Lectures on Topics in Analysis

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Chapter 1

Differentiable functions in $\mathbb{R}^n$

1 Taylor's formula

Let Ω be an open set in $\mathbb{R}^n$, and for $0 \leq k < \infty$ let $C^k(\Omega)$ denote the **1**
set of real valued functions on Ω whose partial derivatives of order $\leq$ k exist and are continuous; $C^\infty(\Omega)$ will stand for the set of functions which belong to $C^k(\Omega)$ for all $k > 0$. We write C^k, $C^\infty, \ldots$ for $C^k(\Omega)$, $C^\infty(\Omega), \ldots$ when no confusion is likely.

We shall use the following notation:

$$\alpha = (\alpha_1, \ldots \alpha_n), \alpha_i \geq 0 \text{ being integers,}$$
$$x = (x_1, \ldots, x_1), x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n},$$
$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \ldots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}, \alpha! = \alpha_1! \ldots \alpha_n!, |\alpha| = \alpha_1 + \cdots + \alpha_n$$
$$|x| = \max_i |x_i|, \|x\| = (|x_1|^2 + \cdots + |x_n|^2)^{\frac{1}{2}}.$$

Similar notation will be used with $\mathbb{R}^n$ replaced by $\mathbb{C}^n$, and for complex valued functions. We shall write $C_0^k(\Omega)$ for the space of C^k functions on Ω which vanish outside a compact subset of Ω (which may depend on the function in question).

Similar notation will be used for q-tuples of functions; $C^{k,q}(\Omega)$ $[C_0^{k,q}(\Omega)]$ is then the set of mappings $f = (f_1, \ldots, f_q)\colon \Omega \to \mathbb{R}^q$ for which $f_i \in C^k(\Omega)[C_0^k(\Omega)]$ for $1 \leq i \leq q$. We write simply C^k, or $C^k(\Omega)$ **2** for $C^{k,p}(\Omega)$ when no confusion is likely; similarly, we sometimes write C_0^k for $C_0^{k,q}(\Omega)$.

A real valued function f defined on Ω is called (*real*) *analytic* (in Ω) if for any $a = (a_1, \ldots, a_n) \in \Omega$, there exists a power series

$$P_a(x) \equiv \sum c_\alpha (x-a)^\alpha \equiv \sum c_{\alpha_1 \cdots \alpha_n} (x_1 - a_1)^{\alpha_1} \cdots (x_n - a_n)^{\alpha_n}$$

which converges to $f(x)$ for x in a neighbourhood of a.

Remark that the power series is uniquely determined by f; in fact $c_\alpha = \dfrac{D^\alpha f(a)}{\alpha!}$ in particular, if $f = 0$ in a neighbourhood of a, then $c_\alpha = 0$ for all α; further $f \in c^\infty$, and, in fact, for any $\beta = (\beta_1, \ldots, \beta_n)$, $D^\beta P_a(x) = \sum\limits_\alpha c_\alpha D^\beta (x-a)^\alpha$.

If U is an open set in $\mathbb{C}^n$, and f a complex valued function in U, then f is called *holomorphic* (in U) if for any $a \in U$, there exists a power series

$$\sum c_\alpha (z-a)^\alpha$$

which converges to f for all z in a neighbourhood of a. We shall assume some elementary properties of holomorphic functions, among them the following. Proofs can be found in Herve' [14].

1 A function f on U is holomorphic if and only if it is continuous and for any γ, $1 \le \gamma \le n$, the partial derivatives

$$\frac{\partial f}{\partial \overline{z}_\nu} \equiv \frac{1}{2}\left(\frac{\partial f}{\partial x_\nu} + \frac{\partial f}{\partial y_\nu}\right)$$

3 exist and zero; here $z_\nu = x_\nu + iy_\nu$, x_ν, y_ν being real.

2 (Principle of analytic continuation.) If f is holomorphic in a connected open set U in $\mathbb{C}^n$, and $D^\alpha f(a) = 0$ for all $\alpha = (\alpha_1 \ldots, \alpha_n)$, and some $a \in U$, then $f \equiv 0$ in U; in particular, if f vanishes on a nonempty open subset of U, $f \equiv 0$.

3 *Weierstrass' theorem.* If $\{f_n\}$ is a sequence of holomorphic functions in U converging uniformly on compact subsets of U to a function f, then f is holomorphic in U; further, for any $\alpha, D^\alpha f_\nu$ converges, uniformly on compact sets, to $D^\alpha f$.

4 *Cauchy's inequalities*. If f is holomorphic in U, and $|f(z)| \leq M$ for $z \in U$, $M > 0$, then for any compact set $K \subset U$, we have, for any α,

$$|D^\alpha f(z)| \leq M\delta^{-|\alpha|}\alpha! \text{ for } z \in K,$$

where δ is the distance of K from the boundary of U.

Lemma 1. *If f is real analytic in $\Omega \subset \mathbb{R}^n$, then there exists an open set $U \subset \mathbb{C}^n$, $U \cap \mathbb{R}^n = \Omega$, in U a holomorphic function F such $F|_\Omega = f$.*

Proof. Suppose, for $a \in \Omega$, $P_a(x) = \sum c_\alpha (x-a)^\alpha$ converges to $f(x)$ for $|x - a| < r_a$, $r_a > 0$. Define

$$U_a = \{z \in \mathbb{C}^n \big| |z - a| < r_a\};$$

then, for $z \in U_a$,

$$P_a(z) = \sum c_\alpha (z-a)^\alpha.$$

converges and is holomorphic in U_a. □ 4

Let $U = \bigcup\limits_{a\in\Omega} U_a$. We assert that if $U_a \cap U_b = U_{a,b} \neq \phi$ then $P_a = P_b$ in $U_{a,b}$. In fact, $U_{a,b}$ is convex, hence connected, and $D^\alpha P_a(c) = D^\alpha P_b(c) = D^\alpha f(c)$ for any α and $c \in U_{a,b} \cap \mathbb{R}^n$ (which is $\neq \phi$ if $U_{a,b} is$). Hence we may define F on U by requiring that $F|U_a = P_a$. Clearly F is holomorphic in U and $F|\Omega = f$.

