Lectures on Cauchy Problem

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Chapter 1

1 Preliminaries and function spaces

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We will be concerned with functions and differential operators defined on the n-dimensional Euclidean space $\underline{\mathrm{R}}^n$. The points of $\underline{\mathrm{R}}^n$ will be denoted by $x = (x_1, \ldots, x_n)$, $\xi = (\xi_1, \ldots, \xi_n)$, etc. and we will use the following abbreviations:

$$|x| = \left(\sum x_j^2\right)^{\frac{1}{2}}, \lambda x = (\lambda x_1, \ldots, \lambda x_n), x \cdot \xi = \sum_j x_j \xi_j;$$

S will denote the sphere $|x| = 1$, dS_x the element of surface area on S, and dx will denote the standard volume element in $\underline{\mathrm{R}}^n$. If $\nu = (\nu_1, \ldots, \nu_n)$is a multi-index of non-negative integers $|\nu| = \nu_1 + \cdots + \nu_n$ is called the (total) order of ν. We will also use the following standard notation:

$$\left(\frac{\partial}{\partial x}\right)^{\nu} = \left(\frac{\partial}{\partial x_1}\right)^{\nu_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\nu_n}, \ \xi^{\nu} = \xi_1^{\nu_1} \ldots \xi_n^{\nu_n},$$
$$a_\nu(x) = a_{\nu_1 \cdots \nu_n}(x).$$

In general $a_\nu(x)$ will be complex valued functions on $\underline{\mathrm{R}}^n$, unless otherwise mentioned. We will also have occasion to use vectors and matrices of complex valued functions. The notation will be obvious from the context.

A general linear partial differential operator can be written in the form

$$a\left(x, \frac{\partial}{\partial x}\right) = \sum_{\nu} a_\nu(x) \left(\frac{\partial}{\partial x}\right)^{\nu}. \tag{1.1}$$

The maximum m of the total orders $|\nu|$ of multi-indices occurring in
2 (1) for which $a_\nu(x) \not\equiv 0$ is called the order of the operator $a\left(x, \frac{\partial}{\partial x}\right)$. The transpose or the formal adjoint of $a\left(x, \frac{\partial}{\partial x}\right)$ is defined by

$$t_a\left(x, \frac{\partial}{\partial x}\right)[u] = \sum_{|\nu| \le m} (-1)^{|\nu|} \left(\frac{\partial}{\partial x}\right)^{\nu} [a_\nu(x) u]. \tag{1.2}$$

The adjoint of $a\left(x, \frac{\partial}{\partial x}\right)$ in L^2 is defined by

$$a^*\left(x, \frac{\partial}{\partial x}\right)[u] = \sum_{|\nu| \le m} (-1)^{|\nu|} \left(\frac{\partial}{\partial x}\right)^{\nu} [\overline{a_\nu(x)} u]. \tag{1.3}$$

In most of our considerations we will be considering systems of linear differential equations of the first order. We refer to these as first order. We refer to these as first order systems. A first order system can therefore be written in the form:

$$\left(A\left(x, \frac{\partial}{\partial x}\right) u\right)_j = \sum_{K=1}^{N} A_{jk}\left(x, \frac{\partial}{\partial x}\right) u_k, \qquad j = 1, \ldots, N, \tag{1.1$'$}$$

where $A_{jk}\left(x, \frac{\partial}{\partial x}\right) = \sum\limits_{\rho=1}^{n} a_{jk,\,\rho}(x) \frac{\partial}{\partial x_\rho} + b_{jk}(x)$ and $u = (u_1, \ldots, u_N)$. The formal adjoint of $A\left(x, \frac{\partial}{\partial x}\right)$ is defined by

$$\left({}^tA\left(x, \frac{\partial}{\partial x}\right) v\right)_j = \sum_{j} {}^tA_{jk}\left(x, \frac{\partial}{\partial x}\right) v_j, \quad k = 1, \ldots, N, \tag{1.2$'$}$$

where ${}^tA_{jk}\left(x, \frac{\partial}{\partial x}\right)u_j = \sum_{\rho=1}^{n}(-1)\frac{\partial}{\partial x_\rho}(a_{jk,\rho}(x)u_j) + b_{jk}(x)u$, and the adjoint in L^2 of $A\left(x, \frac{\partial}{\partial x}\right)$ is defined by

$$(1.3') \qquad \left(A^*\left(x, \frac{\partial}{\partial x}\right)v\right)_k = \sum_j A^*_{jk}\left(x, \frac{\partial}{\partial x}\right)v_j, \quad k = 1, \ldots, N$$

where $A^*_{jk}\left(x, \frac{\partial}{\partial x}\right)v_j = \sum_\rho(-1)(\frac{\partial}{\partial x_\rho})(\overline{a_{jk,\rho}(x)}v_j) + \overline{b_{jk}(x)}v_j$.

We shall now introduce some function spaces used in the sequel. 3
U will denote an open set in $\underline{R}^n$. $\mathscr{D}(U)$, $\mathscr{E}(U)$, $\mathscr{E}^m(U)$, $\mathscr{D}'(U)$, $\mathscr{E}'(U)$, $\mathscr{S}(\underline{R}^n)$, $\mathscr{S}'(\underline{R}^n)$ will denote the function spaces of Schwartz provided with their usual topologies. The space of m times continuously differentiable functions which are bounded together with all their derivatives up to order m in U will be denoted by $\mathscr{B}^m(U)$. $\mathscr{B}^m(U)$ is provided with the topology of convergence in $L^\infty(U)$ of all the derivatives up to order m.$\mathscr{E}^m_{L^p}(U)$ stands for the space of functions in $L^p(U)$ whose distribution derivatives up to order m are functions in $L^p(U)$. For $f \in \mathscr{E}^m_{L^p}(U)$ we define

$$\|f\|_{\mathscr{E}^m_{L^p}(U)} = \|f\|_{p,m} = (\sum_{|\nu|\le m} \|\left(\frac{\partial}{\partial x}\right)^\nu f\|^p_{L^p(U)})^{1/p}.$$

$\mathscr{E}^m_{L^p}(U)$ is a Banach space with this norm. Clearly $\mathscr{E}^m_{L^p}(U) \subset \mathscr{E}^k_{L^p}(U)$ for $k \le m$ and the inclusion mapping is continuous. The space of distributions $f \in \mathscr{D}'(U)$ which are in $\mathscr{E}^m_{Lp}(U')$ for every relatively compact subset U' of U is denoted by $\mathscr{E}^m_{L^p(loc)}(U)$. This space is topologized by the following sequence of semi-norms. If $\{U_n\}$ is a sequence of relatively compact subsets of U, covering U, we define

$$p_n(f) = \|f\|_{\mathscr{E}^m_{L^p}(U_n)} \text{ for } f \in \mathscr{E}^m_{L^p(\text{loc})}(U).$$

$\mathscr{E}^m_{L^p(\text{loc})}(U)$ is a Frechet space with this topology. This space can also be considered as the space of distributions $f \in \mathscr{D}'(U)$ such that $\alpha f \in \mathscr{E}^m_{L^p}(U)$ for every $\alpha \in \mathscr{D}(U)$. Evidently $\mathscr{E}^m_{L^p}(U) \subset \mathscr{E}^m_{L^p(\text{loc})}(U)$ with 4
continous inclusion for $m \ge 0$. The closure of $\mathscr{D}(U)$ in $\mathscr{E}^m_{L^p}(U)$ is denoted by $\mathscr{D}^m_{L^p}(U)$ and is provided with the induced topology. As before

$\mathscr{D}^m_{L^p}(U) \subset \mathscr{D}^k_{L^p}(U)$ for every $k \leq m$ with continuous inclusion. In general $\mathscr{D}^m_{L^p}(U) \neq \mathscr{E}^m_{L^p}(U)$ (for a detailed study of these spaces see Seminaire Schwartz 1954 for the case $p = 2$). However $\mathscr{D}^m_{L^p}(\underline{R}^n) = \mathscr{E}^m_{L^p}(\underline{R}^n)$.

When we consider spaces of vectors or matrices of functions we use the obvious notations, which, however will be clear from the context. For instance, if $f = (f_1, \ldots, f_N)$ where $f_j \in \mathscr{E}^m_{L^2}(U)$ then $\|f\|_{\mathscr{E}^m_{L^2}}$ stands for $\left(\sum\limits_j \|f_j\|^2_{\mathscr{E}^2_{L^2}(U)}\right)^{\frac{1}{2}}$.

When $U = \underline{R}^n$ we simply write $\mathscr{D}$, $\mathscr{E}$, $\mathscr{E}^m \mathscr{D}^m_{L^2}$ etc. for $\mathscr{D}(U), \ldots,$

We will denote the space of all continuous functions of t in an interval $[0, T]$ with values in the topological vector space $\mathscr{E}^m$ by $\mathscr{E}^m[0, T]$. It is provided with the topology of uniform convergence (uniform with respect to t in $[0, T]$) for the topology of $\mathscr{E}^m$. Similar definitions hold for $\mathscr{E}^m_{L^2}[0, T]$, $\mathscr{D}^m_{L^2}[0, T]$, $\mathscr{D}^m_{L^2(loc)}[0, T]$, $\mathscr{B}^m[0, T]$, etc.

We now recall, without proof, a few well-known results on the spaces $\mathscr{E}^m_{L^p}(U)$ and $\mathscr{E}^m_{L^p(\text{loc})}(U)$.

5 **Proposition 1** (Rellich)**.** *Every bounded set in* $\mathscr{E}^m_{L^p}(U)$ *is relatively compact in* $\mathscr{E}^{m-1}_{L^p(\text{loc})}(U)$ *for* $m \geq 1$.

In other words, the proposition asserts that the inclusion mapping of $\mathscr{E}^m_{L^p}(U)$ *into* $\mathscr{E}^{m-1}_{L^p(\text{loc})}(U)$ *is completely continuous.*

The following is a generalization due to Sobolev of a result of F. Riesz.

Proposition 2. *Let* $g \in L^p$, $h \in L^q$ *for* $p, q > 1$ *such that* $\dfrac{1}{p} + \dfrac{1}{q} > 1$. *Then the following inequality holds:*

$$\left| \iint\limits_{\underline{R}^n \times \underline{R}^n} \frac{g(x)h(y)}{|x - y|^\lambda} dx\, dy \right| \leq K \|g\|_{L^p} \cdot \|h\|_{L^p} \tag{1.4}$$

where $\lambda = n\left(2 - \dfrac{1}{p} - \dfrac{1}{q}\right)$ *and* K *is a constant depending only on* p, q, n *but not on* g *and* h.

Proposition 3 (Sobolev)**.** *If $h \in L^p$ for $p > 1$ then the function*

$$f(x) = \int \frac{h(y)}{|x-y|^\lambda} dy, \tag{1.5}$$

where $n > \lambda > \dfrac{n}{p'} = n(1 - \dfrac{1}{p})$, is in L^q where $\dfrac{1}{q} = \dfrac{1}{p} + \dfrac{\lambda}{n} - 1$.

Theorem 1 (Sobolev)**.** *Let U be an open set with smooth boundary ∂U (for instance $\partial U \in C^2$). Then any function $\varphi \in \mathscr{E}^m_{L^p}(U)$ with $pm \leq n$ itself belongs to $L^q(U)$ where q satisfies $\dfrac{1}{q} = \dfrac{1}{p} - \dfrac{m}{n}$. Further we have an estimate*

$$||\varphi||_{L^q(U)} \leq C||\varphi||_{\mathscr{E}^m_{L^p(U)}} \tag{1.6}$$

The contant C depends only on p, q, r and n but not on the function 6
u.

For the study of this inequality and delicate properties of the inclusion mapping see S. Sobolev: Sur un Théorème d'analyse fonctionnelle, Mat. Sbornik, 4(46), 1938.

2 Cauchy Problem

In this section we formulate the Cauchy problem for a linear differential operator $a\left(x, \dfrac{\partial}{\partial x}\right)$. To begin with we make a few formal reductions.

Let S be a hypersurface in $\underline{\mathrm{R}}^n$ defined by an equation $\varphi(x) = 0$ where φ is a sufficiently often continuously differentiable function with its gradient $\varphi_x(x_0) \equiv \left(\dfrac{\partial\varphi}{\partial x_1}(x_0), \ldots, \dfrac{\partial\varphi}{\partial x_n}(x_0)\right) \neq 0$ at every point x_0 of S. Let n denote the normal at the point x_0 to S and $\dfrac{\partial}{\partial n}$ denote the derivation along the normal n.

Suppose x_0 is a point on S; let $u_0, \ldots, u_{m-1}$ be functions on S defined in a neighbourhood of x_0. A set $\psi = (u_0, \ldots, u_{m-1})$ of such functions is called a set of Cauchy data on S for any differeential operator of order m. The Cauchy data ψ are said to be analytic (resp. of class $\mathscr{E}^m$, resp. of class $\mathscr{E}$) if each of the functions $u_0, u_1, \ldots, u_{m-1}$ is an analytic (resp. m times continuously differentiable function resp. infinitely differentiable function) in their domain of definition.

Let there be given a function f defined in a neighbourhood U in $\underline{\mathrm{R}}^n$
7 of a point x_0 of S and Cauchy data ψ in a neighbourhood V of x_0 on S.

The Cauchy problem for the differential operator $a\left(x, \dfrac{\partial}{\partial x}\right)$ with the Cauchy data ψ on S consists in finding a function u defined in a neighbourhood U' of x_0 in $\underline{\mathrm{R}}^n$ satisfying

$$a\left(x, \frac{\partial}{\partial x}\right) u = f \text{ in } U' \tag{2.1}$$

and $u(x) = u_0(x)$, $\dfrac{\partial}{\partial n} u(x) = u_1(x)$; $\ldots$, $\left(\dfrac{\partial}{\partial n}\right)^{m-1} u(x) = u_{m-1}(x)$ for $x \in V \cap U'$. When such a u exists we call it a solution of the Cauchy problem.

In the study of the Cauchy problem the following questions arise: the existence of a solution u and its domain of definition, uniqueness when the solution exists, dependence of the solution on the Cauchy data and the existence of the solution in the large. The answers to these questions will largely depend on the nature of the differential operator and of the surface S (supporting the Cauchy data) in relation to the differential operator besides the Cauchy data ψ and f. In order to facilitate the formulation and the study of the above questions we first make a preliminary reduction.

By a change of variables

$$(x_1, \ldots, x_n) \to (x'_1, \ldots, x'_n)$$

with $x'_1 = x_1, \ldots, x'_{n-1} = x_{n-1}$ and $x'_n = \varphi(x)$ the equation

$$a\left(x, \frac{\partial}{\partial x}\right) u = f \tag{2.1}$$

is transformed into an equation of the form

$$h(x, \varphi_x) \left(\frac{\partial}{\partial x'_n}\right)^m u + \sum \cdots = f$$

where $\varphi_x = \left(\dfrac{\partial\varphi}{\partial x_1}, \ldots, \dfrac{\partial\varphi}{\partial x_n}\right)$ and $h(x,\xi) = \sum\limits_{|\nu|=m} a_\nu(x)\xi^\nu$, $\xi = (\xi_1, \ldots, \xi_n)$. **8**
The summation above contains derivatives of u of orders $< m$ in the x'_n-direction.

(1) If $h(x, \varphi_x(x)) \neq 0$ in a neighbourhood of the point under consideration we can divide the above expression for the equation by the factor $h(x, \varphi_x)$ and write

$$\left(\frac{\partial}{\partial x'_n}\right)^m u + \sum_{\substack{|\nu|\leq m\\ \nu_n\leq m-1}} a'_\nu(x')\left(\frac{\partial}{\partial x'}\right)^\nu u = \frac{f}{h(x,\varphi_x)}. \tag{2.2}$$

This is called the normal form of the equation.

$$a\left(x, \frac{\partial}{\partial x}\right)u = f.$$

The Cauchy problem is now given by

$$\left(\frac{\partial}{\partial x'_n}\right)^j a(x'_1 \ldots, x'_{n-1}, 0) = u_j(x'_1, \ldots, x'_{n-1}) \text{ for } j = 0, 1, \ldots, m-1.$$

(2) In the case in which $h(x, \varphi_x) = 0$ at a point x_0 of S the study of the Cauchy problem in the neighbourhood of x_0 becomes considerably more difficult. In what follows we only study the case (1) where the equation can bebrought to the normal form by a suitable change of variables. This motivates the following

Definition. A surface S defined by an equation $\varphi(x) = 0$ (ψ being once continuously differentiable) in $\underline{\mathrm{R}}^n$ is said to be a characteristic variety or characteristic hypersurface of the operator $a\left(x, \dfrac{\partial}{\partial x}\right)$ if $h(x, \operatorname{grad}\varphi(x)) = 0$ for all the points x on S. **9**

A vector $\xi \in \underline{\mathrm{R}}^n$ is said to be a characteristic direction at x with respect to the differential operator $a\left(x, \dfrac{\partial}{\partial x}\right)$ if $h(x,\xi) = 0$.

Clearly, if S is a characteristic variety of a differential operator $a\left(x, \frac{\partial}{\partial x}\right)$ then the vector normal to S at any point on it will be a characteristic direction at that point. For any point $x \in S$ the set of vectors ξ which are characteristic directions at x form a cone in the ξ-space with vertex at the origin called the characteristic cone of the operator $a\left(x, \frac{\partial}{\partial x}\right)$ at the point x. In the following we restrict ourselves to the case where S is not characteristic for the differential operator at any point and hence assume the operator to be in the normal form.

3 Cauchy - Kowalevsky theorem and Holmgren's theorem

The first general result concerning the Cauchy problem (local) is the following theorem due to Cauchy and Kowalevsky. This we recall without proof. For a proof see for example Petrousky [1].

From now on we change slightly the notation and denote a point of $\underline{R}^{n+1}$ by $(x, t) = (x_1, \ldots, X_n, t)$ and a point of $\underline{R}^n$ by $x = (x_1, \ldots, x_n)$.

Let

$$L \equiv \left(\frac{\partial}{\partial t}\right)^m + \sum_{\substack{|\nu|+j\leq m\\ j\leq m-1}} a_{\nu,j}(x,t)\left(\frac{\partial}{\partial x}\right)^\nu \left(\frac{\partial}{\partial t}\right)^j \tag{3.1}$$

be a differential operator of order m written in the normal form with variable coefficients.

10 **Theorem 1** (Cauchy-Kowalevsky)**.** *Let the coefficients $a_{\nu,j}$ of L be defined and analytic in a neighbourhood U of the origin in the (x, t) space. Suppose that f is an analytic function on U and ψ is an analytic Cauchy datum in a neighbourhood V of the origin in the x-space. Then there exists a neighbourhood W of the origin in the (x, t)-space and a unique solution u of the Cauchy problem*

$$\begin{aligned} &Lu = f \text{ in } W \text{ and}\\ &\left(\frac{\partial}{\partial t}\right)^{j_u} = u_j \text{ on } W \cap \{t = 0\} \text{ for } j = 0, 1, \ldots, m-1, \end{aligned} \tag{3.2}$$

which is defined and analytic in W.

Remark. The domain W of existence of u depends on U, V and the maximum moduli of $a_{\nu,j}$.

It is not in general, possible to assert the existence of a solution of the Cauchy problem when the Cauchy data are only of class $\mathscr{E}$. Howevr for a certain class of differential operators-such as Hyperbolic operators - the existence (even in the large) of solutions of the Cauchy problem can be established under some conditions. This will be done in the subsequent sections.

If u_1 and u_2 are two analytic solutions of the Cauchv problem in a neighbourhood of the origin with the same analytic Cauchy data the theorem of Cauchy-Kowalevsky asserts that $u_1 \equiv u_2$. Holmgren showed that for an operator with analytic coefficients the solution is unique, if it exists, in the class $\mathscr{E}^m$ (m, we recall., is the order of L). More precisely we have the

Theorem 2 (Holmgren)**.** *If the coefficients $a_{\nu,j}$ of the differential operator L are analytic functions in a neighbourhood U of the origin then there exists a number $\varepsilon_0 > 0$ satisfying the following: for any $0 < \varepsilon < \varepsilon_0$ if the Cauchy data ψ vanish on $(t = 0) \cap D_\varepsilon$ then any solution $u \in \mathscr{E}^m$ of the Cauchy problem* 11

$$Lu = 0 \text{ in } D_\varepsilon \text{ and}$$
$$\left(\frac{\partial}{\partial t}\right)^j u = 0 \text{ on } (t = 0) \cap D_\varepsilon \text{ for } j = 0, 1, \ldots, m - 1,$$

itself vanishes identically in D_ε, where D_ε denotes the set

$$\left\{(x, t) \in \underline{R}^{n-1} \middle| |x|^2 + |t| < \varepsilon\right\}.$$

Proof. By a change of variables $(x, t) \to (x', t')$ where $x'_k = x_k (k = 1, \ldots, n)$ and $t' = t + x_1^2 + \cdots + x_n^2$ the half space $t \geq 0$ is mapped into the domain

$$\Omega = \left\{(x', t') \in \underline{R}^{n+1} \middle| t' - |x'|^2 \geq 0\right\}$$

in the (x'_1, t') space. The transformed function $u'(x', t')$ and its derivatives upto order $(m-1)$ in the direction of the interior normal to the hypersurface $\{t' - |x'|^2 = 0\}$ vanish identically on the hyper-surface. Hence extending u' by zero outside the domain Ω we obtain a function in $\mathscr{E}^m$, which we again denote by u, with support contained in Ω. The differential operator is transformed into another differential operator of order m with analytic coefficients. □

Thus we may assume that u is a solution of an equation

$$Lu \equiv \left(\frac{\theta}{\partial t}\right)^m u + \sum_{\substack{|\nu|+j\leq m \\ j\leq m-1}} a_{\nu,j}(x,t)\left(\frac{\partial}{\partial x}\right)^\nu \left(\frac{\partial}{\partial t}\right)^j u = 0 \tag{3.3}$$

12 with support contained in Ω. Let ${}^tL$ be the transpose operator of L and V be a solution of ${}^tL[v] = 0$ in $\Omega_h = \Omega \cap \{0 \leq t \leq h\}$ satisfying the conditions

$$v(x,h) = \frac{\partial}{\partial t} v(x,h) = \ldots = \left(\frac{\partial}{\partial t}\right)^{m-2} v(x,h) = 0 \tag{3.4}$$

on the hyperplane $(t = h)$. Then we have

$$\int_{\Omega_h} (u^t L[v] - v\, L[u]) dx\, dt = 0. \tag{3.5}$$

On the other hand, integrating by parts with respect to the variables t and x yields

$$\int_{\Omega_h} (u^t L[v] - vL[u]) dx\, dt = \int_{t=h} (-1)^m u(x,t) \left(\frac{\partial}{\partial t}\right)^{m-1} v(x,t) dx$$

because of the conditions (3.4).

$$\text{Hence} \quad \int_{t=h} (-1)^m u(x,t) \left(\frac{\partial}{\partial t}\right)^{m-1} v(x,t) dx = 0. \tag{3.6}$$

Now consider the Cauchy problems

$$t_L[v] = 0$$

$$\left(\frac{\partial}{\partial t}\right)^j v(x,0) = 0,\ j = 1, \ldots m,\ \left(\frac{\partial}{\partial t}\right)^{m-1} v(x,0) = P(x),$$

$P(x)$ running through polynomials. By the Cauchy Kowalevsky Theorem, there exists solutions $v(x)$, in a *fixed neighbourhood* $|t| \leq h$ satisfying the above Cauchy problems. Hence there is a $h > 0$ such **13**
that, for every polynomial $P(x)$ there exist $\underline{\text{v}}$ in Ω_h satisfying (3.4) with $\left(\frac{\partial}{\partial t}\right)^{m-1} u(x,h) = P(x)$. Hence by (3.6) $u(x,t)$ is orthogonal to every polynomial $P(x)$ for $t \leq h$. Hence $u(x,t) \equiv 0$ for $0 \leq t \leq h$. Replacing t, by $-t$ we obtain $u(x,t) \equiv 0$ for $-h \leq t \leq 0$. Hence $u(x,t) \equiv 0$ in D_ε which finishes the prove of the theorem.

Further general results on the uniqueness of the solution of the Cauchy problem were proved by Calderon [1]. We restrict ourselves to stating one of his results ([3]).

Theorem 3 (Calderon)**.** *Let L be an operator of the form* (3.1) *with real coefficients. Assume that in a neighbourhood of the origin all the coefficients $a_{\nu,j}(x,t)$, for $|\nu| + j = m$, belong to $C^{1+\sigma}(\sigma > 0)$ and the other coefficients are bounded. Further suppose that the characteristic equation at the origin*

$$P(\lambda,\xi) \equiv \lambda^m + \sum_{|\nu|+j=m} a_{\nu,j}(0,0)\xi^\nu \lambda^j = 0 \tag{3.6}$$

has distinct roots for any real $\xi \neq 0$. If the solution u belong to C^m and has zero Cauchy data (more precisely, Cauchy data, zero in a neighbourhood of the hyperplane $t = 0$) then $u \equiv 0$ in a neighbourhood of the origin.

4 Solvability of the Cauchy problem in the class $\mathscr{E}^m$

In this section we make a few remarks on the existence of solutions of the Cauchy problem in the class $\mathscr{E}^m$ under weaker regularity conditions on the coefficients of the differential operator. We begin with the fol- **14**
lowing formal definition.

Let

$$(4.1) \qquad L \equiv \left(\frac{\partial}{\partial t}\right)^m + \sum_{\substack{|\nu|+j\leq m\\ j\leq m-1}} a_{\nu,j}(x,t)\left(\frac{\partial}{\partial x}\right)^\nu\left(\frac{\partial}{\partial t}\right)^j$$

be a differential operator of order m in the normal form.

Definition. The Cauchy problem for L is said to be solvable at the origin in the class if for any given $f \in \mathscr{E}_{x,t}$ and any Cauchy datum ψ of class $\mathscr{E}_x$ there exists a *neighbourhood* $D_{\psi,f}$ of the origin in the (x,t) space and a solution $u \in \mathscr{E}_{x,t}(D_{(\psi,f)})$ of the Cauchy problem for L with ψ as the Cauchy datum.

Remark. The Cauchy problem for a general linear differential operator L is not in general solvable in the class $\mathscr{E}$ as is shown by the following counter example due to Hadmard.

Counter example (Hadamard). Let L be the Laplacian Δ in $\underline{R}^3$

$$(4.2) \qquad \Delta \equiv \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 + \left(\frac{\partial}{\partial z}\right)^2$$

and $(z = 0)$ be the hyperplane supporting the Cauchy data. Consider for the Cauchy data the conditions

$$u(x,y,0) = u_0(x,y) \text{ and } \frac{\partial u}{\partial z}(x,y,0) = 0.$$

Suppose $u(x,y,z) \equiv u$ is a solution of $\Delta u = 0$ in $z \geq 0$ with the Cauchy data $(u_0, 0)$. Extend u to the whole of $\underline{R}^3$ by setting

$$\begin{aligned} \tilde{u}(x,y,z) &= u(x,y,z) \text{ for } z \geq 0 \text{ and} \\ &= u(x,y,-z) \text{ for } z \leq 0. \end{aligned}$$

15 $\tilde{u}$ satisfies the equation $\Delta\tilde{u} = 0$ in the sense of distributions. In fact, for any $\varphi \in \mathscr{D}(\underline{R}^3)$ we have

$$\langle \tilde{u}, \Delta\varphi\rangle = \int_{\underline{R}^3} \tilde{u}(x,y,z)\Delta\varphi(x,y,z)dx\,dy\,dz$$

$$= \lim_{\varepsilon\to 0}\left\{ -\int\limits_{|z|\geq\varepsilon} \frac{\partial\varphi}{\partial z}\frac{\partial\tilde{u}}{\partial z} dx\, dy\, dz + \int\limits_{|z|\geq\varepsilon}\left(\frac{\partial^2\tilde{u}}{\partial x^2}+\frac{\partial^2\tilde{u}}{\partial y^2}\right)\varphi dx\, dy\, dz\right.$$

and

$$\int\limits_{|z|\geq\varepsilon} \frac{\partial\tilde{u}}{\partial z}\frac{\partial\varphi}{\partial z} dx\ dy\ dz = \int\left[\varphi\frac{\partial\tilde{u}}{\partial z}\right]_{-\varepsilon}^{\varepsilon} dx\, dy - \int_{|z|\geq\varepsilon}\frac{\partial^2\tilde{u}}{\partial z^2}\varphi dz\, dx\, dy$$

Hence

$$\begin{aligned}\langle\tilde{u},\Delta\varphi\rangle &= \lim_{\varepsilon\to 0}\left\{\int\varphi(x,y,\varepsilon)\frac{\partial\tilde{u}}{\partial z}(x,y,\varepsilon)dx\, dy - \int\varphi(x,y,-\varepsilon)\frac{\partial\tilde{u}}{\partial z}(x,y,-\varepsilon)dx\, dy\right\}\\ &= 0\end{aligned}$$

By the regularity of solutions of elliptic equations u is an analytic function of x, y, z in $\underline{\mathrm{R}}^3$. Since $u_0(x,y) = u(x,y,0) = \tilde{u}(x,y,0), u_0$ is an analytic function of (x,y). Thus, if u_0 is taken to be in $\mathscr{E}_x$ but non analytic there does not exist a solution of the Cauchy problem for $\Delta u = 0$ with the Cauchy data $(u_0, 0)$.

As far as the domain of existence of a solution of the Cauchy problem is concerned we know by the Cauchy Kowalevsky theorem that, whenever the coefficients of L, f and the Cauchy data ψ are of analytic classes, there exists a neighbourhood of the origin and an analytic function u on it satisfying $L[u] = f$ with Cauchy data ψ. However it is not in **16**
general possible to continue this local solution u to the whole space as a solution of $L[u] = f$. This is domonstrated by the following counter example which is again due to Hadamard.

Counter example. Let the differential operator be

$$L \equiv \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2.$$

A solution of $L[u] = 0$ is provided by

$$u(x,y) = Re\frac{1}{z-a} = \frac{x-a}{(x-a)^2+y^2} \quad \text{where} \quad a > 0.$$

Clearly $u(0,y)$ and $\dfrac{\partial u}{\partial x}(0,y)$ are analytic functions of Y. However this solution can not be continued to the half plane $x \geq a$ as can be easily seen.

For a class of differential operators the existence of soluctions in the large has been established by Hadamard, Petrowsky, Leray, Garding and others. We shall prove some of these results later by using the method of singular integral operators introduced by Calderon and Zygmund.

Chapter 2

In this chapter as well as in the next chapter we will be mainly concerned with the study of the Cauchy problem for systems of differential equations of the first order, which will be referred to as first order systems. 17

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If $u(x,t) = (u_1(x,t),\ldots,u_N(x,t))$ and $f(x,t) = (f_1(x,t),\ldots,f_N(x,t))$ denote vector valued functions with N components, a first order system of equations can be written in the form

$$M[u] \equiv \frac{\partial}{\partial t}u - \sum_1^n A_K(x,t)\frac{\partial}{\partial x_k}u - B(x,t)u = f \tag{1.1}$$

where $A_k(x,t)$, $B(x,t)$ are matrices of order N of functions whose rigularity conditions will be made precise in each of the problems under consideration.

Definition. The Cauchy problem for a first order system $M[u] = 0$ is said to be locally solvable at the origin in the space $\mathscr{E}$ (resp. $\mathscr{B}$, resp. $D^\infty_{L^2}$) if for any given $\psi \in \mathscr{E}(U)$ (resp. $\mathscr{B}(U)$, resp. $D^\infty_{L^2}(U)$)U being an arbitrary open set in the x-space containing the origin there exists a neighbourhood V of the origin in $\underline{R}^{n+1}$ and a function $u \in \mathscr{E}(V)$ (resp. $\mathscr{B}(V)$, resp $D^\infty_{L^2}(V)$) satisfying

$$M[u] = 0 \text{ and } u(x,0) = \psi(x)$$

(V may depend on ψ).

The following proposition shows that when the system M has an-
18 alytic coefficients the local solvability of the Cauchy problem implies the existence of a neighbourhood V independent of ψ such that for any $\psi \in \mathscr{E}_x$ there exists a unique solution $u \in \mathscr{E}^1(V)$.

We define a family of open sets D_ε of $\underline{\mathrm{R}}^{n+1}$ by

$$D_\varepsilon = \left\{(x,t) \in \underline{\mathrm{R}}^{n+1} \middle| |t| + |x|^2 < \varepsilon\right\}. \tag{1.2}$$

Proposition 1 (P.D. Lax)**.** *[1]. Assume that the coefficients of M are analytic and the Cauchy problem for M is locally solvable at the origin. Then there exists a $\delta > 0$ such that for any given $\psi \in \mathscr{E}_x(U)$ there exists a unique solution $u \in \mathscr{E}^1(D_\delta)$ of $M[u] = 0$, $u(x,0) = \psi(x)$.*

Proof. By Holmgren's theorem there exists an $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$ a solution, $u \in \mathscr{E}^1_{x,t}$ with $u(x,0) = \psi(x)$ on $D_\varepsilon \cap (t = 0)$ is uniquely determined in D_ε. Let $\varepsilon_0 > \varepsilon_1 \ldots$ be a sequence of positive numbers $\varepsilon_n \to 0$. Denote by $A_{k,m}$ the set of all $\psi \in \mathscr{E}_x(U)$ such that the solution u of $M[u] = 0$ with $u(x,0) = \psi(x)$ for $x \in D_{\varepsilon_k} \cap (t = 0)$ is in $\mathscr{E}^{[\frac{n}{2}]+2}_{L^2}(D\varepsilon_k)$ and satisfies

$$||u||_{[\frac{n}{2}]+2} \leq m.$$

The sets $A_{k,m}$ are symmetric and convex. Further $\mathscr{E}(U) = \bigcup\limits_{k,m} A_{k,m}$, by the local solvability at the origin. We shall now show tht $A_{k,m}$ is *closed* for every k, m.

Let ψ_j be a sequence in $A_{k,m}$ converging to ψ_0 in $\mathscr{E}(U)$. The corresponding sequence of solutions u_j is a bounded set in $\mathscr{E}^{[\frac{n}{2}]+2}_{L^2}(D_{\varepsilon_k})$ and hence has a subsequence $u_{j_p}(x,t)$ weakly convergent in $\mathscr{E}^{[\frac{n}{2}]+2}_{L^2}(D_{\varepsilon_k})$. In
19 view of the Prop. 1 of Chap. 1 § 1 we can, if necessary by choosing a subsequence, assume that $u_{j_p}(x,t)$ converges in $\mathscr{E}^{[\frac{n}{2}]+1}_{L^2(\mathrm{loc})}(D_{\in_k})$. Let this limit be u_0. Since $u_{j_p} \to u_0$ weakly in $\mathscr{E}^{[\frac{n}{2}]+2}_{L^2}(D_{\in_k})$ we have $||u_0||_{[\frac{n}{2}]+2} \leq m$. By prop **??** of Chap 1 § 1 (Sobolev's lemma) $u_0 \in \mathscr{E}^1(D_{\in_k})$ and further $M[u_0] = 0$. Again $u_{j_p} \to u_0$ in $\mathscr{E}^{[\frac{n}{2}]+1}_{L^2(\mathrm{loc})}(D_{\in_k})$ implies that this conver-

gence is uniform on every compact subset of $D_{\in_k}$ and hence $u_0(x, 0) = \psi_0(x)$. Thus $A_{k,m}$ is a closed subset of $\mathcal{E}_x(U)$.

Now by Baire's category theorem one of the $A_{k,m}$, let us say A_{k_0,m_0}, contains an open set of $\mathscr{E}_x(U)$. A_{k_0,m_0}, being symmetric and convex contains therefore a neighbourhood of 0 in $\mathscr{E}_x(U)$. Since any $\psi \in \mathscr{E}_x(U)$ has a homothetic image $\lambda\psi$ in this neighbourhood, there is a unique solution $u \in \mathscr{E}^{[\frac{n}{2}]+2}_{L^2}(D_{\in_{k_0}})$, a fortiori, in $\mathscr{E}^1(D_{\mathscr{E}_{k_0}})$ of $M[u] = 0$ with $u(x, 0) = \psi(x)$. $\in_{k_0}$ can be taken to be the required δ. □

Theorem 1. *Let the coefficients $A_k(x, t)$, $B(x, t)$ of M be analytic. If the Cauchy problem is locally solvable at the origin in the space $\mathscr{E}$ then the linear mapping $\psi(x) \to u(x, t)$ is continuous from $\mathscr{E}(U)$ in to $\mathscr{E}^1(D_{\in_0})$.*

Proof. The graph of the mapping $\psi \to u$ is closed in $\mathscr{E}(U)x\mathscr{E}^1(D_{\in_0})$ because of the uniqueness of the solution of $M[u] = 0$, with $u(x, 0) = \psi(x)$ in $D_{\in_0}$. Hence by the closed graph theorem of Banach the mapping is continuous. □

This leads us to the notion of well-posedness of the Cauchy problem **20**
in the sence of Hadamard. This we consider in the following section.

2 Well-posedness and uniform-well posedness of the Cauchy problem

By a k-times differentiable function on a closed interval $[0, h]$ we mean the restriction to $[0, h]$ of a k-times continuously defferentiable function on an open interval containing $[0, h]$.

The space of continuous functions of t in $[0, h]$ with values in the space $\mathscr{E}^m_x$ is denoted by $\mathscr{E}^m[0, h]$. It is provided with the topology of uniform convergence in the topology of $\mathscr{E}^m_x$ (uniform with respect to t in $[0, h]$). In other words, a sequence $\varphi_n \in \mathscr{E}^m[0, h]$ converges to 0 in the topology of $\mathscr{E}^m[0, h]$ if $\varphi_n(t) = \varphi_n(x, t) \to 0$ in $\mathscr{E}^m_x$ uniformly with respect to t in $[0, h]$. A vector valued function $u = (u_1, \ldots, u_N)$ is said to belong to $\mathscr{E}^m[0, h]$ if each of its components u_j belong to $\mathscr{E}^m[0, h]$.

Similarly one can define the spaces $\mathscr{B}^m[0, h] \cdot D^s_{L^2}[0, h]$, $L^2[0, h] = D^0_{L^2}[0, h]$ etc. These will be the spaces which we shall be using in our discussions hereafter. We also write $\mathscr{B}[0,h]$, $\mathscr{E}[0,h]$, $D_{L^2}[0,h]$ instead of $\mathscr{B}^\infty[0,h]$, $\mathscr{E}^\infty[0,h]$, $D^\infty_{L^2}[0,h]$. Following Petrowesky [2] we give the

Definition. The forward Cauchy problem for a first order system M is said to be well posed in the space $\mathscr{E}$ in an interval $[0,h]$ if

(1) for any given function f belonging to $\mathscr{E}[0,h]$ and any Cauchy data $\psi \in \mathscr{E}_x$ there exists a unique solution u belonging to $\mathscr{E}[0,h]$ and
21 once continuously differentiable with respect to t in $[0,h]$ (with its first derivative w.r.t. t having its values in $\mathscr{E}_x$) of $M[u] = f$ with $u(x,0) = \psi(x)$; and

(2) the mapping $(f,\psi) \to u$ is continuous from $\mathscr{E}[0,h] \times \mathscr{E}_x$ into $\mathscr{E}[0,h]$.

Definition. The forward Cauchy problem for a first order system M is said to be uniformly well posed in the space $\mathscr{E}$ if for every $t_0 \in [0,h]$ the following condition is satisfied:

(1) for any given function f belonging to $\mathscr{E}[0,h]$ and any Cauchy data $\psi \in \mathscr{E}_x$ there exists a unique solution $u = u(x,t,t_0)$ belonging to $\mathscr{E}[t_0,h]$ and once continuously differentiable with respect to t in $[t_0,h]$ (the first derivative having its values in $\mathscr{E}_x$) of $M[u] = f$ with $u(x,t_0,t_0) = \psi(x)$; and

(2) the mapping $(f,\psi) \to u$ is uniformly continuous from $\mathscr{E}[0,h]$, $\mathscr{E}_x$ into $\mathscr{E}[t_0,h]$.

The condition of uniform continuity can also be analytically described as follows: given an integer l and a compact set K of $\underline{\mathrm{R}}^n$ there exists an integer l', a compact set K' of $\underline{\mathrm{R}}^n$ and a constant C *(all independent of* t_0 *in* $[0,h]$*) such that*

$$\sup_{t_0 \le t \le h} |u(x,t,t_0)|_{\mathscr{E}^l_K} \le C(|\psi(x)|_{\mathscr{E}^{l'}_{K'}} + \sup_{0 \le t \le h} |f(x,t)|_{\mathscr{E}^{l'}_{K'}} \tag{2.1}$$

where $|g(x)|_{\mathscr{E}^r_K} = \sup\limits_{\substack{x \in K \\ 0 \le |\nu| \le r}} |\left(\dfrac{\partial}{\delta x}\right)^\nu g(x)|$.

22 Similar statements hold also for the spaces $\mathscr{B}$ and $D^\infty_{L^2}$.

We shall now give some criteria for the well posedness of the forward Cauchy problem for first order systems M. For this purpose we introduce the notions of characteristic equation and of the characteristic roots of a first order system M.

The polynomical equation

$$\det\Big(\lambda I - i\sum A_k(x,t)\xi_k - B(x,t)\Big) = 0 \tag{2.2}$$

is called the characteristic equation of M and the roots $\lambda_1(x,t,\xi),\ldots,$ $\lambda_N(x,t,\xi)$ of this equation are called the characteristic roots of M.

It will be useful for our future considerations to introduce the notions of characteristic equation and of characteristic roots for a single equation of order m of the form

$$L = \left(\frac{\partial}{\partial t}\right)^m + \sum_{\substack{|\nu|+j\leq m\\ j\leq m-1}} a_{\nu,j}(x,t)\left(\frac{\partial}{\partial x}\right)^\nu\left(\frac{\partial}{\partial t}\right)^j. \tag{2.3}$$

Consider the principal part of L and write it in the form

$$\left(\frac{\partial}{\partial t}\right)^m + \sum_{j=0}^{m-1} a_j\left(x,t,\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}\right)^j \tag{2.4}$$

where $a_j(x,t,\xi) = \sum\limits_{|\nu|=m-j} a_{\nu,j}(x,t)\xi^\nu$ is a homogeneous polynomial in ξ of degree $m-j$. The characteristic equation of L is defined to be

$$\lambda^m + \sum_{j=0}^{m-1} a_j(x,t,\xi)\lambda^j = 0 \tag{2.5}$$

and its roots are called the characteristic roots of L. **23**

We remark here that if we take $\left(u,\dfrac{\partial u}{\partial t},\ldots,\left(\dfrac{\partial}{\partial t}\right)^{m-1}u\right)$ as a system of unknown functions, say $(u_1,u_2,\ldots,u_m)$, we have

(2.6)

$$\frac{\partial}{\partial t}\begin{pmatrix}u_1\\ \vdots\\ u_m\end{pmatrix} = \begin{pmatrix} 0 & 1 & 0\ldots & 0\\ 0 & 0 & 1\ldots & 0\\ & & \ddots & \\ 0 & 0 & 0\ldots & 1\\ -a_0-a_1 & -a_2 & \ldots & -a_{m-1}\end{pmatrix}\begin{pmatrix}u_1\\ \vdots\\ u_m\end{pmatrix} \equiv H\left(x,t,\frac{\partial}{\partial x}\right)\begin{pmatrix}u_1\\ \vdots\\ u_m\end{pmatrix}$$

and $\det(\lambda I - H(x,t,\xi)) = \lambda^m + \sum_{j=0}^{m-1} a_j(x,t,\xi)\lambda^j$. Thus the characteristic roots of L are the same as those of the system (2.6).

We now obtain necessary and sufficient condition for the well posedness of the Cauchy problem for first order systems in the case where the coefficients depend only on t:

$$\frac{\partial u}{\partial t} = \sum A_k(t)\frac{\partial u}{\partial x_k} + B(t)u. \tag{2.7}$$

These conditions depend on the nature of the roots of its characteristic equation

$$\det(\lambda I - i\sum A_k(t)\xi_k - B(t)) = 0 \tag{2.8}$$

In the case where A_k and B are constant matrices, we have the following proposition.

Proposition 1 (Hadamard)**.** *Let the coefficients A_k and B of M be constants. A necessary condition in order that the forward Cauchy problem for M be well posed in the space $\mathscr{B}$ is that there exist constants c and p such that*

$$\operatorname{Re}\lambda_j(\xi) \leq p\log(1+|\xi|) + c \quad (j = 1,\ldots,N). \tag{2.9}$$

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Proof. Assume that the forward Cauchy problem for M is well posed but the condition (2.9) is not satisfied. First of all we observe that, if $\lambda(\xi)$ is any characteristic root of M there exists a non-zero vector $P(\xi) \in \underline{C}^N$ with $|P(\xi)| = 1$ such that

$$\Big(\lambda(\xi)I - i\sum A_k\xi_k - B\Big)P(\xi) = 0.$$

Then $u(x,t) = exp(\lambda(\xi)t + ix\cdot\xi)$. $P(\xi)$ is a solution of $M[u] = 0$. By assumption for any $p > 0$ there exists a vector ξ, $|\xi| \geq 2$, and a characteristic root $\lambda(\xi)$ such that,

$$\operatorname{Re}\lambda(\xi) \geq p\log(1+|\xi|).$$

For this $\lambda(\xi)$ we have

(i) $M[u] \equiv M[\exp(\lambda(\xi)t + ix.\xi) \cdot P(\xi)] = 0$;

(ii) $|u(x,t)| = \exp(\mathrm{Re}\,\lambda(\xi)t) \cdot |P(\xi)| \geq (1 + |\xi|)^{pt}$ for $t > 0$; and

(iii) $\sum_{|\nu|\leq l} |\left(\frac{\partial}{\partial x}\right)^{\nu} u(x,0)| \leq C(1)(1 + |\xi|)^{l}$.

The inequalities (ii) and (iii) show that the forward Cauchy problem is not well posed which contradicts the assumption. Hence Proposition 1 is proved. □

For a smooth function u (for instance a function in L^2 or $\mathscr{S}$) the Fourier transform $\widehat{u}$ with respect to x is defined by

$$\widehat{u}(\xi,t) = \int u(x,t)\exp(-2\pi i x.\xi)dx. \tag{2.10}$$

More precisely if u belongs to $\mathscr{S}'$ then its Fourier image is denoted by $\widehat{u}$ and $\widehat{u}$ belongs to $\mathscr{S}'$.

Let us now assume that the coefficients A_k and B of M are continu- **25**
ous functions of t in $[0,h]$ but do not depend on x. Consider the system of ordinary differential equations

$$\frac{d}{dt}\widehat{u}(\xi,t) = \left(2\pi i \sum_k A_k(t)\xi_k + B(t)\right)\widehat{u}(\xi,t). \tag{2.11}$$

If v_0^j denotes the vector in $\underline{\mathrm{R}}^N$ whose j^{th} component is 1 and the other companents are 0, let $v^j(\xi,t,t_0)$ be the fundamental system of solutions of the system (2.11) (defined in $[t_0,h]$) with the initial conditions $v^j(\xi,t_0,t_0) = v_0^j$. Then we have the

Proposition 2 (Petrowsky)**.** *Let the coefficients A_k and B of M be continuous functions of t in $[0,h]$. A necessary condition in order that the forward Cauchy problem for M be uniformly well posed in the spaces $\mathscr{B}$ and $\mathscr{D}^{\infty}_{L^2}$ is that there exist constant c and p, both independent of t_0 in $[0,h]$, such that*

$$|V^j(\xi,t,t_0)| \leq c(1 + |\xi|)^p. \tag{2.12}$$

Proof. Necessity in the space $\mathscr{B}$. Assume that the forward Cauchy problem is uniformly well posed in the space $\mathscr{B}$ but the condition (2.12) is not fulfilled. Then for any p, one can find ξ^*, t^*, t_0^* and k such that we have the inequality

$$|V^k(\xi^*, t^*, t_0^*)| \geq p(1 + |\xi^*|)^p.$$

The function $u(x, t, t_0^*) = (u_1(x, t, t_0^*), \ldots, u_N(x, t, t_0^*))$ with

$$u(x, t, t_0^*) = \exp(ix.\xi^*) \cdot v^k(\xi^*, t, t_0^*), t \in [t_0, h] \tag{2.13}$$

26 is a solution of $M[u] = 0$ and satisfies the inequalities

(i) $|u(x, t^*, t_0^*)| \geq p(1 + |\xi^*|)^p$ where $t_0^* \leq t^* \leq h$, and

(ii) $\sum_{|\nu|\in 1} \left|\left(\dfrac{\partial}{\partial x}\right)^\nu u\left(x, t_0^*, t_0^*\right)\right| \leq c(l)(1 + |\xi^*|)^l$,

$c(l)$ being a constant depending only on l which again show that the forward Cauchy problem is not uniformly well posed, thus arriving at a contradiction to the assumption. □

Necessity in the space $\mathscr{D}^\infty_{L^2}$. Again assume that the forward Cauchy problem is uniformly well posed in $\mathscr{D}^\infty_{L^2}$ but the condition (2.12) does not hold. We can therefore assume that for any p, there exist ξ^*, t^*, t_0^* and k such that we have the inequality

$$|V^k(\xi, t^*, t_0^*)| \geq p(1 + |\xi|)^p, t^* \geq t_0^*.$$

holds for all ξ in a neighbourhood U of ξ_0^* in $\underline{\mathrm{R}}^n$. Let $f \in L^2$ with its support contained in U and $\|f\| = 1$. Then the function $u(x, t, t_0^*) = (u_1(x, t, t_0^*), \ldots, u_N(x, t, t_0^*))$, with

$$u(x, t, t_0^*) = \int exp(ix.\xi) v^k(\xi, t, t_0^*) f(\xi) d\xi \text{ for } t \geq t_0^*, \tag{2.14}$$

is a solution of $M[u] = 0$. By Plancheral's theorem we have

$$\|u\| = (2\pi)^{n/2} (\int |v^k(\xi, t, t_0^*)|^2 |f(\xi)|^2 d\xi)^{\frac{1}{2}}$$

$$(2.15) \qquad \geq (2\pi)^{n/2} p(1+|\tilde{\xi}^*|)^p$$

where $|\tilde{\xi}^*| = \text{dist}\ (0, \text{supp}\,.f)$. On the other hand again by applying Plancheral's theorem we have, for any 1, that

$$\sum_{|\nu|\leq 1} \left\| \left(\frac{\partial}{\partial x}\right)^\nu u(x, t_0^*, t_0^*) \right\| = \sum_{\substack{|\nu|\leq \ell \\ \leq c(l)(1+|\xi^*|^l}} (2\pi)^{n/2} \left(\int |\xi^\nu v^k(\xi, t_0^*, t_0^*)^2 | f(\xi)|^2 d\xi \right)^{\frac{1}{2}}$$

(2.16)

where $c(l)$ is a constant depending only on 1. The two inequalities 27
(2.15), (2.16) together show that the forward Cauchy problem is not uniformly well posed leading to a contradiction to the assumption.

Proposition 3 (Petrowsky). *Let the coefficients A_k and B of M be continuous functions of t. Then the condition* (2.12) *is sufficient in order that the forward Cauchy problem be uniformly well posed in the spaces $\mathscr{D}^\infty_{L^2}$, $\mathscr{B}$ and $\mathscr{C}$.*

Proof. **Sufficiency in the space $\mathscr{D}^\infty_{L^2}$.** The inequality (2.12)

$$|v^j(\xi, t, t_0)| \leq c(1+|\xi|)^p$$

shows that there exists a σ such that $(1+|\xi|)^\sigma v^j(\xi, t, t_0) \in \mathscr{B}^0_\xi$ and this depends continuously on (t, t_0). In fact, $v^j(\xi, t, t_0)$ satisfies (2.11)

$$\frac{d}{dt} v^j(\xi, t, t_0) = (i \wedge \cdot \xi + B) v^j(\xi, t, t_0), A.\xi = \sum A_k \xi_k$$

consider

$$V^j(\xi, t, t_o) - v^j(\xi, t_0, t_0) = \int_{t_0}^{t} (iA(s)\cdot\xi + B(s)) v^j(\xi, s, t_0) ds.$$

This implies that $(1+|\xi|)^{-p-1} v^j(\xi, t, t_0)$ is continuous in (t, t_0) in the space $\mathscr{B}^0_\xi$. Hence the inverse Fourier image $R^j_x(t, t_0)$ of $V^j(\xi, t, t_0)$ with respect to ξ belongs to $\mathscr{S}'$ and the operator $R^j_x(t, t_0) *_{(x)}$ has the following properties:

(1) for any $\varphi \in \mathscr{D}^{\mathscr{S}}_{L^2}$, $R^j_x(t, t_0) *_{(x)} \varphi \in \mathscr{D}^{s+\sigma}_{L^2}[t_0, h]$ and 28

(2) for any $f \in \mathscr{D}^s_{L^2}[0, h]$, the integral

$$\int_{t_0}^{t} R^j_x(t, \tau) *_{(x)} f(x, \tau) d\tau$$

belongs to $\mathscr{D}^{s+\sigma}_{L^2}[t_0, h]$. Further the linear mappings

$$\varphi \to R^j_x(t, t_0) *_{(x)} \varphi, \; f \to \int_{t_0}^{t} R^j_x(t, \tau) *_{(x)} f(x, \tau) d\tau \tag{2.17}$$

are continuous. Now given $\psi = (\varphi_1, \ldots, \varphi_N)$ with $\varphi_j \in \mathscr{D}^s_{L^2}$ and $f = (f_1, \ldots, f_N)$ with $f_j \in \mathscr{D}_{L^2}[0, h]$ define $u(x, t, t_0) = (u_1(x, t, t_0), \ldots, u_N(x, t, t_0))$ by

$$u(x, t, t_0) = \sum_j R^j_x(t, t_0) *_{(x)} \varphi_j(x) + \int_{t_0}^{t} R^j_x(t, \tau) *_{(x)} f_j(x, \tau) d\tau. \tag{2.18}$$

Then $u(x, t, t_0)$ is a solution of $M[u] = f$ with the Cauchy data $u(x, t_0, t_0) = \psi(x)$. In view of (2.18) we conclude that the forward Cauchy problem is uniformly well posed in the space $\mathscr{D}^\infty_{L^2}$.

