det(\lambda' I - iA_\xi^0 - \frac{B}{|\xi|}) = 0\) such that

\[
|\lambda_j'(\xi) - i\lambda_j(\xi^0)| < \frac{C}{|\xi|}.
\]
or, what is the same, there exists a root $\tilde{\lambda}_j(\xi)$ of
\[
\det(\lambda I - iA_\xi - B) = 0
\]
such that $|\tilde{\lambda}_j(\xi) - i\lambda_j(\xi)| < C$. Since $\lambda_j(\xi)$ are real it therefore follows that
\[
\Re \tilde{\lambda}_j \xi \leq C \quad (j = 1, \ldots, N)
\]
and by prop. 1 of §2 the forward Cauchy problem is well posed for the system $M - B$. This proves that $M$ is strongly hyperbolic.

Next let us assume that the matrices $A_k$ are Hermitian. By Fourier transforms in the $x$-space we obtain the first order system of ordinary differential equations.

Now consider
\[
\frac{d}{dt}|v(\xi, t)|^2 = \frac{d}{dt} v(\xi, t) \cdot v(\xi, t) + v(\xi, t) \overline{\frac{d}{dt} v(\xi, t)} = (iA_\xi + B)v(\xi, t) \cdot v(\xi, t) + v(\xi, t) \overline{(iA_\xi + B)v(\xi, t)}
\]
Since the $A_k$ are Hermitian, we obtain, $B$ being bounded,
\[
\frac{d}{dt}|v(\xi, t)|^2 = 2 \Re Bv(\xi, t) \cdot v(\xi, t) \leq 2c|v(\xi, t)|^2.
\]
We obtain therefore
\[
(5.7) \quad |v(\xi, t)|^2 \leq |v(\xi, 0)|^2 e^{2ct}.
\]
which shows that the forward Cauchy problem is well posed for the system $M - B$ and so $M$ is strongly hyperbolic. This completes the proof of the proposition. Let us now remark the following fact:
\[
\frac{d}{dt}|v(\xi, t)|^2 = \frac{d}{dt} \langle v(\xi, t), v(\xi, t) \rangle = \langle \frac{d}{dt} v(\xi, t), v(\xi, t) + v(\xi, t), \frac{d}{dt} v(\xi, t) \rangle
\]
\[
= \langle (iA_\xi + B)v(\xi, t), v(\xi, t) \rangle + \langle v(\xi, t), (iA_\xi + B)v(\xi, t) \rangle.
\]
Since $A_k$ are Hermitian we obtain
\[ \frac{d}{dt} ||v(\xi, t)||^2 = 2 \text{Re}(Bv(\xi, t), \overline{v(\xi, t)}) \leq 2c ||v(\xi, t)||^2 \]
with a constant $c$ independent of $\xi$. Integrating both sides of the inequality over $[0, t]$ we obtain
\[ ||v(\xi, t)||^2 \leq ||v(\xi, 0)||^2 e^{2ct}. \]
Hence
\[ (5.8) \quad ||u|| \leq ||u(x, 0)|| e^{ct}. \]

We remark that the notions of hyperbolicity and strong hyperbolicity can be analogously defined for a single differential operator of order $m$. Consider a differential operator of order $m$

\[ (5.9) \quad L = \left( \frac{\partial}{\partial t} \right)^m + \sum_{|\nu|+|j|=m \atop j \leq m-1} a_{\nu,j}(x, t) \left( \frac{\partial}{\partial t} \right)^{\nu} \left( \frac{\partial}{\partial t} \right)^{j}. \]

$L$ is said to be hyperbolic if the Cauchy problem (both the forward and the backward) is well posed for $L$. It is said to be strongly hyperbolic if the Cauchy problem (both the forward and the backward) is well posed for $L - B$ for any choice of the lower order operator $B$. Let
\[ (5.10) \quad P(\lambda, \xi) = \lambda^m + \sum_{|\nu|+|j|=m \atop j \leq m-1} a_{\nu,j}(x, t) \xi^\mu \lambda^j. \]

Proposition 3. A necessary and sufficient condition in order that a differential operator $L$ of order $m$ with constant coefficients be strongly hyperbolic is that for every real vector $\xi(\neq 0)$ in $\mathbb{R}^n$ all the roots of the equation $P(\lambda, \xi) = 0$ are real and distinct.

Proof. The proof of the fact that the roots of $P(\lambda, \xi) = 0$ for all real $\xi(\neq 0)$ are real runs on the same lines as in Prop. We shall now show that for all real $\xi \neq 0$ these roots are all distinct.

If the roots of $P(\lambda, \xi) = 0$ are not distinct for all real $\xi \neq 0$ let us suppose that for some real $\xi' \neq 0$ at least two roots of $P(\lambda, \xi') = 0$
coincide. Writing $P(\lambda, \xi^*)$ explicitly

$$P(\lambda, \xi^*) = (\lambda - \lambda_1(\xi^*))^p \prod_{j=2}^{m-p+1} (\lambda - \lambda_j(\xi^*)), \, P \geq 2,$$

where $\lambda_2(\xi^*), \ldots, \lambda_{m-p+1}(\xi^*)$ are real, and different from $\lambda_1(\xi^*)$. Take for $\xi$ the vector $\tau \xi^*$ with a real parameter $\tau$ and set $\lambda' = \frac{\lambda}{\tau} - i \lambda_1(\xi^*)$. Now consider the equation

$$P(\lambda, i \tau \xi^*) + C \tau^{m-1} = 0$$

with a constant $C$ to be chosen later suitably. From this equation we obtain

$$\lambda'^p \prod_{j=2}^{m-p} \left\{ \lambda' + i(\lambda_1(\xi^*) - \lambda_j(\xi^*)) \right\} + \frac{C}{\tau} \lambda'^p(a_0(\xi^*) + a_1(\xi^*) \lambda' + \cdots + a_{m-p-1}(\xi^*) \lambda^{m-p-1} + \lambda^{m-p}) + \frac{C}{\tau} = 0$$

where $a_0(\xi^*) \neq 0$. Expanding this in a Puiseux series in a neighbourhood of $\tau = \infty$ we see that there exist $p$ roots

$$\lambda'_k(\tau) = \exp \left( \frac{2\pi i}{p} k \right) \left( \frac{-C}{a_0(\xi^*)} \right)^{\frac{1}{p}} \tau^{-\frac{1}{p}} + O(\tau^{-\frac{1}{p}}) \quad (k = 1, \ldots, p)$$

$p$ being at least 2 we can choose the constant $C$ such that there exists a root with positive real part; that is there exists a $k_0$ such that

$$\text{Re} \lambda'_{k_0}(\tau) \geq C_0 \tau^{-1/p} \quad \text{for large } \tau.$$

($C_0$ being a positive constant). Hence

$$\text{Re} \lambda_{k_0}(\tau) \geq C_0 \tau^{1-\frac{1}{p}} \quad \text{for large } \tau.$$

There exist constants $b_\nu$ such that $C = \sum_{|\nu|=m-1} b(\nu) \nu^\nu$. Thus it follows
from prop. 2 § 2 that the Cauchy problem is not well posed for the operator
\[ L + \sum_{|\nu|=m-1} b_{\nu} \left( \frac{\partial}{\partial x} \right)^{\nu}. \]

This contradicts the assumption that the operator \( L \) is strongly hyperbolic.

The sufficiency follows as in the proof of the prop. 2(i).

Finally we mention the following fact: Consider the following equation with coefficients in \( \mathcal{E} \).
\[ M[u] = \partial_t u - \sum A_k(x, t) \partial_x u - B(x, t)u = 0. \]

If, at the origin, for some \( \xi^* \) real \( \neq 0 \), one of the characteristic roots of \( \det(A(t) - \lambda I) = 0 \) is not real, then the Cauchy problem for \( M \) is never well posed in \( \mathcal{E} \) in any small neighbourhoud of the origin. (See Mizohata [3]). We shall prove this fact later, in a simple case. Here we add an important remark: Garding has shown in his paper (G˚ arding [1]), that the condition 9 of § 2 of Hadmard is equivalent to the following:
\[ \Re \lambda_j(\xi) \text{ is bounded from above when } \xi \text{ runs through } \mathbb{R}^n \text{ for } j = 1, \ldots, N. \]

Next Hörmander has systematized such inequalities by using Seidenberg’s lemma (see Hörmande [1]).  

**Proposition 4.** Let the coefficients \( A_k \) and \( B \) of \( M \) be continuous functions of \( t \) in an interval \([0, T]\). If the forward Cauchy problem is uniformly well posed then the backward Cauchy problem is also uniformly well posed.

**Proof.** As before denoting \( \xi_j/|\xi| \) by \( \xi^0 \) let \( v^j(\xi, t, t_0) \) be a fundamental system of solutions of the system of ordinary differential equations
\[ \frac{d}{dt} v(\xi, t) = (i|\xi|A(t)\xi^0 + B(t))v(\xi, t), 0 \leq t \leq t_0 \]
with initial conditions \( v^j(\xi, t_0, t_0) = v^j \equiv (v^j_1, \ldots, v^j_N) \) where \( v^j_j = 1 \) and \( v^j_k = 0 \) for \( k \neq j \). First of all we remark that if \( W(t, \xi) \) is the Wronskian of
this system then \( v^j(\xi, t, t_0) \) define its columns. Since the forward Cauchy problem is uniformly well posed we have

\[ |v^j(\xi, t, t_0)| \leq C(1 + |\xi|)^p, \quad j = 1, \ldots, N. \]

Hence \( W(t, \xi) \) is also majorized by a polynomial in \(|\xi|\). From the theory of ordinary differential equations we know that

\[ W(T, \xi) = W(t, \xi) \exp \left\{ \int_t^T a_{jj}(s, \xi^0) ds + \int_t^T b_{jj}(s) ds \right\}. \]

Now as in Prop. \& it follows that \( \sum_j \int_t^T a_{jj}(s, \xi^0) ds \) is real for any \( t \) and \( \xi^0 \). Thus we have

\[ |W(T, \xi)| \geq |w(t, \xi)| \exp \left\{ - \sum_j \int_t^T b_{jj}(s) ds \right\}. \]

That is, \( |W(T, \xi)| \geq \delta > 0 \) for all \( t \) and \( \xi \). Further we observe that as \( v^j(\xi, t, t_0) \) form a basis for solutions of the system of equations we can write

\[ v^j(\xi, t, t_0) = \sum_k c^j_k(\xi) v^k(\xi, t, T). \]

Putting \( t = t_0 \) and solving for \( c^j_k(\xi) \) we see that \( c^j_k(\xi) \) are majorized by polynomials in \(|\xi|\) since the determinant of this system of linear equations is the Wronskian \( W(\xi, T) \) which is minorized by \( \delta > 0 \) and \( v^j(\xi, t, t_0) \) are majorized by polynomials in \(|\xi|\). Hence \( v^j(\xi, t, t_0) \) are majorized by polynomials in \(|\xi|\) independent of \( t \) and \( t_0 \) in \([0, T]\) which proves that the backward Cauchy problem is uniformly well posed. This completes the proof of the proposition. \( \square \)
Chapter 3

There are obvious analogues of the function spaces introduced at the beginning of Chapter 1 for vector and matrix valued functions. We shall use the same notations for these spaces and norms and scalar products on them. For example, for two vectors \( u = (u_j) \) and \( v = (v_j) \) in \( L^2[0, h] \), we define
\[
(u(t), v(t)) = \sum_j (u_j(x, t), v_j(x, t)).
\]

1 Energy inequalities for symmetric hyperbolic systems

Let \( A_k(x, t) \) and \( B(x, t) \) be matrices (of order \( N \)) of functions. Consider the following system of first order equations.

\[
\frac{\partial}{\partial t} u - \sum A_k(x, t) \frac{\partial}{\partial x_k} u - B(x, t) u = f
\]

where \( A_k(x, t) \) are Hermitian matrices. Suppose that
\[
A_k(x, t) \in \mathcal{B}^1[0, h], B(x, t) \in \mathcal{B}^0[0, h] \quad \text{and} \quad f \in \mathcal{D}_L^1[0, h].
\]

**Proposition 1** (Friedrichs). Let \( u \) be a solution of (1.1) belonging to \( \mathcal{D}^1_L[0, h] \). Then we have

\[
\|u(t)\| \leq \exp(\gamma t) \cdot \|u(0)\| + \int_0^t \exp(\gamma(t-s)) \|f(s)\| ds
\]
where \( \gamma \) is a constant depending only on the bounds of \( A_k, B \).

**Proof.** Differentiating \( ||u(t)||^2 = u(t), u(t) \) with respect to \( t \) we have the identity

\[
\frac{d}{dt}||u(t)||^2 = \left( \frac{du}{dt}(t), u(t) \right) + \left( u(t), \frac{du}{dt}(t) \right).
\]

Since \( A_k \) are Hermitian matrices and since \( u \in D_{L^2}(0,h) \) we obtain from (1.1) the relation

\[
\left( u, \frac{du}{dt} \right) = \sum_k \left( u, A_k \frac{\partial u}{\partial x_k} \right) + (u, Bu + f)
\]

\[
= -\sum_k \left( \frac{\partial}{\partial x_k} (A_k u), u \right) + (u, Bu + f)
\]

\[
= -\left\{ \sum_k \left( A_k \frac{\partial u}{\partial x_k}, u \right) + \sum_k \left( \frac{\partial A_k}{\partial x_k} u, u \right) \right\} + (u, Bu + f).
\]

Hence

\[
\frac{d}{dt}||u(t)||^2 = -\sum_k \left( \frac{\partial A_k}{\partial x_k} \cdot u, u \right) + 2 \text{Re}(u, Bu + f)
\]

\[
\leq 2\gamma||u||^2 + 2||u||||f||
\]

where \( \gamma \) is a constant depending only on the bounds of \( \frac{\partial A_k}{\partial x_k} \) and \( B \). Hence

\[
\frac{d}{dt}||u(t)|| \leq \gamma ||u(t)|| + ||f||
\]

which on integration with respect to \( t \) yields the required inequality

\[
||u(t)|| \leq \exp(\gamma t) \cdot ||u(0)|| + \int_0^t \exp(\gamma(t-s))||f(s)||ds.
\]

The energy inequality involves the \( L^2 \)-norm of the solution \( u \) of the system in the \( x \)-space. It is possible to derive the energy inequality under the weaker assumption that \( u \in L^2(0,h) \). For this we use the method of regularization in the \( x \)-space of the function \( u \) by mollifiers introduced by Friedrichs. We recall the notion of mollifiers and a few of their properties which we need.
1. Energy inequalities for symmetric hyperbolic systems

**Definition.** Mollifiers of Friedrichs. Let $\varphi \in \mathcal{D}$ with its support contained in the unit ball $|x| < 1$ such that $\varphi(x) \geq 0$ and $\int \varphi(x) \, dx = 1$. Then for a $\delta > 0$ define

$$\varphi_\delta(x) = \frac{1}{\delta^n} \varphi \left( \frac{x}{\delta} \right).$$

are called mollifiers.

**Proposition 2** (Friedrichs). Let $a \in \mathcal{B}^1$ and $u \in L^2$. Denote by $C_\delta u$ the commutator defined by

$$C_\delta u = \varphi_\delta * \left( a(x) \frac{\partial u}{\partial x_j} \right) - a(x) \left( \varphi_\delta * \frac{\partial u}{\partial x_j} \right),$$

(3.1) $$= \left[ \varphi_\delta *, a \frac{\partial}{\partial x_j} \right] u.$$ (1.3)

Then we have

(i) $\|C_\delta u\| \leq c\|u\|$ where $c$ is a constant depending only on $\varphi$ and $a$

(ii) $C_\delta u \to 0$ in $L^2$ as $\delta \to 0$.

Before proving this proposition it will be useful to prove the following

**Lemma 1.** If $u \in L^p$ then $\varphi_\delta * u \to u$ in $L^p$ as $\delta \to 0$. More generally, if $u \in \mathcal{D}_L^m, (m = 0, 1, \ldots)$ then $\varphi_\delta * u \to u$ in $\mathcal{D}_L^m$.

**Proof.** Let $\psi_\delta = \varphi_\delta * u - u$. Since $\int \varphi_\delta(x) \, dx = 1$ we have

$$\psi_\delta(x) = \int \varphi_\delta(x - y)u(y) \, dy - u(x) = \int \varphi_\delta(x - y)(u(y) - u(x)) \, dy.$$ \(\square\)

If $p'$ is such that $\frac{1}{p} + \frac{1}{p'} = 1$ by Hölder’s inequality we have

$$|\psi_\delta(x)| \leq \left( \int \varphi_\delta(x - y) \, dy \right)^{1/p'} \left( \int \varphi_\delta(x - y) |u(y) - u(x)|^p \, dy \right)^{1/p}.$$
Here we use $\varphi_\delta = \frac{1}{\delta} \cdot \frac{1}{x - y}$.

Now since $\int \varphi_\delta(x - y)dy = 1$ we have

$$\int |\varphi_\delta(x)|^p dx \leq \int |\varphi_\delta(x - y)|u(y) - u(x)|^p dx dy \leq \int |\varphi_\delta(x - y)|u(y) - u(x)|^p dx dy.$$ By a change of variables $x' = x - y$ we obtain

$$\int |\varphi_\delta(x)|^p dx \leq \int |\varphi_\delta(x')|^p dx' \int |u(y) - u(x' - v)|^p dy.$$ 

If $\epsilon(\delta)$ denotes sup $\int |u(y) - u(y + h)|^p dy$ then $\int |\varphi_\delta(x)|^p dx \leq \epsilon(\delta)$ which tends to 0 as $\delta \rightarrow 0$. The second part is an immediate consequence of this result since $\left(\frac{\partial}{\partial x}\right) (\varphi_\delta \ast u) = \varphi_\delta \ast \left(\frac{\partial}{\partial x}\right) u$ for $|v| \leq m$ if $u \in \mathcal{D}_m^{1, p}$.

**Proof of Proposition 2**

(1.4) $C_\delta u(x) = -\int \varphi_\delta(x - y)(a(x) - a(y)) \frac{\partial u}{\partial y_j}(y)dy$

where the integral on the right is taken in the sense of distributions. Now we have

(1.5) $C_\delta u = \int \frac{\partial}{\partial y_j} (\varphi_\delta(x - y)(a(x) - a(y))) u(y)dy$

where the integral is taken in the usual sense. In fact the integral in (1.5) is equal to

$$-\int \frac{\partial a}{\partial y_j}(y) \varphi_\delta(x - y)u(y)dy + \int (a(x) - a(y)) \frac{\partial \varphi_\delta}{\partial y_j}(x - y)u(y)dy;$$

we now note that

$$|a(x) - a(y)| \leq |a|_\infty \delta_1 |x - y|, \int |x - y||\frac{\partial \varphi_\delta}{\partial y_j}(x - y)|dx \leq c,$$

with $c$ independent of $\delta$. Thus it follows from the Hausdorff-Young theorem that the function represented by the above integral is majorized in the $L^2$-norm by $c_1|a|_\infty \|u\|$. Now we see that the integration by parts
is justified. In fact, the two integrals are equal for \( u \in \mathcal{D} \). Then for any \( u \in L^2 \) the equality is proved by taking a sequence \( u_j \in \mathcal{D} \) having for its limit \( u \) in \( L^2 \). Then \( C_\delta u_j \) tends to the second integral in the sense of \( L^2 \).

On the other hand \( C_\delta u_j \to C_\delta u \) in the sense of distributions. This proves (i).

Since \( (a(x) - a(y))\varphi_\delta(x - y) \) considered, for fixed \( x \), as a function of \( y \) has compact support we see that

\[
\int \frac{\partial}{\partial y_j} \{(a(x) - a(y))\varphi_\delta(x - y)\} \, dy = 0.
\]

Hence

\[
C_\delta u(x) = \int \frac{\partial}{\partial y_j} \{(a(x) - a(y))\varphi_\delta(x - y)\} (u(y) - u(x)) \, dy
= - \int \frac{\partial a}{\partial y_j}(y)\varphi_\delta(x - y)(u(y) - u(x)) \, dy
- \int (a(x) - a(y))\frac{\partial \varphi_\delta}{\partial x_j}(x - y)(u(y) - u(x)) \, dy
= \phi_1(x) + \phi_2(x), \text{ say.}
\]

Now as in the proof of lemma 1, we see that

\[
\|\phi_i(x)\| \to 0 \text{ as } \delta \to 0 (i = 1, 2).
\]

In fact, for instance,

\[
|\phi_2(x)| \leq |a|_{B^1} \int |x - y| \frac{\partial \varphi_\delta(x - y)}{\partial x_j} ||u(y) - u(x)|| \, dv.
\]

Since \( \int \frac{\partial \varphi_\delta}{\partial x_j} |dx \leq c \) (independent of \( \delta \)) we obtain the desired property by the same reasoning as earlier. As an immediate consequence, we have

**Corollary 1.** If we assume \( a \in B^m \) and \( u \in D^m_{L^2} \) in proposition 2 then

\[
(1) \quad \|C_\delta u\|_{D^m_{L^2}} \leq c\|u\|_{D^m_{L^2}},
\]
(2) $C_\delta u \to 0$ in $\mathcal{D}_x^{m}$ as $\delta \to 0$, $m = 1, 2, \ldots$

**Proposition 3** (Friedrichs). Let $u$ be a solution of (1.1) belonging $L^2[0, h]$ then the inequality (1.2)

$$||u(t)|| \leq \exp(\gamma t)||u(0)|| + \int_0^t \exp(\gamma(t-s)) ||f(s)|| ds,$$

holds, where $\gamma$ is the same constant as in prop. 1.

**Proof.** By regularizing $u$ in the $x$-space by mollifiers $\varphi_\delta$ we obtain a function belonging to $D^1_x L^2[0, h]$ to which we can apply the Prop. 1. Let $u_\delta = \varphi_\delta * (u)$. Then

$$\frac{\partial u_\delta}{\partial t} = \frac{\partial}{\partial t}(\varphi_\delta * (u)) = \varphi_\delta * (\frac{\partial u}{\partial t}).$$

Form the equation (1.1) we obtain the following equation for $u_\delta$

$$\frac{\partial u_\delta}{\partial t} = \sum_k \varphi_\delta * (A_k \frac{\partial u}{\partial x_k}) + \varphi_\delta * (Bu) + \varphi_\delta * f,$$

that is

$$\frac{\partial u_\delta}{\partial t} = \sum_k A_k \frac{\partial u_\delta}{\partial x_k} + Bu_\delta + f_\delta + C_\delta u$$

where $C_\delta u = \sum \{ \varphi_\delta * (A_k \frac{\partial u}{\partial x_k}) - A_k(\varphi_\delta * (\frac{\partial u}{\partial x_k})) \}$

$$+ \{ \varphi_\delta * (Bu - R(\varphi_\delta * u)) \}$$

$$= \sum \{ \varphi_\delta * (A_k \frac{\partial}{\partial x_k}) u + [\varphi_\delta * (\varphi_\delta * f), B] u. \}$$

Applying prop. 1 to the equation in $u_\delta$ we obtain since $u_\delta \subset \mathcal{D}_x^{1}[0, h]$

$$||u_\delta(t)|| \leq \exp(\gamma t)||u_\delta(0)|| + \int_0^t \exp(\gamma(t-s)) \int ||f_\delta(s)|| + ||C_\delta(u)(s)|| ds.$$
Now it follows from the Friedrichs lemma (Prop. 2) that
\[ ||C_{\delta}u(s)|| \leq c||u(s)|| \]
where \( c \) is a constant independent of \( \delta \) and \( C_{\delta}u(s) \to 0 \) as \( \delta \to 0 \). By Lebesgue’s bounded convergence theorem it follows that
\[ \int_0^t \exp(\gamma(t - s)) \cdot (||f_\delta(s)|| + ||C_{\delta}u(s)||) \, ds \]
tends to \( \int_0^t \exp(\gamma(t - s))||f(s)|| \, ds \). Thus passing to the limits as \( \delta \to 0 \) we obtain
\[ ||u(t)|| \leq \exp(\gamma t)||u(0)|| + \int_0^t \exp(\gamma(t - s))||f(s)|| \, ds. \]
\[ \square \]

2 Some remarks on the energy inequalities

In the previous section we obtained estimates for the solutions of symmetric hyperbolic systems in \( L^2 \)-norm in terms of the \( L^2 \)-norms of the initial values and of the second member. One can ask whether such estimates can be proved in the maximum norm and \( L^p \)-norm for \( p \neq 2 \). Littman [1] has proved that such an energy inequality cannot hold in the \( L^p \)-norm for \( p \neq 2 \). The existence of such an inequality with the maximum norms of functions and of their derivatives is related to the propagation of regularity, a form of Huygens principle for differentiability. For instance, if \( u(0) \) is \( m \) times continuously differentiable is \( u(t) \) also \( m \) times continuously differentiable? In general an energy inequality in the maximum norm does not hold as we shall show by a counter example due to Sobolev. However, when the dimension of the \( x \)-space is one an inequality for solutions of strongly hyperbolic systems is valid in the maximum norm. This result is due to T. Haar. We indicate his result briefly.
Haar's inequality. Consider the system of equations of the first order

\[
\frac{\partial u}{\partial t} - A(x, t) \frac{\partial u}{\partial x} - B(x, t)u = f
\]

where the matrix \( A(x, t) \) is such that \( \text{det} (\lambda I - A) \) has real and distinct roots. Then we have the inequality

\[
|u(t)|_0 \leq c(T) \left( |u(0)|_0 + \sup_{0 \leq t \leq T} |f(t)|_0 \right)
\]

where \( |u(t)|_0 = \sup_{x \in D_0} |u(x, t)| \), \( D \) being a neighbourhood of the origin and \( D_0 = D \cap \{ t = 0 \} \).

In fact, let \( \lambda_1(x, t), \ldots, \lambda_N(x, t) \) be the roots of \( \text{det} (\lambda I - A) = 0 \). \( A(x, t) \) being diagonalizable there exists a non-singular matrix \( N(x, t) \) such that

\[
N(x, t)A(x, t) = D(x, t)N(x, t)
\]

where \( D(x, t) \) is the diagonal matrix

\[
\begin{pmatrix}
\lambda_1(x, t) & 0 \\
\vdots & \ddots \\
0 & \lambda_N(x, t)
\end{pmatrix}
\]

and such that \( |\text{det} N(x, t)| > \delta > 0 \). We have the identity

\[
\frac{\partial}{\partial t}(Nu) = \frac{\partial N}{\partial t}u + N \frac{\partial u}{\partial t}.
\]

Substituting for \( \frac{\partial u}{\partial t} \) from the given system the right hand side becomes

\[
\frac{\partial N}{\partial t}u + N.A \frac{\partial u}{\partial x} + N.Bu + N.f = \frac{\partial N}{\partial t}u + D N \frac{\partial u}{\partial x} + N.Bu + N.f
\]

\[
= D \frac{\partial}{\partial x}(Nu) + B_1u + N.f,
\]

where \( B_1 = -D \frac{\partial N}{\partial x} + NB + \frac{\partial N}{\partial t} \). If \( B_2 \) denotes \( B_1N^{-1} \) then \( v = Nu \).
2. Some remarks on the energy inequalities

satisfies the system.
\[ \frac{\partial v}{\partial t} = D \frac{\partial v}{\partial x} + B_2 v + Nf \]

which can be reduced to an integral equation of the Volterra type and then can be solved by successive approximation. Let \((x_0, t_0)\) by any point in the \((x, t)\)-plane. Let \(D\) be the domain enclosed by \((t = 0)\), the characteristic curves passing through \((x_0, t_0)\) and having the maximum and minimum slopes. Let \(D_0 = D \cap (t = 0)\). One can then show from the integral equation that

\[ |u(x_0, t_0)| \leq c \left\{ \sup_{x \in D_0} |u(x, 0)| + \sup_{(x, t) \in D} |f(x, t)| \right\} \]

with a constant \(c\) independent of \(u\).

That the energy inequality with the supremum norms does not hold in general is shown by the following counter example due to Sobolev.

**Counter example (Sobolev).** We consider the wave operator

\[ \Box \equiv \frac{\partial^2}{\partial t^2} - \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2} \]

in \(\mathbb{R}^3\). We set

\[ E_1(t, u) = \sup_x \left\{ \left| \frac{\partial u}{\partial t} \right| + \sum_j \left| \frac{\partial u}{\partial x_j} \right| \right\}. \]

We shall show that if \(t_0 > 0\) then an inequality

\[ E_1(t_0, u) \leq c E_1(0, u) \]

does not hold, which proves that the differentiability of the solution is not propagated in the \(t\)-direction. For this purpose, let \(\Gamma(x, t)\) be a fundamental solution of \(\Box\) such that

\[ \Gamma(x, 0) = 0, \quad \frac{\partial \Gamma}{\partial t}(x, 0) = \delta \]
\( \delta \) being the Dirac distribution. Let \( \varphi \in \mathcal{D} \). For an \( \varepsilon > 0 \) define
\[
\varphi^{(\varepsilon)}(x) = \varphi \left( \frac{x}{\varepsilon} \right).
\]
Extending \( \Gamma(x, t) \) to the whole space by setting
\[
\tilde{\Gamma}(x, t) = \Gamma(x, t) \text{ for } t \geq 0
\]
\[
= -\Gamma(x, -t) \text{ for } t \leq 0.
\]
We obtain a distribution solution of \( \frac{\partial^2}{\partial t^2} - \sum_j \frac{\partial^2}{\partial x_j^2} \) in the whole space \( (-\infty < t < \infty) \times \mathbb{R}^3 \). Setting
\[
u(e)(x, t) = \tilde{\Gamma}(x, t - t_0) * \varphi^{(\varepsilon)}(x)
\]
we obtain a solution of the homogeneous equation which satisfies
\[
\frac{\partial}{\partial t} \nu(e)(x, t) = \frac{\partial \tilde{\Gamma}}{\partial t}(x, t_0 - t) * \varphi^{(\varepsilon)}(x) = \delta * \varphi^{(\varepsilon)}(x) = \varphi^{(\varepsilon)}(x)
\]
and
\[
\frac{\partial}{\partial x_j} \nu(e)(x, t) = \tilde{\Gamma}(x, 0) * \varphi^{(\varepsilon)}(x) = 0.
\]
Hence \( E_1(t_0, u_\varepsilon) = \sup \varphi^{(\varepsilon)}(x) \).

On the other hand we first observe that \( \Gamma(x, t) \) can be taken to be
\[
\frac{1}{4\pi t} \delta_{|x| - t}. \quad \text{Let us choose } a\varphi \in \mathcal{D} \text{ with its support contained in the unit ball in } \mathbb{R}^3 \text{ such that } \varphi(0) = 1 \text{ and } |\varphi(x)| \leq 1. \text{ Then } \left| \frac{\partial}{\partial x_j} \varphi^{(e)}(x) \right| \leq \frac{1}{e}.
\]
Thus for \( t = 0 \) we see that
\[
\frac{\partial}{\partial t} u_\varepsilon(x, 0) = \frac{\partial \Gamma}{\partial t}(x, t_0) * \varphi^{(\varepsilon)}(x) = \frac{\partial}{\partial t} \left( \frac{t}{4\pi} \int_{|\xi|=1} \varphi^{(\varepsilon)}(x - t) d\xi \right)
\]
Now since \( \int_{B_{\varepsilon}(x_0) \cap \{|\xi|=1\}} d\xi = O(\varepsilon^2) \) it follows that \( \frac{\partial}{\partial t} u_\varepsilon(x, 0) = O(\varepsilon) \).
We also have \( \frac{\partial}{\partial x_j} u_\varepsilon(x, 0) = O(\varepsilon) \) and so \( E_1(t_0, u_\varepsilon) = O(\varepsilon) \) which together with \( E_1(t_0, u_\varepsilon) = \sup |\varphi^{(\varepsilon)}(x)| = 1 \) shows that an energy inequality of the type \( E_1(t_0, u_\varepsilon) \leq cE_1(0, u_\varepsilon) \) does not hold.
3 Singular integral operators

In this section we introduce the notion of singular integral operators and recall some of their properties which will be useful in the study of the existence and uniqueness of solutions of the Cauchy problem. The following considerations lead us to the notion of singular integral operators.

Consider the system of equations

\[ \frac{\partial u}{\partial t} - \sum A_k \frac{\partial u}{\partial x_k} - Bu = f \]

where \( A_k \) and \( B \) are matrices whose entries are constants and \( f \in L^2[0,h] \). We assume (3.1) to be strongly hyperbolic in the sense that the roots of the equation \( \det (\lambda I - A_\xi) = 0 \) are real and distinct. Let the roots be \( \lambda_1(\xi), \ldots, \lambda_N(\xi) \) for \( \xi \neq 0 \). We have the following.

**Lemma 1.** There exists a non-singular matrix \( N(\xi) \) which is homogeneous of degree zero and bounded such that

1. \( |\det N(\xi)| \leq \delta > 0 \) for all \( \xi \).
2. \( N(\xi)(A_\xi) = D(\xi)N(\xi) \) where \( D(\xi) \) is the diagonal matrix

\[ D(\xi) = \begin{pmatrix} \lambda_1(\xi) & 0 \\ \vdots & \ddots \\ 0 & \lambda_N(\xi) \end{pmatrix} \]

Assume that there exists a solution \( u \in L^2[0,h] \). Then, denoting for every fixed \( t \), the Fourier transform of \( u \) in the \( x \)-space \( \hat{u}(\xi,t) \), we obtain the following system of ordinary differential equations:

\[ \frac{d}{dt}(\hat{u}(\xi,t)) = (2\pi i A_\xi + B)\hat{u}(\xi,t) + \hat{f}(\xi,t) \]

Multiplying both sides of this system by \( N(\xi) \) and using lemma 1 we have

\[ \frac{d}{dt}(N\hat{u})(\xi,t) = (2\pi i D(\xi) \cdot N(\xi) + N(\xi)B)\hat{u}(\xi,t) + N(\xi)\hat{f}(\xi,t). \]
\( v(\xi, t) = N(\xi)\hat{u}(\xi, t) \) satisfies the system of equations

\[
(3.3) \quad \frac{dv}{dt}(\xi, t) = (2\pi iD(\xi) + B'(\xi))v(\xi, t) + N(\xi)\hat{f}(\xi, t),
\]

where

\[
B'(\xi) = N(\xi)BN(\xi)^{-1}.
\]

Now

\[
\frac{d}{dt}\|v(\xi, t)\|^2 = \int \left( \frac{dv}{dt} \cdot \bar{v} + v \cdot \frac{dv}{dt} \right) d\xi
\]

\[
= \int \left( 2\pi(iD(\xi)v \cdot \bar{v} + v \cdot iD(\xi)v) + 2\text{Re}(B'v, \bar{v}) + 2\text{Re}N(\xi)f \cdot \bar{v} \right) d\xi
\]

\[
= 2 \int \text{Re}(B'(\xi)v \cdot \bar{v} + N(\xi)f \cdot \bar{v})(\xi, t) d\xi.
\]

Because \( N(\xi) \) is bounded and condition (1) of lemma holds. The operators \( B' \) is bounded and hence

\[
\frac{d}{dt}\|v(\xi, t)\|^2 \leq 2\gamma\|v\|^2 + 2\text{Re}(N(\xi)\hat{f}, v)
\]

\[
\leq 2\gamma\|v\|^2 + 2\|N(\xi)\hat{f}\|\|v\|
\]

Thus we obtain

\[
\|v\| \leq \exp(\gamma t) \cdot \|v(\xi, 0)\| + \int_0^t \exp(\gamma(t-s))\|N(\xi)\hat{f}(\xi, s)\| ds.
\]

By Plancheral’s formula’s formula we have

\[
\|v(\xi, t)\| = \|N(\xi)\hat{u}(\xi, t)\| \leq c\|u(t)\|
\]

and again since \( N(\xi) \) has a bounded inverse by condition (1) we see that

\[
(3.4) \quad \|u(t)\| \leq c(h)\left(\|u(0)\| + \int_0^t \|f(s)\| ds\right)
\]
where \( c \) is a constant depending only on \( h \).

Now we look at this reasoning without explicitly using the notion of Fourier transforms.

\( N(\xi) \) is homogeneous of degree 0 in \( \xi \) and so the convolution operators \( N(x)* \) defines a bounded operator in the space \( L^2 \) since
\[
\|N(x)*u\| = \|N(\xi)\hat{u}\| \leq c\|u\|
\]
by Plancherel’s formula. Here \( N(x) \) is the inverse Fourier image of \( N(\xi) \). Let \( D(x) \) be the distribution whose Fourier image is \( D\left(\frac{\xi}{|\xi|}\right) \). Define the operators \( \wedge \) by
\[
(\wedge u)(\xi) = |\xi|\hat{u}(\xi).
\]

Then we obtain
\[
\frac{d}{dt}(N(x)*u) = 2\pi i D(x)*\wedge(N(x)*u) + N(x)*Bu + N(x)*f.
\]

In other words \( v = N*(x)u \) satisfies the system
\[
\frac{dv}{dt} = 2\pi i D*(x)\wedge v + B_1v + N*(x)f,
\]
where \( B_1 \in \mathcal{L}(L^2, L^2) \) because of condition (1). Integrating with respect to \( t \) in the interval \([0, t]\) we have the inequality
\[
\|N*(x)u\| \leq \exp(\gamma t)\|N*(x)u(x, 0)\| + \int_0^t \exp(\gamma(t-s))\|N*(x)f(x, s)\|ds
\]
where \( \gamma \) is a constant depending only on \( A \) and \( B \). But there exists a constant \( k \) (depending on \( A \)) such that
\[
\frac{1}{k}\|u(x, t)\| \leq \|N*(x)u(x, t)\| \leq k\|u(x, t)\|
\]
which gives an energy inequality for \( u \).

Now in the case of systems with variable coefficients even though we cannot apply Fourier transforms we may, however, write the system in a form similar to (3.2) to which we can apply the above method to get an energy inequality. For this purpose we introduce the singular integral operators.
For a function $f \in L^2(\mathbb{R}^1)$ consider the integral transform defined by

$$g(x) = v.p. \int_{-\infty}^{\infty} \frac{f(t)}{x-t} \, dt.$$  \hspace{1cm} (4.1)

M. Riesz \[1\] has proved that the Cauchy principal value defining $g$ exists and $g \in L^2(\mathbb{R}^1)$. $f \rightarrow g$ is a continuous linear mapping of $L^2(\mathbb{R}^1)$ into itself. In the language of the theory of distributions we can write

$$g = v.p. \left( \frac{1}{x} \right) * f.$$  \hspace{1cm} (4.2)

$v.p. \left( \frac{1}{x} \right)$ is a tempered distribution whose Fourier image is $\chi(\xi) = -\pi i$ for $\xi > 0$ and $\pi i$ for $\xi < 0$. We observe that $\frac{1}{x}$ is homogeneous of degree $-1$ and has mean value 0. If $\hat{g}$ and $\hat{f}$ are the Fourier images of $g$ and $f$ respectively then $\hat{g} = \partial \chi \hat{f}$ and $\|g\| = \pi \|f\|$ by Plancheral’s formula.

Calderon and Zygmund \[1\] generalized this theory to functions on $\mathbb{R}^n$. Let $N(x)$ be a homogeneous function of degree $-n$ on $\mathbb{R}^n(N(\lambda x) = \lambda^{-n}N(x))$ which is smooth in the complement of the origin and has mean value $\int |x| = 1 N(x) \, d\sigma_x = 0$. Then they proved that $g = v.p. N(x) * f \in L^p$ if $f \in L^p$. In particular $f \rightarrow g$ is a continuous linear map of $L^2$ into itself. This latter fact can be seen observing that $v.p. N(x)$ is a tempered distribution, its Fourier transform $h(\xi)$ is a homogeneous function of degree 0 and has mean value $\int |\xi| = 1 h(\xi) \, d\sigma_\xi = 0$. In this paragraph $d\sigma_x$ and $d\sigma_\xi$ stand for normalized volume element of the unit sphere; viz. $d\sigma_x = dS_x/\text{vol } S$.

Conversely, given any homogeneous function $h(\xi)$ of degree 0 with mean value 0, if $\gamma(x)$ is its inverse Fourier image we can define an integral operators $\gamma$ by

$$(\gamma * f)(x) = \int \exp(2\pi ix \cdot \xi) h(\xi) \hat{f}(\xi) d\xi.$$
Now consider the differential operators

$$L \left( x, \frac{\partial}{\partial x} \right) = \sum a_j(x) \frac{\partial}{\partial x_j}. $$

For a function $u \in \mathcal{S}$ we can write

$$(Lu)(x) = \int \exp(2\pi ix\xi)(\sum_j a_j(x)\xi_j)\hat{u}(\xi)d\xi.$$ 

Denote $h(x, \xi) = 2\pi i \sum_j a_j(x)\xi_j/|\xi|$. If we define

$$(Hf)(x) = \int \exp(2\pi ix\xi)h(x, \xi)\hat{f}(\xi)d\xi$$

$H$ will be a bounded operator in $L^2$. In fact, $H$ can be written

(4.2) 

$$Hf = 2\pi i \sum a_j(x)(R_j * f)$$

where $R_j$ is the inverse Fourier image of $\xi j/|\xi|$. It follows that

$$\| Hf \| \leq 2\pi \sum |a_j(x)|_0 \| R_j * f \| \leq (2\pi \sum |a_j|_0)\|f\|.$$ 

Now $L$ can be written in the form

$$Lu = H \wedge u.$$ 

We introduce the notation used by Calderon-Zygmund [1], [2].

Let $U$ be an open set in $\mathbb{R}^n$. A function $u$ defined on $U$ is said to satisfy a uniform Holder condition of order $\beta (0 \leq \beta \leq 1)$ if for any $x, x' \in U$ we have

(4.3) 

$$|u(x) - u(x')| \leq c|x - x'|^\beta.$$ 

c is called the Hölder constant for $u$. We shall denote by $C_\beta(U), \beta \geq 0$, the class of complex valued continuous bounded functions on $U$ with bounded continuous derivatives upto order $[\beta]$ (the integral part of $\beta$) and with the derivatives of order $[\beta]$ satisfying a Hölder condition of
order $\beta - [\beta]$. $\mathcal{E}_\xi(\mathbb{R}^n - \{0\})$ will denote the space consisting of complex valued functions $h(\xi)$, $\xi \in \mathbb{R}^n$, homogeneous of degree 0 and infinitely differentiable in $\mathbb{R}^n - \{0\}$ with respect to $\xi$. This space $\mathcal{E}_\xi(\mathbb{R}^n - \{0\})$ is topologized by the family of seminorms defined by

$$p_s(h) = \sum_{|\nu| \leq s} \sup_{|\xi| \geq 1} |(\partial^{\nu}_{\xi}) h(\xi)|.$$

We say that $h(x, \xi) \in \mathcal{C}_\beta^\infty, \beta \geq 0$, if

1. for $\beta = 0$ the function $x \mapsto h(x, \xi) \in \mathcal{E}_\xi(\mathbb{R}^n - \{0\})$ is continuous and bounded;

2. for $0 < \beta < 1$, $h(x, \xi) \in \mathcal{C}_0^\infty$ and the function $x \mapsto h(x, \xi) \in \mathcal{E}_\xi(\mathbb{R}^n - \{0\})$ is uniformly Hölder continuous of order $\beta$ in the sense that for any $\nu$

$$\sup_{|\xi| \geq 1} \left| \left( \frac{\partial}{\partial x} \right)^\nu h(x, \xi) - \left( \frac{\partial}{\partial x} \right)^\nu h(x', \xi) \right| \leq c |x - x'|^\beta;$$

3. if $\beta \geq 1$, $\left( \frac{\partial}{\partial x} \right)^\nu h(x, \xi) \in \mathcal{C}_0^\infty$ for $|\nu| \leq \beta$ and $\left( \frac{\partial}{\partial x} \right)^\nu h(x, \xi) \in \mathcal{C}_\beta^\infty - [\beta]$ for $|\nu| = [\beta]$.