Let N be a neighbourhood of the closed unit interval $0 \leq t \leq 1$ in $\mathbb{R}$, and let $f \in C^k(N)$. Then, we prove the

Lemma 2. $f(1) = \sum\limits_{\nu=0}^{k-1} \dfrac{f^{(\nu)}(0)}{\nu!} + \dfrac{f^{(k)}(\xi)}{k!}$, *where* $0 \leq \xi \leq 1$.

Proof. For continuous g, define

$$I_0(g, t) = g(t), I_r(g, t) = \int\limits_0^t I_{r-1}(g, \tau) d\tau, \ r \geq 1.$$

Clearly, if $g \in C^k(N)$ and $g^{(\nu)}(0) = 0$ for $0 \leq r \leq k - 1$, we have

$$g(t) = I_k(g^{(k)}, t).$$

If we apply this to $g(t) = f(t) - \sum\limits_{\nu=0}^{k-1} \frac{f^{(\nu)}(0)}{\nu!} t^\nu$, we obtain

$$f(1) - \sum_{\nu=0}^{k-1} \frac{f^{(\nu)}(0)}{\nu!} = I_k(g^{(k)}, 1) = I_k(f^{(k)}, 1). \tag{1.1}$$

Now, if m, M denote the lower and upper bounds of $f^{(k)}$ in [0, 1], we obviously have

$$\frac{m}{k!} \leq I_k(f^{(k)}, 1) \leq \frac{M}{k!}$$

5 Since $f^{(k)}$, being continuous, assumes all values between m and M, there is ξ, $0 \leq \xi \leq 1$ with

$$I_k(f^{(k)}, 1) = \frac{f^{(k)}(\xi)}{k!}.$$

This proves lemma 2. □

It is easy to prove, by induction, that

$$I_k(g, t) = \frac{1}{(k-1)!} \int_0^t g(\tau)(t - \tau)^{k-1} d\tau.$$

Hence (1.1) can be written

$$f(1) - \sum_{\nu=0}^{k-1} \frac{f^{(\nu)}(0)}{\nu!} = \frac{1}{(k-1)!} \int_0^1 (1-t)^{k-1} f^{(k)}(t) dt. \tag{1.2}$$

Theorem 1 (Taylor's formula)**.** *Let Ω be open in $\mathbb{R}^n$, and $f \in C^k(\Omega)$. Then, if $x, y \in \Omega$ and the closed line segment $[x, y]$ joining x to y is also contained in Ω, we have*

$$f(x) = \sum_{|\alpha| \leq k-1} \frac{D^\alpha f(y)}{\alpha!}(x - y)^\alpha + \sum_{|\alpha| = k} \frac{D^\alpha f(\xi)}{\alpha!}(x - y)^\alpha,$$

where ξ is a point of $[x, y]$.

This theorem follows at once from Lemma 2 applied to the function

$$g(t) = f(y + t(x - y))$$

which belongs to $C^k(N)$, N being a neighbourhood of [0, 1].

If $f \in C^k(\Omega)$(k being finite), K is a compact set in Ω and $0 \leq m \leq k$, **6**
we set

$$\|f\|_m^K = \sum_{|\alpha| \leq m} \sup_{x \in K} |D^\alpha f(x)|.$$

We define a topology on $C^k(\Omega)$ as follows: a fundamental system of neighbourhoods of $f_0 \in C^k(\Omega)$ is given by the sets

$$B(f_0, K, \varepsilon, k) = \{f \in C^k \big| \|f - f_0\|_k^K < \varepsilon\};$$

here ε runs over the positive real numbers, and K over all compact subsets of Ω. The topology on $C^\infty(\Omega)$ is obtained by taking for a fundamental system of neighbourhoods of f_0 the sets

$$B(f_0, K, \varepsilon, k) \cap C^\infty(\Omega)$$

with $\varepsilon > 0$, K compact in Ω and $k > 0$ an arbitrary integer.

The space $C^k(\Omega)$ is metrisable; we may take, for example, as metric the function

$$d(f, g) = \sum_{\nu=0}^{\infty} 2^{-\nu} \frac{\|f - g\|_k^{K_\nu}}{1 + \|f - g\|_k^{K_\nu}};$$

here $\{K_\nu\}$ is a sequence of compact sets with $K_\nu \subset \overset{\circ}{K}_{\nu+1}$, $\cup K_\nu = \Omega$. [$On C^\infty(\Omega)$], a metric can be defined by replacing $\|f - g\|_k^{K_\nu}$ by $\|f - g\|_\nu^{K_\nu}$ in the function above]

Theorem 2. $C^k(\Omega)$ is a complete metric space for $0 \leq k \leq \infty$.

Proof. We have only to prove that if $\{g_\nu\}$ is a sequence of functions in C^k and $\|g_\nu - g_\mu\|_m^K \to$ as $\mu, \nu \to \infty$ for all integers m, $0 \leq m \leq k$ and all **7**
compact $K \subset \Omega$, then there exists $g \in C^k$ for which $\|g_\nu - g_\mu\|_m^K \to 0$ as $\nu \to \infty$, $0 \leq m \leq k$, k compact.