Sufficiency in the space $\mathscr{B}$. We recall that $(v^j(\xi, t, t_0))$ is a fundamental system of solutions of the system (2.11)

$$\frac{d}{dt} v = (2\pi i \sum A_k(t)\xi_k + B(t))V.$$

Hence each $v^j(\zeta, t, t_0)$ is an entire function of exponential type for complex $\zeta \in \underline{C}^n$. In fact, if $|v(J, t, t_0)|^2$ stands for $\sum_{j=1}^{N} v^j(\zeta, t, t_0)|^2$, we have since $A_k(t)$ and $B(t)$ are bounded

$$|(2\pi i \sum A_k(t)\zeta_k + B(t))v(\zeta, t, t_0)| \leq c(1 + |\zeta|)|v(\zeta, t, t_0)| \tag{2.19}$$

29 with a constant c independent of ζ and v, Further

$$\begin{aligned}\frac{d}{dt}|v(\zeta,t,t_0)|^2 &= \sum_j (\frac{dv^j}{dt}(\zeta,t,t_0).\overline{v^j(\zeta,t,t_0)} + v^j(\zeta,t,t_0)\overline{\frac{dv^j}{dt}(\zeta,t,t_0)})\\ &\le 2\Big|\frac{d}{dt}v(\zeta,t,t_0)||v(\zeta,t,t_0)\Big|\\ &= 2|(2\pi i\sum A_k(t)\zeta + B(t))v(\zeta,t,t_0)||v(\zeta,t,t_0)|\\ &\le 2c'|v(\zeta,t,t_0)|^2(1+|\zeta|).\end{aligned}$$

Hence $|v(\zeta,t,t_0)| \le c'' e^{c'(1+|\zeta|)|t-t_0|}$ and consequently for large ζ, we have, for each $j = 1,\ldots,N$ the inequality

$$|v^j(\zeta,t,t_0)| \le c_1 e^{c_2|\zeta||t-t_0|}$$

Hence by Paley-Wiener's theorem $R^j_x(t,t_0)$ is a distribution with compact support contained in $\{(x,t) \in \underline{R}^{n+1} \| x| < c_2|t-t_0|\}$ and depends continuousuly on (t,t_0). By the structure of distribution with compact supports we can wrte

$$R^j_x(t,t_0) = \sum_{|\nu|\le s_j} (\frac{\partial}{\partial x})[g^j_\nu(x,t,t_0)](j = 1,\ldots,N), \tag{2.20}$$

where $g^j_\nu(x,t,t_0) \in \mathscr{B}^0_x[t_0,t]$ with support contained in $\{x\Big||x| < c_3\}$ and the derivatives are taken in the sense of distributions. This implies that

(1) for any $\varphi \in \mathscr{B}$ we have $R^j_x(t,t_0) *_{(x)} \varphi \in \mathscr{B}[t_0,h]$,

(2) for any $f \in \mathscr{B}[0,h]$ the integral

$$\int_{t_0}^{t} R^j_x(t,\tau)^*_{(x)} f(x,\tau)d\tau \in \mathscr{B}[t_0,h].$$

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Further the linear maps

$$\varphi \to R^j_x(t,t_0) *_{(x)} \varphi,\ f \to \int_{t_0}^{t} R^j_x(t,\tau) *_{(x)} f(x,\tau)d\tau \tag{2.21}$$

are continuous. Now the same argument as in the first part of the proposition shows that the Cauchy problem is uniformly well posed in the base $\mathscr{B}$.

Sufficiency in the space $\mathscr{E}$. In the above proof we observe that, since $R_x^j(t,t_0)$ is a distribution with compact support, we have

(1) for any $\varphi \in \mathscr{E}$, $R_x^j(t,t_0) *_{(x)} \varphi \in \mathscr{E}[t_0,h]$,

(2) for any $f \in \mathscr{E}[O,h]$ the integral

$$\int_t^{t_0} R_x^j(t,\tau) *_{(x)} f(x,\tau)d\tau$$

belongs to $\mathscr{E}[t_0,h]$. Again the linear maps

$$\varphi \to R_x^j(t,t_0) *_{(x)} \varphi,\ f \to \int_{t_0}^{t} R_x^j(t,\tau) *_{(x)} f(x,\tau)d\tau$$

are continuous and an argument similar to the one used earlier shows that the forward Cauchy problem is uniformly well posed in the space $\mathscr{E}$.

This completes the proof of the proposition. □

3 Cauchy problem for a single equation of order *m*

By an argument similar to thye ones used in the previous section we shall presently prove a necessary and sufficient condition in order that the forward Cauchy problem for a single equation of order m be uniformly well posed in the space $\mathscr{E}$. Let

$$L \equiv \left(\frac{\partial}{\partial t}\right)^m + \sum_{\substack{|\nu|+j\leq m\\ j\leq m-1}} a_{\nu,j}(t)\left(\frac{\partial}{\partial x}\right)^\nu \left(\frac{\partial}{\partial t}\right)^j \tag{3.1}$$

31 be a linear differential operators of order m whose coefficients $a_{\nu,j}(t)$ are

$(m-1)$ times continuously differentiable functions of t in an interval $[0, h]$. By Fourier transforms in the x-space we are lead to the following oridinary differential equation of order m with $(m-1)$-times continuously differentiable coefficients in t:

$$\tilde{L}[V] \equiv \left(\frac{d}{dt}\right)^m v(\xi, t) + \sum_{\substack{|\nu|+j\leq m\\ j\leq m-1}} a_{\nu,j}(t)(i\xi)^\nu \left(\frac{d}{dt}\right)^j v(\xi, t) = 0. \tag{3.2}$$

Let $v(\xi, t, t_0)$ be a solution of $\tilde{L}[v] = 0$ satisfying the initial conditions on $(t = t_0)$.

$$v(\xi, t_0, t_0) = 0, \ldots, (\frac{d}{dt})^{m-2} v(\xi, t_0, t_0) = 0, (\frac{d}{dt})^{m-1} v(\xi, t_0, t_0) = 1.$$

Then we have the

Proposition 1. *If the coefficients $a_{\nu,j}$ of L are $m-1$ times continuously diffenentiable functions of t in an interval $[0, h]$ the forward Cauchy problem for L is uniformly well posed in the space $\mathscr{E}$ if and only if there exist constants c and p both independent of t_0 such that*

$$|v(\xi, t, t_0)| \leq c(1 + |\xi|)^p. \tag{3.3}$$

Proof. Suppose the Cauchy problem for L is uniformly well posed for the future in the space $\mathscr{E}$ but the condition (3.3) does not hold. Then for any given $p > 0$ there exist ξ^*, t_0^* and t, $t \geq t_0^*$, such that we have the inequality

$$|v(\xi^*, t, t_0^*)| \geq p(1 + |\xi^*|)^p.$$

Then The function $u(x, t, t_0^*) = \exp(ix.\xi^*) v(\xi^*, t, t_0^*)$ is a solution of $Lu = 0$ and has the properties.

(i) $u(x, t, t_0^*) \in \mathscr{E}[t_0^*, h]$ and once continuously differentiable in t with values in $\mathscr{E}_x$,

(ii) $|u(x, t, t_0^*)| = |v(\xi^*, t, t_0^*)| \geq p(1 + |\xi^*|)^p$, and **32**

(iii) $\sum_{|\nu|\leq 1} \left|\left(\frac{\partial}{\partial x}\right)^\nu u(x, t_0^*, t_0^*)\right| = \sum_{|\nu|\leq 1} \left|(i\xi^*)^\nu v(\xi^*, t_0^*, t_0^*)\right| \leq c(l)(1 + |\xi^*|)^l$

The last two inequalities together show that forward Cauchy problem is not uniformly well posed in the space $\mathscr{E}$ which contradiction the assumption.

Conversely, assume that the condition (3.3) is satisfied. The forward Cauchy problem is uniformly well posed in the space $\mathscr{E}$. First of all we prove that the condition (3.3) implise that $v(\xi, t, t_0)$ and all its derivatives upto order $(m-1)$ with respect to t are uniformly majorized in $[t_0, h]$ by polynominals in ξ. For this purpose we rewrite the equation $\tilde{L}[V] = 0$ in the form

$$(3.4) \qquad (\frac{d}{dt})^m v(\xi, t, t_0) + \sum_{j=0}^{m-1} a_j(t, \xi)(\frac{d}{dt})^j v(\xi, t, t_0) = 0$$

where $a_j(t, \xi) = \sum\limits_{|\nu|=m-j} a_{\nu,j}(t)(i\xi)^\nu$ for $j = 0, 1, \ldots, (m-1)$ $a_j(t, \xi)$ are hence polynominals of degree at most $(m-j)$ in ξ with coefficients which are $(m-1)$-times continuously differentiable functions of t in the interval $[0, h]$. Hence we may assume that there exists a constant c such that

$$(3.5) \qquad |a_j(t, \xi)| \leq c(1 + |\xi|)^{m-j},\ j = 0, 1, \ldots, (m-1) \text{ for } t \in [0, h]$$

Integrating (3.4) once with respect to t over the interval $[t_0,\ h]$ we obtain, after using the initial conditions at $t = t_0$,

$$\left(\frac{d}{dt}\right)^{m-1} v(\xi, t, t_0) - 1 = -\sum_{j=0}^{m-1} \int_{t_0}^{t} a_j(\tau, \xi) \left(\frac{d}{d\tau}\right)^j v(\xi, \tau, t_0) d\tau.$$

33 Integrating by parts the terms in the right hand side in view of the initial conditions satisfied by $v(\xi, t, t_0)$ we obatain

$$\left(\frac{d}{dt}\right)^{m-1} v(\xi, t, t_0) - 1 = -\sum_{j=0}^{m-1} \left\{ \sum_{p=0}^{j-1} (-1)^p \left(\frac{\alpha}{dt}\right)^p (a_j(t, \xi)) \left(\frac{d}{dt}\right)^{j-1-p} v(\xi, t, t_0) \right.$$
$$\left. +(-1)^j \int_{t_0}^{t} \left(\frac{d}{d\tau}\right)^j (a_j(\tau, \xi)) v(\xi, \tau, t_0) d\tau \right\}.$$

By successive integration with respect to t over the interval $[t_0, h]$ $(m-1)$-times, using the initial conditions and the inequality (3.5) we show that $\frac{d}{dt}v(\xi, t, t_0), \ldots, \left(\frac{d}{dt}\right)^{m-1} v(\xi, t, t_0)$ are all majorized by polynominals of the form $c_j(1+|\xi|)^{p_j}(j = 1, 2, \ldots, m)$, C_j, p_j being independent of t_0.

Thus it follows that there exist $\sigma_0, \ldots, \sigma_m$ such that $(1+|\xi|)^{\sigma j}\left(\frac{d}{dt}\right)^j v(\xi, t, t_0) \in \mathscr{B}^0_\xi[t_0, h]$ for $j = 0, 1, \ldots, (m-1)$. Let $R^j_x(t, t_0)$ denote the inverse Fourier image of $\left(\frac{d}{dt}\right)^j v(\xi, t, t_0)$ in the ξ-space.

We shall show that each $R^j_x(t, t_0)$ has compact support in the x-space. In view of the theorem of Paley-Wiener we have only to show that each $\left(\frac{d}{dt}\right)^j v(\zeta, t, t_0)$ are of exponential type for complex $\zeta \in \underline{C}^n$.

Denoting $(1+|\zeta|)$ for $\zeta \in \underline{C}^n$ by K we have $|a_j(t, \zeta)| \leq cK^{m-j}$ for all $j = 0, 1, \ldots, m-1$. The equation (3.4) can now be written in the form

$$\left(\frac{d}{dt}\right)^m v(\zeta, t, t_0) + a_{m-1}(t, \zeta)\left(\frac{d}{dt}\right)^{m-1} v(\zeta, t, t_0) + \frac{a_{m-2}}{K}K\left(\frac{d}{dt}\right)^{m-2} v(\zeta, t, t_0)$$
$$+ \ldots + \frac{a_0(t, \zeta)}{k^{m-1}}K^{m-1}v(\zeta, t, t_0) = 0$$

Taking for the new set of function $w = (w_0, w_1, \ldots, w_{m-1})$ where

$$\begin{aligned}
w_0(\zeta, t, t_0) &= K^{m-1}v(\zeta, t, t_0),\\
w_1(\zeta, t, t_0) &= K^{m-2}\frac{dv}{dt}(\zeta, t, t_0)\\
w_{m-2}(\zeta, t, t_0) &= K\left(\frac{d}{dt}\right)^{m-2} v(\zeta, t, t_0)\\
w_{m-1}(\zeta, t, t_0) &= \left(\frac{d}{dt}\right)^{m-1} v(\zeta, t, t_0).
\end{aligned}$$

the above equation can be written as a system of oridinary differential equations in the following way:

(3.6)

$$\frac{d}{dt}\begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_{m-1} \end{pmatrix} = K\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ -\frac{a_0}{K^m} & -\frac{a_1}{K^{m-1}} & -\frac{a_2}{K^{m-2}} & \dots & -\frac{a_{m-2}}{K^2} & -\frac{a_{m-1}}{K} \end{pmatrix}\begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_{m-1} \end{pmatrix}$$

Denote the matrix of the system (3.6) by $H(t,\zeta)$. Since $|a_j(t,\zeta)| \le cK^{m-j}$ the elements of the matrix $H(t,\zeta)$ are bounded in modulus by a constant C_1 independent of ζ in $\underline{C}^n$ and hence $H(t,\zeta)$ as a linear transformation in an m-dimensional vector space is bounded in norm by a constant C_2 which depends only on m but not on ζ in $\underline{C}^n$. Denoting by $|w(\zeta,t,t_0)|^2$ the sum $\sum\limits_j |w_j(\zeta,t,t_0)|^2$ and by $w(\zeta,t,t_0)\cdot\overline{w'(\zeta,t,t_0)}$ the sum $\sum\limits_j w_j(\zeta,t,t_0)\cdot\overline{w'_j(\zeta,t,t_0)}$ we have

$$\begin{aligned} \frac{d}{dt}|w(\zeta,t,t_0)|^2 &= \frac{d}{dt}w(\zeta,t,t_0)\cdot\overline{w(\zeta,t,t_0)} + w(\zeta,t,t_0)\overline{\frac{d}{dt}w(\zeta,t,t_0)} \\ &= K(H(t,\zeta)+\overline{H(t,\zeta)})|w(\zeta,t,t_0)|^2 \end{aligned}$$

34 on account of the system of equation (3.6) satisfied by $w(\zeta,t,t_0)$. Hence

$$\left(\frac{d}{dt}\right)|w(\zeta,t,t_0)|^2 \le 2C_2K|w(\zeta,t,t_0)|^2$$

which, by integration with respect to t over the interval $[t_0,t]$ implies that

$$|w(\zeta,t,t_0)|^2 \le \exp(2C_2K|t-t_0|) = \exp 2C_2(1+|\zeta|)|t-t_0|$$

since $|w(\zeta,t_0,t_0)| = 1$ consequently we have, since $k \ge 1$,

$$\left|\left(\frac{d}{dt}\right)^j v(\zeta,t,t_0)\right| \le \exp[C_2(1+|\zeta|)|t-t_o|].$$

Hence, by the theorem of Paley-Wiener it follows that $R^j_x(t,t_0)$ are distributions with compact support in the x-space and depend continuously on (t,t_0).

Let $\Psi = (\varphi_0, \ldots, \varphi_{m-1})$ with $\varphi_j \in \mathscr{E}_x$ and $f \in \mathscr{E}[0, h]$ be given.

The above argument can be modified a little in order to get convolution operators $\tilde{R}^j_x(t, t_0)$ similar to $R^j_x(t, t_0)$. This we do as follows:

Let $v_j(\xi, t, t_0)$ be the solution of $\tilde{L}[v_j] = 0$ with the initial values given by

$$\left(\frac{\partial}{\partial t}\right)^i v_j(\xi, t, t_0)\Big|_{t=t_0} = \delta^j_i.$$

(δ^j_i are Kronecker's symobls). We see that $v_j(\xi, t, t_0)$ is connected with the solution $v(\xi, t, t_0)$ in the following way.

Let $w_j(\xi, t, t_0) = v_j(\xi, t, t_0) - \dfrac{(t - t_0)}{j!}$, $t \geq t_0$. Then w_j vanishes at $t = t_0$ together with derivatives upto order $(m - 1)$. Now w_j satisfies the equation.

$$\tilde{L}\left[w_j + \frac{(t - t_0)^j}{j!}\right] = 0 \text{ or } \tilde{L}[w_j] = -\frac{1}{j!}\tilde{L}[(t - t_0)^j] =^{\mu}_j (\xi, t, t_0).$$

$\mu_j(\xi, t, t_0)$ are obviously polynomials in ξ and we have **35**

$$|\mu_j(\xi, t, t_0)| \leq c_3(1 + |\xi|)^m \text{ for } 0 \leq t_0 \leq t \leq h,$$

here c_3 is a constant. Hence

$$w_j(\xi, t, t_0) = \int\limits_{t_0}^{t} v(\xi, t, \tau)\mu_j(\xi, \tau, t_0) d\tau.$$

This implies that

$$\begin{aligned}|w_j(\zeta, t, t_0)| &\leq \int\limits_{t_0}^{t} |v(\zeta, t, \tau)||\mu_j(\zeta, \tau, t_0) d\tau \\ &\leq c_3(t - t_0)(1 + |\zeta|)^m \exp[c_4(1 + |\zeta|)(t - t_0)].\end{aligned}$$

Hence the inverse Fourier image $\tilde{R}^j_x(t, t_0)$ of $v_j(\xi, t, t_0) = w_j(\xi, t, t_0) + \dfrac{(t - t_0)^j}{j!}$ has its support in $|x| \leq c'_4(t - t_0)$.

Then the function

$$(3.7)\qquad u(x,t,t_0)=\sum_{j=0}^{m-1}\tilde{R}_x^j(t,t_0)*_{(x)}\varphi_j+\int_{t_0}^{t}R_x(t,\tau)*_{(x)}f(x,\tau)d\tau$$

is a solution of $L[u]=f$ with Cauchy data Ψ on $t=t_0$. (Here $R_x(t,t_0)$) stand for the inverse Fourier image of $v(\xi,t,t_0)$). The linear mappings

$$(3.8)\qquad \varphi_j\to R_x^j(t,t_0)*_{(x)}\varphi_j,\ f\to\int_{t_0}^{t}R_x(t,\tau)*_{(x)}f(s,\tau)d\tau$$

being continuous the forward Cauchy problem is uniformly well posed in the space $\mathscr{E}$. This completes the proof of the proposition. □

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4

Proposition 1. *Let the coefficients A_k and B of a first order system of differential operators M be continuous functions of t in an interval* $[0,h]$. *If the forward Cauchy problem is well posed in the space $\mathscr{E}$ then it is uniformly well posed in $\mathscr{E}$.*

Proof. In view of Prop. of § 2 it is sufficient to prove that if $v^j(\xi,t,t_0)$ is the fundamental system of solutions of the system of oridinary differential equations

$$(4.1)\qquad \frac{d}{dt}v(\xi,t,t_0)=(iA(t)\xi+B(t))v(\xi,t,t_0),\ A(t)\cdot\xi=\sum A_k(t)\xi_k$$

with intial conditions $v^j(\xi,t_0,t_0)=v_0^j$ then $v^j(\xi,t,t_0)$ are majorized by polynominals in $|\xi|$. (We recall that v_0^j denotes the vector in $\underline{\mathrm{R}}^N$ having 1 for the j^{th} component and 0 for the others). If $\xi^0=\dfrac{\xi}{|\xi|}$ we can write the above system as

$$(4.1')\qquad \frac{d}{dt}v(\xi,t,t_0)=(i|\xi|A(t)\cdot\xi^0+B(t))v(\xi,t,t_0).$$

The element $a_{kl}(t,\xi^0)$ of the matrix $A(t)\cdot\xi^0$ are homogeneous functions of ξ^0 of degree one having for coefficients continuous functions of t in $[0,h]$. We remark that $v^j(\xi,t,0)$ define the columns of the Wronskian $W(t,\xi)$ of the above system of differential equations. From the theory of linear ordinary differential equations we know that

$$w(t,\xi) = W(0,\xi)\exp\left\{i|\xi|\sum_j\int_0^t a_{jj}(\tau,\xi^0)d\tau + \sum_j\int_0^t b_{jj}(\tau)d\tau.\right. \tag{4.2}$$

The forward Cauchy problem being well posed we can assume that $\sum_j\int_0^t a_{jj}(\tau,\xi^0)d\tau$ is real for every (t,ξ^0), ξ^0 real. For otherwise we may **37**
assume, if necessary by changing ξ^0 to - ξ^0 that

$$\operatorname{Re} i\sum_j\int_0^t a_{jj}(\tau,\xi^0)d\tau > 0.$$

By the assumption of the well posedness of the forward Cauchy problem it follows that

$$|v^j(\xi,t,0)| \le c(1+|\xi|)^p \tag{4.3}$$

for suitable constants c and p, and so $W(t,\xi)$ is majorized by a polynomial in $|\xi|$. On the other hand, as $\rho\to+\infty$,

$$|w(t,\xi)| \sim |W(0,\xi)|\exp\left\{\rho|\xi^0|\sum_j\operatorname{Re} i\int_0^t a_{jj}(\tau,\xi^0)d\tau\right\},\ \xi=\rho\xi^0.$$

Thus $W(t,\xi)$ tends to $+\infty$ exponentially as $\rho\to+\infty$ contradicting the inequality (4.3). Hence it follows that $\sum_j\int_0^t a_{jj}(\tau,\xi^0)d\tau$ is real for every (t,ξ^0) with real ξ^0. We now have

$$|W(t,\xi)| = |W(0,\xi)|\exp\left\{\sum_j\operatorname{Re}\int_0^t b_{jj}(\tau)d\tau\right.$$

and hence

$$|W(t,\xi)| \geq |W(0,\xi)| \exp\left\{-\sum_j \int_0^t |b_{jj}(\tau)| d\Gamma\right\} \geq \delta > 0 \text{ for all } (t,\xi).$$

ξ real. Further we observe that, as $v^j(\xi,t,0)$ form a basis for the solutions of the system of ordinary differential equations

$$v^i(\xi,t,t_0) = \sum c^i_j(\xi) v^j(\xi,t,0).$$

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Putting $t = t_0$ solving for $c^i_j(\xi)$ we see that, since $\det(v^i_j(\xi,t_0,0))$ is the Wronskian $W(t_0,\xi)$ which is minorized by a polynomial in $|\xi|$ and since $v^j(\xi,t,0)$ are majorized by polynomials in $|\xi|$, $c^i_j(\xi)$ are themselves majorized by polynomials. Hence $v^j(\xi,t,t_0)$ are majorized by polynomials in $|\xi|$ independently of t and t_0 which implies that the forward Cauchy problem is uniformly well posed for M. Hence proposition 1 is proved.

Correspondingly we have the following result for a single differential equation of order m. Let

$$L \equiv \left(\frac{\partial}{\partial t}\right)^m + \sum_{\substack{|\nu|+j\leq m\\ j\leq m-1}} a_{\nu,j}(t) \left(\frac{\partial}{\partial x}\right)^\nu \left(\frac{\partial}{\partial t}\right)^j \tag{4.4}$$

be a linear differential operator of order m with the oefficients depending only on t in the interval $[0,h]$. □

Proposition 2. *Let the coefficients $a_{\nu,j}$ of L be $(m-1)$ times continuously differentiable fucntions of t in an interval $[0,h]$. If the forward Cauchy problem for L is well posed then it is uniformly well posed for the future for L.*

Proof. Writing the operator L in the form

$$\left(\frac{\partial}{\partial t}\right)^m + \sum_{j=0}^{m-1} a_j\left(t, \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}\right)^j \tag{4.5}$$

where $a_j(t,\xi) = \sum\limits_{|\nu|=m-j} a_{\nu,j}(t)(i\xi)^\nu$ $(j = 0,1,\ldots,m-1)$, we are lead to the following oridinary differential equation of order m:

$$\left(\frac{d}{dt}\right)^m v(\xi,t) + \sum_{j=0}^{m-1} a_j(t,\xi)\left(\frac{d}{dt}\right)^j v(\xi,t) = 0. \tag{4.6}$$

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Denoting the Wronskian of the equation (4.6) by $w(t,\xi)$ we have from the theory of ordinary differential equations

$$W(t,\xi) = W(0,\xi)\exp\left\{-\int_0^t a_{m-1}(\tau,\xi)d\tau\right\}. \tag{4.7}$$

Write $a_{m-1}(\tau,\xi) = a^{(1)}_{m-1}(\tau,\xi)+b(\tau)$ where $a^{(1)}_{m-1}(\tau,\xi)$ is homogeneous in ξ of degree one with coefficients continuous functions of t in $[0,h]$. Then

$$a^{(1)}_{m-1}(\tau,\xi) = |\xi| a^{(1)}_{m-1}(\tau,\xi^0), \xi = |\xi|\xi^0$$

and so we can write

$$W(t,\xi) = W(0,\xi)\exp\left\{-|\xi|\int_0^t a^{(1)}_{m-1}(\tau,\xi^0 d\tau - \int_0^t b(\tau)d\tau\right\}.$$

□

Now arguing as in the proof of the proposition 1 one can show that the Cauchy problem is uniformly well posed using again the prop. 36 of § 2. Finally we shall show that for first order systems with constant coefficients the condition of Hadamard implies the condition of Petrowsky. This will prove that for first order systems with constant coefficients these two conditons are equivalent. For this we need the

Lemma 1 (Petrowsky)**.** *Let a system of differential equations with constant coefficients*

$$\frac{d}{dt}v(t) = Av(t) \tag{4.8}$$

where $A = (a_{jk})$ *and* $v(t) = \begin{pmatrix} v_1(t) \\ \vdots \\ v_N(t) \end{pmatrix}$ *with* $|a_{jk}| \leq K$ *be given. Then,*
40 *given any positive number* ε *such that* $\varepsilon \leq (N-1)!2^N K$ *we can find a non-singular matrix C such that*

$$CA = DC \text{ where } D = \begin{pmatrix} a^*_{11} & & & 0 \\ & a^*_{22} & & \\ & & \ddots & \\ a^*_{jk} & & & a^*_{nn} \end{pmatrix} \tag{4.9}$$

where all a^*_{jk}, $k < j$ *satisfy* $|a^*_{jk}| < \varepsilon$. *Moreover*

$$|\det C| = \left[\frac{(N-1)!2^N K}{\in}\right]^{\frac{N(N-1)}{2}} \tag{4.10}$$

and the elements c_{jk} *of C satisfy*

$$|c_{jk}| \leq \left[\frac{(N-1)!2^N K}{\varepsilon}\right]^{(N-1)}. \tag{4.10'}$$

For a proof see Petrowesky [2].

Proposition 3. *Let the coefficients* A_k *and B of M be constants. Then the condition 9 of* § *2 of Hadamard implies the condition 12 of* § *2.*

Proof. Consider the system of ordinary differential equations

$$\frac{d}{dt} v(\xi, t) = (iA.\xi + B) v(\xi, t). \tag{4.11}$$

Let us fix ξ^0. Taking $(iA.\xi^0 + B)$ as the given matrix in the lemma 1 there exist constants c_0, c_1 such that

$$|ia_{jk}(\xi^0) + b_{jk}| < c_0|\xi|^0 + c_1 \tag{4.12}$$

(c_0, c_1 are independent of ξ^0). We take $K = c_0|\xi|^0 + c_1$ and $\xi = (N-1)!2^N K = (N-1)!2^N(c_0|\xi|^0 + c_1)$. Then, by the lemma 1, we can find a matrix $C(\xi^0)$ such that $(|\det C(\xi^0)| = 1$ and its elements $c_{jk}(\xi^0)$ satisfy $|c_{jk}(\xi^0)| \leq 1$. So denoting $c(\xi^0)v$ by w we have

$$\frac{d}{dt} w(\xi^0, t) = \begin{pmatrix} \lambda_1(\xi^0) & & & \\ & \lambda_2(\xi^0) & & \\ & & \ddots & \\ a^*_{jk}(\xi^0) & & & \lambda_N(\xi^0) \end{pmatrix} w(\xi^0, t) \tag{4.13}$$

where $\lambda_1(\xi^0), \ldots, \lambda_N(\xi^0)$ are the roots of the equation **41**

$$\det(\lambda I - iA.\xi^0 - B) = 0 \tag{4.14}$$

and $\left|a^*_{jk}(\xi)\right| \leq (N-1)!2^N (c_0|\xi^0| + c_1)$. by Hadamard's condition we have

$$\operatorname{Re} \lambda_j(\xi^0) < plog(1 + |\xi^0| + logc.$$

Now since $w(\xi^0, t, t_0)$ is a solution of the above system it follows that

$$|w(\xi^0, t, t_0)| \leq c'(1 + |\xi^0|)^{p_o h} \text{ for } 0 \leq t_0 \leq t \leq h$$

with the constants c', p_0 independent of t, t_o, ξ^0. Finally since $v(\xi^0, t, t_0)$ $= c(\xi^0)^{-1} w(\xi^0, t, t_0)$ we have desired property. □

5 Hyperbolic and strongly hyperbolic systems

The notion of well posedness of the Cauchy problem is closely related to the nature of the given system of differential equations. In this section we introduce hyperbolic and strongly hyperbolic systems of differential equations. We give criteria, in order that a given system of differential operators be of this type, in terms of the characteristic roots of the system.

$A_k \equiv A_k(x, t)$, $B \equiv B(x, t)$ will be matrices of order N of functions on $\underline{\mathrm{R}}^n \times [0, h]$ the regularity conditions of which will be prescribed later in each case. Consider the first order system of differential operators

$$M \equiv \frac{\partial}{\partial t} - \sum_k A_k(x, t) \frac{\partial}{\partial x_k} \tag{5.1}$$

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Definition. A system of differential operators M is said to be hyperbolic if the forward and backward Cauchy problems are well posed.

Definition. A first order system of differential operators M is said to be strongly hyperbolic if for any choice of the matrix $B(x,t)$ the Cauchy problem (forward as well as backward) is well posed for the system

$$\frac{\partial}{\partial t} - \sum_k A_k(x,t)\frac{\partial}{\partial x_k} - B(x,t) \tag{5.2}$$

Let $\lambda_1(x,\xi,t),\ldots,\lambda_N(x,\xi,t)$ be the roots of the equation

$$\det(\lambda I - A(x,t)\cdot\xi) = 0 \tag{5.3}$$

where $A(x,t)\cdot\xi$ denotes the matrix $\sum\limits_k A_k(x,t)\cdot\xi_k$.

Proposition 1. *If the coefficient matrices A_k of M are constant matrices then a necessary condition in order that M be strongly hyperbolic is that*

(1) *$\lambda_j(\xi)$ is real for all real $\xi \neq 0$ $(j = 1,\ldots,N)$*

(2) *the matrix $A.\,\xi$ is diagonalizable for all ξ.*

We shall actually prove a slightly stronger result: If one of the $\lambda_j(\xi)$ is not real for some real $\xi \neq 0$, then for any choice of B (a constant matrix) the Cauchy problem for

$$\frac{\partial}{\partial t} - \sum_k A_k\frac{\partial}{\partial x_k} - B$$

is not well posed.

43 *Proof.* If the condition (1) is not satisfied for same real $\xi^* \neq 0$, there exists a root, say $\lambda_1(\xi^*)$, with non vanishing imaginary part of the equation $\det(\lambda I - A.\xi) = 0$. For $\xi = \tau\xi^*$, $\lambda = \tau\lambda'$ we can write

$$\det(\lambda I - iA.\xi - B) = \tau^N \det\left(\lambda' I - iA.\xi^* - \frac{B}{\tau}\right)$$

for any matrix B. Denoting $\det(\lambda' I - iA.\xi^*)$ by $P(\lambda')$ we have

$$\det(\lambda I - iA\cdot\xi - B) = \tau^N \left\{ P(\lambda') + \frac{1}{\tau} Q(\lambda', \tau) \right\} \tag{5.4}$$

where $Q(\lambda', \tau)$ is a polynomial in λ' of degree at most $N - 1$ having for coefficients polynomials in τ^{-1}. Since $\lambda_1(\xi^*)$ is not real we may, without loss of generality, assume that $\operatorname{Im}\lambda_1(\xi^*) < 0$ (if necessary after changing ξ^* by $-\xi^*$ in the equation). Then $i\lambda_1(\xi^*)$ is a root of $P(\lambda') = 0$. By continuity of the roots there exists a root of $P(\lambda') + \frac{1}{\tau} Q(\lambda', \tau) = 0$ in a neighbourhood of $i\lambda_1(\xi^*)$ in the complex plane. More precisely there exists a root $\lambda_1'(\tau)$ for large τ of the equation $P(\lambda') + \frac{1}{\tau} Q(\lambda', \tau) = 0$ such that $\lambda_1'(\tau) = i\lambda_1(\xi^*) + \in (\frac{1}{\tau})$ where $\in(\tau) \to 0$ as $\tau \to +\infty$. Hence Re $\lambda_1'(\tau) \geq \frac{1}{2}(-\operatorname{Im}\lambda_1(\xi^*))$ for large τ. In other words there exists a root $\lambda_1(\tau)$ of the equation

$$\det(\lambda I - iA.\xi - B) = 0$$

such that $\operatorname{Re}\lambda_1(\tau) \leq c\tau$ (with a positive constant c) , which tends to $+\infty$ as $\tau \to \infty$. Hence the forward Cauchy problem is not well posed for the system $M - B$ by prop. 2 of § 2.

(2) Assume again that the system M is strongly hyperbolic, but that for a certain ξ^* the matrix A. ξ^* is not diagonalizable. There exists a non-singular matrix N_0 such that $N_0(A.\xi^*)N_0^{-1}$ has the Jordan canonical form **44**

$$\begin{pmatrix} \lambda_1 & 0\ldots & 0 \\ 1 & \lambda_1\ldots & 0 \\ & * & \ddots \end{pmatrix} \tag{5.5}$$

Consider for B a matrix determined by

$$N_0 B N_0^{-1} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \cdot & \cdot & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}$$

We shall show that the Cauchy problem is not well posed for the system of differential operators

$$\frac{\partial}{\partial t} - \sum A_k \frac{\partial}{\partial x_k} - B.$$

Consider the characteristic equation of this system, namely

$$\det(\lambda I - iA.\xi - B) = 0.$$

Taking for ξ the vector $\tau\xi^*$ (τ a real parameter $\to \infty$) this equation becomes

$$\det(\lambda I - i\tau A.\xi^* - B) = \det(\lambda I - i\tau N_0(A.\xi^*)N_0^{-1} - N_0 B N_0^{-1})$$

$$= \begin{vmatrix} \lambda - i\tau\lambda_1 & -1 & 0\ldots 0 \\ -i\tau\lambda_1 & \lambda - i\tau\lambda_1 & 0\ldots 0 \\ \ldots\ldots & \ldots\ldots & \ldots X \\ \ldots\ldots & \ldots\ldots & \ldots X \end{vmatrix}$$

45 Hence $(\lambda - i\tau\lambda_1)^2 - i\tau = 0$, the roots of which are $\lambda(\tau) = i\tau\lambda_1 \pm \sqrt{i\tau}$ whose real part $\mathcal{R}e\lambda(\tau) \to \infty$ along with τ. Hence the Cauchy problem for the system $M - B$ is not well posed by prop 2 of § 2, which contradicts the assumption. □

Proposition 2. *A sufficient condition in order that the system M be strongly hyperbolic is that one of the following two conditions is satisfied:*

(i) *the characteristic roots $\lambda_i(\xi)$ are real and distinct for all real $\xi \neq 0$;*

(ii) *A_k are Hermitian.*

Proof. Supposing the condition (i) is satisfied. We shall show that this implies that the Cauchy problem is well posed for the system $M - B$ for any choice of B. Consider the equation $\det(\lambda I - iA.\xi - B) = 0$. Denoting the projection $\dfrac{\xi}{|\xi|}$ of ξ on the unit sphere by ξ^0 and $\dfrac{\lambda}{|\xi|}$ by $\lambda'(\xi)$ we can write this equation in the form

$$\det(\lambda' I - iA.\xi^0 - \frac{B}{|\xi|}) = 0.$$

If $\lambda_1(\xi^0), \ldots, \lambda_N(\xi^0)$ are the roots of the equation $(\det \lambda I - A.\xi^0) = 0$ we can write

$$\det(\lambda' I - iA.\xi^0 - \frac{B}{|\xi|}) = \prod_{j=1}^{N}(\lambda' - i\lambda_j(\xi^0)) + \frac{Q(\lambda', \xi^0)}{|\xi|} = 0, \tag{5.6}$$

where $Q(\lambda', \xi)$ is a polynomial $a_0(\xi)\lambda'^{N-1} + \cdots + a_{N-1}(\xi)$ with coefficients bounded for $|\xi| \geq 1$. If Ω_0 is the projection of Ω on the unit sphere we have

$$\inf_{\substack{\xi^0 \in \Omega^0 \\ j \neq k}} |\lambda_j(\xi^0) - \lambda_k(\xi^0)| \geq d > 0$$

since $\lambda_1(\xi^0) \ldots \lambda_N(\xi^0)$ are all distinct. 46

Let $K = \sup\limits_{\substack{\xi^0 \in \Omega^0 \\ 1 \leq j \leq N}} |\lambda_j(\xi^0)$ and $m = \sup\limits_{\substack{|\xi \geq 1| \\ |\lambda| \geq K+1}} |Q(\lambda', \xi)|$.

Let C be a positive number such that $C\left(\frac{d}{2}\right)^{N-1} \geq 2m$ and $\Gamma_1, \ldots, \Gamma_N$ be circles in the complex plane of radic $\frac{C}{|\xi|}\left(\leq \frac{d}{2}\right)$ with centres $\lambda_1(\xi^0)$, $\ldots, \lambda_N(\xi^0)$ respectively. On Γ_K we have

$$\left|\prod_j (\lambda' - i\lambda_j(\xi^0))\right| \geq \frac{C}{|\xi|}\left(\frac{d}{2}\right)^{N-1} \geq \frac{2m}{|\xi|} \text{ and } \frac{|Q(\lambda', \xi)|}{|\xi|} \leq \frac{m}{|\xi|}.$$

Hence by Rouche's theorem there exists a unique root of

$$\prod_j (\lambda' - i\lambda_j(\xi^0) + \frac{Q(\lambda', \xi)}{|\xi|} = 0$$

in the dise enclosed by Γ_k. More precisely there exists a root $\lambda'_j(\xi)$ of $\det(\lambda' I - iA.\xi^0 - \frac{B}{|\xi|}) = 0$ such that

$$\left|\lambda'_j(\xi) - i\lambda_j(\xi^0)\right| < \frac{C}{|\xi|}$$

or, what is the same, there exists a root $\tilde{\lambda}_j(\xi)$ of

$$\det(\lambda I - iA.\xi - B) = 0$$

such that $|\tilde{\lambda}_j(\xi) - i\lambda_j(\xi)| < C$. Since $\lambda_j(\xi)$ are real it therefore follows that

$$\operatorname{Re}\tilde{\lambda}_j\xi \leq C \ (j = 1, \ldots, N)$$

and by prop. 1 of § 2 the forward Cauchy problem is well posed for the
47 system $M - B$. This proves that M is strongly hyperbolic.

Next let us assume that the matrices A_k are Hermitian. By Fourier transforms in the x-space we obtain the first order system of ordinary differential equations.

Now consider

$$\begin{aligned}\frac{d}{dt}|v(\xi,t)|^2 &= \frac{d}{dt}v(\xi,t)\cdot\overline{v(\xi,t)} + v(\xi,t)\overline{\frac{d}{dt}v(\xi,t)}\\ &= (iA.\xi + B)v(\xi,t)\cdot\overline{v(\xi,t)} + v(\xi,t)\overline{(iA.\xi + B)v(\xi,t)}\end{aligned}$$

Since the A_k are Hermitian, we obtain, B being bounded,

$$\frac{d}{dt}|v(\xi,t)|^2 = 2\operatorname{Re} Bv(\xi,t)\cdot\overline{v(\xi,t)} \leq 2c|v(\xi,t)|^2.$$

We obtain therefore

$$|v(\xi,t)|^2 \leq |v(\xi,0)|^2 e^{2ct}. \tag{5.7}$$

which shows that the forward Cauchy problem is well posed for the system $M - B$ and so M is strongly hyperbolic. This completes the proof of the proposition. Let us now remark the following fact:

$$\begin{aligned}\frac{d}{dt}||v(\xi,t)||^2 &= \frac{d}{dt}\langle v(\xi,t), \overline{v(\xi,t)}\rangle\\ &= \langle\frac{d}{dt}v(\xi,t), \overline{v(\xi,t)} + \rangle v(\xi,t), \overline{\frac{d}{dt}v(\xi,t)}\rangle\\ &= \langle(iA.\xi + B)v(\xi,t), \overline{v(\xi,t)}\rangle + \langle v(\xi,t), \overline{(iA.\xi + B)v(\xi,t)}\rangle.\end{aligned}$$

Since A_k are Hermitian we obtain

$$\frac{d}{dt}\|v(\xi,t)\|^2 = 2\,\mathrm{Re}\langle Bv(\xi,t), \overline{v(\xi,t)}\rangle \le 2c\|v(\xi,t)\|^2$$

48 with a constant c independent of ξ. Integrating both sides of the inequality over $[0,t]$ we obtain
$\|v(\xi,t)\|^2 \le \|v(\xi,0)\|^2 e^{2ct}$. Hence

$$\|u\| \le \|u(x,0)\|e^{ct}. \tag{5.8}$$

We remark that the notions of hyperbolicity and strong hyperbolicity can be anologously defined for a single differential operator of order m. Consider a differential operator of order m

$$L = \left(\frac{\partial}{\partial t}\right)^m + \sum_{\substack{|\nu|+j=m \\ j\le m-1}} a_{\nu,j}(x,t)\left(\frac{\partial}{\partial t}\right)^\nu \left(\frac{\partial}{\partial t}\right)^j. \tag{5.9}$$

L is said to be hyperbolic if the Cauchy problem (both the forward and the backward) is well posed for L. It is said to be strongly hyperbolic if the Cauchy problem (both the forward and the backward) is well posed for $L - B$ for any choice of the lower order operator B. Let

$$P(\lambda,\xi) = \lambda^m + \sum_{\substack{|\nu|+j=m \\ j\le m-1}} a_{\nu,j}(x,t)\xi^\mu \lambda^j \tag{5.10}$$

□

Proposition 3. *A necessary and sufficient condition in order that a differential operator L of order m with constant coefficients be strongly hyperbolic is that for every real vector $\xi(\neq 0)$ in $\underline{R}^n$ all the roots of the equation $P(\lambda,\xi) = 0$ are real and distinct.*

Proof. The proof of the fact that the roots of $P(\lambda,\xi) = 0$ for all real $\xi(\neq 0)$ are real runs on the same lines as in Prop. 1. We shall now show that for all real $\xi \neq 0$ these roots are all distinct.

It the roots of $P(\lambda,\xi) = 0$ are not distinct for all real $\xi \neq 0$ let us suppose that for some real $\xi^* \neq 0$ at least two roots of $P(\lambda,\xi^*) = 0$ **49** coincide. Writing $P(\lambda, \xi^*)$ explicitly

$$P(\lambda, \xi^*) = (\lambda - \lambda_1(\xi^*))^p \prod_{j=2}^{m-p+1} (\lambda - \lambda_j(\xi^*)), P \geq 2,$$

where $\lambda_2(\xi^*), \ldots, \lambda_{m-p+1}$ (ξ^*) are real, and different from $\lambda_1(\xi^*)$. Take for ξ the vector $\tau\xi^*$ with a real parameter τ and set $\lambda' = \dfrac{\lambda}{\tau} - i\lambda_1(\xi^*)$. Now consider the equation

$$P(\lambda, i\tau\xi^*) + C\tau^{m-1} = 0$$

with a constant C to be chosen later suitably. From this equation we obtain

$$\lambda'^p \prod_{j=2}^{m-p} \left\{\lambda' + i(\lambda_1(\xi^*) - \lambda_j(\xi^*))\right\} + \frac{C}{\tau}$$
$$= \lambda'^p (a_0(\xi^*) + a_1(\xi^*)\lambda' + \cdots + a_{m-p-1}(\xi^*)\lambda'^{m-p-1} + \lambda'^{m-p}) + \frac{C}{\tau} = 0$$

where $a_0(\xi^*) \neq 0$. Expanding this in a Puiseux series in a neighbourhood of $\tau = \infty$ we see that there exist p roots

$$\lambda'_K(\tau) = \exp\left(\frac{2\pi i}{p} k\right) \cdot \left(\frac{-C}{a_0(\xi^*)}\right)^{\frac{1}{p}} \tau^{-\frac{1}{p}} + 0\left(\tau^{-\frac{1}{p}}\right) \; (k = 1, \ldots, p)$$

p being at least 2 we can choose the constant C such that there exists a root with positive real part; that is there exists a k_0 such that

$$\operatorname{Re} \lambda'_{k_0}(\tau) \geq C_0 \tau^{-1/p} \text{ for large } \tau.$$

(C_0 being a positive constant). Hence

$$\operatorname{Re} \lambda_{k_0}(\tau) \geq C_0 \tau^{1-\frac{1}{p}} \text{ for large } \tau.$$

50 There exist constants b_ν such that $C = \sum\limits_{|\nu|=m-1} b(i\xi^*)^\nu$. Thus it follows

from prop. 2 § 2 that the Cauchy problem is not well posed for the operator

$$L + \sum_{|\nu|=m-1} b_\nu \left(\frac{\partial}{\partial x}\right)^\nu.$$

This contradicts the assumption that the operator L is strongly hyperbolic.

The sufficiency follows as in the proof of the prop. 2(i).

Finally we mention the following fact: Consider the following equation with coefficients in $\mathscr{E}$.

$$M[u] = \frac{\partial}{\partial t}u - \sum A_k(x,t)\frac{\partial}{\partial x_k}u - B(x,t)u = 0.$$

If, at the origin, for some ξ^* real $\neq 0$, one of the characteristic roots of $\det(\lambda I - A(0,0)\xi^*) = 0$ is not real, then the Cauchy problem for M is never well posed in $\mathscr{E}$ in any small neighbourhood of the origin. (See Mizohata [3]). We shall prove this fact later, in a simple case. Here we add an important remark: Garding has shown in his paper (Gårding [1]), that the condition 9 of § 2 of Hadmard is equivalent to the following:

$\operatorname{Re}\lambda_j(\xi)$ is bounded from above when ξ runs through $\underline{R}^n$ for $j = 1,\ldots,N$.

Next Hörmander has systematized such inequalities by using Seidenberg's lemma (see Hörmande [1]). □

Proposition 4. *Let the coefficients A_k and B of M be continuous functions of t in an interval $[0,T]$. If the forward Cauchy problem is uniformly well posed then the backward Cauchy problem is also uniformly well posed.* **51**

Proof. As before denoting $\dfrac{\xi}{|\xi|}$ by ξ^0 let $v^j(\xi,t,t_0)$ be a fundamental system of solutions of the system of ordinary differential equations

$$\frac{d}{dt}v(\xi,t) = (i|\xi|A(t)\xi^0 + B(t))v(\xi,t), 0 \leq t \leq t_0$$

with initial conditions $v^j(\xi,t_0,t_0) = v^j \equiv (v_1^j,\ldots,v_N^j)$ where $v_j^j = 1$ and $v_k^j = 0$ for $k \neq j$. First of all we remark that if $W(t,\xi)$ is the Wronskian of this system then $v^j(\xi, t, t_0)$ define its colums. Since the forward Cauchy problem is uniformly well posed we have

$$|v^j(\xi, t, t_0)| \leq C(1 + |\xi|)^p, \ j = 1, \ldots, N.$$

Hence $W(t, \xi)$ is also majorized by a polynomial in $|\xi|$. From the theory of ordinary differential equations we know that

$$W(T, \xi) = W(t, \xi) \exp \left\{ i|\xi| \sum_j (\int_t^T a_{jj}(s, \xi^0) ds + \int_t^T b_{jj}(s) ds \right\}.$$

Now as in Prop. 1 it follows that $\sum\limits_j \int\limits_t^T a_{jj}(s, \xi^0) ds$ is real for any t and ξ^0. Thus we have

$$|W(T, \xi)| \geq |w(t, \xi)| \exp \left\{ - \sum_j \int_t^t b_{jj}(s) ds \right\}.$$

That is, $|W(T, \xi)| \geq \delta > 0$ for all t and ξ. Further we observe that as
52 $v^j(\xi, t, t_0)$ form a basis for solutions of the system of equations we can write

$$v^j(\xi, t, t_0) = \sum_k c^j_k(\xi) v^k(\xi, t, T).$$

Putting $t = t_0$ and solving for $c^j_k(\xi)$ we see that $c^j_k(\xi)$ are majorized by polynomials in $|\xi|$ since the determinant of this system of linear equations is the Wronskian $W(\xi, T)$ which is minorized by $\delta > 0$ and $v^j(\xi, t, t_0)$ are majorized by polynomials in $|\xi|$. Hence $v^j(\xi, t, t_0)$ are majorized by polynomials in $|\xi|$ independent of t and t_0 in $[0, T]$ which proves that the backward Cauchy problem is uniformly well posed. This completes the proof of the proposition. □

Chapter 3

There are obvious analogues of the function spaces introduced at the begining of Chapter 1 for vector and matrix valued functions. We shall use the same notations for these spaces and norms and scalar products on them. For example, for two vectors $u = (u_j)$ and $v = (v_j)$ in $\mathscr{E}^s_{L^2}[0, h]$, we define 53

$$(u(t),\ v(t)) = \sum_j (u_j(x,t),\ v_j(x,t))_s.$$

1 Energy inequalities for symmetric hyperbolic systems

Let $A_k(x,t)$ and $B(x,t)$ be matrices (of order N) of functions. Consider the following system of first order equations.

$$\frac{\partial}{\partial t}u - \sum A_k(x,t)\frac{\partial}{\partial x_k}u - B(x,t)u = f \tag{1.1}$$

where $A_k(x,t)$ are Hermitian matrices. Suppose that

$$A_k(x,t) \in \mathscr{B}^1[0,h], B(x,t) \in \mathscr{B}^0[0,h] \text{ and } f \in \mathscr{D}^0_{L^2}[0,h].$$

Proposition 1 (Friedrichs)**.** *Let u be a solution of* (1.1) *belonging to $\mathscr{D}^1_{L^2}[0,h]$. Then we have*

$$\|u(t)\| \leq \exp(\gamma t)\cdot\|u(0)\| + \int_0^t \exp(\gamma(t-s))\|f(s)\|ds \tag{1.2}$$

where γ is a constant depending only on the bounds of A_k, B.

Proof. Differentiating $||u(t)||^2 = u(t), u(t))$ with respect to t we have the identity

$$\frac{d}{dt}||u(t)||^2 = \left(\frac{du}{dt}(t), u(t)\right) + \left(u(t), \frac{du}{dt}(t)\right).$$

54 Since A_k are Hermitian matrices and since $u \in \mathscr{D}^1_{L^2}[0,h]$ we obtain from (1.1) the relation

$$\begin{aligned}
\left(u, \frac{du}{dt}\right) &= \sum_k \left(u, A_k \frac{\partial u}{\partial x_k}\right) + (u, Bu + f)\\
&= -\sum_k \left(\frac{\partial}{\partial x_k}(A_k u), u\right) + (u, Bu + f)\\
&= -\left\{\sum_k \left(A_k \frac{\partial u}{\partial x_k}, u\right) + \sum_k \left(\frac{\partial A_k}{\partial x_k} u, u\right)\right\} + (u, Bu + f).
\end{aligned}$$

Hence $$\begin{aligned}\frac{d}{dt}||u(t)||^2 &= -\sum_k \left(\frac{\partial A_k}{\partial x_k} \cdot u, u\right) + 2\,\mathrm{Re}(u, Bu + f)\\ &\leq 2\gamma||u||^2 + 2||u||||f||\end{aligned}$$

where γ is a constant depending only on the bounds of $\dfrac{\partial A_k}{\partial x_k}$ and B. Hence

$$\frac{d}{dt}||u(t)|| \leq \gamma||u(t)|| + ||f||$$

which on integration with respect to t yields the required inequality

$$||u(t)|| \leq \exp(\gamma t) \cdot ||u(0)|| + \int_0^t \exp(\gamma(t-s))||f(s)||ds.$$

The energy inequality involves the L^2-norm of the solution u of the system in the x-space. It is possible to derive the energy inequality under the weaker assumption that $u \in L^2(0,h]$. For this we use the method of regularization in the x-space of the function u by mollifiers introduced by Friedrichs. We recall the notion of mollifiers and a few of their properties which we need. □

Definition . *Mollifiers of Friedrichs.* Let $\varphi \in \mathscr{D}$ with its support contained in the unit ball $\{|x| < 1\}$ such that $\varphi(x) \geq 0$ and $\int \varphi(x)\, dx = 1$. 55
Then for a $\delta > 0$ define

$$\varphi_\delta(x) = \frac{1}{\delta^n}\varphi\left(\frac{x}{\delta}\right).$$

are called mollifiers. 56

Proposition 2 (Friedrichs). *Let $a \in \mathscr{B}^1$ and $u \in L^2$. Denote by C_δ the commutator defined by*

$$C_\delta u = \varphi_\delta * \left(a(x)\frac{\partial u}{\partial x_j}\right) - a(x)\left(\varphi_\delta * \frac{\partial u}{\partial x_j}\right) \tag{3.1}$$

$$= \left[\varphi_\delta *, a\frac{\partial}{\partial x_j}\right] u. \tag{1.3}$$

Than we have

(i) $||C_\delta u|| \leq c||u||$ where c is a constant depending only on φ and a

(ii) $C_\delta u \to 0$ in L^2 as $\delta \to 0$.

Before proving this proposition it will be useful to prove the following

Lemma 1. *If $u \in L^p$ then $\varphi_\delta * u \to u$ in L^p as $\delta \to 0$. More generally, if $u \in \mathscr{D}^m_{L^p}(m = 0, 1, \ldots)$ then $\varphi_\delta * u \to u$ in $\mathscr{D}^m_{L^p}$.*

Proof. Let $\psi_\delta = \varphi_\delta * u - u$. Since $\int \varphi_\delta(x) dx = 1$ we have

$$\psi_\delta(x) = \int \varphi_\delta(x-y)u(y)dy - u(x) = \int \varphi_\delta(x-y)(u(y) - u(x))dy.$$

□

If p' is such that $\frac{1}{p} + \frac{1}{p'} = 1$ by Hölder's inequality we have

$$|\psi_\delta(x)| \leq \left(\int \varphi_\delta(x-y)dy\right)^{1/p'} \left(\int \varphi_\delta(x-y)\big|u(y) - u(x)\big|^p dy\right)^{1/p}.$$

Here we use $\varphi_\delta = \varphi_\delta^{\frac{1}{p'}} \cdot \varphi_\delta^{\frac{1}{p}}$. Now since $\int \varphi_\delta(x-y)dy = 1$ we have $\int \left|\psi_\delta(x)\right|^p dx \leq \iint \varphi_\delta(x-y)|u(y)-u(x)|^p dx\, dy = \iint\limits_{|x-y|\leq\delta} \varphi_\delta(x-y)|u(y)-u(x)|^p dx\, dy$. By a change of variables $x' = x-y$ we obtain

$$\int |\psi_\delta(x)|^p dx \leq \int_{|x'|<\delta} \varphi_\delta(x')dx' \int |u(y)-u(x'-y)|^p dy.$$

57 If $\varepsilon(\delta)$ denotes $\sup\limits_{|h|\leq\delta} \int |u(y)-u(y+h)|^p dy$ then $\int |\psi_\delta(x)|^p dx \leq \varepsilon(\delta)$ which tends to 0 as $\delta \to 0$. The second part is an immediate consequence of this result since $\left(\frac{\partial}{\partial x}\right)^\nu (\varphi_\delta \star u) = \varphi_\delta \star \left(\frac{\partial}{\partial x}\right)^\nu u$ for $\nu| \leq m$ if $u \in \mathscr{D}^m_{L^p}$.