$h(x, \xi)$ being a homogeneous function of $\xi$ can be expanded as a series in spherical harmonics. Let $Y_l(\xi)$ be a normalized real spherical harmonic of degree $l$, that is such that

(4.5) $\int_{|\xi|=1} Y_l(\xi)^2 d\sigma_\xi = 1$

and $Y_{lm}(\xi)$ be a complete orthogonal system of normalized spherical harmonics of degree $l$. Then we can write

(4.6) $h(x, \xi) = a_0(x) + \sum_{l \geq 1, m} a_{lm}(x)Y_{lm}(\xi)$
in terms of the spherical harmonics. Then

(4.7) \[ a_{lm}(x) = \int_{|\xi|=1} h(x, \xi) Y_{lm}(\xi) d\sigma_\xi. \]

Let \( \widetilde{Y}_{lm} \) denote the inverse Fourier image of \( Y_{lm}(\xi) \)

\[ \widetilde{Y}_{lm}(x) = \int e^{2\pi i x \cdot \xi} Y_{lm}(\xi) d\xi. \]

We define

(4.8) \[ (H f)(x) = a_0(x) f(x) + \sum_{l,m} a_{lm}(x) (\widetilde{Y}_{lm} * f)(x). \]

Now we have the following estimates due to Calderon and Zygmund:

(a) \( |Y_{lm}(\xi)| \leq c l^{(n-2)}, c \) being a positive constant;

(b) the number of distinct spherical harmonics \( Y_{lm}(\xi) \) of degree \( l \) is of the order \( l^{n-2} \);

(c) \( |a_{lm}(x)| \leq c M l^{3 - \frac{3}{2} n} \) where \( M = \sup_{x \in \mathbb{R}^n, |\xi| \geq 1} |(\frac{\partial}{\partial \xi})^r h(x, \xi)| \).

More generally we have the following sharper estimates. Let \( L \) be the operator defined by

\[ L(F) = |\xi|^2 (\Delta_\xi F) \] where \( \Delta_\xi = \sum_{j=1}^n \left( \frac{\partial}{\partial \xi_j} \right)^2 \).

Then

(4.7) \[ a_{lm}(x) = (-1)^r r^{-\frac{3}{2} n} \int_{|\xi|=1} L_{\xi}^r (h(x, \xi) Y_{lm}(\xi) d\sigma_\xi. \]

From this it follows that
(d) \(|a_{lm}(x)| \leq c(n, r)M_{2r - 2} n^{-2r} \frac{n}{2} \)

where \(M_{2r} = \sup_{x \in \mathbb{R}^n, |\xi| \geq 1} \left| (\frac{\partial}{\partial \xi})^r h(x, \xi) \right| \)

(e) \(\sup_{|\xi| \geq 1} \left| (\frac{\partial}{\partial \xi})^r Y_{lm}(\xi) \right| \leq c(n, r) \frac{1}{2} |n - 2r + |\nu||. \)

These estimates show that the series defining \(Hf\) is convergent in the \(L^2\)-sense.

In fact,

\[
\|Hf\| \leq (|a_0(x)| + \sum |a_{lm}(x)| |Y_{lm}(\xi)|) \|f\|.
\]

From (a), (b) and (c) it follows that

\[
\sum |a_{lm}| |Y_{lm}| \leq cM \sum l^{-\frac{n}{2} + \frac{3(n - 2) + n - 2}{2}} = cM \sum l^{-3} < \infty.
\]

Hence

\[
\|H\| \leq cM, M \text{ being defined in (c)}.
\]

A singular integral operator was defined by Calderon and Sigmund by the following equation

\[
(Hu)(x) = a(x)u(x) + \int k(x, x - y)u(y)dy,
\]

where \(k(x, z)\) is a complex valued homogeneous function of degree \(-n\) in \(z\), of class \(E\) in \(\mathbb{R}^n - \{0\}\) in the \(z\)-variable for every fixed \(x\) and the function \(k(x, z)\) has mean value zero in the \(z\)-space for every fixed \(x\). Let us expand \(k(x, z)\) in terms of spherical harmonics:

\[
k(x, z) = \sum a_{lm}(x)Y_{lm}(z)z^{-n}, \quad z = \frac{z}{|z|},
\]

where \(a_{lm}(x) = \int k(x, z')d\sigma_{z'}\).

Then, taking into account the fact that \(\mathcal{F}[Y_{lm}(z')z^{-n}] = \gamma_1 Y_{lm}(\xi)\),
\( \gamma_1 \) being a constant, we define the symbol \( \sigma(H) \) as

\[
\sigma(H) = a_0(x) + \sum_{l,m} a_{lm}(x) \gamma_1 Y_{lm}(\xi).
\]

(4.11)

We start from this \( \sigma(H) \) in our definition. However the two definitions are identical since there exists a one to one linear mapping \( \sigma \) of the class of singular integral operators of the class \( C^\infty_\beta \) into the class of functions \( h(x, \xi), x, \xi \in \mathbb{R}^n \) homogeneous of degree zero with respect to \( \xi \) and in \( C^\infty_\beta \). \( \sigma(H) \) is called the symbol of the singular integral operator \( H \). Thus the series \( \sum_{l,m} a_{lm}(x) Y_{lm}(\xi) \) represents in a sense the Fourier transform of \( k(x, z) \) with respect to \( z \). We recall without proof the following important theorems on these operators, which we shall require for later use. For proofs see Calderon-Zygmund [1, 2].

**Theorem 1** (Calderon-Zygmund [1]). If \( H \) is a singular integral operator of type \( C^\infty_\beta \) then its symbol is a homogeneous function of degree zero and of class \( C^\infty_\beta \) with respect to \( \xi \) in \( |\xi| \geq 1 \). Conversely every function of \( x \) and \( \xi \) which is homogeneous of degree zero and belongs to the class \( C^\infty_\beta \) in \( |\xi| \geq 1 \) is the symbol of a unique singular integral operator of type \( C^\infty_\beta \). If

\[
M = \sup_{|x| \leq 2n, |\xi| \geq 1} \left| \left( \frac{\partial}{\partial \xi} \right)^r \sigma(H)(x, \xi) \right|
\]

then

\[
(4.12) \quad \| Hf \|_p \leq MA_p \| f \|_p
\]

where \( A_p \) depends only on \( p \) and \( n \).

If \( h_1(x, \xi), h_2(x, \xi) \) are of class \( C^\infty_\beta \) in \( |\xi| \geq 1 \) then it is easy to see that \( h_1(x, \xi) + h_2(x, \xi) \) and \( h_1(x, \xi)h_2(x, \xi) \) are also of class \( C^\infty_\beta \) and further if \( |h_2(x, \xi)| \geq \delta > 0 \) then \( \frac{h_1(x, \xi)}{h_2(x, \xi)} \) is also of class \( C^\infty_\beta \).

**Theorem 2** (Calderon-Zygmund [2]). Let \( h(x, \xi) = \sigma(H) \) be of type \( C^\infty_\beta \), homogeneous of degree zero in \( \xi \) then

1. for \( r \leq \beta \), \( Hf \in \mathcal{D}_L^r \) for \( f \in \mathcal{D}_L^r \) \((1 < p < \infty)\), and
(2) if $f \in L^p$ and Hölder continuous of order $\alpha (\alpha < \beta$ then $H f \in L^p$
and Hölder continuous of order $\alpha(1 < p < \infty)$.

Let $H^\#$ and $H_1 H_2$ be singular integral operators whose symbols are respectively $\overline{\sigma(H)}$ and $\sigma(H_1), \sigma(H_2)$.

**Theorem 3** (Calderon-Zygmund). If $\sigma(H_1), \sigma(H_2)$ are independent of $x$ then

$$H_1 \circ H_2 = H_1 H_2 = H_2 \circ H_1 = H_2 H_1$$

and if $\sigma(H)$ is independent of $x$ and $|\sigma(H)(\xi)| \geq \delta > 0$ then $H$ is invertible and its inverse $H^{-1}$ is also a singular integral operator. We illustrate by a simple example the motivation for the definition of the singular integral operators $H_1 \circ H_2$ and $H^\#$. Consider the differential operators

$$L = \sum_j a_j(x) \frac{\partial}{\partial x_j}, M = \sum_j b_j(x) \frac{\partial}{\partial x_j}, a_j, b_j \in \mathcal{B}^1.$$  

Then

$$LM = \sum_{jk} a_j(x)b_k(x) \frac{\partial^2}{\partial x_j \partial x_k} + \sum_{jk} a_j(x) \frac{\partial b_k}{\partial x_j} \frac{\partial}{\partial x_k}.$$  

Therefore, if we define

$$L \circ M = \sum_{jk} a_j(x)b_k(x) \frac{\partial^2}{\partial x_j \partial x_k}$$

then $LM = L \circ M$ modulo first order operators. Next if we define

$$L^\# = - \sum a_j(x) \frac{\partial}{\partial x_j}$$

then $L^\ast \equiv L^\#$ modulo bounded operators.

These considerations suggest that the product of two singular integral operators and the conjugate operator $H^\ast$ will be approximated, in some sense, by the singular integral operators $H_1 \circ H_2$ and $H^\#$ respectively. More precisely we have the following:
Theorem 4 (Calderon-Zygmund). Let $H$ be a singular integral operator of type $C^\infty_{\beta} (\beta > 1)$ and $M$ be a bound for $\sigma(H)(x, \xi)$ and its derivatives with respect to the coordinates of $\xi$ of order $2n$, the first derivatives of these with respect to the coordinates of $x$ and Hölder constants of the latter. Then for every $f \in \mathcal{D}_{L^p}^1 (1 < p < \infty)$ we have

\[ \| (H \wedge - H)f \|_{L^p} \leq A_p M \| f \|_{L^p}, \]

\[ \| (H^* \wedge - H^*)f \|_{L^p} \leq A_p M \| f \|_{L^p}, \]

\[ \| \wedge (H^* H^*)f \|_{L^p} \leq A_p M \| f \|_{L^p} \]  

(4.13)

where $A_p$ depends only on $p, n, \beta$. Further if $H_1$ and $H_2$ are two singular integral operators of type $C^\infty_{\beta}$ and $f \in \mathcal{E}^1_{L^p} (1 < p < \infty)$ then $H_1 \circ H_2$ is an operator of type $C^\infty_{\beta}$ and

\[ \| (H_1 \circ H_2 - H_1 H_2)f \|_{L^p} \leq A_p M_1 M_2 \| f \|_{L^p}, \]

\[ \| \wedge (H_1 \circ H_2 - H_1 H_2)f \|_{L^p} \leq A_p M_1 M_2 \| f \|_{L^p} \]

(4.14)

where again $A_p$ depends only on $p, n, \beta$ and $M_1, M_2$ being defined in the same way as $M$.

We can write differential operators in the form of singular integral operators as follows: Let $A = \sum_{|\alpha| = m} a_\alpha(x) \left( \frac{\partial}{\partial x} \right)^\alpha$ be a homogeneous differential operator of order $m$ with coefficients $a_\alpha(x)$ in $C^\beta$, $\beta \geq 0$. If $u \in \mathcal{D}_{L^2}^m$, then $\wedge^m u$ is well defined,

\[ (\wedge^m u)(\xi) = |\xi|^m \hat{u}(\xi) \]

and $Au = H \wedge^m u$ where $H$ is a singular integral operator of type $C^\infty_{\beta}$ and

\[ \sigma(H) = \bar{r}^m \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha |\xi|^{-m}. \]

(4.15)

Similarly any general linear differential operator of order $m$

\[ A = \sum_{k \leq m} A_k, A_k = \sum_{|\nu| = k} a_{k, \nu}(x) \left( \frac{\partial}{\partial x} \right)^\nu \]

with $a_{k, \nu}(x)$ of class $C^\beta$ can be written as

\[ Au = \sum H_k \wedge^k u \]

(4.16)
where $H_k$ is a singular integral operator of class $C^n$ and
\begin{align}
\sigma(H_k) = i^k \sum_{|\nu| = k} a_{k,\nu}(x) \gamma^{\nu} |\xi|^{-k}
\end{align}
for every $u \in \mathcal{D}_{L^2}$.

A matrix of operators is called a singular integral matrix if its elements are singular integral operators and its symbol is the matrix whose elements are the symbols of the corresponding elements of the singular integral matrix. A system of differential operators can be written as a singular integral matrix.

## 5 Extension of Gårding’s inequality to singular integral operators

In this section we prove an inequality for the singular integral operators whose symbol satisfies a condition of positivity. This is an analogue of the well known inequality of Gårding for elliptic differential operators. Before stating the inequality we prove some preliminary results needed in the proof of this inequality. These results are also of independent interest.

The following lemma corresponds to the local property of differential operators, namely, that differential operators decrease supports.

**Lemma 1** (Quasi localisation lemma). Let $\Omega$ be the ball of radius $2\eta$ and of centre a point $x_0$ in $\mathbb{R}^n$. Let $H$ be a singular integral operator whose symbol $\sigma(H)(x,\xi) \in C^n_\beta$, with $\beta > 0$. If $u \in \mathcal{D}_{L^2}$ has its support in the ball of radius $\eta$ and of centre $x_0$ then
\begin{align}
\|H \wedge u\|_{L^2(\Omega)} \leq c(n,\eta) M' \|u\|
\end{align}
where $M' = \sum_{|\nu| \leq 3n+3} \sup_{x \in \mathbb{R}^n, |\xi| \geq 1} |(\frac{\partial}{\partial \xi})^\nu \sigma(H)(x,\xi)|$ and $c(n,\eta)$ is a constant depending only on $n$ and $\eta$.

**Proof.** We decompose the operator $\wedge$ as $\wedge = \wedge_1 + \wedge_2$ with $\wedge_1(\xi) = \alpha(\xi) |\xi|$ and $\wedge_2(\xi) = (1 - \alpha(\xi)) |\xi|$ where $\alpha(\xi) \in \mathcal{D}$ such that $\alpha(\xi) \equiv 1$ on
5. Extension of Gårding’s...

\[ |\xi| \leq 1, 0 \leq \alpha(\xi) \leq 1 \] and vanishes outside \(|\xi| \leq 2\). \(\tilde{\Lambda}_2(\xi)\) is an infinitely differentiable function, \(\tilde{\Lambda}_1(\xi)\) has compact support and hence \(\Lambda_1\) is a bounded operator in \(L^2\). So it is enough to prove that

\[ \| H \wedge_2 u \|_{L^2(C\Omega)} \leq C(n, \eta)M' \| u \| . \]

Let

\[ \sigma(H)(x, \xi) = a_0(x) + \sum_{l,m} a_{lm}(x) Y_{lm}(\xi) \]

be the expansion of \(\sigma(H)\) in terms of a complete system of spherical harmonics \(Y_{lm}(\xi)\). Let \(Y'_{lm}(x)\) be the singular integral operator such that

\[ Y'_{lm}(x) \rightarrow Y_{lm}(\xi) \wedge_2(\xi) \]

by Fourier transforms. Then we can write

\[ (H \wedge_2 u)(x) = a_0(x) \wedge_2 u(x) + \sum_{l,m} a_{lm}(x)(Y'_{lm}(x) * u) . \]

First we show that

\[ |Y'_{lm}(x)| \leq |x|^{-2p}c(p, n)|Y_{lm}(\xi)|_{2p} \]

for \(x \in \mathbb{C}\) for \(2p \geq n + 2\)

where \(|Y_{lm}(\xi)|_{2p} = \sum_{|\nu| \leq 2p, |\xi| \geq 1} \sup |(\frac{\partial}{\partial \xi})^\nu Y_{lm}(\xi)|.|\)

In fact,

\[ Y'_{lm}(x) = |x|^{-2p}\left\{ |x|^{2p}Y'_{lm}(x) \right\} \]

and \(|x|^{2p}Y'_{lm}(x)\) is the inverse Fourier image of const

\[ \Delta_\xi^p(Y_{lm}(\xi)(1 - \alpha(\xi))|\xi|. \]

Hence we have the estimate

\[ |Y'_{lm}(x)| \leq |x|^{-2p}\left( \frac{1}{2\pi} \right)^p \int |\Delta_\xi^p((1 - \alpha(\xi))Y_{lm}(\xi)|\xi)|d\xi \]

\[ \leq |\xi|^{-2p}c(n, p)|Y_{lm}(\xi)|_{2p}f or 2p \geq n + 2. \]
This establishes the assertion (5.4).

Now we show that for any $u \in \mathcal{D}$ with support contained in $\omega = B_{\eta}(x_0)$
\begin{equation}
\| |x|^{-2p} \star u \|_{L^2(C\Omega)} \leq c(n, p, \eta)\|u\|
\end{equation}
holds for $p$ satisfying $4p > n$.

In fact, for $x \in \mathcal{C} \Omega$, $\| |x|^{-2p} \star u \| = \left| \int \frac{u(y)}{|x-y|^{2p}} dy \right|$ by Schwarz inequality,
\[ \leq \|u\| (\text{vol } \omega)^{1/2} (\text{dist. } (x, \omega))^{-2p} \]

Hence $\| |x|^{-2p} \star u \|_{L^2(C\Omega)} \leq (\text{vol } \omega)^{1/2} \|u\| (\int_{|x| \geq 2\eta} \frac{dx}{(|x| - \eta)^{4p}})^{1/2}$. The integral in the right hand side converges for $4p > n$ which proves the assertion (5.5). Now (5.4) and (5.5) together assert that
\[ \|H \wedge_2 u\|_{L^2(C\Omega)} \leq C'(p, n, \eta)M'\|u\|. \]

This completes the proof of lemma [1]. In the proofs of the following results we use a $C^\infty$ partition of unity in $\mathbb{R}^n$:
\[ \alpha \in \mathcal{D}, \quad \alpha_j \geq 0, \quad \sum_j \alpha_j^2 = 1. \]

To simplify the arguments we take a partition of unity satisfying the following conditions: Let $\alpha_0 \in \mathcal{D}$ whose support is contained in the ball of radius $\varepsilon$, $\varepsilon$ being a small number to be determined by the singular integral operator $H$. Let $\{x^{(j)}\}$ be a sequence of points of $\mathbb{R}^n$ whose coordinates are multiples of $\varepsilon = \varepsilon n - \frac{1}{2}$, $\alpha_j(x) = \alpha_0(x - x^{(j)})$, $j = 0, 1, \ldots, x^{(0)} = 0$. The support of $\alpha_0$ will be denoted by $\omega_0$ and the ball of centre $x^{(j)}$ and of radius $2\varepsilon$ will be denoted by $\Omega_j$. Let
\[ \alpha(p) = \sum_{|j| \leq p} \sup_x \left| \frac{\partial}{\partial x} \right| \alpha_0(x). \]
5. Extension of Gårding’s...

Lemma 2. Let $H$ be a singular integral operator with its symbol $\sigma(H)(x,\xi) \leq \beta \in C^\infty_\beta$, with $\beta > 0$ and $(\alpha_j)$ be a $C^\infty$ partition of unity as constructed above. Then for any $u \in D^{1,2}$

$$
\sum_j ||((H \wedge \alpha_j - \alpha_j(H \wedge))u||^2 \leq \gamma ||u||^2
$$

In particular, taking $\sigma(H) = 1$ this would imply

$$
\sum_j ||[\wedge, \alpha_j]u||^2 \leq \gamma ||u||^2.
$$

Let $\beta \in \mathcal{D}_\xi$, $0 \leq \beta(\xi) \leq 1$ with support contained in $|\xi| < 1$ which takes the value 1 in a neighbourhood of the origin. Decompose $\wedge$ into $\wedge = \wedge_1 + \wedge_2$ where $\hat{\wedge}_1(\xi) = \beta(\xi)|\xi|$ and $\hat{\wedge}_2(\xi) = (1 - \beta(\xi))|\xi|$. Clearly $||\wedge_1 u|| \leq ||u||$ and hence

$$
||H \wedge_1 \alpha_j u|| \leq ||H|| ||\alpha_j u||_{L^2(\Omega_j)} \leq \sup_x ||\alpha_j(x)|| ||H|| ||u||_{L^2(\Omega_j)}
$$

Hence

$$
\sum_j ||H \wedge_1 \alpha_j u||^2 \leq \alpha(0)^2 ||H||^2 k ||u||^2
$$

where $k$ is the maximum number of sets $\{\omega_h\}$ intersecting at any point and

$$
\sum_j ||\alpha_j H \wedge_1 u||^2 = ||H \wedge_1 u||^2 \leq ||H||^2 ||u||^2.
$$

So we have only to consider $\sum_j [||H \wedge_2, \alpha_j||u||].$ Consider the term

$$
\varphi_j(x) = [H \wedge_2, \alpha_j]u(x).
$$

$$
\varphi_j(x) = \sum_{l,m} a_{lm}(x) \int \tilde{Y}_{lm}(x-y) \wedge_2 (x-y)(\alpha_j(y) - \alpha_j(x))u(y)dy.
$$
Let us denote the operator $\tilde{Y}_{lm} \ast \wedge_2$ by $Y'_{lm}$. First of all we consider $||\phi_j||_{L^2(\Omega_j)}$. Expanding $\alpha_j(y) - \alpha_j(x)$ in a Taylor series, we obtain

$$\alpha_j(y) - \alpha_j(x) = \sum_{1 \leq |\nu| \leq q-1} \frac{1}{\nu!} \left( \frac{\partial}{\partial x} \right)^\nu \alpha_j(x)(y-x) + \sum_{|\nu| = q} \alpha_{j,\nu}(x, y)(x-y)^\nu$$

where $q$ will be determined later. It follows that

$$\phi_j(x) = \sum_{|\nu| \leq q-1} \frac{1}{\nu!} (-1)^{|\nu|} \left( \frac{\partial}{\partial x} \right)^\nu \alpha_j(x) \sum_{lm} a_{lm}(x)(x^\nu Y'_{lm})u + \varphi_j^{(2)}(x)$$

where

$$(5.10) \quad \varphi_j^{(2)}(x) = \sum a_{lm}(x) \int \alpha_{j,\nu}(x, y)(x-y)^\nu Y'_{lm}(x-y)u(y)dy.$$ 

Now the operators $H_\nu = \sum a_{lm}(x)(x^\nu Y'_{lm})$ are singular integral operators which operate on $L^2$ as continuous linear operators since $\sup_x |a_{lm}(x)|$ is a rapidly decreasing sequence (more precisely, for any positive integer $\sigma$ we have

$$\sum_{l \geq 0} R^\sigma \sup_x |a_{lm}(x)| < \infty$$

(see Calderon-Zygmund [1].)

Hence for the first sum,

$$(5.11) \quad \varphi_j^{(1)}(x) = \sum_{|\nu| \leq q-1} \frac{(-1)^{|\nu|}}{\nu!} \left( \frac{\partial}{\partial x} \right)^\nu \alpha_j(x) \cdot H_\nu u$$

and we have

$$(5.12) \quad \|\varphi_j^{(1)}\|^2_{L^2(\Omega_j)} \leq c(q)q(q-1) \sum_{1 \leq |\nu| \leq q-1} \|H_\nu u\|^2_{L^2(\Omega_j)}.$$ 

To majorize the second sum $\varphi_j^{(2)}(x)$ we begin by considering a typical term $(x^\nu \cdot Y') \ast u$. We have

$$|(x^\nu Y') \ast u| = \left| \int (x-y)^\nu Y'(x-y) \cdot u(y) \, dy \right|$$
5. Extension of Gårding’s...

\[ \leq \int |(x - y)^{Y'}(x - y)||u(y)|dy \]

\[ = \left( \int_{\Omega_j} + \int_{\partial\Omega_j} \right) |(x - y)^{Y'}(x - y)||u||(y)|dy \]

where \( \Omega'_j \) is a sphere of radius \( 6\varepsilon \) about \( x^{(j)} \). The first integral is majorized by

\[ \sup_x |x^Y(x)||u||_{\Omega'_j} (\text{vol } \Omega'_0)^{\frac{1}{2}} \]

and the second integral is majorized by

\[ \sup_x |x^{2p}\cdot x^Y(x)|\int_{\partial\Omega'_j} \frac{|u(y)|}{|x - y|^{2p}}dy. \]

Now \( I \equiv \int_{\partial\Omega'_j} \frac{|u(y)|}{|x - y|^{2p}}dy \leq \sum_k \int_{\partial\Omega'_j} \frac{|u(y)|}{|x - y|^{2p}}dy \) where the sum is taken is taken over all the \( \omega_k \) such that \( d(\Omega_j, \omega_k) \geq 3\varepsilon, \Omega_j \) being the support of \( \alpha_j \). Hence

\[ I \leq \sum_k 2^{2p} d(\omega_k, \Omega_j)^{-2p} ||u||_{\omega_k} (\text{vol } \omega_0)^{\frac{1}{2}}. \]

Hence the second integral is majorized by

\[ \sup_x (|x^{2p}\cdot x^Y(x)|)(\text{vol } \omega_0)^{\frac{1}{2}} 2^{2p} \left( \sum_k d(\omega_k, \Omega_j)^{-2p} ||u||_{\omega_k} \right) \]

where the \( \omega_k \) occurring in the summation are such that \( d(\omega_k, \Omega_j) \geq 3\varepsilon \). For \( |\nu| = q \) sufficiently large it can be shown that

\[ K(\nu) = \sum_{l \geq 0} \sup_x |a_{lm}(x)| \cdot \sup_x |x^Y_{lm}(x)| < \infty \]

and

\[ K(\nu, p) = \sum_{l \geq 0} \sup_x |a_{lm}(x)| \cdot \sup_x |x^{2p}\cdot x^Y_{lm}(x)| < \infty \]
for $p$ sufficiently large. So we have

$$
\|\varphi_j^{(2)}\|_{\Omega_k}^2 \leq \int_{\Omega_k} \left( \sum_{p \leq q} \sum_{m} |u_{lm}(x)| \int \alpha_j(x,y)(x-y)^p Y_{lm}^*(x-y) |u(y)| \, dy \right) \, dx
$$

(5.13) \hspace{1cm} \leq c \left( \sum_{p \leq q} K(v) \|u\|_{\Omega_k}^2 + K(v, p) \left( \sum_k d(\omega_k, \Omega_j)^{-2p} \|u\|_{\Omega_k}^2 \right) \right)^2.

But by Schwarz inequality we have

$$
\sum_k d(\omega_k, \Omega_j)^{-2p} \|u\|_{\omega_k} \leq \left( \sum_k d(\omega_k, \Omega_j)^{-2p} \right)^{\frac{1}{2}} \left( \sum_k d(\omega_k, \Omega_j)^{-2p} \|u\|_{\omega_k}^2 \right)^{\frac{1}{2}}
$$

and since $(\sum_k d(\omega_k, \Omega_j)^{2p}) < K$, a constant we obtain after summing over $j$

$$
\sum_k d(\omega_k, \Omega_j)^{-2p} \|u\|_{\omega_k} \leq K_p \sum_k \|u\|_{\omega_k}^2 \leq K_p r \|u\|^2
$$

where $K_p$ is a constant depending on $p$ and $r$ is the maximum number of balls $\omega_k$ containing a point of $\mathbb{R}^n$. Substituting in (5.13)

$$
\sum_k \|\varphi_j^{(2)}\|_{\Omega_k}^2 \leq c \|u\|^2
$$

which together with (5.12) gives the estimate

(5.14) \hspace{1cm} \sum_k \|\varphi_k\|_{\Omega_k}^2 \leq c' \|u\|^2.

It remains to estimate $\|\varphi_j\|_{C\Omega_k}$ in order to complete the proof of the lemma. For $x \in C\Omega_k$ a typical term in the expression for $\varphi_k(x)$ is of the form

$$
\psi(x) = \int_{\omega_k} Y'_{lm}(x-y) \alpha_j(y) u(y) \, dy
$$
from which we obtain as before the estimate
\[
|\psi(x)| \leq \sup_x |x|^{2p} Y_{lm}(x) \cdot \int_{\Omega_j} \frac{|u(y)|}{|x-y|^{2p}} dy
\]
\[
\leq \sup_x |x|^{2p} Y_{lm}(x) \cdot \|u\|_{\Omega_j} d(x, \Omega_j)^{-2p} (\text{vol } \omega_0)^{\frac{1}{2}}.
\]
Hence
\[
\|\psi\|_{C_{\Omega_k}} \leq \sup_x |x|^{2p} Y_{lm}(x) \cdot \|u\|_{\Omega_j} (\text{vol } \omega_0)^{\frac{1}{2}} \left( \int_{|x| \leq 2e} \frac{1}{d(x, \omega_1)^{4p}} dx \right)^{\frac{1}{2}}.
\]

Taking $4p > n$ and observing that $K(n, p) < \infty$ we see that
\[
\|\psi\|_{C_{\Omega_k}}^2 \leq c'' \|u\|_{\Omega_j}^2,
\]
and again, summing over $k$,
\[
(5.15) \quad \sum_k \|\psi_k\|_{C_{\Omega_k}}^2 \leq c'' \cdot r \|u\|^2.
\]

This completes the proof of the lemma.

The following is an extension to singular integral operators of Gårding’s inequality for elliptic differential operators.

**Proposition 1.** Let $H$ be a singular integral operator such that its symbol $\sigma(H) = h(x, \xi) \in C_\infty^\beta$ with $\beta > 0$ satisfies
\[
(5.16) \quad |h(x, \xi)| \geq \tau > 0
\]
for every $x \in \mathbb{R}^n$ and every vector $\xi$, $\delta$ being a positive constant. Then there exists a $\delta' > 0$ such that
\[
(5.17) \quad \|H \land u\|^2 \geq \delta' \|u\|^2 - \gamma \|u\|^2
\]
for every $u \in \mathcal{D}^1_{L^2}$ where $\gamma$ is a positive constant.
Proof. $H$ being a singular integral operator we know that $||Hu|| \leq AM||u||$ where $A$ is a constant depending only on $n$ and

$$M = \sum_{|\nu| \leq n} \sup_{x \in \mathbb{R}^n, |\xi| \geq 1} \left| \left( \frac{\partial}{\partial \xi} \right)^{\gamma} \gamma(H)(x, \xi) \right|.$$ 

Given a $\delta > 0$ there exists a number $\epsilon > 0$ such that for every $x_0 \in \mathbb{R}^n$ and for every $u \in L^2$ (5.18)

$$||\left( H - H(x_0) \right) u ||^2 \leq \frac{\delta^2}{4} ||u||^2$$

where $H(x_0)$ is the singular integral operator with constant coefficients such that $\sigma(H(x_0))(\xi) = \sigma(H)(x_0, \xi)$. $H(x_0)$ is the tangential operator at $x_0$. $\epsilon$ can be chosen independent of the position of $x_0$. Consider the $C^\infty$ partition of unity introduced earlier,

$$\alpha_j(x) \geq 0, \quad \alpha_j \in \mathcal{D}, \quad \sum \alpha_j^2(x) \equiv 1.$$

As we have

$$||H \wedge u||^2 = \sum ||\alpha_j \wedge H \wedge u||^2$$

it is sufficient to prove the inequality for $\alpha_j \wedge H \wedge u$.

$$||\alpha_j \wedge u||^2 \geq \frac{1}{2}||H\alpha_j \wedge u||^2 - ||(H\alpha_j - \alpha_jH) \wedge u||^2$$

$$\geq \frac{1}{2}||H\alpha_j \wedge u||^2 - 2||H(\wedge \alpha_j - \alpha_j \wedge)u||^2$$

$$= 2||H(\wedge \alpha_j - \alpha_j \wedge)u||^2.$$

Now we have

$$\sum_j H(\wedge \alpha_j - \alpha_j \wedge)u||^2 \leq \sum_j ||H||^2 ||(\wedge \alpha_j - \alpha_j \wedge)u||^2 \leq c_1||H||^2 ||u||^2$$

$$\leq c_1||u||^2$$

and by lemma $2$

$$\sum_j ||(H \wedge \alpha_j - \alpha_j (H \wedge))u||^2 \leq c_2||u||^2.$$
where \( c_1 \) and \( c_2 \) are constants depending only on the norm of \( H \) and \( n \).

Hence

\[
\|H \wedge u\|^2 \geq \frac{1}{2} \sum_j \|H \alpha_j \wedge u\|^2 - c_3\|u\|^2
\]

and we have only to consider \( \|H \alpha_j \wedge u\|^2 \).

For this purpose let \( H(x^{(j)}) \) be the singular integral operator whose symbol is \( h(x^{(j)}, \xi) \), so that

\[
\sigma(H - H(x^{(j)})) = h(x^{(j)}, \xi) - h(x^{(j)}, \xi).
\]

So we have

\[
\|H \alpha_j \wedge u\|^2 \geq \frac{1}{2}\|H(x^{(j)})\alpha_j \wedge u\|^2 - \|(H - H(x^{(j)}))\alpha_j \wedge u\|^2.
\]

From the condition that \( |h(x, \xi)| > \delta \) we have

\[
\frac{1}{2}\|H(x^{(j)})\alpha_j \wedge u\|^2 \geq \frac{\delta^2}{2}\|\alpha_j \wedge u\|^2.
\]

As in lemma 2, let \( \Omega_j \) denote the ball of radius \( 2\varepsilon \) and centre \( x^{(j)} \).

We decompose the second term into a sum

\[
\|(H - H(x^{(j)}))\alpha_j \wedge u\|^2 = \|(H - H(x^{(j)}))\alpha_j \wedge u\|_{\Omega_j}^2 + \|(H - H(x^{(j)}))\alpha_j \wedge u\|_{\Omega_j}^2.
\]

As mentioned at the beginning of the proof, the first term is majorized by \( \frac{\delta^2}{4}\|\alpha_j \wedge u\|^2 \). For the second term we have

\[
\|(H - H(x^{(j)}))\alpha_j \wedge u\|_{\Omega_j}^2 \leq 2\|(H - H(x^{(j)}))(\alpha_j \wedge \alpha_j)u\|_{\Omega_j}^2 + 2\|(H - H(x^{(j)})) \wedge \alpha_j u\|_{\Omega_j}^2.
\]

By lemma 1, \( \|(H - H(x^{(j)})) \wedge \alpha_j u\|_{\Omega_j}^2 \leq c(n, \eta)M'\|\alpha_j u\|^2 \) and since
(H − H(ε₁)) is a singular integral operator we obtain from lemma 2 the inequality

$$\sum_j (H - H(\varepsilon_j))(\alpha_j \wedge u)_{\mathcal{C}_j}^2 \leq c\|u\|^2.$$ 

Hence

$$\|H \wedge u\|^2 \geq \frac{\delta^2}{4} \sum_j \|\alpha_j \wedge u\|^2 - c(n, \eta)\|u\|^2 \geq \frac{\delta^2}{4} \|u\|^2 - \gamma \|u\|^2$$

which completes the proof of the inequality. □

**Proposition 2.** Let H be a singular integral operator whose symbol $$\sigma(H) = h(x, \xi) \in C^\infty$$ with $$\beta > 0$$. Let $$h(x, \xi)$$ satisfy the condition

$$\text{Re} h(x, \xi) \leq -\delta, \delta > 0 \text{ for every } x \in \mathbb{R}^n \text{ and every vector } \xi.$$ (5.19)

Then there exists a $$\delta' > 0$$ such that

$$((H + H^* \wedge u, \wedge u) \leq -\delta' \|u\|^2 + \gamma \|u\|^2 \text{ for } u \in \mathcal{D}_1$$ (5.20)

where $$\gamma$$ is a constant depending only on $$M, \delta$$ and $$n, \delta' (\delta' < \delta)$$ can be chosen as near $$\delta$$ as one wishes.