Since by assumption, for $|\alpha| \leq k$, $D^\alpha(g_\nu - g_\mu) \to 0$, uniformly on any compact set, there exist continuous functions g_α, $|\alpha| \leq k$, for which

$\|D^\alpha g_\nu - g_\alpha\|_0^K \to 0$. If we prove that $g_0 \in C^k$ and $D^\alpha g_0 = g_\alpha$ then clearly $\|g_\alpha - g\|_m^K \to 0$, $0 \le m \le k$, where $g = g_0$. To prove this assertion. we have only to show that if $|\alpha| \le k-1$, and $\beta = (\beta_1, \ldots, \beta_n)$ is such that $|\beta| = 1$, then $g_\alpha \in C^1$ and $D^\beta g_\alpha = g_{\alpha+\beta}$ in Ω. □

Now, if $a \in \Omega$ and x is sufficiently near a we have

$$D^\alpha g_\nu(x) - D^\alpha g_\nu(a) = \sum_{|\beta|=1} D^{\alpha+\beta} g_\nu(\xi_\nu)(x-a)^\beta, \tag{1.3}$$

where ξ_ν is a point on the segment $[a, x]$. We may choose a subsequence $\{\nu_p\}$ such that $\xi_{\nu_p} \to \xi \in [a, x]$. Clearly, if we replace ν by ν_p in (1.3) and let $p \to \infty$, we obtain

$$\begin{aligned} g_\alpha(x) - g_\alpha(a) &= \sum_{|\beta|=1} g_{\alpha+\beta}(\xi)(x-a)^\beta \\ &= \sum_{|\beta|=1} g_{\alpha+\beta}(a)(x-a)^\beta + o(|x-a|). \end{aligned}$$

where $o(|x-a|)$ tends to zero faster than $|x-a|$ as $x \to a$. (The last equality is a consequence of the continuity of $g_{\alpha+\beta}$.) But this implies that $g_\alpha \in C^1$ and that for $|\beta| = 1$, $D^\beta g_\alpha(a) = g_{\alpha+\beta}(a)$.

8 **Remark.** If we write

$$\|f\|_m^K = \sum_{|\alpha|\le m} \sum_{i=1}^{q} \sup_{x\in K} |D^\alpha t_1(x)|$$

for $f = (f_1, \ldots, f_q) \in C^{k,q}(\Omega)$, $m \le k$, we may replace $C^k(\Omega)$ by $C^{k,q}(\Omega)$ in Theorem 2. Another consequence of Taylor's formula is the following:

Proposition 1. *If $f \in C^\infty(\Omega)$, then f is analytic if and only if for any compact $K \subset \Omega$, there exists $M_K > 0$ such that*

$$|D^\alpha f(x)| \le M_K^{|\alpha|+1} \alpha! \, for x \in \text{ and all } \alpha.$$

Proof. The necessity follows at once from Lemma 1 and Cauchy's inequalities (Property 4. of holomorphic functions stated at the beginning). For the sufficiency, we remark that if x is in a compact, convex neighbourhood K of $a \in \Omega$, and $\xi \in [a, x]$, then

$$\Big| \sum_{|\alpha|=k+1} \frac{D^\alpha f(\xi)}{\alpha!}(x-a)^\alpha \Big| \leq (k+1)^n M_K^{K+2} |x-a|^{k+1}.$$

□

If $|x - a| < M_K^{-2}$, Taylor's formula implies that

$$\sum \frac{D^\alpha f(\xi)}{\alpha!}(x-a)^\alpha$$

converges to $f(x)$.

Remark. As is easily verified, the above condition is equivalent with the existence of $M'_K > 0$ such that

$$|D^\alpha f(x)| \leq M_K'^{|\alpha|+1} |\alpha| \text{ for } x \in K \text{ and all } \alpha.$$

2 Partitions of unity

The support of a function φ defined on the open set $\Omega \subset \mathbb{R}^n$, written supp $.\varphi$, is the closure in Ω of the set of points a where $\varphi(a) \neq 0$.

A family of sets $\{E_i\}$ is called locally finite if any point $a \in \Omega$ has 9
a neighbourhood which meets E_i only for finitely many i.

A family of sets $\{E'_j\}_{j\in J}$ is called a refinement of the family $\{E_j\}_{j\in J}$ if there exists a map τ: $J \to I$ for which $E'_j \subset E_{\tau(j)}$.

We shall use the following proposition due to J. Dieudonne [9].

Proposition. *If X is a locally compact, hausdorff space which is a countable union of compact sets, then X is paracompact, i.e. any open covering has a locally finite refinement. Further, for any locally finite open covering* $\{U_i\}_{i\in I}$ *of X, there exists an open covering* $\{V_i\}_{i\in I}$ *for which* $\bar{V}_i \subset U_i$.

Theorem 1. *If Ω is an open subset of $\mathbb{R}^n$ and $\Omega = \bigcup_{i\in I} U_i$, where the U_i are open, then there exists a family of C^∞ functions, say $\{\varphi_i\}'_{i\in I}$ such that*

(i) $0 \leq \varphi_i \leq 1$, supp. $\varphi_i \subset U_i$, (ii) $\{supp.\varphi_i\}$ is a locally finite family, and (ii) $\sum_{i\in I} \varphi_i(x) = 1$ for any $x \in \Omega$.

Lemma 1. *There exists a C^∞ function k in $\mathbb{R}^n$ with $k \geq 0$, $k(0) > 0$,* supp $.k \subset \{x \big| \|x\| < 1\}$.

Proof. Let $s(r)$ be the C^∞ function on $\mathbb{R}^1$ defined by

$$s(r) = \begin{cases} 0^{-1/(c-r)} & \text{if } r < c, \\ 0 & \text{if } r \geq c, \end{cases}$$

where $0 < c < 1$. We have only to take $k(x) = s(x_1^2 + \cdots + x_n^2)$. □

10 **Lemma 2.** *If K is a compact set in $\mathbb{R}^n$, $U \supset K$ is open, then there exists a C^∞ function ψ with $\psi(x) \geq 0$, $\psi(x) > 0$ if $x \in K$, supp. $\psi \subset U$.*