Proof of Proposition 2:

$$C_\delta u(x) = -\int \varphi_\delta(x-y)(a(x)-a(y))\frac{\partial u}{\partial y_j}(y)dy \tag{1.4}$$

where the integral on the right is taken in the sense of distributions. Now we have

$$C_\delta u = \int \frac{\partial}{\partial y_j}\{\varphi_\delta(x-y)(a(x)-a(y))\}\, u(y)dy \tag{1.5}$$

where the integral is taken in the usual sense. In fact the integral in (1.5) is equal to

$$-\int \frac{\partial a}{\partial y_j}(y)\varphi_\delta(x-y)u(y)dy + \int (a(x)-a(y)\frac{\partial \varphi_\delta}{\partial y_j}(x-y)u(y)dy;$$

we now note that

$$|a(x)-a(y)| \leq |a|_{\mathscr{B}}\delta_1|x-y|, \int |x-y||\frac{\partial\varphi_\delta}{\partial y_j}(x-y)|dx \leq c,$$

with c independent of δ. Thus it follows from the Hausdorff-Young theorem that the function represented by the above integral is majorized in the L^2-norm by $c_1|a|_{\mathscr{B}^1}\|u\|$. Now we see that the integration by parts

is justified. In fact, the two integrals are equal for $u \in \mathscr{D}$. Then for any
58 $u \in L^2$ the equality is proved by taking a sequence $u_j \varepsilon \mathscr{D}$ having for its limit u in L^2. Then $C_\delta u_j$ tends to the second integral in the sense of L^2. On the other hand $C_\delta u_j \to C_\delta u$ in the sense of distributions. This proves (i).

Since $(a(x) - a(y))\varphi_\delta(x - y)$ considered, for fixed x, as a function of y has compact support we see that

$$\int \frac{\partial}{\partial y_j} \{(a(x) - a(y))\varphi_\delta(x - y)\} \, dy = 0.$$

Hence

$$\begin{aligned} C_\delta u(x) &= \int \frac{\partial}{\partial y_j} \{(a(x) - a(y))\varphi_\delta(x - y)\} \, (u(y) - u(x))dy \\ &= -\int \frac{\partial a}{\partial y_j}(y)\varphi_\delta(x - y)(u(y) - u(x))dy \\ &\quad - \int (a(x) - a(y))\frac{\partial \varphi_j}{\partial x_j}(x - y)(u(y) - u(x))dy \\ &= \phi_1(x) + \phi_2(x), \text{ say.} \end{aligned}$$

Now as in the proof of lemma 1, we see that

$$\|\phi_i(x)\| \to 0 \text{ as } \delta \to 0 (i = 1, 2).$$

In fact, for instance,

$$|\phi_2(x)| \leq |a|_{\mathscr{B}^1} \int |x - y| |\frac{\partial \varphi_\delta(x - y)}{\partial x_j}| |u(y) - u(x)| dv.$$

Since $\int |x| |\frac{\partial \varphi_\delta}{\partial x_j}| dx \leq c$ (independent of δ) we obtain the desired property by the same reasioning as earlier. As an immediate consequence, we have

Corollary 1. *If we assume $a \in \mathscr{B}^m$ and $u \in \mathscr{D}^m_{L^2}$ in proposition 2 then*

(1) $\|C_\delta u\|_{\mathscr{D}^m_{L^2}} \leq c\|u\|_{\mathscr{D}^m_{L^2}}$, 59

(2) $C_\delta u \to 0$ *in* $\mathscr{D}^m_{L^2}$ *as* $\delta \to 0$, $m = 1, 2, \ldots$

Proposition 3 (Friedrichs)**.** *Let u be a solution of* (1.1) *belonging* $L^2[0, h]$ *then the inequality* (1.2)

$$\|u(t)\| \leq \exp(\gamma t)\|u(0)\| + \int_0^t \exp(\gamma(t-s))\|f(s)\|ds,$$

holds, where γ is the same constant as in prop. 1.

Proof. By regularizing u in the x-space by mollifiers φ_δ we obtain a function belonging to $\mathscr{D}^1_{L^2}[0, h]$ to which we can apply the Prop. 1. Let $u_\delta = \varphi_\delta *_{(x)} u$. Then

$$\frac{\partial u_\delta}{\partial t} = \frac{\partial}{\partial t}(\varphi_\delta *_{(x)} u) = \varphi_\delta *_{(x)} \frac{\partial u}{\partial t}.$$

Form the equation (1.1) we obtain the following equation for u_δ

$$\frac{\partial u_\delta}{\partial t} = \sum_k \varphi_\delta *_{(x)} \left(A_k \frac{\partial u}{\partial x_k}\right) + \varphi_\delta *_{(x)} Bu + \varphi_\delta * f,$$

that is

$$\frac{\partial u_\delta}{\partial t} = \sum_k A_k \frac{\partial u_\delta}{\partial x_k} + Bu_\delta + f_\delta + C_\delta u$$

$$\text{where } C_\delta u = \sum \left\{\varphi_\delta *_{(x)} (A_k \frac{\partial u}{\partial x_k}) - A_k(\varphi_\delta *_{(x)} \frac{\partial u}{\partial x_k})\right\}$$
$$+ \left\{\varphi_\delta *_{(x)} Bu - R(\varphi_\delta * u)\right\}$$
$$= \sum \left[\varphi_\delta *_{(x)}, A_k \frac{\partial}{\partial x_k}\right] u + [\varphi_\delta *_{(x)}, B]\, u.$$

Applying prop. 1 to the equation in u_δ we obtain since $u_\delta \subset \mathscr{D}'_{L^2}[0, h]$

$$\|u_\delta(t)\| \leq \exp(\gamma t)\|u_\delta(0)\| + \int \exp(\gamma(t-s)) \int \|f_\delta(s)\| + \|C_\delta(u)(s)\|ds.$$

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Now it follows from the Friedrichs lemma (Prop. 2) that

$$||(C_\delta u)(s)|| \leq c||u(s)||$$

where c is a constant independent of δ and $C_\delta u(s) \to 0$ as $\delta \to 0$. By Lebesgue's bounded convergence theorem it follows that

$$\int_0^t \exp(\gamma(t-s)) \cdot (||f_\delta(s)|| + ||(C_\delta u)(s)||)\, ds$$

tends to $\int_0^t \exp(\gamma(t-s))||f(s)||ds$. Thus passing to the limits as $\delta \to 0$ we obtain

$$||(u(t)|| \leq \exp(\gamma t)||u(0)|| + \int_0^t \exp(\gamma(t-s)||f(s)||ds.$$

□

2 Some remarks on the energy inequalities

In the previous section we obtained estimates for the solutions of symmetric hyperbolic systems in L^2-norm in terms of the L^2-norms of the initial values and of the second member. One can ask whether such estimates can be proved in the maximum norm and L^p-norm for $p \neq 2$. Littman [1] has proved that such an energy inequality cannot hold in the L^p-norm for $p \neq 2$. The existence of such an inequality with the maximum norms of functions and of their derivatives is related to the propagation of regularity, a form of Huygens principle for differentiablity. For instance, if $u(0)$ is m times continuously differentiable is $u(t)$ also m times continuously differentiable? In general an energy inequality in the maximum norm does not hold as we shall show by a counter example due to Sobolev. However, when the dimension of the x-space is one **61**
an inequality for solutions of strongly hyperbolic systems is valid in the maximum norm. This result is due to T. Haar. We indicate his result briefly.

Haar's inequality. Consider the system of equations of the first order

$$\frac{\partial u}{\partial t} - A(x,t)\frac{\partial u}{\partial x} - B(x,t)u = f \tag{2.1}$$

where the matrix $A(x,t)$ is such that det $(\lambda I - A)$ has real and distinct roots. Then we have the inequality

$$|u(t)|_0 \leq c(T)\left\{|u(0)|_0 + \sup_{0\leq t\leq T} |f(t)|_0\right\} \tag{2.2}$$

where $|u(t)|_0 = \sup_{x\in D_0} |u(x,t)|$, D being a neighbourhood of the origin and $D_0 = D \cap \{t = 0\}$.

In fact, let $\lambda_1(x,t), \ldots, \lambda_N(x,t)$ be the roots of det $(\lambda I - A) = 0$. $A(x,t)$ being diagonalizable there exists a non-singular matrix $N(x,t)$ such that

$$N(x,t)A(x,t) = D(x,t)N(x,t)$$

where $D(x,t)$ is the diagonal matrix

$$\begin{pmatrix} \lambda_1(x,t) & & 0 \\ & \ddots & \\ 0 & & \lambda_N(x,t) \end{pmatrix}$$

and such that $|\det N(x,t)| > \delta > 0$. We have the identity

$$\frac{\partial}{\partial t}(Nu) = \frac{\partial N}{\partial t}u + N\frac{\partial u}{\partial t}.$$

Substituting for $\frac{\partial u}{\partial t}$ from the given system the right hand side becomes

$$\begin{aligned}\frac{\partial N}{\partial t}u + N.A\frac{\partial u}{\partial x} + N.Bu + Nf &= \frac{\partial N}{\partial t}u + DN\frac{\partial}{\partial x}u + N.Bu + N.f \\ &= D\frac{\partial}{\partial x}.(Nu) + B_1u + N.f,\end{aligned}$$

62 where $B_1 = -D\frac{\partial N}{\partial x} + NB + \frac{\partial N}{\partial t}$. If B_2 denotes B_1N^{-1} then $v = Nu$

satisfies the system.

$$\frac{\partial v}{\partial t} = D\frac{\partial v}{\partial x} + B_2 v + Nf$$

which can be reduced to an integral equation of the Volterra type and then can be solved by successive approximation. Let (x_0, t_0) by any point in the (x, t)-plane. Let D be the domain enclosed by $(t = 0)$, the characteristic curves passing through (x_0, t_0) and having the maximum and minimum slopes. Let $D_0 = D \cap (t = 0)$. One can then show from the integral equation that

$$|u(x_0, t_0)| \leq c \left\{ \sup_{x \in D_0} |u(x, 0)| + \sup_{(x,t) \in D} |f(x, t)| \right\}$$

with a constant c independent of u.

That the energy inequality with the supremum norms does not hold in general in shown by the following counter example due to Sobolev.

Counter example (Sobolev). We consider the wave operator

$$\square \equiv \frac{\partial^2}{\partial t^2} - \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2} \tag{2.3}$$

in $\underline{R}^3$. We set

$$E_1(t, u) = \sup_x \left\{ \left| \frac{\partial u}{\partial t} \right| + \sum_j \left| \frac{\partial u}{\partial x_j} \right| \right\}.$$

We shall show that if $t_0 > 0$ then an inequality

$$E_1(t_0, u) \leq c E_1(0, u)$$

does not hold, which proves that the differentiability of the solution **63**
is not propagated in the t-direction. For this purpose, let $\Gamma(x, t)$ be a fundamental solution of $\square$ such that

$$\Gamma(x, 0) = 0, \ \frac{\partial \Gamma}{\partial t}(x, 0) = \delta$$

δ being the Dirac distribution. Let $\varphi \in \mathscr{D}$. For an $\in > 0$ define

$$\varphi^{(\in)}(x) = \varphi\left(\frac{x}{\in}\right).$$

Extending $\Gamma(x,t)$ to the whole space by setting

$$\begin{aligned}\tilde{\Gamma}(x,t) &= \Gamma(x,t) \text{ for } t \geq 0\\ &= -\Gamma(x,-t) \text{ for } t \leq 0.\end{aligned}$$

We obtain a distribution solution of $\dfrac{\partial^2}{\partial t^2} - \sum_j \dfrac{\partial^2}{\partial x_j^2}$ in the whole space $(-\infty < t < \infty) \times \underline{R}^3$. Setting

$$u_\in(x,t) = \tilde{\Gamma}(x, t - to) *_{(x)} \varphi^{(\in)}(x)$$

we obtain a solution of the homogeneous equation which satisfies

$$\frac{\partial}{\partial t} u_\in(x, to) = \frac{\partial\tilde{\Gamma}}{\partial t}(x, t_0 - to) *_{(x)} \varphi^{(\in)}(x) = \delta * \varphi^{(\in)}(x) = \varphi^{(\in)}(x)$$

and $\dfrac{\partial}{\partial x_j} u_\in(x, t_0) = \tilde{\Gamma}(x, 0) *_{(x)} \dfrac{\partial}{\partial x_j}\varphi^{(\in)}(x) = 0.$

Hence $E_1(t_0, u_\in) = \sup\limits_x |\varphi^{(\in)}(x)|$.

On the other hand we first observe that $\Gamma(x,t)$ can be taken to be $\dfrac{1}{4\pi t}\delta_{|x|-t}$. Let us choose $a\varphi \in \mathscr{D}$ with its support contained in the unit
64 ball in $\underline{R}^3$ such that $\varphi(0) = 1$ and $|\varphi(x)| \leq 1$. Then $|\dfrac{\partial}{\partial x_j}\varphi^{(\in)}(x)| \leq \dfrac{\gamma}{\in}$.
Thus for $t = 0$ we see that

$$\frac{\partial}{\partial t} u_\in(x, 0) = +\frac{\partial\Gamma}{\partial t}(x, t_0) *_{(x)} \varphi^{(\in)}(x) = \left[\frac{\partial}{\partial t}\left(\frac{t}{4\pi}\right) \int\limits_{|\xi|=1} \varphi^{(\in)}(x - t) ds_\xi)\right]_{t=t_0}$$

Now since $\displaystyle\int\limits_{B_\in(x_0)\cap\{|\xi|=1\}} ds_\xi = O(\epsilon^2)$ it follows that $\dfrac{\partial}{\partial t} u_\epsilon(x,0) = O(\epsilon)$.
We also have $\dfrac{\partial u_\epsilon}{\partial x_j}(x, 0) = O(\epsilon)$ and so $E_1(0, u_\epsilon) = O(\epsilon)$ which together with $E_1(t_0, u_\epsilon) = \sup |\varphi^{(\epsilon)}(x)| = 1$ shows that an energy inequality of the type $E_1(t_0, u_\epsilon) \leq cE_1(0, u_\epsilon)$ does not hold.

3 Singular integral operators

In this section we introduce the notion of singular integral operators and recall some of their properties which will be useful in the study of the existence and uniqueness of solutions of the Cauchy problem. The following considerations lead us to the notion of singular integral operators.

Consider the system of equations

$$\frac{\partial u}{\partial t} - \sum A_k \frac{\partial u}{\partial x_k} - Bu = f \tag{3.1}$$

where A_k and B are matrices whose entires are constants and $f \in L^2[0.h]$. We assume (3.1) to be strongly hyperbolic in the sense that the roots of the equation. $\det(\lambda I - A.\xi) = 0$ are real and distinct. Let the roots be $\lambda_1(\xi), \ldots, \lambda_N(\xi)$ for $\xi \neq 0$. We have the following.

Lemma 1. *There exists a non-singular matrix $N(\xi)$ which is homogeneous of degree zero and bounded such that*

(1) $|\det N(\xi)| \leq \delta > 0$ *for all* ξ.

(2) $N(\xi)(A.\xi) = D(\xi)N(\xi)$ *where* $D(\xi)$ *is the diagonal matrix*

$$D(\xi) = \begin{pmatrix} \lambda_1(\xi) & & 0 \\ & \ddots & \\ 0 & & \lambda_N(\xi) \end{pmatrix}$$

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Assume that there exists a solution $u \in L^2[0, h]$. Then, denoting for every fixed t, the Fourier transform of u in the x-space $\hat{u}(\xi, t)$, we obtain the following system of ordinary diffenential equations:

$$\frac{d}{dt}\hat{u}(\xi, t) = (2\pi i A.\xi + B)\hat{u}(\xi, t) + \hat{f}(\xi, t) \tag{3.2}$$

Multiplying both sides of this system by $N(\xi)$ and using lemma 1 we have

$$\frac{d}{dt}(N\hat{u})(\xi, t) = (2\pi i D(\xi) \cdot N(\xi) + N(\xi)B)\hat{u}(\xi, t) + N(\xi)\hat{f}(\xi, t).$$

$v(\xi, t) = N(\xi)\hat{u}(\xi, t)$ satisfies the system of equations

(3.3) $$\frac{dv}{dt}(\xi, t) = (2\pi iD(\xi) + B'(\xi))v(\xi, t) + N(\xi)\hat{f}(\xi, t), \quad \text{where}$$

$$B'(\xi) = N(\xi)BN(\xi)^{-1}. \quad \text{Now}$$

$$\begin{aligned}
\frac{d}{dt}\|v(\xi, t)\|^2 &= \int \left(\frac{dv}{dt} \cdot \overline{v} + v \cdot \overline{\frac{dv}{dt}}\right) d\xi \\
&= \int \Big\{ 2\pi(iD(\xi)v \cdot \overline{v} + v \cdot \overline{iD(\xi)v}) + 2\,\mathrm{Re}(B'v, \overline{v}) \\
&\qquad + 2\,\mathrm{Re}\, N(\xi)\overline{f} \cdot \overline{v} \Big\} d\xi \\
&= 2 \int \mathrm{Re}(B'(\xi)v \cdot \overline{v} + N(\xi)\hat{f} \cdot \overline{v})(\xi, t) d\xi.
\end{aligned}$$

Because $N(\xi)$ is bounded and condition (1) of lemma 1 holds. The operators B' is bounded and hence

$$\begin{aligned}
\frac{d}{dt}\|v(\xi, t)\|^2 &\leq 2\gamma\|v\|^2 + 2\,\mathrm{Re}(N(\xi)\hat{f}, v) \\
&\leq 2\gamma\|v\|^2 + 2\|N(\xi)\hat{f}\|\|v\|
\end{aligned}$$

66 Thus we obtain

$$\|v\| \leq \exp(\gamma t) \cdot \|v(\xi, 0)\| + \int_0^t \exp(\gamma(t - s))\|N(\xi)\hat{f}(\xi, s)\| ds.$$

By Plancheral's formula's formula we have

$$\|v(\xi, t)\| = \|N(\xi)\hat{u}(\xi, t)\| \leq c\|u(t)\|.$$

and again since $N(\xi)$ has a bounded inverse by condition (1) we see that

(3.4) $$\|u(t)\| \leq c(h)\Big\{\|u(0)\| + \int_0^t \|f(s)\| ds\Big\}$$

where c is a constant depending only on h.

Now we look at this reasoning without explicitly using the notion of Fourier transforms.

$N(\xi)$ is homogeneous of degree 0 in ξ and so the convolution operators $\mathscr{N}(x)*$ defines a bounded operator in the space L^2 since

$$||\mathscr{N}(x) * u|| = ||N(\xi)\hat{u}|| \leq c||u||$$

by Plancherel's formula. Here $\mathscr{N}(x)$ is the inverse Fourier image of $N(\xi)$. Let $\mathscr{D}(x)$ be the distribution whose Fourier image is $D\left(\dfrac{\xi}{|\xi|}\right)$. Define the operators $\wedge$ by

$$(\widehat{\wedge u}) = |\xi|\hat{u}.$$

Then we obtain

$$\begin{aligned} \frac{d}{dt}(\mathscr{N}(x) *_{(x)} u) &= 2\pi i \mathscr{D}(x) *_{(x)} \wedge(\mathscr{N}(x) *_{(x)} u) \\ &\quad + \mathscr{N}(x) *_{(x)} (Bu) + \mathscr{N}(x) *_{(x)} f. \end{aligned}$$

In other words $v = \mathscr{N} *_{(x)} u$ satisfies the system

$$\frac{dv}{dt} = 2\pi i \mathscr{D} *_{(x)} \wedge v + B_1 v + \mathscr{N} *_{(x)} f,$$

where $B_1 \in \mathscr{L}(L^2, L^2)$ because of condition (1). Integrating with respect to t in the interval $[0, t]$ we have the inequality 67

$$||\mathscr{N} *_{(x)} u|| \leq \exp(\gamma t)||\mathscr{N} *_{(x)} u(x,0)|| + \int_0^t \exp(\gamma(t-s))||\mathscr{N} *_{(x)} f(x,s)||ds$$

where γ is a constant depending only on A and B. But there exists a constant k (depending on A) such that

$$\frac{1}{k}||u(x,t)|| \leq ||\mathscr{N} *_{(x)} u(x,t)|| \leq k||u(x,t)||$$

which gives an energy inequality for u.

Now in the case of systems with variable coefficients even though we cannot apply Fourier transforms we may, however, write the system in a form similar to (3.2) to which we can apply the above method to get an energy inequality. For this purpose we introduce the singular integral operators.

4

For a function $f \in L^2(\underline{R}^1)$ consider the integral transform defined by

$$g(x) = v.p. \int_{-\infty}^{\infty} \frac{f(t)}{x-t} dt. \tag{4.1}$$

M. Riesz [1] has proved that the Cauchy principal value defining g exists and $g \in L^2(\underline{R}^1)$. $f \to g$ is a continuous linear mapping of $L^2(\underline{R}^1)$ into itself. In the language of the theory of distributions we can write
68 $g = v.p.\left(\frac{1}{x}\right) * f$. $v.p.\left(\frac{1}{x}\right)$ is a tempered distribution whose Fourier image is $\chi(\xi) = -\pi i$ for $\xi > 0$ and πi for $\xi < 0$. We observe that $\frac{1}{x}$ is homogeneous of degree -1 and has mean value 0. If $\hat{g}$ and $\hat{f}$ are the Fourier images of g and f respectively then $\hat{g} = \partial\chi\hat{f}$ and $||g = \pi||f||$ by Plancheral's formula.

Calderon and Zygmund [1] generalized this theory to functions on $\underline{R}^n$. Let $N(x)$ be a homogeneous function of degree -n on $\underline{R}^n(N(\lambda x) = \lambda^{-n}N(x))$ which is smooth in the complement of the origin and has mean value $\int_{|x|=1} N(x)d\sigma_x = 0$. Then they proved that $g = v.p.N(x) * f \in L^p$ if $f \in L^p$. In particular $f \to g$ is a continuous linear map of L^2 into itself. This latter fact can be seen observing that $v.p.N(x)$ is a tempered distribution, its Fourier transform $h(\xi)$ is a homogeneous function of degree 0 and has mean value $\int_{|\xi|=1} h(\xi)d\sigma_\xi = 0$. In this paragraph $d\sigma_x$ and $d\sigma_\xi$ stand for normalized volume element of the unit sphere; viz. $d\sigma_x = dS_x/\text{vol } S$.

Conversely, given any homogeneous function $h(\xi)$ of degree 0 with mean value 0, if $\gamma(x)$ is its inverse Fourier image we can define an integral operators $\gamma *$ by

$$(\gamma * f)(x) = \int \exp(2\pi i x \cdot \xi) h(\xi) \hat{f}(\xi) d\xi.$$

Now consider the differential operators

$$L\left(x, \frac{\partial}{\partial x}\right) = \sum a_j(x) \frac{\partial}{\partial x_j}.$$

For a function $u \in \mathscr{S}$ we can write

$$(Lu)(x) = \int \exp(2\pi i x\xi)(\sum_j a_j(x)\xi_j)(2\pi i)\hat{u}(\xi)d\xi.$$

Denote $h(x, \xi) = 2\pi i \sum_j a_j(x)\xi_j/|\xi|$. If we define 69

$$(Hf)(x) = \int \exp(2\pi i x.\xi)h(x, \xi)\hat{f}(\xi)d\xi$$

H will be a bounded operator in L^2. In fact, H can be written

$$Hf = 2\pi i \sum a_j(x)(R_j * f) \tag{4.2}$$

where R_j is the inverse Fourier image of $^{\xi} j/|\xi|$. It follows that

$$\| Hf \| \leq 2\pi \sum |a_j(x)|_0 \| R_j * f \| \leq (2\pi \sum |a_j|_0) \| f \|.$$

Now L can be written in the form

$$Lu = H \wedge u.$$

We introduce the notation used by Calderon-Zygmund [1], [2].

Let U be an open set in $\underline{\mathrm{R}}^n$. A function u defined on U is said to satisfy a uniform Holder condition of order $\beta(0 \leq \beta \leq 1)$ if for any x, $x' \in U$ we have

$$|u(x) - u(x')| \leq c|x - x'|^{\beta}. \tag{4.3}$$

c is called the Hölder constant for u. We shall denote by $C_\beta(U)$, $\beta \geq 0$, the class of complex valued continuous bounded functions on U with bounded continuous derivatives upto order $[\beta]$ (the integral part of β) and with the derivatives of order $[\beta]$ satisfying a Hölder condition of

order $\beta - [\beta]$. $\mathscr{E}_\xi(\underline{R}^n - \{0\})$ will denote the space consisting of complex valued functions $h(\xi)$, $\xi \in \underline{R}^n$, homogeneous of degree 0 and infinitely differentiable in $\underline{R}^n - \{0\}$ with respect to ξ. This space $\mathscr{E}_\xi(\underline{R}^n - \{0\})$ is 70
topologized by the family of seminorms defined by

$$p_s(h) = \sum_{|\nu|\leq s} \sup_{|\xi|\geq 1} |(\frac{\partial}{\partial\xi})^\nu h(\xi)|.$$

We say that $h(x,\xi) \in C^\infty_\beta$, $\beta \geq 0$, if

(1) for $\beta = 0$ the function $x \to h(x,\xi) \in \mathscr{E}_\xi(\underline{R}^n - \{0\})$ is continuous and bounded;

(2) for $0 < \beta < 1$, $h(x,\xi) \in C^\infty_0$ and the function $x \to h(x,\xi) \in \mathscr{E}_\xi(\underline{R}^n - \{0\})$ is uniformly Hölder continuous of order β in the sense that for any ν

$$\sup_{|\xi|\geq 1} \left| \left(\frac{\partial}{\partial\xi}\right)^\nu h(x,\xi) - \left(\frac{\partial}{\partial\xi}\right)^\nu h(x',\xi) \right| \leq c_\nu |x - x'|^\beta; \tag{4.4}$$

(3) if $\beta \geq 1$, $\left(\frac{\partial}{\partial x}\right)^\nu h(x,\xi) \in C^\infty_0$ for $|\nu| \leq \beta$ and $\left(\frac{\partial}{\partial x}\right)^\nu h(x,\xi) \in C^\infty_{\beta-[\beta]}$ for $|\nu| = [\beta]$.

$h(x,\xi)$ being a homogeneous function of ξ can be expanded as a series in spherical harmonics. Let $Y_l(\xi)$ be a normalized real spherical harmonic of degree l, that is such that

$$\int_{|\xi|=1} Y_l(\xi)^2 d\sigma_\xi = 1 \tag{4.5}$$

and $Y_{lm}(\xi)$ be a complete orthogonal system of normalized spherical harmonics of degree l. Then we can write

$$h(x,\xi) = a_0(x) + \sum_{l\geq 1,m} a_{lm}(x) Y_{lm}(\xi) \tag{4.6}$$

in terms of the spherical harmonics. Then

$$a_{lm}(x) = \int\limits_{|\xi|=1} h(x,\xi) Y_{lm}(\xi) d\sigma_\xi. \tag{4.7}$$

Let $\widetilde{Y}_{lm}$ denote the inverse Fourier image of $Y_{lm}(\xi)$ 71

$$\widetilde{Y}_{lm}(x) = \int e^{2\pi i x.\xi} Y_{lm}(\xi) d\xi.$$

We define

$$(Hf)(x) = a_0(x) f(x) + \sum_{l,m} a_{l,m}(x)(\widetilde{Y}_{lm} * f)(x). \tag{4.8}$$

Now we have the following estimates due to Calderon and Zygmund:

(a) $|Y_{lm}(\xi)| \leq c l^{\frac{1}{2}(n-2)}$, c being a positive constant;

(b) the number of distinct spherical harmonics $Y_{lm}(\xi)$ of degree l is of the order l^{n-2};

(c) $|a_{lm}(x)| \leq cMl^{-\frac{3}{2}n}$ where $M = \sup\limits_{\substack{x \in \underline{\mathbb{R}}^n, |\xi| \geq 1 \\ |\nu| \leq 2n}} |(\frac{\partial}{\partial \xi})^\nu h(x,\xi)|$.

More generally we have the following sharper estimates. Let L be the operator defined by

$$L(F) = |\xi|^2 (\triangle_\xi F) \text{ where } \triangle_\xi = \sum_{j=1}^{n} \left(\frac{\partial}{\partial \xi_j}\right)^2.$$

Then

$$a_{lm}(x) = (-1)^r l^{-r} (l+n-2)^{-r} \int\limits_{|\xi|=1} L^r_\xi (h(x,\xi) Y_{lm}(\xi) d\sigma_\xi. \tag{4.7$'$}$$

From this it follows that

(d) $|a_{lm}(x)| \leq c(n,r)M_{2r} l^{-2r+\frac{n}{2}}$

where $M_{2r} = \sup_{\substack{x\in \underline{R}^n, |\xi|\geq 1\\ |\nu|\leq 2r}} |(\frac{\partial}{\partial\xi})^\nu h(x,\xi)|$

72 (e) $\sup_{|\xi|\geq 1} |(\frac{\partial}{\partial \xi})^\nu Y_{lm}(\xi)| \leq c(\nu,n) l^{\frac{1}{2}(n-2)+|\nu|}$.

These estimate show that the series defining Hf is convergent in the L^2-sense.

In fact,

(4.9) $$\| Hf \| \leq (|a_0(x)| + \sum |a_{lm}(x)|_o |y_{lm}(\xi)|_o) \parallel f \parallel .$$

From (a), (b) and (c) it follows that

$$\sum |a_{lm}|_o |Y_{lm}|_o \leq cM \sum_l l^{-\frac{3}{2}n+\frac{1}{2}(n-2)+n-2} = cM \sum_l l^{-3} < \infty.$$

Hence

$$\| H \| \leq cM, M \quad \text{being defined in (c).}$$

A singular integral operator was defined by Calderon and Sygmund by the following equation

(4.10) $$(Hu)(x) = a(x)u(x) + \int k(x, x-y)u(y)dy,$$

where $k(x,z)$ is a complex valued homogeneous function of degree $-n$ in z, of class $\mathscr{E}$ in $\underline{R}^n - \{0\}$ in the z-variable for every fixed x and the function $k(x,z)$ has mean value zero in the z-space for every fixed x. Let us expand $k(x,z)$ in terms of spherical harmonics:

$$k(x,z) = \sum a_{lm}(x) Y_{lm}(z')|z|^{-n}, \quad z' = \frac{z}{|z|},$$

where $a_{lm}(x) = \int_{|z'|=1} k(x,z')d\sigma_{z'}$.

73 Then, taking into account the fact that $\mathscr{F}[Y_{lm}(z')|z|^{-n}] = \gamma_1 Y_{lm}(\xi)$,

γ_1 being a constant, we define the symbol $\sigma(H)$ as

$$\sigma(H) = a_0(x) + \sum a_{lm}(x)\gamma_1 Y_{lm}(\xi). \tag{4.11}$$

We start from this $\sigma(H)$ in our definition. However the two definitions are identical since there exists a one to one linear mapping σ of the class of singular integral operators of the class C^∞_β into the class of functions $h(x,\xi)$, x, $\xi \in \underline{R}^n$ homogeneous of degree zero with respect to ξ and in C^∞_β. $\sigma(H)$ is called the symbol of the singular integral operator H. Thus the series $\sum\limits_{l,m} a_{lm}(x)Y_{lm}(\xi)$ represents in a sense the Fourier transform of $k(x,z)$ with respect to z. We recall without proof the following important theorems on these operators, which we shall require for later use. For proofs see Calderon-Zygmund [1, 2].

Theorem 1 (Calderon-Zygmund [1]). *If H is a singular integral operator of type C^∞_β then its symbol is a homogeneous function of degree zero and of class C^∞_β with respect to ξ in $|\xi| \geq 1$. Conversely every function of x and ξ which is homogeneous of degree zero and belongs to the class C^∞_β in $|\xi| \geq 1$ is the symbol of a unique singular integral operator of type C^∞_β. If*

$$M = \sup_{\substack{x\in \underline{R}^n, |\xi|\geq 1\\ |\nu|\leq 2n}} \left|\left(\frac{\partial}{\partial \xi}\right)^\nu \sigma(H)(x,\xi)\right|$$

then

$$\| Hf \|_p \leq MA_p \| f \|_p \tag{4.12}$$

where A_p depends only on p and n. 74

If $h_1(x,\xi), h_2(x,\xi)$ are of class $C^\infty_\beta in |\xi| \geq 1$ then it is easy to see that $h_1(x,\xi) + h_2(x,\xi)$ and $h_1(x,\xi)h_2(x,\xi)$ are also of class C^∞_β and further if $|h_2(x,\xi)| \geq \delta > 0$ then $\dfrac{h_1(x,\xi)}{h_2(x,\xi)}$ is also of class C^∞_β.

Theorem 2 (Calderon-Zygmund [2]). *Let $h(x,\xi) = \sigma(H)$ be of type C^∞_β, homogeneous of degree zero in $\mathscr{E}$ then*

(1) *for $r \leq \beta$, $Hf \in \mathscr{D}^r_{L^p}$ for $f \in \mathscr{D}^r_{L^p} (1 < p < \infty)$, and*

(2) *if $f \in L^p$ and Hölder continuous of order $\alpha(\alpha < \beta$ then $Hf \in L^p$ and Hölder continuous of order $\alpha (1 < p < \infty)$.*

Let $H^\#$ and $H_1 0 H_2$ be singular integral operators whose symbols are respectively $\overline{\sigma(H)}$ and $\sigma(H_1)$. $\sigma(H_2)$.

Theorem 3 (Calderon-Zygmund)**.** *If $\sigma(H_1)$, $\sigma(H_2)$ are independent of x then*

$$H_1 \circ H_2 = H_1 H_2 = H_2 \circ H_1 = H_2 H_1$$

and if $\sigma(H)$ is independent of x and $|\sigma(H)(\xi)| \geq \delta > 0$ then H is invertible and its inverse H^{-1} is also a singular integral operator. We illustrate by a simple example the motivation for the definition of the singular integral operators $H_1 \circ H_2$ and $H^\#$. Consider the differential operators

$$L = \sum_j a_j(x) \frac{\partial}{\partial x_j}, M = \sum_j b_j(x) \frac{\partial}{\partial x_j}, a_j, b_j \in \mathscr{B}^1.$$

Then

$$LM = \sum_{j,k} a_j(x) b_k(x) \frac{\partial^2}{\partial x_j \partial x_k} + \sum_{j,k} a_j(x) \frac{\partial b_k}{\partial x_j} \frac{\partial}{\partial x_k}.$$

75 Therefore, if we define

$$L \circ M = \sum_{j,k} a_j(x) b_k(x) \frac{\partial^2}{\partial x_j \partial x_k}$$

then $LM = L \circ M$ modulo first order operators. Next if we define

$$L^\# = - \sum \overline{a_j(x)} \frac{\partial}{\partial x_j}$$

then $L^* \equiv L^\#$ modulo bounded operators.

These considerations suggest that the product of two singular integral operators and the conjugate operator H^* will be approximated, in some sense, by the singular integral operators $H_1 \circ H_2$ and $H^\#$ respectively. More precisely we have the following:

Theorem 4 (Calderon-Zygmund). *Let H be a singular integral operator of type $C^\infty_\beta(\beta > 1)$ and M be a bound for $\sigma(H)(x,\xi)$ and its derivatives with respect to the coordinates of ξ of order $2n$, the first derivatives of these with respect to the coordinates of x and Hölder constants of the latter. Then for every $f \in \mathscr{D}^1_{L^p}(1 < p < \infty)$ we have*

$$\begin{aligned} &\| (H\wedge - \wedge H)f \|_{L^p} \leq A_p M \| f \|_{L^p}, \ \| (H^* \wedge - \wedge H^*)f \|_{L^p} \leq A_p M \| f \|_{L^p} \\ &\| (H^* - H^\#)f \|_{L^p} \leq A_p M \| f \|_{L^p}, \| \wedge (H^* H^\#) \|_{L^p} \leq A_p M \| f \|_{L^p} \end{aligned} \tag{4.13}$$

where A_p depends only on p, n, β. Further if H_1 and H_2 are two singular integral operators of type C^∞_β and $f \in \mathscr{E}^1_{L^p}(1 < p < \infty)$ then $H_1 \circ H_2$ is an operator of type C^∞_β and

$$\begin{aligned} &\| (H_1 \circ H_2 - H_1H_2)\wedge f \|_{L^p} \leq A_p M_1 M_2 \| f \|_{L^p}, \\ &\| \wedge(H_1 \circ H_2 - H_1H_2)f \|_{L^p} \leq A_p M_1 M_2 \| f \|_{L^p} \end{aligned} \tag{4.14}$$

where again A_p depends only on p, n, β and M_1, M_2 being defined in the same way as M. 76

We can write differential operators in the form of singular integral operators as follows: Let $A = \sum_{|\alpha|=m} a_\alpha(x)\left(\frac{\partial}{\partial x}\right)^\alpha$ be a homogeneous differential operator of order m with coefficients $a_\alpha(x)$ in C_β, $\beta \geq 0$. If $u \in \mathscr{D}^m_{L^2}$ then $\wedge^m u$ is well defined,

$$(\widehat{\wedge^m u})(\xi) = |\xi|^m \hat{u}(\xi)$$

and $Au = H\wedge^m u$ where H is a singular integral operator of type C^∞_β and

$$\sigma(H) = i^m \sum_{|\propto|=m} a_\propto(x)\xi^\propto |\xi|^{-m}. \tag{4.15}$$

Similarly any general linear differential operator of order m

$$A = \sum_{k\leq m} A_k, A_k = \sum_{|\nu|=k} a_{k,\nu}(x)\left(\frac{\partial}{\partial x}\right)^\nu$$

with $a_{k,\nu}(x)$ of class C_β can be written as

$$Au = \sum H_k \wedge^k u \tag{4.16}$$

where H_k is a singular integral operator of class C^∞_β and

$$\sigma(H_k) = i^k \sum_{|\nu|=k} a_{k,\nu}(x)\xi^\nu |\xi|^{-k} \tag{4.17}$$

for every $u \in \mathscr{D}^m_{L^2}$.

A matrix of operators is called a singular integral matrix if its elements are singular integral operators and its symbol is the matrix whose
77 elements are the symbols of the corresponding elements of the singular integral matrix. A system of differential operators can be written as a singular integral matrix.

5 Extension of Gårding's inequality to singular integral operators

In this section we prove an inequality for the singular integral operators whose symbol satisfies a condition of positivity. This is an analogue of the well know inequality of Garding for elliptic differential operators. Before stating the inequality we prove some preliminary results needed in the proof of this inequality. These results are also of independent interest.

The following lemma corresponds to the local property of differential operators, namely, that differential operators decrease supports.

Lemma 1 (Quasi localisation lemma)**.** *Let Ω be the ball of radius 2η and of centre a point x_0 in $\underline{R}^n$. Let H be a singular integral operator whose symbol $\sigma(H)(x, \xi) \in C^\infty_\beta$, with $\beta > 0$. If $u \in \mathscr{D}^1_{L^2}$ has its support in the ball of radius η and of centre x_0 then*

$$\| H \wedge u \|_{L^2(C\Omega)} \leq c(n, \eta) M' \| u \| \tag{5.1}$$

where $M' = \sum_{|\nu| \leq 3n+3} \sup_{x \in \underline{R}^n, |\xi| \geq 1} \left|(\frac{\partial}{\partial \xi})^\nu \sigma(H)(x, \xi)\right|$ *and $c(n, \eta)$ is a constant depending only on n and η.*

Proof. We decompose the operator $\wedge$ as $\wedge = \wedge_1 + \wedge_2$ with $\widehat{\wedge_1}(\xi) = \alpha(\xi)|\xi|$ and $\widehat{\wedge_2}(\xi) = (1 - \alpha(\xi))|\xi|$ where $\alpha(\xi) \in \mathscr{D}$ such that $\alpha(\xi) \equiv 1$ on

78 $|\xi| \leq 1, 0 \leq \alpha(\xi) \leq 1$ and vanishes outside $|\xi| \leq 2$. $\widehat{\wedge}_2(\xi)$ is an infinitely differentiable function, $\widehat{\wedge}_1(\xi)$ has compact support and hence $\wedge_1$ is a bounded operator in L^2. So it is enough to prove that

$$\| H \wedge_2 u \|_{L^2(C\Omega)} \leq C(n, \eta) M' \| u \| .$$

Let

$$\sigma(H)(x, \xi) = a_0(x) + \sum_{l,m} a_{lm}(x) Y_{lm}(\xi) \tag{5.2}$$

be the expansion of $\sigma(H)$ in terms of a complete system of spherical harmonics $Y_{lm}(\xi)$. Let $Y'_{lm}(x)$ be the singular integral operator such that

$$Y'_{lm}(x) \to Y_{lm}(\xi)\widehat{\wedge}_2(\xi)$$

by Fourier transforms. Then we can write

$$(H \wedge_2 u)(x) = a_0(x) \wedge_2 u(x) + \sum_{l,m} a_{lm}(x)(Y'_{lm}(x) * u). \tag{5.3}$$

First we show that

$$|Y'_{lm}(x) \leq |x|^{-2p} c(p, n) |Y_{lm}(\xi)|_{2p} \text{ for } x \in^c \{0\} \text{ for } 2p \geq n + 2 \tag{5.4}$$

where $|Y_{lm}(\xi)|_{2p} = \sum\limits_{|\nu| \leq 2p} \sup\limits_{|\xi| \geq 1} |(\dfrac{\partial}{\partial \xi})^\nu Y_{lm}(\xi)|$.

In fact,

$$Y'_{lm}(x) = |x|^{-2p} \left\{ |x|^{2p} Y'_{lm}(x) \right\}$$

and $|x|^{2p} Y'_{lm}(x)$ is the inverse Fourier image of const

$$\triangle^p_\xi (Y_{lm}(\xi)(1 - \alpha(\xi))|\xi|.$$

Hence we have the estimate

$$\begin{aligned} |Y'_{lm}(x)| &\leq |x|^{-2p} \left(\frac{1}{2\pi} \right)^p \int |\triangle^p_\xi ((1 - \alpha(\xi)) Y_{lm}(\xi)|\xi|)| d\xi \\ &\leq |\xi|^{-2p} c(n, p) |Y_{lm}(\xi)|_{2p} for 2p \geq n + 2. \end{aligned}$$

This establishes the assertion (5.4). 79

Now we show that for any $u \epsilon \mathscr{D}$ with support contained in $\omega = B_\eta(x_0)$

(5.5) $$\| |x|^{-2p} * u\|_{L^2(C\Omega)} \leq c(n, p, \eta)\|u\|$$

holds for p satisfying $4p > n$.

In fact, for $x \epsilon \complement \Omega$, $\| |x|^{-2p} * u\| = \left| \int \frac{u(y)}{|x-y|^{2p}} dy \right|$ by Schwarz inequality,

$$\leq \|u\|(\text{ vol } \omega)^{1/2}(\text{ dist. } (x, \omega))^{-2p}$$

Hence $\| |x|^{-2p} * u\|_{L^2(^C\Omega)} \leq (\text{ vol}\omega)^{\frac{1}{2}} \|u\|(\int\limits_{|x| \geq 2\eta} \frac{dx}{(|x| - \eta)^{4p}})^{\frac{1}{2}}$. The integral in the right hand side converges for $4p > n$ which proves the assertion (5.5). Now (5.4) and (5.5) together assert that

$$\begin{aligned} \|H \wedge_2 u\|_{L^2(C_\Omega)} &\leq (\text{ vol } \omega)^{\frac{1}{2}} c(p, n, \eta) \left(\sum_{l,m} |a_{lm}(x)|_\circ |Y_{lm}(\xi)|_{2p} \right) \|u\| \\ &\leq C'(p, n, \eta) M' \|u\|. \end{aligned}$$

This completes the proof of lemma 1. In the proofs of the following results we use a C^∞ partition of unity in $\underline{\mathrm{R}}^n$.

$$\alpha_j \xi \mathscr{D}, \quad \alpha_j \geq 0, \quad \sum_j \alpha_j^2 = 1.$$

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To simplify the arguments we take a partition of unity satisfying the following conditions: Let $\alpha_0 \varepsilon \mathscr{D}$ whose support is contained in the ball of redius ε, ε being a small number to be determined by the singular integral operator H. Let $\{x^{(j)}\}$ be a sequence of points of $\underline{\mathrm{R}}^n$ whose coordinates are multiples of $\varepsilon' (= \varepsilon n^{-\frac{1}{2}})$, $\alpha_j(x) = \alpha_0(x - x^{(j)})$, $j = 0, 1, \ldots, x^{(0)} = (0)$. The support of α_0 will be denoted by ω_0 and the ball of centre $x^{(j)}$ and of radius 2ε will be denoted by Ω_j. Let

$$\alpha(p) = \sum_{|\nu| \leq p} \sup_x \left| \left(\frac{\partial}{\partial x} \right)^\nu \alpha_0(x) \right|.$$

□

Lemma 2. *Let H be a singular integral operator with its symbol $\sigma(H)$ $(x, \xi) \varepsilon c^{\infty}_{\beta}$, with $\beta > 0$ and (α_j) be a C^{∞} partition of unity as constructed above. Then for any $u \varepsilon \mathscr{D}^1_L 2$*

$$\sum_j \|((H\wedge)\alpha_j - \alpha_j(H\wedge))u\|^2 \leq \gamma \|u\|^2 \tag{5.6}$$

In particular, taking $\sigma(H) = 1$ this wouls imply

$$\sum_j \|[\wedge, \alpha_j]u\|^2 \leq \gamma \|u\|^2. \tag{5.7}$$

Let $\beta \varepsilon \mathscr{D}_{\xi}$, $0 \leq \beta(\xi) \leq 1$ with support contained in $|\xi| < 1$ which takes the value 1 in a neighbourhood of the origin. Decompose $\wedge$ into $\wedge = \wedge_1 + \wedge_2$ where $\hat{\wedge}_1(\xi) = \beta(\xi)|\xi|$ and $\hat{\wedge}_2(\xi) = (1 - \beta(\xi))|\xi|$. Clearly $\| \wedge_1 u\| \leq \|u\|$ and hence

$$\|H \wedge_1 \alpha_j u\| \leq \|H\| \, \|\alpha_j u\|_{L^2(\Omega_j)} \leq \sup_x \left|\alpha_j(x)\right| \|H\| \;\; \|u\|_{L^2(\Omega_j)}$$

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Hence

$$\sum_j \|H \wedge_1 \alpha_j u\|^2 \leq \alpha(0)^2 \|H\|^2 k \|u\|^2$$

where k is the maximum number of sets $\{\omega_h\}$ intersecting at any point and

$$\sum_j \|\alpha_j H \wedge_1 u\|^2 = \|H \wedge_1 u\|^2 \leq \|H\|^2 \|u\|^2.$$

So we have only to consider $\sum_j \|[H\wedge_2, \ \alpha_j]u\|$. Consider the term

$$\varphi_j(x) = [H\wedge_2, \alpha_j]u(x). \tag{5.8}$$

$$\varphi_j(x) = \sum_{l,m} a_{lm}(x) \int \tilde{Y}_{lm}(x - y) \wedge_2 (x - y)(\alpha_j(y) - \alpha_j(x))u(y)dy.$$

Let us denote the operator $\tilde{Y}_{lm} * \wedge_2$ by Y'_{lm}. First of all we consider $\|\varphi_j\|_{L^2(\Omega_j)}$. Expanding $\alpha_j(y) - \alpha_j(x)$ in a Taylor series, we obtain
(5.9)
$$\alpha_j(y) - \alpha_j(x) = \sum_{1\le|\nu|\le q-1} \frac{1}{\nu!}\left(\frac{\partial}{\partial x}\right)^{\nu} \alpha_j(x)(y-x)^{\nu} + \sum_{|\nu|=q} \alpha_{j,\nu}(x,y)(x-y)^{\nu}$$

where q will be determined later. It follows that

$$\varphi_j(x) = \sum_{|\nu|\le q-1} \frac{1}{\nu!}(-1)^{|\nu|}(\frac{\partial}{\partial x})^{\nu}\alpha_j(x) \sum_{l,m} a_{lm}(x)(x^{\nu} Y'_{lm})u + \varphi_j^{(2)}(x)$$

where

$$\varphi_j^{(2)}(x) = \Sigma a_{lm}(x) \int \alpha_{j,\nu}(x,y)(x-y)^{\nu} Y'_{lm}(x-y)u(y)dy. \tag{5.10}$$

82 Now the operators $H_\nu = \sum a_{lm}(x)(x^{\nu} Y'_{lm})$ are singular integral operators which operate on L^2 as continuous linear operators since $\sup_x |a_{lm}(x)|$ is a rapidly decreasing sequence (more precisely, for any positive integer σ we have

$$\sum_{1\ge 0} l^{\sigma} \sup_x \left|a_{lm}(x)\right| < \infty) \text{ (see Calderon-Zygmund [1].}$$

Hence for the first sum,

$$\varphi_j^{(1)}(x) = \sum_{|\nu|\le q-1} \frac{(-1)^{|\nu|}}{\nu!}(\frac{\partial}{\partial x})^{\nu}\alpha_j(x)\cdot H_\nu u \tag{5.11}$$

and we have

$$\left\|\varphi_j^{(1)}\right\|^2_{L^2(\Omega_j)} \le c(q)\alpha(q-1) \sum_{1\le|\nu|q-1} \left\|H_\nu u\right\|^2_{L^2(\Omega_j)}. \tag{5.12}$$

To majorize the second sum $\varphi_j^{(2)}(x)$ we begin by considering a typical term $(x^{\nu}\cdot Y') * u$. We have

$$|(x^{\nu} Y') * u| = \Big|\int (x-y)^{\nu} Y'(x-y)\cdot u(y)\,|dy\Big|$$

$$\leq \int |(x-y)^{\nu} Y'(x-y)||u(y)|dy$$

$$= \left(\int_{\Omega'_j} + \int_{C\Omega'_j} \right) |(x-y)^{\nu} Y'(x-y)||u||(y)|dy$$

where Ω'_j is a sphere of radius 6ε about $x^{(j)}$. The first integral is majorized by

$$\sup_x |x^{\nu} Y'(x)| \ \ ||u||_{\Omega'_j} (\text{vol } \Omega'_0)^{\frac{1}{2}}$$

and the second integral is majorized by

$$\sup_x \left| |x|^{2p} x^{\nu} Y'(x) \right| \int_{C\Omega'_j} \frac{|u(y)|}{|x-y|^{2p}} dy.$$

Now $I \equiv \int\limits_{C\Omega'_j} \dfrac{|u(y)|}{|x-y|^{2p}} dy \leq \sum\limits_k \int\limits_{\omega_k} \dfrac{|u(y)|}{|x-y|^{2p}} dy$ where the sum is taken **83**
is taken over all the ω_k such that $d(\Omega_j, \omega_k) \geq 3\varepsilon, \Omega_j$ being the support of α_j. Hence

$$I \leq \sum_k 2^{2p} d(\omega_k, \Omega_j)^{-2p} |||u||_{\omega_k} (\text{vol } \omega_0)^{\frac{1}{2}}.$$

Hence the second integral is majorized by

$$\sup_x (|x|^{2p} |x^{\nu} Y'(x)|)(\text{vol } \omega_0)^{\frac{1}{2}} 2^{2p} \left(\sum_k d(\omega_k, \Omega_j)^{-2p} ||u||_{\omega_k} \right)$$

where the ω_k occuring in the summation are such that $d(\omega_k, \Omega_j) \geq 3\varepsilon$. For $|\nu| = q$ sufficiently large it can be shown that

$$K(\nu) = \sum_{1 \geq 0} \sup_x |a_{lm}(x)| \cdot \sup_x |x^{\nu} Y'_{lm}(x)| < \infty$$

and

$$K(\nu, p) = \sum_{1 \geq 0} \sup_x |a_{lm}(x)\big| \cdot \sup_x \big| |x|^{2p} x^{\nu} Y'_{lm}(x)\big| < \infty$$

for p sufficiently large. So we have

$$\|\varphi_j^{(2)}\|^2_{\Omega_j} \leq \int_{\Omega_j}\left(\sum_{|\nu|=q}\sum_{l,m}|a_{lm}(x)|\int \alpha_{j,\nu}(x,y)(x-y)^\nu Y'_{lm}(x-y)\big|u(y)\big|dy\right)dx$$

$$\leq c\Big(\sum_{|\nu|=q}K(\nu)\|u\|^2_{\Omega'_j}+K(\nu,p)\left(\sum_k d(\omega_k,\Omega_j)^{-2p}\|u\|_{\Omega_j}\right)^2. \tag{5.13}$$

But by Schwarz inequality we have

$$\sum_k d(\omega_k,\Omega_j)^{-2p}\|u\|_{\omega_k} \leq \left(\sum_k d(\omega_k,\Omega_j)^{-2p}\right)^{\frac{1}{2}}\left(\sum_k d(\omega_k,\Omega_j)^{-2p}\|u\|^2_{\omega_k}\right)^{\frac{1}{2}}$$

84 and since $(\sum_k d(\omega_k,\Omega_j)^{2p}) < K$, a constant we obtain after summing over j

$$\sum_{k,j}\|u\|^2_{\omega_k}d(\omega_k,\Omega_j)^{-2p} = \sum_k \|u\|^2_{\omega_k}\sum_j d(\omega_k,\Omega_j)^{-2p}$$
$$\leq K_p\sum_k\|u\|^2_{\omega_k} \leq K_p r\|u\|^2$$

where K_p is a constant depending on p and r is the maximum number of balls ω_k containing a point of $\underline{\mathrm{R}}^n$. Substituting in (5.13)

$$\sum_k \|\varphi_k^{(2)}\|^2_{\Omega_k} \leq c\|u\|^2$$

which together with (5.12) gives the estimate

$$\sum_k \|\varphi_k\|^2_{\Omega_k} \leq c'\|u\|^2. \tag{5.14}$$

It remains to estimate $\|\varphi_k\|_{{}^C\Omega_k}$ in order to complete the proof of the lemma. For $x \in {}^C\Omega_k$ a typical term in the expression for $\varphi_k(x)$ is of the form

$$\psi(x) = \int_{\omega_k} Y'_{lm}(x-y)\alpha_j(y)u(y)dy$$

from which we obtain as before the estimate

$$|\psi(x)| \leq \sup_x \left||x|^{2p} Y'_{lm}(x)\right| \cdot \int_{\Omega_j} \frac{|u(y)|}{|x-y|^{2p}} dy$$
$$\leq \sup_x \left||x|^{2p} Y'_{lm}(x)\right| \cdot \|u\|_{\Omega_j} d(x, \Omega_j)^{-2p} |(\text{vol}\, \omega_0)^{\frac{1}{2}}.$$

Hence

$$\|\psi\|_{C\Omega_k} \leq \sup_x \left||x|^{2p} Y'_{lm}(x)\right| \cdot \|u\|_{\Omega_j} (\text{vol}\, \omega_0)^{\frac{1}{2}} \Big(\int_{|x| \leq 2\varepsilon} \frac{1}{d(x, \omega_1)^{4p}} dx \Big)^{\frac{1}{2}}.$$

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Taking $4p > n$ and observing that $K(v, p) < \infty$ we see that

$$\|\varphi_k\|^2_{C\Omega_k} \leq c'' \|u\|^2 \Omega_j,$$

and again, summing over k,

(5.15) $$\sum_k \|\varphi_k\|^2_{C\Omega_k} \leq c'' \cdot r \|u\|^2.$$

This completes the proof of the lemma.

The following is an extension to singular integral operators of Gårding's inequality for elliptic differential operators.

Proposition 1. *Let H be a singular integral operator such that its symbol $\sigma(H) = h(x, \xi) \in C^\infty_\beta$ with $\beta > 0$ satisfies*

(5.16) $$|h(x, \xi)| \geq \tau > 0$$

for every $x \in \underline{R}^n$ and every vector ξ, δ being a positive constant. Then there exists a $\delta' > 0$ such that

(5.17) $$\|H \wedge u\|^2 \geq \delta' \| \wedge u\|^2 - \gamma \|u\|^2$$

for every $u \in \mathscr{D}^1_{L^2}$ where γ is a positive constant.