**Proof.** One can write $$H^* \wedge = H^* \wedge + (H^* - H^*) \wedge$$. By Th. 4 of § 4 $$(H^* - H^*)$$ is a bounded operator in $$L^2$$ and hence it is enough to prove that for $$P = H + H^*, (P \wedge u, \wedge u)$$ satisfies an inequality of the required kind. The symbol $$\sigma(P) = h(x, \xi) + h(x, \xi)$$ is real and $$\leq -2\delta$$. Let $$\alpha_j \in \mathcal{D}, \alpha_j(x) \geq 0, \sum \alpha_j^2(x) \equiv 1$$ be a $$C^\infty$$ partition of unity as in lemma 1. Then

$$(P \wedge u, \wedge u) = \sum_j (\alpha_j P \wedge u, \alpha_j \wedge u) = \sum_j (P \alpha_j \wedge u, \alpha_j \wedge u) - \sum_j ((P \alpha_j - \alpha_j P) \wedge u, \alpha_j \wedge u).$$
For any $\epsilon' > 0$ we have, by Schwarz’s inequality
\[
(P\alpha_j - \alpha_j P) \wedge u, \alpha_j \wedge u \leq \|(P\alpha_j - \alpha_j P) \wedge u\| \cdot \|\alpha_j \wedge u\|
\leq \epsilon' \|\alpha_j \wedge u\|^2 + \frac{1}{\epsilon'}\|(P\alpha_j - \alpha_j P) \wedge u\|^2.
\]

From lemma 2 we have
\[
\sum_j \|(P\alpha_j - \alpha_j P) \wedge u\|^2 \leq 2 \sum_j P(\alpha_j \wedge - \wedge \alpha_j) u||^2 + 2 \sum_j (P \wedge \alpha_j - \alpha_j (P \wedge) u||^2
\leq \epsilon' ||u||^2
\]
and we have only to estimate $(P\alpha_j \wedge u, \alpha_j \wedge u)$. Write $P = P(\chi^j) + (P - P(\chi^j))$ where, as before, $P(\chi^j)$ is the singular integral operator whose symbol is $\sigma(P(\chi^j), \xi)$. Since $\sigma(P(\chi^j), \xi) \leq -2\delta$ we have
\[
(P(\chi^j))\alpha_j \wedge u, \alpha_j \wedge u \leq -2\delta \|\alpha_j \wedge u\|^2.
\]

Again by Schwarz’s inequality
\[
\left\| (P - P(\chi^j))\alpha_j \wedge u, \alpha_j \wedge u \right\| \leq \left\| (P - P(\chi^j)) \alpha_j \wedge u \right\| \cdot \|\alpha_j \wedge u\|
\leq \frac{\epsilon''}{4}||\alpha_j \wedge u||^2 + \frac{4}{\epsilon''}||\epsilon'(P - P(\chi^j))\alpha_j \wedge u||^2.
\]

Now, as in Prop. 1
\[
\| (P - P(\chi^j))\alpha_j \wedge u \|^2 \leq \eta(\epsilon)||\alpha_j \wedge u||^2 + \mu||(\alpha_j \wedge - \wedge \alpha_j) u||^2 + \mu||\alpha_j u||^2.
\]

Putting all these inequalities together one sees that
\[
(P \wedge u, \wedge u) \leq \left( -2\delta + \epsilon' + \frac{\epsilon''}{4} + \frac{4}{\epsilon''} \eta(\epsilon) \right) \sum_j \|\alpha_j \wedge u\|^2 + \mu \sum_j \|\alpha_j \wedge - \wedge \alpha_j) u||^2 + ||u||^2.
\]

Choosing $\epsilon' = \frac{\epsilon''}{2}$, near $\delta$ and fixing $\epsilon$ to have $\frac{4\eta(\epsilon)}{\epsilon''}$ small enough to make $-2\delta + \epsilon' + \frac{\epsilon''}{4} + \frac{4}{\epsilon''} \eta(\epsilon)$ as near $\delta$ as required and using lemma 2 the desired inequality follows.
We shall now prove a lemma which we require later. It is analogous to lemma 2. We define, for any real $s$, $\wedge^s$ by $\left(\wedge^s u\right) = |\xi|^s \hat{u}$.

\[ \square \]

**Lemma 3.** Let $H$ be a singular integral operator whose symbol $\sigma(H) = h(x, \xi) \in C^\infty_{\beta}$, with $\beta = \infty$. Then for any $u \in L^2$

\begin{equation}
\lVert (H \wedge^s - \wedge^s H) \wedge^\sigma u \rVert \leq c \lVert u \rVert \text{ for } s, \sigma \geq 0 \text{ with } s + \sigma \leq 1.
\end{equation}

\[ \text{(5.21)} \]

**Proof.** Let $\alpha \in D_\xi$ be such that $0 \leq \alpha(\xi) \leq 1$, $\alpha(\xi) \equiv 1$ on $|\xi| \leq 1$ and vanish outside $|\xi| \geq 2$. Writing $|\xi|^s = |\xi|^s \alpha(\xi) + |\xi|^s (1 - \alpha(\xi))$ we decompose the operator into a sum $\wedge^s = \wedge^s_0 + \wedge^s_1$ with $\sigma(\wedge^s_0) = |\xi|^s \alpha(\xi)$ and $\sigma(\wedge^s_1) = |\xi|^s (1 - \alpha(\xi))$. As $|\xi|^s \alpha(\xi)$ has compact support $\wedge^s_0$ defines a continuous linear operator in $L^2$ and hence it is enough to prove that

\[ \lVert (H \wedge^s_1 - \wedge^s_1 H) \wedge^\sigma u \rVert \leq c \lVert u \rVert. \]

Expanding $\sigma(H)$ in terms of spherical harmonics $Y_{lm}$ as in lemma 2 and taking the inverse Fourier image we have

\[ H = a_0(x) + \sum a_{lm}(x) \hat{Y}_{lm} \ast. \]

Let $P = a(x) \cdot \hat{Y} \ast$ be a term in the sum. We consider

\[ (P \wedge^s_1 - \wedge^s_1 P) \wedge^\sigma u = \int (a(x) - a(y)) \wedge^s_1 (x - y) \wedge^\sigma \varphi(y) dy \]

where $\varphi(y) = (\hat{Y} \ast u)(y)$. Expand $a(x) - a(y)$ in Taylor series upto order $q, q$ to be determined later:

\[ a(x) - a(y) = - \sum \frac{1}{v!} \left( \frac{\partial}{\partial x} \right)^v a(x) \cdot (y - x)^v - \sum a_v(x, y) (y - x)^v. \]

This gives

\[ (P \wedge^s_1 - \wedge^s_1 P) \wedge^\sigma u = \sum \frac{(-1)^{v+1}}{v!} \left( \frac{\partial}{\partial x} \right)^v a(x) \cdot (x^v \wedge^s_1 \ast \wedge^\sigma \varphi) \]

\[ + \sum (-1)^{v+1} \int \frac{a_v(x, y)}{v!} (x - y)^v \wedge^s_1 (x - y) (\wedge^\sigma \varphi)(y) dy. \]

\[ \text{(5.22)} \]
We estimate the first sum in (5.22). We have
\[
\| (x^\nu \wedge s_1) \| = \left\| \frac{\partial}{\partial \xi} (1 - \alpha(\xi)) |\xi|^{s_1} \right\| \leq c_\nu (1 + |\xi|)^{s-|\nu|}.
\]
Hence
\[
\|(x^\nu \wedge s_1) \ast (\wedge^\sigma \varphi)\| \leq C_\gamma \|(1 + |\xi|)^{s-|\nu|} \varphi\| \leq c_\nu \|\varphi\|
\]
since \(s + \sigma \leq 1\) and \(|\nu| \geq 1\). Summing over \(|\nu| \leq q - 1\), we obtain
\[
(5.23) \sum_{|\nu| \leq q-1} \left| \frac{(-1)^{|\nu|+1}}{\nu!} \left[ \left( \frac{\partial}{\partial x} \right)^{\nu} a \right] \left[ (x^\nu \wedge s_1) \ast (\wedge^\sigma \varphi) \right] \right| \leq c(q) \|\varphi\| |a|_{q-1}
\]
where
\[
|a|_p = \sup_{x,|\nu| \leq p} \left| \frac{\partial}{\partial x} \right|^\nu a(x).
\]
Since \(\|\varphi\| = \|\hat{\varphi}\| = \|Y(\xi)\hat{u}\| \leq |Y|_{0}\), \(\|\varphi\|\) the right hand side of the inequality (5.23) is less than or equal to
\[
c(q)|a|_{q-1}|Y|_{0} \cdot \|\varphi\|.
\]

Now we estimate the second sum. Write \(|\xi|^{\nu}\) as
\[
|\xi|^{\nu} = \alpha(\xi)|\xi|^{\nu} + (1 - \alpha(\xi))|\xi|^{\nu} = \alpha(\xi)|\xi|^{\nu} + |\xi| \left( (1 - \alpha(\xi))|\xi|^{\nu-1} \right)
\]
where \(\alpha(\xi) \in \mathcal{D}_+, \alpha(\xi) \equiv 1\) in a neighbourhood of the origin. Thus \(\wedge^{\nu} = B_0 \wedge B_1\) where \(B_0\) and \(B_1\) are bounded operators in \(L^2\). Hence we have only to consider the part containing \(\wedge B_1\). Denote by \(\psi_{\nu}\) the integral
\[
\psi_{\nu}(x) = \int a_{\nu}(x,y)(x-y)^\nu \wedge_1^s (x-y) \wedge B_1 \varphi(y) dy.
\]

Now we can write \(|\xi| = \Sigma \xi_j |\xi_j|\) and if \(R_j\) denote the Riesz operators defined by \((\widehat{R_j f}) = \frac{\xi_j}{|\xi|} \hat{f}\) we can write \(\wedge = \Sigma \frac{\partial}{\partial x_j} R_j\). Substituting for \(\wedge\) in \(\psi_{\nu}(x)\)
\[
\psi_{\nu}(x) = -\Sigma_j \int \frac{\partial}{\partial x_j} \left[ a_{\nu}(x,y)(x-y)^\nu \wedge_1^s (x-y) \right] \cdot (R_j B_1 \varphi)(y) dy.
\]
We observe that \((x^\nu \wedge s_1)\) is a bounded function together with its derivatives of the first order for \(|\nu| \geq n + 2\). In fact its Fourier image is
\[
\left( \frac{\partial}{\partial \xi} \right)^\nu \{(1 - \alpha(\xi))|\xi|^v\}
\]
and
\[
|x^\nu \wedge s_1| \leq \int |(x^\nu \wedge s_1)| d\xi \leq c_\gamma \int (1 + |\xi|)^{s-|\nu|} d\xi < \infty.
\]

We can write
\[
\psi_\nu(x) = \int a_\nu(x,y)(x - y)^\nu \wedge s_1 (x - y)(\wedge B_1 \varphi)(y) dy
\]
\[
= -\Sigma \int \left[ \frac{\partial a_\nu}{\partial y_j}(x,y) \right] (x - y)^\nu \wedge s_1 (x - y)(R_j B_1 \varphi)(y) dy + \int a_\nu(x,y) \left( \frac{\partial (x - y)^\nu \wedge s_1 (x - y)}{\partial y_j} \right) (R_j B_1 \varphi)(y) dy.
\]

Set
\[
(5.24) \quad \psi_\nu(x) = I_1 + I_2.
\]

We estimate \(I_1\) and \(I_2\) separately.
\[
|I_1| \leq \sum_j \left| \int \left[ \frac{\partial a_\nu}{\partial y_j}(x,y) \right] (x - y)^\nu \wedge s_1 (x - y)(R_j B_1 \varphi)(y) dy \right|
\]
\[
\leq |a|_{q+1} \sum_j \int |(x - y)^\nu \wedge s_1 (x - y)| |(R_j B_1 \varphi)(y)| dy.
\]

The Fourier image of \((1 + |x|^2)^p x^\nu \wedge s_1 (x)\) is
\[
\left( \frac{1}{2\pi} \right)^|v| \left( \frac{\partial}{\partial \xi} \right)^{2p+|v|} \left[ (1 - \alpha(\xi)|\xi|^v \right] + \left( \frac{1}{2\pi} \right)^{2p+|v|} \Delta_\xi^{p} \left( \frac{\partial}{\partial \xi} \right)^{v} \left[ (1 - \alpha(\xi)|\xi|^v\right]
\]
and hence
\[
|x^\nu \wedge s_1 (x)| \leq \frac{1}{1 + |x|^2} \left\{ \left( \frac{1}{2\pi} \right)^{|v|} \int |(x^\nu \wedge s_1 (x)| \right\} d\xi
\]
Thus we have

\[
\frac{1}{2\pi} \int |\Delta f| \left( \frac{\partial}{\partial \xi} \right)^\nu [(1 - \alpha(\xi))|\xi|^\nu] \, d\xi
\]

and similarly we have

\[
\left| \left( \frac{\partial}{\partial x_j} \right) (x^\nu \wedge \psi) (x) \right| \leq \frac{1}{1 + |x|^{2p}} (C_1(\nu) + C_2(p, \nu))
\]

For sufficiently large \( p \) the quantities \( C_2(p, \nu), \ C_2'(p, \nu) \) are finite. Thus we have

(5.25) \quad |J_1| < |a_{q+1}| \sum \int \frac{|(R_jB_1\varphi)(y)|}{1 + |x - y|^{2p}} \, dy

\[
|J_2| \leq |a_q| \sum \int \frac{\partial}{\partial y_j} \left[ (x - y)^\nu \wedge \psi (x - y) \right] \cdot (R_jB_1\varphi)(y) \, dy
\]

\[
\leq |a_q| \sum \int \frac{\partial}{\partial y_j} \left[ (x - y)^\nu \wedge \psi (x - y) \right] (R_jB_1\varphi)(y) \, dy
\]

(5.26) \quad |J_2| \leq M(p)|a_q| \sum \int \frac{(R_jB_1\varphi)(y)}{1 + |x - y|^{2p}} \, dy.

This leads to the inequality

\[
||I_1(x)||_{L^2} \leq |a_{q+1}| \sum_{j=1}^n ||R_jB_1\varphi||_{L^2} \left( \int \frac{1}{1 + |x|^{2p}} \, dx \right)
\]

because of the Hausdorff-Young theorem. We have the same kind estimate for \( ||I_2(x)||_{L^2} \).

Hence

\[
||\psi|| \leq C_3(n) ||(R_jB_1\varphi)|| \cdot |a_{q+1}| \leq C_4(n)|a_{q+1}||\varphi|
\]
Now summing up for all \( l, m \) we have for any \( u \in L^2 \)

\[
\|(H \wedge \xi \mathring{H})^{\sigma} u\| \leq \sum_{l,m} \|(a_{lm} \wedge \xi a_{lm})^{\sigma} (\mathring{Y}_{lm} * u)\|
\]

\[
\leq C_5(n) \left( \sum_{l,m} |a_{lm}| + |a_{lm}| \sum_{l,m} |Y_{lm}(\xi)|^\beta \right) \|u\|
\]

\[
\leq C_5(n, s, \sigma) M \|u\|_{L^2}
\]

and this completes the proof of the lemma. \( \square \)

The following is a generalization of Friedrichs’ lemma to singular integral operators (see Mizohota [1]).

**Proposition 3.** Let \( H \) be a singular integral operator such that its symbol \( \sigma(H) = h(x, \xi) \in C^\infty_{1+\sigma} \), \( \sigma > 0 \). Let \( C_\delta u \) denote, for \( u \in L^2 \), the commutator \([H \wedge \phi_\delta] u \) where \( \phi_\delta \) is the mollifier of Friedrichs.

Then

1. \( \|C_\delta u\| \leq c M'\|u\| \)
   where \( M' = |a_0| + \sum |a_{lm}| |Y_{lm}|^\beta \) and \( c \) depends only on \( \phi \) and \( n \)

2. \( C_\delta u \to 0 \) weakly in \( L^2 \) as \( \delta \to 0 \).

**Proof.** We expand \( h(x, \xi) \) in spherical harmonics \( Y_{lm}(\xi) \)

\[
h(x, \xi) = a_0(x) + \sum_{l,m} a_{lm}(x) Y_{lm}(\xi)
\]

and hence we can write, denoting the inverse Fourier image of \( Y_{lm} \) by \( \mathring{Y}_{lm} \)

\[
Hu(x) = a_0(x) u(x) + \sum_{l,m} a_{lm}(x) (\mathring{Y}_{lm} * u)(x).
\]

To prove (1) it is sufficient to prove it for \( u \in \mathcal{D} \). Now \n
\[
C_\delta u = [H \wedge \phi_\delta] u = H \wedge (u * \phi_\delta) - (H \wedge u) * \phi_\delta
\]
Consider a typical term of this sum:
\[ a_{lm}(x)(\tilde{Y}_{lm} \wedge (u \ast \varphi_\delta)) - a_{lm}(x)(\tilde{Y}_{lm} \wedge u) \ast \varphi_\delta \]
and substitute \( \sum \frac{\partial}{\partial x_j} R_j \) for \( \wedge \) where \( R_j \) are the Riesz operators. Put \( \psi_{lm}(x) = \tilde{Y}_{lm} \ast R_j \ast u \). We have
\[
\begin{align*}
&= a_{lm}(x) \left( \frac{\partial}{\partial x_j} (\tilde{Y}_{lm} \ast R_j \ast u) \ast \varphi_\delta \right) - a_{lm}(x) \left( \frac{\partial}{\partial x_j} (\tilde{Y}_{lm} \ast u) \ast \varphi_\delta \right) \\
&= a_{lm}(x) \left( \frac{\partial}{\partial x_j} \psi_{lm}(x) \ast \varphi_\delta \right) - a_{lm}(x) \left[ \frac{\partial}{\partial x_j} \psi_{lm}(y) \right] \varphi_\delta(x - y) dy \\
&= \int \left[ a_{lm}(x) - a_{lm}(y) \right] \left[ \frac{\partial}{\partial y_j} \psi_{lm}(y) \right] \varphi_\delta(x - y) dy
\end{align*}
\]
where the integral is taken in the sense of distributions. By definition this is
\[
- \int \frac{\partial}{\partial y_j} \left[ (a_{lm}(x) - a_{lm}(y)) \varphi_\delta(x - y) \right] \psi_{lm}(y) dy
\]
where the integral is taken in the usual sense.

Now,
\[
\begin{align*}
&\int \left| \frac{\partial}{\partial y_j} \left[ (a_{lm}(x) - a_{lm}(y)) \varphi_\delta(x - y) \right] \psi_{lm}(y) \right| dy \\
&\leq \left| \int \psi_{lm}(y) (a_{lm}(x) - a_{lm}(y)) \frac{\partial \varphi_\delta}{\partial y_j} (x - y) dy \right| + \left| \int \psi_{lm}(y) \varphi_\delta(x - y) \frac{\partial a_{lm}(y)}{\partial y_j} dy \right| \\
&\leq \|\psi_{lm}\| \left\{ 2|a_{lm}||\varphi_\delta|_1 + |a_{lm}|_1 \cdot c_1(\delta, n) \right\} \\
&\leq c(\delta, n)|a_{lm}|_1 \cdot \|Y(\xi)\|_0 \cdot \|u\|
\end{align*}
\]
which proves (1).
To prove (2) let \( v \in L^2 \) and consider
\[
\int v(x) \int \psi_{lm}(y) \frac{\partial}{\partial y_j}((a_{lm}(x) - a_{lm}(y))\varphi_\delta(x - y)) \, dy \, dx
\]
\[
= \int v(x) \int \psi_{lm}(y) \left\{ \frac{\partial \varphi_\delta}{\partial y_j} (x - y) \cdot (a_{lm}(x) - a_{lm}(y)) - \varphi_\delta(x - y) \frac{\partial a_{lm}}{\partial y_j} (y) \right\} \, dy \, dx
\]
\[
= \int v(x) \int \psi_{lm}(y) \left\{ \sum_k (x_k - y_k) \frac{\partial a_{lm}}{\partial y_k} (y) \cdot \frac{\partial \varphi_\delta}{\partial y_j} (x - y) - \varphi_\delta(x - y) \frac{\partial a_{lm}}{\partial y_j} (y) \right\} \, dy \, dx
\]

where \( \sigma(x,y) = a_{lm}(x) - a_{lm}(y) - \sum_k (x_k - y_k) \frac{\partial a_{lm}}{\partial y_k} (y) \). Let
\[
k_1(y, x - y) = \sum_k (x_k - y_k) \frac{\partial a_{lm}}{\partial y_k} (y) \cdot \frac{\partial \varphi_\delta}{\partial y_j} (x - y) - \varphi_\delta(x - y) \frac{\partial a_{lm}}{\partial y_j} (y)
\]

(5.28)

and (5.28')
\[
k_2(y, x - y) \geq \sigma(x,v) \frac{\partial \varphi_\delta}{\partial y_j} (x - v)
\]

Then \(|k_2(y, x - y)| \leq c |a_{lm}(x)|_1 \sigma |x - y|^{1+\sigma} \left| \frac{\partial \varphi_\delta}{\partial x_j} (x - y) \right| \).

Applying the Hausdorff-Young inequality we have
\[
\| v \int k_2(y, x - y) \, dx \| \leq c |a_{lm}|_1 \sigma \left( \sum \int |x - y|^{1+\sigma} \left| \frac{\partial \varphi_\delta}{\partial x_j} (x - y) \right| \, dx \right) \cdot \|v\|
\]

(5.29)

where \( \varepsilon(\delta) = \sum \int |x|^{1+\sigma} \left| \frac{\partial \varphi_\delta}{\partial x_j} \right| \, dx \to 0 \) as \( \delta \to 0 \). On the other hand we observe that
\[
\int k_1(y, z) \, dz = \int \frac{\partial}{\partial z_j} \left\{ \sum_r z_r \frac{\partial a_{lm}}{\partial y_r} (y) \cdot \varphi_\delta \right\} \, dz = 0,
\]
since \( \varphi_\delta \) has compact support. Now consider
\[
\int \int \int k_1(y, x - y) v(x) \psi_{lm}(y) \, dy \, dx = \int \psi_{lm}(y) \, dy \int \int k_1(y, x - y) v(x) \, dx.
\]
The right hand side can be written after a change of variables $z = x - y$ in the form

$$\int \psi_{lm}(y)dy \int v(y + z)k_1(y, z)dz.$$ 

Schwarz inequality gives

$$|\int \psi_{lm}(y)dy \int v(y + z)k_1(y, z)dz| \leq \|\psi_{lm}\| \left|\int k_1(y, z)v(y + z)dz\right|.$$ 

Since $\int k_1(y, z)dz = 0$ we can write

$$\left|\int k_1(y, z)v(y + z)dz\right| = \left|\int k_1(y, z)v(y + z) - v(y)dz\right|.$$ 

We shall now evaluate the right hand side. Let us set

$$\epsilon'(\delta) = \sup_{|h| \leq \delta} \left(\int |v(y + h) - v(y)|^2 dx\right)^{\frac{1}{2}}.$$ 

Schwarz inequality shows that

$$\left|\int k_1(y, x - y)(v(x) - v(y))dx\right|^2 \leq \left(\int |k_1(y, x - y)|dx\right) \left(\int |k_1(y, x - y)||v(x) - v(y)|^2 dx\right).$$ 

Clearly $\int |k_1(y, x - y)|dx \leq c|a_{lm}||\beta|$ where $c$ is a constant depending only on $\varphi$ and $\delta$. Hence integrating both sides of this inequality with respect to $y$ we have

$$\|\int k_1(y, x - y)(v(x) - v(y))dx\|^2 \leq c|a_{lm}||\beta| \int \int |k_1(y, x - y)||v(x) - v(y)|^2 dx dy.$$
\[ = c|a_{lm}|^{2} |z| \leq \delta \int_{|z| \leq \delta} |k_{1}(x-z, z)||v(x) - v(x-z)|^{2} \, dx \, dz. \]

Since \( k_{1}(y, x-y) \) is a bounded function the right side is less than

\[ (5.30) \quad c'|a_{lm}|^{2} (\delta) (\text{vol } \omega_{\delta}). \]

where \( \epsilon'(\delta) \to 0 \) as \( \delta \to 0 \) and \( \omega_{\delta} \) is the ball \(|z| \leq \delta\). Combining the inequalities (5.29) and (5.30) we obtain

\[ \left| \int \int v(x)\psi_{lm}(y)\{k_{1}(y, x-y) + k_{2}(y, x-y)\} \, dy \, dx \right| \]
\[ \leq \|\psi_{lm}\| (c|a_{lm}|^{2}) |v| (\delta) + c''|a_{lm}| |v| (\delta) \]
\[ \leq c''|u| (|a_{lm}| |Y_{lm}|) |v| (\delta) + |a_{lm}| |Y_{lm}| \, \epsilon''(\delta), \]

which tends to 0 as \( \delta \to 0 \). This completes the proof of the proposition. \( \square \)

**Corollary 1.** If we assume \( u \in D^{[1]}_{L^{2}} \) in proposition 3 then

1. \( ||C_{\delta}u||_{D^{[1]}_{L^{2}}} \leq c||u||_{D^{[1]}_{L^{2}}} \)

2. \( C_{\delta}u \to 0 \) weakly in \( D^{[1]}_{L^{2}} \) as \( \delta \to 0 \).

**Proof.** We remark that

\[ \left( \frac{\partial}{\partial x_{j}} (C_{\delta}u) = C_{\delta} \left( \frac{\partial}{\partial x_{j}} \right) + \left[ H_{A}^{(j)}, \varphi_{\delta} \right] u \right. \]

where \( H_{A}^{(j)} \) denotes the singular integral operator defined by

\[ H_{A}^{(j)}u = a^{(j)}_{0} u + \sum_{m} a^{(j)}_{1m} (Y_{lm} * u), a^{(j)}_{1m} = \frac{\partial}{\partial x_{j}} a_{im}, \]

or equivalently

\[ \sigma(H^{(j)} = a^{(j)}_{0}(x) + \sum a^{(j)}_{1m}(x)Y_{lm}(\xi) \in C^{\infty}_{\sigma} \text{ with } \sigma > 0. \]
Now, the latter term of the right hand side in (\(\ast\)) tends to 0 in \(L^2\) as \(\delta \to 0\). In fact,
\[
\left[ H^{(j)} \wedge, \varphi_\delta \ast \right] u = H^{(j)}(\varphi_\delta \wedge u) - H^{(j)}_\delta \\
+ H^{(j)} \wedge u - \varphi_\delta \ast (H^{(j)}_\delta) \text{ and } u \in L^2.
\]

Now applying Proposition 3 to (\(\ast\)) we have the corollary.

From Prop. 1 it can be easily seen that the following proposition holds. This plays the same role as Gårding’s inequality for differential operators.

**Proposition 4.** Let \(\mathcal{H}\) be a square matrix whose elements \(H_{jk}\) are singular integral operators (belonging to \(C^\infty_\beta\)) with their symbols \(\sigma(H_{jk}) = h_{jk}(x, \xi) \in C^\infty_\beta\) with \(\beta > 0\) \((j, k = 1, \ldots, N)\). Suppose \(\sigma(\mathcal{H})\) is the matrix whose element are \(\sigma(H_{jk})(x, \xi)\) and satisfies the hypothesis

\[
|\sigma(\mathcal{H})\alpha| \geq \delta|\alpha| \text{ for every } x, \xi \in \mathbb{R}^n, \delta > 0
\]

where \(\alpha = (\alpha_1, \ldots, \alpha_N)\) is a complex vector in \(\mathbb{C}^N\). Then for every \(u = (u_1, \ldots, u_N) \in \pi D^1_{L^2}\)

\[
||\mathcal{H} \wedge u||^2 \geq \frac{\delta^2}{8} ||\wedge u||^2 - \gamma_1 ||u||^2,
\]

where \(\gamma_1\) is a positive constant.

**Remark.** \(||u||^2\), for \(u = (u_1, \ldots, u_N) \in \pi D^1_{L^2}\), denotes \(||u_1||^2 + \cdots ||u_N||^2\).

The proof runs on the same lines as in the proof of the Prop. 1.

6 Energy inequalities for regularly hyperbolic systems

Let \(\Omega\) denote the subset \(\mathbb{R}^n \times [0, h]\) of \(\mathbb{R}^{n+1}\).

**Definition.** A first order system of differential operators

\[
M = \frac{\partial}{\partial t} \sum A_k(x, t) \frac{\partial}{\partial x_k}
\]

is said to be regularly hyperbolic in \(\Omega\) if
(1) \( A_k(x, t) \) are bounded,

(2) for every \((x, t) \in \Omega \) and \( \xi \in \mathbb{R}^n \) the roots of the systems

\[
\begin{align*}
  \text{(6.2)} & \quad \det \left( \lambda I - \sum A_k(x, t) \cdot \xi_k \right) = 0 \\
  \text{are real and distinct; further if} \lambda_1(x, t, \xi) \cdots \lambda_N(x, t, \xi) \text{are these roots then}
\end{align*}
\]

\[
\text{(6.3)} & \quad \inf_{(x, t) \in \Omega} |\xi| = 1 \quad |\lambda_j(x, t, \xi) - \lambda_k(x, t, \xi)| > 0
\]

We write the system (6.1) in terms of singular integral operators, by putting

\[
\sum A_k(x, t) \frac{\partial}{\partial x_k} = i\mathcal{H}(t) \wedge
\]

where \( \mathcal{H}(t) \) is a matrix of order \( N \) of singular integral operators whose symbol is the matrix

\[
\sigma(\mathcal{H}(t)) = 2\pi \sum A_k(x, t) \frac{\xi_k}{|\xi|}.
\]

Thus (6.1) is written in the form

\[
M = \frac{\partial}{\partial t} - i\mathcal{H}(t) \wedge
\]

If the coefficients are such that \( A_k = A_k(x, t) \in \beta^{1+\sigma}[0, h] \) with \( \sigma > 0 \) then for each fixed \( t, \sigma(H)(x, t, \xi) \in C^{1+\sigma}_0, \sigma > 0. \)

**Proposition 1** (Petrowsky). Let \( M \) be a regularly hyperbolic system with \( A_k \in \beta^{1+\sigma}[0, h] \). Suppose \( A_k(x, t) \) are real. Then there exists a matrix \( \sigma(\mathcal{R}(t)) = \sigma(\mathcal{R})(x, t, \xi) \) except possibly when \( n = 2 \) such that

(i) \( \sigma(\mathcal{R}(t)) \sigma(\mathcal{H}(t)) = \sigma(\mathcal{D}(t)) \sigma(\mathcal{R}(t)) \) where

\[
\sigma(\mathcal{D}(t)) = \begin{pmatrix}
\lambda_1(x, t, \xi) & 0 \\
0 & \ddots \\
0 & 0 & \lambda_N(x, t, \xi)
\end{pmatrix}
\]
6. Energy inequalities for regularly hyperbolic systems

(ii) \( \sigma(\mathcal{H}(t)) = \sigma(\mathcal{H})(x, t, \xi) \) is of class \( C_{1+\sigma} \) for every fixed \( t \), has real elements and further

\[
|\det \sigma(\mathcal{H}(t))| \geq \delta' > 0 \quad \text{for every } (x, t) \in \Omega, \xi \in \mathbb{R}^n.
\]

(iii) the mapping \( t \to \sigma(\mathcal{H}(t)) \in C_{1+\sigma} \) is once continuously differentiable

**Proof.** Since the roots of \( (6.2) \) \( \det (\lambda I - \sum A_k \cdot \xi_k) = 0 \) are real and distinct it follows that \( \lambda_j(x, t, \xi) \) are single valued functions on \( |\xi| = 1 \) for every fixed \( (x, t) \in \Omega \). This follows by the principle of monodromy in the case \( n > 2 \) and in the case \( n = 2 \) by virtue of hyperbolicity.

To see that \( \lambda_j(x, t, \xi) \in C_{1+\sigma}, \sigma > 0 \) for fixed \( t \) denoting by \( P(\lambda, x, t, \xi) = 0 \) the characteristic equation \( \det (\lambda I - \sum A_k \cdot \xi_k) = 0 \) we have from the implicit function theorem

\[
\frac{\partial \lambda_j}{\partial x_k} = -\left( \frac{\partial P}{\partial \lambda} \right) \bigg|_{\lambda = \lambda_j}
\]

and further \( \left| \frac{\partial P}{\partial \lambda} \right|_{\lambda = \lambda_j} \geq d^{N-1} \) where \( d = \inf_{j \neq k, (x, t) \in \Omega, |\xi| = 1} |\lambda_j - \lambda_k| \). \( \Box \)

**Construction of** \( \sigma(\mathcal{H}(t)) \). Suppose \( n \geq 3 \). To find \( \sigma(\mathcal{H}(t)) \) such that \( \sigma(\mathcal{H}(t)) = \sigma(\mathcal{H}) \) is the same, if we write \( \sigma(\mathcal{H}) = (n_{jk}) \), \( \sigma(\mathcal{H}) = (a_{jk}) \), as finding a matrix solution of

\[
\lambda_j n_{jl} = \sum_k n_{jk} a_{kl}.
\]

For a fixed \( j \) the vector \( (n_{j1}, \ldots, n_{jN}) \) is an eigenvector of the matrix \( A = (a_{jk}) \) corresponding to the eigenvalue \( \lambda_j \). Consider the case \( \lambda_j = \lambda_1 \). We assert that the space of eigenvectors at the point \( (x, t, \xi) \) can be given
by explicit expressions (the space of eigenvectors is one dimensional)
in such a way that this vector is continuous in \((x, t, \xi)\) is class \(C_{1+r}^{\infty}\) and
continuously differentiable in \(t\). In fact, if \(M_{jk}(t)\) is the \((j, k)\)-cofactor of \((\lambda_I - A)\) then \((M_{1j}, M_{2j}, \ldots, M_{Nj})\) \((j = 1, \ldots, N)\) span the space of
eigenvectors. As the rank of \((\lambda_I - A)\) is \((N - 1)\) everywhere one of these
is not trivial.

**Remark.** In the case where the coefficients \(A_k(x, t)\) are not real there
will be topological difficulties in the above reasoning which proves the
existence of smooth \(\sigma_{\text{\text{H}}}(x, t, \xi)\). It should however be observed that the
theorem of local existence of smooth \(\sigma_{\text{\text{H}}}(x, t, \xi)\) remains valid. Therefore it would be better to use a partition of unity to derive energy in-
equalities for such systems. Moreover this argument can be applied for
more general hyperbolic systems. (See: Le problème de Cauchy pour
les systèmes hyperboliques et paraboliques, Mem. Coll. Sc., Kyoto

**Proposition 2** (Energy inequality). Let

\[
M = \frac{\partial}{\partial t} - \sum A_k(x, t) \frac{\partial}{\partial x_k}
\]

be a regularly hyperbolic system in \(\Omega\) with the coefficients \(A_k(x, t)\) satis-
fying

\[
A_k \in B^{1+\sigma}[0, h], \quad \frac{\partial}{\partial t} A_k \in B^0[0, h].
\]

Suppose \(B \in \mathbb{B}^0[0, h]\), \(f \in L^2[0, h]\) given. Then, if \(u \in L^2[0, h]\) is a
solution of

\[
\frac{\partial u}{\partial t} - \sum A_k(x, t) \frac{\partial u}{\partial x_k} - B(x, t)u = f
\]

we have the inequality

\[
\|u(t)\| \leq c(h) \left\{ \|u(0)\| + \int_0^t \|f(s)\|ds \right\}.
\]
Proof. First we assume this \( u \in D_{L^2}^1[0,h] \). The given system is written in singular-integral-operator form as

\[
(6.7) \quad \frac{\partial u}{\partial t} - i\mathcal{H}(t) \wedge u - B(t)u = f.
\]

Multiplying this system by the matrix \( \mathfrak{R} \) obtained in Prop. 1 we obtain

\[
\frac{\partial}{\partial t}(\mathfrak{R}u) - i\mathfrak{R}(t)\mathcal{H}(t) \wedge u - (\mathfrak{R}B + \frac{\partial \mathfrak{R}}{\partial t})u = \mathfrak{R}f.
\]

By Prop. \( \mathfrak{R} \circ \mathfrak{R} = D \circ \mathfrak{R} \) which implies that

\[
\mathfrak{R}\mathcal{H} \equiv D\mathfrak{R} \wedge \text{(mod. bounded operators)}
\]

because \( (\mathfrak{R}\mathcal{H}) \wedge \equiv (\mathfrak{R}) \circ \mathcal{H} \wedge \text{(mod bounded operators)} \)

\[
(D\mathfrak{R}) \wedge \equiv (D \circ \mathfrak{R}) \wedge \text{(mod. bounded operators)}
\]

Also \( (D\mathfrak{R}) \wedge = D \wedge \mathfrak{R} + \text{a bounded operator} \), and hence the new system becomes

\[
\frac{\partial}{\partial t}(\mathfrak{R}u) = iD \wedge (\mathfrak{R}u) + (\mathfrak{R}B + \frac{\partial \mathfrak{R}}{\partial t})u + \mathfrak{R}f.
\]

In other words \( \nu = \mathfrak{R}u \) satisfies

\[
\frac{\partial \nu}{\partial t} = iD \wedge \nu + B_1\nu + \mathfrak{R}f
\]

where \( B_1 = \begin{pmatrix} \mathfrak{R}B + \frac{\partial \mathfrak{R}}{\partial t} \end{pmatrix} \) is a bounded operator in view of Prop. 1. Now

\[
\frac{\partial}{\partial t}(v,\nu) = (iD \wedge \nu, v) + (v, iD \wedge \nu) + 2\text{Re}(B_1\nu + \mathfrak{R}f, v)
\]

\[
= i(D \wedge - \wedge D^*)\nu, v) + 2\text{Re}(B_1\nu + \mathfrak{R}f, v).
\]

But \( \wedge D^* = \wedge D \wedge + \text{a bounded operator} \), and since \( D \) is real \( D^\# = D \) and \( \wedge D = D \wedge + \text{a bounded operator} \). Hence \( D \wedge - \wedge D \) is a bounded operator and

\[
\frac{\partial}{\partial t}|v|^2 \leq 2\gamma_1||v||^2 + 2c||u||||v|| + 2||\mathfrak{R}f||||v||.
\]
that is
\[ \frac{\partial}{\partial t} ||v|| \leq \gamma ||v|| + c||u|| + ||f||. \]

By the regular hyperbolicity we have in view of Prop. 1
(6.4) \quad |\text{det}(\mathcal{R}(t))| \geq \delta’ > 0.

Hence by the generalized Garding inequality applied to \( \mathcal{R} \) there exist \( \delta'' > 0 \) and \( \beta > 0 \) such that
(6.8) \quad ||\mathcal{R} \wedge u|| \geq \delta'' ||\wedge u|| - \beta ||u||.

Define
(6.9) \quad ||u|| = ||\mathcal{R} u|| + \beta(\wedge + 1)^{-1} u||

where \((\wedge + 1)^{-1} u \xrightarrow{\mathcal{F}} \frac{1}{(1 + |\xi|)} \hat{u} \). It is clear that \(||u|| \leq c_1 ||u||\) since \( \mathcal{R} \) and \((\wedge + 1)^{-1}\) are bounded. On the other hand
\[ \mathcal{R} u = \mathcal{R} \wedge (\wedge + 1)^{-1} u + \mathcal{R}(\wedge + 1)^{-1} u \]
implies
\[ ||\mathcal{R} u|| \geq ||\mathcal{R} \wedge (\wedge + 1)^{-1} u|| - ||\mathcal{R}(\wedge + 1)^{-1} u|| \]
\[ \geq \delta'' ||\wedge (\wedge + 1)^{-1} u|| - \beta ||(\wedge + 1)^{-1} u|| - ||\mathcal{R}(\wedge + 1)^{-1} u|| \]
\[ \geq \delta'' ||u|| - (\beta' + 1) ||(\wedge + 1)^{-1} u|| \]
which proves that \(||u|| \geq c_2 ||u||\) consequently the norms \(||u||\) and \(||u||\) are equivalent. It is therefore sufficient to prove the energy inequality for the norm \(||u||\).
\[ \frac{\partial}{\partial t} ||u(t)|| = \frac{\partial}{\partial t} (||\mathcal{R} u|| + \beta||\wedge + 1)^{-1} u||) \]
(6.10) \quad \leq \gamma ||\mathcal{R} u|| + c||u|| + ||f|| + \beta \frac{\partial}{\partial t} ||(\wedge + 1)^{-1} u||.
6. Energy inequalities for regularly hyperbolic systems

Considering \( \frac{\partial u}{\partial t} = i \mathcal{H} \wedge u + Bu + f \)

\( (\wedge + 1)^{-1} \frac{\partial u}{\partial t} = i(\wedge + 1)^{-1} \mathcal{H} \wedge u + (\wedge + 1)^{-1}(Bu + f) \)

but \( (\wedge + 1)^{-1} \mathcal{H} \wedge = (\wedge + 1)^{-1} \wedge \mathcal{H} + (\wedge + 1)^{-1}B_2 \) where \( B_2 \) is a bounded operator in \( L^2 \) and hence

\[ \frac{\partial}{\partial t} \| (\wedge + 1)^{-1} u \| \leq \delta_o \| u \| + \| (\wedge + 1)^{-1} f \|. \]

Substituting in the inequality (6.10) we obtain

\[ \frac{\partial}{\partial t} \| u(t) \| \leq \gamma' \| u(t) \| + \| f \|, \]

which, on integration with respect to \( t \), gives

\[ \| u(t) \| \leq \| u(0) \| \exp(\gamma' t) + \int_0^t \| f(S) \| \exp(\gamma' (t - s)) ds. \]

Since \( \| u(t) \| \sim \| u(t) \| \) we obtain the required inequality

\[ \| u(t) \| \leq c(h)\| u(0) \| + \int_0^t \| f(s) \| ds. \]

In the general case in which \( u \in L^2[0,h] \) we regularize it by the the mollifiers \( \varphi_\delta \) of Friendliche and apply the above argument to the function \( u_\delta = \varphi_\delta \ast (\epsilon_j) \) and pass to the limits as \( \delta \to 0 \) in the inequality for \( u_\delta \) to obtain the energy inequality for \( u \). \( \square \)

**Remark.** In the above proof the norm \( \| u \| \) depends a priori on the parameter \( t \) since it involves the operator \( \mathfrak{H}(t) \). When \( t \) runs through a bounded set the constant \( \beta \) in the definition of \( \| u \| \) can be chosen to be independent of \( \mathfrak{H} \).

In the following proposition we prove that, if \( A_k \) and \( B \) are differentiable of sufficiently high order, then there exists an energy inequality for higher order derivatives.
Proposition 3. Let $M$ be a regularly hyperbolic system with $A_k(x, t) \in \mathcal{B}^{\max(1+\sigma, m)}[0, h]$, $0 < \sigma < 1$, $\frac{\partial}{\partial t} A_k(x, t) \in \mathcal{B}^0[0, h]$. Suppose $B(x, t) \in \mathcal{B}^m[0, h]$, and $f(x, t) \in \mathcal{D}_L^m[0, h]$ are given. If $u \in \mathcal{D}^m_2[0, h]$ is a solution of

$$(M-B)u = f$$

then

$$||u(t)||_m \leq c_m(h) \left\{ ||u(0)||_m + \int_0^t ||f(s)||_m ds \right\}.$$  

Proof. It is sufficient to prove the proposition for the case $m = 1$ and the general case will follow by repeated application of the argument. Let $\frac{\partial u}{\partial x_j} = u^{(j)}$. Then

$$M[u^{(j)}] = \sum_k \frac{\partial A_k}{\partial x_j}(x, t) \frac{\partial u}{\partial x_k} + \frac{\partial B}{\partial x_j}(x, t)u + \frac{\partial f}{\partial x_j}, \ j = 1, 2, \ldots, n$$

that is $u^{(j)}$ satisfy a regularly hyperbolic system with new $B$ and $f$. Denoting $\sum_{j=1}^n ||u^{(j)}||$ by $\varphi_1(t)$ we obtain

$$\frac{d\varphi_1}{dt}(t) \leq \gamma_1 \varphi_1(t) + \sum_j ||\frac{\partial f}{\partial x_j}|| + \sum_j ||\frac{\partial B}{\partial x_j}||u||$$

which on integration yields the required inequality

$$||u(t)||_1 \leq c_1(h) \left\{ ||u(0)||_1 + \int_0^t ||f(s)||_1 ds \right\}.$$  

In the following we deduce on energy inequality for solutions of a single regularly hyperbolic differential equation of order $m$. Consider the evolution equation

$$(6.12) \quad \left( \frac{\partial}{\partial t} \right)^m u + \sum_{j=1}^n a_{j, \nu}(x, t) \left( \frac{\partial}{\partial x} \right)^\nu \left( \frac{\partial}{\partial t} \right)^j u = g.$$
The principal part of this is by definition the homogeneous differential operator of order \( m \)

\[
\left( \frac{\partial}{\partial t} \right)^m + \sum_{|\nu|+j=m \atop j \leq m-1} a_{j,\nu}(x, t) \left( \frac{\partial}{\partial x} \right)^\nu \left( \frac{\partial}{\partial t} \right)^j \equiv L
\]

which we write in the form

\[
L \equiv \left( \frac{\partial}{\partial t} \right)^m + \sum_{j=1}^{m} h_j(x, t, \frac{\partial}{\partial x}) \left( \frac{\partial}{\partial t} \right)^{m-j}
\]

where \( h_j(x, t, \frac{\partial}{\partial x}) = \sum_{|\nu|+j=m} a_{m-j,\nu}(x, t) \left( \frac{\partial}{\partial x} \right)^\nu \). The given operator is said to be regularly hyperbolic if the polynomial equation

\[
\lambda^m + \sum_{j} h_j(x, t, \xi) \lambda^{m-j} = 0
\]

has real and distinct roots for every \((x, t) \in \Omega; |\xi| = 1\). \( h_j(x, t, \frac{\partial}{\partial x}) \) can be considered as the symbol of a singular integral operator \( H_j \) and hence we can represent

\[
h_j(x, t, \frac{\partial}{\partial x}) = H_j(i\lambda)^j
\]

and

\[
L \equiv \left( \frac{\partial}{\partial t} \right)^m + \sum_{j=1}^{m} H_j(i\lambda)^j \left( \frac{\partial}{\partial t} \right)^{m-j}.
\]

Setting

\[
v_1 = \left( \frac{\partial}{\partial t} \right)^{m-1} u
\]

\[
v_2 = i(\lambda + 1) \left( \frac{\partial}{\partial t} \right)^{m-2} u
\]
\[ v_j = [i(\land + 1)]^{j-1} \left( \frac{\partial}{\partial t} \right)^{m-j} u \]
\[ v_m = [i(\land + 1)]^{m-1} u \]

We see that \((i\land)^{j-1} = (i\land)^{j-1} [i(\land + 1)]^{-j} (i(\land + 1))^{j-1}\)
\[ = (1 + S_{j-1}) (i(\land + 1))^{j-1} \]

where \(\sigma(S_{j-1}) = \left( \frac{|\xi|}{1 + |\xi|} \right)^{j-1} - 1\). \(S_{j-1}\) is a bounded operator in \(L^2\).

Then the principal part is rewritten as
\[ L[u] = \left( \frac{\partial}{\partial t} \right)^m u + i \sum H_j \land (1 + S_{j-1}) (i(\land + 1))^{j-1} \left( \frac{\partial}{\partial t} \right)^{m-j} u \]
\[ = \frac{\partial}{\partial t} v_1 + i \sum H_j \land v_j + i \sum H_j \land S_{j-1} v_j. \]

Then \(v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}\) satisfies the system of first order equations
\[ (6.16) \quad \frac{\partial}{\partial t} v = i\mathcal{H} \land v + Bv + f \]

where
\[ (6.17) \quad \sigma(\mathcal{H}) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ -h_1 & -h_2 & \cdots & -h_{m-1} & -h_m \end{pmatrix}, \]

\(B\) a bounded operator and \(f = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ g \end{pmatrix}\).

Let \(P(\lambda) = \det(\lambda I - \sigma(\mathcal{H})) = \lambda^m + \sum_j h_j \left( x, t, \frac{\xi}{|\xi|} \right)^{m-j} \). Thus the given equation is regularly hyperbolic if and only if the associated first order system is.
Proposition 4. Suppose \( P(\lambda) = 0 \) has real and distinct roots \( \lambda_1(x, t, \xi) < \cdots < \lambda_N(x, t, \xi) \) such that
\[
\inf_{(x, t) \in \Omega, \xi \neq 0} |\lambda_j(x, t, \xi) - \lambda_k(x, t, \xi)| = d > 0
\]
and further the coefficients are such that
\[
a_{j, \nu} \in \mathcal{B}^{1+\sigma}[0, h], \quad \frac{\partial}{\partial t} a_{j, \nu} \in \mathcal{B}^0[0, h] \text{ for } j + |\nu| = m
\]
\[
a_{j, \nu} \in \mathcal{B}^0[0, h] \text{ for } j + |\nu| \leq m - 1.
\]
Let \( g \in L^2[0, h] \) be given. If \( u \in \mathcal{D}^m[0, h] \) is a solution of (6.12) then
\[
\|v(t)\|' \leq C_0(h) \left\{ \|v(0)\|' + \int_0^t \|f(s)\|' ds \right\}
\]
where \( \|v(t)\|_{12} = \sum_{j=1}^m \|\frac{\partial}{\partial t}^{m-j} u\|_{j-1}^2 \).