Proof. Let δ be the distance of K from $\mathbb{R}^n - U$; for $a \in K$, let $\psi_a(x) = k\left(\dfrac{x-a}{\delta}\right)$, where k is as in Lemma 1. Let $V_a = \{x \in \mathbb{R}^n | \psi_a(x) > 0\}$. Then $a \in V_a \subset U$. Since K is compact, there exist finitely many points $a_1, \ldots, a_p \in K$ for which $v_{a_i} \cap \ldots \cap v_{a_p} \supset K$. Define $\psi(x) = \sum_{i=1}^{p} \psi_{a_i}(x)$. □

Proof of theorem 1. Let $\{V_j\}_{j\in J}$ be a locally finite refinement of$\{U_i\}_{i\in I}$ by relatively compact open subset of Ω(which exists by Dieudonne's proposition). Let $\{W_j\}_{j\in J}$ be an open covering of Ω such that $\overline{W}_j \subset V_j$. By Lemma 2, there exists $\psi_j \in C^\infty(\Omega)$, $\psi_j(x) > 0$ for $x \in W_j$ and supp $\psi_j \subset V_j$, $\psi_j \geq 0$. Let $\varphi'_j = \psi_j / \sum_{k\in J} \psi_k$. (Since V_j is locally finite, $\sum_{k\in J} \psi_k$ is defined and $\in C^\infty(\Omega)$ and is everywhere > 0 since $\psi_j > 0$ on W_j and $\cup W_j = \Omega$.) Clearly $0 \leq \varphi'_j \leq 1$, supp $.\varphi'_j \subset V_j$ and $\sum_{j\in J} \varphi'_j = 1$. Let $\tau\colon J \to I$ be a map so that $V_j \subset U_{\tau(j)}$. Let $J_i \subset J$ be the set $\tau^{-1}(i)$, $i \in I$. Define $\varphi_i = \sum_{j\in J_i} \varphi'_j$(an empty sum stands for 0). Since the sets J_i are mutually disjoint and cover J, we have $\sum \varphi_i = 1$. It is clear that supp $\varphi_i \subset U_i$ and that $\{$supp $.\varphi_i\}$ form a locally finite family.

Corollary. *Let Ω be open in $\mathbb{R}^n$, X a closed subset of Ω, U an open subset of Ω containing X. Then there exists a C^∞ function ψ on Ω such* 11 *that $\psi(x) = 1$ for $x \in X$, $\psi(x) = 0$ for $x \in \Omega - U$, $0 \leq \psi \leq 1$ everywhere.*

Proof. By Theorem 1, there exist C^∞ functions $\varphi_1, \varphi_2 \geq 0, \operatorname{supp}.\varphi_1 \subset U$, $\operatorname{supp}\varphi_2 \subset \Omega - X$ with $\varphi_1 + \varphi_2 = 1$ on Ω. We have only to take $\psi = \varphi_1$. □

Lemma 3. *If $\{U_i\}$ is an open covering of Ω, then there exist C^∞ functions ψ_i with $\operatorname{supp}\psi_i \subset U_i$, $0 \leq \psi_i \leq 1$, and $\sum \psi_i^2 = 1$ on Ω.*

In fact, if φ_i is a partition of unity relative to $\{U_i\}$, we may set $\psi_i = \varphi_i/(\sum \varphi_i^2)^{\frac{1}{2}}$.

3 Inverse functions, implicit functions and the rank theorem

Let Ω be an open set in $\mathbb{R}^n$ and $f\colon \Omega \to \mathbb{R}^m$ a map which is in $C^1(\Omega)$ [i.e. its components are in $C^1(\Omega)$]. Let $a \in \Omega$.

Definition. $(df)(a)$ is defined to be the linear map of $\mathbb{R}^n$ in $\mathbb{R}^m$ for which

$$(df)(a)(v_1, \ldots, v_n) = (w_1, \ldots, w_m),$$

with
$$w_j = \sum_{i=1}^{n} \left(\frac{\partial f_j}{\partial x_i}\right) v_i.$$

We shall call $(df)(a)$ the differential of f at a.

Theorem 1. *If f is a C^1 maps of Ω into $\mathbb{R}^n$ and for $a \in \Omega, (df)(a)$ is nonsingular, then there exist neighbourhoods U of a and V of $f(a)$ such that $f|U$ maps U homeomorphically onto V.*

Proof. Without loss of generality we may assume that $a = 0$, $f(a) = 0$. Since $(df)(a)$ is nonsingular we may assume, by composing f with a 12 non-singular linear map of $\mathbb{R}^n$ into itself, that $(df)(a) =$ identity. Let g be defined on Ω by

$$g(x) = f(x) - x. \text{ Then obviously } (dg)(a) = 0.$$

□

This implies that there exists a neighbourhood W of 0, $\overline{W} \subset \Omega$, $W = \{x \big| |x_i| < r\}$ such that $x, y \in \overline{W}$ implies $|g(x) - g(y)\big| \leq \frac{1}{2}|x - y|$. We remark that $|f(x) - f(y)| \geq \frac{1}{2}|x - y|$ if $x, y \in W$, so that f is injective on W. Let $V = \{x \big| |x_i| < \frac{1}{2}r\}$, $U = W \cap f^{-1}(V)$. Define $\varphi_0\colon V \to W$ to be $\varphi_0(y) = 0$ and by induction $\varphi_k(y) = y - g[\varphi_{k-1}(y)]$. It is easily verified by induction that $\varphi_k(y) \in W$ for each k, and further that $|\varphi_k(y) - \varphi_{k-1}(y)| = |g(\varphi_{k-1}(y)) - g(\varphi_{k-2}(y))| \leq \frac{r}{2^k}$. Hence φ_k is uniformly convergent to a function $\varphi\colon V \to \mathbb{R}^n$. Since $\varphi_k(y) \in W$ for each k, $\varphi(y) \in \overline{W}$ and

$$\varphi(y) = y - g[\varphi(y)]. \tag{3.1}$$

Since $|y| < r/_2$ and $\big|g[\varphi(y)]\big| \leq r/_2$ we have $\varphi(y) \in W$. From (3.1) it follows that $f[\varphi(y)] = y$. Since $f\big|W$ is injective φ is the inverse of f. The continuity of φ follows from that of φ_k and the uniform convergence.