Proof. H being a singular integral operator we know that $||Hu|| \leq AM||u||$ where A is a constant depending only on n and

$$M = \sum_{|\nu|\leq 2n} \sup_{x\varepsilon \underline{R}^n,\ |\xi|\geq 1} \left|(\frac{\partial}{\partial\xi})^\nu \gamma(H)(x,\xi)\right|.$$

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Given a $\delta > 0$ there exists a number $\epsilon > 0$ such that for every $x_0 \in \underline{R}^n$ and for every $u \in L^2$

$$(5.18) \qquad ||(H - H(x_0))u||^2_{\omega_0} \leq \frac{\delta^2}{4}||u||^2$$

where $H(x_0)$ is the singular integral operator with constant coefficients such that $\sigma(H(x_0))(\xi) = \sigma(H)(x_0, \xi)$. ($H(x_0)$ is the tangential operator at x_0). ε can be chosen independent of the position of x_0. Consider the C^∞ partition of unity introduced earlier,

$$\alpha_j(x) \geq 0, \quad \alpha_j \in \mathscr{D}, \quad \sum \alpha_j^2(x) \equiv 1.$$

As we have

$$||H \wedge u||^2 = \sum ||\alpha_j H \wedge u||^2$$

it is sufficient to prove the inequality for $\alpha_j H \wedge u$.

$$\begin{aligned} ||\alpha_j H \wedge u||^2 &\geq \frac{1}{2}||H\alpha_j \wedge u||^2 - ||(H\alpha_j - \alpha_j H) \wedge u||^2 \\ &\geq \frac{1}{2}||H\alpha_j \wedge u||^2 - 2||H(\wedge\alpha_j - \alpha_j\wedge)u||^2 \\ &\quad - 2||((H\wedge)\alpha_j - \alpha_j(H\wedge))u||^2. \end{aligned}$$

Now we have

$$\begin{aligned} \sum_j H(\wedge\alpha_j - \alpha_j\wedge)u||^2 &\leq \sum_j ||H||^2\ ||(\wedge\alpha_j - \alpha_j\wedge)u||^2 \leq c_1'||H||^2||u||^2 \\ &\leq c_1||u||^2 \end{aligned}$$

and by lemma 2

$$\sum_j ||(H\wedge)\alpha_j - \alpha_j(H\wedge))u||^2 \leq c_2||u||^2$$

87 where c_1 and c_2 are constants depending only on the norm of H and n.

Hence

$$\|H \wedge u\|^2 \geq \frac{1}{2} \sum_j \|H\alpha_j \wedge u\|^2 - c_3\|u\|^2 \tag{5.21}$$

and we have only to consider $\|H\alpha_j \wedge u\|^2$.

For this purpose let $H(x^{(j)})$ be the singular integral operator whose symbol is $h(x^{(j)}, \xi)$, so that

$$\sigma(H - H(x^{(j)})) = h(x, \xi) - h(x^{(j)}, \xi).$$

So we have

$$\|H\alpha_j \wedge u\|^2 \geq \frac{1}{2}\|H(x^{(j)})\alpha_j \wedge u\|^2 - \|(H - H(x^{(j)}))\alpha_j \wedge u\|^2.$$

From the condition that $|h(x, \xi)| > \delta$ we have

$$\frac{1}{2}\|H(x^{(j)})\alpha_j \wedge u\|^2 \geq \frac{\delta^2}{2}\|\alpha_j \wedge u\|^2.$$

As in lemma 2, let Ω_j denote the ball of radius 2ε and centre $x^{(j)}$. We decompose the second term into a sum

$$\begin{aligned}\|(H - H(x^{(j)}))\alpha_j \wedge u\|^2 = \|(H - H(x^{(j)}))\alpha_j \wedge u\|^2_{\Omega_j} \\ + \|(H - H(x^{(j)}))\alpha_j \wedge u\|^2_{^c\Omega_j}\end{aligned}$$

As mentioned at the begining of the proof, the first term is majorized by $\dfrac{\delta^2}{4}\|\alpha_j \wedge u\|^2$. For the second term we have

$$\begin{aligned}\|(H - H(x^{(j)}))\alpha_j \wedge u\|^2_{^c\Omega_j} \leq 2\|(H - H(x^{(j)}))(\alpha_j \wedge - \wedge \alpha_j)u\|^2_{^c\Omega_j} \\ + 2\|(H - H(x^{(j)})) \wedge \alpha_j u\|^2_{^c\Omega_j}.\end{aligned}$$

By lemma 11, $\|(H - H(x^{(j)})) \wedge \alpha_j u\|^2_{^c\Omega_j} \leq c(n, \eta)M'\|\alpha_j u\|^2$ and since 88

$(H - H(x^{(j)}))$ is a singular integral operator we obtain from lemma 2 the inequality

$$\sum_j (H - H(x^{(j)}))(\alpha_j \wedge - \wedge \alpha_j) u\|^2_{c\Omega_j} \leq c\|u\|^2.$$

Hence

$$\begin{aligned}\|H \wedge u\|^2 &\geq \frac{\delta^2}{4} \sum_j \|\alpha_\delta \wedge u\|^2 - c(n, \eta) M'' \sum_j \|\alpha_j u\|^2 - c\|u\|^2 \\ &\geq \frac{\delta^2}{4} \| \wedge u\|^2 - \gamma \|u\|^2\end{aligned}$$

which completes the proof of the inequality. □

Proposition 2. *Let H be a singular integral operator whose symbol $\sigma(H) = h(x, \xi) \in C^\infty_\beta$ with $\beta > 0$. Let $h(x, \xi)$ satisfy the condition*

(5.19) $\operatorname{Re} h(x, \xi) \leq -\delta,\ \delta > 0$ *for every* $x \in \underline{R}^n$ *and every vector* ξ.

Then there exists a $\delta' > 0$ such that

(5.20) $((H + H^*) \wedge u, \wedge u) \leq -\delta' \| \wedge u\|^2 + \gamma \|u\|^2$ *for* $u \in \mathscr{D}^1_{L^2}$

where γ is a constant depending only on M, δ and n, $\delta'(\delta' < \delta)$ can be chosen as near δ as one wishes.

Proof. One can write $H^* \wedge = H^\# \wedge + (H^* - H^\#)\wedge$. By Th. 4 of § 4, $(H^* - H^\#)$ is a bounded operator in L^2 and hence it is enough to prove that for $P = H + H^\#$, $(P \wedge u, \wedge u)$ satisfies an inequality of the required kind. The symbol $\sigma(P) = h(x, \xi) + \overline{h(x, \xi)}$ is real and $\leq -2\delta$. Let $\alpha_j \in \mathscr{D}$,
89 $\alpha_j(x) \geq 0$, $\sum \alpha_j^2(x) \equiv 1$ be a C^∞ partition of unity as in lemma 1. Then

$$\begin{aligned}(P \wedge u, \wedge u) = \sum_j (\alpha_j P \wedge u, \alpha_j \wedge u) &= \sum_j (P\alpha_j \wedge u, \alpha_j \wedge u) \\ &\quad - \sum_j ((P\alpha_j - \alpha_j P) \wedge u, \alpha_j \wedge u).\end{aligned}$$

For any $\epsilon' > 0$ we have, by Schwarz's inequality

$$
\begin{aligned}
(P\alpha_j - \alpha_j P) \wedge u, \alpha_j \wedge u) &\leq \|(P\alpha_j - \alpha_j P) \wedge u\| \cdot \|\alpha_j \wedge u\| \\
&\leq \epsilon' \|\alpha_j \wedge u\|^2 + \frac{1}{\epsilon'} \|(P\alpha_j - \alpha_j P) \wedge u\|^2.
\end{aligned}
$$

From lemma 2 we have

$$
\begin{aligned}
\sum_j \|(P\alpha_j - \alpha_j P) \wedge u\|^2 &\leq 2 \sum_j P(\alpha_j \wedge - \wedge \alpha_j) u\|^2 + 2 \sum_j (P\wedge)\alpha_j - \alpha_j (P\wedge) u\|^2 \\
&\leq c' \|u\|^2
\end{aligned}
$$

and we have only to estimate $(P\alpha_j \wedge u, \alpha_j \wedge u)$. Write $P = P(x^{(j)}) + (P - P(x^j))$ where, as before, $P(x^{(j)})$ is the singular integral operator whose symbol is $\sigma(P)(x^{(j)}, \xi)$. Since $\sigma(P(x, \xi)) \leq -2\delta$ we have

$$
(P(x^j))\alpha_j \wedge u, \alpha_j \wedge u) \leq -2\delta \|\alpha_j \wedge u\|^2.
$$

Again by Schwarz's inequality

$$
\begin{aligned}
\left|(P - P(x^{(j)}))\alpha_j \wedge u, \alpha_j \wedge u)\right| &\leq \| \left\{P - P(x^{(j)})\right\} \alpha_j \wedge u\| \cdot \|\alpha_j \wedge u\| \\
&\leq \frac{\varepsilon''}{4} \|\alpha_j \wedge u\|^2 + \frac{4}{\varepsilon''} \|(P - P(x^{(j)}))\alpha_j \wedge u\|^2.
\end{aligned}
$$

Now, as in Prop. 1,

$$
|(P - P(x^{(J)}))\alpha_j \wedge u\|^2 \leq \eta(\varepsilon) \|\alpha_j \wedge u\|^2 + \mu \|(\alpha_j \wedge - \wedge \alpha_j) u\|^2 + \mu \|\alpha_j u\|^2.
$$

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Putting all these inequalities together one sees that

$$
\begin{aligned}
(P \wedge u, \wedge u) \leq \left(-2\delta + \varepsilon' + \frac{\varepsilon''}{4} + \frac{4}{\varepsilon''}\eta(\varepsilon)\right) \sum_j \|\alpha_j \wedge u\|^2| \\
+\mu \sum_j \|(\alpha_j \wedge - \wedge \alpha_j) u\|^2 + \|u\|^2.
\end{aligned}
$$

Choosing $\varepsilon' \dfrac{\varepsilon''}{4}$, near δ and fixing ε to have $\dfrac{4\eta(\varepsilon)}{\varepsilon''}$ small enough to make $-2\delta + \varepsilon' + \dfrac{\varepsilon''}{4} + \dfrac{4}{\varepsilon''}\eta(\varepsilon)$ as near δ as required and using lemma 2 the desired inequality follows.

We shall now prove a lemma which we require later. It is analogous to lemma 2. We define, for any real $s, \wedge^s$ by $\widehat{(\wedge^s u)} = |\xi|^s \hat{u}$. □

Lemma 3. *Let H be a singular integral operator whose symbol $\sigma(H) = h(x,\xi)\varepsilon C^\infty_\beta$, with $\beta = \infty$. Then for any $u \in L^2$*

(5.21) $\|(H \wedge^s - \wedge^s H) \wedge^\sigma u\| \leq c\|u\|$ *for* $s, \sigma \geq 0$ *with* $s + \sigma \leq 1$.

Proof. Let $\alpha \in \mathscr{D}_\xi$ be such that $0 \leq \alpha(\xi) \leq 1$, $\alpha(\xi) \equiv 1$ on $|\xi| \leq 1$ and vanish outside $|\xi| \geq 2$. Writing $|\xi|^s = |\xi|^s\alpha(\xi) + |\xi|^s(1-\alpha(\xi))$ we decompose the operator into a sum $\wedge^s = \wedge^s_0 + \wedge^s_1$ with $\sigma(\wedge^s_0) = |\xi|^s\alpha(\xi)$ and $\sigma(\wedge^s_1) = |\xi|^s(1-\alpha(\xi))$. As $|\xi|^s\alpha(\xi)$ has compact support $\wedge^s_0$ defines a continuous linear operator in L^2 and hence it is enough to prove that

$$\|(H \wedge^s_1 - \wedge^s_1 H) \wedge^\sigma u\| \leq c\|u\|.$$

Expanding $\sigma(H)$ in terms of spherical harmonics Y_{lm} as in lemma 2 and taking the inverse Fourier image we have

$$H = a_0(x) + \Sigma a_{lm}(x)\tilde{Y}_{lm} * .$$

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Let $P = a(x) \cdot \tilde{Y}*$ be a term in the sum. We consider

$$(P \wedge^s_1 - \wedge^s_1 P) \wedge^\sigma u = \int (a(x) - a(y)) \wedge^s_1 (x-y) \wedge^\sigma \varphi(y) dy$$

where $\varphi(y) = (\tilde{Y} * u)(y)$. Expand $a(x) - a(y)$ in Taylor series upto order q, q to be determined later:

$$a(x) - a(y) = -\sum_{|\leq|\nu|\leq q-1} \frac{1}{\nu!}\left(\frac{\partial}{\partial x}\right)^\nu a(x)\cdot(y-x)^\nu - \sum_{|\nu|=q}\frac{a_\nu(x,y)}{\nu!}(y-x)^\nu.$$

This gives

$$(P \wedge^s_1 - \wedge^s_1 P) \wedge^\sigma u = \sum_{1\leq|\nu|\leq q-1} (-1)^{|\nu|+1}\left(\frac{\partial}{\partial x}\right)^\nu a(x)\cdot(x^\nu \wedge^s_1) * (\wedge^\sigma\varphi)$$

(5.22) $$+\sum_{|\nu|=q}(-1)^{|\nu|+1}\int \frac{a_\nu(x,y)}{\nu!}(x-y)^\nu \wedge^s_1 (x-y)(\wedge^\sigma\varphi)(y)dy.$$

We estimate the first sum in (5.22). We have

$$|\widehat{(x^\nu \wedge_1^s)}| = \left|\left(\frac{\partial}{\partial\xi}\right)^\nu (1-\alpha(\xi))|\xi|^s\right| \leq c_\nu(1+|\xi|)^{s-|\nu|}.$$

Hence

$$||(x^\nu \wedge_1^s) * (\wedge^\sigma \varphi)|| \leq C_\gamma ||(1+|\xi|)^{s-|\nu|}|\xi|^\sigma \hat{\varphi}|| \leq c_\nu ||\varphi||$$

since $s+\sigma \leq 1$ and $|\nu| \geq 1$. Summing over ν with $|\nu| \leq q-1$, we obtain

$$\sum_{|\nu|\leq q-1} ||\frac{(-1)^{|\nu|+1}}{\nu!}\left[\left(\frac{\partial}{\partial x}\right)^\nu a\right]\left[(x^\nu \wedge_1^s) * (\wedge^\sigma \varphi)\right]|| \leq c(q)||\varphi||\ |a|_{q-1} \tag{5.23}$$

where

$$|a|_p = \sup_{x,|\nu|\leq p} \left|\left(\frac{\partial}{\partial x}\right)^\nu a(x)\right|.$$

Since $||\varphi|| = ||\hat{\varphi}|| = ||Y(\xi)\hat{u}|| \leq |Y|_0.\ ||u||$ the right hand side of the inequality (5.23) is less than or equal to 92

$$c(q)|a|_{q-1}|Y|_0 \cdot ||u||.$$

Now we estimate the second sum. Write $|\xi|^\sigma$ as

$$|\xi|^\sigma = \alpha(\xi)|\xi|^\sigma + (1-\alpha(\xi))|\xi|^\sigma = \alpha(\xi)|\xi|^\sigma + |\xi|\left\{(1-\alpha(\xi))|\xi|^{\sigma-1}\right\}$$

where $\alpha(\xi) \in \mathscr{D}$, $\alpha(\xi) \equiv 1$ in a neighbourhood of the origin. Thus $\wedge^\sigma = B_0 + \wedge B_1$ where B_0 and B_1 are bounded operators in L^2. Hence we have only to consider the part containing $\wedge B_1$. Denote by ψ_ν the integral

$$\psi_\nu(x) = \int a_\nu(x,y)(x-y)^\nu \wedge_1^s (x-y) \wedge B_1\varphi(y)dy.$$

Now we can write $|\xi| = \Sigma \xi_j \dfrac{\xi_j}{|\xi|}$ and if R_j denote the Riesz operators defined by $\widehat{(R_j f)} = \dfrac{\xi_j}{|\xi|}\hat{f}$ we can write $\wedge = \Sigma \dfrac{\partial}{\partial x_j} R_j$. Substituting for $\wedge$ in $\psi_\nu(x)$

$$\psi_\nu(x) = -\Sigma_j \int \frac{\partial}{\partial x_j}\left\{a_\nu(x,y)(x-y)^\nu \wedge_1^s (x-y)\right\} \cdot (R_j B_1 \varphi)(y)dv.$$

We observe that $(x^\nu \wedge_1^s)$ is a bounded function together with its derivatives of the first order for $|\nu| \geq n+2$. In fact its Fourier image is $\left(\dfrac{\partial}{\partial\xi}\right)^\nu \{(1-\alpha(\xi))|\xi|^s\}$ and

$$|x^\nu \wedge_1^s| \leq \int |\widehat{(x^\nu \wedge_1^s)}| d\xi \leq c_\gamma \int (1+|\xi|)^{s-|\nu|} d\xi < \infty.$$

We can write

$$\begin{aligned}\psi_\nu(x) &= \int a_\nu(x,y)(x-y)^\nu \wedge_1^s (x-y)(\wedge B_1\varphi)(y)dy \\ &= -\Sigma\Big\{\int \left[\frac{\partial a_\nu}{\partial y_j}(x,y)\right](x-y)^\nu \wedge_1^s (x-y)(R_jB_1\varphi)(y)dy \\ &\quad + \int a_\nu(x,y)\left[\frac{\partial}{\partial y_j}((x-y)^\nu \wedge_1^s (x-y))\right](R_jB_1\varphi)(y)dy.\end{aligned}$$

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Set

$$\psi_\nu(x) = I_1 + I_2. \tag{5.24}$$

We estimate I_1 and I_2 separately.

$$\begin{aligned}|I_1| &\leq \sum_j \Big| \int \left[\frac{\partial a_\nu}{\partial y_j}(x,y)\right](x-y)^\nu \wedge_1^s (x-y)(R_jB_1\varphi)(y)dy\Big| \\ &\leq |a|_{q+1} \sum_j \int |(x-y)^\nu \wedge_1^s (x-y)|\,|(R_jB_1\varphi)(y)dy.\end{aligned}$$

The Fourier image of $(1+|x|^{2P})x^\nu \wedge_1^s (x)$ is

$$\left(\frac{1}{2\pi i}\right)^{|\nu|}\left(\frac{\partial}{\partial\xi}\right)[(1-\alpha(\xi))|\xi|^s] + \left(\frac{1}{2\pi i}\right)^{2p+|\nu|} \triangle_\xi^p \left(\frac{\partial}{\partial\xi}\right)^\nu [(1-\alpha(\xi))|\xi|^s]$$

and hence

$$|x^\nu \wedge_1^s (x)| \leq \frac{1}{1+|x|^{2p}}\left\{\left(\frac{1}{2\pi}\right)^{|\nu|} \int |\left(\frac{\partial}{\partial\xi}\right)^\nu [(1-\alpha(xi))|\xi^s|]\,|d\xi\right.$$

$$+\left(\frac{1}{2\pi}\right)^{2p+|\nu|}\int |\triangle_{\xi}^{p}\left(\frac{\partial}{\partial\xi}\right)^{\nu}[(1-\alpha(\xi))|\xi|^{s}]\,|d\xi\Bigg\}$$

$$\leq \frac{1}{1+|x|^{2p}}(C_1(\nu)+C_2(p,\nu))$$

and similarly we have

$$\left|\left(\frac{\partial}{\partial x_j}\right)(x^{\nu}\wedge_1^s(x))\right| = \frac{1}{1+|x|^{2p}}\left|(1+|x|^{2p})\frac{\partial}{\partial x_j}(x^{\nu}\wedge_1^s(x))\right|$$
$$\leq \frac{1}{1+|x|^{2p}}\left\{\left|\frac{\partial}{\partial x_j}(x^{\nu}\wedge_1^s(x))\right| + \left||x|^{2p}\left(\frac{\partial}{\partial x_j}\right)(x^{\nu}\wedge_1^s(x))\right|\right\}$$
$$\leq \frac{1}{1+|x|^{2p}}(C_2(\nu)+C_1^2(p,\nu)).$$

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For sufficiently large p the quantities $C_2(p,\nu)$, $C_2'(p,\nu)$ are finite. Thus we have

(5.25) $$|I_1| < |a|_{q+1}\sum_j\int\frac{|(R_jB_1\varphi)(y)|}{1+|x-y|^{2p}}dy$$

$$|I_2| \leq \sum_j\left|\int a_{\nu}(x,y)\frac{\partial}{\partial y_j}\left[(x-y)^{\nu}\wedge_1^s(x-y)\right]\cdot(R_jB_1\varphi)(y)\right|dy$$
$$\leq |a|_q\Sigma\int\left|\frac{\partial}{\partial y_j}\left[(x-y)^{\nu}\wedge_1^s(x-y)\right]\right|(R_jB_1\varphi)(y)dy$$

(5.26) $$|I_2| \leq M(p)|a|_q\Sigma\int\frac{(R_jB_1\varphi)(y)}{1+|x-y|^{2p}}dy.$$

This leads to the inequality

$$||I_1(x)||_{L^2} \leq |a|_{q+1}\sum_{j=1}^{n}||R_jB_1\varphi||_{L^2}\left(\int\frac{1}{(1+|x|^{2p})}dx\right)$$

because of the Hausdorff-Young theorem. We have the same kind estimate for $||I_2(x)||_{L^2}$.

Hence

$$||\psi_{\nu}|| \leq C_3(n)||(R_jB_1\varphi)||\cdot|a|_{q+1} \leq C_4(n)|a|_{q+1}||\varphi||$$

$$\leq C_4(n)|a|_{q+1}|Y(\xi)|_0 \cdot ||u||.$$

95 Now summing up for all l, m we have for any $u \in L^2$

$$\begin{aligned}||(H \wedge_1^s - \wedge_1^s H) \wedge^\sigma u|| &\leq \sum_{l,m} ||(a_{lm} \wedge_1^s - \wedge_1^s a_{lm}) \wedge^\sigma (\tilde{Y}_{lm} * u)|| \\ &\leq C_5(n)\left(\sum_{l,m} |a_{lm}|_{n+3}|Y_{lm}(\xi)||_0\right)||u|| \\ &\leq C_5(n, s, \sigma)M||u||_{L^2}\end{aligned}$$

and this completes the proof of the lemma. □

The following is a generalization of Friedrichs' lemma to singular integral operators (see Mizohota [1]).

Proposition 3. *Let H be a singular integral operator such that its symbol $\sigma(H) = h(x, \xi) \in C^\infty_{1+\sigma}$, $\sigma > 0$. Let $C_\delta u$ denote, for $u \in L^2$, the commutator $[H\wedge, \varphi_\delta *]u$ where φ_δ is the mollifier of Friedrichs.*

Then

(1) $||C_\delta u|| \leq cM'||u||$

where $M' = |a_0|_{\beta^{1+\sigma}} + \sum_{l,m} |a_{lm}|_{\beta^{1+\sigma}}|Y_{lm}|_{\beta^0}$ and c depends only on φ and n

(2) $C_\delta u \to 0$ *weakly in L^2 as $\delta \to 0$.*

Proof. We expand $h(x, \xi)$ in spherical harmonics $Y'_{lm}(\xi)$

$$h(x, \xi) = a_0(x) + \sum_{l,m} a_{lm}(x)Y'_{lm}(\xi)$$

and hence we can write, denoting the inverse Fourier image of Y'_{lm} by $\tilde{Y}_{lm}$

$$Hu(x) = a_0(x)u(x) + \sum_{l,m} a_{lm}(x)(\tilde{Y}_{lm} * u)(x).$$

To prove (1) it is sufficient to prove it for $u \in \mathscr{D}$. Now

$$C_\delta u = [H\wedge, \varphi_\delta *]u = H \wedge (u * \varphi_\delta) - (H \wedge u) * \varphi_\delta$$

$$= \sum_{l,m} \left\{ a_{lm}(x)(\tilde{Y}_{lm} * \wedge(u * \varphi_\delta)) - a_{lm}(x)(\tilde{Y}_{lm} * \wedge u) * \varphi_\delta \right\}. \tag{5.27}$$

96 Consider a typical term of this sum:

$$a_{lm}(x)(\tilde{Y}_{lm} * \wedge(u * \varphi_\delta)) - (a_{lm}(x)(\tilde{Y}_{lm} * \wedge u)) * \varphi_\delta$$

and substitute $\Sigma \dfrac{\partial}{\partial x_j} R_j$ for $\wedge$ where R_j are the Riesz operators. Put $\psi_{lm}(x) = \tilde{Y}_{lm} * R_j * u$. We have

$$\begin{aligned}
&a_{lm}(x)(\tilde{Y}_{lm} * \frac{\partial}{\partial x_j} R_j * (u * \varphi_\delta) - (a_{lm}(x)(\tilde{Y}_{lm} * \frac{\partial}{\partial x_j} R_j * u)) * \varphi_\delta \\
&= a_{lm}(x) \left[\frac{\partial}{\partial x_j} (\tilde{Y}_{lm} * R_j * u) * \varphi_\delta \right] - (a_{lm} \frac{\partial}{\partial x_j} (\tilde{Y}_{lm} * R_j * u)) * \varphi_\delta \\
&= a_{lm}(x) \left[\frac{\partial}{\partial x_j} \psi_{lm}(x) * \varphi_\delta \right] - \left[a_{lm} \frac{\partial}{\partial x_j} \psi_{lm} \right] * \varphi_\delta \\
&= \int [a_{lm}(x) - a_{lm}(y)] \left[\frac{\partial}{\partial y_j} \psi_{lm}(y) \right] \varphi_\delta(x - y) dy
\end{aligned}$$

where the integral is taken in the sense of distributions. By definition this is

$$- \int \frac{\partial}{\partial y_j} \{[a_{lm}(x) - a_{lm}(y)] \varphi_\delta(x - y)\} \psi_{lm}(y) dy$$

where the integral is taken in the usual sense.

Now,

$$\begin{aligned}
&\int \left| \frac{\partial}{\partial y_j} \{[a_{lm}(x) - a_{lm}(y)] \varphi_\delta(x - y)\} \psi_{lm}(y) dy \right| \\
&\leq \left| \int \psi_{lm}(y)(a_{lm}(x) - a_{lm}(y)) \frac{\partial \varphi_\delta}{\partial y_j}(x - y) dy \right| + \left| \int \psi_{lm}(y) \varphi_\delta(x - y) \frac{\partial_{lm}(y)}{\partial y_j} dy) \right| \\
&\leq ||\psi||_{lm} \left\{ 2|a_{lm}|_0 \left\| \frac{\partial \varphi_\delta}{\partial x_j} \right\|_{L^1} + |a_{lm}|_1 \cdot ||\varphi_\delta||_{L^1} \right\} \\
&\leq ||\psi_{lm}|| \{2|a_{lm}|_0 c_1(\delta, n) + |a_{lm}|_1 c_2(\delta, n)\} \\
&\leq c(\delta, n)|a_{lm}|_1 \cdot |Y'(\xi)|_0 \cdot ||u||
\end{aligned}$$

which proves (1). 97

To prove (2) let $v \in L^2$ and consider

$$\int v(x) \int \psi_{lm}(y) \frac{\partial}{\partial y_j} \{(a_{lm}(x) - a_{lm}(y))\varphi_\delta(x-y)\}\, dy\, dx$$
$$= \int v(x) \int \psi_{lm}(y) \left\{ \frac{\partial \varphi_\delta}{\partial y_j}(x-y) \cdot (a_{lm}(x) - a_{lm}(y)) - \varphi_\delta(x-y) \frac{\partial a_{lm}}{\partial y_j}(y) \right\} dy\, dx$$
$$= \int v(x) \int \psi_{lm}(y) \left\{ \sum_k (x_k - y_k) \frac{\partial a_{lm}}{\partial y_k}(y) \cdot \frac{\partial \varphi_\delta}{\partial y_j}(x-y) + \sigma(x,y) \frac{\partial \varphi_\delta}{\partial y_j}(x-y) \right.$$
$$\left. - \varphi_\delta(x-y) \frac{\partial a_{lm}}{\partial y_j}(y) \right\} dy\, dx$$

where $\sigma(x,y) = a_{lm}(x) - a_{lm}(y) - \sum\limits_k (x_k - y_k) \dfrac{\partial a_{lm}}{\partial y_k}(y)$. Let

$$k_1(y, x-y) = \sum_k (x_k - y_k) \frac{\partial a_{lm}}{\partial y_k}(y) \cdot \frac{\partial \varphi_\delta}{\partial y_j}(x-y) - \varphi_\delta(x-y) \frac{\partial a_{lm}}{\partial y_j}(y)$$
$$(5.28) \qquad = -\frac{\partial}{\partial x_j} \left\{ \sum (x_k - y_k) \frac{\partial a_{lm}}{\partial y_k}(y) \cdot \varphi_\delta(x-y) \right\}$$

and (5.28)′ $\qquad k_2(y, x-y) \geq \sigma(x,v) \dfrac{\partial \varphi_\delta}{\partial y_j}(x-v)$

Then $|k_2(y, x-y)| \leq c|a_{lm}(x)|_{1+\sigma} |x-y|^{1+\sigma} \left| \dfrac{\partial \varphi_\delta}{\partial x_j}(x-y) \right|$.

Applying the Hausdorff-Young inequality we have

$$|| \int v(x) k_2(y, x-y) dx || \leq c|a_{lm}|_{1+\sigma} \left(\sum \int |x-y|^{1+\sigma} \left| \frac{\partial \varphi_\delta}{\partial x_j}(x-y) \right| dx \right) \cdot ||v||$$
$$(5.29) \qquad = c|a_{lm}|_{1+\sigma} ||v|| \varepsilon(\delta)$$

98 where $\varepsilon(\delta) = \sum \int |x|^{1+\sigma} \left| \dfrac{\partial \varphi_\delta}{\partial x_j} \right| dx \to 0$ as $\delta \to 0$. On the other hand we observe that

$$\int k_1(y,z) dz = \int \frac{\partial}{\partial z_j} \left\{ \sum_r z_r \frac{\partial a_{lm}}{\partial y_r}(y) \cdot \varphi_\delta z \right\} dz = 0,$$

since φ_δ has compact support. Now consider

$$\int \int k_1(y, x-y) v(x) \psi_{lm}(y) dy\, dx = \int \psi_{lm}(y) dy \int k_1(y, x-y) v(x) dx.$$

The right hand side can be written after a change of variables $z = x - y$ in the form

$$\int \psi_{lm}(y)dy \int v(y+z)k_1(y,z)dz.$$

Schwarz inequality gives

$$\Big| \int \psi_{lm}(y)dy \int v(y+z)k_1(y,z)dz \Big| \leq \|\psi_{lm}\| \left\| \int k_1(y,z)v(y+z)dz \right\|.$$

Since $\int k_1(y,z)dz = 0$ we can write

$$\left\| \int k_1(y,z)v(y+z)dz \right\| = \left\| \int k_1(y,z)\{v(y+z) - v(v)\}dz \right\|.$$

We shall now evaluate the right hand side. Let us set

$$\varepsilon'(\delta) = \sup_{|h| \leq \delta} \left(\int |v(y+h) - v(y)|^2 dx \right)^{\frac{1}{2}}.$$

Schwarz inequality shows that

$$\Big| \int k_1(y, x-y)(v(x) - v(y))dx \Big|^2$$
$$\leq \left(\int |k_1(y, x-y)dx \right) \left(\int |k_1(y, x-y)||v(x) - v(y)|^2 dx \right).$$

Clearly $\int |k_1(y, x-y)|dx \leq c|a_{lm}|_{\beta^1}$ where c is a constant depending only on φ and δ. Hence integrating both sides of this inequality with respect to y we have **99**

$$\| \int k_1(y, x-y)(v(x) - v(y))dx\|^2$$

$$\leq c|a_{lm}|_{\beta^1} \iint |k_1(y, x-y)||v(x) - v(y)|^2 dx\, dy$$

$$= c|a_{lm}|_{\beta^1} \int\limits_{|z|\le\delta} dz \int |k_1(x-z,z)||v(x)-v(x-z)|^2 dx\, dz.$$

Since $k_1(y, x-y)$ is a bounded function the right side is less than

$$c'|a_{lm}|_{\beta^1}\varepsilon'(\delta)^2(\text{vol }\omega_\delta). \tag{5.30}$$

where $\varepsilon'(\delta) \to 0$ as $\delta \to 0$ and ω_δ is the ball $|z| \le \delta$. Combining the inequalities (5.29) and (5.30) we obtain

$$\begin{aligned}
&\Big|\iint v(x)\psi_{lm}(y)\{k_1(y,x-y)+k_2(y,x-y)\}dy\, dx\Big| \\
&\le \|\psi_{lm}\|(c|a_{lm}|_{\beta^{1+\sigma}}\|v\|\varepsilon(\delta) + c''|a_{lm}|_1\varepsilon'(\delta)) \\
&\le c''\|u\|(|a_{lm}|_{1+\sigma}|Y_{lm}|_o\|v\|\varepsilon(\delta) + |a_{lm}|_1|Y_{lm}|_0\varepsilon'(\delta)),
\end{aligned}$$

which tends to 0 as $\delta \to 0$. This completes the proof of the proposition. ◻

Corollary 1. *If we assume $u \in \mathscr{D}^1_{L^2}$ in proposition 3 then*

(1) $\|C_\delta u\|_{\mathscr{D}^1_{L^2}} \le c\|u\|_{\mathscr{D}^1_{L^2}}$

(2) $C_\delta u \to 0$ *weakly in* $\mathscr{D}^1_{L^2}$ *as* $\delta \to 0$.

100 *Proof.* We remark that

$$\frac{\partial}{\partial x_j}(C_\delta u) = C_\delta\left(\frac{\partial}{\partial x_j}\right) + \left[H^{(j)}_\wedge, \varphi_\delta *\right] u \tag{$*$}$$

where $H^{(j)}$ denotes the singular integral operator defined by

$$H^{(j)}_u = a^{(j)}_0 u + \sum a^{(j)}_{1m}(\tilde{Y}_{1m} * u), a^{(j)}_{1m} = \frac{\partial}{\partial x_j} a_{lm},$$

or equivalently

$$\sigma(H^{(j)}) = a^{(j)}_0(x) + \sum a^{(j)}_{1m}(x)Y_{lm}(\xi) \in C^\infty_\sigma \text{ with } \sigma > 0.$$

Now, the latter term of the right hand side in (∗) tends to 0 in L^2 as $\delta \to 0$. In fact,

$$\left[H^{(j)}\wedge, \varphi_\delta *\right] u = H^{(j)}(\varphi_{\delta^*} \wedge u) - H^{(j)}_{\wedge^u} + H^{(j)} \wedge u - \varphi_\delta * (H^{(j)}_{\wedge^u}) \text{ and } \wedge u \in L^2.$$

Now applying Proposition (3) to (∗) we have the corollary.

From Prop. 1 it can be easily seen that the following proposition holds. This plays the same role as Gårding's inequalitv for differential operators.

Proposition 4. *Let $\mathscr{H}$ be a square matrix whose elements H_{jk} are singular integral operators (belonging to C^∞_β) with their symbols $\sigma(H_{jk}) = h_{jk}(x, \xi) \in C^\infty_\beta$ with $\beta > 0$ $(j, k = 1, \ldots, N)$. Suppose $\sigma(\mathscr{H})$ is the matrix whose element are $\sigma(H_{jk})(x, \xi)$ and satisfies the hypothesis*

(5.31) $$|\sigma(\mathscr{H})\alpha| \geq \delta|\alpha| \quad \textit{for every} \quad x, \xi \in \underline{R}^n, \delta > 0$$

where $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a complex vector in $\underline{C}^N$. Then for every
$u = (u_1, \ldots, u_N) \in \pi\mathscr{D}'_{L^2}$ 101

(5.32) $$\|\mathscr{H} \wedge u\|^2 \geq \frac{\delta^2}{8}\| \wedge u\|^2 - \gamma_1\|u\|^2,$$

where γ_1 is a positive constant.

Remark. $\|u\|^2$, for $u = (u_1, \ldots, u_N) \in \pi\mathscr{D}^1_{L^2}$, denotes $\|u_1\|^2 + \cdots \|u_N\|^2$. The proof runs on the same lines as in the proof of the Prop. 1.

6 Energy inequalities for regularly hyperbolic systems

Let Ω denote the subset $\underline{R}^n \times [0, h]$ of $\underline{R}^{n+1}$.

Definition. A first order system of differential operators

(6.1) $$M = \frac{\partial}{\partial t} - \sum A_k(x, t)\frac{\partial}{\partial x_k}$$

is said to be regularly hyperbolic in Ω if

(1) $A_k(x,t)$ are bounded,

(2) for every $(x,t) \in \Omega$ and $\xi \in \underline{R}^n$ the roots of the systems

$$\det\left(\lambda I - \sum A_k(x,t)\cdot\xi_k\right) = 0 \tag{6.2}$$

are real and distinct; further if $\lambda_1(x,t,\xi)\cdots\lambda_N(x,t,\xi)$ are these roots then

$$\inf_{\substack{(x,t)\in\Omega\\ j\neq k}}, |\xi| = 1^{|\lambda_j(x,t,\xi)-\lambda_k(x,t,\xi)|>0} \tag{6.3}$$

We write the system (6.1) in terms of singular integral operators, by putting $\sum A_k(x,t)\dfrac{\partial}{\partial x_k} = i\mathscr{H}(t)\wedge$ where $\mathscr{H}(t)$ is a matrix of order N of singular integral operators whose symbol is the matrix

$$\sigma(\mathscr{H}(t)) = 2\pi\sum A_k(x,t)\frac{\xi_k}{|\xi|}.$$

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Thus (6.1) is written in the form

$$M = \frac{\partial}{\partial t} - i\mathscr{H}(t)\wedge \tag{6.1)'}$$

If the coefficients are such that $A_k = A_k(x,t) \in \beta^{1+\sigma}[0,h]$ with $\sigma > 0$ then for each fixed $t, \sigma(H)(x,t,\xi) \in C^\infty_{1+\sigma}, \sigma > 0$. □

Proposition 1 (Petrowsky). *Let M be a regularly hyperbolic system with $A_k \in \beta^{1+\sigma}[0,h]$. Suppose $A_k(x,t)$ are real. Then there exists a matrix $\sigma(\mathfrak{R}(t)) = \sigma(\mathfrak{R})(x,t,\xi)$ except possibly when $n = 2$ such that*

(i) $\sigma(\mathfrak{R}(t))\sigma(\mathscr{H}(t)) = \sigma(\mathscr{D}(t))\sigma(\mathfrak{R}(t))$ *where*

$$\sigma(\mathscr{D}(t)) = \begin{pmatrix} \lambda_1(x,t,\xi) & & 0 \\ & \ddots & \\ 0 & & \lambda_N(x,t,\xi) \end{pmatrix}$$

(ii) *$\sigma(\mathfrak{N}(t)) = \sigma(\mathfrak{N})(x, t, \xi)$ is of class $C_{1+\sigma}$ for every fixed t, has real elements and further*

$$(6.4) \qquad |\det \sigma(\mathfrak{N}(t))| \geq \delta' > 0 \ \textit{for every} \ (x, t)\varepsilon\Omega, \xi\varepsilon \underline{R}^n.$$

(iii) *the mapping $t \to \sigma(\mathfrak{N}(t)) \in C^{\infty}_{1+\sigma}$ is once continuously differentiable*

Proof. Since the roots of (6.2) det $(\lambda I - \sum A_k \cdot \xi_k) = 0$ are real and distinct it follows that $\lambda_j(x, t, \xi)$ are single valued functions on $|\xi| = 1$ for every fixed $(x, t) \in \Omega$. This follows by the principle of monodromy **103**
in the case $n > 2$ and in the case $n = 2$ by virtue of hyperbolicity.

To see that $\lambda_j(x, t, \xi) \in C^{\infty}_{1+\sigma}$, $\sigma > 0$ for fixed t denoting by

$$P(\lambda, x, t, \xi) = 0$$

the characteristic equation

$$\det\left(\lambda I - \sum A_k \cdot \xi_k\right) = 0$$

we have from the implicit function theorem

$$\frac{\partial \lambda_j}{\partial x_k} = -\left(\frac{\partial P}{\partial x_k}\bigg|\frac{\partial P}{\partial \lambda}\right)_{\lambda=\lambda_j}$$

and further $\left|\left(\dfrac{\partial P}{\partial \lambda}\right)_{\lambda=\lambda_j}\right| \geq d^{N-1}$ where $d = \inf\limits_{\substack{(x,t)\in\Omega, |\xi|=1 \\ j\neq k}} |\lambda_j - \lambda_k|$. □

Construction of $\sigma(\mathfrak{N}(t))$. Suppose $n \geq 3$. To find $\sigma(\mathfrak{N}(t))$ such that $\sigma(\mathfrak{N}(t))\sigma(\mathscr{H}(t)) = \sigma(\mathscr{D}(t))\sigma(\mathfrak{N}(t))$ is the same, if we write $\sigma(\mathfrak{N} = (n_{jk})$, $\sigma(\mathfrak{N}) = (a_{jk})$, as finding a matrix solution of

$$\lambda_j n_{jl} = \sum_k n_{jk} a_{kl}.$$

For a fixed j the vector $(n_{j1}, \ldots, n_{jN})$ is an eigenvector of the matrix $A = (a_{jk})$ corresponding to the eigenvalue λ_j. Consider the case $\lambda_j = \lambda_1$. We assert that the space of eigenvectors at the point (x, t, ξ) can be given

by explicit expressions (the space of eigenvectors is one dimensional) in such a way that this vector is continuous in (x, t, ξ) is class $C^{\infty}_{1+\sigma}$ and continuously differentiable in t. In fact, if $M_{jk}(t)$ is the (j, k)-cofactor of $(\lambda_1 I - A)$ then $(M_{1j}, M_{2j}, \ldots, M_{Nj})$ $(j = 1, \ldots, N)$ span the space of 104
eigenvectors. As the rank of $(\lambda_1 I - A)$ is $(N - 1)$ everywhere one of these is not trival.

Remark. In the case where the coefficients $A_k(x, t)$ are not real there will be topological difficulties in the above reasoning which proves the existence of smooth $\sigma\mathfrak{R}(x, t, \xi)$. It should however be observed that the theorem of local existence of smooth $\sigma\mathfrak{R}(x, t, \xi)$ remains valid. Therefore it would be better to use a partition of unity to derive energy inequalities for such systems. Moreover this argument can be applied for more general hyperbolic systems. (See: Le problème de Cauchy pour les systèmes hyperboliques et paraboliques, Mem. Coll. Sc., Kyoto Univ,. Ser. A. Math., 1959).

Proposition 2 (Energy inequality). *Let*

$$M = \frac{\partial}{\partial t} - \sum A_k(x, t) \frac{\partial}{\partial x_k}$$

be a regularly hyperbolic system in Ω *with the coefficients* $A_k(x, t)$ *satisfying*

$$A_k \in \mathbb{B}^{1+\sigma}[0, h], \ \frac{\partial}{\partial t} A_k \in B^0[0, h].$$

Suppose $B \in \mathbb{B}^0[0, h]$, $f \in L^2[0, h]$ *given. Then, if* $u \in L^2[0, h]$ *is a solution of*

$$\frac{\partial u}{\partial t} - \sum A_k(x, t) \frac{\partial u}{\partial x_k} - B(x, t) u = f \tag{6.5}$$

we have the inequality

$$\|u(t)\| \leq c(h) \left\{ \|u(0)\| + \int_0^t \|f(s)\| ds \right\}. \tag{6.6}$$

105 *Proof.* First we assume this $u \in \mathscr{D}^1_{L^2}[0, h]$. The given system is written in singular-integral-operator form as

$$\frac{\partial u}{\partial t} - i\mathscr{H}(t) \wedge u - B(t)u = f. \tag{6.7}$$

Multiplying this system by the matrix $\mathfrak{R}$ obtained in Prop. 1 we obtain

$$\frac{\partial}{\partial t}(\mathfrak{R}u) - i\mathfrak{R}(t)\mathscr{H}(t)) \wedge u - (\mathfrak{R}B + \frac{\partial \mathfrak{R}}{\partial t})u = \mathfrak{R}f.$$

By Prop. 1 $\mathfrak{R} \circ \mathfrak{R} = D \circ \mathfrak{R}$ which implies that

$$\mathfrak{R}\mathscr{H} \wedge \equiv \mathscr{D}\mathfrak{R} \wedge \text{(mod. bounded operators)}$$

because $(\mathfrak{R}\mathscr{H})\wedge \equiv (\mathfrak{R}) \circ \mathscr{H} \wedge$ (mod bounded operators)

$$(\mathscr{D}\mathfrak{R})\wedge \equiv (\mathscr{D} \circ \mathfrak{R}) \wedge \text{(mod. bounded operators)}$$

Also $(\mathscr{D}\mathfrak{R})\wedge = \mathscr{D} \wedge \mathfrak{R} +$ a bounded operator, and hence the new system becomes

$$\frac{\partial}{\partial t}(\mathfrak{R}u) = i\mathscr{D} \wedge (\mathfrak{R}u) + (\mathfrak{R}B + \frac{\partial \mathfrak{R}}{\partial t}u + \mathfrak{R}f.$$

In otherwords $v = \mathfrak{R}u$ satisfies

$$\frac{\partial v}{\partial t} = i\mathscr{D} \wedge v + B_1 u + \mathfrak{M}f$$

where $B_1 = \left(\mathfrak{R}B + \dfrac{\partial \mathfrak{M}}{\partial t}\right)$ is a bounded operator in view of Prop. 1. Now

$$\begin{aligned}\frac{\partial}{\partial t}(v, v) &= (i\mathscr{D} \wedge v, v) + (v, i\mathscr{D} \wedge v) + 2\operatorname{Re}(B_1 u + \mathfrak{R}f, v)\\ &= i(\mathscr{D} \wedge - \wedge \mathscr{D}^*)v, v) + 2\operatorname{Re}(B_1 u + \mathfrak{R}f, v).\end{aligned}$$

But $\wedge\mathscr{D}^* = \wedge\mathscr{D}^\# +$ a bounded operator, and since $\mathscr{D}$ is real $\mathscr{D}^\# = \mathscr{D}$ and $\wedge\mathscr{D} = \mathscr{D} \wedge +$ a bounded operator. Hence $\mathscr{D} \wedge - \wedge \mathscr{D}$ is a bounded 106
operator and

$$\frac{\partial}{\partial t}||v||^2 \leq 2\gamma_1||v||^2 + 2c||u||\ ||v|| + 2||\mathfrak{R}f||\ ||v||,$$

that is

$$\frac{\partial}{\partial t}||v|| \leq \gamma||v|| + c||u|| + ||\mathfrak{R} f||.$$

By the regular hyperbolicity we have in view of Prop. 1

(6.4) $$|det\sigma(\mathfrak{R}(t))| \geq \delta' > 0.$$

Hence by the generalized Garding inequality applied to $\mathfrak{R}$ there exist $\delta'' > 0$ and $\beta > 0$ such that

(6.8) $$||\mathfrak{R} \wedge u|| \geq \delta''|| \wedge u|| - \beta||u||.$$

Define

(6.9) $$|||u||| = ||\mathfrak{M}u|| + \beta||(\wedge + 1)^{-1}u||$$

where $(\wedge + 1)^{-1}u \xrightarrow{\mathscr{F}} \dfrac{1}{(1 + |\xi|)}\hat{u}$. It is clear that $|||u||| \leq c_1||u||$ since $\mathfrak{R}$ and $(\wedge + 1)^{-1}$ are bounded. On the other hand

$$\mathfrak{R}u = \mathfrak{R} \wedge (\wedge + 1)^{-1}u + \mathfrak{R}(\wedge + 1)^{-1}u$$

implies

$$\begin{aligned}
||\mathfrak{R}u|| &\geq ||\mathfrak{R} \wedge (\wedge + 1)^{-1}u|| - ||\mathfrak{R}(\wedge + 1)^{-1}u||\\
&\geq \delta''|| \wedge (\wedge + 1)^{-1}u|| - \beta||(\wedge + 1)^{-1}u|| - ||\mathfrak{R}(\wedge + 1)^{-1}u||\\
&\geq \delta''|| \wedge (\wedge + 1)^{-1}u|| - \beta'||(\wedge + 1)^{-1}u||\\
&\geq \delta''||u|| - (\beta' + 1)||(\wedge + 1)^{-1}u||
\end{aligned}$$

107 which proves that $|||u||| \geq c_2||u||$ consequently the norms $|||u|||$ and $||u||$ are equivalent. It is therefore sufficient to prove the energy inequality for the norm $|||u|||$.

$$\frac{\partial}{\partial t}|||u(t)||| = \frac{\partial}{\partial t}(||\mathfrak{R}u|| + \beta||(\wedge + 1)^{-1}u||)$$

(6.10) $$\leq \gamma||\mathfrak{R}(u)|| + c||u|| + ||\mathfrak{R} f|| + \beta\frac{\partial}{\partial t}||(\wedge + 1)^{-1}u||.$$

Considering $\dfrac{\partial u}{\partial t} = i\mathscr{H} \wedge u + Bu + f$

$$(\wedge + 1)^{-1} \frac{\partial u}{\partial t} = i(\wedge + 1)^{-1} \mathscr{H} \wedge u + (\wedge + 1)^{-1}(Bu + f)$$

but $(\wedge+1)^{-1}\mathscr{H}\wedge = (\wedge+1)^{-1} \wedge \mathscr{H} + (\wedge+1)^{-1}B_2$ where B_2 is a bounded operator in L^2 and hence

$$\frac{\partial}{\partial t} ||(\wedge + 1)^{-1} u|| \leq \delta_o ||u|| + ||(\wedge + 1)^{-1} f||.$$

Substituting in the inequality (6.10) we obtain

$$\frac{\partial}{\partial t} |||u(t)||| \leq \gamma' |||u(t)||| + |||f|||,$$

which, on integration with respect to t, gives

$$|||u(t)||| \leq |||u(0)||| \exp(\gamma' t) + \int_0^t |||f(S)||| \exp(\gamma'(t-s)) ds.$$

Since $|||u(t)||| \sim ||u(t)||$ we obtain the required inequality

$$||u(t)|| \leq c(h)\{||u(0)|| + \int\limits_0^t ||f(s)|| ds.$$

In the general case in which $u \in L^2[0, h]$ we regularize it by the the mollifiers φ_δ of Friendriche and apply the above argument to the function $u_\delta = \varphi_\delta *_{(x)} u$ and pass to the limits as $\delta \to 0$ in the inequality **108**
for u_δ to obtain the energy inquality for u. □

Remark. In the above proof the norm $|||u|||$ depends a priori on the parameter t since it involves the operator $\mathfrak{N}(t)$. When t runs through a bounded set the constant β in the definition of $|||u|||$ can be chosen to be independent of $\mathfrak{N}$.

In the following proposition we prove that, if A_k and B are differentiable of sufficiently high order, then there exists an energy inequality for higher order derivatives.

Proposition 3. *Let M be a regularly hyperbolic system with $A_k(x,t) \in \mathscr{B}^{\max(1+\sigma,m)}[0,h]$, $0 < \sigma < 1$, $\frac{\partial}{\partial t} A_k(x,t) \in \mathscr{B}^0[0,h]$. Suppose $B(x,t) \in \mathscr{B}^m[0,h]$, and $f(x,t) \in \mathscr{D}^m_{L^2}[0,h]$ are given. If $u \in \mathscr{D}^m_{L^2}[0,h]$ is a solution of*

$$(M-B)u = f$$

then

$$\|u(t)\|_m \leq c_m(h)\left\{\|u(0)\|_m + \int_0^t \|f(s)\|_m ds\right\}. \tag{6.11}$$

Proof. It is sufficient to prove the proposition for the case $m = 1$ and the general case will follow by repeated application of the argument. Let $\frac{\partial u}{\partial x_j} = u^{(j)}$. Then

$$M[u^{(j)}] = \sum_k \frac{\partial A_k}{\partial x_j}(x,t)\frac{\partial u}{\partial x_k} + \frac{\partial B}{\partial x_j}(x,t)u + \frac{\partial f}{\partial x_j},\ j = 1,2,\ldots,n$$

that is $u^{(j)}$ satisfy a regularly hyperbolic system with new B and f. De-
109 noting $\sum\limits_{j=1}^n |||u^{(j)}|||$ by $\varphi_1(t)$ we obtain

$$\frac{d\varphi_1}{dt}(t) \leq \gamma_1\varphi_1(t) + \sum_j |||\frac{\partial f}{\partial x_j}||| + \sum_j |||\frac{\partial B}{\partial x_j}u|||$$

which on integration yields the required inequality

$$\|u(t)\|_1 \leq c_1(h)\left\{\|u(0)\|_1 + \int\limits_0^t \|f(s)\|_1 ds\right\}.$$

In the following we duduce on energy inequality for solutions of a single regularly hyperbolic differential equation of order m.

Consider the evolution equation

$$\left(\frac{\partial}{\partial t}\right)^m u + \sum_{\substack{j+|\nu|\leq m\\ j\leq m-1}} a_{j,\nu}(x,t)\left(\frac{\partial}{\partial x}\right)^\nu \left(\frac{\partial}{\partial t}\right)^j u = g. \tag{6.12}$$

The principal part of this is by definition the homogeneous differential operator of order m

$$\left(\frac{\partial}{\partial t}\right)^m + \sum_{\substack{|\nu|+j=m\\ j\leq m-1}} a_{j,\nu}(x,t)\left(\frac{\partial}{\partial x}\right)^\nu \left(\frac{\partial}{\partial t}\right)^j \equiv L \tag{6.13}$$

which we write in the form

$$L \equiv \left(\frac{\partial}{\partial t}\right)^m + \sum_{j=1}^{m} h_j\left(x,t,\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}\right)^{m-j}$$

where $h_j\left(x,t,\frac{\partial}{\partial x}\right) = \sum_{|\nu|=j} a_{m-j,\nu}(x,t)\left(\frac{\partial}{\partial x}\right)^\nu$. The given operator is said to be regularly hyperbolic if the polynomial equation

$$\lambda^m + \sum_j h_j(x,t,\xi)\lambda^{m-j} = 0 \tag{6.14}$$

has real and distinct roots for every $(x,t)\epsilon\Omega$; $|\xi| = 1$. $h_j\left(x,t,\frac{\xi}{|\xi|}\right)$ can be **110**
considered as the symbol of a singular integral operator H_j and hence we can represent

$$h_j\left(x,t,\frac{\partial}{\partial x}\right) = H_j(i\wedge)^j$$

and

$$L \equiv \left(\frac{\partial}{\partial t}\right)^m + \sum_{j=1}^{m} H_j(i\wedge)^j\left(\frac{\partial}{\partial t}\right)^{m-j}. \tag{6.15}$$

Setting

$$v_1 = \left(\frac{\partial}{\partial t}\right)^{m-1} u$$

$$v_2 = i(\wedge + 1)\left(\frac{\partial}{\partial t}\right)^{m-2} u$$

$$v_j = \{i(\wedge + 1)\}^{j-1}\left(\frac{\partial}{\partial t}\right)^{m-j} u$$
$$v_m = \{i(\wedge + 1)\}^{m-1} u$$

We see that $(i\wedge)^{j-1} = (i\wedge)^{j-1}\{i(\wedge+1)\}^{-(j-1)}\{i(\wedge+1)\}^{j-1}$
$= (1 + S_{j-1})\{i(\wedge+1)\}^{j-1}$

where $\sigma(S_{j-1}) = \left(\frac{|\xi|}{1+|\xi|}\right)^{j-1} - 1$. $S_{j-1}\wedge$ is a bounded operator in L^2. Then the principal part is rewritten as

$$L[u] = \left(\frac{\partial}{\partial t}\right)^m u + i\sum H_j \wedge (1+S_{j-1})\{i(\wedge+1)\}^{j-1}\left(\frac{\partial}{\partial t}\right)^{m-j} u$$
$$= \frac{\partial}{\partial t} v_1 + i\sum H_j \wedge v_j + i\sum H_j \wedge S_{j-1} v_j.$$

Then $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ satisfies the system of first order equations

(6.16) $$\frac{\partial}{\partial t} v = i\mathscr{H} \wedge v + Bv + f$$

111 where

(6.17) $$\sigma(\mathscr{H}) = \begin{pmatrix} 1 & & & & \\ & & 1 & & \\ & & & \ddots & \\ -h_1 & -h_2 & \cdots & -h_{m-1} & -h_m \end{pmatrix},$$

B a bounded operator and $f = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ g \end{pmatrix}$.