This proposition is proved easily using the energy inequality for the associated first order system.

\[\Box\]

7 Uniqueness theorems

From the energy inequalities obtained in the previous section some results on the local uniqueness follow immediately. We shall show that a solution of a homogeneous regularly hyperbolic system of equations vanishes identically in a cone if the cauchy data is zero. This was first proved by Holmgren and later made precise by F. John [1].

Consider the first order system of equations
\[
M[u] \equiv \frac{\partial u}{\partial t} - \sum A_k(x, t) \frac{\partial u}{\partial x_k} - B(x, t) u = 0
\]
where \( M \) is regularly hyperbolic in \( \Omega = \mathbb{R}^n \times [0, h] \).

Proposition 1. Let \( M \) be regularly hyperbolic in \( \Omega \) with \( A_k \in \mathcal{B}^{1+\sigma}_{x,t} \), \( B \in \mathcal{B}^0_{x,t} \). If \( u \in \mathcal{D}^1_{x,t} \) satisfies \( M[u] = 0 \) and \( u(x, 0) \equiv 0 \) in a neighbourhood \( U \) of the origin in \( \mathbb{R}^n \), then \( u \equiv 0 \) in a neighbourhood of the origin in \( \Omega \).
Proof. Let $D_\epsilon \subset \Omega$ be the set $\{(x,t)\in \Omega : |x|^2 + t < \epsilon, t \geq 0\}$. We first make a change of variables

$$t' = t + \sum x^2_j, x'_j = x(j = 1, \ldots, n).$$

(7.2)

Under this transformation let $\tilde{u}(x', t') = u(x, t)$ then the system of equations is transformed into the system

$$\partial \tilde{u} \partial t' = \sum A_k \partial \tilde{u} \partial x'_k + B\tilde{u}.$$  

(7.3)

$D_\epsilon$ is transformed into a strictly convex domain $\tilde{D}_\epsilon$ bounded by $t' = \sum x^2_j, t' = \epsilon$. $\tilde{u}$ is defined in the domain $\tilde{D}_\epsilon$ and we extend $\tilde{u}$ outside $\tilde{D}_\epsilon$ by 0 and we denote this again by $\tilde{u}$. Clearly $\tilde{u}$ vanishes identically in a neighbourhood of $t' = \sum x^2_j$. Thus $\tilde{u}$ has its support in $\tilde{D}_\epsilon$. It follows from lemma 1 that if $x'$ is in a small neighbourhood of the origin (it is sufficient to take $2|x'|A$, $(I - 2 \sum x'_j A_k)$ is invertible and the eigenvalues of $(I - 2 \sum x'_j A_k)^{-1} \sum A_k \cdot \xi_k$ are real and distinct since those of $\sum A_k \cdot \xi_k$ are. Thus the transformed system remains regularly hyperbolic in $\tilde{D}_\epsilon$. Extending $A_k(x, t), B(x, t)$ to the whole of $\mathbb{R}^n \times [0, h]$ in such a way that the system remains regularly hyperbolic we obtain $\tilde{M}[u] = 0$ in $\mathbb{R}^n \times [0, h]$ (this can be achieved by taking the inverse image by a suitable differentiable retraction of $\mathbb{R}^n \times [0, h]$ to $\tilde{D}_\epsilon$.

(7.4) $\partial \tilde{u} \partial t' = \sum \left(I - 2 \sum x'_j A_k\right)^{-1} \left(A_k \partial \tilde{u} \partial x'_k \right) + \left(I - 2 \sum x'_j A_k\right)^{-1} B\tilde{u}$.

$\tilde{u}$ has Cauchy data zero and hence the energy inequality shows that $\tilde{u}(x', t') \equiv 0$ and hence $u$ vanishes on $D_\epsilon$. □

Similarly it can be proved that $u$ vanishes in $D_\epsilon^{-1} = \{(x,t) : t \leq 0, \sum x^2_j + t < \epsilon$ and this completes the proof. We now prove the following lemma due to H.F. Weinberger (Weinberger [1]).

**Lemma 1.** Suppose $A$ is a constant matrix such that for all real $\xi \neq 0$, $\det(AI - \sum A_k \cdot \xi_k) = 0$ has real and distinct roots $\lambda_1(\xi) < \ldots < \lambda_N(\xi)$. If $\lambda_{\max}$ denotes $\sup_{|\xi|=1} \lambda_N(\xi))$ and $\alpha = \left(\begin{array}{c} a_1 \\ \vdots \\ a_N \end{array}\right)$ is a real vector $\neq 0$ with
Uniqueness theorems

114

|α| ≤ \frac{1}{\lambda_{\max}} then \det(\mu B - \sum A_k \cdot \xi_k) = 0, B = I - A \cdot \alpha, has real and distinct roots for any real \xi \neq 0.

Remark. From the choice of \alpha it follows that B is invertible.

Proof. First we assert that all the eigen values \nu_k of B are positive. For, they are the roots of
\[
\det(\nu I - B) = \det(\nu I - (I - A \cdot \alpha)) = (-1)^N \det((1 - \nu)I - A \cdot \alpha) = 0.
\]
and hence
\[
1 - \nu_k = \lambda_k(\alpha) = |\alpha| \frac{\lambda_k}{|\alpha|}
\]
which implies that
\[
(7.5) \quad \nu_k = 1 - |\alpha| \frac{\lambda_k}{|\alpha|} > 0
\]
since \lambda_k(\xi) < \frac{1}{|\alpha|} on |\xi| = 1. Consider
\[
\det(\mu B - \lambda I - A \cdot \xi) = (-1)^N \det((\lambda - \mu)I + A(\xi + \mu \alpha)) = 0
\]
and let \varphi_1(\mu), \ldots, \varphi_N(\mu) be the roots of the equation (with respect to \lambda)
\[
\det((\lambda - \mu)I + A(\xi + \mu \cdot \alpha)) = 0
\]
for a fixed \xi. We can write
\[
\det((\lambda - \mu)I + A(\xi + \mu \cdot \alpha)) = (\lambda - \varphi_1(\mu)) \cdots (\lambda - \varphi_N(\mu)).
\]
Now we assert that
(i) \varphi_j(\mu) \to I\infty as \mu \to \pm\infty
(ii) \varphi_j(\mu) are strictly increasing functions of \mu. Since we have
\[
\varphi_k(\mu) - \mu = \lambda_k(-\xi - \mu - \alpha) or
\]
it follows that for each fixed \( \mu \), \( \varphi_k(\mu) \) are real and distinct. To show (i) consider \( \det (\mu B - \lambda I - A \cdot \xi) = 0 \) which implies that \( \det \left( B - \frac{\lambda}{\mu} I - \frac{A}{\mu} \xi \right) = 0 \). For a fixed \( \xi \), \( \varphi_k(\mu) \) tends to the eigenvalues of \( B \) as \( \mu \to \infty \) and hence for large \( \mu \varphi_k(\mu) \sim \mu \cdot \nu_k \). Since \( \nu_k \) are positive, \( \varphi_k(\mu) \) behaves like \( \mu \) for large \( \mu \).

As for (ii), suppose on the contrary there exists a \( j_0 \) and \( \mu_1, \mu_2 \) with \( \mu_1 < \mu_2 \) such that \( \varphi_{j_0}(\mu_1) > \varphi_{j_0}(\mu_2) \). Then there exists \( \lambda_0 \) such that for three distinct \( \mu_1', \mu_2', \mu_3' \) we have

\[
\varphi_{j_0}(\mu_1') = \varphi_{j_0}(\mu_2') = \varphi_{j_0}(\mu_3') = \lambda_0.
\]

Since each \( \varphi_j(\mu) (j \neq j_0) \) contributes at least one root of \( \det (\mu B - \lambda_0 I - A \cdot \xi) = 0 \) it will have at least \( N + 2 \) roots. This being an equation of degree \( N \) we are lead to a contradiction. Now putting \( \lambda = 0 \),

\[
\det (\mu B - A \xi) = (-1)^N \varphi_1(\mu) \varphi_2(\mu) \cdots \varphi_N(\mu).
\]

Since every \( \varphi_j(\mu) \) has only one zero and the zeros are distinct, we have the lemma. \( \square \)

**Remark.** Since \( \lambda_j(-\xi) = -\lambda_j(\xi) \) for every \( j \), \( \lambda_{\max} \) is positive and equal to \( \sup_{1 \leq j \leq N} |\lambda_j(\xi)| \).

**Corollary 1.** Let \( M \) be a regularly hyperbolic system in \( \Omega = \mathbb{R}^n \times [0, h] \), \( \lambda_j(x, t, \xi) \) be the roots of \( \det (\lambda I - A \cdot \xi) = 0 \) and let

\[
(7.7) \quad \lambda_{\max} = \sup_{1 \leq j \leq N} |\lambda_j(x, t, \xi)|.
\]

Suppose \( S \) is a hypersurface in \( \Omega \) passing through a point \( (x_0, t_0) \) and defined by an equation \( \varphi(x, t) = 0 \), \( \varphi \in \mathcal{C}^2 \) with

\[
(7.8) \quad \left( \frac{\partial \varphi}{\partial t} \right)^2 \geq \lambda_{\max}^2 \sum_{j} \left( \frac{\partial \varphi}{\partial x_j} \right)^2.
\]

If \( u \) is a \( \mathcal{C}^1 \) solution of \( M[u] = 0 \) with \( u(x, t) = 0 \) for \( (x, t) \in S \) then \( u(x, t) \equiv 0 \) in a neighbourhood of \( (x_0, t_0) \).
Proof. By a change of coordinates \( x'_j = x_j (1 \leq j \leq n) \) \( r' = \varphi(x, t) \) the system \( M \) is transformed into the system

\[
(7.9) \quad \left( \frac{\partial \varphi}{\partial t} I - \sum A_k \frac{\partial \varphi}{\partial x_k} \right) \frac{\partial \tilde{u}}{\partial t'} = \sum A_k(x, t) \frac{\partial \tilde{u}}{\partial x'_k} + \cdots
\]

where \( \tilde{u} \) is, as before, the image of \( u \) by this mapping. \( S \) is mapped into \( t' = 0 \). Taking

\[
\alpha = \left( \frac{\partial \varphi}{\partial x_1}, \frac{\partial \varphi}{\partial x_2}, \ldots, \frac{\partial \varphi}{\partial x_n} \right)
\]

the conditions of the lemma are satisfied because of the assumptions on \( \alpha \) and hence \( \left( \frac{\partial \varphi}{\partial t} I - \sum A_k \frac{\partial \varphi}{\partial x_k} \right) \) is invertible. Thus \( \tilde{u} \) satisfies

\[
(7.10) \quad \frac{\partial \tilde{u}}{\partial t'} = \left( \frac{\partial \varphi}{\partial t} I - \sum A_k \frac{\partial \varphi}{\partial x_k} \right)^{-1} \sum A_k \frac{\partial \tilde{u}}{\partial x'_k} + \cdots
\]

This is again a regularly hyperbolic system since

\[
\det \left( A I - \left( \frac{\partial \varphi}{\partial t} I - \sum A_k \frac{\partial \varphi}{\partial x_k} \right)^{-1} \sum A_k \cdot \xi_k \right) = \det \left( \frac{\partial \varphi}{\partial t} I - \sum A_k \frac{\partial \varphi}{\partial x_k} \right)^{-1} \det \left( A \left( \frac{\partial \varphi}{\partial t} I - \sum A_k \frac{\partial \varphi}{\partial x_k} \right) - A \cdot \xi \right)
\]

and by the lemma its roots are real and distinct for

\[
\alpha = \left( \frac{\partial \varphi}{\partial x_1}, \frac{\partial \varphi}{\partial x_2}, \ldots, \frac{\partial \varphi}{\partial x_n} \right).
\]

Thus by the local uniqueness (Prop. \( \Box \) \( \tilde{u} \) vanishes in a neighbourhood of the origin and hence \( u \) vanishes identically in a neighbourhood of \((x_0, t_0)\). \( \Box \)

Proposition 2. Let \( M \) be a regularly hyperbolic system in \( \Omega = \mathbb{R}^n \times [0,h] \), \((x_0, t_0) \in \Omega \) and \( C \) be the backward cone defined by \( \{ t - t_0 = \alpha_0 | x - x_0 |, t < t_0 \} \) where \( \alpha_0 = \frac{1}{\lambda_{\max}} \). Let \( D \) be the interior of this backward cone.
belonging to \( \Omega \). If \( u \) is a \( C^1 \) solution of \( M[u] = 0 \) in \( D \), continuous up to the cone, and vanishing on \( D_0 = D \cap (t = 0) \), then \( u \) vanishes identically in \( D + C \) in particular \( u(x_0, t_0) = 0 \).

Proof: (F. John [1]) we first remark that \( u(x, t) \) vanishes identically in a neighbourhood of the hyperplane \( t = 0 \). Let \( S_\theta(0 < \theta \leq t_0^2) \) be a one parameter family of hyper-surfaces \( \varphi(x, t, \theta) = 0 \) where

\[
\varphi(x, t, \theta) = (t - t_0)^2 - \alpha_0^2|x - x_0|^2 - \theta
\]

Then \( \cup S_\theta \supset D \) and

\[
\left( \frac{\partial \varphi}{\partial t} \right)^2 \sum \left( \frac{\partial \varphi}{\partial x_k} \right)^2 = \frac{(t - t_0)^2}{\alpha_0^2|x - x_0|^2} = \frac{\alpha_0^2|x - x_0|^2 + \theta}{\alpha_0^2|x - x_0|^2} > \frac{1}{\alpha_0^2} = \lambda_{\text{max}}^2
\]

Hence, it follows from the lemma that if \( u \) vanishes on \( S_\theta_0 \) for some \( \theta_0 \) then it vanishes on \( S_\theta \) for \( \theta \) in a neighbourhood of \( \theta_0 \). The set of \( \theta \) for which \( u \) vanishes on \( S_\theta \) is therefore open. It is also closed and non-empty. Hence it is the whole set. Thus \( u \) vanishes in the whole cone \( D + C \).

Remark 1. This result holds also for a single equation of order \( m \) and can be proved by writing it as a system by means of singular integral operators and applying the above arguments.

Remark 2. Form Prop. 2 above it follows that if the Cauchy data has for support a small set containing the origin then the support of the solution lies in some cone limited by lines whose slope \( \frac{1}{\alpha} \geq \lambda_{\text{max}} \). This is interpreted as follows: the maximum speed of propagation of the disturbance is less than \( \lambda_{\text{max}} \).

Remark 3. The above proposition gives a unique continuation theorem for solutions of systems of some semi linear equations:

\[
M[u] = \frac{\partial u}{\partial t} - \sum A_k(x, t) \frac{\partial u}{\partial x_k} - f(x, t, u)
\]

where \( A_k(x, t) \) satisfy the same conditions as in Prop.1 and \( f \in C^1_{x,t} \). More precisely if \( u_1 \) and \( u_2 \) are two solutions of \( M[u] = 0 \) such that \( u_1(x, 0) = u_2(x, 0) = 0 \).
7. Uniqueness theorems

$u_2(x,0)$ for $x \in D_0$ then $u_1 \equiv u_2$ in the whole of the cone $D$ with $D_0$ as base. For, $v = u_1 - u_2$ satisfies

\[ \frac{\partial v}{\partial t} - \sum A_k \frac{\partial v}{\partial x_k} - \{ f(x,t,u_1) - f(x,t,u_2) \} = 0. \]

$v(x,0) = 0$ for $x \in D_0$

By the mean value theorem $f(x,t,u_1) - f(x,t,u_2) = B(x,t)(u_1 - u_2)$. By Prop. 2 we have $v \equiv 0$ in $C$ and hence $u_1 \equiv u_2$ in $D$.

Finally we apply the method of sweeping a cone by a one parameter family of surfaces to show that the solutions of second order parabolic equations have no lacuna.

Consider a parabolic equation of the second order

\[ \left( \frac{\partial}{\partial t} - L \right) [u] = 0 \]

where $L = \sum_{j,k=1}^{n} a_{jk}(x,t) \frac{\partial^2}{\partial x_j \partial x_k} + \sum_{j} b_j(x,t) \frac{\partial}{\partial x_j} + c(x,t)$ with infinitely differentiable real coefficients and $a_{jk}$ satisfy further the condition

\[ \sum_{j,k=1}^{n} a_{jk}(x,t) \xi_j \xi_k \geq \delta(x,t) |\xi|^2, \]

$\delta(x,t) > 0$, for real $\xi \neq 0$. It is known that the unique continuation across time like hyperplanes holds in the sense that if $u$ is a $C^2$ solution of the above parabolic equation with

\[ u(x,t) \big|_{x_1=0} = 0, \quad \frac{\partial u}{\partial x_1}(x,t) \big|_{x_1=0} = 0. \]

in some neighbourhood of the origin in $X_1 = 0$ then $u(x,t) \equiv 0$ in a neighbourhood of the origin in the $(x,t)$- space (see Mizohatai \cite{Mizohata}, Memoires of the college of Science, Kyoto University, 1958)

**Proposition 3.** Suppose $M$ is a parabolic operator of the second order
defined in $\Omega = \mathbb{R}^n \times [0, h]$ and suppose a $C^1$ solution $u$ of $M[u] = 0$ vanishes on a non-empty open set $\theta$ of $\Omega$ then $u \equiv 0$ in a horizontal component $T$ of $\Omega$ containing $\theta$.

By horizontal component $T$ of $\theta$ in $\Omega$ we mean the set $\{ (x, t) \in \Omega \}$ such that there exists an $x'$ with $(x', t) \in \theta$.

Proof. Suppose $S$ is a hypersurface defined by an equation

$$\varphi(x, t) = 0, \varphi \in \mathbb{E}^2$$

such that the tangent space of $S$ at the origin is not parallel to $t = 0$. Then $\sum |\frac{\partial \varphi}{\partial x_j}| \neq 0$. Suppose $\frac{\partial \varphi}{\partial x_j} \neq 0$; then one can solve for $x_1$ in a neighbourhood of the origin as $x_1 = \psi(x_2, \ldots, x_n, t)$. By a change of variables

$$t' = t, x'_1 = x_1 - \psi(x_2, \ldots, x_n, t), x'_j = x_j (j = 2, \ldots, n)$$

$S$ will be transformed into $(x'_1 = 0)$ and the form of the equation remains unaltered. Hence by the remark above the transformed function $\tilde{u}$ vanishes in a neighbourhood of the origin and hence $u$ vanishes in a neighbourhood of the origin on $S$. We may assume $\theta$ to be a neighbourhood of the origin and consider a one-parameter family of ellipsoids $S_\theta$ defined by

$$\varphi(x, t, \theta) = \frac{t^2}{a^2} + \frac{|x|^2}{b^2} - 1 = 0 (0 < \theta < \infty)$$

with the condition that the tangent space to this is not parallel to $(t = 0)$. Again by the argument of connectedness, as before, we obtain the proposition. \qed

8 Existence theorems

In this section we prove some theorems on the existence of solutions of the Cauchy problem for hyperbolic equation. To begin with we recall the Hille-Yosida theorem on the infinitesimal generator of a semi group of operators on a Banach space. This is used to assert the existence of solutions.
Theorem 1 (Hille-Yosida). Let $X$ be a Banach space and $A$ be a linear operator on $X$ with domain of definition $D_A$ dense in $X$. Assume that $A$ has the following property:

(P) there exists a real number $\varepsilon_0 > 0$ such that for every real number $\lambda$ with $|\lambda| < \varepsilon_0$ we have

1. $(I - \lambda A)$ is a one to one surjective mapping of $D_A$ onto $X$,
2. there exists a constant $\gamma > 0$ such that

$$
||(I - \lambda A)u|| \geq (1 - \gamma |\lambda|)||u||
$$

for every $u \in D_A$. Then for any given $u_0 \in D_A$ there exists in $-\infty < t < \infty$ a once continuously differentiable solution

$$
\frac{du}{dt}(t) = Au(t) \text{ with } u(0) = u_0
$$

(8.1) with values in $D_A$.

Corollary. Let $A$ be a linear operator with domain of definition $D_A$ dense in $X$ and possessing the property $(P)$ of Th. 1. If $t \rightarrow f(t) \in D_A$ is a continuous function of $t$ such that $t \rightarrow Af(t) \in X$ is a continuous function of $t$ and a $u_0 \in D_A$ is given there exists a once continuously differentiable solution $u(t)$ (with values in $D_A$) of

$$
\frac{du}{dt}(t) = Au(t) + f(t) \text{ with } u(0) = u_0
$$

(8.2)

We first consider the case of systems whose coefficients do not depend on $t$.

We remark that for a differential operator it is not in general possible to secure the condition $P(2)$ when we take $L^2$ for the Banach space $X$ even when (8.1) is well posed in the space $L^2$. For, suppose the condition $P(2)$ is satisfied.

$$
||(I - \lambda A)u||^2 = ||u||^2 + \lambda^2 ||Au||^2 - \lambda ((A + A^*)u, u) \\
\geq (1 - \gamma |\lambda|)||u||^2.
$$
As $|\lambda|$ can be taken arbitrarily small this would imply if $|\lambda|$ is small that

\[(A + A^*)u, u) \leq \gamma ||u||^2 \text{ for } \lambda > 0 \text{ and} \]
\[(A + A^*)u, u) \geq -\gamma ||u||^2 \text{ for } \lambda < 0 \]

which together imply

\[|(A + A^*)u, u)| \leq \gamma ||u||^2\]

This would mean, when we take

\[A = \sum A_k(x) \frac{\partial}{\partial x_k}, A_k \in \mathbb{B}, \text{ that} \]

\[A_k = A_k^*. \text{ In fact, } A + A^* = \sum (A_k - A_k^*) \frac{\partial}{\partial x_k} - \frac{\partial A_k^*}{\partial x_k}, \text{ and it is easy to see that the above inequality holds if and only if } A_k \equiv A_k^*(k = 1, 2, \ldots, n).\]

We then proceed to study the system

\[\partial u(t) = \sum A_k(x) \frac{\partial u}{\partial x_k} + B(x)u + f\]

We take for the operator $A$ the differential operator

\[A = \sum A_k(x) \frac{\partial}{\partial x} + B(x)\]

in $D_{L^2}^1$. We take for the domain of definition of $A$ the set

\[D_A = \{u \in D_{L^2}^1 : Au \in D_{L^2}^1\}.\]

We remark that $D_{L^2}^2 \subset D_A$ and consequently $D_A$ is dense in $D_{L^2}^1$. $A$ is a closed operator in the sense that its graph is closed. In fact, let $u_p \in D_A$ be a sequence such that $u_p \rightarrow u_0$, $Au_p \rightarrow v_0$ in $D_{L^2}^1$. Since $A$ is a continuous operator from $D_{L^2}^1$ into $L^2$ we have $Au_0 = v_0$ in $L^2$ and since the injection of $D_{L^2}^1$ into $L^2$ is bi-unique $Au_0 = v_0$ in $D_{L^2}^1$, that is $u_0 \in D_A$.

Proposition 1. Let

\[\frac{\partial u}{\partial t} = \sum A_k(x) \frac{\partial u}{\partial x_k} + B(x)u + f\]
be a regularly hyperbolic system in \( \Omega = \mathbb{R}^n \times [0, h] \) with \( A_k \in \mathbb{B}^{i+\sigma} \), \( B \in \mathbb{B}^1 \) and \( f \in \mathcal{D}_A[0, h] \). Then, given \( u_0 \in \mathcal{D}_A \) there exists a unique solution \( u \in \mathcal{D}_A[0, h] \), which is a differentiable function of \( t \) in the sense of \( L^2 \) with values in \( \mathcal{D}_A \) of (8.3) for which \( u(0) = u_0 \).

**Proof.** We write the system in the singular integral operator form

\[
\frac{d}{dt} u = (i\mathcal{H} \wedge + B)u + f
\]

and \( A = i\mathcal{H} \cap +B \). By the condition of regular hyperbolicity of (8.3) there exists a bounded singular integral operator \( N \) such that

\[
N^0 H = D^0 H
\]

where \( D \) is a singular integral matrix whose symbol is

\[
\sigma(D) = \begin{pmatrix}
\lambda_1(x, \xi) & 0 \\
& \ddots \\
0 & \lambda_N(x, \xi)
\end{pmatrix}
\]

and \( |\det \sigma(D)| > \delta > 0 \).

Define a bilinear form by

\[
(Lu, v) = (D \wedge \mathcal{H}, D \wedge v) + \beta(u, v) = ((D \mathcal{H} + \beta I)u, v).
\]

for \( u, v \in \mathcal{D}^1_{L^2} \) with a \( \beta \) to be chosen later. \( (Lu, u) \) defines a norm equivalent to that of \( \mathcal{D}^1_{L^2} \) for sufficiently large \( \beta \). In fact, since \( \mathcal{H} \) is a bounded operator in \( L^2 \) we have

\[
(Lu, u) \leq ||\mathcal{H}||^2_{L^2(L^2; L^2)} ||u||^2 + \beta ||u||^2 \leq M||u||^2_{\mathcal{D}^1_{L^2}}.
\]

On the other hand by Gårding’s inequality there exists a \( \gamma > 0 \) such that

\[
(Lu, u) \geq \delta' ||u||^2 - \gamma ||u||^2 + \beta ||u||^2,
\]

then for sufficiently large \( \beta(\geq \gamma) \) this would imply that

\[
(Lu, u) \geq c||u||^2_{\mathcal{D}^1_{L^2}}
\]
which proves the assertion. We provide $\mathcal{D}_1^1$ with the norm $(Lu, u)$. We proceed to verify conditions 1, 2, of the Hille-Yosida Theorem. To prove condition $P(2)$ we must prove that for real $\lambda$ near the origin

$$\text{(8.8)} \quad (L(I - \lambda A)u, (I - \lambda A)u) \geq (1 - \gamma |\lambda|)(Lu, u) \quad \text{for every} \quad u \in \mathcal{D}_A.$$

To do this we assume at first that $u \in \mathcal{D}_2^1$ we have then,

$$(L(I - \lambda A)u, (I - \lambda A)u) = (Lu, u) + \lambda^2(LAu, Au) - \lambda((LA + A^*L)u, u) \geq (Lu, u) - \lambda((LA + A^*L)u, u).$$

Since $A = i\mathcal{H} \Lambda + B$ we have

$$(LA + A^*L) = (\Lambda R^*R \Lambda + \beta I)(\mathcal{R} \wedge + B) + (-i \cap \mathcal{N}^* + B^*)(\wedge R^* \mathcal{R} \wedge + \beta I)$$

But $\mathcal{R} \wedge \mathcal{N} \equiv D \wedge \mathcal{R} \mod (\wedge 0)$ where $P_1 \equiv P_2 \mod (\wedge 0)$ means that $P_1 - P_2$ is a bounded operator in $\mathcal{D}_2^1$.

In fact,

$$\mathcal{R} \wedge \mathcal{N} \equiv \mathcal{R} \wedge \mathcal{N} \equiv (\mathcal{R} \circ \mathcal{N}) \wedge (\mathcal{R} \circ \mathcal{N}) \wedge \equiv (\mathcal{R} \circ \mathcal{N}) \wedge \equiv \mathcal{R} \wedge \equiv \mathcal{N} \mod (\wedge 0).$$

Hence

$$((LA + A^*L)u, u) = i((D \wedge \mathcal{R} \wedge u, \mathcal{R} \wedge u) - (\mathcal{R} \wedge u, D \wedge \mathcal{R} \wedge u) + 2 \text{Re}(B_1 \wedge u, \mathcal{R} \wedge u),$$

where $B_1$ is a bounded operator in $L^2$. Now

$$D \wedge - \wedge D^* \equiv D \wedge - \wedge D^* \equiv \wedge \equiv (\wedge - \wedge D^*) \equiv (\wedge - \wedge D^*) \wedge .$$

Since $D$ is a diagonal matrix and $\sigma(D)$ is real, we see that $D = D^*$. Hence $\mathcal{R} \wedge - \wedge D^* \equiv \mod(\wedge 0)$. Hence there exists a constant $\gamma_1$ such that

$$-\gamma_1 ||u||_{L^2}^2 \leq ((LA + A^*L)u, u) \leq \gamma_1 ||u||_{L^2}^2.$$
or equivalently we write following Leray [11]

\[-\gamma_1(\lambda + 1)^2 \leq LA + A^*L \leq \gamma_1(\lambda + 1)^2\]

and thus, as \(\|v\|^2_{\mathcal{L}_2^1}\) and \((Lu, u)\) are equivalent we obtain

\[(L(I - \lambda A)u, (I - \lambda A)u) \geq (1 - \gamma_1|\lambda|)(Lu, u)\]

for \(|\lambda| < \frac{1}{\gamma_1}\).

Next the inequality (8.8) holds for all \(u \in \mathcal{D}_A\) also. Suppose \(u \in \mathcal{D}_A\). If \(\varphi_0\) are mollifiers of Friedrichs then the function \(u_\delta = u * \varphi_0\) belongs to \(\mathcal{D}_A^1\) and it follows from (8.8) that there exists a constant \(\gamma_1\) such that for some real near the origin

\[(L(I - \lambda A)u_\delta, (I - \lambda A)u_\delta) \geq (1 - \gamma_1|\lambda|)(Lu_\delta, u_\delta)\]

But

\[Au_\delta \rightarrow Au\] in \(\mathcal{D}_A^1\) as \(\delta \rightarrow 0\).

In fact, \(Au_\delta - Au = (Au_\delta - \varphi_0 * (Au)) + (\varphi_0 * (Au) - Au)\) in which the first term tends to 0 in \(\mathcal{D}_A^1\) by Friderich’s lemma and the latter term tends to 0 in \(\mathcal{D}_A^1\) since \(Au \in \mathcal{D}_A^1\). Thus condition \(P(2)\) of Hille-Yosida Theorem is verified. To prove condition \(P(1)\) we must prove that \((I - \lambda A)\) is a one-to-one surjective mapping of \(\mathcal{D}_A\) onto \(\mathcal{D}_A^1\) for sufficiently small \(\lambda\). From (8.8) it follows that \((I - \lambda A)\) is one-to-one for \(|\lambda| < \frac{1}{\gamma_1}\).

Next \((I - \lambda A)\mathcal{D}_A\) is closed in \(\mathcal{D}_A^1\). For, \((I - \lambda A)u_n \rightarrow v_0\) in \(\mathcal{D}_A^1\) for \(u_n \in \mathcal{D}_A\) means by (8.8) that \(u_n\) is a Cauchy sequence for the new norm hence has a unique limit \(u_0\) in \(\mathcal{D}_A^1\). Hence \(-\lambda Au_n \rightarrow v_0 - u_0\) in \(\mathcal{D}_A^1\). As \(A\) is a closed mapping \(u_0 \in \mathcal{D}_A\) and \((I - \lambda A)u_0 = v_0\).

Finally we prove that \((I - \lambda A)\mathcal{D}_A\) is dense in \(\mathcal{D}_A^1\). The proof is by contradiction. Suppose \((I - \lambda A)\mathcal{D}_A\) is not dense in \(\mathcal{D}_A^1\). Then there exists \(\alpha \psi \in \mathcal{D}_A^1\). \(\psi \neq 0\) such that \((I - \lambda A)u_\psi, \psi_1 = 0\) i.e. \((\lambda + 1)(I - \lambda A)u_\psi, (\lambda + 1)\psi_1 = 0\) for all \(u \in \mathcal{D}_A\), that is, \((I - \lambda A^*)(\lambda + 1)\psi_1 = 0\) where \(A^* = -i \wedge \mathcal{H}^* + B^*\) and \(\psi_1 = (\lambda + 1)\psi \in \mathcal{L}_2^1\).

Now \(A^*(\lambda + 1)\psi_1 = (\lambda \wedge \mathcal{H}^* + B^*)(\lambda + 1)\psi_1 = (\lambda + 1)(-i \wedge \mathcal{H}^* + B^*)(\lambda + 1)\psi_1 + B_0\psi_1\), where \(B_0 = -i \wedge (\mathcal{H}^* \wedge \lambda \wedge \mathcal{H}^* + (B^* \wedge \lambda + B^*)\).
Further $A^*(\wedge + 1)\psi_1 = (\wedge + 1)(-i\mathcal{H}^\# \wedge + B^* + B_1)\psi_1 + B_0\psi_1$ where $B_1 = (-i\mathcal{H}^\# \wedge + i\mathcal{H}^\# \wedge)\in \mathcal{L}(L^2, L^2)$.

But $B = B_1 + (\wedge + 1)^{-1}B_0 + B^*$ is a bounded operator in $L^2$ and hence $(I - \lambda A^*)((\wedge + 1)\psi_1 = 0$ is equivalent to saying that $[I - \lambda (-i\mathcal{H}^\# \wedge + \tilde{B})]\psi_1 = 0$, which in turn is equivalent to saying that $[I - \lambda (-i\mathcal{H}^\# \wedge + \tilde{B})]\psi = 0$.

Starting from the equation

$$\frac{\partial}{\partial t} u = - \sum A_k(x) \frac{\partial}{\partial x_k} u - \tilde{B} u$$

and using (8.8) after observing that $\psi \in \mathcal{D}_A$ we obtain an inequality

$$\frac{1}{2} (L_1(I - \lambda (-i\mathcal{H}^\# \wedge + \tilde{B}))\psi, (I - \lambda (-i\mathcal{H}^\# \wedge + \tilde{B}))\psi) \geq (1 - \gamma |\lambda|) (L_1\psi, \psi)$$

which implies that $||\psi|| = 0$ and hence $\psi = 0$ which is a contradiction to the assumption.

Now all the conditions of Hille-Yosida theorem for $A = i\mathcal{H}^\# \wedge + B$ are verified and hence there exists a solution of the equation

$$\frac{d}{dt} u = (i\mathcal{H}^\# \wedge + B)u + f \text{ with } u(0) = u_0$$

with the required properties. \( \square \)

In the above proposition we proved the existence of solutions of regularly hyperbolic systems when $u_0 \in \mathcal{D}_A$ in particular when $u_0 \in \mathcal{D}_{L^2}$ and $f \in \mathcal{D}_A[0, h]$ and so in particular when $f \in \mathcal{D}_{L^2}$. This result can be improved as follows.

**Proposition 2.** Suppose (8.3) is a regularly hyperbolic system in $\Omega = R^n \times [0, h]$ with $A_k \in B^{1,q}$, $B \in B^1$, $u_0 \in \mathcal{D}_{L^2}$ and $f \in \mathcal{D}_{L^2}[0, h]$. Then there exists $u \in \mathcal{D}_{L^2}[0, h]$ (once differentiable in $t$ in the sense of $L^2$) satisfying the system in the $L^2$-sense and $u(0) = u_0$. Also the following energy inequality holds:

$$(Lu(t), u(t)) \leq \exp(\gamma t) \cdot (Lu(0), u(0))$$
(8.12) \[ \frac{d}{dt} u_\delta = \sum A_k(x) \frac{\partial}{\partial x_k} u_\delta + B u_\delta + f_\delta \]

where \((Lu, u)\) is defined in Prop. 1.

Proof. We regularize \(u_0\) and \(f\) by mollifiers of Friedrichs \(\varphi_\delta\) to obtain \(u_0 * \varphi_\delta = u_0^\delta \in D^2_{L^2}, \ f * \varphi_\delta = f_\delta \in D^2_{L^2}[0, h]\). By prop. 1 applied to \(u_0^\delta, f_\delta\) there exists a \(u_\delta\) continuous and with values in \(D \subset D^1_{L^2}\) satisfying

\[ \frac{\partial}{\partial t} u_\delta = \sum A_k(x) \frac{\partial}{\partial x_k} u_\delta + B u_\delta + f_\delta \]

and \(u_\delta(0) = u_0^\delta\). Further \(u_\delta(t) - u_\delta' (t)\) satisfies the equation

\[ \frac{\partial}{\partial t} [u_\delta(t) - u_\delta' (t)] = \sum A_k(x) \frac{\partial}{\partial x_k} [u_\delta(t) - u_\delta' (t)] + B[u_\delta(t) - u_\delta' (t)] + (f_\delta - f_\delta') \]

and hence by the energy inequality

\[ ||u_\delta(t) - u_\delta' (t)||_1 \leq c(h) \left( ||u_\delta(0) - u_\delta' (0)||_1 + \int_0^h ||f_\delta(s) - f_\delta'(s)||_1 ds \right), \]

which shows that \(\{u_\delta(t)\}\) is a Cauchy sequence in the space of continuous functions with values in \(D^1_{L^2}\). Hence \(u_\delta(t) \to u(t)\) in the space of continuous functions with values in \(D^1_{L^2}\). On the other hand the equation

\[ u_\delta(t) - u_0^\delta = \int_0^t [Au_\delta(s) + f_\delta(s)]ds, \quad A = \sum A_k \frac{\partial}{\partial x_k} + B \]

holds in \(L^2\). Passing to the limits in \(L^2\) we obtain

\[ u(t) - u_0 = \int_0^t [Au(s) + f(s)]ds. \]

Differentiating this, we see that the relation

\[ \frac{d}{dt} u(t) = Au(t) + f(t) \]
holds in the sense of $L^2$ where $u \in D^1_{L^2}[0, h], \frac{\partial u}{\partial t} \in L^2[0, h]$ respectively. Consider now

\[
\frac{d}{dt}(Lu_\delta, u_\delta) = \left(\frac{d}{dt}u_\delta, u_\delta\right) + \left(Lu_\delta, \frac{\partial u_\delta}{\partial t}\right) + (L'u_\delta, u_\delta)
\]

\[
\leq ((LA + A'L)u_\delta, u_\delta) + 2\text{ Re}(Lf_\delta, f_\delta) + \gamma'(Lu_\delta, u_\delta)
\]

\[
\leq \gamma(Lu_\delta, u_\delta) + (Lf_\delta, f_\delta).
\]

Since $u_\delta(t)$ and $f_\delta(t)$ converge, uniformly in $t$, to $u(t)$, $f(t)$ respectively in $D^1_{L^2}$ as $\delta \to 0$ we have (8.12). This completes the proof of the proposition.

\[
\square
\]

Remark. The above equation is a particular case of one involving singular integral operators. If in fact we consider an equation

\[
\frac{d}{dt}u(t) = i\mathcal{H} \wedge u(t) + Bu(t) + f(t)
\]

(8.14) \equiv Au(t) + f(t),

with $\sigma(\mathcal{H}) \in C^\infty_{1+\sigma}, B \in \mathcal{L}(L^2, L^2) \cap \mathcal{L}(D^1_{L^2}, D^1_{L^2}),$ which is regularly hyperbolic, we could prove an analogous proposition in the same way. We would have to use the Fridrichs' lemma for singular integral operators, namely,

\[
[H \wedge, \varphi_\delta^*] \to 0 \text{ weakly in } D^1_{L^2}.
\]

Now we consider the general case of regularly hyperbolic systems when the coefficients are functions of the variable $t$ also. We use a method similar to the one of Cauchy-Peano for ordinary differential equations.

\[
\text{Theorem 2. Let } \Omega = \mathbb{R}^n \times [0, h] \text{ and}
\]

(8.16) \[
\frac{\partial u}{\partial t} = \sum A_k(x, t) \frac{\partial}{\partial x_k} u + B(x, t) u + f
\]

be a regularly hyperbolic system in $\Omega$ with $A_k \in \mathbb{B}^{1+\sigma}[0, h], B \in \mathbb{B}^1[0, h], f \in D^1_{L^2}[0, h]$. Given a $u_0 \in D^1_{L^2}$ there exists a unique solution $u$ of (8.16), in the sense of $L^2$, which belongs to $D^1_{L^2}[0, h]$ and is differentiable in the sense of $L^2$ for which $u(0) = u_0$.\]
8. Existence theorems

Proof. Consider a subdivision

$$\Delta : 0 = t_0 < t_1 \ldots < t_s = h.$$  

We define a function $u$ inductively as follows: For $t_{j-1} \leq t \leq t_j$, $u_\Delta(t) = u_j(t)$ where $u_j$ satisfies the system

$$\frac{\partial}{\partial t} u_j = \sum A_k(x, t_{j-1}) \frac{\partial}{\partial x_k} u_j + B(x, t_{j-1}) u_j + f, u_j(t_{j-1}) = u_{j-1}(t_{j-1})$$  

for $j = 1, \ldots, s$. By Prop. ?? there exists a unique solution $u_j \in \mathcal{D}_{L}^1$ for this system for $j = 1, \ldots, s$. Thus $u_\Delta(t)$ is uniquely determined. We shall show that $u_\Delta$ is uniformly bounded for small subdivisions (subdivisions of small norms), that is,

$$\sup_{t \in [0, h]} \|u_\Delta(t)\|_1 \leq M < \infty.$$  

It follows from (8.18) using the given conditions on the coefficients that

$$\sup_{t \in [0, h]} \left\| \frac{d}{dt} u_\Delta(t) \right\|_{L^2} \leq M' < \infty.$$  

Hence $\{u_\Delta(t)\}$ is a bounded set in $\mathcal{D}_{L}^1(\Omega)$ as $\Delta$ runs through subdivisions of small norm. Thus by choosing a suitable subsequence of $\Delta$, $u_\Delta \rightarrow u$ weakly in $\mathcal{D}_{L}^1(\Omega)$ and $u$ satisfies

$$\frac{\partial u}{\partial t} = \sum A_k(x, t) \frac{\partial u}{\partial x_k} + B(x, t) u + f$$

$$u_\Delta \rightarrow u, \quad \frac{\partial u_\Delta}{\partial x_k} \rightarrow \frac{\partial u}{\partial x_k}, \quad \frac{\partial u_\Delta}{\partial t} \rightarrow \frac{\partial u}{\partial t} \text{ weakly in } L^2(\Omega)$$

and these derivatives are taken in the sense of distribution in $\Omega$.