Remark. The theorem has an analogue for functions from $\mathbb{C}^n$ to $\mathbb{C}^n$. If Ω is an open set in $\mathbb{C}^n$, f a holomorphic map of Ω into $\mathbb{C}^n$ and if $(df)(a)$ is nonsingular at some $a \in \Omega$, then there exist neighbourhood U, V of a and $f(a)$ respectively such that (i) $f\big|U$ maps U homeomorphically onto V and (ii) the inverse mapping of $f\big|U$ is holomorphic on V. The proof
13 is identical with that given above; since each φ_k is holomorphic and φ_k converges uniformly to φ, φ is holomorphic.

Definition. Let f be a C^1 map of $\Omega_1 \times \Omega_2$ into $\mathbb{R}^p$, and let $(a, b) \in \Omega_1 \times \Omega_2$. Let $f(a, y) = g(y)$. Then $(d_2 f)(a, b)$ is defined by $(d_2 f)(a, b) = (dg)(b)$; $(d_1 f)(a, b)$ is defined similarly.

Theorem 2. *Let $\Omega_1 \times \Omega_2$ be an open set in $\mathbb{R}^{m+n}$ and $f\colon \Omega_1 \times \Omega_2 \to \mathbb{R}^n$ a function in C^1. Suppose that for some $(a, b) \in \Omega_1 \times \Omega_2$, we have $f(a, b) = 0$ and $(d_2 f)(a, b)$ has rank n. Then there exists a neighbourhood $U \times V$ of (a, b) such that for any $x \in U$ there is a unique $y = y(x) \in V$ for which $f(x, y) = 0$; the map $x \to y(x)$ is continuous.*

Proof. Consider $F\colon \Omega_1 \times \Omega_2 \to \mathbb{R}^{m+n}$ defined by $F(x,y) = (x, f(x,y))$. Then the statement that $(d_2 f)(a,b)$ has rank n is equivalent to saying that $(dF)(a,b)$ is nonsingular. Therefore by Theorem 1, there exists a neighbourhood $U' \times V$ of (a,b) and a neighbourhood W of $(a,0)$ such that $F\big|U' \times V \to W$ is a homeomorphism. Let $\varphi\colon W \to U' \times V$ be the continuous inverse of F. Then there exists a neighbourhood U of a such that $x \in U$ implies $(x, 0 \in W$. Then for $x \in U$, let $y(x)$ be the projection of $\varphi(x,0)$ on V. Clearly if $y \in V$ is such that $f(x,y) = 0$ then $y = y(x)$; moreover $y(x)$ is a continuous map with $f(x,y(x)) = 0$. □

Remark. The above theorem can be extended to a holomorphic map $f : \mathbb{C}^{m+n} \to \mathbb{C}^n$; $y(x)$ it then a holomorphic function of x.

Lemma 1. *With the same notation as in Theorem 2, if* $A(x) = (d_2 f)$ 14
$(x,y(x))$ *and* $B(x) = (d_1 f)(x,y(x))$ *and if* U *is so small that* $A(x)$ *is so small that* $A(x)$ *is invertible for* $x \in U$ *then* $y \in C^1(U)$ *and*

(3.2) $$(dy)(x) = -A(x)^{-1} \circ B(x)$$

Proof. Let $x, x+\xi \in U$ and $\eta = y(x+\xi) - y(x)$. Then $f(x+\xi, y(x)+\eta) = 0$ and by Taylor's formula

$$0 = f(x.y(x)) + B(x)\xi + A(x)\eta + \circ(|\xi| + |\eta|)$$

and $$\eta \to 0 \text{ as } \xi \to 0.$$

□

Hence $A(x)\eta = -B(x)\xi + \circ(|\xi| + |\eta|)$. If $x \in K$ compact $\subset U$ then $A(x)^{-1}$ is bounded on K and

$$\eta = -A(x)^{-1} \circ B(x)\xi + \circ(|\xi| + |\eta|).$$

This implies that $|\eta| = \circ(|\xi|)$ and hence

$$y(x+\xi) - y(x) = -A(x)^{-1} \circ B(x)\xi + \circ(|\xi|).$$

Hence $y(x)$ is differentiable and (3.2) holds.

Corollary. *If in Theorem 2,* $f \in C^k$ *then* $y \in C^k$.

Proof. We proceed by induction. If $f \in C^k$ and $y \in C^r$, $r < k$ then $A(x)$, $B(x) \in C^r$ and by (3.2), $y \in C^{r+1}$. □

From the remark about holomorphic mappings made after Theorem 2 we deduce the following

Corollary 1. *If f is real analytic so is y.*

Corollary 2. *In Theorem 1, if f is C^k (or analytic) then so is f^{-1}*

15 In fact we have only to apply the above corollaries to the map F: $\mathbb{R}^n \times \Omega \to \mathbb{R}^n$ defined by $F(x, y) = x - f(y)$.

The statement in Corollary 2 above is known as the inverse function theorem: those contained in the corollary to Lemma 1 and Corollary 1 above form the content of the implicit function theorem.

Definition. A cube in $\mathbb{R}^n$ is a set of the form $\{x \big| |x_i - a_i| < r_i\}$. A polycylinder in $\mathbb{C}^n$ is set of the form $\{z \big| |z_i - a_i| < r_i\}$.

Theorem 3 (The rank theorem)**.** *If Ω is an open set in $\mathbb{R}^n$ and $f \colon \Omega \to \mathbb{R}^m$, $f \in C^1$ and if rank $(df)(x) = r$ is an integer independent of x then there exist*

(i) *an open neighbourhood U of a,*

(ii) *an open neighbourhood V of $b = f(a)$,*

(iii) *cubes Q_1, Q_2 in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively,*

(iv) *homomorphisms $u_1 \colon Q_1 \to U$ $u_2 \colon V \to Q_2$ such that u_1, u_2 and their inverses are C^1*

with the property that if $\varphi = u_2 . \ f . \ u_1$, then

$$\varphi(x_1, x_2 \ldots, x_n) = (x_1, x_2 \ldots x_r, 0 \cdots 0).$$

Moreover if $f \in C^k$ or is analytic, u_1, u_2 may be chosen to have the same property.