Let $P(\lambda) = \det(\lambda I - \sigma(\mathscr{H})) = \lambda^m + \sum_j h_j \left(x, t, \frac{\xi}{|\xi|}\right)^{m-j}$. Thus the given equation is regularly hyperbolic if and only if the associated first order system is.

Proposition 4. *Suppose* $P(\lambda) = 0$ *has real and distinct roots* $\lambda_1(x, t, \xi) < \cdots < \lambda_N(x, t, \xi)$ *such that*

$$\inf_{\substack{(x,t)\epsilon\Omega,|\xi|=1\\ j\neq k}} \left|\lambda_j(x, t, \xi) - \lambda_k(x, t, \xi)\right| = d > 0 \tag{6.18}$$

and further the coefficients are such that

$$a_{j,\nu}\epsilon\mathscr{B}^{1+\sigma}[0, h], \frac{\partial}{\partial t}a_{j,\nu}\epsilon\mathscr{B}^0[0, h] \text{ for } j + |\nu| = m$$
$$a_{j,\nu}\epsilon\mathscr{B}^0[0, h] \text{ for } j + |\nu| \leq m - 1.$$

Let $g\epsilon L^2[0, h]$ be given. If $u\epsilon\mathscr{D}^m_{L^2}[0, h]$ is a solution of (6.12) then

$$||v(t)||' \leq C_0(h)\left\{||v(0)||' + \int_0^t ||f(s)||' ds\right\} \tag{6.19}$$

where $||v(t)||^{12} = \sum\limits_{j=1}^{m} ||(\frac{\partial}{\partial t})^{m-j}u||^2_{j-1}$.

This proposition is proved easily using the energy inequality for the associated first order system. □

7 Uniqueness theorems

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From the energy inequalities obtained in the previous section some results on the local uniqueness follow immediately. We shall show that a solution of a homogeneous regularly hyperbolic system of equations vanishes identically in a cone if the cauchy data is zero. This was first proved by Holmgren and later made precise by F. John [1].

Consider the first order system of equations

$$M[u] \equiv \frac{\partial u}{\partial t} - \sum A_k(x, t)\frac{\partial u}{\partial x_k} - B(x, t)u = 0 \tag{7.1}$$

where M is regularly hyperbolic in $\Omega = \underline{R}^n \times [0, h]$.

Proposition 1. *Let* M *be regularly hyperbolic in* Ω *with* $A_k\epsilon\mathscr{B}^{1+\sigma}_{x,t}$, $B\epsilon\mathscr{B}^0_{x,t}$. *If* $u\epsilon\mathscr{E}^1_{x,t}$ *satisfies* $M[u] = 0$ *and* $u(x, 0) \equiv 0$ *in a neighbourhood* U *of the origin in* $\underline{R}^n_x$ *then* $u \equiv 0$ *in a neighbourhood of the origin in* Ω.

Proof. Let $D_\epsilon \subset \Omega$ be the set $\left\{(x,t)\epsilon\Omega : |x|^2 + t < \epsilon, t \geq 0\right\}$. We first make a change of variables

$$t' = t + \sum x_j^2, x'_j = x_j (j = 1, \ldots, n). \tag{7.2}$$

Under this transformation let $\tilde{u}(x'_k, t') = u(x,t)$ then the system of equations is transformed into the system

$$\left(I - 2\sum x'_k \cdot A_k\right)\frac{\partial \tilde{u}}{\partial t'} = \sum A_k \frac{\partial \tilde{u}}{\partial x'_k} + B\tilde{u}. \tag{7.3}$$

D_ϵ is transformed into a strictly convex domain $\tilde{D}_\epsilon$ bounded by $t' =$
113 $\sum x'^2_j$, $t' = \epsilon$. $\tilde{u}$ is defined in the domain $\tilde{D}_\epsilon$ and we extend $\tilde{u}$ outside $\tilde{u}_\epsilon$ by 0 and we denote this again by $\tilde{u}$. Clearly $\tilde{u}\epsilon\mathscr{E}^1$ since it vanishes identically in a neighbourhood of $t' = \sum x'^2_j$. Thus $\tilde{u}$ has its support in $\tilde{D}_\epsilon$. It follows from lemma 1 that if x' is in a small neighbourhood of the origin (it is sufficient to take $2|x'|A$), $(I - 2 \in x'_k A_k)$ is invertible and the eigenvalues of $(I - 2\sum x'_k A_k)^{-1}\sum A_k \cdot \xi_k$ are real and distinct since those of $\sum A_k \cdot \xi_k$ are. Thus the transformed system remains regularly hyperbolic in $\tilde{D}_\epsilon$. Extending $A_k(x,t)$, $B(x,t)$ to the whole of $\underline{R}^n_x[0,h]$ in such a way that the system remains regularly hyperbolic we obtain $\tilde{M}[u] = 0$ in $\underline{R}^n \times [0,h]$ (this can be achieved by taking the inverse image by a suitable differentiable *retraction* of $\underline{R}^n \times [0,h]$ to $\tilde{D}_\epsilon$.

$$\frac{\partial \tilde{u}}{\partial t'} = \sum \left(I - 2\sum x'_k \cdot A_k\right)^{-1}\left(A_k \frac{\partial \tilde{u}}{\partial x'_k}\right) + \left(I - 2\sum x'_k A_k\right)^{-1} B\tilde{u}. \tag{7.4}$$

$\tilde{u}$ has Cauchy data zero and hence the energy inequality shows that $\tilde{u}(x', t') \equiv 0$ and hence u vanishes on D_ϵ. □

Similarly it can be proved that u vanishes in $D_\epsilon^{-1} = \{(x,t) : t \leq 0$, $\sum x_j^2 + t < \epsilon$ and this completes the proof. We now prove the following lemma due to H.F. Weinberger (Weinberger [1]).

Lemma 1. *Suppose A is a constant matrix such that for all real $\xi \neq 0$, $\det(\lambda I - \sum A_k \cdot \xi_k) = 0$ has real and distinct roots $\lambda_1(\xi) < \ldots < \lambda_N(\xi)$. If $\lambda_{\max}$ denotes $\sup\limits_{|\xi|=1}(\lambda_N(\xi))$ and $\alpha = \begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_N\end{pmatrix}$ is a real vector $\neq 0$ with*

114 $|\alpha| \leq \dfrac{1}{\lambda_{\max}}$ *then* $\det(\mu B - \sum A_k \cdot \xi_k) = 0$, $B = I - A \cdot \alpha$, *has real and distinct roots for any real* $\xi \neq 0$.

Remark. From the choice of α it follows that B is invertible.

Proof. First we assert that all the eigen values ν_k of B are positive. For, they are the roots of

$$\begin{aligned}\det(\nu I - B) &= \det(\nu I - (I - A \cdot \alpha)) \\ &= (-1)^N \det((1-\nu)I - A \cdot \alpha) = 0.\end{aligned}$$

and hence

$$1 - \nu_k = \lambda_k(\alpha) = |\alpha| \lambda_k(\frac{\alpha}{|\alpha|})$$

which implies that

$$\nu_k = 1 - |\alpha| \lambda_k(\frac{\alpha}{|\alpha|}) > 0 \tag{7.5}$$

since $\lambda_k(\xi) < \dfrac{1}{|\alpha|}$ on $|\xi| = 1$. Consider

$$\det(\mu B - \lambda I - A \cdot \xi) = (-1)^N \det((\lambda - \mu)I + A(\xi + \mu\alpha)) = 0$$

and let $\varphi_1(\mu), \ldots, \varphi_N(\mu)$ be the roots of the equation (with respect to λ)

$$\det((\lambda - \mu)I + A(\xi + \mu \cdot \alpha)) = 0$$

for a fixed ξ. We can write

$$\det((\lambda - \mu)I + A(\xi + \mu \cdot \alpha)) = (\lambda - \varphi_1(\mu)) \cdots (\lambda - \varphi_N(\mu)).$$

Now we assert that

(i) $\varphi_j(\mu) \to I\infty$ as $\mu \to \pm\infty$

(ii) $\varphi_j(\mu)$ are strictly increasing functions of μ. Since we have

$$\varphi_k(\mu) - \mu = \lambda_k(-\xi - \mu - \alpha) \text{ or}$$

(7.6) $$\varphi_k(\mu) = \mu - \lambda_k(\xi + \mu \cdot \alpha)$$

it follows that for each fixed $\mu, \varphi_k(\mu)$ are real and distinct. To show (i) consider det $(\mu B - \lambda I - A \cdot \xi) = 0$ which implies that det $(B - \frac{\lambda}{\mu} I - \frac{A \cdot \xi}{\mu}) = 0$. For a fixed ξ, $\frac{\varphi_k(\mu)}{\mu}$ tends to the eigen values of B as $\mu \to \infty$ and hence for large $\mu \varphi_k(\mu) \sim \mu \cdot \nu_k$. Since ν_k are positive, $\varphi_k(\mu)$ behaves like μ for large μ. 115

As for (ii), suppose on the contrary there exists a j_0 and μ_1, μ_2 with $\mu_1 < \mu_2$ such that $\varphi_{j_0}(\mu_1) > \varphi_{j_0}(\mu_2)$. Then there exists $a\lambda_0$ such that for three distinct μ'_1, μ'_2, μ'_3 we have

$$\varphi_{j_0}(\mu'_1) = \varphi_{j_0}(\mu'_2) = \varphi_{j_0}(\mu'_3) = \lambda_0.$$

Since each $\varphi_j(\mu)(j \neq j_0)$ contributes at least one root of det $(\mu B - \lambda_0 I - A \cdot \xi) = 0$ it will have at least $N + 2$ roots. This being an equation of degree N we are lead to a contradiction. Now putting

$$\lambda = 0, \det(\mu B - A.\xi) = (-1)^N \varphi_1(\mu)\varphi_2(\mu) \cdots \varphi_N(\mu).$$

Since every $\varphi_j(\mu)$ has only one zero and the zeros are distinct, we have the lemma. □

Remark. Since $\lambda_j(-\xi) = -\lambda_j(\xi)$ for every $j, \lambda_{\max}$ is positive and equal to $\sup\limits_{\substack{|\xi|=1 \\ 1 \leq j \leq N}} |\lambda_j(\xi)|$.

116 **Corollary 1.** *Let M be a regularly hyperbolic system in $\Omega = \underline{R}^n \times [0, h]$, $\lambda_j(x, t, \xi)$ be the roots of det $(\lambda I - A \cdot \xi) = 0$ and let*

(7.7) $$\lambda_{\max} = \sup_{\substack{|\xi|=1, (x,t) \epsilon \Omega \\ 1 \leq j \leq N}} |\lambda_j(x, t, \xi)|.$$

Suppose S is a hypersurface in Ω passing through a point (x_0, t_0) and defined by an equation $\varphi(x, t) = 0$, $\varphi \epsilon \mathscr{E}^2$ with

(7.8) $$\left(\frac{\partial \varphi}{\partial t}\right)^2 \geq \lambda^2_{\max} \sum \left(\frac{\partial \varphi}{\partial x_j}\right)^2.$$

If u is a C^1 solution of $M[u] = 0$ with $u(x, t) = 0$ for $(x, t) \epsilon S$ then $u(x, t) \equiv 0$ in a neighbourhood of (x_0, t_0).

Proof. By a change of coordinates $x'_j = x_j(1 \leq j \leq n)t' = \varphi(x,t)$ the system M is transformed into the system

$$\left(\frac{\partial\varphi}{\partial t}I - \sum A_k \frac{\partial\varphi}{\partial x_k}\right)\frac{\partial\tilde{u}}{\partial t'} = \sum A_k(x,t)\frac{\partial\tilde{u}}{\partial x'_k} + \cdots \tag{7.9}$$

where $\tilde{u}$ is, as before, the image of u by this mapping. S is mapped into $t' = 0$. Taking

$$\alpha = \left(\frac{\partial\varphi}{\partial x_1}\middle|\frac{\partial\varphi}{\partial t}, \ldots, \frac{\partial\varphi}{\partial x_n}\middle|\frac{\partial\varphi}{\partial t}\right)$$

the conditions of the lamma 1 are satisfied because of the assumptions on α and hence $\left(\frac{\partial\varphi}{\partial t}I - \sum A_k\frac{\partial\varphi}{\partial x_k}\right)$ is invertible. Thus $\tilde{u}$ satisfies

$$\frac{\partial\tilde{u}}{\partial t'} = \left(\frac{\partial\varphi}{\partial t}I - \sum A_k \cdot \frac{\partial\varphi}{\partial x_k}\right)^{-1}\sum A_k\frac{\partial\tilde{u}}{\partial x'_k} + \ldots \tag{7.10}$$

This is again a regularly hyperbolic system since **117**

$$\det\left(\lambda I - \left(\frac{\partial\varphi}{\partial t}I - \sum A_k(\frac{\partial\varphi}{\partial x_k})\right)^{-1}\sum A_k \cdot \xi_k\right)$$
$$= \det\left(\frac{\partial\varphi}{\partial t}I - \sum A_k\frac{\partial\varphi}{\partial x_k}\right)^{-1} \cdot \det\left(\lambda\left(\frac{\partial\varphi}{\partial t}I - \sum A_k\frac{\partial\varphi}{\partial x_k}\right) - A \cdot \xi\right)$$

and by the lemma its roots are real and distinct for

$$\alpha = (\frac{\partial\varphi}{\partial x_1}\bigg|\frac{\partial\varphi}{\partial t}, \ldots, \frac{\partial\varphi}{\partial x_n}\bigg|\frac{\partial\varphi}{\partial t}).$$

Thus by the local uniqueness (Prop. 1) $\tilde{u}$ vanishes in a neighbourhood of the origin and hence u vanishes identically in a neighbourhood of (x_0, t_0). □

Proposition 2. *Let M be a regularly hyperbolic system in $\Omega = \underline{R}^n \times [0,h]$, $(x_0,t_0)\epsilon\Omega$ and C be the backward cone defined by $\left\{t - t_0 = \alpha_0|x - x_0|, t < t_0\right.$ where $\left.\alpha_0 = \frac{1}{\lambda_{\max}}\right\}$. Let D be the interior of this backward cone* *belonging to Ω. If u is a $\mathscr{C}^1$ solution of $M[u] = 0$ in D, continuous upto the cone, and vanishing on $D_0 = D \cap (t = 0)$, then u vanishes identically in $D + C$ in particular $u(x_0, t_0) = 0$.*

Proof: (F. John [1])we first remark that $u(x, t)$ vanishes identically in a neighbourhood of the hyperplane $t = 0$. Let $S_\theta(0 < \theta \leq t_0^2)$ be a one parameter family of hyper-surfaces $\varphi(x, t, \theta) = 0$ where

$$\varphi(x, t, \theta) = (t - t_0)^2 - \alpha_0^2 |x - x_0|^2 - \theta \tag{7.11}$$

Then $\cup S_\theta \supset D$ and

(7.12)
$$\left(\frac{\partial\varphi}{\partial t}\right)^2 \Bigg| \sum \left(\frac{\partial\varphi}{\partial x_k}^2\right) = \frac{(t - t_0)^2}{\alpha_0^4 |x - x_0|^2} = \frac{\alpha_0^2 |x - x_0|^2 + \theta}{\alpha_0^4 |x - x_0|^2} > \frac{1}{\alpha_0^2} = \lambda_{\max}^2$$

118 Hence, it follows from the lemma that if u vanishes on S_{θ_0} for some θ_0 then it vanishes on S_θ for θ in a neighbourhood of θ_0. The set of θ for which u vanishes on S_θ is therefore open. It is also closed and non-empty. Hence it is the whole set. Thus u vanishes in the whole cone $D + C$. ◻

Remark 1. This result holds also for a single equation of order m and can be proved by writing it as a system by means of singular integral operators and applying the above arguments.

Remark 2. Form Prop. 2 above it follows that if the Cauchy data has for support a small set containing the origin then the support of the solution lies in some cone limited by lines whose slope $\dfrac{1}{\alpha} \geq \lambda_{\max}$. This is interpreted as follows: the maximum speed of propagation of the disturbance is less than $\lambda_{\max}$.

Remark 3. The above proposition gives a unique continuation theorem for solutions of systems of some semi linear equations:

$$M[u] \equiv \frac{\partial u}{\partial t} - \Sigma A_k(x, t) \frac{\partial u}{\partial x_k} - f(x, t, u) \tag{7.13}$$

where $A_k(x, t)$ satisfy the same conditions as in Prop.1 and $f \epsilon \mathscr{E}^1_{x,t}$. More precisely if u_1 and u_2 are two solutions of $M[u] = 0$ such that $u_1(x, 0) =$

$u_2(x, 0)$ for $x\epsilon D_0$ then $u_1 \equiv u_2$ in the whole of the cone D with D_0 as base. For, $v = u_1 - u_2$ satisfies

$$\frac{\partial v}{\partial t} - \sum A_k \frac{\partial v}{\partial x_k} - \{f(x, t, u_1) - f(x, t, u_2)\} = 0. \tag{7.14}$$
$$v(x, 0) = 0 \text{ for } x\epsilon D_0$$

By the mean value theorem $f(x, t, u_1) - f(x, t, u_2) = B(x, t)(u_1, -u_2)$ 119
$= B(x, t)v$, $B(x, t) = \dfrac{\partial f}{\partial u}(x, t, u_2 + \theta(u_1 - u_2)$. By Prop. 2 we have $v \equiv 0$ in C and hence $u_1 \equiv u_2$ in D.

Finally we apply the method of sweeping a cone by a one parameter family of surfaces to show that the solutions of second order parabolic equations have no lacuna.

Consider a parabolic equation of the second order

$$\left(\frac{\partial}{\partial t} - L\right)[u] = 0 \tag{7.15}$$

where $L = \sum\limits_{j,k=1}^{n} a_{jk}(x, t)\dfrac{\partial^2}{\partial x_j \partial x_k} + \sum\limits_{j} b_j(x, t)\dfrac{\partial}{\partial x_j} + c(x, t)$ with infinitely differentiable real coefficients and a_{jk} satisfy further the condition

$$\sum_{j,k=1}^{n} a_{jk}(x, t)\xi_j\xi_k \geq \delta(x, t)|\xi|^2, \tag{7.16}$$

$\delta(x, t) > 0$, for real $\xi \neq 0$. It is known that the unique continuation across time like hyperplanes holds in the sense that if u is a C^2 solution of the above parabolic equation with

$$u(x, t)\Big|_{x_1=0} = 0, \quad \frac{\partial u}{\partial x_1}(x, t)\Big|_{x_1=0} = 0.$$

in some neighbourhood of the origin in $X_1 = 0$ then $u(x, t) \equiv 0$ in a neighbourhood of the origin in the $(x, t)-$ space (see Mizohatai [4], Memoines of the college of Science, Kyoto University, 1958)

Proposition 3. *Suppose M is a parabolic operator of the second order* 120 *defined in* $\Omega = \underline{R}^n \times [0,h]$ *and suppose a* C^1 *solution* u *of* $M[u] = 0$ *vanishes on a non-empty open set* θ *of* Ω *then* $u \equiv 0$ *in a horizontal component* T *of* Ω *containing* θ*.*

By horizontal component T *of* θ *in* Ω *we mean the set* $\{(x,t) \in \Omega\}$ *such that there exists an* x' *with* $(x',t) \in \theta$*.*

Proof. Suppose S is a hypersurface defined by an equation

$$\varphi(x,t) = 0, \varphi \varepsilon \mathscr{E}^2_{x,t}$$

such that the tangent space of S at the origin is not paralled to $t = 0$. Then $\sum \left|\frac{\partial \varphi}{\partial x_j}\right| \neq 0$. Suppose $\frac{\partial \varphi}{\partial x_j} \neq 0$; then one can solve for x_1 in a neighbourhood of the origin as $x_1 = \psi(x_2, \ldots, x_n, t)$. By a change of variables

$$t' = t, x'_1 = x_1 - \psi(x_2, \ldots, x_n, t),\ x'_j = x_j (j = 2, \ldots, n)$$

S will be transformed into $(x'_1 = 0)$ and the form of the equation remains unaltered. Hence by the remark above the transformed function $\tilde{u}$ vanishes in a neighbourhood of the origin and hence u vanishes in a neighbourhood of the origin on S. We may assume $\mathscr{O}$ to be a neighbourhood of the origin and consider a one-parameter family of ellipsoods S_θ defined by

$$\varphi(x,t,\theta) = \frac{t^2}{a^2} + \frac{|x|^2}{\theta^2} - 1 = 0 (0 < \theta < \infty)$$

with the condition that the tangent space to this is not parallet to $(t = 0)$. Again by the argument of connectedness, as before, we obtain the proposition. ◻

8 Existence theorems

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In this section we prove some theorems on the existence of solutions of the Cauchy problem for hyperbolic equation. To begin with we recall the Hille-Yosida theorem on the infinitesimal generator of a semi group of operators on a Banach space. This is used to assert the existence of solutions.

Theorem 1 (Hille-Yosida). *Let X be a Banach space and A be a linear operator on X with domain of definition $\mathscr{D}_A$ dense in X. Assume that A has the following property:*

(P) *there exists a real number $\varepsilon_0 > 0$ such that for every real number λ with $|\lambda| < \varepsilon_0$ we have*

(1) $(I - \lambda A)$ *is a one to one surjective mapping of $\mathscr{D}_A$ onto X,*

(2) *there exists a constant $\gamma > 0$ such that*

$$||(I - \lambda A)u|| \geq (1 - \gamma|\lambda|)||u||$$

for every $u \in \mathscr{D}_A$. Then for any given $u_0 \in \mathscr{D}_A$ there exists in $-\infty < t < \infty$ a once continuously differentiable solution

$$\frac{du}{dt}(t) = Au(t) \text{ with } u(0) = u_0 \tag{8.1}$$

with values in $\mathscr{D}_A$.

Corollary . *Let A be a linear operator with domain of definition $\mathscr{D}_A$ dense in X and possessing the property (P) of Th. 1. If $t \to f(t) \in \mathscr{D}_A$ is a continuous function of t such that $t \to Af(t) \in X$ is a continuous function of t and a $u_0 \in \mathscr{D}_A$ is given there exists a once continuously differentiable solution $u(t)$ (with values in $\mathscr{D}_A$) of*

$$\frac{du}{dt}(t) = Au(t) + f(t) \text{ with } u(0) = u_0 \tag{8.2}$$

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We first consider the case of systems whose coefficients do not depend on t.

We remark that for a differential operator it is not in general possible to secure the condition $P(2)$ when we take L^2 for the Banach space X even when (8.1) is well posed in the space L^2. For, suppose the condition $P(2)$ is satisfied.

$$\begin{aligned}||(I - \lambda A)u||^2 &= ||u||^2 + \lambda^2||Au||^2 - \lambda((A + A^*)u, u)\\ &\geq (1 - \gamma|\lambda|)||u||^2.\end{aligned}$$

As $|\lambda|$ can be taken arbitrarily small this would imply if $|\lambda|$ is small that

$$((A + A^*)u, u) \leq \gamma\|u\|^2 \text{ for } \lambda > 0 \text{ and}$$
$$((A + A^*)u, u) \geq -\gamma\|u\|^2 \text{ for } \lambda < 0$$

which togeter imply

$$|((A + A^*)u, u)| \leq \gamma\|u\|^2$$

This would mean, when we take $A = \sum A_k(x)\dfrac{\partial}{\partial x_k}$, $A_k \in \mathbb{B}'$, that $A_k = A_k^*$. In fact, $A + A^* = \sum(A_k - A_k^*)\dfrac{\partial}{\partial x_k} - \dfrac{\partial A_k^*}{\partial x_k}$, and it is easy to see that the above inequality holds if and only if $A_k \equiv A_k^*(k = 1, 2, \ldots, n)$. We then proceed to study the system

$$\frac{\partial u}{\partial t}(t) = \sum A_k(x)\frac{\partial u}{\partial x_k} + B(x)u + f \tag{8.3}$$

We take for the operator A the differential operator

$$A = \sum A_k(x)\frac{\partial}{\partial x} + B(x) \tag{8.4}$$

123 in $\mathscr{D}^1_{L^2}$. We take for the domain of definition of A the set

$$\mathscr{D}_A = \left\{u \in \mathscr{D}^1_{L^2} : Au \in \mathscr{D}^1_{L^2}\right\}. \tag{8.5}$$

We remark that $\mathscr{D}^2_{L^2} \subset \mathscr{D}_A$ and consequently $\mathscr{D}_A$ is dense in $\mathscr{D}^1_{L^2}$. A is a closed operator in the sense that its graph is closed. In fact, let $u_p \in \mathscr{D}_A$ be a sequence such that $u_p \to u_0$, $Au_p \to v_0$ in $\mathscr{D}^1_{L^2}$. Since A is a continuous operator from $\mathscr{D}^1_{L^2}$ into L^2 we have $Au_0 = v_0$ in L^2 and since the injection of $\mathscr{D}^1_{L^2}$ into L^2 is bi-unique $Au_0 = v_0$ in $\mathscr{D}^1_{L^2}$, that is $u_0 \in \mathscr{D}_A$.

Proposition 1. *Let*

$$\frac{\partial u}{\partial t} = \sum A_k(x)\frac{\partial u}{\partial x_k} + B(x)u + f \tag{8.3}$$

be a regularly hyperbolic system in $\Omega = \underline{R}^n \times [0,h]$ *with* $A_k \in \mathbb{B}^{i+\sigma}$, $B \in \mathbb{B}^1$ *and* $f \in \mathscr{D}_A[0,h]$*. Then, given* $u_0 \in \mathscr{D}_A$ *there exists a unique solution* $u \in \mathscr{D}_A[0,h]$*, which is a differentiable function of* t *in the sense of* L^2 *with values in* $\mathscr{D}_A$ *of* (8.3) *for which* $u(0) = u_0$*.*

Proof. We write the system in the singular integral operator form

$$\frac{d}{dt}u = (i\mathscr{H} \wedge +B)u + f \tag{8.6}$$

and $A = i\mathscr{H} \cap +B$. By the condition of regular hyperbolicity of (8.3) there exists a bounded singular integral operator $\mathfrak{N}$ such that

$$\mathfrak{N}0\mathscr{H} = \mathscr{D}0\mathscr{H}$$

where $\mathscr{D}$ is a singular integral matrix whose symbol is 124

$$\sigma(\mathscr{D}) = \begin{pmatrix} \lambda_1(x,\xi) & & 0 \\ & \ddots & \\ 0 & & \lambda_N(x,\xi) \end{pmatrix}$$

and $|\det \sigma|(\mathfrak{N})| > \delta > 0$.

Define a bilinear form by

$$(Lu, v) = (\mathfrak{N} \wedge u, \mathfrak{N} \wedge v) + \beta(u, v) = ((\lambda\mathfrak{N} * \mathfrak{N}\Lambda + \beta I)u, v). \tag{8.7}$$

for $u, v\varepsilon\mathscr{D}^1_{L^2}$ with a β to be chosen later. (Lu, u) defines a norm equivalent to that of $\mathscr{D}^1_{L^2}$ for sufficiently large β. In fact, since $\mathfrak{N}$ is a bounded operator in L^2 we have

$$(Lu, u) \leq \|\mathfrak{N}\|^2_{\mathscr{L}(L^2,L^2)}\| \wedge u\|^2 + \beta\|u\|^2 \leq M\|u\|^2_{\mathscr{D}^1_{L^2}}.$$

On the other hand by Gårding's inequality there exists *a* $\gamma > 0$ such that

$$(Lu, u) \geq \delta'\| \wedge u\|^2 - \gamma\|u\|^2 + \beta\|u\|^2,$$

then for sufficiently large $\beta(> \gamma)$ this would imply that

$$(Lu, u) \geq c\|u\|^2_{\mathscr{D}^1_{L^2}}$$

which proves the assertion. We provide $\mathscr{D}^1_{L^2}$ with the norm (Lu, u). We proceed to verify conditions 1, 2, of the Hille-Yosida Theorem. To prove **125**
condition $P(2)$ we must prove that for real λ near the origin

$$(8.8)\quad (L(I-\lambda A)u, (I-\lambda A)u) \geq (1-\gamma|\lambda|)(Lu, u) \text{ for every } u \in \mathscr{D}_A.$$

To do this we assume at first that $u \in \mathscr{D}^2_{L^2}$ we have then,

$$\begin{aligned}(L(I-\lambda A)u, (I-\lambda A)u) &= (Lu, u) + \lambda^2(LAu, Au) - \lambda((LA + A^*L)u, u)\\ &\geq (Lu, u) - \lambda((LA + A^*L)u, u).\end{aligned}$$

Since $A = i\mathscr{H}\Lambda + B$ we have

$$(LA + A^*L) = (\Lambda\mathfrak{R}^*\mathfrak{R}\Lambda + \beta I)(i\mathfrak{R}\wedge + B) + (-i \cap \mathscr{H}^* + B^*)(\wedge\mathfrak{R}^*\mathfrak{R}\wedge + \beta I)$$

But $\mathfrak{R}\wedge\mathscr{H} \equiv D\wedge\mathfrak{R} \mod (\wedge^0)$ where $P_1 \equiv P_2 \mod (\wedge^0)$ means that $P_1 - P_2$ is a bounded operator in $\mathscr{D}^1_{L^2}$.

In fact,

$$\begin{aligned}\mathfrak{R}\wedge\mathscr{H} &\equiv \mathfrak{R}\mathscr{H}\wedge \equiv (\mathfrak{R}\circ\mathscr{H})\wedge - (\mathfrak{R}\circ\mathscr{H} - \mathfrak{R}\mathscr{H})\wedge\\ &\equiv (\mathfrak{R}\circ\mathscr{H})\wedge \mod (\Lambda^), \text{ (since } \mathfrak{R}\circ\mathscr{H} = \mathscr{D}\circ\mathfrak{R})\\ &\equiv (\mathscr{D}\circ\mathfrak{R})\wedge \equiv \mathscr{D}\mathfrak{R}\wedge \equiv \mathscr{D}\wedge\mathfrak{R}\mathrm{mod}(\wedge^0).\end{aligned}$$

Hence

$$\begin{aligned}((LA + A^*L)u, u) = i\{&(D\wedge\mathfrak{R}\wedge u, \mathfrak{R}\wedge u) - (\mathfrak{R}\wedge u, \mathscr{D}\wedge\mathfrak{R}\wedge u)\\ &+ 2\,\mathrm{Re}(\mathbb{B}_1\wedge u, \mathfrak{R}\wedge u),\end{aligned}$$

where $\mathbb{B}_1$ is a bounded operator in L^2. Now

$$\mathscr{D}\wedge - \wedge\mathscr{D}* \equiv \mathscr{D}\wedge - \wedge\mathscr{D}^\# \equiv (\mathscr{D}\wedge - \mathscr{D}^\#\wedge) \equiv (\mathscr{D} - \mathscr{D}^\#)\wedge .$$

Since $\mathscr{D}$ is a diagonal matrix and $\sigma(\mathscr{D})$ is real, we see that $\mathscr{D} = \mathscr{D}^\#$.
126 Hence $\mathscr{D}\wedge - \wedge\mathscr{D}^* \equiv \mathrm{mod}(\wedge^0)$. Hence there exists a constant γ_1 such that

$$-\gamma_1\|u\|^2_{\mathscr{D}^1_{L^2}} \leq ((LA + A^*L)u, u) \leq \gamma_1\|u\|^2_{\mathscr{D}^1_{L^2}}$$

or equivalently we write following Leray [1]

$$-\gamma_1(\wedge+1)^2 \leq LA + A^*L \leq \gamma_1(\wedge+1)^2$$

and thus, as $||u||^2_{\mathscr{D}^1_{L^2}}$ and (Lu, u) are equivalent we obtain

$$(L(I-\lambda A)u,\ (I-\lambda A)u) \geq (1-\gamma_1|\lambda|)(Lu, u)$$

for $|\lambda| < \dfrac{1}{\gamma_1}$.

Next the inequality (8.8) holds for all $u \in \mathscr{D}_A$ also. Suppose $u \in \mathscr{D}_A$. If φ_δ are mollifiers of Friedrichs then the function $u_\delta = u * \varphi_\delta$ belongs to $\mathscr{D}^2_{L^2}$ and it follows from (8.8) that there exists a constant γ_1 such that for some real near the origin

$$(L(I-\lambda A)u_\delta,\ (I-\lambda A)u_\delta) \geq (1-\gamma_1|\lambda|)(Lu_\delta, u_\delta).$$

But

$$Au_\delta \to Au \text{ in } \mathscr{D}^2_{L^2} \text{ as } \delta \to 0.$$

In fact, $Au_\delta - Au = (Au_\delta - \varphi_\delta * (Au)) + (\varphi_\delta * (Au) - Au)$ in which the first term tends to 0 in $\mathscr{D}^1_{L^2}$ by Friderich's lemma and the latter term tends to 0 in $\mathscr{D}^1_{L^2}$ since $Au \in \mathscr{D}^1_{L^2}$. Thus condition $P(2)$ of Hille-yosida Theorem is verified. To prove condition $P(1)$ we must prove that $(I-\lambda A)$ is a one-to-one surjective mapping of $\mathscr{D}_A$ onto $\mathscr{D}^1_{L^2}$ for sufficiently small **127**
λ. From (8.8) it follows that $(I-\lambda A)$ is one-to-one for $|\lambda| < \dfrac{1}{\gamma_1}$.

Next $(I-\lambda A)\mathscr{D}_A$ is closed in $\mathscr{D}^1_{L^2}$. For, $(I-\lambda A)u_n \to v_0$ in $\mathscr{D}^1_{L^2}$ for $u_n \in \mathscr{D}_A$ means by (8.8) that u_n is a Cauchy sequence for the new norm hence has a unique limit u_0 in $\mathscr{D}^1_{L^2}$. Hence $-\lambda Au_n \to v_0 - u_0$ in $\mathscr{D}^1_{L^2}$. As A is a closed mapping $u_0 \in \mathscr{D}_A$ and $(I-\lambda A)u_0 = v_0$.

Finally we prove that $(I-\lambda A)\mathscr{D}_A$ is dense in $\mathscr{D}^1_{L^2}$. The proof is by contradiction. Suppose $(I-\lambda A)\mathscr{D}_A$ is not dense in $\mathscr{D}^1_{L^2}$. Then there exists a$\psi \in \mathscr{D}^1_{L^2}$, $\psi \neq 0$ such that $((I-\lambda A)u, \psi)_1 = 0$ i.e. $((\wedge+1)(I-\lambda A)u,$ $(\Lambda+1)\psi) = 0$ for all $u \in \mathscr{D}_A$, that is, $(I-\lambda A^*)(\wedge+1)\psi_1 = 0$ where $A^* = -i\wedge\mathscr{H}^* + B^*$ and $\psi_1 = (\wedge+1)\psi\varepsilon L^2$.

Now $A^*(\wedge+1)\psi_1 = (-i\wedge\mathscr{H}^* + B^*)(\wedge+1)\psi_1 = (\wedge+1)(-i\wedge\mathscr{H}^* + B^*)\psi_1 + B_0\psi_1$. where $B_0 = -i\wedge(\mathscr{H}^*\wedge - \wedge\mathscr{H}^*) + (B^*\wedge - \wedge B^*)$.

Further $A^*(\wedge + 1)\psi_1 = (\wedge + 1)(-i\mathscr{H}^\# \wedge + B^* + B_1)\psi_1 + B_0\psi_1$ where $B_1 = (-i\mathscr{H}^\# \wedge + i\mathscr{H}^\# \wedge)\varepsilon\mathscr{L}(L^2, L^2)$.

But $B = B_1 + (\wedge + 1)^{-1}B_0 + B^*$ is a bounded operator in L^2 and hence $(I - \lambda A^*)(\wedge + 1)\psi_1 = 0$ is equivalent to saying that $[I - \lambda(-i\mathscr{H}^\# \wedge + \tilde{B}]\psi_1 = 0$, which in turn is equivalent to saying that $[I - \lambda(-i\mathscr{H}^\# \wedge + \tilde{B}]\psi = 0$. Starting from the equation

$$\frac{\partial}{\partial t}u = -\sum {}^t\overline{A}_k(x)\frac{\partial}{\partial x_k}u - \tilde{B}u \tag{8.9}$$

128 and using (8.8) after observing that $\psi \in \mathscr{D}_A$ we obtain an inequality

$$\begin{aligned}(L_1(I - \lambda(-i\mathscr{H}^\# \wedge + \tilde{B}))\psi, (I - \lambda(-i(\mathscr{H}^\# \wedge + \tilde{B}))\psi) \\ \geq (1 - \gamma|\lambda|)\,(L_1\psi, \psi)\end{aligned} \tag{8.10}$$

which implies that $||\psi|| = 0$ and hence $\psi = 0$ which is a contradiction to the assumption.

Now all the conditions of Hille-Yosida theorem for $A = i\mathscr{H} \wedge + B$ are verified and hence there exists a solution of the equation

$$\frac{d}{dt}u = (i\mathscr{H} \wedge + B)u + f \text{ with } u(0) = u_0$$

with the required properties. □

In the above proposition we proved the existence of solutions of regularly hyperbolic systems when $u_0 \in \mathscr{D}_A$ in particular when $u_0 \in \mathscr{D}^2_{L^2}$ and $f \in \mathscr{D}_A[0, h]$ and so in particular when $f \in \mathscr{D}^2_{L^2}$. This result can be improved as follows.

Proposition 2. *Suppose* (8.3) *is a regularly hyperbolic system in* $\Omega = \underline{R}^n \times [0, h]$ *with* $A_k \in \mathbb{B}^{1+\sigma}$, $B \in \mathbb{B}^1$, $u_0 \in \mathscr{D}^1_{L^2}$ *and* $f \in \mathscr{D}^1_{L^2}[0, h]$. *Then there exists* $u \in \mathscr{D}^1_{L^2}[0, h]$ *(once differentiable in t in the sense of* L^2*) satisfying the system in the* L^2*-sense and* $u(0) = u_0$. *Also the following energy inequality holds:*

$$(Lu(t), u(t)) \leq \exp(\gamma t) \cdot (Lu(0), u(0))$$

(8.12) $$+\int_0^t (L(f(s)), f(s)) \exp(\gamma(t-s)) ds$$

where (Lu, u) *is defined in Prop. 1.*

Proof. We regularize u_0 and f by mollifiers of Friedrichs φ_δ to obtain $u_0 * \varphi_\delta = u_0^\delta \in \mathscr{D}^2_{L^2}$, $f * \varphi_\delta = f_\delta \in \mathscr{D}^2_{L^2}[0, h]$. By prop. 1 applied to u_0^δ, f_δ there exists a u_δ continuous and with values in $\mathscr{D}_A$ satisfying **129**

(8.12) $$\frac{\partial}{\partial t} u_\delta = \sum A_k(x) \frac{\partial}{\partial x_k} u_\delta + Bu_\delta + f_\delta$$

and $u_\delta(0) = u_0^\delta$. Further $u_\delta(t) - u_{\delta'}(t)$ satisfies the equation

$$\frac{\partial}{\partial t}[u_\delta(t) - u_{\delta'}(t)] = \sum A_k(x) \frac{\partial}{\partial x_k}[u_\delta(t) - u_{\delta'}(t)] + B[u_\delta(t) - u_{\delta'}(t)] + (f_\delta - f_{\delta'})$$

and hence by the energy inequality

(8.13)

$$\|u_\delta(t) - u_{\delta'}(t)\|_1 \leq c(h) \left\{ \|u_\delta(0) - u_{\delta'}(0)\|_1 + \int_0^h \|f_\delta(s) - f_{\delta'}(s)\|_1 ds \right\},$$

which shows that $\{u_\delta(t)\}$ is a Cauchy sequence in the space of continues functions with values in $\mathscr{D}^1_{L^2}$. Hence $u_\delta(t) \to u(t)$ in the space of continuous functions with values in $\mathscr{D}^1_{L^2}$. On the other hand the equation

$$u_\delta(t) - u_0^\delta = \int_0^t \{Au_\delta(s) + f_\delta(s)\} ds, \quad A = \sum A_k \frac{\partial}{\partial x_k} + B$$

holds in L^2. Passing to the limits in L^2 we obtain

$$u(t) - u_0 = \int_0^t \{Au(s) + f(s)\} ds.$$

Differentiating this, we see that the relation

$$\frac{d}{dt} u(t) = Au(t) + f(t)$$

holds in the sense of L^2 where $u \in \mathscr{D}^1_{L^2}[0,h]$, $\dfrac{\partial u}{\partial t} \in L^2[0,h]$ respectively. 130
Consider now

$$\begin{aligned}\frac{d}{dt}(Lu_\delta, u_\delta) &= \left(L\frac{d}{dt}u_\delta, u_\delta\right) + \left(Lu_\delta, \frac{d}{dt}u_\delta\right) + (L'_t u_\delta, u_\delta)\\ &\le ((LA + A^* L)u_\delta, u_\delta) + 2\operatorname{Re}(Lf_\delta, f_\delta) + \gamma'(Lu_\delta, u_\delta)\\ &\le \gamma(Lu_\delta, u_\delta) + (Lf_\delta, f_\delta).\end{aligned}$$

Since $u_\delta(t)$ and $f_\delta(t)$ converge, uniformly in t, to $u(t)$, $f(t)$ respectively in $\mathscr{D}^1_{L^2}$ as $\delta \to 0$ we have (8.12). This completes the proof of the proposition. □

Remark. The above equation is a particular case of one involving singular integral operators. If in fact we consider an equation

$$\begin{aligned}\frac{d}{dt}u(t) &= i\mathscr{H} \wedge u(t) + Bu(t) + f(t)\\ &\equiv Au(t) + f(t),\end{aligned} \tag{8.14}$$

with $\sigma(\mathscr{H}) \in C^\infty_{1+\sigma}$, $B \in \mathscr{L}(L^2, L^2) \cap \mathscr{L}(\mathscr{D}^1_{L^2}, \mathscr{D}^1_{L^2})$, which is regularly hyperbolic, we could prove an analogous proposition in the same way. We would have to use the Fridrichs' lemma for singular integral operators, namely,

$$[\mathscr{H}\wedge, \varphi_\delta *] \to 0 \text{ weakly in } \mathscr{D}^1_{L^2}. \tag{8.15}$$

Now we consider the general case of regularly hyperbolic systems when the coefficients are functions of the variable t also. We use a method similar to the one of Cauchy-Peano for ordinary differential equations.

131 **Theorem 2.** *Let* $\Omega = \underline{R}^n \times [0,h]$ *and*

$$\frac{\partial}{\partial t}u = \sum A_k(x,t)\frac{\partial}{\partial x_k}u + B(x,t)u + f \tag{8.16}$$

be a regularly hyperbolic system in Ω *with* $A_k \in \mathbb{B}^{1+\sigma}[0,h]$, $B \in \mathbb{B}^1[0,h]$, $f \in \mathscr{D}^1_{L^2}[0,h]$. *Given a* $u_0 \in \mathscr{D}^1_{L^2}$ *there exists a unique solution* u *of* (8.16), *in the sense of* L^2, *which belongs to* $\mathscr{D}^1_{L^2}[0,h]$ *and is differentiable in the sense of* L^2 *for which* $u(0) = u_0$.

Proof. Consider a subdivision

$$\Delta : 0 = t_0 < t_1 \ldots < t_s = h.$$

We define a function u inductively as follows: For $t_{j-1} \leq t \leq t_j$, $u_\Delta(t) = u_j(t)$ where u_j satisfies the system

(8.18)

$$\frac{\partial}{\partial t} u_j = \sum A_k(x, t_{j-1}) \frac{\partial}{\partial x_k} u_j + B(x, t_{j-1}) u_j + f, u_j(t_{j-1}) = u_{j-1}(t_{j-1})$$

for $j = 1, \ldots, s$. By Prop. **??** there exists a unique solution $u_j \in \mathscr{D}^1_{L^2}$ for this system for $j = 1, \ldots, s$. Thus $u_\Delta(t)$ is uniquely determined. We shall show that u_Δ is uniformly bounded for small subdivisions (subdivisions of small norms), that is,

$$\sup_{t \in [0,h]} \|u_\Delta(t)\|_1 \leq M < \infty.$$

It follows from (8.18) using the given conditions on the coeficients that

$$\sup_{t \in [0,h]} \|\frac{d}{dt} u_\Delta(t)\|_{L^2} \leq M' < \infty.$$

Hence $\{u_\Delta(t)\}$ is a bounded set in $\mathscr{E}^1_{L^2}(\Omega)$ as Δ runs through subdivisions of small norm. Thus by choosing a suitable subsequence of Δ, $u_\Delta \to u$ weakly in $\mathscr{E}^1_{L^2}(\Omega)$ and u satisfies **132**

(8.18)
$$\frac{\partial u}{\partial t} = \sum A_k(x, t) \frac{\partial u}{\partial x_t} + B(x, t) u + f$$

$$u_\Delta \to u, \ \frac{\partial u_\Delta}{\partial x_k} \to \frac{\partial u}{\partial x_k}, \ \frac{\partial u_\Delta}{\partial t} \to \frac{\partial u}{\partial t} \text{ weakly in } L^2(\Omega)$$

and these derivatives are taken in the sense of distribution in Ω.

Next we shall show that $u \in \mathscr{D}^1_{L^2}[0, h]$ and $u(0) = u_0$. For almost all t, $u(x, t)$, as a function of t for each fixed x, is absolutely continuous (see Sehwartz [1]). Hence we can write

$$u(x, t') - u(x, t'') = \int_{t'}^{t''} \frac{\partial u}{\partial t}(x, t) \, dt$$

the derivative in the right hand side is taken in the distribution sense. By the Schwarz inequality

$$|u(x,t') - u(x,t'')|^2 \leq |t' - t''| \int\limits_{t'}^{t''} |\frac{\partial u}{\partial t}(x,t)|^2 \, dt$$

which on integration with respect to x gives

$$\|u(x,t') - u(x,t'')\|_{L^2(\underline{R}^n)} \leq |t' - t''|^{\frac{1}{2}} \|\frac{\partial u}{\partial t}(x,t)\|_{L^2(\Omega)}$$

proving that $u \in L^2[0,h]$. If φ_δ denote mollifiers of Friedrichs, the function $u = u * \varphi_\delta$ satisfies $u_\delta \in \mathscr{D}^1_{L^2}[0,h]$ and

$$\frac{\partial}{\partial t} u_\delta(t) = \sum A_k(x,t) \frac{\partial u}{\partial x_k} u_\delta + B(x,t)u + f + C_\delta u \tag{8.19}$$

133 where

$$C_\delta = \sum \left[A_k \frac{\partial}{\partial x_k}, \varphi_\delta * \right] + [B, \varphi_\delta *]. \tag{8.20}$$

By Friedrichs' lemma $\|C_\delta u\|_1 \leq c\|u\|_1$ and $\|C_\delta u\|_1 \to 0$ as $\delta \to 0$ for fixed t. Since $\int\limits_0^h \|u(x,t)\|_1 dt < \infty$, it follows that $|C_\delta u\|_1$ is integrable, and from Lebesgue's bounded convergence theorem, we deduce that

$$\int\limits_0^h \|C_\delta u(x,t)\|_1 \, dt \to 0 \text{ as } \delta \to 0.$$

Now from the energy inequality for the system (8.19)

$$\|u_\delta(t)\|_1 \leq c(h) \left\{ \|u_\delta(0)\|_1 + \int\limits_0^h (\|f_\delta(s)\|_1 + \|C_\delta u(s)\|_1) ds \right\}$$

it follows that $\sup\limits_{t \in [0,h]} \|u_\delta(t)\|_1 \leq M < \infty$. Again $u_\delta(t) - u_{\delta'}(t)$ satisfies an equation

$$\frac{\partial}{\partial t}(u_\delta(t) - u_{\delta'}(t)) = \sum A_k(x,t) \frac{\partial}{\partial x_k}(u_\delta(t) - u_{\delta'}(t)$$

$$+ B(x,t)(u_\delta(t) - u_{\delta'}(t)) + C_\delta u(t) - C_{\delta'} u(t)$$

and we have the energy inequality

$$\|u_\delta(t) - u_{\delta'}(t)\|_1 \leq c'(h)\left\{\|u_\delta(0) - u_{\delta'}(0)\|_1 + \int\limits_0^h \|(C_\delta - C_{\delta'})u(s)\|_1 ds\right\}$$

which shows that $\|u_\delta(t) - u_{\delta'}(t)\|_1 \to 0$ as δ, $\delta' \to 0$. So $\{u_\delta(t)\}$ is a Cauchy sequence in $\mathscr{D}^1_{L^2}[0,h]$ and hence its limit is in $\mathscr{D}^1_{L^2}[0,h]$. By the **134**
uniqueness of limits in $L^2[0,h]$, $u_\delta \to u$ and $u \in \mathscr{D}^1_{L^2}[0,h]$. Since the operation of restriction is continuous and the restriction of u_Δ to $t = 0$, namely $u_\Delta(x,0)$), is u_0 we see that $u(x,0) = u_0$.

Now it only remains to show that $\{u_\Delta(t)\}$ is a bounded set in $\mathscr{E}^1_{L^2}$. For this we proceed as follows. We use the norm defined by

$$(Lu,u) = (\mathfrak{R} \wedge u, \mathfrak{R} \wedge u) + \beta(u,u)$$

for suitable $\beta > 0$ (see (8.7)). $\mathscr{D}^1_{L^2}$ is provided with this norm.

By the energy inequalities we have, for $j = 1,\ldots,s$

$$\begin{aligned}(8.21)\qquad (L(t_{j-1})u_\Delta, u_\Delta) &= (L(t_{j-1})u_j(t), u_j(t)) \\ &\leq \exp(\gamma(t - t_{j-1}))(L(t_{j-1})u_j(t_{j-1}), u_j(t_{j-1})) \\ &\quad + \int\limits_{t_{j-1}}^{t_j} \exp(\gamma(t-s))(L(t_{j-1})f(s), f(s))\, ds.\end{aligned}$$

The $L(t)$ depends on the $\mathfrak{R}(t)$ which form a bounded set of singular integral operators and hence by the remark ater prop. 2, § 6 we can use the same constant β to the new norm in $D^1_{L^2}$. Further letting $L_k = L(t_k)$

$$\begin{aligned}(L_j u_\Delta, u_\Delta) - (L_{j-1}u_\Delta, u_\Delta) &= \|\mathfrak{R}(t_j) \wedge u_\Delta\|^2 - \|^{\mathfrak{R}}(t_{j-1}) \wedge u_\Delta\|^2 \\ &\leq C\|(\mathfrak{R}(t_j) - \mathfrak{R}(t_{j-1})\|_{\alpha(L^2,L^2)}\|u_\Delta\|_1^\alpha.\end{aligned}$$

Since $(L_{j-1}u_\Delta, u_\Delta) \sim \|u_\Delta\|_1^2$ we have $\|u_\Delta\|_1^2 \leq k(L_{j-1}u_\Delta, u_\Delta)$ and hence

$$(L_j u_\Delta, u_\Delta) \leq (1 + k\|\mathscr{H}(t_j) - \mathfrak{R}(t_{j-1}\|_{\alpha(L^2,L^2)})(L_{j-1}u_\Delta, u_\Delta)$$

$$= (1 + \varepsilon(t_{j-1}, t_j))(L_{j-1}u_\Delta, u_\Delta).$$

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Using this in the above inequality (8.21) we have

$$(L_{s-1}u_\Delta, u_\Delta) \leq \exp(\gamma t)(L_0 u(t_0), u(t_0)) + \int_0^h \exp(\gamma(t-s))(L_0 f(s), f(s))ds \prod_{j=1}^{\mathscr{S}}\{1 + \varepsilon(t_{j-1}), t_j\}.$$

But we have, by a will-known inequality,

$$\prod_{j=1}^{\mathscr{S}}\{1 + \varepsilon(t_{j-1}, t_j)\} \leq \left(1 + \frac{1}{S}\sum \in (t_{j-1}, t_j)\right)^s \leq e^{\gamma_0}$$

where

$$\gamma_0 = \sup \sum \varepsilon(t_{j-1}, t_j) = \sup_\Delta k \sum \|\mathfrak{N}(t_j) - \mathfrak{N}(t_{j-1})\|_{\alpha(L^2, L^2)})$$
$$\leq k \int_0^h \sup_{x \in R^n} \sum_{|\nu| \leq 2n} \sup_{|\xi| \geq 1} \left|\left(\frac{\partial}{\partial \xi}\right)^\nu \frac{\partial}{\partial t}\sigma(\mathfrak{N})(x, t, \xi)\right|.$$

Hence $\{u_\Delta(t)\}$ is a bounded set in $\mathscr{E}^1_{L^2}$, this completes the proof.

If we assume taht that coefficients and the initial data u_0 and f are sufficiently smooth we can improve Theorem 2. We indicate this briefly.

We assume

$$A_k \in \mathbb{B}^2[0, h], \frac{\partial A_k}{\partial t} \in \mathbb{B}^0[0, h], B \in \mathbb{B}^2[0, h], u_0 \in \mathscr{D}^2_{L^2}, f \in \mathscr{D}^2_{L^2}, [0, h]$$

From theorem 2, we know that there exists a unique solution $u \in \mathscr{D}^1_{L^2}[0, h]$ of

(8.16) $$M[u] = f.$$

136 Differentiating with respect to x_j (denoting $\dfrac{\partial}{\partial x_j}$ by D_j) we have

$$(8.22) \quad M[D_j u] - \sum_k (D_j A_k)[D_k u] = D_j f + (D_j B)[u] \; (j = 1, 2, \ldots, n)$$

where the second member $D_j f + (D_j B)[u] \in \mathscr{D}^1_{L^2}[0, h]$ and $D_j u(0) \in \mathscr{D}^1_{L^2}$. Now (8.22) is a system of equations with unknown functions $(D_1 u, \ldots, D_n u)$ which has the same principal part M. We can show, without any singnificant modification in the previous argument, that there exists a unique solution $(D_1 u, \ldots, D_n u) \in \mathscr{D}^1_{L^2}[0, h]$. On the other hand, by the energy inequality, we can see that the system:

$$M[v_j] - \Sigma_k \, (D_j A_k)[v_k] = g_j \varepsilon \, L^2[0, h]$$

has atmost one solution v in $L^2[0, h]$. This shows that $u \in \mathscr{D}^2_{L^2}[0, h]$. ◻

Corollary 1. *Let* (8.16) *be a regularly hyperbolic system in the set* $\Omega = \underline{R}^n \times [0, T]$ *with*

$$\left(A_k(x, t), \; \frac{\partial}{\partial t} A_k(x, t)\right) \in (\mathscr{B}^2[0, T], \; \mathscr{B}^1[0, T]),$$
$$\left(B(x, t), \; \frac{\partial}{\partial t} B(x, t)\right) \in (\mathscr{B}^2[0, T], \; \mathscr{B}^1[0, T])$$

and $f(x, t) \varepsilon \mathscr{D}^2_{L^2}[0, T]$.