Next we shall show that $u \in \mathcal{D}_{L}^1[0, h]$ and $u(0) = u_0$. For almost all $t$, $u(x, t)$, as a function of $t$ for each fixed $x$, is absolutely continuous (see Schwart [11]). Hence we can write

$$u(x, t') - u(x, t'') = \int_{t'}^{t''} \frac{\partial u}{\partial t}(x, t) \, dt.$$
the derivative in the right hand side is taken in the distribution sense. By the Schwarz inequality
\[ |u(x', t') - u(x, t'')|^2 \leq |t' - t''|^2 \int_{t'}^{t''} |\frac{\partial u}{\partial t}(x, t)|^2 \, dt, \]
which on integration with respect to \( x \) gives
\[ \|u(x', t') - u(x, t'')\|_{L^2(\mathbb{R}^n)} \leq |t' - t''|^2 \|\frac{\partial u}{\partial t}(x, t)\|_{L^2(\Omega)}, \]
proving that \( u \in L^2[0, h] \). If \( \varphi_\delta \) denote mollifiers of Friedrichs, the function \( u = u \ast \varphi_\delta \) satisfies \( u_\delta \in D^1 L^2[0, h] \) and
\begin{equation}
\frac{\partial}{\partial t} u_\delta(t) = \sum A_k(x, t) \frac{\partial u}{\partial x_k} u_\delta + B(x, t)u + f + C_\delta u \tag{8.19}
\end{equation}
where
\begin{equation}
C_\delta = \sum [A_k(\frac{\partial}{\partial x_k}, \varphi_\delta^*) + [B, \varphi_\delta^*]]. \tag{8.20}
\end{equation}

By Friedrichs’ lemma \( \|C_\delta u\|_1 \leq c \|u\|_1 \) and \( \|C_\delta u\|_1 \to 0 \) as \( \delta \to 0 \) for fixed \( t \). Since \( \int_0^h \|u(x, t)\|_1 \, dt < \infty \), it follows that \( \|C_\delta u\|_1 \) is integrable, and from Lebesgue’s bounded convergence theorem, we deduce that
\[ \int_0^h \|C_\delta u(x, t)\|_1 \, dt \to 0 \text{ as } \delta \to 0. \]

Now from the energy inequality for the system (8.19)
\[ \|u_\delta(t)\|_1 \leq c(h) \left( \|u_\delta(0)\|_1 + \int_0^h (\|f_\delta(s)\|_1 + \|C_\delta u(s)\|_1) \, ds \right) \]
it follows that \( \sup_{t \in [0, h]} \|u_\delta(t)\|_1 \leq M < \infty \). Again \( u_\delta(t) - u_\delta'(t) \) satisfies an equation
\[ \frac{\partial}{\partial t}(u_\delta(t) - u_\delta'(t)) = \sum A_k(x, t) \frac{\partial u}{\partial x_k}(u_\delta(t) - u_\delta'(t)) \]
and we have the energy inequality

$$\|u_\delta(t) - u_{\delta'}(t)\|_1 \leq c'(h) \left\{ \|u_\delta(0) - u_{\delta'}(0)\|_1 + \int_0^h \| (C_\delta - C_{\delta'}) u(s) \|_1 ds \right\}$$

which shows that \(\|u_\delta(t) - u_{\delta'}(t)\|_1 \to 0\) as \(\delta, \delta' \to 0\). So \(u_\delta(t)\) is a Cauchy sequence in \(D^1_{L^2}[0,h]\) and hence its limit is in \(D^1_{L^2}[0,h]\). By the uniqueness of limits in \(L^2[0,h]\), \(u_\delta \to u\) and \(u \in D^1_{L^2}[0,h]\). Since the operation of restriction is continuous and the restriction of \(u_\Delta\) to \(t = 0\), namely \(u_\Delta(x,0)\), is \(u_0\) we see that \(u(x,0) = u_0\).

Now it only remains to show that \(\{u_\Delta(t)\}\) is a bounded set in \(E^1_{L^2}\). For this we proceed as follows. We use the norm defined by

$$(Lu, u) = (\mathcal{R} \wedge u, \mathcal{R} \wedge u) + \beta(u, u)$$

for suitable \(\beta > 0\) (see (8.7)). \(D^1_{L^2}\) is provided with this norm.

By the energy inequalities we have, for \(j = 1, \ldots, s\)

$$\begin{align*}
(L(t_{j-1})u_\Delta, u_\Delta) & = (L(t_{j-1})u_j(t), u_j(t)) \\
& \leq \exp(\gamma(t - t_{j-1}))(L(t_{j-1})u_j(t_{j-1}), u_j(t_{j-1})) \\
& \quad + \int_{t_{j-1}}^{t_j} \exp(\gamma(t - s))(L(t_{j-1})f(s), f(s)) \, ds.
\end{align*}$$

The \(L(t)\) depends on the \(\mathcal{R}(t)\) which form a bounded set of singular integral operators and hence by the remark after prop. \(\mathcal{R}\) we can use the same constant \(\beta\) to the new norm in \(D^1_{L^2}\). Further letting \(L_k = L(t_k)\)

$$\begin{align*}
(L_j u_\Delta, u_\Delta) - (L_{j-1} u_\Delta, u_\Delta) & = \|\mathcal{R}(t_j) \wedge u_\Delta\|^2 - \|\mathcal{R}(t_{j-1}) \wedge u_\Delta\|^2 \\
& \leq C \|\mathcal{R}(t_j) - \mathcal{R}(t_{j-1})\|_{\alpha(L^2, L^2)} \|u_\Delta\|_{\alpha}^2.
\end{align*}$$

Since \(L_{j-1} u_\Delta, u_\Delta\) \(\sim \|u_\Delta\|^2\) we have \(\|u_\Delta\|_1^2 \leq k(L_{j-1} u_\Delta, u_\Delta)\) and hence

$$L_j u_\Delta, u_\Delta) \leq (1 + k\|\mathcal{R}(t_j) - \mathcal{R}(t_{j-1})\|_{\alpha(L^2, L^2)}) (L_{j-1} u_\Delta, u_\Delta)$$
\[(1 + \varepsilon(t_{j-1}, t_j))(L_{j-1}u_\Delta, u_\Delta)\].

Using this in the above inequality (8.21) we have
\[(L_{x-1}u_\Delta, u_\Delta) \leq \exp(\gamma t) (L_0 u(0), u(0))\]
\[+ \int_0^h \exp(\gamma(t-s))(L_0 f(s), f(s))ds \prod_{j=1}^{\infty} [1 + \varepsilon(t_{j-1}, t_j)].\]

But we have, by a well-known inequality,
\[\prod_{j=1}^{\infty} [1 + \varepsilon(t_{j-1}, t_j)] \leq \left(1 + \frac{1}{S} \sum_{j \in (t_{j-1}, t_j)} \right)^S \leq e^{\gamma_0}\]
where
\[\gamma_0 = \sup_{j} \sum_{j \in (t_{j-1}, t_j)} \varepsilon(t_{j-1}, t_j) = \sup_k \sum_{i \in (t_{j-1}, t_j)} \|\mathcal{R}(t_i) - \mathcal{R}(t_{j-1})\|_{\mathcal{L}_2}\]
\[\leq k \int_0^h \sup_{x \in \mathbb{R}^n} \sum_{|\xi| \leq 2\pi |k|} \left| \left( \frac{\partial}{\partial x} \right)^{\gamma_0} \frac{\partial}{\partial t} \sigma(\mathcal{R})(x, t, \xi) \right| .\]

Hence \(\{u_\Delta(t)\}\) is a bounded set in \(\mathcal{D}_2^1\), this completes the proof.

If we assume that coefficients and the initial data \(u_0\) and \(f\) are sufficiently smooth we can improve Theorem 2. We indicate this briefly.

We assume
\[A_k \in \mathcal{B}^2[0, h], \frac{\partial A_k}{\partial t} \in \mathcal{B}^0[0, h], B \in \mathcal{B}^2[0, h], u_0 \in \mathcal{D}_2^2, f \in \mathcal{D}_2^2, [0, h]\]
\[\text{From theorem } 2 \text{ we know that there exists a unique solution } u \in \mathcal{D}_2^1[0, h] \text{ of}
\[M[u] = f.\]

Differentiating with respect to \(x_j\) (denoting \(\frac{\partial}{\partial x_j}\) by \(D_j\)) we have
(8.22) \[ M[D_ju] - \sum_k (D_j A_k)[D_k u] = D_j f + (D_j B)[u] \quad (j = 1, 2, \ldots, n) \]

where the second member \( D_j f + (D_j B)[u] \in \mathcal{D}^1_{L^2}[0, h] \) and \( D_j u(0) \in \mathcal{D}^1_{L^2} \). Now (8.22) is a system of equations with unknown functions \( (D_1 u, \ldots, D_n u) \) which has the same principal part \( M \). We can show, without any significant modification in the previous argument, that there exists a unique solution \( (D_1 u, \ldots, D_n u) \in \mathcal{D}^1_{L^2}[0, h] \). On the other hand, by the energy inequality, we can see that the system:

\[
M[v_j] - \sum_k (D_j A_k)[v_k] = g_j \in L^2[0, h]
\]

has at most one solution \( v \in L^2[0, h] \). This shows that \( u \in \mathcal{D}^2_{L^2}[0, h] \). □

**Corollary 1.** Let (8.16) be a regularly hyperbolic system in the set \( \Omega = \mathbb{R}^n \times [0, T] \) with

\[
\begin{align*}
(A_k(x, t), \frac{\partial}{\partial t} A_k(x, t)) & \in (\mathcal{B}^2[0, T], \mathcal{B}^1[0, T]), \\
(B(x, t), \frac{\partial}{\partial t} B(x, t)) & \in (\mathcal{B}^2[0, T], \mathcal{B}^1[0, T])
\end{align*}
\]

and \( f(x, t) \in \mathcal{D}^2_{L^2}[0, T] \).

Then, given an element \( u_0 \in \mathcal{D}^2_{L^2} \) there exists a unique solution \( u \in \mathcal{D}^2_{L^2}[0, T] \) of (8.3) with \( u(0) = u_0 \).

**Proof.** Differentiating both sides of the equation (8.3) with respect to \( x_j \) in the sense of distributions we have

\[
\frac{\partial}{\partial x_j} M[u] = \frac{\partial}{\partial x_j} \frac{\partial}{\partial t} u - \frac{\partial}{\partial x_j} \left( \sum_k A_k(x, t) \frac{\partial}{\partial x_k} \right) = \frac{\partial}{\partial x_j} \frac{\partial}{\partial t} (B(x, t) u) = \frac{\partial}{\partial x_j} \frac{\partial}{\partial t} f
\]

which can be rewritten as

\[
\frac{\partial}{\partial t} \left( \frac{\partial u}{\partial x_j} \right) - \sum_k A_k(x, t) \frac{\partial u}{\partial x_j} \left( \frac{\partial u}{\partial x_j} \right) - \sum_k \frac{\partial A_k(x, t)}{\partial x_k} \frac{\partial u}{\partial x_j} - B(x, t) \frac{\partial u}{\partial x_j} = \frac{\partial}{\partial x_j} f(x, t) + \frac{\partial f}{\partial x_j}
\]
That is,
\begin{align*}
(8.22) \quad M \left[ \frac{\partial u}{\partial x_j} \right] &= -\sum \frac{\partial A_k}{\partial x_j}(x,t) \frac{\partial u}{\partial x_k} = \frac{\partial f}{\partial x_j} + \frac{\partial B}{\partial x_j}(x,t)u.
\end{align*}

Setting $\frac{\partial u}{\partial x_j} = v_j$ for $j = 1, \ldots, n$ we obtain a new system
\begin{align*}
M[v_j] - \sum \frac{\partial A_k}{\partial x_j}(x,t)v_k &= \varphi_j
\end{align*}
and we can take for $v_j(0)$ the function $\frac{\partial u_0}{\partial x_j} \in \mathcal{D}^1_{L^2}$ (the derivative being taken in the sense of distributions) since $u_0 \in \mathcal{D}^2_{L^2}$. If we assume $f \in \mathcal{D}^2_{L^2}[0,T]$ and $B \in \mathcal{B}^2[0,T]$. Then by Th. 2 there exists a unique solution (in $L^2$) $v = (v_1, \ldots, v_n)$ with $v_j \in \mathcal{D}^1_{L^2}[0,T]$. Hence $u \in \mathcal{D}^2_{L^2}[0,T]$.

**Corollary 2.** Let \textit{(8.16)} be regularly hyperbolic in the set $\Omega$ with
\begin{align*}
\left( A_k, \frac{\partial A_k}{\partial t}, \ldots, \left( \frac{\partial}{\partial t} \right)^m A_k \right) &\in (\mathcal{B}^m [0,T], \ldots, \mathcal{B}^0 [0,T]), \quad B \in \mathcal{B}^m [0,T]
\end{align*}
and $f \in \mathcal{D}^m [0,T]$; then given $u_0 \in \mathcal{D}^m$ there exists a unique solution $u$ in $\mathcal{D}^m_{L^2}[0,T]$ of \textit{(8.3)} with $u(0) = u_0$.

This can be proved by successively applying the argument of Corollary 1.

Taking $m = \left[ \frac{n}{2} \right] + 2$ we obtain, using Sobolev's lemma, the following

**Corollary 3.** Let \textit{(8.16)} be regularly hyperbolic with
\begin{align*}
\left( A_k, \frac{\partial A_k}{\partial t}, \ldots \right) &\in \left( \mathcal{D}^{[n/2]+2}_{L^2} [0,T], \mathcal{D}^{[n/2]+1}_{L^2}, \ldots \right), \quad B \in \mathcal{D}^{[n/2]+1}_{L^2}[0,T]
\end{align*}
and $f \in \mathcal{D}^{[n/2]+2}_{L^2}[0,T]$ then, given $u_0 \in \mathcal{D}^{[n/2]+2}$ there exists a solution $u \in \mathcal{D}^1$ of \textit{(8.16)} with $u(0) = u_0$, unique in $L^2$. 
Corollary 4. Assume that (8.16) is regularly hyperbolic in an open neighbourhood $U$ of 0 in $\mathbb{R}^{n+1}$, $A_k, B \in \mathcal{E}(U)$ then there exists a neighbourhood $U' \subset U$ such that for any $u_0 \subset \mathcal{E}(U \cap \{t = 0\})$, $f \in \mathcal{E}(U)$ there exists a solution $u \in \mathcal{E}(U')$ of (8.16), unique in $L^2$.

Remark. If we use a partition of unity the above arguments can be used to proved results analogous to the above corollaries in the spaces $\mathcal{E}^m_{L^2(\text{loc})}$ in place of $\mathcal{D}^m_{L^2}$.

Finally we have the following result on the existence of solutions of a single regularly hyperbolic equation of order $m$.

Corollary 5. Let

\begin{equation}
(8.23) \quad L[u] \equiv \left( \frac{\partial}{\partial t} \right)^m u + \sum_{j+|\nu| \leq m} a_{j,\nu}(x,t) \left( \frac{\partial}{\partial x} \right)^\nu \left( \frac{\partial}{\partial t} \right)^j u = g
\end{equation}

be a regularly hyperbolic equation of order $m$ in a neighbourhood of the origin with infinitely differentiable coefficients $a_{j,\nu}$. Let $g$ be infinitely differentiable in a neighbourhood $U$ of the origin. Then given the initial conditions

\[(u_0, u_1, \ldots, u_{m-1}) \in \prod \mathcal{E}(U \cap \{t = 0\})\]

there exists a solution $u \in \mathcal{E}(U')$ in a neighbourhood $U'$ such that

\[\left( \frac{\partial}{\partial t} \right)^j u(x, 0) = u_j(x), \quad j = 0, 1, \ldots, (m - 1).\]

9 Necessary condition for the well posedness of the Cauchy problem

In chapter 2 we considered necessary condition for well posedness of the Cauchy problem when the coefficients were independent of $x$. In Chapter 3 we considered some sufficiency condition for well posedness e.g. hyperbolicity, when the coefficients depended on $x$. 
Now we consider some necessary conditions in this later case. For simplicity we shall consider single, first order differential operator,

\[
\frac{\partial u}{\partial t} = \sum a_k(x, t) \frac{\partial u}{\partial x_k} + b(x, t)u
\]

(for a fuller treatment see Mizohata [3]).

If all \(a_k(x, t), b(x, t)\) are real the classical method of characteristics establishes the well posedness of the Cauchy problem. However if the \(a_k\) are complex the question of existence was not settled till recently. The characteristic polynomial of the above equation is \(\sum a_k(2\pi \xi_k) - \lambda\).

If this has real eigenvalues i.e. \(a_k\)'s are real, the Cauchy problem is well posed as shown by the results of Chap III. We shall prove that if there is \(\xi^0\) such that \(\text{im} \sum a_k(0, 0)\xi^0 \neq 0\) (say \(\neq 0\)), the problem is not well posed. The idea of the proof is as follows: we construct a sequence of solution \(u_n(x), n = 1, 2\) for which, on the hypothesis of well-posedness, we must have \(\sup |u_n(x, t)| = 0(nh)\) while on the other hand by using an energy inequality for a suitable operator we, must have a minorization by \(\exp(n)\) for some functions closely related to \(u_n\) above which will give a contradiction. More precisely we shall prove

**Proposition 3.** Let

\[
\frac{\partial u}{\partial t} = H \wedge u + b(x, t)u + f
\]

be an equation in the singular integral form with \(\sigma(H) = h(x, t, \xi)\) satisfying

\[
\text{Re} h(x, t, \xi) \leq 0 \text{ for all } (x, t, \xi) \text{ and } t \to h(x, t, \xi) \in C_1^{\infty}
\]

is continuous. Then given any \(u_o \in \mathcal{D}_{L^2}^1\) and \(f \in \mathcal{D}_{L^2}^1[0, h]\) there exists a unique solution

\[
u \in \mathcal{D}_{L^2}^1[0, h] \text{ of } (9.2) \text{ with } u(x, 0) = u_o(x).
\]

On the contrary, if there exists a \(\xi^0\) such that \(\text{Re} h(x, t, \xi^0) > 0\) then the energy inequality cannot be obtained in the \(L^2\)-space. Of course this
does not immediately imply that the Cauchy problem is not well posed in $\mathcal{D}_L^\infty$.

We see that $a(x,t,-\xi) = -a(x,t,\xi)$ which shows that in the case of a differential operator (9.1) the condition $\text{Re} a(x,t,\xi) \equiv 0$ will be necessary for the existence theorem. We analyse this situation more clearly.

**Theorem 1.** Suppose there exists a real vector $\xi^o \in \mathbb{R}^n$, $\xi^o \neq 0$ and $x^o$ such that $\text{Im} \sum a_k(x^o,0)\xi^o_k < 0$. Then the forward Cauchy problem is not well posed for (9.1) in $\mathcal{E}$ or in $\mathcal{D}_L^\infty$ or in $\mathcal{B}$.

**Remark.** P.D. Lax [1] also proved a similar theorem, by using the characteristic method, that if eigenvalues are simple for the well posedness of the Cauchy problem it is necessary that the eigenvalues be real.

We first prove an energy inequality for a suitably modified operator and then establish two lemmas for commutators which together will prove the theorem. Suppose $x^0 = 0$.

First of all we localize the differential operator given in (9.1). Suppose $u$ is a solution of (9.1) of class $E^1$. Let $\beta(x) \in \mathcal{D}$ with support contained in a small neighbourhood of the origin. Now

\[(9.4) \quad \frac{\partial}{\partial t}(\beta u) = \sum a_k \frac{\partial}{\partial x_k}(\beta u) + b(\beta u) - \sum a_k \frac{\partial \beta}{\partial x_k} u\]

Since the support of $\beta u$ and of $\frac{\partial}{\partial x_k}(\beta u)$ are contained in the support of $\beta$ we can modify $a_k$ and $b$ outside the support of $\beta$. We can write

\[(9.4)' \quad \frac{\partial}{\partial t}(\beta u) - \sum \tilde{a}_k(x,t) \frac{\partial}{\partial x_k}(\beta u) - \tilde{b}(x,t)(\beta u) = - \sum \tilde{a}_k(x,t) \frac{\partial \beta}{\partial x_k} \cdot u\]

where $\tilde{a}_k$ and $\tilde{b}$ are equal to $a_k$ and $b$ respectively on the support of $\beta$ and

\[(9.5) \quad \text{im} \sum \tilde{a}_k(x,t)\xi^o_k < -\delta, \quad \delta > 0 \text{ for all } (x,t) \text{ with } x \in \mathbb{R}^n \text{ and } 0 \leq t \leq t_o\]
We can assume $|\xi_0| = 1$ if necessary by multiplying by a suitable constant. There exists a neighbourhood $V$ of $\xi_0$ such that

\[(9.6) \quad \text{im} \sum \tilde{a}_k(x, t)\xi_k \leq -\delta \text{ for } 0 \leq t \leq t_0, \xi \in V.\]

Let $\hat{\alpha} \in \mathcal{D}$ with the support contained on $V$ and $\hat{\alpha}(\xi) \equiv 1$ in a neighbourhood of $\xi_0$. Define $\hat{\alpha}_p$ by

\[(9.7) \quad \hat{\alpha}_p(\xi) = \hat{\alpha}(\frac{\xi}{p}), \alpha_p(x) = \mathcal{F}[\hat{\alpha}_p(\xi)].\]

Convolving both sides of (9.4)' with $\alpha_p$ we obtain

\[(9.8) \quad \left(\frac{\partial}{\partial t} - \sum \tilde{a}_k \frac{\partial}{\partial x_k} - \hat{b}\right)\left(\alpha_p * (\beta u)\right) = L [\alpha_p * (\beta u)] - \sum [\alpha_p * (\beta_k u)] - \sum [\alpha_p * (\beta_k u)].\]

where $\beta_k = \frac{\partial \beta}{\partial x_k}$.

We rewrite

\[\sum \tilde{a}_k \frac{\partial}{\partial x_k} (\alpha_p * (\beta u)) = H \wedge (\alpha_p * (\beta u))\]

where $v = \beta u$ and $\sigma(H) = 2\pi i \sum \tilde{a}_k \frac{\partial}{\partial |\xi|} = h(x, t, \xi)$. That is

\[H \wedge (\alpha_p * (\beta u)) = \int \exp(2\pi i x \cdot \xi) \cdot h(x, t, \xi) \hat{\alpha}_p(\xi) \hat{\alpha}(\xi) d\xi.\]

This operator depends only on the value of the symbol $h$ on the set $\{\lambda V\}_{\lambda \geq 0}$ since the support of $[\xi] \hat{\alpha}_p \hat{v}$ is contained in the set $[\lambda V]$. Hence we can modify the symbol $h$ to $\tilde{h}$ outside $\{\lambda V\}$ as follows:

(i) $\tilde{h}(x, t, \xi) \equiv h(x, t, \xi)$ for $\xi \in \lambda V$

(ii) $\text{Re} \tilde{h}(x, t, \xi) \geq \delta', \delta' > 0.$
Thus we have finally an equation

\[(9.9) \quad \left(\frac{\partial}{\partial t} - \tilde{H} \wedge \tilde{b}(\alpha_p * (\beta u))\right) = f\]

where \(\tilde{H}\) is the singular integral operator whose symbol \(\sigma(\tilde{H})\) is \(\tilde{h}\), \(f\) being the right-hand side of (9.8).

**Lemma 1.** Suppose \(H(t)\) is a singular integral operator of class \(C^\infty_\beta\), \(\beta = \infty\) such that

\[(9.10) \quad \sigma(H)(x, t, \xi) \geq \delta' > 0.\]

Suppose \(f \in L^2[0, h]\) is given. If \(u \in D^1_{L^2}[0, h]\) satisfies

\[(9.11) \quad \frac{\partial}{\partial t}(\alpha_p * (\beta u)) = H \wedge (\alpha_p * (\beta u)) + b(x, t)(\alpha_p * (\beta u)) + f\]

then there exists a \(\delta'' > 0\) such that

\[(9.12) \quad \frac{d}{dt} \|\alpha_p * (\beta u)\| \geq \delta'' \|\alpha_p * (\beta u)\| - \|f\|\]

for sufficiently large \(p\).

**Proof.** Let us denote \(\alpha_p * (\beta u)\) by \(v_p\). Then we have

\[d\frac{dt}{dt}(v_p, v_p) = ((H \wedge + \wedge H^*)v_p, v_p) + 2 \text{Re}(bv_p, v_p) + 2 \text{Re}(v_p, f).\]

But \(\wedge H^* = H^# \wedge (\text{mod} \wedge^o)\) implies

\[\frac{d}{dt}(v_p, v_p) = ((H + H^#) \wedge v_p, v_p) + 2 \text{Re}(bv_p, v_p) + 2 \text{Re}(v_p, f) + (Bv_p, v_p),\]

with \(B\) a bounded operator. If \(P\) denotes the singular integral operator \(H + H^#\) then \(\sigma(P) \geq 2\delta'\). We remark that \((P \wedge^s - \wedge^s P) \wedge^r\) is a bounded operator if \(s, \sigma \geq 0\) and \(s + \sigma \leq 1\). Taking \(s = \sigma = \frac{1}{2}\)

\[P \wedge \equiv \wedge^{\frac{1}{2}} P \wedge^2 \text{ (mod } \wedge^0\).\]
Hence

\[ (P \wedge v_p, v_p) = P \wedge ^\frac{1}{2} v_p, ^\frac{1}{2} v_p) + ((C v_p, v_p) \]

where \( C = P \wedge - \wedge ^\frac{1}{2} P \wedge ^\frac{1}{2} \) is a bounded operator. Thus

\[
\frac{d}{dt} (v_p, v_p) \geq \text{Re}((H + H^\#) ^\frac{1}{2} v_p, ^\frac{1}{2} v_p) - \gamma_1 \|v_p\|^2 - 2\|v_p\| \|f\|
\]

on the other hand we have by Gårding’s lemma that

\[
\frac{d}{dt} (v_p, v_p) \geq \delta'(\wedge ^\frac{1}{2} p, ^\frac{1}{2} v_p) - \gamma_2 \|v_p\|^2.
\]

Since the distance of the support of \( \hat{v}_p(\xi, t) \equiv \hat{\alpha}_p(\xi) \hat{u}(\xi, t) \) from the origin is larger than \( \sigma p, \sigma > 0 \), we have by Plancheral’s formula

\[
(\wedge ^\frac{1}{2} p, ^\frac{1}{2} v_p) = \int |\xi|^2 |\hat{v}_p(\xi, t)|^2 d\xi 
\geq \sigma p \int |\hat{v}_p(\xi, t)|^2 d\xi = \sigma p \|v_p\|^2
\]

Thus we have

\[
\frac{d}{dt} \|v_p\|^2 \geq \sigma p \|v_p\|^2 - (\gamma_1 + \gamma_2) \|v_p\|^2 - 2\|v_p\| \|f\|
\]

which implies

\[
\frac{d}{dt} \|v_p\| \geq (\delta_1 - \gamma) \|v_p\| - \|f\|
\]

where \( \delta > 0, \gamma > 0 \) are constants. Therefore for large \( p(\delta p - \gamma) \geq \delta''_p, \delta'' > 0. \)

For such \( p \) we have

\[
\frac{d}{dt} \|v_p\| \geq \delta'' p \|v_p\| - \|f\|.
\]

In other words we have

\[
\frac{d}{dt} \|\alpha_p * (x) u\| \geq \delta'' p \|\alpha_p * (x) u\| - \|f\|
\]

completing the proof of the lemma. \( \square \)
Lemma 2. If \( a \in \mathcal{B} \) and \( u \in D_{L^2}^1 \) there exists a constant \( c > 0 \) such that

\[
\left\| \left[ \alpha_p^*, a(x) \frac{\partial}{\partial x} \right] u \right\| \leq \left\{ c \left\| \frac{\partial}{\partial x} (x^\rho \alpha_p) * u \right\| + \left( \sum_{|\rho| = m} \left\| \frac{\partial}{\partial x^j} (x^\rho \alpha_p) \right\|_{L^1} + \left\| (x^\rho \alpha_p) \right\|_{L^1} \right) \right\} [u]
\]

(9.15)

Proof. Let \( v = \left[ \alpha_p^*, a(x) \frac{\partial}{\partial x^j} \right] u \); then

\[
v(x) = \int \left( a(y) - a(x) \right) \alpha_p(x - y) \frac{\partial u}{\partial y_j}(y) dy.
\]

Expanding \( a(y) - a(x) \) by mean value theorem up to order \( m \), to be determined later,

\[
a(y) - a(x) = \sum_{1 \leq |\rho| \leq m-1} \frac{(y - x)^\rho}{\rho!} \frac{\partial}{\partial x} a(x) + \sum_{|\rho| = m} a_p(x, y)(x - y)^\rho
\]

and hence

\[
v(x) = \sum_{1 \leq |\rho| \leq m-1} \frac{(-1)^\rho}{\rho!} \frac{\partial}{\partial x} a(x) \frac{\partial}{\partial x^j} (x^\rho \alpha_p) * u
\]

\[
+ \sum_{|\rho| = m} \int (x - y)^\rho \alpha_p(x - y) a_p(x, y) \frac{\partial u}{\partial y_j}(y) dy.
\]

Now

\[
\varphi(x) = \int (x - y)^\rho \alpha_p(x - y) a_p(x, y) \frac{\partial u}{\partial y_j}(y) dy
\]

\[
= - \int \frac{\partial}{\partial y_j} \left( (x - y)^\rho \alpha_p(x - y) a_p(x, y) \right) u(y) dy
\]

\[
= - \int \left\{ \frac{\partial}{\partial y_j} \left( (x - y)^\rho \alpha_p(x - y) \right) a_p(x, y) + (x - y)^\rho \alpha_p(x - y) \frac{\partial}{\partial y_j} a_p(x, y) \right\} u(y) dy.
\]
Hence \( |\varphi(x)| \leq c\|x^\rho \alpha_p| * u + \frac{\partial}{\partial x_j} (x^\rho \alpha_p) * |u|\). Applying Hausdorff-Young inequality to the right hand side we obtain the desired inequality.

Similarly one can prove that if \( a \in \mathcal{B} \) and \( u \in L^2 \) then

\[
(9.16) \quad \| [\alpha_p * a] u \| \leq c \left( \sum_{1 \leq |\delta| \leq m-1} \|(x^\rho \alpha_p) * u\| + \left( \sum_{|\rho|=m} \|x^\rho \alpha_p\|_{L^1} \right) \|u\| \right)
\]

where \( c \) is a positive constant.

Now we look at the terms appearing in the right hand side of (9.15).

First of all \( \frac{\partial}{\partial x_j} (x^\rho \alpha_p) * u \) has its Fourier image \( (2\pi i \xi_j) \hat{\alpha}_p(\xi) \hat{u} \) = \((2\pi i \xi_j) \hat{\alpha}_p(\xi) \hat{u} \). const. \( \hat{\alpha}_p(\xi) \) which shows, since the support of \( \hat{\alpha}_p(\xi) \) has diameter \( \sigma' p \) where \( \sigma' \) is a constant depending only on \( \hat{\alpha}_p \), that,

\[
(9.17) \quad \left\| \frac{\partial}{\partial x_j} (x^\rho \alpha_p) * u \right\| \leq c \rho \|x^\rho \alpha_p\|_{L^1} \|
\]

Next consider \( \|x^\rho \alpha_p\|_{L^1} \), for \( |\rho| = m \)

\[
\sup |x^\rho \alpha_p| \leq \text{const.} \int |\hat{\alpha}_p(\xi)| d\xi = \text{const.} \int \left| \frac{\partial}{\partial \xi} \hat{\alpha}_p(\xi) \right| d\xi = \text{const.} \left( \frac{1}{p} \right)^{|\rho|-n} \int \left| \frac{\partial}{\partial \xi} \hat{\alpha}_p(\xi) \right| d\xi.
\]

Similarly \( |x^{2n}|x^\rho \alpha_p| \leq \text{const.} \left( \frac{1}{p} \right)^{|\rho|+n} \int \left| \Delta^n_{\xi} \frac{\partial}{\partial \xi} \hat{\alpha}_p(\xi) \right| d\xi \) which implies that

\[
(1 + |x|^{2n})|x^\rho \alpha_p| \leq \text{const.} \left( \frac{1}{p} \right)^{|\rho|-n}.
\]

Hence \( \|x^\rho \alpha_p\|_{L^1} \leq \text{const.} \int \frac{dx}{1 + |x|^{2n}} \cdot \left( \frac{1}{p} \right)^{|\rho|-n} \leq c \left( \frac{1}{p} \right)^{|\rho|-n} \). In the same way one can show that

\[
\left\| \frac{\partial}{\partial x_j} (x^\rho \alpha_p) \right\|_{L^1} \leq c \left( \frac{1}{p} \right)^{|\rho|-n-1}.
\]

\[ \square \]
Thus we have proved the

**Corollary of Lemma 2.** If \( a \in B \) and \( u \in D_{L^2} \) then

\[
(9.18) \quad \| \alpha_p^* \cdot a(x) \frac{\partial}{\partial x_j} u \| \leq c_p \sum_{1 \leq \rho \leq m} \| (x^\rho \alpha_p)^* u \| + O \left( \frac{1}{p^{m-n}} \right) \| u \|.
\]

This follows from (9.15).

Similarly it follows from (9.16) that

\[
(9.18)' \quad \| \alpha_p^* \cdot a(x) u \| \leq c \sum_{1 \leq \rho \leq m} \| (x^\rho \alpha_p)^* u \| + O \left( \frac{1}{p^{m-n}} \right) \| u \|.
\]

**Lemma 3.** If \( L \) is a differential operator of the first order with its coefficients in \( B \)

\[
(9.19) \quad L = \sum a_k(x) \frac{\partial}{\partial x_k} + b(x)
\]

then for any \( u \in D_{L^2} \)

\[
(9.20) \quad \| \alpha_p^* \cdot L u \| \leq c \sum_{1 \leq \rho \leq m} \| (x^\rho \alpha_p)^* u \| + O \left( \frac{1}{p^{m-n}} \right) \| u \|.
\]

This is an immediate consequence of the inequalities (9.18) and (9.18)'. More generally one can prove exactly in the same way

\[
(9.21) \quad \| (x^\nu \alpha_p)^* \cdot L u \| \leq c \sum_{|\nu|+1 \leq \rho \leq m} \| (x^\rho \alpha_p)^* u \| + O \left( \frac{1}{p^{m-n+|\nu|}} \right) \| u \|.
\]

and

\[
(9.22) \quad \| (x^\nu \alpha_p)^* \cdot L u \| \leq c_p \sum_{|\nu|+1 \leq \rho \leq m} \| (x^\rho \alpha_p)^* u \| + O \left( \frac{1}{p^{m+|\nu|-n}} \right) \| u \|.
\]

for every \( u \in D_{L^2} \).
Now we can complete the

**Proof of Theorem 1.** We prove this theorem in the spaces $E$. As we shall see from the method of proof the same will be valid for the spaces $D_{L^2}$ and $B$. The proof is by contradiction.

Suppose the Cauchy problem is well posed in the spaces. We construct a sequence of initial conditions $\psi_q(x)$ and consider the corresponding sequence of solutions $\psi_q(x)$ are defined as follows:

Let $V$ be a small a neighbourhood of $\xi_o$ and $\hat{\alpha} \in D$ have its support in $V$ with $\hat{\alpha}(\xi) \equiv 1$ in neighbourhood $V'$ of $\xi_o$, $V' \subset V$. Take a $\hat{\psi} \in D$, $\hat{\psi}(\xi) \neq 0$ with support contained in $V'$. Denoting

$$\hat{\psi}_q(\xi) = \hat{\psi}(\xi - q\xi_o)$$

we have by taking inverse Fourier transforms

$$(9.23) \quad \psi_q(x) = \exp(2\pi iq\cdot \xi_o)\psi(x)$$

$\psi_q \in E$ (also in $D_{L^2}$, $B$). Further

$$(9.24) \quad \|\psi_q\|_{\psi \psi} = O(q^h).$$

(We remark that (9.24) holds for the semi-norms in $D_{L^2}$ and $B$ also).

By hypothesis of the well posedness, the corresponding solution $u_q(x,t)$ of (9.1) having $\psi_q(x)$ as the initial data is estimated by

$$(9.25) \quad \sup_K |u_q(x,t)| = O(q^h)$$

for some fixed $h$ where $K$ is a compact set in the $(x,t)$-space. Also we see that

$$(9.26) \quad \|\alpha_p * (\beta \psi_p)\| \geq c > 0.$$
by using Plancheral’s formula and the fact that \( \hat{\alpha}_p \equiv 1 \) on the support of \( \hat{\psi}_p, \psi \) being an analytic function and \( \| \beta \hat{\psi} \| > 0 \). On the other hand it is easy to see that \( \| [\alpha_p \ast, \beta] \hat{\psi} \| = O \left( \frac{1}{p} \right) \). Now we prove this leads to contradiction as follows. Instead of \( \alpha_p \ast (\beta u) \) in (9.8) we consider \( (x^\nu \alpha_p) \ast (\beta^{\mu} u_p) \) with \( |\nu| \leq m - 1 \), \( |\nu| \leq m + mh \) which form a system of localisers, \( \beta^{\mu} = \left( \frac{\partial}{\partial x^{\mu}} \right) \beta(x) \). Then we have

\[
\left( \frac{\partial}{\partial t} - \sum \tilde{a}_k \frac{\partial}{\partial x_k} - b \right) ((x^\nu \alpha_p) \ast (\beta^{\mu} u_p)) \]

\[
= [L, (x^\nu \alpha_p) \ast (\beta^{\mu} u_p)] - \sum \tilde{a}_k ((x^\nu \alpha_p) \ast (\beta^{\nu_k} u_p)) \]

\[
- \sum ((x^\nu \alpha_p) \ast \tilde{a}_k (\beta^{\nu_k} u_p)) \]

where \( \mu_k = \mu + e_k, \ell_k = (0, \ldots, 1, \ldots, 0) \) the \( k \)th component is 1.

Applying inequality (9.12) for \( (x^\nu \alpha_p) \ast (\beta^{\mu} u_p) \) and using inequalities (9.21), (9.22) with \( m = h + n + 2 \), we have

\[
\frac{d}{dt} ||(x^\nu \alpha_p) \ast (\beta^{\mu} u_p)|| \geq \delta'' p ||(x^\nu \alpha_p) \ast (\beta^{\mu} u_p)|| - \|f\| \]

\[
\geq \delta'' p ||(x^\nu \alpha_p) \ast (\beta^{\mu} u_p)|| - cp \sum_{|\nu|+1 \leq \mu \leq m-1} ||(x^\nu \alpha_p) \ast (\beta^{\mu} u_p)|| \]

\[
- c \sum_{|\nu|+1 \leq \mu \leq m-1, |\nu| \leq |\mu|+1} ||(x^\nu \alpha_p) \ast (\beta^{\nu_k} u_p)|| - c \sum_{|\nu| = |\mu|+1} ||(x^\nu \alpha_p) \ast (\beta^{\mu} u_p)|| - O \left( \frac{1}{p} \right) \]

Now consider the functions \( \theta_{(\nu, \mu)} u_p \) defined by

\[
\theta_{(\nu, \mu)} u_p = p^{\theta|\nu|+|\mu|}(x^\nu \alpha_p) \ast (\beta^{\mu} u_p) \]

where \( 0 < \theta < 1 \). In fact we take \( \theta = \frac{1}{m} \). We have from above inequality

\[
\frac{d}{dt} \|\theta_{(\nu, \mu)} u_p\| \geq \delta'' p \|\theta_{(\nu, \mu)} u_p\| - cp^{1-\theta} \sum_{|\nu|+1 \leq |\mu| \leq m-1} \|\theta_{(\rho, \mu)} u_p\| \]
\[ -cp^\theta \sum_{|\mu| = |\mu'| + 1} ||\theta_p(v, \mu')u_p|| - c \sum_{|\nu| + 1 \leq |\mu| \leq m-1} ||\theta_p(\rho, \mu')u_p|| \]

(9.27)

\[ -p^{\theta(|v| - |\mu|)} O\left(\frac{1}{p}\right). \]

Now if \(|\mu'| = m + mh\) we have, by (9.25)

\[ ||\theta_p(v, \mu)u_p|| \leq cp^{\theta(|v| - |\mu|)} ||u_p|| \leq c'p^{\theta(|v| - |\mu|) + h}. \]

But \(\theta(|v| - |\mu|) \leq \theta(m+1 - m - mh - 1) = \theta(-mh - 2) = -h - 2\theta\) since \(\theta = \frac{1}{m}\). Thus \(||\theta_p(v, \mu)u_p|| \leq c\theta^{-2}\). Denoting

\[ S_p(t) = \sum_{0 \leq |\nu| \leq m-1, 0 \leq |\mu| \leq m+mh} ||\theta_p(v, \mu)u_p(t)|| \]

we have from (9.27) that

\[ \frac{d}{dt} S_p(t) \geq \delta'' pS_p(t) - cp^{1-\theta}S_p(t) - O(1) \]

\[ \geq \gamma'' pS_p(t) - O(1) \text{ for large } p, r'' > 0. \]

Integrating this with respect to \(t\)

\[ S_p(t) \geq \exp(\gamma'' pt)S_p(0) - \int_0^t \exp(r'' p(t - s))O(1) ds \]

\[ = \exp(\gamma'' pt) \left[ S_p(0) - O\left(\frac{1}{p}\right)\right]. \]

152

But \(S_p(0) = \sum_{0 \leq |\nu| \leq m-1, 0 \leq |\mu| \leq m+mh} ||\theta_p(v, \mu)u_p(0)|| \geq ||\alpha_p * (\beta u_p)(0)|| \geq c > 0\) by (9.25) for large \(p\). Hence for every fixed \(t\) the function \(S_p(t)\) increases exponentially with respect to \(p\) i.e. \(S_p(t) \geq ce^{r'' pt}\). On the other hand

\[ ||\theta_p(v, \mu)u_p(t)|| = p^{\theta(|v| - |\mu|)} ||(x^\nu \alpha_p) * (\beta u_p)|| \]

and \(||\beta u_p(t)|| = O(p)^h\). Hence \(S_p(t) \geq cp^k\) for a large \(k\). In fact \(||\theta_p(v, \mu)u_p(t)|| \leq O(p^{k+1}) \text{ since } \theta |v| < 1\). This is a contradiction. This completes the proof of the theorem.
Chapter 4

In this chapter we briefly discuss the existence of solutions of the Cauchy problem for parabolic equations. In section 1 we introduce parabolic equations of order \( m \) in the \( x \)-variables and prove an existence theorem when coefficients do not depend on \( t \). In section 2 we obtain an energy inequality for parabolic equations which we use to prove the existence of solutions of the Cauchy problem for parabolic equation with sufficiently smooth initial conditions when coefficients depend on \( t \) as well.

1 Parabolic equations

Consider the differential equation

\[
\frac{\partial}{\partial t} u = \sum_{|\nu| \leq 2m} a_\nu(x) \left( \frac{\partial}{\partial x} \right)^\nu u + f = A \left( x, \frac{\partial}{\partial x} \right) u + f
\]

where \( A \) is negative elliptic of order \( 2m \) in \( \mathbb{R}^n \) in the sense that

\[
\text{Re} \sum_{|\nu| = 2m} a_\nu(x)(i\xi)^\nu \leq -\delta |\xi|^{2m}
\]

\( \delta \) being a positive constant. We assume that the coefficients \( a_\nu \) belong to \( \mathbb{B}^{2m} \).