Proof. By affine automorphisms of $\mathbb{R}^n$ and $\mathbb{R}^m$ we may suppose that $a = 0$, $b = 0$ and that $(df)(0)$ is the linear map

$$(v_1, \ldots, v_n) \to (v_1, \ldots, v_r, 0, \cdots, 0)$$

Consider the map $u\colon \Omega \to \mathbb{R}^n$ defined by

$$u(x) = (f_1(x), \ldots, f_r(x), n_{r+1}, \ldots, x_n).$$

□

Then $(du)(0) =$ identity, hence by the inverse function theorem there exists a neighbourhood U of 0 and a cube Q_1 such that $u|U \to Q_1$ is a C^1 homeomorphism and its inverse is in C^1. Let $u^{-1}|Q_1 = u_1$. Clearly $f(u_1)(y) = (y_1, \ldots, y_r, \varphi_{r+1}(y), \ldots \varphi_m(y))$. If $\psi(y) = f(u_1(y))$, obviously rank $(d\psi)(y) = r$ and hence **16**

$$\frac{\partial \varphi_j}{\partial y_k} = 0, \ j, \ k, > r,$$

i.e.,
$$\varphi_j = \varphi_j(y_i, \ldots, y_r) j > r$$

suppose that $Q_1 = I^r \times I^{n-r}$, where I^r, I^{n-r} are cubes in $\mathbb{R}^r$, $\mathbb{R}^{n-r}$. Define $u_2'\colon I^r \times \mathbb{R}^{m-r} \to \mathbb{R}^{m-r}$ by

$$u_2(y_1, \ldots, y_r, \ldots, y_m) = (y_1, \ldots, y_r, y_{r+1} - \varphi_{r+1}(y), \ldots, y_m - y_m(y)).$$

Trivially u_2 is bijective and its inverse is $u_2^{-1}(y_1, \ldots, y_r, \ldots, y_m) = (y_1, \ldots y_r, y_{r+1} + \varphi_{r+1}(u), \ldots, y_m + \varphi_m(y))$. Let Q_2 be a cube such that $u_2\psi(Q_1) \subset Q_2$ and $V = u_2^{-1}(Q_2)$ and clearly we have

$$\varphi(x_1, \ldots, x_n) = (x_1, x_2, \ldots, x_r, 0, \ldots, 0)$$

4 Sard's theorem and functional dependence

Lemma 1. *Let Ω be an open set $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^n$, aC^1 map. Then f carries sets of measure zero into sets of measure zero.*

Remark. If in Lemma 1, the condition that $f \in C'$ is replaced by the condition that f satisfies a Lipschitz condition on every compact $K \subset \Omega$, i.e., $|f(x) - f(y)| \leq M_k|x - y|$ for $x, y \in K$, then f carriers sets of measure 17
zero into sets of measure zero. This fact is trivial.

Lemma 2. *If Ω is an open set in $\mathbb{R}^n$, $f\colon \Omega \to \mathbb{R}^m$ is a C^1 map and if $m > n$, then $f(\Omega)$ has measure zero in $\mathbb{R}^m$.*

Proof. If we define $g\colon \Omega \times \mathbb{R}^{m-n} \to \mathbb{R}$ by $g(x_1, x_2, \ldots, x_m) = f(x_1, \ldots, x_n)$. Then by Lemma 1 $f(\Omega) = g(\Omega \times 0)$ has measure zero. □

Let be an open set in $\mathbb{R}^n$ and $f : \omega \to \mathbb{R}^n, aC^1$ map

Definition. A point $a \in \Omega$ is called a critical point of f if rank $(df)(a) < m$.

Remark. (1) If $m > n$, each point of Ω is clearly a critical point of f.

(2) The set A of critical points of f is closed in Ω.

(3) If $m > n$, $f(a)$ has measure zero in $\mathbb{R}^m$.

We shall prove the following

Theorem 1 (Sard)**.** *If Ω is an open set in $\mathbb{R}^n$, $f\colon \Omega \to \mathbb{R}^m$ is a C^∞ map $n \geq m$, and if A is the set of critical points of f then $f(A)$ has measure zero in $\mathscr{R}^m$*

In what follows Ω will denote an open set in $\mathbb{R}^n$, f, a map of Ω in some $\mathbb{R}^m$ and A, the set of critical points of f in Ω.

Actually the theorem of Sard states that if $f\colon \Omega \to \mathbb{R}^m$ and $f \in C^{n-m+1}(\Omega)$, then $f(A)$ has measure zero. The proof of this, however, requires more delicate analysis; see A. Sard [38] and A. P. Morse [30]. We shall prove this stronger statement when $m = n$ before proving Theorem
18 1. H . Whitney [47] has given an example of an $f \in C^{n-m}(\Omega)$, $n > m$, for which $f(A)$ has positive measure (even covers $\mathbb{R}^m$).