Then, given an element $u_0 \in \mathscr{D}^2_{L^2}$ there exists a unique solution $u \in \mathscr{D}^2_{L^2}[0, T]$ of (8.3) with $u(0) = u_0$.

Proof. Defferentiating both sides of the equation (8.3) with respect to x_j in the sense of distributions we have

$$\frac{\partial}{\partial x_j} M[u] = \frac{\partial}{\partial x_j} \frac{\partial}{\partial t} u - \frac{\partial}{\partial x_j} \left(\sum A_k(x, t) \frac{\partial}{\partial x_k}\right) - \frac{\partial}{\partial x_j} (B(x, t) u) = \frac{\partial f}{\partial x_j}$$

which can be rewritten as **137**

$$\frac{\partial}{\partial t} \left(\frac{\partial u}{\partial x_j}\right) - \sum A_k(x, t) \frac{\partial}{\partial x_j} \left(\frac{\partial u}{\partial x_j}\right) - \sum \frac{\partial A_k}{\partial x_j}(x, t) \frac{\partial u}{\partial x_k} - B(x, t) \frac{\partial u}{\partial x_j}$$
$$= \frac{\partial B}{\partial x_j}(x, t) u + \frac{\partial f}{\partial x_j}.$$

That is,

$$M\left[\frac{\partial u}{\partial x_j}\right] = -\sum \frac{\partial A_k}{\partial x_j}(x,t)\frac{\partial u}{\partial x_k} = \frac{\partial f}{\partial x_j} + \frac{\partial B}{\partial x_j}(x,t)u. \tag{8.22}$$

Setting $\dfrac{\partial u}{\partial x_j} = v_j$ for $j = 1, \ldots, n$ we obtain a new system

$$M\left[v_j\right] - \sum \frac{\partial A_k}{\partial x_j}(x,t)v_k = \varphi_j$$

and we can take for $v_j(0)$ the function $\dfrac{\partial u_0}{\partial x_j} \in \mathscr{D}^1_{L^2}$ (the derivative being taken in the sense of distributions) since $u_o \in \mathscr{D}^2_{L^2}$. If we assume $u \in \mathscr{D}^1_{L^2}[0,T]$ it follows then that $\varphi_j \in \mathscr{D}^1_{L^2}[0,T]$ since $f \in \mathscr{D}^2_{L^2}[0,T]$ and $B \in \mathscr{B}^2[0,T]$. Then by Th. 2 there exists a unique solution (in L^2) $v = (v_1, \ldots, v_n)$ with $v_j \in \mathscr{D}^1_{L^2}[0,T]$. Hence $u \in \mathscr{D}^2_{L^2}[0,T]$. □

Corollary 2. *Let* (8.16) *be regularly hyperbolic in the set* Ω *with*

$$\left(A_k, \frac{\partial A_k}{\partial t}, \ldots, \left(\frac{\partial}{\partial t}\right)^m A_k\right) \in (\mathscr{B}^m[0,T], \ldots, \mathscr{B}^0[0,T]),\ B \in \mathscr{B}^m[0,T]$$

and $f \in \mathscr{D}^m_{L^2}[0,T]$*; then given* $u_0 \in \mathscr{D}^m_{L^2}$ *there exists a unique solution* u
138 *in* $\mathscr{D}^m_{L^2}[0,T]$ *of* (8.3) *with* $u(0) = u_0$.

This can be proved by successively applying the argument of Corollary 1.

Taking $m = \left[\frac{n}{2}\right] + 2$ *we obtain, using Sobolev's lemma, the following*

Corollary 3. *Let* (8.16) *be regularly hyperbolic with*

$$\left(A_k, \frac{\partial A_k}{\partial t}, \ldots\right) \in \left(\mathscr{D}^{[\frac{n}{2}]+2}_{L^2}[0,T],\ \mathscr{D}^{[\frac{n}{2}]+1}_{L^2}, \ldots\right),\ B \in \mathscr{D}^{[\frac{n}{2}]+1}_{L^2}[0,T]$$

and $f \in \mathscr{D}^{[\frac{n}{2}]+2}_{L^2}[0,T]$ *then, given* $u_0 \in \mathscr{D}^{[\frac{n}{2}]+2}_{L^2}$ *there exists a solution* $u \in \mathscr{E}^1$ *of* (8.16) *with* $u(0) = u_0$*, unique in* L^2.

Corollary 4. *Assume that* (8.16) *is regularly hyperbolic in an open neighbourhood U of 0 in* $\underline{R}^{n+1}$, A_k, $B \in \mathscr{E}(U)$ *then there exists a neighbourhood* $U' \subset U$ *such that for any* $u_0 \subset \mathscr{E}(U \cap \{t = 0\})$, $f \in \mathscr{E}(U)$ *there exists a solution* $u \in \mathscr{E}(U')$ *of* (8.16), *unique in* L^2.

Remark. If we use a partition of unity the above arguments can be used to proved results analogous to the above corollaries in the spaces $\mathscr{E}^m_{L^2(\text{loc})}$ in place of $\mathscr{D}^m_{L^2}$.

Finally we have the following result on the existence of solutions of a single regularly hyperbolic equation of order m.

Corollary 5. *Let*

$$L[u] \equiv \left(\frac{\partial}{\partial t}\right)^{m_u} + \sum_{\substack{j+|\nu|\leq m \\ j<m}} a_{j,\nu}(x,t)\left(\frac{\partial}{\partial x}\right)^{\nu}\left(\frac{\partial}{\partial t}\right)^{j} u = g \tag{8.23}$$

be a regularly hyperbolic equation of order m in a neighbourhood of the origin with infinitely differentiable coefficients $a_{j,\nu}$. *Let g be infinitely diffrentiable in a neighbourhood U of the origin. Then given the initial conditions* 139

$$(u_o, u_1, \ldots, u_{m-1}) \in \prod \mathscr{E}(U \cap \{t = 0\})$$

there exists a solution $u \in \mathscr{E}(U')$ *in a neighbourhood* U' *such that*

$$\left(\frac{\partial}{\partial t}\right)^{j} u(x,0) = u_j(x), \quad j = 0, 1, \ldots, (m-1).$$

9 Necessary condition for the well posedness of the Cauchy problem

In chapter 2 we considered necessary condition for well posedness of the Cauchy problem when the coefficients were inepependent of x. In Chapter 3 we considered some sufficiency condition for well posedness e.g, hyperbolicity, when the coefficients depended on x.

Now we consider some necessary conditions in this later case. For simplicity we shall consider single, first order differential opearator,

$$\frac{\partial u}{\partial t} = \sum a_k(x,t)\frac{\partial u}{\partial x_k} + b(x,t)u \tag{9.1}$$

(for a fuller treatement see Mizohata [3]).

If all $a_k(x,t)$, $b(x,t)$ are real the classical method of characteristics establishes the well posedness of the Cauchy problem. However if the a_k are complex the question of existence was not settled till recently. The characteristic polynomial of the above equation is $\sum a_k(2\pi\xi_k) - \lambda$. If this has real eigenvalues i.e. a_k's are real, the Cauchy problem is well
140 posed as shown by the results of Chap III. We shall prove that if there is ξ^o such that im $\sum a_k(0,0)\xi^o \neq 0$ (say $\neq 0$), the problem is not well posed. The idea of the proof is as follows: we construct a sequence of solution $u_n(x)$, $n = 1, 2$ for which, on the hypothesis of well-posedness, we must have $\sup |u_n(x,t)| = 0(n^h)$ while on the other hand by using an energy inequality for a suitable operator we, must have a minorization by $\exp(n)$ for some functions closely related to u'_n above which will give a contradiction. More precisely we shall prove

Proposition 3. *Let*

$$\frac{\partial u}{\partial t} = H \wedge u + b(x,t)u + f \tag{9.2}$$

be an equation in the singular integral form with $\sigma(H) = h(x,t,\xi)$ *satisfying*

$$\operatorname{Re} h(x,t,\xi) \leq 0 \textit{ for all } (x,t,\xi) \textit{ and } t \to h(x,t,\xi) \in C^{\infty}_{1+\sigma} \tag{9.3}$$

is continuous. Then given any $u_o \in \mathscr{D}^1_{L^2}$ *and* $f \in \mathscr{D}^1_{L^2}[0,h]$ *there exists a unique solution*

$$u \in \mathscr{D}^1_{L^2}[0,h] \textit{ of } (9.2) \textit{ with } u(x,0) = u_o(x).$$

On the contrary, if there exists a ξ^o *such that* $\operatorname{Re} h(x,t,\xi^o) > 0$ *then the energy inequality cannot be obtained in the* L^2*-space. Of course this* *does not immediately imply that the Cauchy problem is not well posed in* $\mathscr{D}^{\infty}_{L^2}$.

141 *We see that* $a(x,t,-\xi) = -a(x,t,\xi)$ *which shows that in the case of a differential operator* (9.1) *the condition* $\operatorname{Re} a(x,t,\xi) \equiv 0$ *will be necessary for the existence theorem. We analyse this situation more clearly.*

Theorem 1. *Suppose there exists a real vector* $\xi^o \in \underline{R}^n$, $\xi^o \neq 0$ *and* x^o *such that* $\operatorname{Im} \sum a_k(x^o, 0)\xi^o_k < 0$. *Then the forward Cauchy problem is not well posed for* (9.1) *in* $\mathscr{E}$ *or in* $\mathscr{D}^{\infty}_{L^2}$ *or in* $\mathscr{B}$.

Remark. P.D. Lax [1] also proved a similar theorem, by using the characteristic method, that if eigenvalues are simple for the well posedness of the Cauchy problem it is necessary that the eigenvalues be real.

We first prove an energy inequlity for a suitably modified operator and then establish two lemmas for commutators which together will prove the theorem. Suppose $x^0 = 0$.

First of all we localize the differential operator given in (9.1). Suppose u is a solution of (9.1) of class $\mathscr{E}^1$. Let $\beta(x) \in \mathscr{D}$ with support contained in a small neighbourhood of the origin. Now

$$(9.4) \qquad \frac{\partial}{\partial t}(\beta u) = \sum a_k \frac{\partial}{\partial x_k}(\beta u) + b(\beta u) - \sum a_k \frac{\partial \beta}{\partial x_k} u$$

Since the support of βu and of $\dfrac{\partial}{\partial x_k}(\beta u)$ are contained in the support of β we can modify a_k and b outside the support of β. We can write

$$(9.4)' \quad \frac{\partial}{\partial t}(\beta u) - \sum \tilde{a}_k(x,t)\frac{\partial}{\partial x_k}(\beta u) - \tilde{b}(x,t)(\beta u) = -\sum \tilde{a}_k(x,t)\frac{\partial \beta}{\partial x_k} \cdot u$$

where $\tilde{a}_k$ and $\tilde{b}$ are equal to a_k and b respectively on the support of β and 142

(i) $\tilde{a}_k,\ \tilde{b} \in \mathscr{B}^{\infty}_{x,t}$

(9.5)

(ii) $\operatorname{im} \sum \tilde{a}_k(x,t)\xi^o_k < -\delta,\ \delta > 0$ for all (x,t) with $x \in \underline{R}^n$ and $0 \leq t \leq t_o$

We can assume $|\xi^o| = 1$ if necessary by multiplying by a suitable constant. There exists a neighbourhood V of ξ^o such that

$$\text{im} \sum \tilde{a}_k(x,t)\xi_k \leq -\delta \text{ for } 0 \leq t \leq t_o, \xi \in V. \tag{9.6}$$

Let $\hat{\alpha} \in \mathscr{D}$ with the support contained on V and $\hat{\alpha}(\xi) \equiv 1$ in a neighbourhood of ξ^o. Define $\hat{\alpha}_p$ by

$$\hat{\alpha}_p(\xi) = \hat{\alpha}(\frac{\xi}{p}), \alpha_p(x) = \bar{\mathscr{F}}[\hat{\alpha}_p(\xi)]. \tag{9.7}$$

Convolving both sides of (9.4)′ with α_p we obtain

$$\left(\frac{\partial}{\partial t} - \sum \tilde{a}_k \frac{\partial}{\partial x_k} - \tilde{b}\right)\left(\alpha_p *_{(x)} (\beta u)\right) \equiv L\left[\alpha_p *_{(x)} (\beta u)\right]$$

$$= -\left[\alpha_p *, L\right](\beta u) - \sum \tilde{a}_k(\alpha_p *_{(x)} (\beta_k u)) - \sum \left[\alpha_p *_{(x)}, \tilde{a}_k\right](\beta_k u). \tag{9.8}$$

where $\beta_k = \dfrac{\partial \beta}{\partial x_k}$.

We rewrite

$$\sum \tilde{a}_k \frac{\partial}{\partial x_k}(\alpha_p *_{(x)} v) = H \wedge (\alpha_p *_{(x)} v)$$

where $v = \beta u$ and $\sigma(H) = 2\pi i \sum \tilde{a}_k \frac{\xi_k}{|\xi|} = h(x,t,\xi)$. That is

$$H \wedge (\alpha_p *_{(x)} v) = \int \exp(2\pi i x.\xi) \cdot h(x,t,\xi)|\xi|\hat{\alpha}_p(\xi)\hat{v}(\xi) d\xi.$$

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This operator depends only on the value of the symbol h on the set $\{\lambda V\}_{\lambda \geq 0}$ since the support of $|\xi|\hat{\alpha_p}\hat{v}$ is contained in the set $\{\lambda V\}$. Hence we can modify the symbol h to h outside $\{\lambda V\}$ as follows:

(i) $\tilde{h}(x,t,\xi) \equiv h(x,t,\xi)$ for $\xi \in \lambda V$

(ii) $\text{Re}\, \tilde{h}(x,t,\xi) \geq \delta', \delta' > 0$.

Thus we have finally an equation

$$(\frac{\partial}{\partial t} - \tilde{H} \wedge -\tilde{b})(\alpha_p *_{(x)} (\beta u)) = f \tag{9.9}$$

where $\tilde{H}$ is the singular integral operator whose symbol $\sigma(\tilde{H})$ is $\tilde{h}$, f being the right-hand side of (9.8).

Lemma 1. *Suppose $H(t)$ is a singular integral operator of class C^∞_β, $\beta = \infty$ such that*

$$\sigma(H)(x, t, \xi) \geq \delta' > 0. \tag{9.10}$$

Suppose $f \in L^2[0, h]$ is given. If $u \in \mathscr{D}^1_{L^2}[0, h]$ satisfies

$$\frac{\partial}{\partial t}(\alpha_p *_{(x)} u) = H \wedge (\alpha_p *_{(x)} u) + b(x, t)(\alpha_p *_{(x)} u) + f \tag{9.11}$$

then there exists a $\delta'' > 0$ such that

$$\frac{d}{dt}||\alpha_p *_{(x)} u|| \geq \delta'' p||\alpha_p *_{(x)} u|| - ||f|| \tag{9.12}$$

for sufficiently large p.

Proof. Let us denote $\alpha_p *_{(x)} u$ by v_p. Then we have **144**

$$\frac{d}{dt}(v_p, v_p) = ((H \wedge + \wedge H^*)v_p, v_p) + 2\,\mathrm{Re}(bv_p, v_p) + 2\,\mathrm{Re}(v_p, f).$$

But $\wedge H^* = H^\# \wedge (\mathrm{mod} \wedge^o)$ implies

$$\frac{d}{dt}(v_p, v_p) = ((H + H^\#) \wedge v_p, v_p) + 2\,\mathrm{Re}(bv_p, v_p) + 2\,\mathrm{Re}(v_p, f) + (Bv_p, v_p),$$

with B a bounded operator. If P denotes the singular integral operator $H + H^\#$ then $\sigma(P) \geq 2s'$. We remark that $(P \wedge^s - \wedge^s P)\wedge^\sigma$ is a bounded operator if $s, \sigma \geq 0$ and $s + \sigma \leq 1$. Taking $s = \sigma = \dfrac{1}{2}$

$$P\wedge \equiv \wedge^{\frac{1}{2}} P \wedge^{\frac{1}{2}} \ (\mathrm{mod} \wedge^0).$$

Hence

$$(P \wedge v_p, v_p) = P \wedge^{\frac{1}{2}} v_p, \wedge^{\frac{1}{2}} v_p) + ((C v_p, v_p)$$

where $C = P \wedge - \wedge^{\frac{1}{2}} P \wedge^{\frac{1}{2}}$ is a bounded operator. Thus

$$\frac{d}{dt}(v_p, v_p) \geq \operatorname{Re}((H + H^{\#}) \wedge^{\frac{1}{2}} v_p, \wedge^{\frac{1}{2}} v_p) - \gamma_1 ||v_p||^2 - 2v_p|| \, ||f|| \tag{9.13}$$

on the other hand we have by Gårding's lemma that

$$\operatorname{Re}((H + H^{\#}) \wedge^{\frac{1}{2}} v_p, \wedge^{\frac{1}{2}} v_p) \geq \delta'(\wedge_p^{\frac{1}{2}}, \wedge^{\frac{1}{2}} v_p) - \gamma_2 ||v_p||^2. \tag{9.14}$$

Since the distance of the support of $\hat{v}_p(\xi, t) \equiv \hat{\alpha}_p(\xi)\hat{u}(\xi, t)$ from the origin is larger than σp, $\sigma > 0$, we have by Plancheral's formula

$$\begin{aligned}(\wedge^{\frac{1}{2}} v_p, \wedge^{\frac{1}{2}} v_p) &= \int |\xi| \hat{v}_p(\xi, t)|^2 d\xi \\ &\geq \sigma p \int |\hat{v}_p(\xi, t)|^2 d\xi = \sigma p ||v_p||^2\end{aligned}$$

145 Thus we have

$$\frac{d}{dt}||v_p||^2 \geq \sigma^{\delta'} p ||v_p||^2 - (\gamma_1 + \gamma_2)||v_p||^2 - 2||v_p|| \;\; ||f||$$

which implies

$$\frac{d}{dt}||v_p|| \geq (\delta p - \gamma)||v_p|| - ||f||$$

where $\delta > 0$, $\gamma > 0$ are constants. Therefore for large $p(\delta p - \gamma) \geq \delta''_p$, $\delta'' > 0$.

For such p we have

$$\frac{d}{dt}||v_p|| \geq \delta'' p ||v_p|| - ||f||.$$

In other words we have

$$\frac{d}{dt}||\alpha_p *_{(x)} u|| \geq \delta'' p ||\alpha_p *_{(x)} u|| - ||f|| \tag{9.12}$$

completing the proof of the lemma. □

Lemma 2. *If $a \in \mathscr{B}$ and $u \in \mathscr{D}^1_{L^2}$ there exists a constant $c > 0$ such that*

$$\left\| \left[\alpha_p *, a(x)\frac{\partial}{\partial x_j}\right] u \right\| \leq c \left\{ \sum_{1 \leq |\rho| \leq m-1} \left\| \frac{\partial}{\partial x_j}(x^\rho \alpha_p) * u \right\| \right.$$

(9.15)
$$\left. + \left(\sum_{|\rho|=m} \left\| \frac{\partial}{\partial x_j}(x^\rho \alpha_p) \right\|_{L^1} + \|(x^\rho \alpha_p)\|_{L^1} \right) \|u\| \right\}.$$

Proof. Let $v = \left[\alpha_p *, a(x)\dfrac{\partial}{\partial x_j}\right] u$; then

$$v(x) = \int (a(y) - a(x))\alpha_p(x-y)\frac{\partial u}{\partial y_j}(y)dy.$$

Expanding $a(y) - a(x)$ by mean value theorem upto order m, to be determine later, 146

$$a(y) - a(x) = \sum_{1 \leq |\rho| \leq m-1} \frac{(y-x)^\rho}{\rho!}\left(\frac{\partial}{\partial x}\right)^\rho a(x) + \sum_{|\rho|=m} a_\rho(x,y)(x-y)^\rho$$

and hence

$$v(x) = \sum_{1 \leq |\rho| \leq m-1} \frac{(-1)^\rho}{\rho!}\left(\frac{\partial}{\partial x}\right)^\rho a(x)\frac{\partial}{\partial x_j}(x^\rho \alpha_p) * u$$
$$+ \sum_{|\rho|=m} \int (x-y)^\rho \alpha_p(x-y) a_\rho(x,y)\frac{\partial u}{\partial y_j}(y)\, dy.$$

Now $\quad \varphi(x) = \int (x-y)^\rho \alpha_p(x-y) a_\rho(x,y)\dfrac{\partial u}{\partial y_j}(y)\, dy$

$$= -\int \frac{\partial}{\partial y_j}\left\{(x-y)^\rho \alpha_p(x-y) a_\rho(x,y)\right\} u(y)\, dy$$
$$= -\int \left\{ \frac{\partial}{\partial y_j}\left[(x-y)^\rho \alpha_p(x-y)\right] a_\rho(x,y) \right.$$
$$\left. + (x-y)^\rho \alpha_p(x-y)\frac{\partial}{\partial y_j} a_\rho(x,y) \right\} u(y)\, dy.$$

Hence $\|\varphi(x)\| \leq c\| \, |x^\rho \alpha_p| * |u| + |\dfrac{\partial}{\partial x_j}(x^\rho \alpha_p)| * |u|\|$. Applying Hausdorff-Young inequality to the right hand side we obtain the desired inequality.

Similarly one can prove that if $a \in \mathscr{B}$ and $u \in L^2$ then

$$\|[\alpha_p *, a]u\| \leq c \left\{ \sum_{1 \leq |\delta| \leq m-1} \|(x^\rho \alpha_p) * u\| + (\sum_{|\rho|=m} \|x^\rho \alpha_p\|_{L^1})\|u\| \right\} \tag{9.16}$$

where c is a positive constant.

147 Now we look at the terms appearing in the right hand side of (9.15).

First of all $\dfrac{\partial}{\partial x_j}(x^\rho \alpha_p) * u$ has its Fourier image $(2\pi i \xi_j)(\widehat{x^\rho \alpha_p})\hat{u} = (2\pi i \xi_j)\hat{u}$. const.$\hat{\alpha}_p^\rho(\xi)$ which shows, since the support of $\hat{\alpha}_p^\rho(\xi)$ has diameter $\sigma' p$ where σ' is a constant depending only on $\hat{\alpha}$, that,

$$\|\frac{\partial}{\partial x_j}(x^\rho \alpha_p) * u\| \leq cp\|(x^\rho \alpha_p) * u\|. \tag{9.17}$$

Next consider $\|x^\rho \alpha_p\|_{L^1}$ for $|\rho| = m$

$$\begin{aligned} \sup \left| x^\rho \alpha_p \right| &\leq \text{ const. } \int |\hat{\alpha}_p^{(\rho)}(\xi)| d\xi = \text{ const. } \int |(\frac{\partial}{\partial \xi})^\rho \hat{\alpha}_p(\xi)| d\xi \\ &= \text{ const. } (\frac{1}{p})^{|\rho|-n} \int |(\frac{\partial}{\partial \xi})^\rho \hat{\alpha}| d\xi. \end{aligned}$$

Similarly $|x|^{2n} |x^\rho \alpha_p| \leq \text{const} \left(\dfrac{1}{p}\right)^{|\rho|+n} \int \left| \Delta_\xi^n (\dfrac{\partial}{\partial \xi})^\rho \hat{\alpha} \right| d\xi$ which implies that

$$(1 + |x|^{2n}) |x^\rho \alpha_p| \leq \text{ const. } \left(\frac{1}{p}\right)^{|\rho|-n}.$$

Hence $\|x^\rho \alpha_p\|_{L^1} \leq$ const. $\displaystyle\int \frac{dx}{1 + |x|^{2n}} \cdot \left(\frac{1}{p}\right)^{|\rho|-n} \leq c \left(\frac{1}{p}\right)^{|\rho|-n}$. In the same way one can show that

$$\|\frac{\partial}{\partial x_j}(x^\rho \alpha_p)\|_{L^1} \leq c \left(\frac{1}{p}\right)^{|\rho|-n-1}.$$

□

Thus we have proved the

Corollary of Lemma 2: If $a \in \mathscr{B}$ and $u \in \mathscr{D}^1_{L^2}$ then 148

$$(9.18) \quad ||[\alpha_p *, a(x)\frac{\partial}{\partial x_j}]u|| \leq cp \sum_{1 \leq |\rho| \leq m-1} ||(x^\rho \alpha_p) * u|| + O\left(\frac{1}{p^{m-n-1}}\right)||u||.$$

This follows from (9.15).

Similarly it follows from (9.16) that

$$(9.18)' \qquad ||[\alpha_p *, a(x)]u|| \leq c \sum_{1 \leq |\rho| \leq m-1} ||(x^\rho \alpha_p) * u|| + O(\frac{1}{p^{m-n}})||u||.$$

Lemma 3. *If L is a differential operator of the first order with its coefficients in $\mathscr{B}$*

$$(9.19) \qquad L = \sum a_k(x)\frac{\partial}{\partial x_k} + b(x)$$

then for any $u \in \mathscr{D}^1_{L^2}$

$$(9.20) \qquad ||[\alpha_p *, L]u|| \leq c \sum_{1 \leq |\rho| \leq m-1} p||(x^\rho \alpha_p) * u|| + O(\frac{1}{p^{m-n-1}})||u||.$$

This is an immediate consequence of the inequalities (9.18) *and* (9.18)′. *More generally one can prove exactlly in the same way*

$$(9.21) \quad ||[(x^\nu \alpha_p) *, L]u|| \leq c \sum_{|\nu|+1 \leq |\rho| \leq m-1} ||(x^\rho \alpha_p) * u|| + O\left(\frac{1}{p^{m-n+|\nu|}}\right)||u||.$$

and

$$(9.22) \quad ||[(x^\nu \alpha_p) *, L]u|| \leq cp \sum_{|\nu|+1 \leq |\rho| \leq m-1} ||(x^\rho \alpha_p) * u|| + O(\frac{1}{p^{m+1|\nu|-n-}})||u||.$$

for every $u \in \mathscr{D}^1_{L^2}$.

Now we can complete the 149

Proof of Theorem 1: We prove this theorem in the spaces $\mathscr{E}$. As we shall see from the method of proof the same will be valid for the spaces $\mathscr{D}^{\infty}_{L^2}$ and $\mathscr{B}$. The proof is by contradiction.

Suppose the Cauchy problem is well posed in the spaces. We construct a sequence of initial conditions $\psi_q(x)$ and consider the corresponding sequence of solutions $\psi_q(x)$ are defined as follows:

Let V be a small a neighbourhood of ξ^o and $\hat{\alpha} \in \mathscr{D}$ have its support in V with $\hat{\alpha}(\xi) \equiv 1$ in neighbourhood V' of ξ^o, $V' \subset V$. Take *a* $\hat{\psi} \in \mathscr{D}$, $\hat{\psi}(\xi) \neq 0$ with support contained in V'. Denoting

$$\hat{\psi}_q(\xi) = \hat{\psi}(\xi - q\xi^o)$$

we have by taking inverse Fourier transforms

$$\psi_q(x) = \exp(2\pi i q x.\xi^o)\psi(x) \tag{9.23}$$

$\psi_q \in \mathscr{E}$ (also in $\mathscr{D}^{\infty}_{L^2}$, $\mathscr{B}$). Further

$$\|\psi_q\|_{\mathscr{E}^h} = O(q^h). \tag{9.24}$$

(We remark that (9.24) holds for the semi-norms in $\mathscr{D}^{\infty}_{L^2}$ and $\mathscr{B}$ also).

By hypothesis of the well posedness, the corresponding solution $u_q(x,t)$ of (9.1) having $\psi_q(x)$ as the initial data is estimated by

$$\sup_K |u_q(x,t)| = O(q^h) \tag{9.25}$$

for some fixed h where K is a compact set in the (x,t)-space. Also we see that

$$\|\alpha_p * (\beta\psi_p)\| \geq c > 0. \tag{9.26}$$

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In fact,

$$\alpha_p * (\beta\psi_p) = \beta(\alpha_p * \psi_p) + [\alpha_p *, \beta]\psi_p.$$

$$\|\beta(\alpha_p * \psi_p)\| = \|\hat{\beta} * (\hat{\alpha}_p\hat{\psi}_p)\| = \|\hat{\beta} * \hat{\psi}_p\| = \|\beta\psi_p\| = \|\beta\psi\| > 0$$

by using Plancheral's formula and the fact that $\hat{\alpha}_p \equiv 1$ on the support of $\hat{\psi}_p, \psi$ being an analytic function and $\|\beta\psi\| > 0$. On the other hand it is easy to see that $\|[\alpha_p *, \beta]\psi_p\| = O\left(\frac{1}{p}\right)$. Now we prove this leads to contradiction as follows. Instead of $\alpha_p * (\beta u)$ in (9.8) we consider $(x^\nu \alpha_p) * (\beta^\nu u_p)$ with $|\nu| \leq m-1$, $|\nu| \leq m + mh$ which form a system of localisers, $\beta^\mu = \left(\frac{\partial}{\partial x}\right)^\mu \beta(x)$. Then we have

$$\left(\frac{\partial}{\partial t} - \sum \tilde{a}_k \frac{\partial}{\partial x_k} - \tilde{b}\right)((x^\nu \alpha_p) * (\beta^\mu u_p))$$

$$= [L, (x^\nu \alpha_p)*](\beta^\mu u_p) - \sum \tilde{a}_k((x^\nu \alpha_p) * (\beta^{\mu_k} u_p)) - \sum [(x^\nu \alpha_p)*, \tilde{a}_k](\beta^{\mu_k} u_p)$$

where $\mu_k = \mu + e_k$, $\ell_k = (0, \ldots, 1, \ldots, 0)$ the kth component is 1.

Applyinginequallity (9.12) for $(x^\nu \alpha_p) * (\beta^\mu u_p)$ and using inequalities (9.21), (9.22) with $m = h + n + 2$, we have

$$\frac{d}{dt}\|(x^\nu \alpha_p) * (\beta^\mu u_p)\| \geq \delta'' p \|(x^\nu \alpha_p) * (\beta^\mu u_p)\| - \|f\|$$
$$\geq \delta'' p \|(x^\nu \alpha_p) * (\beta^{(\mu)} u_p)\| - cp \sum_{|\nu|+1 \leq |\rho| \leq m-1} \|(x^\rho \alpha_p) * (\beta^\mu u_p)\|$$
$$-c \sum_{\substack{|\nu|+1 \leq |\rho| \leq m-1 \\ |\mu'| = |\mu|+1}} \|(x^\rho \alpha_p) * (\beta^{(\mu')} u_p)\| - c \sum_{|\mu'| = |\mu|+1} \|(x^\nu \alpha_p) * (\beta^{\mu'} u_p)\| - O\left(\frac{1}{p}\right)$$

Now consider the functions $\theta_p(\nu, \mu) u_p$ defined by **151**

$$\theta_p(\nu, \mu) u_p = p^{\theta(|\nu| - |\mu|)}(x^\nu)\alpha_p) * (\beta^{(\mu)} u_p)$$

where $0 < \theta < 1$. In fact we take $\theta = \frac{1}{m}$. We have from above inequality

$$\frac{d}{dt}\|\theta_p(\nu, \mu) u_p\| \geq \delta'' p \|\theta_p(\nu, \mu) u_p\| - cp^{1-\theta} \sum_{|\nu|+1 \leq |\rho| \leq m-1} \|\theta_p(\rho, \mu) u_p\|$$

$$-cp^{\theta}\sum_{|\mu|=|\mu|+1}\|\theta_p(\nu,\mu')u_p\| - c\sum_{\substack{|\nu|+1|\leq 1|\rho|\leq m-1\\ |\mu'|=|\mu|+1}}\|\theta_p(\rho,\mu')u_p\|$$

$$-p^{\theta(|\nu|-|\mu|)}O(\frac{1}{p}). \tag{9.27}$$

Now if $|\mu'| = m + mh$ we have, by (9.25)

$$\|\theta_p(\nu,\mu)u_p\| \leq cp^{\theta(|\nu|-|\mu|)}\|u_p\| \leq c' p^{\theta(|\nu|-|\mu|)+h}.$$

But $\theta(|\nu| - |\mu|) \leq \theta(m - 1 - m - mh - 1) = \theta(-mh - 2) = -h - 2\theta$ since $\theta = \dfrac{1}{m}$. Thus $\|\theta_p(\nu,\mu)u_p\| \leq cp^{-2\theta}$. Denoting

$$S_p(t) = \sum_{\substack{0\leq|\nu|\leq m-1\\ 0\leq|\mu|\leq m+mh}}\|\theta_p(\nu,\mu)u_p(t)\|$$

we have from (9.27) that

$$\begin{aligned}\frac{d}{dt}S_p(t) &\geq \delta'' pS_p(t) - cp^{1-\theta}S_p(t) - O(1)\\ &\geq \gamma'' pS_p(t) - O(1) \text{ for large } p, r'' > 0.\end{aligned}$$

Integrating this with respect to t

$$\begin{aligned}S_p(t) &\geq \exp(\gamma'' pt)S_p(0) - \int_0^t \exp(r'' p(t-s))O(1)ds\\ &= \exp(\gamma''\ pt)\left[S_p(0) - O\left(\frac{1}{p}\right)\right].\end{aligned}$$

152 But $S_p(0) = \sum_{\substack{0\leq|\nu|\leq m-1\\ 0\leq|\mu|\leq m+mh}}\|\theta_p(\nu,\mu)u_p(0)\| \geq \|\alpha_p * (\beta u_p)(0)\| \geq c > 0$ by (9.26) for large p. Hence for every fixed t the function $S_p(t)$ increases exponentially with respect to p i.e. $S_p(t) \geq ce^{\gamma''\mathrm{pt}}$. On the other hand

$$\|\theta_p(\nu,\mu)u_p(t)\| = p^{\theta(|\nu|-|\mu|)}\|(x^{\nu}\alpha_p) * (\beta^{\mu}u_p)\|$$

and $\|\beta^{\mu}u_p(t)\| = 0(p)^h$. Hence $S_p(t) \geq cp^k$ for a large k. In fact $\|\theta_p(\nu,\mu)u_p(t)\| \leq 0(p^{h+1})$ since $\theta|\nu| < 1$). This is a contradiction. This completes the proof of the theorem 1.

Chapter 4

In this chapter we briefly discuss the existence of solutions of the Cauchy 153
problem for parabolic equations.

In section 1 we introduce parabolic equations of order m in the x-variables and prove an existence theorem when coefficinets do not depend on t. In section 2 we obtain an energy inequality for parabolic equations which we use to prove the existence of solutions of the Cauchy problem for parabolic equation with sufficiently smooth initial conditions when coefficients depend on t as well.

1 Parabolic equations

Consider the differential equation

$$\frac{\partial}{\partial t}u = \sum_{|\nu|\leq 2m} a_\nu(x)\left(\frac{\partial}{\partial x}\right)^\nu u + f = A\left(x, \frac{\partial}{\partial x}\right)u + f \tag{1.1}$$

where A is negative elliptic of order $2m$ in $\bar{R}^n$ in the sense that

$$\text{Re}\sum_{|\nu|=2m} a_\nu(x)(i\xi)^\nu \leq -\delta|\xi|^{2m} \tag{1.2}$$

δ being a positive constant. We assume that the coefficients a_ν belong to $\mathbb{B}^{2m}$.

We prove the existence of a solution of (1.1) in the space L^2. We take for the domain of definition $\mathscr{D}_A$ of A the space $\mathscr{D}^{2m}_{L^2}$.

Proposition 1. *For small $\lambda > 0$ the operator $(I - \lambda A)$ defines a one-to-one surjective mapping of $\mathscr{D}^{2m}_{L^2}$ onto L^2.*

154 *Proof.* For $u \in \mathscr{D}^{2m}_{L^2}$ and $\lambda > 0$

$$\|(I - \lambda A)u\|^2 = \|u\|^2 - \lambda((A + A^*)u, u) + \lambda^2\|Au\|^2. \tag{1.3}$$

Since A is negatively elliptic we have, from Gårding's lemma, that

(i) $-((A + A^*)u, u) \geq \delta\|u\|^2_m - \gamma_1\|u\|^2$

(ii) $\|Au\|^2 \geq \dfrac{\delta^2}{2}\| \wedge^{2m} u\|^2 - \gamma_2\|u\|^2$.

where γ_1, γ_2 are positive constants depending on δ. Hence it follows from (1.3) that

$$\|(I - \lambda A)u\|^2 \geq (1 - \gamma_1\lambda - \gamma_2\lambda^2)\|u\|^2 + \frac{\delta^2}{2}\lambda^2\| \wedge^{2m} u\|^2, \tag{1.4}$$

which show that for sufficiently small $\lambda, (I - \lambda A)$ is one-t-one from $\mathscr{D}^{2m}_{L^2}$ to L^2 and that the image is closed.

Next we show that the image $(I - \lambda A)\mathscr{D}^{2m}_{L^2}$ is dense in L^2, for $\lambda > 0$ small. This is done by contradiction. Suppose the image is not dense in L^2. Then there exists a $\psi \in L^2$, $\psi \neq 0$ such that

$$((I - \lambda A)u, \psi) = 0 \text{ for all } u \in \mathscr{D}^{2m}_{L^2},$$

a fortiori for all $u \in \mathscr{D}$. This implies that

$$(I - \lambda A^*)\psi = 0. \quad \text{Let } \psi_1 = (1 - \Delta)^{-m}_{\psi}. \tag{1.6}$$

Then $\psi_1 \in \mathscr{D}^{2m}_{L^2}$, $\psi_1 \neq 0$ and

$$(I - \lambda A^*)(1 - \Delta)^m\psi_1 = 0$$

155 Hence $((I - \lambda A^*)(1 - \Delta)^m\psi_1, \psi_1) = \|\psi_1\|^2_m - \lambda(A^*(1 - \Delta)^m\psi_1, \psi_1) = 0$. Now the real part of $(A^*(1 - \Delta)^m\psi_1, \psi_1)$ is

$$\frac{1}{2}(\{A^*(1 - \Delta)^m + (1 - \Delta)^m A\}\psi_1, \psi_1),$$

and since $\{A^*(1-\Delta)^m + (1-\Delta)^m A\}$ is an elliptic operator of order $4m$, we have by Gårding's lemma,

$$(1.7)\quad \frac{1}{2}(\{A^*(1-\Delta)^m + (1-\Delta)^m A\})\psi_1, \psi_1) \le -\frac{\delta}{2}\| \wedge^{2m} \psi_1\|^2 + \gamma_3\|\psi_1\|^2.$$

Hence, we have

$$\begin{aligned} &\mathrm{Re}\{\|\psi_1\|_m^2 - \lambda(A^*(1-\Delta)^m\psi_1, \psi_1)\} \\ &\ge \|\psi_1\|_m^2 + \lambda(\frac{\delta}{2}\| \wedge^{2m} \psi_1\|^2 - \gamma_3\|\psi_1\|^2) \\ (1.8)\qquad &\ge (1-\lambda\gamma_3)\|\psi_1\|_m^2. \end{aligned}$$

This implies that $\psi_1 = 0$ contrary to the assumption, which proves that $(I-\lambda A)$ is surjective for sufficiently small λ. □

Corollary 1. *If $u \in L^2$ such that $A[u] \in L^2$ then $u \in \mathscr{D}^{2m}_{L^2}$.*

Proof. Since from the Theorem for sufficiently small $\lambda, (I-\lambda A)$ is surjective it follows that there exists $w \in \mathscr{D}^{2m}_{L^2}$ such that $(I-\lambda A)w = (I-\lambda A)u$. Hence $(I-\lambda A)(w-u) = 0$. Now in the course of the proof of the theorem we have shown that $(I-\lambda A)v = 0$, $v \in L^2$ implies $v = 0$. Hence $u = w \in \mathscr{D}^{2m}_{L^2}$. □

Proposition 2. *Given any initial data $u_0 \in \mathscr{D}^{2m}_{L^2}$ and any second member* **156**
$f \in \mathscr{D}^{2m}_{L^2}[0,h]$ then there exists a solution $u \in \mathscr{D}^{2m}_{L^2}[0,h]$ of (1.1) *such that $u(0) = u_0$ where the deriative $\dfrac{\partial}{\partial t}u$ is taken in the sense of L^2.*

Proof. The prop. 1 asserts that all the conditions of Hille-Yosida theorem are satisfied taking $X = L^2$, $\mathscr{D}_A = \mathscr{D}^{2m}_{L^2}$. Hence we have the proposition by the application of Hille-Yosida theorem. Let us remark that $u, Au \in L^2[0,h]$ implies $u \in \mathscr{D}^{2m}_{L^2}[0,h]$.

We have proved the Proposition 2 under the assumption that $f \in \mathscr{D}^{2m}_{L^2}[0,h]$. We shall improve it by proving it assuming only

$$f \in \mathscr{D}^m_{L^2}[0,h].$$

For this purpose we establish an energy inequality for the parabolic equation (1.1). □

2 Energy inequality for parabolic equations

Consider the parabolic equation

$$\frac{\partial}{\partial t}u = \sum_{|\nu|\leq 2m} a_\nu(x)\left(\frac{\partial}{\partial x}\right)^\nu u + f = A\left(x, \frac{\partial}{\partial x}\right)u + f \tag{2.1}$$

Proposition 1. *Let* (2.1) *be a parabolic equation with the coefficients* $a_\nu(x)$ *of* A *belonging to* $\mathscr{B}^{2m}$ *and the second member* $f \in \mathscr{D}^{2m}_{L^2}[0, h]$. *If* $u \in \mathscr{D}^{3m}_{L^2}[0, h]$ *satisfies* (2.1) *then*

$$\|u(t)\|^2_{2m} \leq \exp(\gamma_1 t)\|u(0)\|^2_{2m} + \gamma_2(\delta)\int\limits_0^t \exp(\gamma(t-s))\|f(s)\|^2_m ds, \tag{2.2}$$

where γ_1, γ_2 *are positive constants.*

157 *Proof.* Consider

$$\begin{aligned}\frac{d}{dt}(u(t), u(t))_{2m} &= \left(\frac{d}{dt}u(t), u(t)\right)_{2m} + \left(u(t)\frac{d}{dt}u(t)\right)_{2m}\\ &= ((A + A^*)u, u)_{2m} + 2\operatorname{Re}(f, u)_{2m}\\ &= (((1-\Delta)^{2m}A + A^*(1-\Delta)^{2m}u, u) + 2\operatorname{Re}(f, u)_{2m}.\end{aligned}$$

The first term in the right hand side is by Gårdings's ineequality less than

$$-\frac{\delta}{2}\|\Lambda^{3m}u\|^2 + \gamma_0\|u\|^2_{2m} \leq -\frac{\delta}{2}\|u\|^2_{3m} + \gamma_1\|u\|^2_{2m} \tag{2.3}$$

since $(1-\Delta)^{2m}A$ is an elliptic operator of order $6m$. Also

$$|(f, u)_{2m}| \leq \|f\|_m\|u\|_{3m} \leq \frac{2}{\delta}\|f\|^2_m + \frac{\delta}{2}\|u\|^2_{3m}$$

by the inequality between the arithmetic and geometric means. Hence

$$\frac{d}{dt}(u, u)_{2m} \leq \left(\frac{\delta}{2} - \delta\right)\|u\|^2_{3m} + \gamma_1\|u\|^2_{2m} + \frac{2}{\delta}\|f\|^2_m$$

that is

$$\frac{d}{dt}\|u(t)\|^2_{2m} \leq -\frac{\delta}{2}\|u\|^2_{3m} + \gamma_1\|u\|^2_{2m} + \frac{2}{\delta}\|f\|^2_m \tag{2.4}$$

a fortiori

$$\frac{d}{dt}\|u(t)\|^2_{2m} \leq \gamma_1\|u(t)\|^2_{2m} + \frac{4}{\delta}\|f(t)\|^2_m \tag{2.4$'$}$$

and hence we obtain after integrating with respect to t in $[0, h]$ the required inequality (2.2). □

Next we obtain the energy inequality of the form (2.2) under the **158**
assumption that $u \in \mathscr{D}^{2m}_{L^2}[0, h]$ instead of $u \in \mathscr{D}^{3m}_{L^2}[0, h]$. In the case of hyperbolic systems such an improvement could be achieved easily by using Friedrichs' lemma. This method will not work in our case since A is not of the first order. However, as we shall show, by a slight modification, we can use this method of regularisation by mollifiers.

As before we estimate the commutators of convolutions with mollifiers φ_ε of Friedrichs.

Lemma 1. *For $a \in \mathscr{B}^{2m}$ and $v \in L^2$ denote by $C_\varepsilon v$ the commutator*

$$C_\varepsilon v = [\varphi^*_\varepsilon, a]v. \tag{2.5}$$

Then there exists a constant γ_0 such that for $|\nu| \leq m$

$$\left\|\left(\frac{\partial}{\partial x}\right)^\nu C_\varepsilon v\right\| \leq \gamma_0|a|_{\mathscr{B}^{2m}}\left\{\sum_{1\leq|\rho|\leq m}\|(x^\rho|\varphi_\varepsilon) * v\|_\nu + \varepsilon\|v\|\right\|. \tag{2.6}$$

Proof. We have,

$$C_\varepsilon v = \int[a(y) - a(x)]\varphi_\varepsilon(x - y)v(y)dy.$$

Developing $a(y) - a(x)$ by Taylor's theorem

$$a(y) - a(x) = \sum_{1\leq|\rho|\leq m}\frac{(y - x)^\rho}{\rho!}\left(\frac{\partial}{\partial x}\right)^\rho a(x) + \sum_{|\rho|=m} a_\rho(x, y)(y - x)^\rho,$$

where since $a \in \mathscr{B}^{2m}$

$$\left|\left(\frac{\partial}{\partial x}\right)^{\nu'} a_\rho(x,y)\right| \leq c_1 |y-x|\ |a|_{\mathscr{B}^{2m}} \text{ for } |\nu'| \leq m-1$$

and

$$\left|\left(\frac{(\partial)}{\partial x}\right)^{\nu} a_\rho(x,y)\right| \leq c_2 |a|_{\mathscr{B}^2} \text{ for } |\nu| = m.$$

159 In fact,

$$a_\rho(x,y) = \frac{m}{\rho!}\int_0^1 (1-\theta)^{m-1}\left\{a^{(\rho)}(x+\theta(y-x)) - a^{(\rho)}(x)\right\} d\theta,$$

$$a^{(\rho)}(x) = \left(\frac{\partial}{\partial x}\right)^{\rho} a(x).$$

Hence

$$C_\varepsilon v = \sum_{1\leq|\rho|\leq m} \frac{(-1)^{|\rho|}}{\rho!}\left(\frac{\partial}{\partial x}\right)^{\rho} a(x)\,[(x^\rho \varphi_\varepsilon) * v]$$

$$+\sum_{|\rho|=m} (-1)^m \int a_\rho(x,y)(x-y)^\rho \varphi_\varepsilon(x-y)v(y)\,dy. \tag{2.7}$$

(2.7) implies the lemma. Obviously the terms of the first sum on the right hand side contribute to the terms of the sum of the right hand side of (2.6). As far as the second sum is concerned we remark that $\int |x|\left|\left(\frac{\partial}{\partial x}\right)^{\nu}(x^\rho \varphi_\varepsilon)\right| dx = O(\varepsilon)$ for $|\nu| \leq m$ and $|\rho| = m$.

By Hausdorff-Young inequality the second sum on the right hand side of (2.7) is less than $O(\varepsilon)\|v\|$ and this completes the proof of the lemma. ◻

More generally we have the

Lemma 2. *Let $a \in \mathscr{B}^{2m}$ and $v \in L^2$. If*

$$C_\varepsilon^\nu v = [(x^\nu \varphi_\varepsilon)*, a]v \text{ for } |\nu| \leq m-1 \tag{2.8}$$

then there exists a constant $\gamma_0 > 0$ *such that*

$$(2.9) \qquad \|C_\varepsilon^\nu v\|_m \leq \gamma_0 |a|_{\mathscr{B}^{2m}} \left(\sum_{|\nu|+1 \leq |\rho| \leq m} \|(x^\rho \varphi_\varepsilon) * v\|_m + \varepsilon \|v\| \right).$$

The proof is completely analogous to that of lemma 1 and hence we do not repeat it here.

As a consequence of lemma 1 and 2 we have **160**

Corollary 1. *If* $A = \sum_{|\nu| \leq 2m} a_\nu(x)(\frac{\partial}{\partial x})^\nu$ *is a differential operator of order* $2m$ *with* $a_\nu \in \mathscr{B}^{2m}$, *then for any* $u \in \mathscr{D}_{L^2}^{2m}$ *and for any* $|\nu| \leq m$

$$(2.10) \qquad \|[A, (x^\nu \varphi_\varepsilon)*]u\|_m \leq c \left(\sum_{|\nu|+1 \leq \rho \leq m} \|(x^\rho \varphi_\varepsilon) * u\|_{3m} + \varepsilon \|u\|_{2m} \right)$$

where $c = \gamma_0, \sup_\mu |a_\mu(x)|_{\mathscr{B}^{2m}} \gamma_0 > 0$, *is a constant. We remark that* (2.10) *asserts also that, for any* $|\gamma| \geq m$,

$$\|[A, (x^\nu \varphi_\varepsilon)*]u\| \leq c\ \varepsilon \|u\|_{2m}.$$

Proposition 2. *Let* (2.1) *be a parabolic equation of order* $2m$ *in* Ω *with* $a_\nu \in \mathscr{B}^{2m}$ *and* $f \in \mathscr{D}_{L^2}^{2m}[0, h]$. *If* $u \in \mathscr{D}_{L^2}^{2m}[0, h]$ *satisfies* (2.1) *then*

$$(2.11) \qquad \|u(t)\|_{2m}^2 \leq \exp(\gamma, t)\|u(0)\|_{2m}^2 + c \int_0^t \exp(\gamma(t-s))\|f(s)\|_m^2 ds,$$

Proof. Consider the function $(x^\nu \varphi_\varepsilon) *_{(x)} u = u_\varepsilon^\nu$ for $0 \leq |\nu| \leq m$. Clearly $u_\varepsilon^\nu \in \mathscr{D}_{L^2}^{3m}[0, h]$ and satisfies the system

$$(2.12) \qquad \frac{\partial}{\partial t} u_\varepsilon^\nu = A u_\varepsilon^\nu + f_\varepsilon^\nu + [(x^\nu \varphi_\varepsilon) * (x), A]u, \quad 0 \leq |\nu| \leq m.$$

Then inequality (2.4) of Prop. 1 applied to this system gives the system of inequalities

$$\frac{d}{dt}\|u_\varepsilon^\gamma(t)\|_{2m}^2 \leq -\delta' \|u_\varepsilon^\nu(t)\|_{3m}^2 + \gamma_1 \|u_\varepsilon^\nu(t)\|_{2m}^2 + \gamma_2 \|f_\varepsilon^\nu(t)\|_m^2$$

$$+\gamma_2 \|[(x^\nu \varphi_\varepsilon)*_{(x)}, A]u\|_m^2 \ \text{ for } \ 0 \leq |\nu| \leq m. \tag{2.13}$$

From the corollary 1 after lemma 2 applied to $[(x^\nu\varphi_\varepsilon)*_{(x)}, A]$ we
161 obtain for all $0 \leq |\nu| \leq m$.

$$\|[(x^\nu\varphi_\varepsilon)*_{(x)}, A]u\|_m \leq C_1\Big(\sum_{|\nu|+1\leq|\rho|\leq m} \|(x^\rho\varphi_\varepsilon) *_{(x)} u\|_{3m} + \varepsilon\|u\|_m$$

$$= C\left(\sum_{|\nu|+1\leq|\rho|\leq m} \|u_\varepsilon^\rho\|_{3m} + \varepsilon\|u\|_{2m}\right). \tag{2.14}$$

We define $v_\varepsilon^\nu = \varepsilon^{-\theta|\nu|}u_\varepsilon^\nu$ where $\theta > 0$ is small constant. Multiplying (2.13) by $\varepsilon^{-2\theta|\nu|}$ and setting $S_\varepsilon(t) = \sum\limits_\nu \|v_\varepsilon^\nu(t)\|_{2m}^2$ we have (after adding for ν over $0 \leq |\nu| \leq m$ from (2.14)

$$\frac{d}{dt}(S_\varepsilon(t)) \leq -\delta' \sum_\nu \|v_\varepsilon^\nu(t)\|_{3m}^2 + \gamma_1 S_\varepsilon(t) + \gamma_2 F_\varepsilon(t)$$

$$+\gamma_2 \sum_\nu \varepsilon^{-2\theta|\nu|} C_2\Big(\sum_{|\nu|+1\leq|\rho|\leq m} \|u_\varepsilon^\rho\|_{3m}^2 + \varepsilon^2\|u\|_{2m}^2\Big). \tag{2.15}$$

But

$$\sum_{0\leq|\nu|\leq m} \varepsilon^{-2\theta|\nu|} \sum_{|\nu|+1\leq|\rho|\leq m} \|u_\varepsilon^\rho\|_{3m}^2 = \sum_\nu \varepsilon^{-2\theta|\nu|} \sum_{|\nu|+1|\leq|\rho|\leq m} \varepsilon^{2\theta|\rho|}\|v_\varepsilon^\rho\|_{3m}^2$$

$$\leq n'\varepsilon^{2\theta} \sum_\nu \sum_\rho \|v_\varepsilon^\rho\|_{3m}^2.$$

Thus

$$\frac{d}{dt}S_\varepsilon(t) \leq \gamma_1 S_\epsilon(t) + \gamma_2 F_\varepsilon(t) + (\gamma_2 C_2 n'\varepsilon^{2\theta} - \delta') \sum_{0\leq|\nu|\leq m} \|v_\varepsilon^\nu\|_{3m}^2$$

$$+ c\,\varepsilon^{2(1-m\theta)}\|u(t)\|_{2m}^2$$

For small $\varepsilon > 0$, $(\gamma_2 C_1 n'\varepsilon^{2\theta} - \delta') < 0$ and hence

$$\frac{d}{dt}S_\varepsilon(t) \leq \gamma_1 S_\varepsilon(t) + \gamma_2 F_\varepsilon(t) + O(\varepsilon^{2(1-m\theta)}),$$

Integrating with respect to t

(2.16)

$$S_\varepsilon(t) \leq \exp(\gamma_1 t) S_\varepsilon(0) + \gamma_2 \int_0^t \exp(\gamma_1(t-s)) \Big\{ F_\varepsilon(s) + O(\varepsilon^{2(1-m\theta)}) \Big\} ds$$

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But

$$\begin{aligned} \|v_\varepsilon^\rho\|_{2m}^2 &= \|u_\varepsilon^\rho\|_{2m}^2 \varepsilon^{-2\theta|\rho|} \\ &= (\widehat{x^\rho \varphi_\varepsilon}) \hat{u}(\xi)(1+|\xi|)^{2m}\|^2 \varepsilon^{-2\theta|\rho|} \end{aligned}$$

by Plancherel's formular where $\hat{g}$ denotes the Fourier image of g in the x-space and

$$\begin{aligned} (\widehat{x^\rho \varphi_\varepsilon})(\xi) &= \int x^\rho \varphi_\varepsilon e^{-2\pi i x.\xi_{dx}} \\ &= \varepsilon^{|\rho|} \int x^\rho \varphi(x) e^{-2\pi i \in x.\xi} dx \end{aligned}$$

Since φ has its support in $|x| < 1$. We have

$$|(\widehat{x^\rho \varphi_\varepsilon})\ (\xi)| \leq \varepsilon^{|\rho|} \int \varphi(x)\ dx = \varepsilon^{|\rho|}$$

Hence

$$\|v_\varepsilon^\rho\|_{2m}^2 \leq \varepsilon^{2|\rho|(1-\theta)} \|u\|_{2m}^2$$

and

$$\sum_{0 \leq |\rho| \leq m} \|v_\varepsilon^\rho\|_{2m}^2 \leq \|u\|_{2m}^2 \sum_{0 \leq |\rho| \leq m} \varepsilon^{2|\rho|(1-\theta)} \leq \|u\|_{2m}^2 (1 + c\varepsilon^{2(1-\theta)})$$

which tends to $\|u\|_{2m}^2$ as $\varepsilon \to 0$. Hence $S_\varepsilon(t) \to \|u(t)\|_{2m}^2$ as $\varepsilon \to 0$. Also $F_\varepsilon(t) \to \|f(t)\|_m^2$. Hence on taking limits as $\varepsilon \to 0$ we have

$$\|u(t)\|_{2m}^2 \leq \exp(\gamma_1 t)\|u(0)\|_{2m}^2 + \gamma_2 \int_0^t \exp(\gamma_1(t-s))\|f(s)\|_m^2 ds.$$

This completes the proof of proposition. □ 163

Finally we consider the case parabolic systems in which the coefficients are functions of (x, t) in Ω. Let

$$\frac{\partial}{\partial t}u - \sum_{|\nu|\leq 2m} a_\nu(x,t)(\frac{\partial}{\partial x})^\nu u = f \tag{2.17}$$

be a parabolic equation of order $2m$. That is we assume that

$$A = \sum_{|\nu|\leq 2m} a_\nu(x,t)(\frac{\partial}{\partial x})^\nu$$

is uniformaly negatively elliptic in Ω ($\Omega = \{(x,t)|x \in \underline{R}^n, 0 \leq t \leq h$). This means that

$$\text{Re} \sum_{|\nu|=2m} a_\nu(x,t)(i\xi)^\nu \leq -\delta|\xi|^{2m}$$

for all $(x, t) \in \Omega$, $\xi \in \underline{R}^n$, $\delta > 0$.