We prove the existence of a solution of (1.1) in the space \( L^2 \). We take for the domain of definition \( \mathcal{D}_A \) of \( A \) the space \( \mathcal{D}_L^{2m} \).
Proposition 1. For small \( \lambda > 0 \) the operator \( (I - \lambda A) \) defines a one-to-one surjective mapping of \( \mathcal{D}_2^{2m} \) onto \( L^2 \).

**Proof.** For \( u \in \mathcal{D}_2^{2m} \) and \( \lambda > 0 \)

\[
|(I - \lambda A)u|^2 = |u|^2 - \lambda((A + A^*)u, u) + \lambda^2||Au||^2.
\]

Since \( A \) is negatively elliptic we have, from Gårding’s lemma, that

\[
(i) \quad -(A + A^*)u, u) \geq \delta||u||^2_m - \gamma_1||u||^2
\]

\[
(ii) \quad ||Au||^2 \geq \frac{\delta^2}{2}||A^{2m}u||^2 - \gamma_2||u||^2.
\]

where \( \gamma_1, \gamma_2 \) are positive constants depending on \( \delta \). Hence it follows from (1.3) that

\[
||(I - \lambda A)u||^2 \geq (1 - \gamma_1\lambda - \gamma_2\lambda^2)||u||^2 + \frac{\delta^2}{2}||A^{2m}u||^2,
\]

which show that for sufficiently small \( \lambda, (I - \lambda A) \) is one-to-one from \( \mathcal{D}_2^{2m} \) to \( L^2 \) and that the image is closed.

Next we show that the image \( (I - \lambda A)\mathcal{D}_2^{2m} \) is dense in \( L^2 \), for \( \lambda > 0 \) small. This is done by contradiction. Suppose the image is not dense in \( L^2 \). Then there exists a \( \psi \in L^2, \psi \neq 0 \) such that

\[
((I - \lambda A)u, \psi) = 0 \text{ for all } u \in \mathcal{D}_2^{2m},
\]

a fortiori for all \( u \in \mathcal{D} \). This implies that

\[
(I - \lambda A^*)\psi = 0. \quad \text{Let } \psi_1 = (1 - \Delta)^{-m} \psi.
\]

Then \( \psi_1 \in \mathcal{D}_2^{2m}, \psi_1 \neq 0 \) and

\[
(I - \lambda A^*)(1 - \Delta)^m\psi_1 = 0
\]

Hence \( ((I - \lambda A^*)(1 - \Delta)^m\psi_1, \psi_1) = ||\psi_1||^2_m - \lambda(A^*(1 - \Delta)^m\psi_1, \psi_1) = 0. \) Now the real part of \( (A^*(1 - \Delta)^m\psi_1, \psi_1) \) is

\[
\frac{1}{2}((A^*(1 - \Delta)^m + (1 - \Delta)^mA)\psi_1, \psi_1),
\]
and since \( A^*(1 - \Delta)^m + (1 - \Delta)^m A \) is an elliptic operator of order \( 4m \), we have by Gårding’s lemma,

\[
\frac{1}{2}(A^*(1 - \Delta)^m + (1 - \Delta)^m A)\psi_1, \psi_1 \leq -\frac{\delta}{2} \| \wedge^{2m} \psi_1 \|^2 + \gamma_3 \| \psi_1 \|^2.
\]

Hence, we have

\[
\text{Re}(\|\psi_1\|^2_m - \lambda(A^*(1 - \Delta)^m \psi_1, \psi_1)) \\
\geq \|\psi_1\|^2_m + \lambda(\frac{\delta}{2} \| \wedge^{2m} \psi_1 \|^2 - \gamma_3 \| \psi_1 \|^2) \\
\geq (1 - \lambda \gamma_3)\|\psi_1\|^2_m.
\]

(1.8)

This implies that \( \psi_1 = 0 \) contrary to the assumption, which proves that \( (I - \lambda A) \) is surjective for sufficiently small \( \lambda \).

\( \square \)

**Corollary 1.** If \( u \in L^2 \) such that \( A[u] \in L^2 \) then \( u \in D^{2m}_{L^2} \).

**Proof.** Since from the Theorem for sufficiently small \( \lambda \), \( (I - \lambda A) \) is surjective it follows that there exists \( w \in D^{2m}_{L^2} \) such that \( (I - \lambda A)w = (I - \lambda A)u \). Hence \( (I - \lambda A)(w - u) = 0 \). Now in the course of the proof of the theorem we have shown that \( (I - \lambda A)v = 0 \), \( v \in L^2 \) implies \( v = 0 \). Hence \( u = w \in D^{2m}_{L^2} \).

\( \square \)

**Proposition 2.** Given any initial data \( u_0 \in D^{2m}_{L^2} \) and any second member \( f \in D^{2m}_{L^2}[0, h] \) then there exists a solution \( u \in D^{2m}_{L^2}[0, h] \) of \((1.1)\) such that \( u(0) = u_0 \) where the derivative \( \partial_t \) is taken in the sense of \( L^2 \).

**Proof.** The prop. \( \square \) asserts that all the conditions of Hille-Yosida theorem are satisfied taking \( X = L^2 \), \( D_A = D^{2m}_{L^2} \). Hence we have the proposition by the application of Hille-Yosida theorem. Let us remark that \( u, Au \in L^2[0, h] \) implies \( u \in D^{2m}_{L^2}[0, h] \).

We have proved the Proposition \( \square \) under the assumption that \( f \in D^{2m}_{L^2}[0, h] \). We shall improve it by proving it assuming only

\[
f \in D^{m}_{L^2}[0, h].
\]

For this purpose we establish an energy inequality for the parabolic equation \((1.1)\).
2 Energy inequality for parabolic equations

Consider the parabolic equation

\[
\frac{\partial}{\partial t} u = \sum_{|\nu|\leq 2m} a_{\nu}(x) \left( \frac{\partial}{\partial x} \right)^\nu u + f = A(x, \frac{\partial}{\partial x})u + f
\]

**Proposition 1.** Let (2.1) be a parabolic equation with the coefficients \(a_{\nu}(x)\) of \(A\) belonging to \(B_{2m}\) and the second member \(f \in D_{2m}^{2m}[0,h]\). If \(u \in D_{3m}^{2m}L^2[0,h]\) satisfies (2.1) then

\[
\|u(t)\|_{2m}^2 \leq \exp(\gamma_1 t)\|u(0)\|_{2m}^2 + \gamma_2 \int_0^t \exp(\gamma(t-s))\|f(s)\|_{2m}^2 ds,
\]

where \(\gamma_1, \gamma_2\) are positive constants.

**Proof.** Consider

\[
\frac{d}{dt} (u(t), u(t))_{2m} = \left( \frac{d}{dt} u(t), u(t) \right)_{2m} + \left( u(t) \frac{d}{dt} u(t) \right)_{2m} = ((A + A^*)u, u)_{2m} + 2 \text{Re}(f, u)_{2m}
\]

\[
= ((1 - \Delta)^{2m}A + A^*(1 - \Delta)^{2m}u, u) + 2 \text{Re}(f, u)_{2m}.
\]

The first term in the right hand side is by Gårdings’s inequality less than

\[
\frac{\delta}{2} \|\Lambda^{3m}u\|^2 + \gamma_0 \|u\|_{2m}^2 \leq \frac{\delta}{2} \|u\|_{3m}^2 + \gamma_1 \|u\|_{2m}^2
\]

since \((1 - \Delta)^{2m}A\) is an elliptic operator of order \(6m\). Also

\[
\|f, u\|_{2m} \leq \|f\|_{2m} \leq \frac{2}{\delta} \|f\|_{3m}^2 + \frac{\delta}{2} \|u\|_{3m}^2
\]

by the inequality between the arithmetic and geometric means. Hence

\[
\frac{d}{dt} (u, u)_{2m} \leq \left( \frac{\delta}{2} - \delta \right) \|u\|_{3m}^2 + \gamma_1 \|u\|_{2m}^2 + \frac{2}{\delta} \|f\|_{2m}^2
\]
that is

\begin{equation}
\frac{d}{dt}|||u(t)|||_{2m}^2 \leq -\frac{\delta}{2}|||u|||_{3m}^2 + \gamma_1|||u|||_{2m}^2 + \frac{2}{\delta}||f|||_{m}^2
\end{equation}

a fortiori

\begin{equation}
\frac{d}{dt}|||u(t)|||_{2m}^2 \leq \gamma_1|||u(t)|||_{2m}^2 + \frac{4}{\delta}||f(t)|||_{m}^2
\end{equation}

and hence we obtain after integrating with respect to \( t \) in \([0, h]\) the required inequality \((2.2)\).

Next we obtain the energy inequality of the form \((2.2)\) under the assumption that \( u \in D_{L^2}^{2m}[0, h] \) instead of \( u \in D_{L^2}^{3m}[0, h] \). In the case of hyperbolic systems such an improvement could be achieved easily by using Friedrichs' lemma. This method will not work in our case since \( A \) is not of the first order. However, as we shall show, by a slight modification, we can use this method of regularisation by mollifiers.

As before we estimate the commutators of convolutions with mollifiers \( \varphi_\varepsilon \) of Friedrichs.

**Lemma 1.** For \( a \in D_{L^2}^{2m} \) and \( v \in L^2 \) denote by \( C_\varepsilon v \) the commutator \((2.5)\)

\[ C_\varepsilon v = [\varphi_\varepsilon, a]v. \]

Then there exists a constant \( \gamma_0 \) such that for \( |v| \leq m \)

\begin{equation}
\left| \left( \frac{\partial}{\partial x} \right)^{\nu} C_\varepsilon v \right| \leq \gamma_0 |a|_{D_{L^2}^{2m}} \left\{ \sum_{1 \leq |\rho| \leq m} ||(x^\rho |\varphi_\varepsilon)| * v||_{v} + \varepsilon ||v|| \right\}.
\end{equation}

**Proof.** We have,

\[ C_\varepsilon v = \int [a(y) - a(x)] \varphi_\varepsilon(x - y)v(y)dy. \]

Developing \( a(y) - a(x) \) by Taylor's theorem

\[ a(y) - a(x) = \sum_{1 \leq |\rho| \leq m} \frac{(y - x)^\rho}{\rho!} \left( \frac{\partial}{\partial x} \right)^{\rho} a(x) + \sum_{|\rho|=m} a_\rho(x, y)(y - x)^\rho, \]
where since $a \in \mathcal{B}^{2m}$

$$\left| \left( \frac{\partial}{\partial x} \right)^\nu a_p(x, y) \right| \leq c_1 |y - x| |a|_{\mathcal{B}^{2m}} \quad \text{for} \quad |\nu| \leq m - 1$$

and

$$\left| \left( \frac{\partial}{\partial x} \right)^\nu a_p(x, y) \right| \leq c_2 |a|_{\mathcal{B}^2} \quad \text{for} \quad |\nu| = m.$$

In fact,

$$a_p(x, y) = \frac{m}{\rho!} \int_0^1 (1 - \theta)^{m-1} \left\{ a^{(\rho)}(x + \theta(y - x)) - a^{(\rho)}(x) \right\} d\theta,$$

$$a^{(\rho)}(x) = \left( \frac{\partial}{\partial x} \right)^\rho a(x).$$

Hence

$$C_{\nu} v = \sum_{1 \leq \rho \leq m} \frac{(-1)^{\rho}}{\rho!} \left( \frac{\partial}{\partial x} \right)^\rho a(x) \left[ (x^\rho \varphi_{\epsilon}) * v \right]$$

$$+ \sum_{\rho = m} (-1)^m \int a_p(x, y)(x - y)^\rho \varphi_{\epsilon}(x - y)v(y) \, dy.$$

(2.7) implies the lemma. Obviously the terms of the first sum on the right hand side contribute to the terms of the sum of the right hand side of (2.6). As far as the second sum is concerned we remark that

$$\int |x|| \left( \frac{\partial}{\partial x} \right)^\rho (x^\rho \varphi_{\epsilon})| \, dx = O(\epsilon) \quad \text{for} \quad |\nu| \leq m \quad \text{and} \quad |\rho| = m.$$

By Hausdorff-Young inequality the second sum on the right hand side of (2.7) is less than $O(\epsilon)||v||$ and this completes the proof of the lemma.

More generally we have the

**Lemma 2.** Let $a \in \mathcal{B}^{2m}$ and $v \in L^2$. If

(2.8) $$C_{\nu} v = [(x^\nu \varphi_{\epsilon}) * a]v \quad \text{for} \quad |\nu| \leq m - 1$$
then there exists a constant \( \gamma_0 > 0 \) such that

\[
|v|_m \leq \gamma_0 |a|_{B^{2m}}^m \left( \sum_{|\nu|+1 \leq |\rho| \leq m} \|(x^\nu \phi_e) * v\|_m + \varepsilon \|v\| \right).
\] (2.9)

The proof is completely analogous to that of lemma 1 and hence we do not repeat it here.

As a consequence of lemma 1 and 2 we have

**Corollary 1.** If \( A = \sum_{|\nu| \leq 2m} a_\nu(x) \frac{\partial}{\partial x}^\nu \) is a differential operator of order \( 2m \) with \( a_\nu \in B^{2m} \), then for any \( u \in D^{2m}_2 \) and for any \( |\nu| \leq m \)

\[
\|(A, (x^\nu \phi_e) \ast u)\|_m \leq c \left( \sum_{|\nu|+1 \leq |\rho| \leq m} \|(x^\rho \phi_e) \ast u\|_m + \varepsilon \|u\|_{2m} \right)
\] (2.10)

where \( c = \gamma_0, \sup |a_\nu(x)|_{B^{2m}} \gamma_0 > 0 \), is a constant. We remark that (2.10) asserts also that, for any \( |\gamma| \geq m \),

\[
\|(A, (x^\nu \phi_e) \ast u)\| \leq c \varepsilon \|u\|_{2m}.
\]

**Proposition 2.** Let \( \phi \) be a parabolic equation of order \( 2m \) in \( \Omega \) with \( a_\nu \in B^{2m} \) and \( f \in D^{2m}_2 [0, h] \). If \( u \in D^{2m}_2 [0, h] \) satisfies (2.1) then

\[
\|u(t)\|_{2m}^2 \leq \exp(\gamma, t)\|u(0)\|_{2m}^2 + c \int_0^t \exp(\gamma(t - s))\|f(s)\|_{2m}^2 ds,
\] (2.11)

**Proof.** Consider the function \( (x^\nu \phi_e) \ast u = u_e^\nu \) for \( 0 \leq |\nu| \leq m \). Clearly \( u_e^\nu \in D^{3m}_2 [0, h] \) and satisfies the system

\[
\frac{\partial}{\partial \tau} u_e^\nu = Au_e^\nu + f_e^\nu + [(x^\nu \phi_e) \ast (x), A]u, \quad 0 \leq |\nu| \leq m.
\] (2.12)

Then inequality (2.4) of Prop. 1 applied to this system gives the system of inequalities

\[
\frac{d}{dt}\|u_e^\nu(t)\|_{2m}^2 \leq -\delta' \|u_e^\nu(t)\|_{2m}^2 + \gamma_1 \|u_e^\nu(t)\|_{2m}^2 + \gamma_2 \|f_e^\nu(t)\|_{2m}^2
\]
(2.13) \[ + \gamma_2 \|[(x^\nu \varphi_e)^{\ast(x)}, A]u\|_m^2 \text{ for } 0 \leq |\nu| \leq m. \]

From the corollary \[1\] after lemma \[2\] applied to \([(x^\nu \varphi_e)^{\ast(x)}, A]\) we obtain for all \(0 \leq |\nu| \leq m.\)

\[
\|[(x^\nu \varphi_e)^{\ast(x)}, A]u\|_m \leq C \left( \sum_{|\nu|+1 \leq |\nu| \leq m} \|[(x^\nu \varphi_e)^{\ast(x)}]u\|_m + \varepsilon \|u\|_m \right)
\]

(2.14) \[= C \left( \sum_{|\nu|+1 \leq |\nu| \leq m} \|u\|_m + \varepsilon \|u\|_m \right). \]

We define \(\nu^\varepsilon = \varepsilon^{-2|\nu|} u^\varepsilon\) where \(\theta > 0\) is small constant. Multiplying (2.13) by \(\varepsilon^{-2|\nu|}\) and setting \(S_\varepsilon(t) = \sum \|\nu^\varepsilon(t)\|_m^2\) we have (after adding for \(\nu\) over \(0 \leq |\nu| \leq m\) from (2.14)

\[
\frac{d}{dt} S_\varepsilon(t) \leq -\delta' \sum \|\nu^\varepsilon(t)\|_m^2 + \gamma_2 S_\varepsilon(t) + \gamma_2 F_\varepsilon(t)
\]

(2.15) \[+ \gamma_2 \sum \varepsilon^{-2|\nu|} C_2 \left( \sum_{|\nu|+1 \leq |\nu| \leq m} \|u\|_m + \varepsilon^2 \|u\|_m \right). \]

But

\[
\sum_{0 \leq |\nu| \leq m} \varepsilon^{-2|\nu|} \sum_{|\nu|+1 \leq |\nu| \leq m} \|u^\varepsilon\|_m^2 = \sum_{\nu} \varepsilon^{-2|\nu|} \sum_{|\nu|+1 \leq |\nu| \leq m} \varepsilon^{2|\nu|} \|\nu^\varepsilon\|_m^2 \leq n' \varepsilon^{2\theta} \sum \|\nu^\varepsilon\|_m^2.
\]

Thus

\[
\frac{d}{dt} S_\varepsilon(t) \leq \gamma_1 S_\varepsilon(t) + \gamma_2 F_\varepsilon(t) + (\gamma_2 C_2 n' \varepsilon^{2\theta} - \delta') \sum_{0 \leq |\nu| \leq m} \|\nu^\varepsilon\|_m^2
\]

\[+ c \varepsilon^{2(1-m\theta)} \|u(t)\|_m^2. \]

For small \(\varepsilon > 0\), \((\gamma_2 C_1 n' \varepsilon^{2\theta} - \delta') < 0\) and hence

\[
\frac{d}{dt} S_\varepsilon(t) \leq \gamma_1 S_\varepsilon(t) + \gamma_2 F_\varepsilon(t) + O(\varepsilon^{2(1-m\theta)}),
\]
Integrating with respect to \( t \)

\[
S_\varepsilon(t) \leq \exp(\gamma_1 t)S_\varepsilon(0) + \gamma_2 \int_0^t \exp(\gamma_1 (t - s)) \left\{ F_\varepsilon(s) + O(\varepsilon^{2(1-m)}) \right\} ds
\]

But

\[
\|v_\varepsilon\|_2^2 = \|u_\varepsilon\|_2^2 e^{-2\theta|\phi|}
\]

\[
= (x^\ell \varphi_\varepsilon)\overline{u}(\xi)(1 + |\xi|)^{2m} e^{-2\theta|\phi|}
\]

by Plancherel’s formular where \( \hat{g} \) denotes the Fourier image of \( g \) in the \( x \)-space and

\[
(\hat{x^\ell \varphi_\varepsilon})(\xi) = \int x^\ell \varphi_\varepsilon e^{-2\pi i x \cdot \xi} dx
\]

= \( \varepsilon^{|\nu|} \int x^\ell \varphi(x) e^{-2\pi i x \cdot \xi} dx \)

Since \( \varphi \) has its support in \( |x| < 1 \). We have

\[
|\hat{x^\ell \varphi_\varepsilon}(\xi)| \leq \varepsilon^{\nu|} \int |\varphi(x)| dx = \varepsilon^{\nu|}
\]

Hence

\[
\|v_\varepsilon\|_2^2 \leq \varepsilon^{2\nu(1-\theta)} \|u\|_2^2
\]

and

\[
\sum_{0 \leq |\nu| \leq m} \|v_\varepsilon\|_2^2 \leq \|u\|_2^2 \sum_{0 \leq |\nu| \leq m} \varepsilon^{2\nu(1-\theta)} \leq \|u\|_2^2 (1 + ce^{2(1-\theta)})
\]

which tends to \( \|u\|_2^2 \) as \( \varepsilon \to 0 \). Hence \( S_\varepsilon(t) \to \|u(t)\|_2^2 \) as \( \varepsilon \to 0 \). Also \( F_\varepsilon(t) \to \|f(t)\|_2^2 \). Hence on taking limits as \( \varepsilon \to 0 \) we have

\[
\|u(t)\|_2^2 \leq \exp(\gamma_1 t)\|u(0)\|_2^2 + \gamma_2 \int_0^t \exp(\gamma_1 (t - s)) \|f(s)\|_2^2 ds.
\]

This completes the proof of proposition. \( \square \)
Finally we consider the case parabolic systems in which the coefficients are functions of \((x, t)\) in \(\Omega\). Let

\[
\frac{\partial}{\partial t} u - \sum_{|\nu| \leq 2m} a_\nu(x, t) \left( \frac{\partial}{\partial x} \right) u = f
\]

be a parabolic equation of order \(2m\). That is we assume that

\[
A = \sum_{|\nu| \leq 2m} a_\nu(x, t) \left( \frac{\partial}{\partial x} \right) u
\]

is uniformly negatively elliptic in \(\Omega\) (\(\Omega = \{(x, t) | x \in \mathbb{R}^n, 0 \leq t \leq h\}\)). This means that

\[
\text{Re} \sum_{|\nu|=2m} a_\nu(x, t)(i\xi) \nu \leq -\delta|\xi|^{2m}
\]

for all \((x, t) \in \Omega, \xi \in \mathbb{R}^n, \delta > 0\).

**Proposition 3.** Let (2.17) be a parabolic system in \(\Omega\) with \(a_\nu \in B_{2m}^{2m}[0, h]\) and \(f \in D_{L^2}^{2m}[0, h]\). Then, given a \(u_0 \in D_{L^2}^{2m}\) there exists \(u \in D_{L^2}^{2m}[0, h]\) satisfying (2.17), with \(u \big|_{t=0} = u_0\), and which satisfies the energy inequality (2.11).

**Proof.** Let \(0 = t_0 < t_1 \cdots < t_k = h\) be a subdivision of \([0, h]\) of equal length. We define \(u_1(t), \ldots, u_k(t)\) in \([t_0, t_1], \ldots, [t_{k-1}, t_k]\) by the following conditions

\[
\frac{du_1}{dt} = A(t_0)u_1 + f, \quad u_1(t_0) = u_0 \quad \text{for} \quad t_0 \leq t \leq t_1
\]

\[
\frac{du_2}{dt} = A(t_1)u_2 + f, \quad u_2(t_1) = u_1(t_1) \quad \text{for} \quad t_1 \leq t \leq t_2
\]

\[
\frac{du_k}{dt} = A(t_{k-1})u_k + f, \quad u_k(t_k) = u_1(t_k) \quad \text{for} \quad t_{k-1} \leq t \leq t_k.
\]

We denote by \(u^{(k)}(t)\) the function which in \(t_{j-1} \leq t \leq t_j\) is equal to \(u_j(t)\). It is easy to see that \(\{u^{(k)}(t)\}\) is a uniformly bounded set. More
2. Energy inequality for parabolic equations

precisely it is a bounded set in the Hilbert space $E^{2m,1}_{L^2}(\Omega)$, consisting of all the functions $u \in L^2$ such that

$$\frac{\partial u}{\partial t} \in L^2, \left(\frac{\partial}{\partial x}\right)^\nu u \in L^2$$

for $|\nu| \leq 2m$,

where the derivatives are taken in the sense of distributions. $E^{2m,1}_{L^2}(\Omega)$ provided with the scalar product

$$(u, v)_{E^{2m,1}_{L^2}(\Omega)} = (u, v)_{L^2(\Omega)} + \sum_{|\nu| \leq 2m} \left(\left(\frac{\partial}{\partial x}\right)^\nu u, \left(\frac{\partial}{\partial x}\right)^\nu v\right)_{L^2(\Omega)}$$

is a Hilbert space. Hence $\{u^{(k)}(t)\}$ has a weak limit in $E^{2m,1}_{L^2}(\Omega)$, say $u(x, t) \cdot u(x, t)$ satisfies the equation

(2.18) $$\frac{\partial u}{\partial t} = Au + f$$

in the sense of distributions. We shall now show that $u \in D^{2m}_{L^2}(0, h]$. We know that $u \in L^2(0, h]$. If $\varphi_\varepsilon$ be mollifiers of Friedrichs consider the equation

$$\frac{\partial}{\partial t}((x^\nu \varphi_\varepsilon) * (x^\nu) u) = A((x^\nu \varphi_\varepsilon) * (x^\nu) u) + (x^\nu \varphi_\varepsilon) * (x^\nu) f + [(x^\nu \varphi_\varepsilon) * (x^\nu) A] u$$

for $|\nu| \leq m$. The functions $u_\varepsilon^\nu = (x^\nu \varphi_\varepsilon) * (x^\nu) u$ form a Cauchy sequence as $\varepsilon \to 0$. This can be proved by an argument similar to the one in Prop. 2. It can also be shown that

$$u_\varepsilon^\nu \to u(t) \text{ in } D^{2m}_{L^2} \text{ for } \nu = 0,$$

$$\to 0 \text{ in } D^{2m}_{L^2} \text{ otherwise}$$

uniformly in $t$. This proves that the energy inequality (2.11) holds in this case also.

Recent work by P. Sobolevskii develops the semi-group theory for the equations of the parabolic type by using fractional powers. Equations of parabolic type in Banach space, Trudy Moscov Mat, Obsc. 10(1961), 297 - 350.
Chapter 5

In this chapter we study non-linear equations. Much of this chapter is inspired by the recent monograph of S.L. Sobolev: Sur les equations aux derivees particlles hyperboliques non-lineaires (Cremonese, Roma 1961).

1 Preliminaries to the study of semi-linear equations

In this section we recall, without giving the proofs, a few results of Sobolev concerning the differentiability properties of functions belonging to the spaces $D^m_{L^2}$. More precisely we give estimates in the $L^p$ norm for the derivatives of these functions in terms of their norms in the space $D^m_{L^2}$. We shall also introduce the functions spaces $D^s_{L^2}$ for any arbitrary real number $s \geq 0$ and obtain $L^2$ estimates of some non-linear functions of derivatives of functions belonging to the spaces $D^s_{L^2}$.

To begin with the state the following important result due to Sobolev (1).

**Proposition 1** (Sobolev’s lemma). Let $p$ and $q$ be positive numbers with $p > 1$, and \( \frac{1}{q} + \frac{1}{q} > 1 \). If \( g \in L^p \) and \( h \in L^q \) then

\[
\left| \int \int \frac{g(x)h(y)}{|x-y|^s} \, dx \, dy \right| \leq K\|g\|_{L^p}\|h\|_{L^q},
\]

(1.1)
where $\lambda = n(2 - \frac{1}{p} - \frac{1}{q})$ and $K$ is a constant depending only on $p, q, n$.

Suppose $u \in L^p$ and a number $\lambda$ such that $0 < \lambda < n$ and $\frac{\lambda}{n} > 1 - \frac{1}{p}$ are given. Then the above inequality implies that the linear mapping

\begin{equation}
(1.2) \quad h \rightarrow \int (u(y) * \frac{1}{|y|^\lambda}) \cdot h(y) \, dy
\end{equation}

is a continuous linear functional on the space $L^q$ for $q > 1$ with $\frac{1}{q} = \left(2 - \frac{1}{p} - \frac{\lambda}{n}\right)$. Hence $u * \frac{1}{|x|^{\lambda}} \in L^{q'}$ where $q'$ satisfies $\frac{1}{q'} = 1 - \frac{1}{q} = \frac{\lambda}{n} + \frac{1}{p} - 1$. This proves the following

**Corollary 1.** Let $u \in L^p$ for a $p > 1$ and $\lambda$ be a positive number such that $0 < \lambda < n$ and $\frac{\lambda}{n} > 1 - \frac{1}{p}$. Then $u * \frac{1}{|x|^{\lambda}} \in L^{q'}$ where $\frac{1}{q'} = \frac{\lambda}{n} + \frac{1}{p} - 1$.

In corollary 1 taking $p = 2$ and $\lambda$ a number such that $\frac{n}{2} < \lambda < n$ we have the following

**Corollary 2.** If $u \in L^2$ then for any positive number $\lambda$ such that $\frac{n}{2} < \lambda < n$ we have $u * \frac{1}{|x|^{\lambda}} \in L^q$ where $\frac{1}{q} = \frac{\lambda}{n} - \frac{1}{2}$ and

\begin{equation}
(1.3) \quad \|u * \frac{1}{|x|^{\lambda}}\|_{L^q} \leq K\|u\|_{L^2}
\end{equation}

where $K$ is a constant depending on $n, \lambda$.

We shall now introduce the function space $\mathcal{D}^s_L \equiv \mathcal{D}^s_L(\mathbb{R}^n)$ for any arbitrary real number $s > 0$.

Let $\Omega$ be an open set in $\mathbb{R}^n$ and $m$ be a non-negative integer. We recall that $\mathcal{E}^m_{L^2}(\Omega)$ denotes the space of all square integrable functions $f$ on $\Omega$ for which all the derivative $\left(\frac{\partial}{\partial x}\right) f$ (in the sense of distributions)
1. Preliminaries to the study of semi-linear equations

of orders $|\nu| \leq m$ are again square integrable functions on $\Omega$. $\mathcal{E}^m_{L^2}(\Omega)$ is provided with the scalar product

\[(1.4) \quad (f, g)_{\mathcal{E}^m_{L^2}(\Omega)} = \sum_{|\alpha| \leq m} \left( \frac{\partial}{\partial x} \right)^\alpha f \cdot \left( \frac{\partial}{\partial x} \right)^\alpha g \quad \text{for } f, g \in \mathcal{E}^m_{L^2}(\Omega)\]

Here $\left( \frac{\partial}{\partial x} \right)^\alpha$ denotes a derivation in the sense of distributions. $\mathcal{E}^m_{L^2}(\Omega)$ is a Hilbert space for this scalar product. Clearly $\mathcal{D}(\Omega) \subset \mathcal{E}^m_{L^2}(\Omega)$. The closure of $\mathcal{D}(\Omega)$ in $\mathcal{E}^m_{L^2}(\Omega)$ is denoted by $\mathcal{D}^m_{L^2}(\Omega)$, with the scalar product which is the restriction of that in $\mathcal{E}^m_{L^2}(\Omega)$, is again a Hilbert space. In general $\mathcal{D}^m_{L^2}(\Omega)$ is not equal to $\mathcal{E}^m_{L^2}(\Omega)$. However when $\Omega = \mathbb{R}^n$ we have $\mathcal{D}^m_{L^2}(\mathbb{R}^n) = \mathcal{E}^m_{L^2}(\mathbb{R}^n)$. We write $\mathcal{D}^m_{L^2}(\mathbb{R}^n) = D^m_{L^2}(\mathbb{R}^n)$ for abbreviation. The elements of $\mathcal{D}^m_{L^2}(\Omega)$ can be considered as functions vanishing up to order $(m - 1)$ (in a generalized sense) on the boundary of $\Omega$.

We observe that $\mathcal{D}^m_{L^2}(\Omega) \subset L^\infty$. Hence by Plancheral’s theorem we have

$$
\|f\|_m^2 = \|f\|^2_{\mathcal{E}^m_{L^2}} = \sum_{|\alpha| \leq m} \left\| \left( \frac{\partial}{\partial x} \right)^\alpha f \right\|^2_{L^2} = \sum_{|\alpha| \leq m} \|2\pi i \xi \|^\alpha \hat{f} \|^2_{L^2}
$$

where $\hat{f}$ is the Fourier image of $f$. Now there exist constants $c_1, c_2 > 0$ such that

$$
c_1^2 (1 + |\xi|)^{2m} \leq \sum_{|\alpha| \leq m} \| (2\pi i \xi)^\alpha \|^2 \leq c_2^2 (1 + |\xi|)^{2m}.
$$

Thus, if $f \in \mathcal{D}^m_{L^2}$ then $(1 + |\xi|)^m \hat{f} \in L^2$ and further

$$
C_1 \|(1 + |\xi|)^m \hat{f}\|_{L^2} \leq \|f\|_m \leq C_2 \|(1 + |\xi|)^m \hat{f}\|_{L^2}.
$$

Hence $\mathcal{D}^m_{L^2}$ can also be defined as the space of all tempered distributions $f$ such that $(1 + |\xi|)^m \hat{f} \in L^2$ where $\hat{f}$ denotes the Fourier image of $f$. This motivates the following.

**Definition.** For any real $s$, $\mathcal{D}^s_{L^2}$ is the space of tempered distributions $f$ such that $(1 + |\xi|)^s \hat{f} \in L^2$. 

\[ \mathcal{D}_{L^2}^s \text{ is provided with the scalar product} \]

\[ (f, g)_s \equiv (f, g)_{\mathcal{D}_{L^2}^s} = ((1 + |\xi|^2)^s \hat{f}, (1 + |\xi|^2)^s \hat{g})_{L^2} \]

For this scalar product \( \mathcal{D}_{L^2}^s \) is a Hilbert space. It is clear that if \( s \geq s' \) then \( \mathcal{D}_{L^2}^s \subset \mathcal{D}_{L^2}^{s'} \) and the inclusion mapping is continuous.

**Remarks.** (1) The dual space of \( \mathcal{D}_{L^2}^s \) is \( \mathcal{D}_{L^2}^{-s} \). \( (\mathcal{D}_{L^2}^s)' = \mathcal{D}_{L^2}^{-s} \).

(2) The mapping \( u \to \frac{\partial u}{\partial x_j} \) from \( \mathcal{D}_{L^2}^s \) into \( \mathcal{D}_{L^2}^{s-1} \) is continuous.

(3) The mappings defined by

\[ (a(x), u) \to a(x)u \]

(i) from \( \mathcal{B}^m \times \mathcal{D}_{L^2}^m \) into \( \mathcal{D}_{L^2}^m \) and (ii) from \( \mathcal{B}^m \times \mathcal{D}_{L^2}^{-m} \) into \( \mathcal{D}_{L^2}^{-m} \) are continuous for \( m = 0, 1, 2, \ldots \).

**Lemma 1.** Let \( s \) be a real number \( \geq 0 \)

(i) If \( u \in \mathcal{D}_{L^2}^s \) for \( 0 \leq s < \frac{n}{2} \) then \( u \in L^p \) where \( \frac{1}{p} = \frac{1}{2} - \frac{s}{n} > 0 \) and

\[ ||u||_{L^p} \leq c(s, n)||u||_s \]

where the constant \( c(s, n) \) depends only on \( s \) and \( n \);

(ii) If \( u \in \mathcal{D}_{L^2}^s \) for \( s > \frac{n}{2} \) then \( u \in \mathcal{B}^0 \) and

\[ ||u||_{\mathcal{B}^0} \leq c(s, n)||u||_s \]

where the constant \( c(s, n) \) depends only on \( s, n \).

More precisely, for any \( \sigma \leq 1 \) with \( 0 < \sigma < s - \frac{n}{2} \) we have

\[ ||u||_{\mathcal{B}^\sigma} \leq c(s, n, \sigma)||u||_s \]

where the constant \( c(s, n, \sigma) \) depends only on \( s, n, \sigma \).
1. Preliminaries to the study of semi-linear equations

Remark. We recall that \( \frac{1}{|x|^s} \) is tempered distribution and we have the formulae giving its Fourier image.

\[
\mathcal{F} \left( \frac{1}{|x|^m} \right) = \frac{1}{\Pi^{m-m}} \frac{\Gamma \left( \frac{m-n}{2} \right)}{\Gamma \left( \frac{m}{2} \right)} \left( \frac{1}{|\xi|^{m-m}} \right) \text{ for } n \leq m < n \quad \text{and} \\
\mathcal{F} \left( \frac{1}{|x|^{n-m}} \right) = \frac{1}{\Pi^{n-m}} \frac{\Gamma \left( \frac{n}{2} \right)}{\Gamma \left( \frac{n-m}{2} \right)} \left( \frac{1}{|\xi|^{m}} \right) \text{ for } 0 < m < \frac{n}{2}.
\]

(1.9)

For proof of these formulae we refer to L. Schwartz, Théorie des distributions, Vol. II, p. 113.

Proof of Lemma 1: (i) The assertion (i) is trivial when \( s = 0 \). Hence we may assume that \( 0 < s < \frac{n}{2} \). Let \( u \in \mathcal{D}^s_L \). Writing \( \hat{u} \) as \( |\xi|^{-s} (|\xi|^s \hat{u}) \) we have

\[ u = c \cdot \frac{1}{|x|^{n-s}} \ast (\wedge^s u) \]

by taking the inverse Fourier images and using the above remark (we note that \( c \) is a positive constant depending only on \( n, s \)). It follows now, from cor. 2 after Prop. 1, that \( u \in L^p \) and

\[ ||u||_{L^p} = c ||\frac{1}{|x|^{n-s}} \ast (\wedge^s u)||_{L^p} \leq c(s, n) ||\wedge^s u||_{L^2} \]

where \( \frac{1}{p} = \frac{1}{2} - \frac{s}{n} \) (the constant \( c(s, n) \) depends only on \( s, n \)). By Plancheral’s theorem we have

\[ ||\wedge^s u||_{L^2} = |||\xi|^s \hat{u}||_{L^2} \leq ||(1 + |\xi|^s)^s \hat{u}||_{L^2} = ||u||_s. \]

This proves the inequality (1.6).

(ii) Let \( u \in \mathcal{D}^s_L \) for \( s > \frac{n}{2} \). We have, using Cauchy-Schwarz inequality

\[ |u(x)| \leq \int |\hat{u}(\xi)| d\xi \leq ||(1 + |\xi|^s)^s \hat{u}||_{L^2} ||(1 + |\xi|^{-s})||_{L^2} \]

which implies that \( |u(x)| \leq c(s, n) ||u||_s \) where \( c(s, n) \) is a constant depending only on \( s, n \).
We shall now prove Hölder continuity of $u$. Consider

$$u(x) - u(x') = \int \exp(2\pi i x \cdot \xi) \hat{u}(\xi) d\xi - \int \exp(2\pi i x' \cdot \xi) \hat{u}(\xi) d\xi = \int \exp(2\pi i x \cdot \xi) (1 - \exp(2\pi i (x' - x) \cdot \xi)) \hat{u}(\xi) d\xi.$$ 

For any real number $\sigma$ such that $0 < \sigma \leq 1$ let

$$M_\sigma = \sup_{-\infty < \lambda < \infty} \left| \frac{\exp(i\lambda) - 1}{\lambda^\sigma} \right|.$$

Clearly $M_\sigma < \infty$. Taking $\lambda' = 2\pi(x - x') \cdot \xi$ we obtain

$$|1 - \exp(2\pi i (x' - x) \cdot \xi)| \leq M_\sigma (2\pi |x - x'| |\xi|)^\sigma.$$

Hence

$$\left| \frac{u(x) - u(x')}{|x - x'|} \right| \leq (2\pi)^\sigma M_\sigma \int |\xi|^\sigma |\hat{u}(\xi)| d\xi \leq (2\pi)^\sigma M_\sigma \| (1 + |\xi|)^{\sigma-s} \hat{u} \|_{L^2} \| (1 + |\xi|)^{\sigma-s} \|_{L^2}.$$

We know that $\sigma - s < -\frac{n}{2}$ implies $\| (1 + |\xi|)^{\sigma-s} \|_{L^2} < \infty$ and this proves the Hölder continuity of $u$. Thus $u \in B^\sigma$ for any $\sigma \leq 1$ with $0 < \sigma < s - \frac{n}{2}$.

Proposition 2. If $u \in \mathcal{D}_2^{[\frac{n}{2}] + 1}$ then, for $1 \leq |\nu| \leq \left[ \frac{n}{2} \right] + 1$, $(\frac{\partial}{\partial x})^\nu u \in L^p$, where $p$ is a positive number such that

(a) $\frac{1}{p} \in \left[ \frac{|\nu|}{n} - \frac{1}{n} \cdot \frac{1}{2} \right] - \{0\}$ when $n$ is even,

(b) $\frac{1}{p} \in \left[ \frac{|\nu|}{n} - \frac{1}{2n} \cdot \frac{1}{2} \right]$ when $n$ is odd.

Further the mapping $u \rightarrow \left( \frac{\partial}{\partial x} \right)^\nu u$ is continuous from $\mathcal{D}_2^{[\frac{n}{2}] + 1}$ into $L^2$ and we have the inequality

$$\| (\frac{\partial}{\partial x})^\nu u \|_{L^p} \leq c(\nu, n, p) \| u \|_{[\frac{n}{2}] + 1}.$$
1. Preliminaries to the study of semi-linear equations

The constant $c(\nu, n, p)$ depends only on $\nu, n, p$.

Before proceeding with the proof of this proposition we introduce the following

**Definition.** The operator $\Lambda^s$. For any $u \in D^{[\frac{n}{2}]+1}_{L_2}$, the operator $\Lambda^s$ is defined by the condition that $\Lambda^s u$ is the inverse Fourier image of $|\xi|^{s-|\nu|}$.

**Proof.** For any real $s \geq 0$ such that $s \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$, $u \in D^{[\frac{n}{2}]+1}_{L_2}$ implies that $u \in D^s_{L_2}$. Since the inverse Fourier image of $\frac{1}{|\xi|^{s-|\nu|}}$ is $c(n, \nu)\frac{1}{|x|^{n-(s-|\nu|)}}$ we can write

$$\left(\frac{\partial}{\partial x}\right)^\nu u = c(n, \nu)\frac{1}{|x|^{n-(s-|\nu|)}} + \left(\Lambda^{s-|\nu|}\left(\frac{\partial}{\partial x}\right)^\nu u\right)$$

by taking inverse Fourier image of

$$\left(\left(\frac{\partial}{\partial x}\right)^\nu u\right) = (2\pi i \xi)^\nu \hat{u} = \frac{1}{|\xi|^{s-|\nu|}}([|\xi|^{s-|\nu|}](2\pi i \xi)^\nu \hat{u})$$.