Proposition 1. *If f $\Omega \to \mathbb{R}^n$ is a C^1 map then $f(A)$ has measure zero in $\mathbb{R}^n$.*

Proof. Let a be in A. □

Since $(df)(a)$ has rank $< n$, $f(a) + (df)(a)(x - a)$ lies in an affine subspace V_a of $\mathbb{R}^n$, the dimension of V_a being $< n$. Choose an orthonormal basis $(u_1, u_2, \ldots, u_n)$ for $\mathbb{R}^n$ with centre $f(a)$ such that V_a lies in the 19
subspace spanned by $u_1, \ldots, u_{n-1}$. Let Q be a closed in cube in Ω. It is enough to show that $f(A \cap Q)$ has measure zero in $\mathbb{R}^n$. For $x \in Q \cap A$, by Taylor's formula, we have

$$f(x) - f(a) = (df)(a)(x - a) + r(x, a)$$

where $r(x, a) = 0(|x - a|)$ uniformly on $Q \times Q$ as $|x - a| \to 0$. Hence there exists a map $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\alpha(t) \to 0$ as $t \to 0$ and

$$\|r(x, a)\| \leq \alpha(|x - a|).|x - a|.$$

Then for sufficiently small $\varepsilon > 0$, of x lies in a cube Q_ε of side ε which contains a, $f(x)$ lies in the region between the hyperplanes $u_n = \alpha(\varepsilon).\ \in$ and $u_n = -\alpha(\varepsilon).\ \in$. Also since an orthonormal change of basis preserves distance, by Taylor's formula, there exists a constant M such that $f(x)$ lies in the cube of side $M\varepsilon$ with $f(a)$ as its centre. The volume of the intersection of the cube of side $M\varepsilon$ and the region between the hyperplanes $u_n = \pm\alpha(\varepsilon).\ \varepsilon$ is $\leq 2m^n\varepsilon^n\alpha(\varepsilon)$. Since an orthonormal change of basis leaves the measure in $\mathbb{R}^n$ invariant, we conclude that $f(Q_\varepsilon)$ has measure $\leq 2M^n\varepsilon^n\alpha(\varepsilon)$. We can assume without loss of generality that Q has side 1. Divide Q into ε^{-n} cubes Q_i of side ε, $i = 1, 2, \ldots, \varepsilon^{-n}$. Then if $Q_i \cap A \neq \phi$, $f(Q_i)$ has measure $\leq 2M^n\varepsilon^n\alpha(\varepsilon)$.

Hence measure of $[f(A \cap Q)] \leq \sum\limits_{\substack{A\cap Q_i \neq \phi \\ \leq 2M^n\alpha(\varepsilon).}} \{\text{ measure } f(Q_i \cap A)\}$

Since $\alpha(\varepsilon) \to 0$ as $\varepsilon \to 0$, $f(A \cap Q)$ has measure zero in $\mathbb{R}^n$.

Proposition 2. *If $f \colon \Omega \to \mathbb{R}^1$ is a C^∞ map, then $f(A)$ has measure zero in $\mathbb{R}^1$.*

Proof. Define A_k by

$$A_k = \{a \in \Omega | D^\alpha f(a) = 0 \text{ for } 0 < |\alpha| \leq k\}.$$

Obviously, $\{A_k\}$ is monotone decreasing and we have

$$A = (A_1 - A_2) \cup (A_2 - A_3) \cup \cdots \cup (A_{n-1} \cup A_n). \tag{4.1}$$

If $a \in A_n$, by Taylor's formula there exists a constant M such that for x in a closed cube Q about a, we have $|f(x) - f(a)| \leq M|x - a|^{n+1}$ so that image of the a cube of side ε about a has measure $\leq \varepsilon^{n+1} M$ in $\mathbb{R}^1$. Hence as in proposition 1, $f(A_n \cap Q)$ has measure $< M\varepsilon$; Whence,

(4.2) $$f(A_n) \text{ has measure zero in } \mathbb{R}^1.$$

20 Note that if $n = 1$, $A = A_n$, so that (4.2) is Prop. 2 with $n = 1$. We now suppose, by indication, that if Ω' is an open set in $\mathbb{R}^{n-1}$, g, a C^∞ map $\Omega' \to \mathbb{R}$ and if A^1 is the set of critical points of g, then $g(A^1)$ has measure zero.

For $k < n$ let $A_k - A_{k+1} = B_k$. Let $a \in B_k$; it is sufficient to show that a has a neighbourhood which goes into a set of measure zero. There exists $\alpha = (\alpha_1, \ldots, \alpha_n)$, with $|\alpha| = k + 1$, such that $D^\alpha f(a) \neq 0$. If $\alpha_i \neq 0$, define $\beta = \alpha - (0, \ldots, 1, \ldots, 0)1$ in the i^{th} place.

Define $$g : B_k \to \mathbb{R}^1 \text{ by } g(x) = D^\beta f(x).$$

$(dg)(a)$ has maximal rank $= 1$. Therefore there exists an open neighbourhood U' of a such that $(dg)(x)$ has rank 1 for x in U'. Applying the rank theorem to U' there exist

(1) a neighbourhood U of a, $U \subset U'$,

(2) a cube Q_1 in $\mathbb{R}^n$,

(3) an invertible map $u : Q_1 \to U$, u, u^{-1} being C^∞,

(4) a neighbourhood V of $g(a)$, such that gou: $Q_1 \to V$ is given by

$$gou(x_1, \ldots, x_n) = x_1 (= p_1(x) \text{say}).$$

Now, $B_k \cap U \subset B' = \{x \in U \big| g(x) = 0\}$ so that

$$u^{-1}(B_k \cap U) \subset B = \{x \in Q_1 | p_1(x) = 0\}.$$

Let $\Omega' = \{(x_2, \ldots, x_n) \in \mathbb{R}^{n-1} | (0, x_2, \ldots, x_n) \in Q_1\}$. Let $v : \Omega' \to U$ be the map $v(x_2, \ldots, x_n) = u(0, x_2, \ldots, x_n)$ and let $\psi = f \circ v$; ψ is a C^∞ map $\Omega' \to \mathbb{R}$. □

21 Let A^1= The set of critical points of ψ. Since $d(\psi)(x) = (df)(v(x))\circ (dv)(x), u^{-1}(B_k \cap U) \subset A^1$. By induction hypothesis, $\psi(A^1)$ has measure zero in $\mathbb{R}^1$. Since $\psi(A^1) \supset f(B_k \cap U)$, $f(B_k \cap U)$ has measure zero in $\mathbb{R}^1$, for each k which by (4.1) and (4.2) implies that $f(A)$ has measure zero in $\mathbb{R}^1$.