Proposition 3. *Let* (2.17) *be a parabolic system in* Ω *with* $a_\nu \in \mathscr{B}^{2m}$ $[0, h]$ *and* $f \in \mathscr{D}^m_{L^2}[0, h]$. *Then, given a* $u_0 \in \mathscr{D}^{2m}_{L^2}$ *there exists* $u \in \mathscr{D}^{2m}_{L^2}[0, h]$ *satisfying* (2.17)*, with* $u\Big|_{t=0} = u_0$*, and which satisfies the energey inequality* (2.11).

Proof. Let $0 = t_0 < t_1 \cdots < t_k = h$ be a subdivision of $[0, h]$ of equal length. We define $u_1(t), \ldots, u_k(t)$ in $[t_0, t_1], \ldots, [t_{k-1}, t_k]$ by the following conditions

$$\begin{aligned}
\frac{du_1}{dt} &= A(t_0)u_1 + f,\ u_1(t_0) = u_0 \quad \text{for} \quad t_0 \leq t \leq t_1\\
\frac{du_2}{dt} &= A(t_1)u_2 + f,\ u_k(t_1) = u_1(t_1) \quad \text{for} \quad t_1 \leq t \leq t_2\\
\frac{du_k}{dt} &= A(t_{k-1})u_k + f,\ u_2(t_1) = u_1(t_1) \quad \text{for} \quad t_{k-1} \leq t \leq t_k.
\end{aligned}$$

164 We denote by $u^{(k)}(t)$ the function which in $t_{j-1} \leq t \leq t_j$ is equal to $u_j(t)$. It is easy to see that $\{u^{(k)}(t)\}$ is a uniformly bounded set. More

precisely it is a bounded set in the Hilbert space $\mathscr{E}^{2m,1}_{L^2}(\Omega)$, consisting of all the functions $u \in L^2$ such that

$$\frac{\partial u}{\partial t} \in L^2, \left(\frac{\partial}{\partial x}\right)^{\nu} u \in L^2 \text{ for } |\nu| \leq 2m,$$

where the derivatives are taken in the sense of distributions. $\mathscr{E}^{2m,1}_{L^2}(\Omega)$ provided with the scalar product

$$(u, v)_{\mathscr{E}^{2m,1}_{L^2}(\Omega)} = (u, v)_{L^2(\Omega)} + \left(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}\right)_{L^2(\Omega)} + \sum_{|\nu| \leq 2m} \left(\left(\frac{\partial}{\partial x}\right)^{\nu} u, \left(\frac{\partial}{\partial x}\right) v\right)_{L^2(\Omega)}$$

is a Hilbert space. Hence $\left\{u^{(k)}(t)\right\}$ has a weak limit in $\mathscr{E}^{2m,1}_{L^2}(\Omega)$, say $u(x, t) \cdot u(x, t)$ satisfies the equation

$$\frac{\partial u}{\partial t} = Au + f \tag{2.18}$$

in the sense of distributions. We shall now show that $u \in \mathscr{D}^{2m}_{L^2}[0, h]$. We know that $u \in L^2[0, h]$. If φ_ε be mollifiers of Friedrichs consider the equation

$$\frac{\partial}{\partial t}((x^\nu \varphi_\varepsilon) *_{(x)} u) = A((x^\nu \varphi_\varepsilon)) *_{(x)} u) + (x^\nu \varphi_\varepsilon) *_{(x)} f + [(x^\nu \varphi_\varepsilon) *_{(x)}, A]\, u$$

for $|\nu| \leq m$. The functions $u^\nu_\varepsilon = (x^\nu \varphi_\in) *_{(x)} u$ form a Cauchy sequence as $\varepsilon \to 0$. This can be proved by an argument similar to the one in Prop. 2. It can also be shown that 165

$$\begin{aligned} u^\nu_\varepsilon &\to u(t) \text{ in } \mathscr{D}^{2m}_{L^2} \text{ for } \nu = 0, \\ &\to 0 \text{ in } \mathscr{D}^{2m}_{L^2} \text{ otherwise} \end{aligned}$$

uniformly in t. This proves that the energy inequality (2.11) holds in this case also.

Recent work by P. Sobolevskii develops the semi-group theory for the equations of the parabolic type by using fractional powers. Equations of parabolic type in Banach space, Trudy Moscov Mat, Obsc. 10(1961), 297 - 350. □

Chapter 5

In this chapter we study non-linear equations. Much of this chapter is inspired by the recent monograph of S.L. Sobolev: Sur les equations aux derivees particlles hyperboliques non-lineaires (Cremonese, Roma 1961). **166**

1 Preliminaries to the study of semi-linear equations

In this section we recall, without giving the proofs, a few results of Sobolev concering the differentiability properties of functions belonging to the spaces $\mathscr{D}^m_{L^2}$. More precisely we give estimates in the L^p norm for the derivatives of these functions in terms of their norms in the space $\mathscr{D}^m_{L^2}$. We shall also introduce the functions spaces $\mathscr{D}^s_{L^2}$ for any aribitrary real number $s \geq 0$ and obtain L^2 estimates of some non-linear functions of derivatives of functions belonging to the spaces $\mathscr{D}^s_{L^2}$.

To begin with the state the following important result due to Sobolev [1].

Proposition 1 (Sobolev's lemma)**.** *Let p and q be positive numbers with $\underset{q>1}{p > 1}$, and $\dfrac{1}{p} + \dfrac{1}{q} > 1$. If $g \in L^p$ and $h \in L^q$ then*

(1.1) $$\left| \iint \frac{g(x)h(y)}{|x-y|^\lambda} dx\, dy \right| \leq K ||g||_{L^p} ||h||_{L^q},$$

where $\lambda = n(2 - \frac{1}{p} - \frac{1}{q})$ *and* K *is a constant depending only on* p, q, n.

Suppose $u \in L^p$ *and a number* λ *such that* $0 < \lambda < n$ *and* $\frac{\lambda}{n} > 1 - \frac{1}{p}$ *are given. Then the above inequality implies that the linear mapping*

$$h \to \int (u(y) * \frac{1}{|y|^\lambda}) \cdot h(y)\, dy \tag{1.2}$$

167 *is a continuous linear functional on the space* L^q *for* $q > 1$ *with* $\frac{1}{q} = \left(2 - \frac{1}{p} - \frac{\lambda}{n}\right)$. *Hence* $u * \frac{1}{|x|^\lambda} \in L^{q'}$ *where* q' *satisfies* $\frac{1}{q'} = 1 - \frac{1}{q} = \frac{\lambda}{n} + \frac{1}{p} - 1$. *This proves the following*

Corollary 1. *Let* $u \in L^p$ *for a* $p > 1$ *and* λ *be a positive number such that* $0 < \lambda < n$ *and* $\frac{\lambda}{n} > 1 - \frac{1}{p}$. *Then* $u * \frac{1}{|x|^\lambda} \in L^{q'}$ *where* $\frac{1}{q'} = \frac{\lambda}{n} + \frac{1}{p} - 1$.

In corollary 1 taking $p = 2$ and λ a number such that $\frac{n}{2} < \lambda < n$ we have the following

Corollary 2. *If* $u \in L^2$ *then for any positive number* λ *such that* $\frac{n}{2} < \lambda < n$ *we have* $u * \frac{1}{|x|^\lambda} \in L^q$ *where* $\frac{1}{q} = \frac{\lambda}{n} - \frac{1}{2}$ *and*

$$\|u * \frac{1}{|x|^\lambda}\|_{L^q} \leq K \|u\|_{L^2} \tag{1.3}$$

where K *is a constant depending on* n, λ.

We shall now introduce the function space $\mathscr{D}^s_{L^2} \equiv \mathscr{D}^s_{L^2}(\underline{R}^n)$ for any arbitrary real number $s > 0$.

Let Ω be an open set in $\underline{R}^n$ and m be a non-negative integer. We recall that $\mathscr{E}^m_{L^2}(\Omega)$ denotes the space of all square integrable functions f on Ω for which all the derivative $\left(\frac{\partial}{\partial x}\right)^\nu f$ (in the sense of distributions) of orders $|\nu| \leq m$ are again square integrable functions on Ω. $\mathscr{E}^m_{L^2}(\Omega)$ is provided with the scalar product

(1.4)
$$(f,g)_{\mathscr{E}^m_{L^2}}(\Omega) \equiv (f,g)_m = \sum_{|\alpha|\leq m} \left(\left(\frac{\partial}{\partial x}\right)^{\alpha}\right) f, \left(\frac{\partial}{\partial x}\right)^{\alpha} g)_{L^2(\Omega)} \text{ for } f,g \in \mathscr{E}^m_{L^2}(\Omega)$$

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Here $\left(\frac{\partial}{\partial x}\right)^{\alpha}$ denotes a derivation in the sense of distributions. $\mathscr{E}^m_{L^2}(\Omega)$ is a Hilbet space for this scalar product. Clearly $\mathscr{D}(\Omega) \subset \mathscr{E}^m_{L^2}(\Omega)$. The closure of $\mathscr{D}(\Omega)$ in $\mathscr{E}^m_{L^2}(\Omega)$ is deoted by $\mathscr{D}^m_{L^2}(\Omega) \cdot \mathscr{D}^m_{L^2}(\Omega)$, with the scalar product which is the restriction of that in $\mathscr{E}^m_{L^2}(\Omega)$, is again a Hilbert space. In general $\mathscr{D}^m_{L^2}(\Omega) \neq \mathscr{E}^m_{L^2}(\Omega)$. However when $\Omega = \underline{R}^n$ we have $\mathscr{D}^m_{L^2}(\underline{R}^n) = \mathscr{E}^m_{L^2}(\underline{R}^n)$. We write $\mathscr{D}^m_{L^2}(\underline{R}^n) = \mathscr{E}^m_{L^2}(\underline{R}^n) = \mathscr{D}^m_{L^2}$ for abbreviation. The elements of $\mathscr{D}^m_{L^2}(\Omega)$ can be considered as functions vanishing upto order $(m-1)$ (in a generalized sense) on the boundary of Ω.

We observe that $\mathscr{D}^m_{L^2} \subset \mathscr{L}'$. Hence by Plancheral's theorem we have

$$\|f\|^2_m = \|f\|^2_{\mathscr{D}^m_{L^2}} = \sum_{|\alpha|\leq m} \|\left(\frac{\partial}{\partial x}\right)^{\alpha} f\|^2_{L^2} = \sum_{|\alpha|\leq m} \|(2\pi i\xi)^{\alpha} \hat{f}\|_{L^2}$$

where $\hat{f}$ is the Fourier image of f. Now there exist constants $c_1, c_2 > 0$ such that

$$c_1^2(1+|\xi|)^{2m} \leq \sum_{|\alpha|\leq m} |(2\pi i\xi)^{\alpha}|^2 \leq c_2^2(1+|\xi|)^{2m}.$$

Thus, if $f \in \mathscr{D}^m_{L^2}$ then $(1+|\xi|)^m f \in L^2$ and further

$$C_1\|(1+|\xi|)^m \hat{f}\|_{L^2} \leq \|f\|_m \leq C_2\|(1+|\xi|)^m \hat{f}\|_{L^2}.$$

Hence $\mathscr{D}^m_{L^2}$ can also be defined as the space of all tempered distribu- 169
tions f such that $(1+|\xi|)^m \hat{f} \in L^2$ where $\hat{f}$ denotes the Fourier image of f. This motivatives the following.

Definition. For any real s, $\mathscr{D}^s_{L^2}$ is the space of tempered distributions f such that $(1+|\xi|)^s \hat{f} \in L^2$.

$\mathscr{D}^2_{L^2}$ is provided with the scalar product

(1.5) $$(f,g)_s \equiv (f,g)_{\mathscr{D}^s_{L^2}} = ((1+|\xi|)^s \hat{f}, (1+|\xi|)^s \hat{g})_{L^2}$$

For this scalar product $\mathscr{D}^s_{L^2}$ is a Hilbert space. It is clear that if $s \geq s'$ then $\mathscr{D}^s_{L^2} \subset \mathscr{D}^{s'}_{L^2}$ and the inclusion mapping is continuous.

Remarks. (1) The dual space of $\mathscr{D}^s_{L^2}$ is $\mathscr{D}^{-s}_{L^2}$: $(\mathscr{D}^s_{L^2})' = \mathscr{D}^{-s}_{L^2}$.

(2) The mapping $u \to \dfrac{\partial u}{\partial x_j}$ from $\mathscr{D}^s_{L^2}$ into $\mathscr{D}^{s-1}_{L^2}$ is continuous.

(3) The mappings defined by

$$(a(x), u) \to a(x)u$$

(i) from $\mathscr{B}^m \times \mathscr{D}^m_{L^2}$ into $\mathscr{D}^m_{L^2}$ and (ii) from $\mathscr{B}^m \times \mathscr{D}^{-m}_{L^2}$ into $\mathscr{D}^{-m}_{L^2}$ are continuous for $m = 0, 1, 2, \ldots$

Lemma 1. *Let s be a real number* ≥ 0

(i) If $u \in \mathscr{D}^s_{L^2}$ *for* $0 \leq s < \dfrac{n}{2}$ *then* $u \in L^p$ *where* $\dfrac{1}{p} = \dfrac{1}{2} - \dfrac{s}{n} > 0$ *and*

(1.6) $$||u||_{L^p} \leq c(s,n)||u||_s$$

170 *where the constant* $c(s,n)$ *depends only on s and n;*

(ii) If $u \in \mathscr{D}^s_{L^2}$ *for* $s > \dfrac{n}{2}$ *then* $u \in \mathscr{B}^0$ *and*

(1.7) $$||u||_{\mathscr{B}^0} \leq c(s,n)||u||_s$$

where the constant $c(s,n)$ *depends only on* s, n.

More precisely, for any $\sigma \leq 1$ *with* $0 < \sigma < s - \dfrac{n}{2}$ *we have*

(1.8) $$||u||_{\mathscr{B}^\sigma} \leq c(s,n,\sigma)||u||_s$$

where the constant $c(s,n,\sigma)$ *depends only on* s, n, σ.

Remark. We recall that $\dfrac{1}{|x|^{\lambda}}$ is tempered distribution and we have the formulae giving its Fourier image.

(1.9)
$$\mathscr{F}\left(\frac{1}{|x|^m}\right) = \frac{1}{\Pi^{\frac{n}{2}-m}} \frac{\Gamma(\frac{m-n}{2})}{\Gamma(\frac{m}{2})} \left(\frac{1}{|\xi|^{n-m}}\right) \text{ for } \frac{n}{2} \leq m < n \text{ and}$$
$$\mathscr{F}\left(\frac{1}{|x|^{n-m}}\right) = \frac{1}{\Pi^{m-\frac{n}{2}}} \frac{\Gamma(\frac{m}{2})}{\Gamma(\frac{n-m}{2})} \left(\frac{1}{|\xi|^{m}}\right) \text{ for } 0 < m < \frac{n}{2}.$$

For proof of these formulae we refer to L. Schwartz, Theorie des distributions, Vol. II, p. 113.

Proof of Lemma 1 : (i) The assertion (i) is trivial when $s = 0$. Hence we may assume that $0 < s < \dfrac{n}{2}$. Let $u \in \mathscr{D}^s_{L^2}$. Writing $\hat{u}$ as $|\xi|^{-s}(|\xi|^s\hat{u})$ we have
$$u = c \cdot \frac{1}{|x|^{n-s}} * (\wedge^s u)$$
by taking the inverse Fourier images and using the above remark (we note that c is a positive constant depending only on n, s). It follows now, from cor. 2 after Prop. 1, that $u \in L^p$ and **171**

$$||u||_{L^p} = c||\frac{1}{|x|^{n-s}} * (\wedge^s u)||_{L^p} \leq c(s, n)|| \wedge^s u||_{L^2}$$

where $\dfrac{1}{p} = \dfrac{1}{2} - \dfrac{s}{n}$ (the constant $c(s, n)$ depends only on s, n) . By Plancheral's theorem we have

$$|| \wedge^s u||_{L^2} = || \, |\xi|^2\hat{u}||_{L^2} \leq ||(1 + |\xi|)^s\hat{u}||_{L^2} = ||u||_s.$$

This proves the inequality (1.6).

(ii) Let $u \in \mathscr{D}^s_{L^2}$ for $s > \dfrac{n}{2}$. We have, using Cauchy-Schwarz inequality

$$|u(x)| \leq \int |\hat{u}(\xi)|d\xi \leq ||(1 + |\xi|)^s\hat{u}||_{L^2}||(1 + |\xi|)^{-s}||_{L^2}$$

which implies that $|u(x)| \leq c(s, n)||u||_s$ where $c(s, n)$ is a constant depending only on s, n.

We shall now prove Hölder continuity of u. Consider

$$u(x) - u(x') = \int \exp(2\pi i\, x.\xi \cdot \xi)\hat{u}(\xi)d\xi - \int \exp(2\pi i x'.\xi)\hat{u}(\xi)d\xi$$
$$= \int \exp(2\pi i x.\xi)\{1 - \exp(2\pi i(x' - x).\xi\}\hat{u}(\xi)d\xi.$$

For any real number σ such that $0 < \sigma \leq 1$ let

$$M_\sigma = \sup_{-\infty<\lambda<\infty} \left| \frac{e^{i\lambda} - 1}{\lambda^\sigma} \right| \tag{1.10}$$

Clearly $M_\sigma < \infty$. Taking $\lambda' = 2\Pi(x - x') \cdot \xi$ we obtain

$$|1 - \exp(2\pi i(x' - x) \cdot \xi)| \leq M_\sigma (2\pi |x - x'|\, |\xi|)^\sigma.$$

172 Hence

$$\frac{|u(x) - u(x')|}{|x - x'|} \leq (2\pi)^\sigma M_\sigma \int |\xi|^\sigma |\hat{u}(\xi)| d\xi$$
$$\leq (2\pi)^\sigma M_\sigma \|(1 + |\xi|)^s \hat{u}\|_{L^2} \|(1 + |\xi|^{\sigma - s})\|_{L^2}.$$

We know that $\sigma - s < -\frac{n}{2}$ implies $\|(1 + |\xi|)^{\sigma - s}\|_{L^2} < \infty$ and this proves the Holder continuity of u. Thus $u \in \mathbb{B}^\sigma$ for any $\sigma \leq 1$ with $0 < \sigma < s - \frac{n}{2}$.

Proposition 2. *If $u \in \mathscr{D}^{[\frac{n}{2}]+1}_{L^2}$ then, for $1 \leq |\nu| \leq \left[\frac{n}{2}\right] + 1$, $\left(\frac{\partial}{\partial x}\right)^\nu u \in L_p$ where $\mathfrak{p}$ is a positive number such that*

(a) $\frac{1}{p} \in \left[\frac{|\nu|}{n} - \frac{1}{n}, \frac{1}{2}\right] - \{0\}$ *when n is even,*

(b) $\frac{1}{p} \in \left[\frac{|\nu|}{n} - \frac{1}{2n}, \frac{1}{2}\right]$ *when n is odd*

Further the mapping $u \to \left(\frac{\partial}{\partial x}\right)^\nu u$ is continuous from $\mathscr{D}^{[\frac{n}{2}]+1}_{L^2}$ into L^2 and we have the inequality

$$\left\| \left(\frac{\partial}{\partial x}\right)^\nu u \right\|_{L^p} \leq c(\nu, n, p) \|u\|_{[\frac{n}{2}]+1}. \tag{1.11}$$

The constant $c(\nu, n, p)$ depends only on ν, n, p.

Before proceeding with the proof of this proposition we introduce the following

Definition. *The operator* Λ^s. For any $u \in \mathscr{D}^{\sigma}_{L^2}$ with $-\infty < \sigma < \infty$ the operator Λ^s is defined by the condition that $\Lambda^s u$ is the inverse Fourier 173
image of $|\xi|^2 \hat{u}$.

Proof. For any real $s \geq 0$ such that $s \leq \left[\frac{n}{2}\right] + 1$, $u \in \mathscr{D}^{[\frac{n}{2}]+1}_{L^2}$ implies that $u \in \mathscr{D}^{s}_{L^2}$. Since the inverse Fourier image of $\dfrac{1}{|\xi|^{s-|\nu|}}$ is $c(n, \nu)\dfrac{1}{|x|^{n-(s-|\nu|)}}$ we can write

$$\left(\frac{\partial}{\partial x}\right)^{\nu} u = c(n, \nu) \frac{1}{|x|^{n-(s-|\nu|)}} * \left(\Lambda^{s-|\nu|} \left(\frac{\partial}{\partial x}\right)^{\nu} u\right)$$

by taking inverse Fourier image of

$$\left(\left(\frac{\widehat{\partial}}{\partial x}\right)^{\nu} u\right) = (2\pi i \xi)^{\nu} \hat{u} = \frac{1}{|\xi|^{s-|\nu|}} \{|\xi|^{s-|\nu|} (2\pi i \xi)^{\nu} \hat{u}\}.$$

Hence it follows, from Cor. 2 of Prop. 1, that

$$\begin{aligned} \| \left(\frac{\partial}{\partial x}\right)^{\nu} u\|_{L^p} &= c(n, \nu) \| \frac{1}{|x|^{n-(s-|\nu|)}} * \Lambda^{s-|\nu|} \left(\frac{\partial}{\partial x}\right)^{\nu} u\|_{L^p} \\ &\leq c(s, n, \nu) \|\Lambda^{s-|\nu|} \left(\frac{\partial}{\partial x}\right)^{\nu} u\|_{L^2} \end{aligned}$$

for $\dfrac{1}{p} = \dfrac{n - (s - |\nu|)}{n} - \dfrac{1}{2} = \dfrac{1}{2} - \dfrac{s - |\nu|}{n}$. On the other hand we know that

$$\|\Lambda^{s-|\nu|} \left(\frac{\partial}{\partial x}\right)^{\nu} u\|_{L^2} \leq \|u\|_s \leq \|u\|_{[\frac{n}{2}]+1}$$

which proves that

$$\| \left(\frac{\partial}{\partial x}\right)^{\nu} u\|_{L^p} \leq c(s, n, \nu) \|u\|_{[\frac{n}{2}]+1}.$$

Using the fact that $|\nu| \leq s \leq \left[\dfrac{n}{2}\right] + 1$ we have, since

$$\frac{1}{p} \in \left[\frac{1}{2} - \frac{[\frac{n}{2}] + 1 - |\nu|}{n}, \frac{1}{2}\right] - \{0\},$$

174 that $\dfrac{1}{p} \in \left[\dfrac{|\nu|}{n} - \dfrac{1}{n}, \dfrac{1}{2}\right] - \{0\}$ when n is even and similarly $\dfrac{1}{p} \in \left[\dfrac{|\nu|}{n} - \dfrac{1}{n}, \dfrac{1}{2}\right]$ when n is odd.

An entirely analogous proof will yield □

Proposition 2′. *If $u \in \mathscr{D}^{[\frac{n}{2}]+N}_{L^2}$ we have $\left(\dfrac{\partial}{\partial x}\right)^{\nu} u \in L^p$ where p is a positive number such that*

(a) $\dfrac{1}{p} \in \left[\dfrac{|\nu|}{n} - \dfrac{N}{n}, \dfrac{1}{2}\right] - \{0\}$ *when n is even and*

(b) $\dfrac{1}{p} \in \left[\dfrac{|\nu|}{n} - \dfrac{2N-1}{2n}, \dfrac{1}{2}\right]$ *when n is odd, where* $1 \leq N \leq |\nu| \leq \left[\dfrac{n}{2}\right] + N$.

Fourther the mapping $u \to \left(\dfrac{\partial}{\partial x}\right)^{\nu} u$ is continuous from $\mathscr{D}^{[\frac{n}{2}]+N}_{L^2}$ into L^p and we have the inequality

$$(1.12) \qquad \|\left(\frac{\partial}{\partial x}\right)^{\nu} u\|_{L^p} \leq c(\nu, \|n, N, p)\|u\|_{[\frac{n}{2}]+N}$$

where the constant $c(\nu, n, N, p)$ depends only ν, n, N, p.

The following result gives estimates in the L^2 norm of some non-linear functions of the derivatives of functions belonging to $\mathscr{D}^s_{L^2}$. The proofs are based essentially on the above result and a generalization of Holder's inequality which we recall without proof.

Proposition 3 (Generalized Hölder's inequality)**.** *Let $\lambda_1, \ldots, \lambda_p$ be positive numbers > 1 such that $\sum \dfrac{1}{\lambda_j} = 1$. If $f_1, \ldots, f_p$ are functions belonging to $L^{\lambda_1}, \ldots, L^{\lambda_p}$ respectively then*

$$(1.13) \qquad \int |f_1(x) \ldots f_p(x)| dx \leq \|f_1\|_L \lambda_1, \ldots \|f_p\|_L \lambda_p.$$

175 **Proposition 4.** *Let l be an arbitrary integer ≥ 1 and $\nu_1, \ldots, \nu_1$ denote multi-indices*

(i) If $u \in \mathscr{D}^{[\frac{n}{2}]+1}_{L^2}$ and $\sum |\nu_j| \leq \left[\frac{n}{2}\right] + 1$ then $\left(\frac{\partial}{\partial x}\right)^{\nu_1} u \ldots \left(\frac{\partial}{\partial x}\right)^{\nu_1} u \in L^2$ and satisfies

$$\|\left(\frac{\partial}{\partial x}\right)^{\nu_1} u \ldots \left(\frac{\partial}{\partial x}\right)^{\nu_l} u\|_{L^2} \leq c\|u\|^l_{[\frac{n}{2}]+1} \tag{1.14}$$

where c depends on n, $\nu_1, \ldots, \nu_1$ only.

(ii) If $u \in \mathscr{D}^{[\frac{n}{2}]+2}_{L^2}$ and $\sum |\nu_j| \leq \left[\frac{n}{2}\right] + 2$ then $\left(\frac{\partial}{\partial x}\right)^{\nu_1} u \ldots \left(\frac{\partial}{\partial x}\right)^{\nu_l} u \in L^2$ and satisfies

$$\|\left(\frac{\partial}{\partial x}\right)^{\nu_1} u \ldots \left(\frac{\partial}{\partial x}\right)^{\nu_1} u\|_{L^2} \leq c\|u\|^{l-1}_{[\frac{n}{2}]+1}\|u\|_{[\frac{n}{2}]+2}; \tag{1.15}$$

the constant c depends only on n, $\nu_1, \ldots, \nu_l$.

(iii) If $u \in \mathscr{D}^{[\frac{n}{2}]+N+1}_{L^2}$ and $\sum |\nu_j| \leq \left[\frac{n}{2}\right]+N+1$ then $\left(\frac{\partial}{\partial x}\right)^{\nu_1} u \ldots \left(\frac{\partial}{\partial x}\right)^{\nu_l} u \in L^2$ and satisfies

$$\|\left(\frac{\partial}{\partial x}\right)^{\nu_1} u \ldots \left(\frac{\partial}{\partial x}\right)^{\nu_l} u\|_{L^2} \leq c\|u\|^{l-1}_{[\frac{n}{2}]+N}\|u\|_{[\frac{n}{2}]+N+1}; \tag{1.16}$$

the constant c depend only on n, N, $\nu_1, \ldots, \nu_l$.

Proof. The case $l = 1$ is trivial. If $\nu_j = 0$ for some j one can majorize u in the maximum norm by $\|u\|_{[\frac{n}{2}]+1}$. Hence we may assume that $l \leq 2$ and $|\nu_j| \geq 1$.

(i) Since $u \in \mathscr{D}^{[\frac{n}{2}]+1}_{L^2}$ it follows, from Prop. 2, that $\left(\frac{\partial}{\partial x}\right)^{\nu_j} u \in L^{pj}$ for 176
$1 \leq |\nu_j| \leq \left[\frac{n}{2}\right] + 1$ where p_j is a real number such that

(a) $\frac{1}{p_j} \in \left[\frac{|\nu_j|}{n} - \frac{1}{n}, \frac{1}{2}\right] - \{0\}$ when n is even and

(b) $\dfrac{1}{p_j} \in \left[\dfrac{|\nu_j|}{n} - \dfrac{1}{2n}, \dfrac{1}{2}\right]$ when n is odd.

Further we have

$$\| \left(\frac{\partial}{\partial x}\right)^{\nu_j} u\|_{L^{p_j}} \leq c(\nu_j, n, p_j)\|u\|_{[\frac{n}{2}]+1} (j = 1, \ldots, l).$$

Let $\dfrac{1}{P_j}$ denote the infimum of $\dfrac{1}{p_j}$ in this range.
If n is even (a) implies that

$$\sum \frac{1}{P_j} = \sum \left(\frac{|\nu_j|}{n} - \frac{1}{n}\right) \leq \frac{\frac{n}{2}+1}{n} - \sum \frac{1}{n} < \frac{1}{2} + \frac{1}{n}$$

and so. One can choose $p_1, \ldots, p_l$ satisfying (a) and such that $\sum \dfrac{1}{p_j} = \dfrac{1}{2}$. Similarly if n is odd (b) implies that

$$\sum \frac{1}{P_j} = \sum \left(\frac{|\nu_j|}{n} - \frac{1}{2n}\right) \leq \frac{\frac{n-1}{2}+1}{n} - \sum \frac{1}{2n} \leq \frac{1}{2} + \frac{1}{2n}.$$

Again one can choose $p_1, \ldots, p_1$ such that $\sum \dfrac{1}{p_j} = \dfrac{1}{2}$ and satisfies (b). Applying the generalized Hölder's inequality with these $p_1, \ldots, p_l$ we obtain

$$\int | \left(\frac{\partial}{\partial x}\right)^{\nu_1} u \ldots \left(\frac{\partial}{\partial x}\right)^{\nu_l} u\Big|^2 dx \leq \Pi_j \left(\int | \left(\frac{\partial}{\partial x}\right)^{\nu_j} u|^{2 \cdot \frac{p_j}{2}} dx\right)^{2/p_j}$$

$$= \prod_j \| \left(\frac{\partial}{\partial x}\right)^{\nu_j} u\|^2_L p_j \leq c(\nu_1, \ldots, \nu_1, n)\|u\|^{2l}_{[\frac{n}{2}]+1}$$

177 (ii) Since $u \in \mathscr{D}^{[\frac{n}{2}]+2}_{L^2}$ it follows, from Prop. 2′ that $\left(\dfrac{\partial}{\partial x}\right)^{\nu_j} u \in L^{p_j} (j = 1, \ldots, l)$ and

$$\| \left(\frac{\partial}{\partial x}\right)^{\nu_1} u\|_{L^{p_1}} \leq c(\nu_1, n, p_1)\|u\|_{[\frac{n}{2}]+2'}$$

$$\| \left(\frac{\partial}{\partial x}\right)^{\nu_j} u\|_{L^{p_j}} \leq c(\nu_j, n, p_j)\|u\|_{[\frac{n}{2}]+1}(j = 2, \ldots, l)$$

for $1 \leq |\nu_1| \leq \left[\frac{n}{2}\right] + 2$, $1 \leq |\nu_j| \leq \left[\frac{n}{2}\right] + 1$ where $p_1, \ldots, p_l$ are real numbers such that

(a$_1$) $\frac{1}{p_1} \in \left[\frac{|\nu_1|}{n} - \frac{2}{n}, \frac{1}{2}\right] - \{0\}$ when n is even,

(b$_1$) $\frac{1}{p_1} \in \left[\frac{|\nu_1|}{n} - \frac{3}{2n}, \frac{1}{2}\right]$ when n is odd

and

(a$_j$) $\frac{1}{p_j} \in \left[\frac{|\nu_j|}{n} - \frac{1}{n}, \frac{1}{2}\right] - \{0\}$ when n is even,

(b$_j$) $\frac{1}{p_j} \in \left[\frac{|\nu_j|}{n} - \frac{1}{2n}, \frac{1}{2}\right]$ when n is odd $(j = 2, \ldots, l)$.

We may without loss of generality assume that $|\nu_1| \geq |\nu_j|$ for $j = 2, \ldots, l$.

(1) Suppose $|\nu_1| = 1$. Since $\sum_2^l |\nu_j| \leq \left[\frac{n}{2}\right] + 1$ we have from lemma 1 that

$$\| \left(\frac{\partial}{\partial x}\right)^{\nu_1} u \ldots \left(\frac{\partial}{\partial x}\right)^{\nu_l} u\|_{L^2} \leq \sup \left| \left(\frac{\partial}{\partial x}\right)^{\nu_1} u \right| \cdot \| \left(\frac{\partial}{\partial x}\right)^{\nu_2} u \ldots \left(\frac{\partial}{\partial x}\right)^{\nu_l} u\|_{L^2}$$
$$\leq c(n)\|u\|_{[\frac{n}{2}]+2} \cdot \| \left(\frac{\partial}{\partial x}\right)^{\nu_2} u \ldots \left(\frac{\partial}{\partial x}\right)^{\nu_l} u\|_{L^2}$$

(iii) Suppose $|\nu_j| \geq 2 (j = 2, \ldots, l)$ then we have the estimates of the type (1.11). As before we denote the infimum of $\frac{1}{p_j}$ by $\frac{1}{P_j} (j = 1, \ldots, l)$.

If n is even (a$_1$), (a$_j$) imply that **178**

$$\sum \frac{1}{P_j} = \frac{|\nu_1|}{n} - \frac{2}{n} + \sum_2^l + \left(\frac{|^{\nu} j|}{n} - \frac{1}{n}\right) < \frac{1}{2}$$

and if n is odd (b$_1$), (b_j) imply that $\sum \frac{1}{P_j} < \frac{1}{2}$. In either of the cases we can choose $p_2, \ldots, p_l$ such that $\sum \frac{1}{p_j} = \frac{1}{2}$.

Again applying the generalized Hölder's inequality we obtain

$$\begin{aligned}\|\left(\frac{\partial}{\partial x}\right)^{\nu_1} u \ldots \|\left(\frac{\partial}{\partial x}\right)^{\nu_1} u\|_{L^2} &\leq \prod_j \|\left(\frac{\partial}{\partial x}\right)^{\nu j} u\|_{L^{p_j}} \\ &\leq c(n, \nu_1, \ldots, \nu_1)\|u\|^{l-1}_{[\frac{n}{2}]+1}\|u\|_{[\frac{n}{2}]+2}.\end{aligned}$$

As before we may assume that $|\nu_1| \geq |\nu_j|$ for $j = 2, \ldots, l$. Let $u \in \mathscr{D}^{[\frac{n}{2}]+N+1}_{L^2}$. We distinguish the following three different cases:

$$(\alpha)|\nu_1| \leq N-1, \quad (\beta)|\nu_1| = N, \quad (\gamma)|\nu_j| \geq N.$$

□

Case (α). Since $|\nu_j| \leq |\nu_1| \leq N-1$ by Sobolev's lemma we have

$$\sup |\left(\frac{\partial}{\partial x}\right)^{\nu_j} u| \leq c\|u\|_{[\frac{n}{2}]+N}.$$

Therefore we have

$$\begin{aligned}\|\left(\frac{\partial}{\partial x}\right)^{\nu_1} u \ldots \left(\frac{\partial}{\partial x}\right)^{\nu_l} u\|_{L^2} &\leq \|\left(\frac{\partial}{\partial x}\right)^{\nu_1} u\|_{L^2} \cdot \prod_{j=2}^{l} \sup |\left(\frac{\partial}{\partial x}\right)^{\nu_j} u| \\ &\leq C\|u\|_{|\nu_1|} \cdot \|u\|^{l-1}_{[\frac{n}{2}]+N} \\ &\leq C\|u\|_{[\frac{n}{2}]+N+1} \cdot \|u|^{l-1}_{[\frac{n}{2}]+N}.\end{aligned}$$

179 *Case* (β). $|\nu_1| = N$ implies that $\sum_{j=2}^{l} |\nu_j \leq \left[\frac{n}{2}\right] + 1$ and we have from lemma 1 that

$$\|\left(\frac{\partial}{\partial x}\right)^{\nu_1} u \ldots \left(\frac{\partial}{\partial x}\right)^{\nu_l} u\|_{L^2} \leq \sup |\left(\frac{\partial}{\partial x}\right)^{\nu_1} u| \, \|\left(\frac{\partial}{\partial x}\right)^{\nu_2} u \ldots \left(\frac{\partial}{\partial x}\right)^{\nu_l} u\|_{L^2}.$$

By Sobolev's lemma we have

$$\sup|\left(\frac{\partial}{\partial x}\right)^{\nu_1} u| \leq c(n, N, \nu_1)\|u\|_{[\frac{n}{2}]+N+1}$$

and on the other hand $\left(\frac{\partial}{\partial x}\right)^{\nu_j} u \in L^{p_j}$ with

$$\|\left(\frac{\partial}{\partial x}\right)^{\nu_j} u\|_L^{p_j} \leq c(n, N, \nu_j, p_j)\|u\|_{[\frac{n}{2}]+N}$$

for $\frac{1}{p_j} \in \left[\frac{|\nu_j|}{n} - \frac{N}{n}, \frac{1}{2}\right] - \{0\}$ if n is even and $\frac{1}{p_j} \in \left[\frac{|\nu_j|}{n} - \frac{2N-1}{n}, \frac{1}{2}\right]$ if n is odd (from Prop. 2′).

Denoting inf $\frac{1}{p_j}$ by $\frac{1}{p_j}$ we see that $\sum\limits_{j=2}^{l} \frac{1}{P_j} = \sum\left(\frac{|\nu_j|}{n} - \frac{N}{n}\right) < \frac{1}{2}$ if n is even and $\sum\limits_{j=2}^{l} \frac{1}{P_j} = \sum\left(\frac{|\nu_j|}{n} - \frac{2N-1}{n}\right) < \frac{1}{2}$ if n is odd. One can choose $p_2, \ldots, p_l$ such that $\sum\limits_{j=2}^{l} \frac{1}{p_j} = \frac{1}{2}$ in both the cases. An application of the generalized Hölder's inequality with these $p_2, \ldots, p_l$ gives

$$\begin{aligned}\|\left(\frac{\partial}{\partial x}\right)^{\nu_2} u \ldots \|\left(\frac{\partial}{\partial x}\right)^{\nu_l} u\|_{L^2} &\leq \prod_{j=2}^{l} \|\left(\frac{\partial}{\partial x}\right)^{\nu_j} u\|_{L^{p_j}} \\ &\leq c(n, \nu_2, \ldots \nu_l, N, p_2, \ldots, p_l)\|u\|^{l-1}_{[\frac{n}{2}]+N}\end{aligned}$$

(γ) If $|\nu_j| \geq N$ for $j = 2, \ldots, l$ we have from Prop. 2′ that **180**

$$\|\left(\frac{\partial}{\partial x}\right)^{\nu_1} u\|_{L^{p_1}} \leq c(\nu_1, p_1, N, n)\|u\|_{[\frac{n}{2}]+N+1}$$

and $\|\left(\frac{\partial}{\partial x}\right)^{\nu_j} u\|_{L^{p_j}} \leq c(\nu_j, p_j, N, n)\|u\|_{\left[\frac{n}{2}\right]+N}$ where $p_1, \ldots, p_l$ are real numbers such that

$$\begin{cases} \frac{1}{p_1} \in \left[\frac{|\nu_1|}{n} - \frac{N+1}{n}, \frac{1}{2}\right] - \{0\}, \\ \frac{1}{p_j} \in \left[\frac{|\nu_j|}{n} - \frac{N}{n}, \frac{1}{2}\right] - 0 \text{ for even } n \text{ and} \end{cases}$$

$$\begin{cases} \frac{1}{p_1} \in \left[\frac{|\nu_1|}{n} - \frac{2N+1}{2n}, \frac{1}{2}\right] \\ \frac{1}{p_j} \in \left[\frac{|\nu_j|}{n} - \frac{2N-1}{2n}, \frac{1}{2}\right] \text{ for odd } n. \end{cases}$$

If $\dfrac{1}{P_j}$ denotes inf $\dfrac{1}{p_j}$ we have

$$\sum_{j=1}^{l} \frac{1}{P_j} = \sum_{j=1}^{l} \left(\frac{|\nu_j|}{n}\right) - \frac{N+1}{n} - \sum_{j=2}^{l} \frac{N}{n} < \frac{1}{2} \text{ for even } n \text{ and}$$

$$\sum_{j=1}^{l} \frac{1}{P_j} = \sum_{j=1}^{l} \left(\frac{|\nu_j|}{n}\right) - \frac{2N+1}{2n} - \sum_{j=2}^{l} < \frac{2N-1}{2n} < \frac{1}{2} \text{ for odd } n.$$

Once again choosing $p_2, \ldots, p_l$ such that $\sum_{j=2}^{l} \dfrac{1}{p_j} = \dfrac{1}{2}$ we obtain the desired inequality after applying the genearlized Hölder's inequality to $\|\left(\dfrac{\partial}{\partial x}\right)^{\nu_1} u \ldots \left(\dfrac{\partial}{\partial x}\right)^{\nu_l} u\|_{L^2}$ with these $p_1, \ldots, p_l$ and using the estimates of the form (1.11).

181 By an argument completely analogous to the one in the prop. 4 one can establish the following more general result.

Proposition 5. *Let l be an arbitrary integer and $\nu_1, \ldots, \nu_1$ be l multi-indices.*

(i) *If $u_1, \ldots, u_l \in \mathscr{D}_{L^2}^{[\frac{n}{2}]+1}$ and $\sum_{j=1}^{l} |\nu_j| \leq \left[\dfrac{n}{2}\right] + 1$ then $\left(\dfrac{\partial}{\partial x}\right)^{\nu_1} u_1 \ldots \left(\dfrac{\partial}{\partial x}\right)^{\nu_l} u_l \in L^2$. Further*

$$\|\left(\frac{\partial}{\partial x}\right)^{\nu_1} u_1 \ldots \left(\frac{\partial}{\partial x}\right)^{\nu_l} u_l\|_{L^2} \leq c \prod_{j=1}^{l} \|u_j\|_{[\frac{n}{2}]+1} \tag{1.17}$$

where the constant c depends only on n, $\nu_1, \ldots, \nu_l$.

(ii) *Let* $|\nu_1| \geq |\nu_j|$ *for* $j = 2, \ldots, l$. *If* $u_1 \in \mathscr{D}_{L^2}^{[\frac{n}{2}]+N+1}$, $u_2, \ldots, u_l \mathscr{D}_{L^2}^{[\frac{n}{2}]+N}$ *and* $\sum |\nu_j| \leq \left[\frac{n}{2}\right] + N + 1$ *then* $\left(\frac{\partial}{\partial x}\right)^{\nu_1} u_1 \ldots \left(\frac{\partial}{\partial x}\right)^{\nu_l} u_l \in L^2$ *and*

$$(1.18) \quad \|\left(\frac{\partial}{\partial x}\right)^{\nu_1} u_1 \ldots \left(\frac{\partial}{\partial x}\right)^{\nu_l} u_l\|_{L^2} \leq c\|u_1\|_{[\frac{n}{2}]+N+1} \prod_{j=2}^{l} \|u_j\|_{[\frac{n}{2}]+N}$$

where c depends only on $n, \nu_1, \ldots, \nu_1, N$.

2 Regularity of some non-linear functions

Here we make a few remarks on the local properties of certain smooth non-linear functions of x, t, u which will be required for the study of some quasi-linear differential equations. Let Ω denote the set

$$\left\{(x, t) \middle| x \in \underline{R}^n, 0 \leq t \leq T\right\}.$$

Let $f(x, t, u)$ be a function belonging to $\mathscr{E}^{[\frac{n}{2}]+2}(\Omega \times \mathbb{C})$. For a fixed function $\alpha \in \mathscr{D}(\underline{R}^n)$ we denote $\alpha(x) f(x, t, u)$ by $\tilde{f}(x, t, u)$. α localizes **182**
$f(x, t, u)$ in the x-space. We use the following abbreviations $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial u}\right)^{\beta}$ stands for a derivation of order $|\beta|$ with respect to x and u; $F(x, t)$, $\tilde{F}(x, t), G(x, t), \ldots$ stand respectively for

$$f(x, t, u(x, t)), \ \tilde{f}(x, t, u(x, t)), \ g(x, t, u(x, t)) \ldots.$$

Let U be the subset of $\Omega \times \mathbb{C}$ defined by

$$(2.1) \qquad U = \{(x, t, u) | (x, t) \in \Omega, |u| \leq \sup_{\Omega} |u(x, t)|\}.$$

Throughout this section $c_1(n), c_2(n), \ldots$ denote constants depending only on n.

Lemma 1. *If* $u \in \mathscr{D}_{L^2}^{[\frac{n}{2}]+1}[0, T]$ *then* $\tilde{F} = \tilde{F}(x, t) \in \mathscr{D}_{L^2}^{[\frac{n}{2}]+1}[0, T]$ *and*

$$(2.2) \qquad \|\tilde{F}\|_{[\frac{n}{2}]+1} \leq c_1(n) M \left\{\|1 + \|u\|_{[\frac{n}{2}]+1}^{[\frac{n}{2}]+1}\right\}$$

where $M = \max_{|\beta| \leq \frac{n}{2}+1} \sup_{U} \left| \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial u} \right)^{\beta} \tilde{f}(x, t, u) \right|$.

Before proceeding with the proof of the lemma 1 we make the following two remarks. Let $u \in \mathscr{D}_{L^2}^{[\frac{n}{2}]+1}[0, T]$. Let φ_ε be the mollifiers in the x-space and let $u_\varepsilon(x, t) = u(x, t) *_{(x)} \varphi_\varepsilon(x)$; then

(i) $u_\varepsilon \in \mathscr{B}_x^0[0, T]$ and

(2.3) $$|u_\varepsilon(x, t)|_{\mathscr{B}_x^0} \leq |u(x, t)|_{\mathscr{B}_x^0}.$$

This is an immediate consequence of lemma 1 § 1 of Chap. 3.

183 (ii) $u_\varepsilon \in \mathscr{D}_{L^2}^s[0, T]$ and

$$\|u_\varepsilon\|_s \leq \|u\|_s \text{ for } 0 \leq s \leq \left[\frac{n}{2}\right] + 1.$$

In fact, we observe that $\hat{\varphi}_\varepsilon(\xi) = \hat{\varphi}(\varepsilon\xi) \to \hat{\varphi}(0) = 1$ as $\varepsilon \to 0$. Consider

$$\begin{aligned} \|u_\varepsilon - u\|_s &= \|(1 + |\xi|)^s (\hat{u}_\varepsilon(\xi, t) - \hat{u}(\xi, t))\|_{L^2} \\ &= \|(1 + |\xi|)^s \hat{u}(\xi, t) - (\hat{\varphi}_\varepsilon(\xi) - 1)\|_{L^2} \end{aligned}$$

which converges to 0 as $\varepsilon \to 0$. Hence

$$\|u_\varepsilon\| \leq \|u\| + \|u_\varepsilon - u\|$$

implies the assertion.

Proof of the Lemma. Through out the proof the derivatives with respect to x are taken in the sense of distributions. Denoting $\tilde{f}(x, t, u_\varepsilon(x, t))$ by $\tilde{F}_\varepsilon(x, t)$ we see that $\tilde{F}_\varepsilon(x, t) \to F(x, t)$ as $\varepsilon \to 0$. For,

$$\|\tilde{F}_\varepsilon(x, t) - F(x, t)\|_{L^2} = \left\| \left[\frac{\partial \tilde{f}}{\partial u}\right] (x, t, u(x, t)) \cdot (u_\varepsilon(x, t) - u(x, t)) \right\|_{L^2}$$

which tends to 0 as $\varepsilon \to 0$. Now, for $1 \leq j \leq n$,

$$\frac{\partial}{\partial x_j} F(x, t) = \lim_{\varepsilon \to 0} \frac{\partial}{\partial x_j} F_\varepsilon(x, t)$$

where the limit is taken in the space L^2. In fact, we can write

$$\frac{\partial}{\partial x_j}\tilde{F}_\varepsilon(x,t) = \left[\frac{\partial \tilde{f}}{\partial x_j}\right](x,t,u_\varepsilon(x,t)) + \left[\frac{\partial \tilde{f}}{\partial u}\right](x,t,u_\varepsilon(x,t))\cdot\frac{\partial u_\varepsilon}{\partial x_j}(x,t)$$

in the sense of distributions. This function tends to

$$\left[\frac{\partial \tilde{f}}{\partial x_j}\right](x,t,u(x,t)) + \left[\frac{\partial \tilde{f}}{\partial u}\right](x,t,u(x,t))\cdot\frac{\partial u}{\partial x_j}(x,t).$$

in the space $L^2[0,T]$, because $u \in \mathscr{D}^{[\frac{n}{2}]+1}_{L^2}[0,T]$ implies that $\left[\frac{\partial \tilde{f}}{\partial x_j}\right]$ **184**
$(x,t,u(x,t))$, $\left[\frac{\partial \tilde{f}}{\partial u}\right](x,t,u(x,t))$ belong to the space $\mathscr{B}^0_x[0,T]$.

For a multi-index ν with $|\nu| \leq [\frac{n}{2}] + 1$ we have

(2.4)
$$\left(\frac{\partial}{\partial x}\right)^\nu \tilde{F}_\varepsilon(x,t) = \sum_{\substack{|\rho_j|\leq|\nu| \\ l\leq|\nu|}} C_{\rho_1\ldots\rho_l} g_{\rho_1\ldots\rho_l}(x,t,u_\varepsilon(x,t)) \prod_{j=1}^{l}\left(\frac{\partial}{\partial x}\right)^{\rho_j} u_\varepsilon(x,t)$$

where $C_{\rho_1\ldots\rho_l}$ are constants and $g_{\rho_1\ldots\rho_l}(x,t,u)$ is one of the derivatives $\left(\frac{\partial}{\partial x},\frac{\partial}{\partial u}\right)^\beta \tilde{f}(x,t,u)$ of orders $|\beta| \leq |\nu|$. This identity is again taken in the sense of distributions in the x-space. In view of the prop. 4 § 1, the function.

(2.5)
$$g_{\rho_1\ldots\rho_l}(x,t,u(x,t)) \prod_{j=1}^{l}\left(\frac{\partial}{\partial x}\right)^{\rho_j} u(x,t) \equiv G_{\rho_1\ldots\rho_l}(x,t) \prod_{j=1}^{1}\left(\frac{\partial}{\partial x}\right)^{\rho_j} u(x,t)$$

belongs to $L^2[0,T]$. Setting

(2.6)
$$J_\varepsilon(x) = G_{\rho_1\ldots\rho_l,\varepsilon}(x,t) \prod_{j=1}^{l}\left(\frac{\partial}{\partial x}\right)^{\rho_j} u_\varepsilon(x,t) - G_{\rho_1\ldots\rho_l}(x,t) \prod_{j=1}^{l}\left(\frac{\partial}{\partial x}\right)^{\rho_j} u(x,t)$$

we have

$$\|J_\varepsilon\|_{L^2} \leq M\left\{\|(u_\varepsilon - u)(x,t)\prod_{j=1}^{l}\left(\frac{\partial}{\partial x_j}\right)^{\rho_j} u(x,t)\|_{L^2}\right.$$

$$+\sum_{j=1}^{l} \|u(x,t)\left(\frac{\partial}{\partial x}\right)^{\rho_l} u(x,t)\ldots\left(\frac{\partial}{\partial x}\right)^{\rho_j} u(x,t)\left(\frac{\partial}{\partial x}\right)^{\rho_{j+1}} (u_\varepsilon - u)(x,t)\left(\frac{\partial}{\partial x}\right)^{\rho_l} u_\varepsilon(x,t)\|_{L^2}.$$

185 The prop. 4 of § 1 implies that

$$\|J_\varepsilon\|_{L^2} \leq c_2(n)M\|(u_\varepsilon - u)\|_{[\frac{n}{2}]+1}\|u\|^{1-1}_{[\frac{n}{2}\|]+1} \tag{2.7}$$

which tends to 0 as $\varepsilon \to 0$. This proves that

$$\left(\frac{\partial}{\partial x}\right)^{\nu} \tilde{f}((x,t),u(x,t)) = \sum c_{\rho_l\ldots\rho_l} G_{\rho_1\ldots\rho_l}(x,t)\prod_{j=1}^{l}\left(\frac{\partial}{\partial x}\right)^{\rho_j} u(x,t). \tag{2.8}$$

Again applying Prop. 4 § 1 to (2.8) it is easy to see that the estimate (2.2) holds. The continuity in t of F is proved as before. This completes the proof of the lemma.

The following results are proved in exactly the same manner as the lemma 1.