Hence it follows, from Cor. 2 of Prop. 1, that

$$\|\left(\frac{\partial}{\partial x}\right)^\nu u\|_{L^p} = c(n, \nu)[\|1|\xi|^{s-|\nu|}|(2\pi i \xi)^\nu \hat{u}\|]_{L^p}$$

for $\frac{1}{p} = \frac{n-(s-|\nu|)}{n} - \frac{1}{2} = \frac{s-|\nu|}{n}$. On the other hand we know that

$$\|\Lambda^{s-|\nu|}\left(\frac{\partial}{\partial x}\right)^\nu u\|_{L^2} \leq \|u\|_{L^2} \leq \|u\|_{L^2}^{(\nu)+1}$$

which proves that

$$\|\left(\frac{\partial}{\partial x}\right)^\nu u\|_{L^p} \leq c(n, \nu)[\|u\|_{L^2}^{(\nu)+1}].$$
Using the fact that $|\nu| \leq s \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$ we have, since

$$\frac{1}{p} \in \left[ \frac{1}{2} - \frac{|\nu|}{n} + 1 - \frac{1}{n}, \frac{1}{2} \right] - \{0\},$$

that $\frac{1}{p} \in \left[ \frac{|\nu|}{n} - \frac{1}{n}, \frac{1}{2} \right] - \{0\}$ when $n$ is even and similarly $\frac{1}{p} \in \left[ \frac{|\nu|}{n} - \frac{1}{n}, \frac{1}{2} \right] - \{0\}$ when $n$ is odd.

An entirely analogous proof will yield □

**Proposition 2'**. If $u \in \mathcal{D}_{L^2}^{[\frac{n}{2}] + N}$ we have $\left( \frac{\partial}{\partial x} \right)^\nu u \in L^p$ where $p$ is a positive number such that

(a) $\frac{1}{p} \in \left[ \frac{|\nu|}{n} - \frac{N}{n} - \frac{1}{2}, \frac{1}{2} \right] - \{0\}$ when $n$ is even and

(b) $\frac{1}{p} \in \left[ \frac{|\nu|}{n} - \frac{2N}{2n} - \frac{1}{2}, \frac{1}{2} \right] - \{0\}$ when $n$ is odd, where $1 \leq N \leq |\nu| \leq \left\lfloor \frac{n}{2} \right\rfloor + N$.

Further the mapping $u \rightarrow \left( \frac{\partial}{\partial x} \right)^\nu u$ is continuous from $\mathcal{D}_{L^2}^{[\frac{n}{2}] + N}$ into $L^p$ and we have the inequality

$$\| \left( \frac{\partial}{\partial x} \right)^\nu u \|_{L^p} \leq c(\nu, |n|, N, p) \| u \|_{\mathcal{D}_{L^2}^{[\frac{n}{2}] + N}}$$

where the constant $c(\nu, n, N, p)$ depends only $\nu, n, N, p$.

The following result gives estimates in the $L^2$ norm of some non-linear functions of the derivatives of functions belonging to $\mathcal{D}_{L^2}^{[\frac{n}{2}] + N}$. The proofs are based essentially on the above result and a generalization of Hölder’s inequality which we recall without proof.

**Proposition 3** (Generalized Hölder’s inequality). Let $\lambda_1, \ldots, \lambda_p$ be positive numbers $> 1$ such that $\sum \frac{1}{\lambda_j} = 1$. If $f_1, \ldots, f_p$ are functions belonging to $L^{\lambda_1}, \ldots, L^{\lambda_p}$ respectively then

$$\int |f_1(x) \cdots f_p(x)| dx \leq \| f_1 \|_{L^{\lambda_1}} \cdots \| f_p \|_{L^{\lambda_p}}.$$
Proposition 4. Let $l$ be an arbitrary integer $\geq 1$ and $\nu_1, \ldots, \nu_l$ denote multi-indices

(i) If $u \in D^{[[\frac{n}{2}]+1}_L$ and $\sum |\nu_j| \leq \left[\frac{n}{2}\right] + 1$ then $\left(\frac{\partial}{\partial x}\right)^{\nu_1} u \ldots \left(\frac{\partial}{\partial x}\right)^{\nu_l} u \in L^2$ and satisfies

\[(1.14) \quad \| \left(\frac{\partial}{\partial x}\right)^{\nu_1} u \ldots \left(\frac{\partial}{\partial x}\right)^{\nu_l} u \|_{L^2} \leq c\|u\|_{[[\frac{n}{2}]+1}^{l}
\]

where $c$ depends on $n, \nu_1, \ldots, \nu_l$ only.

(ii) If $u \in D^{[[\frac{n}{2}]+2}_L$ and $\sum |\nu_j| \leq \left[\frac{n}{2}\right] + 2$ then $\left(\frac{\partial}{\partial x}\right)^{\nu_1} u \ldots \left(\frac{\partial}{\partial x}\right)^{\nu_l} u \in L^2$ and satisfies

\[(1.15) \quad \| \left(\frac{\partial}{\partial x}\right)^{\nu_1} u \ldots \left(\frac{\partial}{\partial x}\right)^{\nu_l} u \|_{L^2} \leq c\|u\|_{[[\frac{n}{2}]+1}^{l-1}\|u\|_{[[\frac{n}{2}]+2}^{l};
\]

the constant $c$ depends only on $n, \nu_1, \ldots, \nu_l$.

(iii) If $u \in D^{[[\frac{n}{2}]+N+1}_L$ and $\sum |\nu_j| \leq \left[\frac{n}{2}\right] + N + 1$ then $\left(\frac{\partial}{\partial x}\right)^{\nu_1} u \ldots \left(\frac{\partial}{\partial x}\right)^{\nu_l} u \in L^2$ and satisfies

\[(1.16) \quad \| \left(\frac{\partial}{\partial x}\right)^{\nu_1} u \ldots \left(\frac{\partial}{\partial x}\right)^{\nu_l} u \|_{L^2} \leq c\|u\|_{[[\frac{n}{2}]+1}^{l-1}\|u\|_{[[\frac{n}{2}]+N+1}^{l};
\]

the constant $c$ depend only on $n, N, \nu_1, \ldots, \nu_l$.

Proof. The case $l = 1$ is trivial. If $\nu_j = 0$ for some $j$ one can majorize $u$ in the maximum norm by $\|u\|_{[[\frac{n}{2}]+1}$. Hence we may assume that $l \leq 2$ and $|\nu_j| \geq 1$.

(i) Since $u \in D^{[[\frac{n}{2}]+1}_L$ it follows, from Prop. 2 that $\left(\frac{\partial}{\partial x}\right)^{\nu_j} u \in L^{p_j}$ for $1 \leq |\nu_j| \leq \left[\frac{n}{2}\right] + 1$ where $p_j$ is a real number such that

(a) $\frac{1}{p_j} \in \left[\frac{|\nu_j|}{n} - \frac{1}{n}, \frac{1}{2}\right] - \{0\}$ when $n$ is even and
(b) \( \frac{1}{p_j} \in \left[ \frac{|\nu_j|}{n} - \frac{1}{2n}, \frac{1}{2} \right] \) when \( n \) is odd.

Further we have

\[
\| (\frac{\partial}{\partial x})^{\nu_j} u \|_{L^{p_j}} \leq c(\nu_j, n, p_j) \| u \|_{\left[ \frac{2}{n} \right] + 1} (j = 1, \ldots, l).
\]

Let \( \frac{1}{P_j} \) denote the infimum of \( \frac{1}{p_j} \) in this range.

If \( n \) is even (a) implies that

\[
\sum \frac{1}{P_j} = \sum \left( \frac{|\nu_j|}{n} - \frac{1}{n} \right) \leq \frac{\frac{n+1}{n} - \frac{1}{n}}{n} - \sum \frac{1}{2n} < \frac{1}{2} + \frac{1}{2n}
\]

and so. One can choose \( p_1, \ldots, p_l \) satisfying (a) and such that \( \sum \frac{1}{p_j} = \frac{1}{2} \). Similarly if \( n \) is odd (b) implies that

\[
\sum \frac{1}{P_j} = \sum \left( \frac{|\nu_j|}{2n} - \frac{1}{2n} \right) \leq \frac{\frac{n+1}{2n} + \frac{1}{2n}}{n} - \sum \frac{1}{2} \leq \frac{1}{2} + \frac{1}{2n}.
\]

Again one can choose \( p_1, \ldots, p_l \) such that \( \sum \frac{1}{p_j} = \frac{1}{2} \) and satisfies (b). Applying the generalized Hölder’s inequality with these \( p_1, \ldots, p_l \) we obtain

\[
\int \left( (\frac{\partial}{\partial x})^{\nu_i} u \ldots (\frac{\partial}{\partial x})^{\nu_l} u \right)^2 dx \leq \Pi_j \left( \int (\frac{\partial}{\partial x})^{\nu_j} u^2 dx \right)^{2/p_j}
\]

\[
= \prod_j \| (\frac{\partial}{\partial x})^{\nu_j} u \|_{L^{p_j}}^2 \leq c(\nu_1, \ldots, \nu_l, n) \| u \|_{\left[ \frac{2}{n} \right] + 1}^{2l}.
\]

(ii) Since \( u \in \mathcal{D}_{L^2}^{[\frac{2}{n}]+2} \) it follows, from Prop. 2 that \( (\frac{\partial}{\partial x})^{\nu_j} u \in L^{p_j}(j = 1, \ldots, l) \) and

\[
\| (\frac{\partial}{\partial x})^{\nu_i} u \|_{L^{p_i}} \leq c(\nu_1, n, p_i) \| u \|_{\left[ \frac{2}{n} \right] + 2^{\nu_i}}.
\]
1. Preliminaries to the study of semi-linear equations

\[
\| \left( \frac{\partial}{\partial x} \right)^{\nu_j} u \|_{L^p_j} \leq c(\nu_j, n, p_j) \| u \|_{L^2}^{1+1} (j = 2, \ldots, l)
\]

for \(1 \leq |\nu_1| \leq \left\lfloor \frac{n}{2} \right\rfloor + 2, 1 \leq |\nu_j| \leq \left\lfloor \frac{n}{2} \right\rfloor + 1\) where \(p_1, \ldots, p_l\) are real numbers such that

(a) \( \frac{1}{p_1} \in \left[ \frac{|\nu_1|}{n} - \frac{2}{n} \frac{1}{2}, \frac{1}{2} \right] \) when \(n\) is even,

(b) \( \frac{1}{p_1} \in \left[ \frac{|\nu_1|}{n} - \frac{3}{2n} \frac{1}{2} \right] \) when \(n\) is odd

and

(a) \( \frac{1}{p_j} \in \left[ \frac{|\nu_j|}{n} - \frac{1}{2n}, \frac{1}{2} \right] \) when \(n\) is even,

(b) \( \frac{1}{p_j} \in \left[ \frac{|\nu_j|}{n} - \frac{1}{2n}, \frac{1}{2} \right] \) when \(n\) is odd \((j = 2, \ldots, l)\).

We may without loss of generality assume that \(|\nu_1| \geq |\nu_j|\) for \(j = 2, \ldots, l\).

(1) Suppose \(|\nu_1| = 1\). Since \(\sum_{j=2}^l |\nu_j| \leq \left\lfloor \frac{n}{2} \right\rfloor + 1\) we have from lemma 1 that

\[
\| \left( \frac{\partial}{\partial x} \right)^{\nu_1} u \ldots \left( \frac{\partial}{\partial x} \right)^{\nu_l} u \|_{L^2} \leq \sup \left\| \left( \frac{\partial}{\partial x} \right)^{\nu_1} u \ldots \left( \frac{\partial}{\partial x} \right)^{\nu_l} u \right\|_{L^2} \\
\leq c(n) \| u \|_{L^2} \left( \left\| \left( \frac{\partial}{\partial x} \right)^{\nu_1} u \ldots \left( \frac{\partial}{\partial x} \right)^{\nu_l} u \right\|_{L^2} \right)
\]

(iii) Suppose \(|\nu_j| \geq 2(j = 2, \ldots, l)\) then we have the estimates of the type (1.11). As before we denote the infimum of \(\frac{1}{p_j}\) by \(\frac{1}{p_j} (j = 1, \ldots, l)\).

If \(n\) is even \((a_1), (a_j)\) imply that

\[
\sum_j \frac{1}{p_j} = \frac{|\nu_1|}{n} - \frac{2}{n} + \sum \left( \frac{|\nu_j|}{n} - \frac{1}{n} \right) < \frac{1}{2}
\]
and if \( n \) is odd \((b_1)\), \((b_j)\) imply that \( \sum \frac{1}{p_j} < \frac{1}{2} \). In either of the cases we can choose \( p_2, \ldots, p_l \) such that \( \sum \frac{1}{p_j} = \frac{1}{2} \).

Again applying the generalized Hölder’s inequality we obtain
\[
\left\| \left( \frac{\partial}{\partial x} \right)^{\nu_j} u \right\|_{L^2} \leq \prod_{j=2}^{l} \left\| \frac{\partial}{\partial x} \right\|^{\nu_j} u_{L^p}
\]
\[
\leq c(n, \nu_1, \ldots, \nu_l) \|u\|_{L^q_1} \|u\|_{L^p_2}.
\]

As before we may assume that \(|\nu_1| \geq |\nu_j|\) for \( j = 2, \ldots, l \). Let \( u \in D^{[2]_1+N+1} \). We distinguish the following three different cases:

\((\alpha)|\nu_1| \leq N - 1\), \((\beta)|\nu_1| = N\), \((\gamma)|\nu_j| \geq N\).

\(\Box\)

Case \((\alpha)\). Since \(|\nu_j| \leq |\nu_1| \leq N - 1\) by Sobolev’s lemma we have
\[
\sup_{x} \left| \left( \frac{\partial}{\partial x} \right)^{\nu_j} u \right| \leq c \|u\|_{[2]^{N+1}}.
\]

Therefore we have
\[
\left\| \left( \frac{\partial}{\partial x} \right)^{\nu_j} u \right\|_{L^2} \leq \left\| \left( \frac{\partial}{\partial x} \right)^{\nu_j} u \right\|_{L^2} \cdot \prod_{j=2}^{l} \sup_{x} \left| \left( \frac{\partial}{\partial x} \right)^{\nu_j} u \right|
\]
\[
\leq C \|u\|_{[2]_1+N+1} \cdot \|u\|_{L^q_1} \|u\|_{L^p_2}
\]
\[
\leq C \|u\|_{[2]_1+N+1} \cdot \|u\|_{L^q_1} \|u\|_{L^p_2}.
\]

179 Case \((\beta)\). \(|\nu_1| = N\) implies that \( \sum_{j=2}^{l} |\nu_j| = \left\lfloor \frac{n}{2} \right\rfloor + 1 \) and we have from lemma that
\[
\left\| \left( \frac{\partial}{\partial x} \right)^{\nu_j} u \right\|_{L^2} \leq \sup_{x} \left| \left( \frac{\partial}{\partial x} \right)^{\nu_j} u \right| \left\| \left( \frac{\partial}{\partial x} \right)^{\nu_1} u \right\|_{L^2} \left( \frac{\partial}{\partial x} \right)^{\nu_2} u \ldots \left( \frac{\partial}{\partial x} \right)^{\nu_l} u \right\|_{L^2}.
\]
1. Preliminaries to the study of semi-linear equations

By Sobolev’s lemma we have

\[ \sup \left| \frac{\partial}{\partial x}^\nu u \right| \leq c(n, N, \nu_1) ||u||^{1+1}_{\frac{2}{N}+1} \]

and on the other hand \( \left( \frac{\partial}{\partial x}^\nu \right) u \in L^p \) with

\[ || \left( \frac{\partial}{\partial x}^\nu \right) u ||_{L^p} \leq c(n, N, \nu, p) ||u||^{1+1}_{\frac{2}{N}+1} \]

for \( \frac{1}{p_j} \in \left[ \frac{|\nu_j|}{n} - \frac{N}{2n}, \frac{1}{2} \right] - \{0\} \) if \( n \) is even and \( \frac{1}{p_j} \in \left[ \frac{|\nu_j|}{n} - \frac{2N-1}{2n}, \frac{1}{2} \right] \) if \( n \) is odd (from Prop. 2').

Denoting \( \inf \frac{1}{p_j} \) by \( \frac{1}{p_j} \) we see that \( \sum \frac{1}{p_j} = \frac{1}{p_j} \left( \frac{|\nu_j|}{n} - \frac{N}{n} \right) < \frac{1}{2} \) if \( n \) is even and \( \sum \frac{1}{p_j} = \frac{1}{p_j} \left( \frac{|\nu_j|}{n} - \frac{2N-1}{n} \right) < \frac{1}{2} \) if \( n \) is odd. One can choose \( p_2, \ldots, p_l \) such that \( \sum \frac{1}{p_j} = \frac{1}{2} \) in both the cases. An application of the generalized Hölder’s inequality with these \( p_2, \ldots, p_l \) gives

\[ \left\| \left( \frac{\partial}{\partial x}^\nu \right) u \ldots \left( \frac{\partial}{\partial x}^\nu \right) u \right\|_{L^2} \leq \prod_{j=2}^l \left\| \left( \frac{\partial}{\partial x}^\nu \right) u \right\|_{L^{p_j}} \]

\[ \leq c(n, \nu_2, \ldots, \nu_l, N, p_2, \ldots, p_l) ||u||^{l-1}_{\frac{2}{N}+1} \]

(γ) If \( |\nu_j| \geq N \) for \( j = 2, \ldots, l \) we have from Prop. 2 that

\[ \left\| \left( \frac{\partial}{\partial x}^\nu \right) u \right\|_{L^{p_1}} \leq c(\nu_1, p_1, N, n) ||u||^{1+1}_{\frac{2}{N}+1} \]

and \( \left\| \left( \frac{\partial}{\partial x}^\nu \right) u \right\|_{L^{p_j}} \leq c(\nu_j, p_j, N, n) ||u||^{n}_{\frac{2}{N}+1} \) where \( p_1, \ldots, p_l \) are real numbers such that

\[ \begin{cases} \frac{1}{p_1} \in \left[ \frac{|\nu_1|}{n} - \frac{N+1}{n}, \frac{1}{2} \right] - \{0\}, \\
\frac{1}{p_j} \in \left[ \frac{|\nu_j|}{n} - \frac{N}{n}, \frac{1}{2} \right] - \{0\} \end{cases} \] for even \( n \) and
\[
\begin{cases}
\frac{1}{p_i} \in \left[ \frac{\nu_i}{n} - \frac{2N+1}{2n}, \frac{1}{2} \right] \\
\frac{1}{p_j} \in \left[ \frac{\nu_j}{n} - \frac{2N-1}{2n}, \frac{1}{2} \right]
\end{cases}
\text{for odd } n.
\]

If \( \frac{1}{p_j} \) denotes \( \inf \frac{1}{p_j} \) we have

\[
\sum_{j=1}^{l} \frac{1}{p_j} = \sum_{j=1}^{l} \left( \frac{\nu_j}{n} - \frac{N+1}{n} \right) - \sum_{j=2}^{l} \frac{N}{2n} < \frac{1}{2} \text{ for even } n \text{ and}
\]

\[
\sum_{j=1}^{l} \frac{1}{p_j} = \sum_{j=1}^{l} \left( \frac{\nu_j}{n} - \frac{2N+1}{2n} \right) - \sum_{j=2}^{l} \frac{2N-1}{2n} < \frac{1}{2} \text{ for odd } n.
\]

Once again choosing \( p_2, \ldots, p_l \) such that \( \sum \frac{1}{p_j} = \frac{1}{2} \) we obtain the desired inequality after applying the generalized Hölder’s inequality to

\[
\left\| \left( \frac{\partial}{\partial x} \right)^{\nu_1} u \cdots \left( \frac{\partial}{\partial x} \right)^{\nu_l} u \right\|_{L^2} \text{ with these } p_1, \ldots, p_l \text{ and using the estimates of the form (1.11).}
\]

By an argument completely analogous to the one in the prop. 4 one can establish the following more general result.

**Proposition 5.** Let \( l \) be an arbitrary integer and \( \nu_1, \ldots, \nu_l \) be \( l \) multi-indices.

(i) If \( u_1, \ldots, u_l \in \mathcal{D}^{\left[\frac{n}{2}\right]+1}_{L^2} \) and \( \sum_{j=1}^{l} |\nu_j| \leq \left[ \frac{n}{2} \right] + 1 \) then

\[
\left( \frac{\partial}{\partial x} \right)^{\nu_1} u_1 \cdots \left( \frac{\partial}{\partial x} \right)^{\nu_l} u_l \in L^2. \text{ Further}
\]

\[
(1.17) \quad \left\| \left( \frac{\partial}{\partial x} \right)^{\nu_1} u_1 \cdots \left( \frac{\partial}{\partial x} \right)^{\nu_l} u_l \right\|_{L^2} \leq c \prod_{j=1}^{l} \|u_j\|_{L^2}^{\nu_j+1}
\]

where the constant \( c \) depends only on \( n, \nu_1, \ldots, \nu_l \).
2. Regularity of some non-linear functions

(i) Let \( |v_1| \geq |v_j| \) for \( j = 2, \ldots, l \). If \( u_1 \in \mathcal{D}'_{L^2}^{n+1}, u_2, \ldots, u_l \mathcal{D}'_{L^2}^{N} \)
and \( \sum |v_j| \leq \left[ \frac{n}{2} \right] + N + 1 \) then
\[
(\frac{\partial}{\partial x})^{v_1} u_1 \cdots (\frac{\partial}{\partial x})^{v_l} u_l \in L^2 \text{ and }
\]
\[
(1.18) \quad \left\| (\frac{\partial}{\partial x})^{v_1} u_1 \cdots (\frac{\partial}{\partial x})^{v_l} u_l \right\|_{L^2} \leq c \left\| u_1 \right\|_{\mathcal{D}'_{L^2}^{n+1}} \prod_{j=2}^l \left\| u_j \right\|_{\mathcal{D}'_{L^2}^{N}}
\]
where \( c \) depends only on \( n, v_1, \ldots, v_l, N \).

2 Regularity of some non-linear functions

Here we make a few remarks on the local properties of certain smooth
non-linear functions of \( x, t, u \) which will be required for the study of
some quasi-linear differential equations. Let \( \Omega \) denote the set
\[
\left\{ (x, t) \mid x \in \mathbb{R}^n, 0 \leq t \leq T \right\}.
\]

Let \( f(x, t, u) \) be a function belonging to \( \mathcal{D}'(\mathbb{R}^n) \). For a fixed
function \( \alpha \in \mathcal{D}(\mathbb{R}^n) \) we denote \( \alpha(x)f(x, t, u) \) by \( \tilde{f}(x, t, u) \). \( \alpha \) localizes \( f(x, t, u) \) in the \( x \)-space. We use the following abbreviations \( (\frac{\partial}{\partial x}, \frac{\partial}{\partial u})^\beta \)
stands for a derivation of order \( |\beta| \) with respect to \( x \) and \( u \); \( F(x, t), \bar{F}(x, t), G(x, t), \ldots \) stand respectively for
\[
f(x, t, u(x, t)), \quad \tilde{f}(x, t, u(x, t)), \quad g(x, t, u(x, t)) \ldots
\]

Let \( U \) be the subset of \( \Omega \times \mathbb{C} \) defined by
\[
(2.1) \quad U = \{(x, t, u) \mid (x, t) \in \Omega, |u| \leq \sup_{\Omega} |u(x, t)|\}.
\]

Throughout this section \( c_1(n), c_2(n), \ldots \) denote constants depending
only on \( n \).

Lemma 1. If \( u \in \mathcal{D}'_{L^2}^{n+1}[0, T] \) then \( \tilde{F} = \tilde{f}(x, t) \in \mathcal{D}'_{L^2}^{n+1}[0, T] \) and
\[
(2.2) \quad \| \tilde{F} \|_{\mathcal{D}'_{L^2}^{n+1}} \leq c_1(n) M \left\{ 1 + \| u \|_{\mathcal{D}'_{L^2}^{n+1}} \right\}
\]
where

\[ M = \max_{|\beta| \leq n^2 + 1} \left| \frac{\partial}{\partial x} \left( \frac{\partial}{\partial u} \right)^{\beta} f(x, t, u) \right|. \]

Before proceeding with the proof of the lemma, we make the following two remarks. Let \( u \in L^2_x[0, T] \). Let \( \varphi_\varepsilon \) be the mollifiers in the \( x \)-space and let \( u_\varepsilon(x, t) = u(x, t) * \varphi_\varepsilon(x) \); then

(i) \( u_\varepsilon \in B^0_x[0, T] \) and

\[ (2.3) \quad \|u_\varepsilon(x, t)\|_{B^0_x} \leq \|u(x, t)\|_{B^0_x}. \]

This is an immediate consequence of lemma \( \text{§} \) of Chap. 3.

(ii) \( u_\varepsilon \in D_x^s[0, T] \) and

\[ \|u_\varepsilon\|_s \leq \|u\|_s \quad \text{for} \ 0 \leq s \leq \left[ \frac{n}{2} \right] + 1. \]

In fact, we observe that \( \hat{\varphi}_\varepsilon(\xi) = \hat{\varphi}(\varepsilon \xi) \to \hat{\varphi}(0) = 1 \) as \( \varepsilon \to 0 \). Consider

\[ \|u_\varepsilon - u\|_s = \|(1 + |\xi|^s)(\hat{u}_\varepsilon(\xi, t) - \hat{u}(\xi, t))\|_L^2 \]

which converges to 0 as \( \varepsilon \to 0 \). Hence

\[ \|u_\varepsilon\| \leq \|u\| + \|u_\varepsilon - u\| \]

implies the assertion.

**Proof of the Lemma.** Throughout the proof the derivatives with respect to \( x \) are taken in the sense of distributions. Denoting \( f(x, t, u_\varepsilon(x, t)) \) by \( \tilde{F}_\varepsilon(x, t) \) we see that \( \tilde{F}_\varepsilon(x, t) \to F(x, t) \) as \( \varepsilon \to 0 \). For,

\[ \|
\tilde{F}_\varepsilon(x, t) - F(x, t)\|_L^2 = \|
\left( \frac{\partial}{\partial u} \right) (x, t, u(x, t)) \cdot (u_\varepsilon(x, t) - u(x, t))\|_L^2 \]

which tends to 0 as \( \varepsilon \to 0 \). Now, for \( 1 \leq j \leq n \),

\[ \frac{\partial}{\partial x_j} F(x, t) = \lim_{\varepsilon \to 0} \frac{\partial}{\partial x_j} F_\varepsilon(x, t) \]
where the limit is taken in the space \( L^2 \). In fact, we can write

\[
\frac{\partial}{\partial x_j} \tilde{F}_\epsilon(x, t) = \left[ \frac{\partial \tilde{f}}{\partial x_j} \right](x, t, u_\epsilon(x, t)) + \left[ \frac{\partial \tilde{f}}{\partial u} \right](x, t, u_\epsilon(x, t)) \cdot \frac{\partial u_\epsilon}{\partial x_j}(x, t)
\]

in the sense of distributions. This function tends to

\[
\left[ \frac{\partial \tilde{f}}{\partial x_j} \right](x, t, u(x, t)) + \left[ \frac{\partial \tilde{f}}{\partial u} \right](x, t, u(x, t)) \cdot \frac{\partial u}{\partial x_j}(x, t).
\]

in the space \( L^2[0, T] \), because \( u \in \mathcal{D}[\frac{\epsilon}{2}]^{1+}[0, T] \) implies that \( \left[ \frac{\partial \tilde{f}}{\partial x_j} \right] \) (2.4) \( (x, t, u(x, t)) \) belong to the space \( \mathcal{D}_x^0[0, T] \).

For a multi-index \( \nu \) with \( |\nu| \leq \left\lfloor \frac{n}{2} \right\rfloor + 1 \) we have

\[
(\frac{\partial}{\partial x})^\nu \tilde{F}_\epsilon(x, t) = \sum_{|\rho_j| \leq |\nu|} C_{\rho_1...\rho_l} g_{\rho_1...\rho_l}(x, t, u_\epsilon(x, t)) \prod_{j=1}^l \left( \frac{\partial}{\partial x_j} \right)^{\rho_j} u_\epsilon(x, t)
\]

where \( C_{\rho_1...\rho_l} \) are constants and \( g_{\rho_1...\rho_l}(x, t, u) \) is one of the derivatives

\[
(\frac{\partial}{\partial x}, \frac{\partial}{\partial u})^\rho \tilde{f}(x, t, u)
\]

of orders \( |\rho| \leq |\nu| \). This identity is again taken in the sense of distributions in the \( x \)-space. In view of the prop. \( \frac{\epsilon}{2} \) the function.

\[
g_{\rho_1...\rho_l}(x, t, u(x, t)) \prod_{j=1}^l \left( \frac{\partial}{\partial x} \right)^{\rho_j} u(x, t) = G_{\rho_1...\rho_l}(x, t) \prod_{j=1}^l \left( \frac{\partial}{\partial x} \right)^{\rho_j} u(x, t)
\]

belongs to \( L^2[0, T] \). Setting

\[
J_\epsilon(x) = G_{\rho_1...\rho_l}(x, t) \prod_{j=1}^l \left( \frac{\partial}{\partial x} \right)^{\rho_j} u_\epsilon(x, t) - G_{\rho_1...\rho_l}(x, t) \prod_{j=1}^l \left( \frac{\partial}{\partial x} \right)^{\rho_j} u(x, t)
\]

we have

\[
||J_\epsilon||_{L^2} \leq M ||u_\epsilon - u(x, t) \prod_{j=1}^l \left( \frac{\partial}{\partial x} \right)^{\rho_j} u(x, t)||_{L^2}
\]
\[ + \sum_{j=1}^{l} ||u(x,t)\left(\frac{\partial}{\partial x}\right)^{\nu} u(x,t) \ldots \left(\frac{\partial}{\partial x}\right)^{\nu_{j}} u(x,t)\left(\frac{\partial}{\partial x}\right)^{n_{j}} (u_{c} - u)(x,t) \left(\frac{\partial}{\partial x}\right)^{n_{j}} u_{c}(x,t)||_{L^{2}}. \]

The prop. 4 of §1 implies that

\[ (2.7) \quad \|J_{e}\|_{L^{2}} \leq c_{2}(n)M\|u_{c} - u\|[\frac{5}{2}]_{*+1}||u||_{[\frac{5}{2}]}^{-1} \]

which tends to 0 as \(e \to 0\). This proves that

\[ (2.8) \quad \left(\frac{\partial}{\partial x}\right)^{\nu} f((x,t),u(x,t)) = \sum_{j=1}^{l} c_{j_{1}...j_{l}} G_{j_{1}...j_{l}}(x,t) \prod_{j=1}^{l} \left(\frac{\partial}{\partial x}\right)^{n_{j}} u(x,t). \]

Again applying Prop. 4 to (2.8) it is easy to see that the estimate (2.2) holds. The continuity in \(t\) of \(F\) is proved as before. This completes the proof of the lemma.

The following results are proved in exactly the same manner as the lemma 1.

**Corollary 1.** If \(f(x,t,u) \in \mathcal{C}^{l}[\frac{5}{2}]^{*N+1}(\Omega \times \mathbb{R})\) and \(u \in \mathcal{D}_{L^{2}}[\frac{5}{2}]^{*N+1}[0,T]\) then

\[ (2.9) \quad \|F(x,t)||_{[\frac{5}{2}]^{*N+1}} \leq C_{3}(n)M_{1} \left\{ 1 + (1 + ||u||_{[\frac{5}{2}]^{*N+1}}) ||u||_{[\frac{5}{2}]^{*N+1}} \right\} \]

where \(M_{1} = \max_{[\frac{5}{2}]^{*N+1}} \sup_{u} \left| \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial u}\right)^{l} f(x,t,u) \right| \).

**Corollary 2.** If \(f(x,t,u_{1},...,u_{s}) \in \mathcal{C}^{l}[\frac{5}{2}]^{*N}(\Omega \times \mathbb{R}^{s})\) and \(u_{j} \in \mathcal{D}_{L^{2}}[\frac{5}{2}]^{*N+1}[0,T](1 \leq j \leq s)\) then \(\alpha(x) \in \mathcal{D}\) implies that

\[ \alpha(x)f(x,t,u_{1}(x,t),...,u_{s}(x,t)) \in \mathcal{D}_{L^{2}}[\frac{5}{2}]^{*N+1}[0,T] \]

and

\[ (2.10) \quad ||\alpha(x)f(x,t,u_{1}(x,t),...,u_{s}(x,t))||_{[\frac{5}{2}]^{*N+1}} \leq C_{4}(n)M_{2}[1 + \sum_{j=1}^{s} ||u_{j}(x,t)||_{[\frac{5}{2}]^{*N+1}} \]

where \(M_{2} = \max_{[\frac{5}{2}]^{*N+1}} \sup_{u_{s}} \left| \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial u}\right)^{l} \alpha(x)f(x,t,u_{1}(x,t),...,u_{s}(x,t)) \right| \).
2. Regularity of some non-linear functions

Here $U_s$ is the subset of $\Omega \times \mathbb{C}^s$ defined by

\[(2.11) \quad U_s = \left\{(x, t, u_1, \ldots, u_s) \mid |u_j| \leq \sup_{\Omega} |u_j(x, t)|, \quad 1 \leq j \leq s \right\}.
\]

**Corollary 3.** If $f(x, t, u)$ is a vector valued function

\[
\begin{pmatrix}
  f_1(x, t, u) \\
  \vdots \\
  f_m(x, t, u)
\end{pmatrix}
\]

with $f_k \in \mathcal{C}^{[m]+2}(\Omega \times \mathbb{C})$ for $1 \leq k \leq m$ and $u \in \mathcal{D}_L^{[m]+1}[0, T]$ then $\alpha \in \mathcal{D}$ implies that $\alpha(x)f_k(x, t, u(x, t))$ belong to the space $\mathcal{D}_L^{[m]+1}[0, T]$ and

\[
\|\alpha(x)f(x, t, u(x, t))\|_{[m]+1} = \sum_k \|\alpha(x)f_k(x, t, u(x, t))\|_{[m]+1}
\]

\[(2.12) \quad \leq C_5(n)M_3(1 + \|u(x, t)\|_{[m]+1})
\]

where $M_3 = \max_{k, |\beta|\leq[m]+1} \sup_{U} \left| \left( \frac{\partial}{\partial x_\beta} \frac{\partial}{\partial u} \right) [\alpha(x)f_k(x, t, u)] \right|$. Similar results hold when $u$ is a vector $(u_1, \ldots, u_s)$ and when $u_j \in \mathcal{D}_L^{[n]+m+1}[0, T]$.

Finally we state a result which is a consequence of these and will be of importance.

**Corollary 4.** Let $f(x, t, u_1, \ldots, u_s) \in \mathcal{C}^{[m]+2}(\Omega \times \mathbb{C}^s)$ and $\nu_1, \ldots, \nu_s$ denote multi-indices. If $u \in \mathcal{D}_L^{[n]+m+1}[0, T]$ and $|\nu_1| + \cdots + |\nu_s| \leq m$ then

\[
\alpha(x)f\left(x, t, \left( \frac{\partial}{\partial x} \right)^{\nu_1} (u(x, t)), \ldots, \left( \frac{\partial}{\partial x} \right)^{\nu_s} u(x, t) \right) \in \mathcal{D}_L^{[n]+1}[0, T]
\]

and

\[
\|\alpha(x)f\left(x, t, \left( \frac{\partial}{\partial x} \right)^{\nu_1} u(x, t), \ldots \right)\|_{[n]+1}
\]
\[ (2.13) \quad \leq M' c(n, m) \left\{ 1 + ||u||_{\frac{n}{2}+1}^{n} \right\}. \]

where \( M' = \max_{[\frac{n}{2}+2]} \sup_{U'_s} \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_s} \right)^{\beta} [\alpha(x)f(x, t, u_1, \ldots, u_s)]. \)

Here again
\[ (2.14) \quad U'_s = \left\{ (x, t, u_1, \ldots, u_2) | (x, t) \in \Omega, |u_j| \leq \sup |(\frac{\partial}{\partial x})^{\nu_j} u(x, t)|, 1 \leq j \leq s \right\}. \]

Proof. From Prop. 4 § 1 we have that, if \( u_1, \ldots, u_s \in D[\frac{n}{2}+1, 0, T] \) and if \( \nu_1, \ldots, \nu_s \) are multi-indices with \( \sum |\nu_j| \leq \left[ \frac{n}{2} \right] + 1 \) then
\[ (2.15) \quad \| \prod_{j=1}^{s} (\frac{\partial}{\partial x})^{\nu_j} u \|_{L^2} \leq C(n, \nu_1, \ldots, \nu_s) \prod_{j=1}^{s} ||u_j||_{\frac{n}{2}+1}. \]

Taking \( u_j = (\frac{\partial}{\partial x})^{\nu_j} u \) we apply this inequality and the rest of the proof is the same as in the previous corollaries. \( \square \)

3 An example of a semi-linear equation

In this section we consider an example of a semi-linear partial differential equation of the second order and we recall a theorem on the existence of solutions of the Cauchy problem for such an equation. This result is due to K. Jörgens (see: Das Anfangswertproblem in Grossen für eine Klasse nichtlinearer Wellengleichungen, Math. Zeit., 77 (1961), 295-308). This theorem will be proved in §5.

Let \( u \to f(u) \) be a real valued infinitely differentiable function defined in \(-\infty < u < \infty\). We consider the following semi-linear wave equation
\[ (3.1) \quad \left( \frac{\partial}{\partial t} \right)^2 u - \Delta u + f(u) = 0. \]
We assume that \( f(0) = 0 \). We shall show that, under certain conditions on the function \( f \), for a given smooth initial data \((u_0, U_1)\) on the hyperplane \( t = 0 \) there exists a unique solution \( u \) of (3.1) in \( t \geq 0 \) with \( u(x, 0) = u_0(x), \frac{\partial}{\partial t} u(x, 0) = u_1(x) \). For instance, we shall show that if \( u_0 \in \mathcal{D}[\mathbb{T}_L^{1+2} \cap \mathcal{E}]) \) and \( u_1 \in \mathcal{D}[\mathbb{T}_L^{1+1} \cap \mathcal{E}) \) then there exists a unique solution \( u \) of (3.1) such that

\[
    u \in \mathcal{D}[\mathbb{T}_L^{1+2} \cap \mathcal{E}), \quad \frac{\partial u}{\partial t} \in \mathcal{D}[\mathbb{T}_L^{1+1} \cap \mathcal{E})
\]

both depending continuously on \( t \) in \( 0 \leq t \leq \infty \) and such that

\[
    u(x, 0) = u_0(x), \quad \frac{\partial u}{\partial t}(x, 0) = u_1(x).
\]

Under the assumption \( f(0) = 0 \) one can also show that if the supports of \( u_0 \) and \( u_1 \) are contained in \( \{ |x| \leq R_0 \} \) then the supports of \( u, \frac{\partial u}{\partial t} \) are contained in \( \{ |x| \leq R_0 + t \} \).

Let \( u_0 \in \mathcal{D}[\mathbb{T}_L^{1+3} \cap \mathcal{E}), u_1 \in \mathcal{D}[\mathbb{T}_L^{1+2} \cap \mathcal{E}) \) be given with their supports contained in \( \{ |x| \leq R_0 \} \). Assume that a solution of (3.1) with the initial data \((u_0, u_1)\) on \( t = 0 \) exists locally. More precisely we assume that there exists a \( t_0 > 0 \) such that there exists a solution \( u \) of (3.1) defined in \( \{ x \in \mathbb{R}^n, 0 \leq t \leq t_0 \} \) with the property that

\[
    (1) \quad u \in (\mathcal{D}[\mathbb{T}_L^{1+3} \cap \mathcal{E})][0, t_0], \quad \frac{\partial u}{\partial t} \in (\mathcal{D}[\mathbb{T}_L^{1+2} \cap \mathcal{E})][0, t_0],
\]

\[
    \left( \frac{\partial}{\partial t} \right)^2 u \in (\mathcal{D}[\mathbb{T}_L^{1+1} \cap \mathcal{E})][0, t_0]) \quad \text{and}
\]

1. \( u(x, 0) = u_0(x), \quad \frac{\partial u}{\partial t}(x, 0) = u_1(x) \).

We say that an \emph{a priori} estimate in the \( L^2 \)-sense for the solution of the Cauchy problem for (3.1) of order \( \left[ \frac{n}{2} \right] + 1 \) holds if the following conditions is satisfied: for any given initial data \((u_0, u_1)\) with \( u_0 \in
for all $0 \leq t \leq T$, where $u$ exists an $u(x,0) = u_0, \frac{\partial u}{\partial t}(x,0) = u_1(x)$. $c$ is called an a priori bound.

The following is a special case of a theorem that will be proved in §5. We state it here to motivate Prop. 1.

**Theorem 1.** Let $f$ be an infinitely differentiable function in $-\infty < u < \infty$ with $f(0) = 0$. Assume that an a priori estimate in the $L^2$-sense for the solution of the Cauchy problem for (3.1) of order $\left[ \frac{n}{2} \right] + 1$ holds. Then, for any initial data $(u_0, u_1)$ with $u_0 \in D^m_{L^2} \cap \mathcal{E}^1, \ u_1 \in D^{m-1}_{L^2} \cap \mathcal{E}^1$, there exists a unique solution $u$ of (3.1) such that

1. $u \in D^m_{L^2} \cap \mathcal{E}^1, \frac{\partial u}{\partial t} \in D^{m-1}_{L^2} \cap \mathcal{E}^1, \left(\frac{\partial}{\partial t}\right)^2 u \in D^{m-2}_{L^2} \cap \mathcal{E}'$ all depending continuously on $t$.

Proposition 1. Let $f$ be an infinitely differentiable function in $-\infty < u < \infty$ with $f(0) = 0$. Then

(i) for $n = 1$ an a priori estimate of order one for the solutions of the Cauchy problem for (3.1) holds when

(a) $\int_0^u f(v) dv \equiv F(u) > -L_0$ ($L_0$ a positive constant),

(ii) assume further that $f(u)$ satisfies the condition

(b) if $n = 2$ there exist $\alpha$ and $k$ such that

$$\left| \frac{df(u)}{du} \right| \leq \alpha (1 + |u|)^k$$
3. An example of a semi-linear equation

and if $n = 3$ there exists an $\alpha$ such that

$$\left| \frac{df(u)}{du} \right| \leq \alpha(1 + u^2).$$

Then an a priori estimate of order 2 for solutions of the Cauchy problem for (3.1) holds.

Proof. Assume that $u_0 \in \mathcal{D}^m_{L^2} \cap \mathcal{E}^1$, $u_1 \in \mathcal{D}^{m-1}_{L^2} \cap \mathcal{E}^1$ ($m \geq \left\lceil \frac{n}{2} \right\rceil + 3$) are given and also that there exists a solution $u$ of the Cauchy problem for (3.1) with initial data $(u_0, u_1)$ such that

$$u \in (\mathcal{D}_{L^2}^m \cap \mathcal{E}^1)[0, T], \quad \frac{\partial u}{\partial t} \in (\mathcal{D}_{L^2}^{m-1} \cap \mathcal{E}^1)[0, T], \quad \left(\frac{\partial}{\partial t}\right)^2 u \in (\mathcal{D}_{L^2}^{m-2} \cap \mathcal{E}^1)[0, T].$$

Let $R$ be a number such that $R_0 + t < R$ for $t \leq T$.