Corollary 1. *If* $f(x,t,u) \in \mathscr{E}^{[\frac{n}{2}]+N+1}(\Omega\times \in)$ *and* $u \in \mathscr{D}^{[\frac{n}{2}]+N+1}_{L^2}[0,T]$ *then*

$$\|\tilde{F}(x,t)\|_{[\frac{n}{2}]+N+1} \leq C_3(n)M_1\left\{1 + (1+\|u\|^{[\frac{n}{2}]+N}_{[\frac{n}{2}]+N+1})\|u\|_{[\frac{n}{2}]+N+1}\right\} \tag{2.9}$$

where $M_1 = \max\limits_{|\beta|\leq[\frac{n}{2}]+N+1}\sup\limits_{U}\left|\left(\frac{\partial}{\partial x},\frac{\partial}{\partial u}\right)^{\beta}\tilde{f}(x,t,u)\right|.$

Corollary 2. *If* $f(x,t,u_1,\ldots,u_s) \in \mathscr{E}^{[\frac{n}{2}]+2}(\Omega\times\mathbb{C}^s)$ *and* $u_j \in \mathscr{D}^{[\frac{n}{2}]+1}$ $[0,T](1\leq j\leq s)$ *then* $\alpha(x)\in\mathscr{D}$ *implies that*

$$\alpha(x)f(x,t,u_1(x,t),\ldots,u_s(x,t)) \in \mathscr{D}^{[\frac{n}{2}]+1}_{L^2}[0,T]$$

and

$$\begin{aligned}&\|\alpha(x)f(x,t,u_1(x,t),\ldots,u_s(x,t))\|_{[\frac{n}{2}]+1}\\ &\leq C_4(n)M_2\{1+\sum_{j=1}^{s}\|u_j(x,t)\|^{[\frac{n}{2}]+1}_{[\frac{n}{2}]+1}\end{aligned} \tag{2.10}$$

186 *where* $M_2 = \max\limits_{|\beta|\leq[\frac{n}{2}]+1}\sup\limits_{U_s}|\left(\frac{\partial}{\partial x},\frac{\partial}{\partial u}\right)^{\beta}[\alpha(x)f(x,t,u_1(x,t),\ldots,u_s(x,t))]|.$

Here U_s is the subset of $\Omega \times \mathbb{C}^s$ defined by

$$U_s = \left\{ (x, t, u_1, \ldots, u_s) \middle| |u_j| \leq \sup_{\Omega} |u_j(x, t)|,\ 1 \leq j \leq s \right\}. \tag{2.11}$$

Corollary 3. *If $f(x, t, u)$ is a vector valued function*

$$\begin{pmatrix} f_1(x, t, u) \\ \vdots \\ f_m(x, t, u) \end{pmatrix}$$

with $f_k \in \mathscr{E}^{[\frac{n}{2}]+2}(\Omega \times \mathbb{C})$ for $1 \leq k \leq m$ and $u \in \mathscr{D}^{[\frac{n}{2}]+1}_{L^2}[0, T]$ then $\alpha \in \mathscr{D}$ implies that $\alpha(x) f_k(x, t, u(x, t))$ belong to the space $\mathscr{D}^{[\frac{n}{2}]+1}_{L^2}[0, T]$ and

$$\begin{aligned} ||\alpha(x) f(x, t, u(x, t))||_{[\frac{n}{2}]+1} &= \sum_k ||\alpha(x) f_k(x, t, u(x, t))||_{[\frac{n}{2}]+1} \\ &\leq C_5(n) M_3 (1 + ||u(x, t)||^{[\frac{n}{2}]+1}_{[\frac{n}{2}]+1}) \end{aligned} \tag{2.12}$$

where $M_3 = \max\limits_{k, |\beta| \leq [\frac{n}{2}]+1} \sup\limits_{U} | \left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial u} \right)^{\beta} [\alpha(x) f_k(x, t, u)]|$.

Similar results hold when u is a vector $(u_1, \ldots, u_s)$ and when $u_j \in \mathscr{D}^{[\frac{n}{2}]+N+1}_{L^2}[0, T]$.

Finally we state a result which is a consequence of these and will be of importance.

Corollary 4. *Let $f(x, t, u_1, \ldots, u_s) \in \mathscr{E}^{[\frac{n}{2}]+2}(\Omega \times \mathbb{C}^s)$ and $\nu_1, \ldots, \nu_s$ denote multi-indices. If $u \in \mathscr{D}^{[\frac{n}{2}]+m+1}[0, T]$ and $|\nu_1| + \cdots + |\nu_s| \leq m$ then* 187

$$\alpha(x) f \left(x, t, \left(\frac{\partial}{\partial x} \right)^{\nu_1} (u(x, t)), \ldots, \left(\frac{\partial}{\partial x} \right)^{\nu_s} u(x, t) \right) \in \mathscr{D}^{[\frac{n}{2}]+1}_{L^2}[0, T]$$

and

$$||\alpha(x) f \left(x, t, \ldots, \left(\frac{\partial}{\partial x} \right)^{\nu_1} u(x, t), \ldots \right) ||_{[\frac{n}{2}]+1}$$

$$\text{(2.13)} \qquad \leq M'c(n,m)\left\{1+\|u\|^{[\frac{n}{2}]+1}_{[\frac{n}{2}]+m+1}\right\},$$

where $M' = \max\limits_{|\beta|\leq[\frac{n}{2}]+2} \sup\limits_{U'_s} \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_s}\right)^\beta [\alpha(x) f(x,t,u_1,\ldots,u_s)]|.$

Here again

(2.14)

$$U'_s = \left\{(x,t,u_1,\ldots,u_2)\Big|(x,t)\in\Omega, |u_j| \leq \sup|\left(\frac{\partial}{\partial x}\right)^{\nu_j} u(x,t)|, 1\leq j\leq s\right\}.$$

Proof. From Prop. 4 § 1 we have that, if $u_1,\ldots,u_s \in \mathscr{D}^{[\frac{n}{2}]+1}_{L^2}[0,T]$ and if $\nu_1,\ldots,\nu_s$ are multi-indices with $\sum|\nu_j| \leq \left[\frac{n}{2}\right]+1$ then

$$\text{(2.15)} \qquad \|\prod_{j=1}^{s}\left(\frac{\partial}{\partial x}\right)^{\nu_j} u\|_{L^2} \leq C(n,\nu_1,\ldots,\nu_s)\prod_{j=1}^{s}\|u_j\|_{[\frac{n}{2}]+1}.$$

Taking $u_j = \left(\frac{\partial}{\partial x}\right)^{\nu_j} u$ we apply this inequality and the rest of the proof is the same as in the previous corollaries. □

3 An example of a semi-linear equation

In this section we consider an example of a semi-linear partial differ-
188 ential equation of the second order and we recall a theorem on the existence of solutions of the Cauchy problem for such an equation. This result is due to K. Jörgens (see: Das Anfangswertproblem in Grossen fur eine Klasse nichtlinearer Wellengleichungen, Math.Zeit., 77 (1961), 295-308). This theorem will be proved in §5.

Let $u \to f(u)$ be a real valued infinitely differentiable function defined in $-\infty < u < \infty$. We consider the following semi-linear wave equation

$$\text{(3.1)} \qquad \left(\frac{\partial}{\partial t}\right)^2 u - \Delta u + f(u) = 0.$$

We assume that $f(0) = 0$. We shall show that, under certain conditions on the function f, for a given smooth initial data (u_0, U_1) on the hyperplane $t = 0$ there exists a unique solution u of (3.1) in $t \geq 0$ with $u(x, 0) = u_0(x), \dfrac{\partial}{\partial t} u(x, 0) = u_1(x)$. For instance, we shall show that if $u_0 \in \mathscr{D}^{[\frac{n}{2}]+2}_{L^2} \cap \mathscr{E}^1$, $u_1 \in \mathscr{D}^{[\frac{n}{2}]+1}_{L^2} \cap \mathscr{E}^1$ then there exists a unique solution u of (3.1) such that

$$u \in \mathscr{D}^{[\frac{n}{2}]+2}_{L^2} \cap \mathscr{E}',\ \frac{\partial u}{\partial t} \in \mathscr{D}^{[\frac{n}{2}]+1}_{L^2} \cap \mathscr{E}'$$

both depnding continuously on t in $0 \leq t \leq \infty$ and such that

$$u(x, 0) = u_0(x), \frac{\partial u}{\partial t}(x, 0) = u_1(x).$$

Under the assumption $f(0) = 0$ one can also show that if the supports of u_0 and u_1 are contained in $\{|x| \leq R_0\}$ then the supports of $u, \dfrac{\partial u}{\partial t}$ are contained in $\{|x| \leq R_0 + t\}$.

Let $u_0 \in \mathscr{D}^{[\frac{n}{2}]+3}_{L^2} \cap \mathscr{E}'$, $u_1 \in \mathscr{D}^{[\frac{n}{2}]+2}_{L^2} \cap \mathscr{E}'$ be given with their supports **189**
contained in $\{|x| \leq R_0\}$. Assume that a solution of (3.1) with the initial data (u_0, u_1) on $t = 0$ exists locally. More precisely we assume that there exists a $t_0 > 0$ such that there exists a solution u of (3.1) defined in $\{x \in \underline{R}^n, 0 \leq t \leq t_0\}$ with the property that

(1) $u \in (\mathscr{D}^{[\frac{n}{2}]+3}_{L^2} \cap \mathscr{E}^1)[0, t_0] \dfrac{\partial u}{\partial t} \in (\mathscr{D}^{[\frac{n}{2}]+2}_{L^2} \cap \mathscr{E}^1)[0, t_0]$,

$\left(\dfrac{\partial}{\partial t}\right)^2 u \in (\mathscr{D}^{[\frac{n}{2}]+1}_{L^2} \cap \mathscr{E}^1)[0, t_0])$ and

1. $u(x, 0) = u_0(x), \dfrac{\partial u}{\partial t}(x, 0) = u_1(x)$.

We say that an *a priori* estimate in the L^2-sense for the solution of the Cauchy problem for (3.1) of order $\left[\dfrac{n}{2}\right] + 1$ holds if the following conditions is satisfied: for any given initial data (u_0, u_1) with $u_0 \in$

$\mathscr{D}_{L^2}^{[\frac{n}{2}]+3} \cap \mathscr{E}^1$, $u_1 \in \mathscr{D}_{L^2}^{[\frac{n}{2}]+2} \cap \mathscr{E}'$ and a number $T > 0$ there exists a constant $c \equiv c(T, u_0, u_1)$ such that

$$||u(t)||_{[\frac{n}{2}]+1} \leq c$$

for all $0 \leq t \leq T$. where u exists an $u(x, 0) = u_0$, $\dfrac{\partial u}{\partial t}(x, 0) = u_1(x)$. c is called an a priori bound.

The following is a special case of a theorem that will be proved in § 5. We state it here to motivate Prop. 1.

190 **Theorem 1.** *Let f be an infinitely differentiable function in $-\infty < u < \infty$ with $f(0) = 0$. Assume that a priori estimate in the L^2-sense for the solution of the Cauchy problem for* (3.1) *of order $\left[\dfrac{n}{2}\right]+1$ holds. Then, for any intial data (u_0, u_1) with $u_0 \in \mathscr{D}_{L^2}^m \cap \mathscr{E}^1$, $u_1 \in D_{L^2}^{m-1} \cap \mathscr{E}^1 (m \geq \left[\dfrac{n}{2}\right]+3)$ there exists a unique solution u of* (3.1) *such that*

(1) $u \in \mathscr{D}_{L^2}^m \cap \mathscr{E}^1, \dfrac{\partial u}{\partial t} \in \mathscr{D}_{L^2}^{m-1} \cap \mathscr{E}^1, \left(\dfrac{\partial}{\partial t}\right)^2 u \in \mathscr{D}_{L^2}^{m-2} \cap \varepsilon'$ *all depending continuously on t,*

(2) $u(x, 0) = u_0(x), \dfrac{\partial u}{\partial t}(x, 0) = u_1(x)$.

Proposition 1. *Let f be an infinitely differentiable function in $-\infty < u < \infty$ with $f(0) = 0$. Then*

(i) *for $n = 1$ an a priori estimate of order one for the solutions of the Cauchy problem for* (3.1) *holds when*

(a) $\int_0^u f(v)dv \equiv F(u) > -L_0$ *(L_0 a positive constant),*

(ii) *assume further that $f(u)$ satisfies the condition*

(b) *if $n = 2$ there exist α and k such that*

$$|\frac{df(u)}{du}| \leq \alpha(1 + |u|)^k$$

and if $n = 3$ there exists an α such that

$$|\frac{df(u)}{du}| \leq \alpha(1 + u^2).$$

Then an a priori estimate of order 2 for solutions of the Cauchy problem for (3.1) *holds.*

Proof. Assume that $u_0 \in \mathscr{D}^m_{L^2} \cap \varepsilon^1$, $u_1 \in \mathscr{D}^{m-1}_{L^2} \cap \mathscr{E}^1 (m \geq \left[\frac{n}{2}\right] + 3)$ are given and also that there exists a solution u of the Cauchy problem for (3.1) with initial data (u_0, u_1) such that **191**

$$u \in (\mathscr{D}^m_{L^2} \cap \mathscr{E}')[0, T], \frac{\partial u}{\partial t} \in (\mathscr{D}^{m-1}_{L^2} \cap \mathscr{E}')[0, T], \left(\frac{\partial}{\partial t}\right)^2 u \in (\mathscr{D}^{m-2}_{L^2} \cap \mathscr{E}')[0, T].$$

Let R be a number such that $R_0 + t < R$ for $t \leq T$.

(i) Set $E_1(t) = \int\limits_{|x|<R} \left[\frac{1}{2}\left\{\left(\frac{\partial u}{\partial t}\right)^2 + \sum_{j=1}^{n}\left(\frac{\partial u}{\partial x_j}\right)^2\right\} + F(u) + c\right] dx$

where c is a constant to be chosen later. Differentiating with respect to t

$$\frac{d}{dt}E_1(t) = \int\limits_{|x|\leq R} \left\{\frac{\partial u}{\partial t}\left(\frac{\partial}{\partial t}\right)^2 u + \sum_j \frac{\partial u}{\partial x_j}\left(\frac{\partial u}{\partial x_j}\right)\left(\frac{\partial}{\partial t}\right) u + f(u)\frac{\partial u}{\partial t}\right\} dx.$$

Since $\frac{\partial u}{\partial x_j}, \left(\frac{\partial}{\partial x_j}\right)\left(\frac{\partial}{\partial t}\right) u$ have compact supports the second term in the right hand side becomes after integration by parts

$$\int \Delta^u \cdot \frac{\partial u}{\partial t} dx$$

and so we have

$$\frac{d}{dt}E_1(t) = \int\limits_{|x|\leq R} (\Box u + f(u))\frac{\partial u}{\partial t} \cdot dx = 0$$

(where $\Box = \left(\frac{\partial}{\partial t}\right)^2 - \Delta$) since $\Box u + f(u) = 0$. Hence $E_1(t)$ is a constant $= E_1 0$. □

Taking $c > L_0$ we have $F(u) + c > 0$ and so

$$\int \frac{1}{2}\left\{\left(\frac{\partial u}{\partial t}\right)^2 + \sum_j \left(\frac{\partial u}{\partial x_j}\right)^2\right\} dx \leq E_1(t) = E_1(0). \tag{3.2}$$

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Since the support of u is compact there exists c_1 such that

$$\|u\|_{L^2} \leq c_1 \sum_j \|\frac{\partial u}{\partial x_j}\|_{L^2}. \tag{3.3}$$

In fact, $u \in \mathscr{D}^m_{L^2} \subset \mathscr{D}^{\frac{n}{2}+3}_{L^2}$ implies that u is in $\mathscr{E}^1$. We can hence write

$$u(x,t) = \int_{-\infty}^{x_j} \frac{\partial u}{\partial x_j}(y,t)dy_j,\ j = 1, \ldots, n.$$

Using Cauchy-Schwarz inequality and calculating the norm of u in L^2 we obtain (3.3). The estimates (3.2), (3.3) together show that an a priori estimate of order one holds thus proving (i).

(ii) Differentiating (3.1) with respect to x_j we have

$$\Box u_j + \frac{df}{du}u_j = 0 \text{ where } u_j = \frac{\partial u}{\partial x_j}. \tag{3.4}$$

Denoting $\dfrac{\partial}{\partial x_j}\dfrac{\partial}{\partial x_k}u$ by u_{jk} and $\dfrac{\partial}{\partial x_j}\dfrac{\partial}{\partial t}u$ by u_{jt} we define

$$E_2(t) = \sum_{j=1}^n \int \frac{1}{2}\left(u_{jt}^2 + \sum_{k=1}^n u_{jk}^2\right) dx.$$

Differentiating $E_2(t)$ with respect to t

$$\begin{aligned}\frac{dE_2}{dt}(t) &= \sum_j \int \left(u_{jt} \cdot u_{jtt} + \sum_k u_{jk} \cdot u_{jkt}\right) dx\\ &= \sum_j \int (\Box u_j) \cdot u_{jt}\, dx\end{aligned}$$

193 since $\sum_k \int u_{jk} \cdot u_{jkt} dx = -\sum_k \int u_{jkk} \cdot u_{jt}\, dx$ by integration by parts. using the equation (3.4) we obtain

$$\frac{dE_2}{dt}(t) = -\sum_j \int \frac{df}{du} \cdot u_j u_{jt}\, dx.$$

From the generalized Hölder's inequality it follows that

$$|\int \frac{df}{du} \cdot u_j \cdot u_{jt}\, dx| \leq \|u_{jt}\|_{L^2} \|u_j\|_{L^6} \cdot \|\frac{df}{du}\|_{L^3}.$$

If $n = 2$ by Prop. 2 § 1 we see that

$$\|u_j\|_{L^6} \leq c_1(n)\|u\|_2$$

where $c_1(n)$ is a constant depending only on n. From (b) we have, with a suitable constant α' depending on α, since u has compact support in $|x| < R$

$$\int\limits_{|x|<R} |\frac{df}{du}|^3\, dx \leq \alpha'^3 \int (u^6 + 1) dx \leq \alpha'^3 \|u\|^6_{L^6} + C_2(\alpha', R, n)$$

$$\leq C_3(n, \alpha', R)(1 + \|u\|^6_1).$$

These estimates together show that

$$\frac{dE_2}{dt}(t) \leq \gamma_1 E_2(t).$$

Multiplying by $e^{-\gamma_1 t}$ and integrating with respect to t we obtain

$$E_2(t) \leq E_2(0) \cdot e^{\gamma_1 t}. \tag{3.5}$$

This proves that there is an a priori bound of order 2. A similar argument holds for the case $n = 3$. This completes the proof of the proposition.

Exercise. Consider the semi-linear hyperbolic equation 194

$$M[u] + f(u) = 0 \tag{3.6}$$

where

$$M = \left(\frac{\partial}{\partial t}\right)^2 - \sum a_{jk}(x,t)\frac{\partial^2}{\partial x_j \partial x_k} - \sum a_j(x,t)\frac{\partial}{\partial x_j} - a_0(x,t)\frac{\partial}{\partial t}$$

with (1°) $a_{jk} \in B^1[0,T]$, $\frac{\partial}{\partial t} a_{jk} \in B^0[0,T], a_0 a_j \in B^0[0,T]$,

(2°) $\sum a_{jk}(x,t)\xi_j\xi_k \geq \delta|\xi|^2$, $\delta > 0$ is a constant.

Prove, under the same hypothesis on f as in Prop. 1, that an a priori estimate of order 2 holds and consequently there exists a global solution of (3.6).

4 Existence theorems for first order systems of semi-linear equations

In this section we establish theorems on the existence of local and global solutions of the Cauchy problem for semi-linear regularly hyperbolic first order systems of differential equations.

Let Ω be the set $\{(x,t) | x \in \underline{R}^n, 0 \leq t \leq T\}$. Consider the semi-linear first order system of equations

(4.1) $$M[u] = \frac{\partial u}{\partial t} - \sum_{k=1}^{n} A_k(x,t)\frac{\partial u}{\partial x_k} = f(x,t,u),$$

where we assume that the coefficients A_k of M and f satisfy the following regularity conditions:

(a) $A_k \in B^{[\frac{n}{2}]+2}[0,T]$, $\frac{\partial A_k}{\partial t} \in \mathscr{B}^0[0,T]$ and

(b) $f \in \mathscr{E}^{[\frac{n}{2}]+3}$ in $\Omega \times \underline{C}$.

195 We also assume that M is regularly hyperbolic. As we shall show later that under stronger differentiabililty conditions on the coefficients A_k and f the Cauchy problem has more regular solutions: For instance we assume

(a′) $A_k \in B^m[0,T]$, $\dfrac{\partial A_k}{\partial t} \in B^0[0,T]$ and

(b′) $f \in \mathscr{E}^{m+1}$ in $\Omega \times \underline{C}$,

where $m \geq \left[\dfrac{n}{2}\right] + 2$.

Although we are interested here mainly in the local existence theorem we consider the following equation (4.1)′ instead of (4.1) in order to elucidate our construction. We decompose f into two parts

$$f(x,t,u) = f(x,t,0) + (f(x,t,u) - f(x,t,0)) = f(x,t,0) + g(x,t,u)$$

where

$$g(x,t,u) = f(x,t,u) - f(x,t,0). \tag{4.2}$$

We remark that $g(x,t,0) \equiv 0$. Define the function $\tilde{f} \in \mathscr{E}^{[\frac{n}{2}]+3}$ in $\Omega \times \underline{C}$ by setting

$$\tilde{f}(x,t,u) = \alpha(x)g(x,t,u) + \beta(x)f(x,t,0)$$

where $\alpha, \beta \in \mathscr{D}$, and consider the first order system of semi-linear equaions

$$M[u] = \tilde{f}. \tag{4.1$'$}$$

Clearly $\tilde{f} = f$ whereever $\alpha(x) = 1 = \beta(x)$. If the initial data $u_0 \in \mathscr{E}'$ **196**
has compact support then, since $\beta(x)\tilde{f}(x,t,u)$ has compact support in the x-space, the solution u also has a fixed compact support for all $0 \leq t \leq T$.

Now we find a sequence of fucntions $\{u_j\}$ which will converge to a limits u giving the solution. Let ψ be the solution of Cauchy problem

$$M[\psi] = \beta(x)f(x,t,0) \text{ with } \psi(0) = u_0. \tag{4.3}$$

Hence by the theory of linear equations, there exists a constant γ_0 depending on T such that

$$\|\psi(t)\|_{[\frac{n}{2}]+2} \leq \gamma_0\{\|u_0\|_{[\frac{n}{2}]+2} + \sup_{0\leq t\leq T} \|\|\beta_f(x,t,0)\|_{\frac{n}{2}} + 2$$

$$\|\psi(t)\|_{[\frac{n}{2}]+1} \leq \gamma_0\{\|u_0\|\|\|_{[\frac{n}{2}]+1} + \sup_{0\leq t\leq T} \|\beta f(x,t,0)\|_{[\frac{n}{2}]+1}. \tag{4.4}$$

The Cauchy problem for (4.1)′ is therefore reduced to the following problem: to find a solution $u \in \mathscr{D}^{[\frac{n}{2}]+2}_{L^2}[0,T]$ of

$$M[u] = \tilde{g}(x,t,\psi+u)$$

with the initial data u_0. Here

$$\tilde{g}(x,t,\psi+u) = \alpha(x)(f(x,t,u+\psi) - f(x,t,0)).$$

Our main interest here is to determine how does the domain of existence $\underline{R}^n \times \{0 \leq t \leq h\}$ of the solution depend on the initial data u_0, after
197 fixing $\alpha, \beta \in \mathscr{D}$. The functions u_j are defined inductively as solutions of the Cauchy problem for the first order system of equations:

$$\begin{aligned}
M[u_1] &= \tilde{g}(x,t,\psi),\ u_1(0) = 0,\\
M[u_2] &= \tilde{g}(x,t,u_1+\psi),\ u_2(0) = 0,\\
&\cdots\cdots\cdots\cdots\cdots\\
M[u_j] &= \tilde{g}(x,t,u_{j-1}+\psi),\ u_j(0) = 0,\\
&\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

Now since $\psi \in \mathscr{D}^{[\frac{n}{2}]+2}_{L^2}[0,T]$ we have $\tilde{g}(x,t,\psi(t)) \in \mathscr{D}^{[\frac{n}{2}]+2}_{L^2}[0,T]$ and hence by the theory of linear equations there exists a solution u_1 of the Cauchy problem

$$M[u_1] = \tilde{g}(x,t,\psi), \quad u_1(0) = 0,$$

and $u_1 \in \mathscr{D}^{[\frac{n}{2}]+2}_{L^2}[0,T]$. Again we have $\tilde{g}(x,t,(\psi+u_1)(x,t)) \in \mathscr{D}^{[\frac{n}{2}]+2}_{L^2}$ $[0,T]$ and hence there exists a solution u_2 of

$$M[u_2] = \tilde{g}(x,t,u_1+\psi),\ u_2(0) = 0$$

and $u_2 \in \mathscr{D}^{[\frac{n}{2}]+2}_{L^2}[0,T]$. This proceedure can be used to obtain u_j inductively.

Now we have the

Proposition 1. *There exists a positive, non-increasing functions $\varphi(\xi)$ of $\xi > 0$ such that*

$$h = \varphi(\|u_0\|_{[\frac{n}{2}]+1}) > 0$$

and the set $\left\{ \sup\limits_{0 \le t \le h} \|u_j(t)\|_{[\frac{n}{2}]+1} \right\}$ *is bounded.* **198**

Proof. Let γ denote the $\sup\limits_{(x,t)\in\Omega} |\psi(x,t)|$. In view of (4.4) γ is less than or equal to $c_0 + c_1\|u_0\|_{[\frac{n}{2}]+1}$ where c_0, c_1 are constants depending on T. If b is a positive number let F be the set

$$F = \{(x,t,u) | (x,t) \in \Omega, |u| < b + \gamma\}$$

and put

$$M = \sup_{F, |\alpha| \le [\frac{n}{2}]+2} \left| \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial u}\right)^{\alpha} \tilde{g}(x,t,u) \right| \tag{4.5}$$

where $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial u}\right)^{\alpha}$ denotes a derivation of order $|\alpha|$ with respect to x and u. $M = M(b+\gamma)$ is an increasing function of the parameter. If $u \in \mathscr{D}^{[\frac{n}{2}]+1}_{L^2}[0,T]$ with $|u(x,t)| \le b$ for $(x,t) \in \Omega$ then we have

$$\|\tilde{g}(x,t,(u+\psi)(x,t))\|_{[\frac{n}{2}]+1} \le Mc\{1 + \|u(t)\|^k_{[\frac{n}{2}]+1}\}, \tag{4.6}$$

$k = \left[\frac{n}{2}\right] + 1$. Now, since $u_j(0) = 0$, we have by the energy inequality

$$\|u_j(t)\|_{[\frac{n}{2}]+1} \le c(T) \int\limits_0^T \|\tilde{g}(x,s,(u_{j-1}+\psi)(x,s))\|_{[\frac{n}{2}]+1} ds.$$

Hence from (4.6) we obtain

$$\|u_j(t)\|_{[\frac{n}{2}]+1} \le Mcc(T) \int\limits_0^T (1 + \|u_{j-1}(s)\|^k_{[\frac{n}{2}]+1}) ds. \tag{4.7}$$

We recall that this was derived with the assumption that $|u_{j-1}(x,t)| < b$ which, we shall show, holds when h is small and $0 \leq t \leq h$. Put 199

$$c_2 = Mc \cdot c(T)$$

(4.8) $$\gamma_1 = 1 + 2^k \sup_{0\leq t\leq T} \|\psi(t)\|^k_{[\frac{n}{2}]+1}.$$

Since $\|(\psi + u_{j-1})(t)\|^k_{[\frac{n}{2}]+1} \leq 2^k \left\{\|u_{j-1}(t)\|^k_{[\frac{n}{2}]+1} + \|\psi(t)\|^k_{\frac{n}{2}+1}\right\}$ (4.7) can be written as

$$\|u_j(t)\|_{[\frac{n}{2}]+1} \leq 2^K c_2 \int_0^t \{\gamma_1 + \|u_{j-1}(s)\|^k_{[\frac{n}{2}]+1} ds,$$

where $u_0(t) \equiv 0$. Putting again $2^k c_2 = c_3$ we have

(4.9) $$\|u_j(t)\|_{[\frac{n}{2}]+1} \leq c_3 \int_0^t \{\gamma_1 + \|u_{j-1}(s)\|^k_{[\frac{n}{2}]+1}\} ds.$$

Let $c_s(n)$ denote the Sobolev's constant, namely the constant in the inequality

$$\sup |\varphi(x)| \leq c_s(n) \|\varphi\|_{[\frac{n}{2}]+1}.$$

Define b' by

(4.10) $$b' = \frac{b}{c_s(n)}$$

and denote $c_3(\gamma_1 + b'^k)$ by $\tilde{M}$. Take

(4.11) $$h = \frac{b'}{\tilde{M}} = \frac{b'}{c_3(\gamma_1 + b'^k)}.$$

200 Consider the sequence $y_j(t)$ defined by the sequence of integral equations

$$y_j(t) = c_3 \int_0^t \left\{\gamma_1 + y_{j-1}(s)^k\right\} ds \text{ for } t \geq 0, \ y_0(t) \equiv 0.$$

Then we assert that

$$0 \leq y_j(t) \leq b' \text{ for } 0 \leq t \leq h, \;\; j = 1, 2, \ldots$$

$$\text{In fact, } \; y_1(t) \leq c_3\gamma_1 t \leq \tilde{M}t \leq \tilde{M}h = b',$$
$$y_2(t) \leq \tilde{M}t \leq \tilde{M}h = b' \;\text{ and so on.}$$

Evidently $||u_j(t)||_{[\frac{n}{2}]+1} \leq y_j(t)$ and

$$||u_j(t)||_{[\frac{n}{2}]+1} \leq b' \text{ for } 0 \leq t \leq h \tag{4.12}$$

which, a fortiori, implies (by using Sobolev's lemma) that

$$\sup |u_j(x,t)| \leq b' c_s(n) = b \quad \text{(see (4.10))}.$$

From (4.11) we obtain

$$\frac{1}{h} = \frac{c_3(\gamma_1 + b'^k)}{b'} = 2^k c \cdot c(T) \frac{b'^k + \gamma_1}{b'} M$$
$$\leq c_0(n,T) \frac{b^k + C'_0(n) + c''_0(n)||\psi(t)||^k_{[\frac{2}{n}]+1}}{b} M,$$

where $M = M(\gamma + b)$. $M(\xi) > 0$ is an increasing function of $\xi > 0$. So, if $||u_0||_{[\frac{n}{2}]+1}$ runs through a bounded set, fixing b, h has a positive infimum (M is taken to be a fixed positive number). This completes the 201
proof. □

Remark . Instead of taking hte initial data to be given at $t = 0$ we can take the initial data to be given at an arbitray $t_0 (0 \leq t_0 \leq T)$. We define $\psi(t, t_0)$ corresponding to $\psi(t)$ in the above arguments. Here $||\psi(t, t_0)||_{[\frac{n}{2}]+1}$ is majorized by $C_0 + C_1 ||u_0||_{[\frac{n}{2}]+1}$, C_0, C_1 can be taken independently. The expression for $\dfrac{1}{h}$ shows that h has a positive infimum independent of t_0 if the initial data u_0 runs through a bouded set in $\mathscr{D}^{[\frac{n}{2}]+1}_{L^2}$.

Next we prove that the sequence $\{u_j(t)\}$ is a Cauchy sequence in $\mathscr{D}^{[\frac{n}{2}]+2}_{L^2}[0,h]$. First of all we shall show that $\{\sup_{0\le t\le h} \|u_j(t)\|_{[\frac{n}{2}]+2}\}$ is bounded. In fact, we have

$$\|u_j(t)\|_{[\frac{n}{2}]+2} \le c(T)\int_0^t \|\tilde{g}(x, s(u_{j-1}+\psi)(x,s))\|_{[\frac{n}{2}]+2} ds$$

$$\le cM'\int_0^t \{1+(1+\|(\psi+u_{j-1})(s)\|^k_{[\frac{n}{2}]+1})\|u_{j-1}+\psi(s))\|_{[\frac{n}{2}]+2} ds,$$

$$k = \left[\frac{n}{2}\right]+1.$$

$$\|u_2(t)-u_1(t)\|_{[\frac{n}{2}]+2} \le Kc't,$$
$$\|u_3(t)-u_2(t)\|_{[\frac{n}{2}]+2} \le K\frac{(c't)^2}{2!},\ldots,$$
$$\|u_{j+1}-u_j(t)\|_{[\frac{n}{2}]+2} \le K\frac{(c't)^j}{j!},\ldots$$

Hence $\{u_j(t)\}$ is a Cauchy sequence in $\mathscr{D}^{[\frac{n}{2}]+2}_{L^2}[0,h]$ and therefore
202 converges to a limit $u(t)$ in $\mathscr{D}^{[\frac{n}{2}]+2}_{L^2}[0,h]$.

If $m \ge [\frac{n}{2}]+3$ we now assume that $A_k \in \mathscr{B}^m[0,T]$, $\frac{\partial A_k}{\partial t} \in \mathscr{B}^0[0,T]$ and $f \in \mathscr{E}^{m+1}(\Omega\times \underline{C})$. Let $u_0 \in \mathscr{D}^m_{L^2}$ be given. Then the limit $u(t)$ in $\mathscr{D}^{[\frac{n}{2}]+2}_{L^2}[0,h]$ of the sequence $\{u_j(t)\}$ obtianed above itself belongs to $\mathscr{D}^m_{L^2}[0,h]$. In fact, it is enough to prove that $\left\{\sup_{0\le t\le h}\|u_j(t)\|_m\right\}$ is bounded and $\{u_j(t)\}$ is a Cauchy sequence in $\mathscr{D}^m_{L^2}[0,h]$. For this we have only to use the following lemma which results by arguments similar to those used in §2.

Lemma 1. *Let $u \in \mathscr{D}^m_{L^2}[0,T]$ and $f \in \mathscr{G}^{m+1}(\Omega\times\underline{C})$ for an $m \ge \left[\frac{n}{2}\right]+2$.*

Then there exists constants C_m, M_m such that

$$\|\left(\frac{\partial}{\partial x}\right)^{\nu} f(x,t,u(x,t))\|_m C_m M_m \Big\{1 + (1 + \|u(t)\|^{m-1}_{m-1})\|u(t)\|_m.$$

Thus we have proved the following:

Theorem 1 (local existence theorem)**.** *Given any intial data $u_0 \in \mathscr{D}^m_{L^2}$, $m \geq \left[\frac{n}{2}\right] + 2$ and any initial time t_0, $0 \leq t_0 \leq T$ there exists a unique solution $u(t) \in \mathscr{D}^m_{L^2}[t_0, t_0 + h]$ of the equation*

(4.1)′ $\quad M[u] = \tilde{f}(x,t,u) = \beta(x) f(x,t,0) + \alpha(x)\{f(x,t,u) - f(x,t,0)\}$

with $u(t_0) = u_0$. Moreover h can be chosen to be independent of t_0 in $[0,T]$ when $\|u_0\|_{[\frac{n}{2}]+2}$ runs through a bounded set.

Now we obtain a global existence theorem for solutions of the Cauchy problem for regularly hyperbolic first order systems of semi-linear equations. For this we assume that an a *priori estimate* of the following type holds. **203**

If $\beta \in \mathscr{D}$ consider the regularly hyperbolic first order system of equations

(4.13) $\quad M[u] = \beta f(x,t,0) + (f(x,t,u) - f(x,t,0)).$

By A *priori estimate* we mean the following: For any initial data u_0 in $\mathscr{D}^{[\frac{n}{2}]+2}_{L^2} \cap \mathscr{E}'$ and any $t_0 (0 \leq t_0 \leq T)$ the solution $u(t) \in \mathscr{D}^{[\frac{n}{2}]+2}_{L^2}[t_0, T]$ of (4.13) satisfies the following condition: there exists a constant $c = c(T)$ such that

(4.14) $\quad \|u(t)\|_{[\frac{n}{2}]+1} \leq c$ for all $t_0 \leq t \leq T$.

Theorem 2 (global existence theorem)**.** *Suppose an a priori estimate of the type* (4.14) *holds for solutions of* (4.13)*. Then, given any initial data $u_0 \in \mathscr{E}^m_{L^2(\text{loc})}$, $m \geq \left[\frac{n}{2}\right] + 2$ there exists a unique solution $u(t)$ of*

(4.1) $\quad M[u] = f$ *with* $f \in \mathscr{E}^{m+1}(\Omega \times \underline{C})$

for $0 \leq t \leq T$ such that $u(0) = u_0$, $u \in \mathscr{E}^m_{L^2(\text{loc})}[0,T]$ and $\dfrac{\partial u}{\partial t} \in \mathscr{E}^{m-1}_{L^2(\text{loc})}[0,T]$.

Proof. As we have seen in the section on dependence domain there exists a retrograde cone K such that the value of a solution u of $M[u] = f$ at a point $(x_0, t_0) \in \Omega$ depends only on the second member in the set
204 $(x_0, t_0) + K$ and on the value of the initial data in the intersection of this translated cone with $(t = 0)$. Let D be the subset of Ω swept by $(x, T)+K$ as x runs through a ball $|x| < R$ and D_0 be the set $D \cap \{t = 0\}$. Let $\beta \in \mathscr{D}$ such that $\beta(x) \equiv 1$ for $x \in D_0$. Given any initial data $u_0 \in \mathscr{E}^m_{L^2(\text{loc})}$ we consider the Cauchy problem

$$\begin{aligned} &M[u_1] = \beta(x) f(x, t, 0) + (f(x, t, u) - f(x, t, 0)) \\ \text{(4.15)} \qquad \text{with} \quad &u_1(x, 0) = \beta(x) u_0(x) \in \mathscr{D}^m_{L^2}. \end{aligned}$$

This solution $u_1(x, t)$ has an a priori estimate $\|u_1(t)\|_{[\frac{n}{2}]+1} \leq C$. On the other hand this solution u_1 has compact support as far as the solution exists. Hence, if we take $\alpha \in \mathscr{D}$ such that $\alpha(x) \equiv 1$ for $|x| \leq R$, (4.15) is equivalent to

$$\text{(4.1)}' \qquad M[u_1] = \beta(x) f(x, t, 0) + \alpha(x)(f(x, t, u) - f(x, t, 0)).$$

Now since u_1 has an a priori estimate $\|u_1(t)\|_{[\frac{n}{2}]+1} \leq C$, it follows, by using theorem 1 to continue the solution step by step, that there exists a solution $u_1(x, t)$ for $0 \leq t \leq T$. Clearly $u(x, t) = u_1(x, t)$ for $(x, t) \in D$ and this completes the proof of theorem 2. □

5 Existence theorems for a single semi-linear equation of higher order

In this section we obtain theorems on existence of solutions, local and global, of the Cauchy problem for a single semi-linear equation of order m.

As before Ω be the set $\{(x, t) | x \in \underline{R}^n, 0 \leq t \leq T$ and

$$\text{(5.1)} \qquad M = \left(\frac{\partial}{\partial t}\right)^m + \sum_{\substack{j+|\nu| \leq m \\ j<m}} a_{j,\nu}(x, t) \left(\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\nu$$

205 be a regularly hyperbolic operator in Ω. Consider the quasi-linear equation

$$(5.2) \qquad M[u] = f\left(x, t, \left(\frac{\partial}{\partial t}\right)^{j_1}\left(\frac{\partial}{\partial x}\right)^{\alpha_1} u, \ldots, \left(\frac{\partial}{\partial t}\right)^{j_s}\left(\frac{\partial}{\partial t}\right)^{\alpha_s} u\right)$$

where $j_k + |\alpha_k| \leq m - 1 (k = 1, \ldots, s)$. We make the following assumptions on the coefficients of M and f:

$$a_{j,\nu} \in \mathscr{B}^{[\frac{n}{2}]+2}[0, T], \frac{\partial}{\partial t} a_{j,\nu} \in \mathscr{B}^0[0, T] \text{ and } f \in \mathscr{E}^{[\frac{n}{2}]+3}(\Omega \times \underline{C}^s).$$

When we consider the regularity properties of higher degrees. We assume for $N \geq [\frac{n}{2}] + 3$

$$a_{j,\nu} \in \mathscr{B}^N[0, T], \frac{\partial}{\partial t} a_{j,\nu} \in \mathscr{B}^0[0, T] \text{ and } f \in \mathscr{E}^{N+1}(\Omega \times \underline{C}^s).$$

The reasoning used in the case of the first order system (see § 4) can be applied to this case without any significant change. We will indicate the necessary modifications very briefly.

The space of all functions u such that

$$u \in \mathscr{D}^{k+m-1}_{L^2}[0, T], \frac{\partial u}{\partial t} \in \mathscr{D}^{k+m-2}_{L^2}[0, T], \ldots, \left(\frac{\partial}{\partial t}\right)^{m-1} u \in \mathscr{D}^k_{L^2}[0, T]$$

is denoted by $\tilde{\mathscr{D}}^k_{L^2}[0, T]$. We introduce a topology on $\tilde{\mathscr{D}}^k_{L^2}[0, T]$ by a norm $|||u(t)|||_k$ defined by

$$(5.3) \qquad |||u|||^2_k = ||u(t)||^2_{k+m-1} + \cdots + \left(\frac{\partial}{\partial t}\right)^{m-1} u(t)||^2_k.$$

Now we recall the result in the linear case. Given the equation 206

$$(5.4) \qquad M[u] = f$$

with $f \varepsilon \mathscr{D}^{[\frac{n}{2}+1]}_{L_2}[0, T]$ (resp. $f \in \mathscr{D}^{[\frac{n}{2}]+2}_{L^2}[0, T]$) and the initial data $u(0) \in \tilde{\mathscr{D}}_{L^2}{}^{[\frac{n}{2}]+1}$ (resp. $u(0) \in \tilde{\mathscr{D}}^{[\frac{n}{2}]+2}_{L^2}$) the solution $u(t)$ of the Cauchy problem belongs to $\tilde{\mathscr{D}}^{[\frac{n}{2}]+1}_{L^2}[0,T]$ (resp. to $\tilde{\mathscr{D}}^{[\frac{n}{2}]+2}_{L^2}[0,T]$) and further we have the energy inequality

$$|||u(t)|||_{[\frac{n}{2}]+1} \leq c(T)\left\{||u(0)||_{[\frac{n}{2}]+1} + \int_0^t ||f(s)||_{[\frac{n}{2}]+1} ds\right\}$$

$$\left(\text{resp. } |||u(t)|||_{[\frac{n}{2}]+2} \leq c(T)\left\{||u(0)||_{[\frac{n}{2}]+2} + \int_0^t ||f(s)||_{[\frac{n}{2}+2]} ds\right\}\right)$$

for $0 \leq t \leq T$.

In the semi-linear case we use the following

Lemma 1. *If* $u(t) \in \tilde{\mathscr{D}}^{[\frac{n}{2}]+1}_{L^2}[0,T]$ *then for any* $\alpha \in \mathscr{D}$ *the function* $\tilde{f} = \alpha f$ *satisfies*

$$\tilde{f}\left(x, t, \left(\frac{\partial}{\partial t}\right)^{j_1}\left(\frac{\partial}{\partial x}\right)^{\alpha_1} u(x,t), \ldots, \left(\frac{\partial}{\partial t}\right)^{j_s}\left(\frac{\partial}{\partial x}\right)^{\alpha_s} u(x,t)\right) \in \mathscr{D}^{[\frac{n}{2}]}_{L^2}[0,T]$$

and

$$\begin{aligned} &||\tilde{f}(x,t,\left(\frac{\partial}{\partial t}\right)^{j_1}\left(\frac{\partial}{\partial x}\right)^{\alpha_1} u(x,t), \ldots, \left(\frac{\partial}{\partial t}\right)^{j_s}\left(\frac{\partial}{\partial x}\right)^{\alpha_s} u(x,t))||_{\frac{n}{2}+1} \\ &\leq C\, M\left\{1 + |||u(t)|||^{[\frac{n}{2}]+1}_{[\frac{n}{2}]+1}\right\}. \end{aligned} \tag{5.5}$$

207 *Proof.* We write $v_k(t)$ for $\left(\frac{\partial}{\partial t}\right)^{j_k}\left(\frac{\partial}{\partial x}\right)^{\alpha_k} u(x,t)$ and $\tilde{f}(x,t,v_1(t),\ldots,v_s(t))$ for $\tilde{f}\left(x,t,\left(\frac{\partial}{\partial t}\right)^{j_1}\left(\frac{\partial}{\partial x}\right)^{\alpha_1} u(x,t),\ldots\right)$. Now we see that $||v_k(t)||_{[\frac{n}{2}]+1} \leq c|||u(t)|||_{[\frac{n}{2}]+1} (k = 1,\ldots,s)$. In fact,

$$||v_k(t)||_{[\frac{n}{2}]+1} = ||\left(\frac{\partial}{\partial t}\right)^{j_k}\left(\frac{\partial}{\partial x}\right)^{\alpha_k} u(t)||_{[\frac{n}{2}]+1} \leq c||\left(\frac{\partial}{\partial t}\right)^{j_k} u(t)||_{[\frac{n}{2}]+|\alpha_k|+1}.$$

Since $j_k + |\alpha_k| \leq m-1$ we have $\left[\frac{n}{2}\right] + |\alpha_k| + 1 \leq \left[\frac{n}{2}\right] + 1 + (m-1-j_k)$ and hence

$$||v_k(t)||_{[\frac{n}{2}]+1} \leq c||\left(\frac{\partial}{\partial t}\right)^{j_{k_u}} ||_{[\frac{n}{2}]+1+(m-1-j_k)} \leq c|||u|||_{[\frac{n}{2}]+1}.$$

The assertion follows from this by an application of Cor. 2 after lemma 1 of § 2.

The following lemma is proved on the same lines and we omit the proof. □

Lemma 2. *If* $u \in \tilde{\mathscr{D}}_{L^2}^{[\frac{n}{2}]+1+N}[0,T]$ *for an integer* $N \geq 1$ *then for any* $\alpha \in \mathscr{D}$

$$\tilde{f}\left(x, t, \left(\frac{\partial}{\partial t}\right)^{j_1}\left(\frac{\partial}{\partial x}\right)^{\alpha_1} u(x,t), \ldots, \left(\frac{\partial}{\partial t}\right)^{j_s}\left(\frac{\partial}{\partial x}\right)^{\alpha_s} u(x,t)\right) \in \tilde{\mathscr{D}}_{L^2}^{[\frac{n}{2}]+1+N}[0,T]$$

and

$$\|\tilde{f}\left(x, t, \left(\frac{\partial}{\partial t}\right)^{j_1}\left(\frac{\partial}{\partial x}\right)^{\alpha_1} u(x,t), \ldots \left(\frac{\partial}{\partial t}\right)^{j_s}\left(\frac{\partial}{\partial x}\right)^{\alpha_s} u(x,t)\right)\|_{[\frac{n}{2}]+1+N}$$

$$\leq cM_n\left\{1 + \left(1 + |||u(t)|||_{[\frac{n}{2}]+N}^{[\frac{n}{2}]+N}\right)|||u(t)|||_{[\frac{n}{2}]+N+1}\right. \tag{5.6}$$

As in the local existence theorem for the first order systems we define **208**

$$\tilde{f}(x, t, v_1, \ldots, v_s) = \beta(x) f(x, t, 0, \ldots, 0) + \alpha(x)\{f(x, t, v_1, \ldots, v_s) - f(x, t, 0, \ldots, 0)\}$$

where $\alpha, \beta \in \mathscr{D}$. Then the same arguments as in the first order systems prove the following

Theorem 1 (local existence theorem)**.** *For fixed* $\alpha, \beta \in \mathscr{D}$ *and* T *let*

$$M[u] = f\left(x, t, \left(\frac{\partial}{\partial t}\right)^{j_1}\left(\frac{\partial}{\partial t}\right)^{\alpha_1} u(x,t), \ldots\right) \tag{5.7}$$

be a semi-linear regularly hyperbolic equation of order m. Given any initial data $u^{(0)} \in \mathscr{D}_{L^2}^N$, $N \geq \left[\frac{n}{2}\right] + 2$ *(more precisely, given*

$$(u_0, u_1, \ldots, u_{m-1})$$

with $u_j \in \mathscr{D}_{L^2}^{N+m-j}$*) and the initial time* $t_0 (0 \leq t_0 \leq T)$ *there exists a unique solution* $u(x,t) = u(t)$ *for* $t_0 \leq t \leq t_0 + h$ *of* (5.7) *such that* $u \in$

$\tilde{\mathscr{D}}^N_{L^2}[t_0, t_0+h]$, $\dfrac{\partial u}{\partial t} \in \tilde{\mathscr{D}}^{N-1}_{L^2}[t_0, t_0+h]$ *taking the initial value* $u^{(0)}$ *at* $t = t_0$. *h can be taken to be a fixed number independent of* t_0 *when* $\left\{|||u^{(0)}|||_{[\frac{n}{2}]+1}\right\}$ *is a bounded set. More precisely, there exists a non-increasing function* $\alpha(\xi) > 0$ *of* $\xi > 0$ *such that*

$$h = \varphi\left(|||u^{(0)}||_{[\frac{n}{2}]+1}\right).$$

Now we state a global existence theorem for a single semi-linear
209 regularly hyperbolic equation of order m. We assume an a *priori estimate* of the following type holds:

For any initial data $u^{(0)} \in \mathscr{D}^{[\frac{n}{2}]+2}_N \cap \mathscr{E}', \beta \in \mathscr{D}$ the solution $u(t)$ of

(5.8) $M[u] = \beta f(x, t, 0, \ldots, 0) + \alpha(f, x, t, v_1, \ldots, v_s) - f(x, t, 0, \ldots, 0))$

(where $v_k = \left(\dfrac{\partial}{\partial t}\right)^{j_k}\left(\dfrac{\partial}{\partial t}\right)^{\alpha_k} u$) satisfies

(5.9) $||u(t)||_{[\frac{n}{2}]+m} + ||\dfrac{\partial}{\partial t}u(t)||_{[\frac{n}{2}]+m-1} + \cdots + ||\left(\dfrac{\partial}{\partial t}\right)^{m-1} u(t)||_{[\frac{n}{2}]+1} \leq ||c.$

Theorem 2 (global existence theorem)**.** *under the assumption that there exists an a priori estimate of the above type, given any initial data* $(u_0, u_1, \ldots, u_{m-1})$ *with* $u_k \in \mathscr{E}^{N+m-k-1}_{L^2(loc)}$, $N \geq \left[\dfrac{n}{2}\right]+2$, *there exists a unique solution* $u(t) = u(x, t)$ *for* $0 \leq t \leq T$ *of* (5.2) *such that*

$$u \in \mathscr{E}^{N+m-1}_{L^2(\text{loc})}[0, T], \frac{\partial u}{\partial t} u \in \mathscr{E}^{N+m-2}_{L^2(\text{loc})}[0, T], \ldots, \left(\frac{\partial u}{\partial}\right)^m u \in \mathscr{E}^{N-1}_{L^2(\text{loc})}[0, T].|$$

Remark 1. As a particular case of the Theorem we have Theorem 1 of § 3.

Remark 2. We assumed an a priori estimate (5.9) for the theorem of exstence of global solutions. If in $f(x, t, v_1, \ldots, v_s)\left(v_k = \left(\dfrac{\partial}{\partial t}\right)^{jk}\left(\dfrac{\partial}{\partial x}\right)^{\alpha k} u\right)$ the orders $j_k + |\alpha_k|$ are less than $(m-1)$ the following remark will be useful. If we have an estimate of derivatives of u of the form

$$||\left(\frac{\partial}{\partial t}\right)^{j_k}\left(\frac{\partial}{\partial x}\right)^{\alpha_k} u(t)||_{[\frac{n}{2}]+1} \leq c \ (k = 1, \ldots, s)$$

210 then we have an a priori estimate of the form (5.9). In fact, first of all we have, if $g(x,t,v_1,\ldots,v_s)$ denotes $f(x,t,v_1,\ldots,v_s) - f(x,t,0,\ldots,0)$ then for any $\alpha \in \mathscr{D}$ the function $\tilde{g} = \alpha g$ satisfies the inequality

$$\|\tilde{g}\left(x, t, \left(\frac{\partial}{\partial t}\right)^{j_1} u(x,t), \ldots, \left(\frac{\partial}{\partial t}^{j_s}\right)\left(\frac{\partial}{\partial x}\right)^{\alpha_s} u(x,t)\right)\|\|_{[\frac{n}{2}]+1} \leq c'$$

with a constant c'. Now as in the case of first order systems this inequality, together with the energy inequality in the linear case, implies (5.9).

We illustrate this by the following simple example. Take for M the operator $\square = \frac{\partial^2}{\partial} \Delta$ and consider the semi-linear equation

$$\square u + f(u) = 0.$$

We assume $f(0) = 0$. We show that it is enough to obtain an estimate of $\|u(t)\|_{[\frac{n}{2}]+1}$. in order to get an a priori estimate of $\|u(t)\|_{[\frac{n}{2}]+2} + \|\frac{\partial u}{\partial t}(t)\|_{[\frac{n}{2}]+1}$. First we obeserve that the conditon $f(0) = 0$ can be removed. In fact, if $C_0 = f(0)$ we consider the equation

$$\square u + (f(u) - f(0))' + \beta(x) f(0) f(0) = 0;$$

that is,

$$\square u + C_0 \beta(x) + (f(u) - C_0) = 0,$$

where $\beta \in \mathscr{D}$.

It is enough to obtain an a priori estimate for solutions of this equation. If u_0, u_1, $\in \mathscr{E}'$ then we know that for $0 \leq t \leq T$ the solution $u(t)$ 211
has its support contained in some compact set: say in $|x| < R$.

Define

$$E_1(t) = \int\limits_{|x|<R} \left[\frac{1}{2}\left\{\left(\frac{\partial u}{\partial t}\right)^2 + \sum_j \left(\frac{\partial u}{\partial x_j}\right)^2\right\} + F(u) - c_0 u + \gamma(u^2 + 1)\right] dx$$

where $F(u) = \int\limits_0^u f(\tau) d\tau$ and γ is chosen so large that $F(u) - c_0 u + \gamma(u^2 + 1) \geq 0$ for any u. This is always possible if we assume $F(u) > -L$.

Differentiating $E_1(t)$ with respect to t and using integration by parts we have

$$\frac{d}{dt}E_1(t) = \int\limits_{|x|<R} \left\{ \frac{\partial u}{\partial t}.\,\Box u + (f(u) - c_0)\frac{\partial u}{\partial t} + 2\gamma u.\frac{\partial u}{\partial t} \right\} dx$$
$$= \int 2\gamma u.\frac{\partial u}{\partial t} - c_0\beta(x)\frac{\partial u}{\partial t} dx \text{ since } \Box u + f(u) - c_0 = -\beta(x)c_0.$$
$$\leq CE_1(t).$$

Hence $E_1(t) \leq e^{ct} \leq e^{cT} = c'$. This, together with the expression for $E_1(t)$, shows that we have the assertion.

By considering the equation obtained by differentiating the equation $\Box u + c_0\beta(x) + (f(u) - c_0) = 0$ with respect to x_j

$$\Box \frac{\partial u}{\partial x_j} + f'(u)\frac{\partial u}{\partial x_j} + c_0\frac{\partial \beta}{\partial x_j} = 0 \; (j = 1, 2, \ldots, n),$$

we can obtain an estimate for $E_2(t)$ in an analogous way. Thus we have the following result:

212 Suppose the function f satisfies the conditions

(1) $F(u) > -L$,

(2) $|f'(u)| < \alpha(u^2 + 1)$ for $n = 3$

$\leq$ a polynomial for $n = 2$.

For any initial data (u_0, u_1) with $u_0 \in \mathscr{E}^m_{L^2(\text{loc})}$, $u_1 \in \mathscr{E}^{m-1}_{L^2(\text{loc})}$, $m \geq [\frac{n}{2}] + 3$, there exists a unique solution $u(t) = u(x, t)$ for $0 \leq t < \infty$ such that

$$u \in \mathscr{E}^m_{L^2(\text{loc})}[0, \infty), \frac{\partial u}{\partial t} \in \mathscr{E}^{m-1}_{L^2(\text{loc})}[0, \infty), \left(\frac{\partial}{\partial t}\right)^2 u \in \mathscr{E}^{m-2}_{L^2(\text{loc})}[0, \infty).$$

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