(i) Set

$$E_1(t) = \int_{|x| \leq R} \left[ \frac{1}{2} \left( \frac{\partial u}{\partial t} \right)^2 + \sum_{j=1}^n \left( \frac{\partial u}{\partial x_j} \right)^2 \right] + F(u) + c \, dx$$

where $c$ is a constant to be chosen later. Differentiating with respect to $t$

$$\frac{d}{dt} E_1(t) = \int_{|x| \leq R} \left( \frac{\partial u}{\partial t} \left( \frac{\partial}{\partial t} \right)^2 u + \sum_{j=1}^n \frac{\partial u}{\partial x_j} \left( \frac{\partial u}{\partial x_j} \right) \left( \frac{\partial}{\partial t} \right)^2 u + f(u) \frac{\partial u}{\partial t} \right) \, dx.$$  

Since $\frac{\partial u}{\partial x_j} \left( \frac{\partial}{\partial x_j} \right) \left( \frac{\partial}{\partial t} \right)^2 u$ have compact supports the second term in the right hand side becomes after integration by parts

$$\int \Delta u \cdot \frac{\partial u}{\partial t} \, dx$$

and so we have

$$\frac{d}{dt} E_1(t) = \int_{|x| \leq R} (\Box u + f(u)) \frac{\partial u}{\partial t} \, dx = 0$$  

(where $\Box = \left( \frac{\partial}{\partial t} \right)^2 - \Delta$) since $\Box u + f(u) = 0$. Hence $E_1(t)$ is a constant $E_10$. \qed
Taking \( c > L_0 \) we have \( F(u) + c > 0 \) and so

(3.2) \[
\int \frac{1}{2} \left( \frac{\partial u}{\partial t} \right)^2 + \sum_j \left( \frac{\partial u}{\partial x_j} \right)^2 \, dx \leq E_1(t) = E_1(0).
\]

Since the support of \( u \) is compact there exists \( c_1 \) such that

(3.3) \[ \|u\|_{L^2} \leq c_1 \sum_j \left\| \frac{\partial u}{\partial x_j} \right\|_{L^2}. \]

In fact, \( u \in D^m_{L^2} \subset D^{2+3}_{L^2} \) implies that \( u \) is in \( D^1 \). We can hence write

\[
u(x, t) = \int_{-\infty}^{x_j} \frac{\partial u}{\partial x_j}(y, t) \, dy, \quad j = 1, \ldots, n.
\]

Using Cauchy-Schwarz inequality and calculating the norm of \( u \) in \( L^2 \) we obtain (3.3). The estimates (3.2), (3.3) together show that an a priori estimate of order one holds thus proving (i).

(ii) Differentiating (3.1) with respect to \( x_j \) we have

(3.4) \[
\Box u_j + \frac{df}{du} u_j = 0 \quad \text{where} \quad u_j = \frac{\partial u}{\partial x_j}.
\]

Denoting \( \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} u \) by \( u_{jk} \) and \( \frac{\partial}{\partial x_j} \frac{\partial}{\partial t} u \) by \( u_{jt} \) we define

\[ E_2(t) = \sum_{j=1}^n \int \frac{1}{2} \left( u_{jt}^2 + \sum_{k=1}^n u_{jk}^2 \right) \, dx. \]

Differentiating \( E_2(t) \) with respect to \( t \)

\[
\frac{d}{dt} E_2(t) = \sum_j \int \left( u_{jt} \cdot u_{jt} + \sum_k u_{jk} \cdot u_{jkt} \right) \, dx
\]

\[ = \sum_j \int (\Box u_j) \cdot u_{jt} \, dx. \]
3. An example of a semi-linear equation

since \( \sum_k \int u_{jk} \cdot u_{jk} \, dx = - \sum_k \int u_{jkk} \cdot u_{jt} \, dx \) by integration by parts. using the equation (3.4) we obtain

\[
\frac{dE_2}{dt}(t) = - \sum_j \int \frac{df}{du} \cdot u_j u_{jt} \, dx.
\]

From the generalized Hölder’s inequality it follows that

\[
| \int \frac{df}{du} \cdot u_j u_{jt} \, dx | \leq ||u_j||_{L^2} ||u_{jt}||_{L^6} \cdot ||\frac{df}{du}||_{L^3}.
\]

If \( n = 2 \) by Prop. 2 we see that

\[
||u_j||_{L^6} \leq c_1(n) ||u||_{L^2}
\]

where \( c_1(n) \) is a constant depending only on \( n \). From (b) we have, with a suitable constant \( \alpha' \) depending on \( \alpha \), since \( u \) has compact support in \( |x| < R \)

\[
\int_{|x|<R} \frac{df}{du}^3 \, dx \leq \alpha'^3 \int (u^6 + 1) \, dx \leq \alpha'^3 ||u||_{L^6}^6 + C_2(\alpha', R, n)
\]

\[
\leq C_3(n, \alpha', R)(1 + ||u||_{L^6}^6).
\]

These estimates together show that

\[
\frac{dE_2}{dt}(t) \leq \gamma_1 E_2(t).
\]

Multiplying by \( e^{-\gamma_1 t} \) and integrating with respect to \( t \) we obtain

(3.5) \( E_2(t) \leq E_2(0) \cdot e^{\gamma_1 t} \).

This proves that there is an a priori bound of order 2. A similar argument holds for the case \( n = 3 \). This completes the proof of the proposition.

**Exercise.** Consider the semi-linear hyperbolic equation

(3.6) \( M[u] + f(u) = 0 \)
where

\[ M = \left( \frac{\partial}{\partial t} \right)^2 - \sum a_{jk}(x, t) \frac{\partial^2}{\partial x_j \partial x_k} - \sum a_j(x, t) \frac{\partial}{\partial x_j} - a_0(x, t) \frac{\partial}{\partial t} \]

with (1°) \( a_{jk} \in B^1[0, T] \), \( \frac{\partial}{\partial t} a_{jk} \in B^0[0, T] \), \( a_0 a_j \in B^0[0, T] \),

(2°) \( \sum a_{jk}(x, t) \xi_j \xi_k \geq \delta |\xi|^2 \), \( \delta > 0 \) is a constant.

Prove, under the same hypothesis on \( f \) as in Prop. 1, that an a priori estimate of order 2 holds and consequently there exists a global solution of (3.6).

4 Existence theorems for first order systems of semi-linear equations

In this section we establish theorems on the existence of local and global solutions of the Cauchy problem for semi-linear regularly hyperbolic first order systems of differential equations.

Let \( \Omega \) be the set \( \{(x, t) | x \in \mathbb{R}^n, 0 \leq t \leq T\} \). Consider the semi-linear first order system of equations

\[ (4.1) \quad M[u] = \frac{\partial u}{\partial t} - \sum_{k=1}^{n} A_k(x, t) \frac{\partial u}{\partial x_k} = f(x, t, u), \]

where we assume that the coefficients \( A_k \) of \( M \) and \( f \) satisfy the following regularity conditions:

(a) \( A_k \in B^{[\frac{3}{2}] + 2}[0, T] \), \( \frac{\partial A_k}{\partial t} \in B^0[0, T] \) and

(b) \( f \in E^{[\frac{3}{2}] + 3} \) in \( \Omega \times \mathbb{C} \).

We also assume that \( M \) is regularly hyperbolic. As we shall show later that under stronger differentiability conditions on the coefficients \( A_k \) and \( f \) the Cauchy problem has more regular solutions: For instance we assume
(a') $A_k \in B^n[0, T], \frac{\partial A_k}{\partial t} \in B^0[0, T]$ and

(b') $f \in \mathcal{D}^{m+1}$ in $\Omega \times \overline{C}$.

where $m \geq \left\lfloor \frac{n}{2} \right\rfloor + 2.$

Although we are interested here mainly in the local existence theorem we consider the following equation (4.1)′ instead of (4.1) in order to elucidate our construction. We decompose $f$ into two parts

$$f(x, t, u) = f(x, t, 0) + (f(x, t, u) - f(x, t, 0)) = f(x, t, 0) + g(x, t, u)$$

where

$$g(x, t, u) = f(x, t, u) - f(x, t, 0).$$

We remark that $g(x, t, 0) \equiv 0$. Define the function $\tilde{f} \in \mathcal{D}^{[\frac{n}{2}]+3}$ in $\Omega \times \overline{C}$ by setting

$$\tilde{f}(x, t, u) = \alpha(x)g(x, t, u) + \beta(x)f(x, t, 0)$$

where $\alpha, \beta \in \mathcal{D}$, and consider the first order system of semi-linear equations

$$(4.1)′ \quad M[u] = \tilde{f}.$$ 

Clearly $\tilde{f} = f$ wherever $\alpha(x) = 1 = \beta(x)$. If the initial data $u_0 \in \mathcal{D}'$ has compact support then, since $\beta(x)\tilde{f}(x, t, u)$ has compact support in the $x$-space, the solution $u$ also has a fixed compact support for all $0 \leq t \leq T$.

Now we find a sequence of functions $\{u_j\}$ which will converge to a limits $u$ giving the solution. Let $\psi$ be the solution of Cauchy problem

$$(4.3) \quad M[\psi] = \beta(x)f(x, t, 0) \text{ with } \psi(0) = u_0.$$ 

Hence by the theory of linear equations, there exists a constant $\gamma_0$ depending on $T$ such that

$$\|\psi(t)\|_{\mathcal{D}} + 2 \leq \gamma_0\|u_0\|_{\mathcal{D}} + \sup_{0 \leq t \leq T} \|\beta f(x, t, 0)\|_{\mathcal{D}} + 2.$$
\[ \|\psi(t)\|_{[\bar{\mathcal{U}}]}^{+1} \leq \gamma_0 \|u_0\|_{[\bar{\mathcal{U}}]}^{+1} + \sup_{0 \leq t \leq T} \|\beta f(x, t, 0)\|_{[\bar{\mathcal{U}}]}^{+1}. \]

The Cauchy problem for (4.1)' is therefore reduced to the following problem: to find a solution \( u \in \mathcal{D}_{L^2}^{[\bar{\mathcal{U}}]}[0, T] \) of
\[
M[u] = \tilde{g}(x, t, \psi + u)
\]
with the initial data \( u_0 \). Here
\[
\tilde{g}(x, t, \psi + u) = \alpha(x)(f(x, t, u + \psi) - f(x, t, 0)).
\]

Our main interest here is to determine how does the domain of existence \( \mathbb{R}^n \times [0 \leq t \leq h] \) of the solution depend on the initial data \( u_0 \), after fixing \( \alpha, \beta \in \mathcal{D} \). The functions \( u_j \) are defined inductively as solutions of the Cauchy problem for the first order system of equations:
\[
M[u_1] = \tilde{g}(x, t, \psi), \quad u_1(0) = 0,
\]
\[
M[u_2] = \tilde{g}(x, t, u_1 + \psi), \quad u_2(0) = 0,
\]
\[
\ldots
\]
\[
M[u_j] = \tilde{g}(x, t, u_{j-1} + \psi), \quad u_j(0) = 0,
\]
\[
\ldots
\]

Now since \( \psi \in \mathcal{D}_{L^2}^{[\bar{\mathcal{U}}]}[0, T] \) we have \( \tilde{g}(x, t, \psi(t)) \in \mathcal{D}_{L^2}^{[\bar{\mathcal{U}}]}[0, T] \) and hence by the theory of linear equations there exists a solution \( u_1 \) of the Cauchy problem
\[
M[u_1] = \tilde{g}(x, t, \psi), \quad u_1(0) = 0,
\]
and \( u_1 \in \mathcal{D}_{L^2}^{[\bar{\mathcal{U}}]}[0, T] \). Again we have \( \tilde{g}(x, t, (\psi + u_1)(x, t)) \in \mathcal{D}_{L^2}^{[\bar{\mathcal{U}}]}[0, T] \) and hence there exists a solution \( u_2 \) of
\[
M[u_2] = \tilde{g}(x, t, u_1 + \psi), \quad u_2(0) = 0
\]
and \( u_2 \in \mathcal{D}_{L^2}^{[\bar{\mathcal{U}}]}[0, T] \). This procedure can be used to obtain \( u_j \) inductively.

Now we have the
Proposition 1. There exists a positive, non-increasing functions \( \varphi(\xi) \) of \( \xi > 0 \) such that

\[
h(\varphi(\|u_0\|_{1+1})) > 0
\]

and the set \( \left\{ \sup_{0 \leq t \leq h} \|u_j(t)\|_{1+1} \right\} \) is bounded.

Proof. Let \( \gamma \) denote the \( \sup_{(x,t) \in \Omega} |\psi(x,t)| \). In view of (4.4) \( \gamma \) is less than or equal to \( c_0 + c_1 \|u_0\|_{1+1} \) where \( c_0, c_1 \) are constants depending on \( T \). If \( b \) is a positive number let \( F \) be the set

\[
F = \{ (x,t,u) | (x,t) \in \Omega, |u| < b + \gamma \}
\]

and put

(4.5) \[
M = \sup_{F,|\alpha| \leq [n/2] + 2} |(\frac{\partial}{\partial x}, \frac{\partial}{\partial u})^{\alpha} g(x,t,u)|
\]

where \( (\frac{\partial}{\partial x}, \frac{\partial}{\partial u})^{\alpha} \) denotes a derivation of order \( |\alpha| \) with respect to \( x \) and \( u \). \( M = M(b + \gamma) \) is an increasing function of the parameter. If \( u \in \mathcal{D}_{1+1} \{0,T\} \) with \( |u(x,t)| \leq b \) for \( (x,t) \in \Omega \) then we have

(4.6) \[
\|g(x,t,(u+\psi)(x,t))\|_{1+1} \leq Mc[1 + \|u(t)\|_{1+1}],
\]

where \( k = \left[ \frac{n}{2} \right] + 1 \). Now, since \( u_j(0) = 0 \), we have by the energy inequality

\[
\|u_j(t)\|_{1+1} \leq c(T) \int_0^T \|g(x,s,(u_j-1+\psi(x,s)))\|_{1+1} ds.
\]

Hence from (4.6) we obtain

(4.7) \[
\|u_j(t)\|_{1+1} \leq Mcc(T) \int_0^T (1 + \|u_{j-1}(s)\|_{1+1}^k) ds.
\]
5.

We recall that this was derived with the assumption that $|u_{j-1}(x, t)| < b$ which, we shall show, holds when $h$ is small and $0 \leq t \leq h$. Put

$$c_2 = Mc \cdot c(T)$$

(4.8)

$$\gamma_1 = 1 + 2^k \sup_{0 \leq t \leq T} \|\psi(t)\|_{\frac{k}{2} + 1}.$$  

Since $\|\psi + u_{j-1}(t)\|_{\frac{k}{2} + 1} < b$ (4.7) can be written as

$$\|u_j(t)\|_{\frac{k}{2} + 1} \leq 2^k \int_0^t \{\gamma_1 + \|u_{j-1}(s)\|_{\frac{k}{2} + 1}\} ds,$$

where $u_0(t) \equiv 0$. Putting again $2^k c_2 = c_3$ we have

(4.9)  

$$\|u_j(t)\|_{\frac{k}{2} + 1} \leq c_3 \int_0^t \{\gamma_1 + \|u_{j-1}(s)\|_{\frac{k}{2} + 1}\} ds.$$

Let $c_s(n)$ denote the Sobolev’s constant, namely the constant in the inequality

$$\sup |\varphi(x)| \leq c_s(n)\|\varphi\|_{\frac{k}{2} + 1}.$$  

Define $b'$ by

(4.10)  

$$b' = \frac{b}{c_s(n)}$$

and denote $c_3(\gamma_1 + b'^k)$ by $\tilde{M}$. Take

(4.11)  

$$h = \frac{b'}{\tilde{M}} = \frac{b'}{c_3(\gamma_1 + b'^k)}.$$  

Consider the sequence $y_j(t)$ defined by the sequence of integral equations

$$y_j(t) = c_3 \int_0^t \{\gamma_1 + y_{j-1}(s)^k\} ds \quad \text{for} \quad t \geq 0, \quad y_0(t) \equiv 0.$$
4. Existence theorems for first order systems of.....

Then we assert that

\[ 0 \leq y_j(t) \leq b' \text{ for } 0 \leq t \leq h, \ j = 1, 2, \ldots \]

In fact, \( y_1(t) \leq c_3 \gamma_1 t \leq \tilde{M} t \leq \tilde{M} h = b' \),
\[ y_2(t) \leq \tilde{M} t \leq \tilde{M} h = b' \text{ and so on.} \]

Evidently \( \| u_j(t) \|_{[\frac{1}{2}]} + 1 \leq y_j(t) \) and

\[ (4.12) \quad \| u_j(t) \|_{[\frac{1}{2}]} + 1 \leq b' \text{ for } 0 \leq t \leq h \]

which, a fortiori, implies (by using Sobolev’s lemma) that

\[ \sup |u_j(x, t)| \leq b' c_s(n) = b \text{ (see (4.10)).} \]

From (4.11) we obtain

\[ \frac{1}{h} = \frac{c_3 (\gamma_1 + b^k)}{b'} = 2^k c \cdot c(T) \frac{b^k + \gamma_1 M}{b'} \]
\[ \leq c_0 (n, T) \frac{b^k + C'_0(n) + C''_0(n) \| \psi(t) \|_{[\frac{1}{2}]} + 1 M}{b}, \]

where \( M = M(\gamma + b) \). \( M(\xi) > 0 \) is an increasing function of \( \xi > 0 \).
So, if \( \| u_0 \|_{[\frac{1}{2}]} + 1 \) runs through a bounded set, fixing \( b, h \) has a positive infimum (\( M \) is taken to be a fixed positive number). This completes the proof. \( \square \)

**Remark.** Instead of taking the initial data to be given at \( t = 0 \) we can take the initial data to be given at an arbitrary \( t_0 (0 \leq t_0 \leq T) \). We define \( \psi(t, t_0) \) corresponding to \( \psi(t) \) in the above arguments. Here \( \| \psi(t, t) \|_{[\frac{1}{2}]} + 1 \) is majorized by \( C_0 + C_1 \| u_0 \|_{[\frac{1}{2}]} + 1 \), \( C_0, C_1 \) can be taken independently. The expression for \( \frac{1}{h} \) shows that \( h \) has a positive infimum independent of \( t_0 \) if the initial data \( u_0 \) runs through a bounded set in \( J_{L^2} \).
Next we prove that the sequence \( \{u_j(t)\} \) is a Cauchy sequence in \( \mathcal{D}_{L^2}^{\frac{n}{2} + 2}[0, h] \). First of all we shall show that \( \{ \sup_{0 \leq t \leq h} ||u_j(t)||_{\mathcal{D}_{L^2}^{\frac{n}{2} + 2}} \) is bounded. In fact, we have

\[
||u_j(t)||_{\mathcal{D}_{L^2}^{\frac{n}{2} + 2}} \leq c(T) \int_0^t ||\tilde{g}(x, s(u_j - 1 + \psi)(x, s))||_{\mathcal{D}_{L^2}^{\frac{n}{2} + 2}} ds
\]

\[
\leq cM' \int_0^t \left\{ 1 + (1 + ||(\psi + u_j - 1)(s)||_{\mathcal{D}_{L^2}^{\frac{n}{2} + 2}}) ||u_j - 1 + \psi(s)||_{\mathcal{D}_{L^2}^{\frac{n}{2} + 2}} \right\} ds,
\]

\[k = \left\lfloor \frac{n}{2} \right\rfloor + 1.
\]

\[
||u_2(t) - u_1(t)||_{\mathcal{D}_{L^2}^{\frac{n}{2} + 2}} \leq Kc't,
\]

\[
||u_3(t) - u_2(t)||_{\mathcal{D}_{L^2}^{\frac{n}{2} + 2}} \leq K\frac{(c')^2}{2!}, \ldots,
\]

\[
||u_{j+1} - u_j(t)||_{\mathcal{D}_{L^2}^{\frac{n}{2} + 2}} \leq K\frac{(c')^j}{j!}, \ldots
\]

Hence \( \{u_j(t)\} \) is a Cauchy sequence in \( \mathcal{D}_{L^2}^{\frac{n}{2} + 2}[0, h] \) and therefore converges to a limit \( u(t) \) in \( \mathcal{D}_{L^2}^{\frac{n}{2} + 2}[0, h] \).

If \( m \geq \left\lceil \frac{n}{2} \right\rceil + 3 \) we now assume that \( A_k \in \mathcal{B}^m[0, T], \frac{\partial A_k}{\partial t} \in \mathcal{B}^0[0, T] \) and \( f \in \mathcal{G}^{m+1}(\Omega \times \mathbb{C}) \). Let \( u_0 \in \mathcal{D}_L^m \) be given. Then the limit \( u(t) \) in \( \mathcal{D}_{L^2}^{\frac{n}{2} + 2}[0, h] \) of the sequence \( \{u_j(t)\} \) obtained above itself belongs to \( \mathcal{D}_L^m[0, h] \). In fact, it is enough to prove that \( \left\{ \sup_{0 \leq t \leq h} ||u_j(t)||_{\mathcal{D}_L^m} \right\} \) is bounded and \( \{u_j(t)\} \) is a Cauchy sequence in \( \mathcal{D}_L^m[0, h] \). For this we have only to use the following lemma which results by arguments similar to those used in \( \S 2 \)

**Lemma 1.** Let \( u \in \mathcal{D}_L^m[0, T] \) and \( f \in \mathcal{G}^{m+1}(\Omega \times \mathbb{C}) \) for an \( m \geq \left\lceil \frac{n}{2} \right\rceil + 2 \).
4. Existence theorems for first order systems of....  

Then there exists constants $C_m$, $M_m$ such that

$$\left\| \frac{\partial}{\partial x}^{\nu} f(x, t, u(x, t)) \right\|_{m} C_m M_m \left[ 1 + \| u(t) \|_{m}^{m-1} \right] \| u(t) \|_m.$$  

Thus we have proved the following:

**Theorem 1** (local existence theorem). Given any initial data $u_0 \in \mathcal{D}^m_{L^2}$, $m \geq \left\lfloor \frac{n}{2} \right\rfloor + 2$ and any initial time $t_0$, $0 \leq t_0 \leq T$ there exists a unique solution $u(t) \in \mathcal{D}^m_{L^2}[t_0, t_0 + h]$ of the equation

$$(4.1)' \quad M[u] = \tilde{f}(x, t, u) = \beta(x) f(x, t, 0) + \alpha(x) \{ f(x, t, u) - f(x, t, 0) \}$$

with $u(t_0) = u_0$. Moreover $h$ can be chosen to be independent of $t_0$ in $[0, T]$ when $\| u_0 \|_{\frac{n}{2} + 2}$ runs through a bounded set.

Now we obtain a global existence theorem for solutions of the Cauchy problem for regularly hyperbolic first order systems of semi-linear equations. For this we assume that an a priori estimate of the following type holds.

If $\beta \in \mathcal{D}$ consider the regularly hyperbolic first order system of equations

$$(4.13) \quad M[u] = \beta f(x, t, 0) + (f(x, t, u) - f(x, t, 0)).$$

By A priori estimate we mean the following: For any initial data $u_0$ in $\mathcal{D}^{\frac{n}{2} + 2} \cap \mathcal{D}'$ and any $t_0(0 \leq t_0 \leq T)$ the solution $u(t) \in \mathcal{D}^{\frac{n}{2} + 2}[t_0, T]$ of (4.13) satisfies the following condition: there exists a constant $c = c(T)$ such that

$$(4.14) \quad \| u(t) \|_{\frac{n}{2} + 1} \leq c \text{ for all } t_0 \leq t \leq T.$$  

**Theorem 2** (global existence theorem). Suppose an a priori estimate of the type (4.14) holds for solutions of (4.13). Then, given any initial data $u_0 \in \mathcal{E}^m_{L^2(\Omega)}$, $m \geq \left\lfloor \frac{n}{2} \right\rfloor + 2$ there exists a unique solution $u(t)$ of

$$(4.1) \quad M[u] = f \text{ with } f \in \mathcal{E}^{m+1}(\Omega \times C)$$

for $0 \leq t \leq T$ such that $u(0) = u_0$, $u \in \mathcal{E}^m_{L^2(\Omega)}[0, T]$ and $\frac{\partial u}{\partial t} \in \mathcal{E}^{m-1}_{L^2(\Omega)}[0, T]$. 

Proof. As we have seen in the section on dependence domain there exists a retrograde cone \( K \) such that the value of a solution \( u \) of \( M[u] = f \) at a point \((x_0, t_0) \in \Omega\) depends only on the second member in the set \((x_0, t_0) + K\) and on the value of the initial data in the intersection of this translated cone with \((t = 0)\). Let \( D \) be the subset of \( \Omega \) swept by \((x, T) + K\) as \( x \) runs through a ball \(|x| < R\) and \( D_0 \) be the set \( D \cap \{t = 0\} \). Let \( \beta \in \mathcal{D} \) such that \( \beta(x) \equiv 1 \) for \( x \in D_0 \). Given any initial data \( u_0 \in \mathcal{D}^m_{L^2(\text{loc})} \) we consider the Cauchy problem

\[
M[u_1] = \beta(x)f(x, t, 0) + (f(x, t, u) - f(x, t, 0))
\]

(4.15) with \( u_1(x, 0) = \beta(x)u_0(x) \in \mathcal{D}^m_{L^2} \).

This solution \( u_1(x, t) \) has an a priori estimate \( \|u_1(t)\|_{L^\infty} \leq C \). On the other hand this solution \( u_1 \) has compact support as far as the solution exists. Hence, if we take \( \alpha \in \mathcal{D} \) such that \( \alpha(x) \equiv 1 \) for \(|x| \leq R\), (4.15) is equivalent to

\[
(4.1)' \quad M[u_1] = \beta(x)f(x, t, 0) + \alpha(x)(f(x, t, u) - f(x, t, 0)).
\]

Now since \( u_1 \) has an a priori estimate \( \|u_1(t)\|_{L^\infty} \leq C \), it follows, by using theorem \( \text{I} \) to continue the solution step by step, that there exists a solution \( u_1(x, t) \) for \( 0 \leq t \leq T \). Clearly \( u(x, t) = u_1(x, t) \) for \((x, t) \in D\) and this completes the proof of theorem \( \text{II} \).

\[\square\]

5 Existence theorems for a single semi-linear equation of higher order

In this section we obtain theorems on existence of solutions, local and global, of the Cauchy problem for a single semi-linear equation of order \( m \).

As before \( \Omega \) be the set \([x, t]|x \in \mathbb{R}^n, 0 \leq t \leq T\) and

\[
M = \left( \frac{\partial}{\partial t} \right)^m + \sum_{j+|\nu| \leq m_{j < m}} a_{j,\nu}(x, t) \left( \frac{\partial}{\partial t} \right)^j \left( \frac{\partial}{\partial x} \right)^\nu
\]

(5.1)
be a regularly hyperbolic operator in $\Omega$. Consider the quasi-linear equation

\begin{equation}
M[u] = f\left(x, t, \left(\frac{\partial}{\partial t}\right)^{j_1} \left(\frac{\partial}{\partial x}\right)^{\alpha_1} u, \ldots, \left(\frac{\partial}{\partial t}\right)^{j_s} \left(\frac{\partial}{\partial t}\right)^{\alpha_s} u\right)
\end{equation}

where $j_k + |\alpha_k| \leq m - 1 (k = 1, \ldots, s)$. We make the following assumptions on the coefficients of $M$ and $f$:

\[a_{j,\nu} \in B^{n_2+1}[0, T], \frac{\partial}{\partial t} a_{j,\nu} \in B^0[0, T]\text{ and } f \in C^N(\Omega \times C^s).\]

When we consider the regularity properties of higher degrees. We assume for $N \geq \left[\frac{n}{2}\right] + 3$

\[a_{j,\nu} \in B^N[0, T], \frac{\partial}{\partial t} a_{j,\nu} \in B^0[0, T]\text{ and } f \in C^{N+1}(\Omega \times C^s).\]

The reasoning used in the case of the first order system (see §4) can be applied to this case without any significant change. We will indicate the necessary modifications very briefly.

The space of all functions $u$ such that

\[u \in D^{k}L^2[0, T], \frac{\partial}{\partial t} u \in D^{k+1}L^2[0, T], \ldots, \left(\frac{\partial}{\partial t}\right)^m u \in D^kL^2[0, T]\]

is denoted by $D^kL^2[0, T]$. We introduce a topology on $D^kL^2[0, T]$ by a norm $\|u(t)\|_k$ defined by

\begin{equation}
\|u\|_k^2 = \|u(t)\|_{k+m-1}^2 + \cdots + \left(\frac{\partial}{\partial t}\right)^{m-1} u(t)\|_{k}^2.
\end{equation}

Now we recall the result in the linear case. Given the equation

\begin{equation}
M[u] = f
\end{equation}

with $f \in D^{\frac{n}{2}+1}L^2[0, T]$ (resp. $f \in D^{\frac{n}{2}+2}L^2[0, T]$) and the initial data $u(0) \in D^{\frac{n}{2}}L^2[0, T]$ (resp. $u(0) \in D^{\frac{n}{2}}L^2[0, T]$) the solution $u(t)$ of the Cauchy problem
Proof. We write (5.5)

\[
\|u(t)\|_{\mathcal{A}^{1}} \leq C(T) \left\{ \|u(0)\|_{\mathcal{A}^{1}} + \int_{0}^{t} \|f(s)\|_{\mathcal{A}^{1}} ds \right\} 
\]

(resp. \(\|u(t)\|_{\mathcal{A}^{2}} \leq C(T) \left\{ \|u(0)\|_{\mathcal{A}^{2}} + \int_{0}^{t} \|f(s)\|_{\mathcal{A}^{2}} ds \right\} \))

for \(0 \leq t \leq T\).

In the semi-linear case we use the following

Lemma 1. If \(u(t) \in \mathcal{G}^{[\mathcal{A}^{1}]}_{L^{2}}[0,T]\) then for any \(\alpha \in \mathcal{D}\) the function

\[
\tilde{f} = \alpha f
\]

satisfies

\[
\tilde{f} \left( x, t, \frac{\partial}{\partial t} \right)^{j_{1}} \left( \frac{\partial}{\partial x} \right)^{\alpha_{1}} u(x, t), \ldots, \left( \frac{\partial}{\partial t} \right)^{j_{s}} \left( \frac{\partial}{\partial x} \right)^{\alpha_{s}} u(x, t) \right) \in \mathcal{G}^{[\mathcal{A}]}[0,T]
\]

and

\[
\|\tilde{f}(x, t, \left( \frac{\partial}{\partial t} \right)^{j_{1}} \left( \frac{\partial}{\partial x} \right)^{\alpha_{1}} u(x, t), \ldots, \left( \frac{\partial}{\partial t} \right)^{j_{s}} \left( \frac{\partial}{\partial x} \right)^{\alpha_{s}} u(x, t))\|_{\mathcal{A}^{1}} \leq C \left[ 1 + \|u(t)\|_{\mathcal{A}^{1}} \right] \cdot \]

(5.5)

Proof. We write \(v_{k}(t)\) for \(\left( \frac{\partial}{\partial t} \right)^{j_{1}} \left( \frac{\partial}{\partial x} \right)^{\alpha_{1}} u(x, t)\) and \(\tilde{f}(x, t, v_{1}(t), \ldots, v_{s}(t))\) for \(\tilde{f} \left( x, t, \left( \frac{\partial}{\partial t} \right)^{j_{1}} \left( \frac{\partial}{\partial x} \right)^{\alpha_{1}} u(x, t), \ldots \right) \). Now we see that \(\|v_{k}(t)\|_{\mathcal{A}^{1}} \leq c\|u(t)\|_{\mathcal{A}^{1}} (k = 1, \ldots, s)\). In fact,

\[
\|v_{k}(t)\|_{\mathcal{A}^{1}} = \| \left( \frac{\partial}{\partial t} \right)^{j_{k}} \left( \frac{\partial}{\partial x} \right)^{\alpha_{k}} u(t)\|_{\mathcal{A}^{1}} \leq c\| \left( \frac{\partial}{\partial t} \right)^{j_{k}} u(t)\|_{\mathcal{A}^{1} + |\alpha_{k}| + 1}.
\]

Since \(j_{k} + |\alpha_{k}| \leq m - 1\) we have \(\left\lfloor \frac{n}{2} \right\rfloor + |\alpha_{k}| + 1 \leq \left\lfloor \frac{n}{2} \right\rfloor + 1 + (m - 1 - j_{k})\) and hence

\[
\|v_{k}(t)\|_{\mathcal{A}^{1}} \leq c\| \left( \frac{\partial}{\partial t} \right)^{j_{k}} \|_{\mathcal{A}^{1} + (m - 1 - j_{k})} \leq c\|u(t)\|_{\mathcal{A}^{1}}.
\]
The assertion follows from this by an application of Cor. 2 after lemma 1 of §2.

The following lemma is proved on the same lines and we omit the proof.

Lemma 2. If \( u \in \mathcal{D}[\frac{n}{2} + 1 + N] \) for an integer \( N \geq 1 \) then for any \( \alpha \in \mathcal{D} \)

\[
\tilde{f} \left( x, t, \left( \frac{\partial}{\partial t} \right)^{j_1} \left( \frac{\partial}{\partial x} \right)^{\alpha_1} u(x, t), \ldots, \left( \frac{\partial}{\partial t} \right)^{j_s} \left( \frac{\partial}{\partial x} \right)^{\alpha_s} u(x, t) \right) \in \mathcal{D}[\frac{n}{2} + 1 + N][0, T]
\]

and

\[
\|\tilde{f} \left( x, t, \left( \frac{\partial}{\partial t} \right)^{j_1} \left( \frac{\partial}{\partial x} \right)^{\alpha_1} u(x, t), \ldots, \left( \frac{\partial}{\partial t} \right)^{j_s} \left( \frac{\partial}{\partial x} \right)^{\alpha_s} u(x, t) \right) \|_{[\frac{n}{2}] + N + 1}
\]

\[
\leq cM_n \left( 1 + \left( 1 + \|u(t)\|_{[\frac{n}{2}] + N} \right)^{\frac{1}{2}} \|u(t)\|_{[\frac{n}{2}] + N + 1} \right)
\]

As in the local existence theorem for the first order systems we define

\[
\tilde{f}(x, t, v_1, \ldots, v_s) = \beta(x)f(x, t, 0, \ldots, 0)
\]

\[
+ \alpha(x)[f(x, t, v_1, \ldots, v_s) - f(x, t, 0, \ldots, 0)]
\]

where \( \alpha, \beta \in \mathcal{D} \). Then the same arguments as in the first order systems prove the following

Theorem 1 (local existence theorem). For fixed \( \alpha, \beta \in \mathcal{D} \) and \( T \) let

\[
M[u] = f \left( x, t, \left( \frac{\partial}{\partial t} \right)^{j_1} \left( \frac{\partial}{\partial x} \right)^{\alpha_1} u(x, t), \ldots \right)
\]

be a semi-linear regularly hyperbolic equation of order \( m \). Given any initial data \( u(0) \in \mathcal{D}_L, N \geq \left\lceil \frac{n}{2} \right\rceil + 2 \) (more precisely, given

\[
(u_0, u_1, \ldots, u_{m-1})
\]

with \( u_j \in \mathcal{D}_L^j \) and the initial time \( t_0(0 \leq t_0 \leq T) \) there exists a unique solution \( u(x, t) = u(t) \) for \( t_0 \leq t \leq t_0 + h \) of (5.7) such that \( u \in \mathcal{D}[\frac{n}{2} + 1 + N] \).
\[ \tilde{D}_L^2[t_0, t_0 + h], \quad \frac{\partial u}{\partial t} \in \tilde{D}_L^{-1}[t_0, t_0 + h] \] taking the initial value \( u(0) \) at \( t = t_0 \).

\( h \) can be taken to be a fixed number independent of \( t_0 \) when \( |||u(0)|||_{[\frac{1}{2}]+1} \) is a bounded set. More precisely, there exists a non-increasing function \( \alpha(\xi) > 0 \) of \( \xi > 0 \) such that

\[ h = \varphi \left( |||u(0)|||_{[\frac{1}{2}]+1} \right). \]

Now we state a global existence theorem for a single semi-linear regularly hyperbolic equation of order \( m \). We assume an a priori estimate of the following type holds:

\begin{align*}
M[u] = & \beta f(x, t, 0, \ldots, 0) + \alpha(f(x, t, v_1, \ldots, v_s)) - f(x, t, 0, \ldots, 0) \\
& (where \( v_k = \left( \frac{\partial}{\partial t} \right)^j \left( \frac{\partial}{\partial x} \right)^{\alpha_k} u \) satisfies

\begin{align*}
||u(t)||_{[\frac{1}{2}]+m} + ||& \frac{\partial u(t)}{\partial t}||_{[\frac{1}{2}]+m-1} + \cdots + || \left( \frac{\partial}{\partial t} \right)^{m-1} u(t) ||_{[\frac{1}{2}]+1} \leq c.
\end{align*}

**Theorem 2** (global existence theorem). under the assumption that there exists an a priori estimate of the above type, given any initial data \( (u_0, u_1, \ldots, u_{m-1}) \) with \( u_k \in \mathcal{C}_N^{N+k-1}, N \geq \left[ \frac{n}{2} \right] + 2 \), there exists a unique solution \( u(t) = u(x, t) \) for \( 0 \leq t \leq T \) of (5.2) such that

\[ u \in \mathcal{C}_L^{N+m-1}[0, T], \quad \frac{\partial u}{\partial t} \in \mathcal{C}_L^{N+m-2}[0, T], \ldots, \left( \frac{\partial u}{\partial t} \right)^{m} u \in \mathcal{C}_L^{N-1}[0, T]. \]

**Remark 1.** As a particular case of the Theorem we have Theorem 1 of §3.

**Remark 2.** We assumed an a priori estimate (5.9) for the theorem of existence of global solutions. If in \( f(x, t, v_1, \ldots, v_s) \) the orders \( j_k + |\alpha_k| \) are less than \( (m - 1) \) the following remark will be useful. If we have an estimate of derivatives of \( u \) of the form

\[ || \left( \frac{\partial}{\partial t} \right)^j \left( \frac{\partial}{\partial x} \right)^{\alpha_k} u(t) ||_{[\frac{1}{2}]+1} \leq c (k = 1, \ldots, s) \]
then we have an a priori estimate of the form \(5.9\). In fact, first of all we have, if \(g(x,t,v_1,\ldots,v_s)\) denotes \(f(x,t,v_1,\ldots,v_s) - f(x,t,0,\ldots,0)\) then for any \(\alpha \in \mathcal{D}\) the function \(\tilde{g} = \alpha g\) satisfies the inequality

\[
\left\| \tilde{g} \left( x,t,\left( \frac{\partial}{\partial t} \right)^j u(x,t),\ldots,\left( \frac{\partial}{\partial t} \right)^s u(x,t) \right) \right\|_{[\frac{s}{2}] + 1} \leq c'
\]

with a constant \(c'\). Now as in the case of first order systems this inequality, together with the energy inequality in the linear case, implies \(5.9\).

We illustrate this by the following simple example. Take for \(M\) the operator \(\Box = \frac{\partial^2}{\partial t^2} \Delta\) and consider the semi-linear equation

\[
\Box u + f(u) = 0.
\]

We assume \(f(0) = 0\). We show that it is enough to obtain an estimate of \(\|u(t)\|_{[\frac{s}{2}] + 1}\) in order to get an a priori estimate of \(\|u(t)\|_{[\frac{s}{2}] + 2} + \|\frac{\partial u(t)}{\partial t}\|_{[\frac{s}{2}] + 1}\). First we observe that the condition \(f(0) = 0\) can be removed. In fact, if \(C_0 = f(0)\) we consider the equation

\[
\Box u + (f(u) - f(0))' + \beta(x)f(0)f(0) = 0;
\]

that is,

\[
\Box u + C_0 \beta(x) + (f(u) - C_0) = 0,
\]

where \(\beta \in \mathcal{D}\).

It is enough to obtain an a priori estimate for solutions of this equation. If \(u_0, u_1, \in \mathcal{E}\) then we know that for \(0 \leq t \leq T\) the solution \(u(t)\) has its support contained in some compact set: say in \(|x| < R\).

Define

\[
E_1(t) = \int_{|x|<R} \left[ \frac{1}{2} \left( \frac{\partial u}{\partial t} \right)^2 + \sum_j \left( \frac{\partial u}{\partial x_j} \right)^2 \right] + F(u) - c_0 u + \gamma (u^2 + 1) \right] dx
\]

where \(F(u) = \int_0^u f(\tau) d\tau\) and \(\gamma\) is chosen so large that \(F(u) - c_0 u + \gamma (u^2 + 1) \geq 0\) for any \(u\). This is always possible if we assume \(F(u) > -L\).
Differentiating $E_1(t)$ with respect to $t$ and using integration by parts we have

$$\frac{d}{dt} E_1(t) = \int_{|x|<R} \left\{ \frac{\partial u}{\partial t} \cdot \Box u + (f(u) - c_0) \frac{\partial u}{\partial t} + 2\gamma u \frac{\partial u}{\partial t} \right\} dx$$

$$= \int 2\gamma u \frac{\partial u}{\partial t} - c_0 \beta(x) \frac{\partial u}{\partial t} dx$$

since $\Box u + f(u) - c_0 = -\beta(x)c_0$.

$$\leq CE_1(t).$$

Hence $E_1(t) \leq e^{ct} \leq e^{cT} = c'$. This, together with the expression for $E_1(t)$, shows that we have the assertion.

By considering the equation obtained by differentiating the equation

$$\Box u + c_0 \beta(x) + (f(u) - c_0) = 0$$

with respect to $x_j$

$$\Box \frac{\partial u}{\partial x_j} + f'(u) \frac{\partial u}{\partial x_j} + c_0 \frac{\partial \beta}{\partial x_j} = 0 \quad (j = 1, 2, \ldots, n),$$

we can obtain an estimate for $E_2(t)$ in an analogous way. Thus we have the following result:

Suppose the function $f$ satisfies the conditions

1. $F(u) > -L$, 
2. $|f'(u)| < \alpha(u^2 + 1)$ for $n = 3$

$\leq$ a polynomial for $n = 2$.

For any initial data $(u_0, u_1)$ with $u_0 \in L^2_{\text{loc}}$, $u_1 \in L^{m-1}_{\text{loc}}$, $m \geq \left\lceil \frac{n}{2} \right\rceil + 3$, there exists a unique solution $u(t) = u(x, t)$ for $0 \leq t < \infty$ such that

$$u \in C^m \left[0, \infty\right), \frac{\partial u}{\partial t} \in C^{m-1} \left[0, \infty\right), \left(\frac{\partial}{\partial t}\right)^2 u \in C^{m-2} \left[0, \infty\right).$$
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