Corollary. *If $f : \Omega \to \mathbb{R}^m$ is a C^∞ function, $B = \{x \big| (df)(x) = 0\}$. then $f(B)$ has measure zero in $\mathbb{R}^m$.*

Proof. Let $f = (f_1, f_2, \ldots, f_m)$

$$B_1 = \left\{x \big| (df_1)(x) = 0\right\}.$$

By prop 2. $f_1(B_1)$ has measure zero in $\mathbb{R}^1$ and clearly $B \subset B_1$. Hence $f(B) \subset f(B_1) \times \mathbb{R}^{m-1}$, so that $f(B)$ has measure zero in $\mathbb{R}^m$. In the proof of Theorem 1, we shall use the following ◻

Theorem (Fubini). *If F is a measurable set in $\mathbb{R}^p$, a point in $\mathbb{R}^p$ denoted by (x, y), $x \in \mathbb{R}^r, y \in \mathbb{R}^{p-r}$, $o < r < p$, then the set of $y \in \mathbb{R}^{p-r}$ such that $(c, y) \in F$ has measurable zero in $\mathbb{R}^{p-r}$ for almost all c if and only if F has measure zero i $\mathbb{R}^p$.*

Proof of theorem 1. Let $E_k = \{x | rank(df)(x) = k\}$. We have

$$A = \bigcup_{k \leq m} E_k.$$

If $a \in E_k, k < \circ$, then by a permutation of $\{f_i\}_{1 \leq i \leq m}$, we may suppose that $i f u = (f_1, \ldots, f_k)$, $(du)(a)$ has rank k. We can then find $v_{k+1}, \ldots, v_n, v_i$: $\Omega \to \mathbb{R}^1$, $k + 1 \leq i \leq n$, such that if w is defined by $w(x) = (f_1, (x), \ldots, f_k(x), v_{k+1}, \ldots, v_n(x))$, then $(dw)(a)$ is invertible. By 22
the inverse function theorem there exist neighbourhoods U and V of a and $w(a)$ respectively, such that $W|U \to V$ is a homeomorphism and $w|U$ and $w^{-1}|V$ are C^∞. We may further suppose that V is a cube in $\mathbb{R}^n$. Define g on V by

$$g(x) = f \circ w^{-1}(x).$$

If $u = (x_1, \ldots, x_k), v = (x_{k+1}, \ldots, x_n)$ we have $g(u, v) = (u, h(u, v))$ where

$$h : \mathbb{R}^n \to \mathbb{R}^{m-k} \text{ is a } C^\infty \text{map.}$$

Let $w(a) = (\alpha, \beta)$. $\alpha \in \mathbb{R}^k, \beta \in \mathbb{R}^{n-k}$. Then $(df)(a)$ has rank k

$$\begin{aligned} &\Leftrightarrow \quad (dg)(w(a)) \text{ has rank } k \\ &\Leftrightarrow \quad (d_2 h)(\alpha, \beta) \text{ has rank } 0. \end{aligned}$$

Let $F_k = g[w(E_k \cap U)] = f(E_k \cap U)$. If suffices to prove that F_k has measure zero. If V' is the projection of V on $\mathbb{R}^{n-k}$, define the map h_c: $V' \to \mathbb{R}^{m-k}$ by $h_c(v) = h(c, v)$, when $(c, v) \in V$. Let $W = \{v \in V^1 | (dh_c)(v) = 0\}$. We have

$$F_k \cap \{u = c\} = \{u = c\} \times \{h_c(W)\}.$$

h_c: $V' \to \mathbb{R}^{m-k}$ is a C^∞ function. Hence, by the corollary to Prop. 2, $h_c(W)$ has measure zero $\mathbb{R}^{m-k}$, i.e. the set of points $y \in \mathbb{R}^{m-k}$ such that $(c, y) \in F_k$, has measure in $\mathbb{R}^{m-k}$, for all c. Hence, by Fubini's theorem,
23 F_k has measure zero in $\mathbb{R}^m$ for every $k < m$ and this proves the theorem.

Definition. If f: $\Omega \to \mathbb{R}^m$ is a C^∞ map and $f = (f_1, f_2, \ldots, f_m)$, then $\{f_i\}_{i \leq i \leq m}$ are said to be functionally dependent if there exists an open set $\Omega' \supset f(\Omega)$, and a C^∞ map g: $\Omega' \to \mathbb{R}^1$ such that

(1) $g^{-1}(0)$ is nowhere dense in Ω'.

(2) $g \circ f = 0$

If g can be chosen real analytic, we say that $\{f_i\}$ are analytically dependent.

Lemma 3. *If E is any closed set in $\mathbb{R}^n$ then there exists a C^∞ function φ: $\mathbb{R}^n \to \mathbb{R}$ such that*

$$\{x \in \mathbb{R}^n | \varphi(x) = 0\} = E.$$

Proof. If E is closed, there exists $\{U_p\}_{p \geq 1}$, open sets in $\mathbb{R}^n$, such that $E = \bigcap_{p \geq 1} U_p$. There exist compact sets $\{K_m\}_{m \geq 1}$ in $\mathbb{R}^n$ such that

$$\bigcup_{m=1}^{\infty} K_m = \mathbb{R}^n \text{ and } K_p \subset K^0_{p+1}.$$

By the corollary to Theorem 1, 2, there exist $\varphi_p\colon \mathbb{R}^n \to \mathbb{R}$, C^∞ maps such that

$$\varphi_p(x) = \begin{cases} 0 & \text{for } x \in E \\ 1 & \text{for } x \in \mathbb{R}^n - U_p \end{cases} \tag{1}$$

and

$$0 \leq \varphi_p(x) \leq 1. \tag{2}$$

Consider $\|\varphi_p\|_p^{K_p} = \sum\limits_{|\alpha| \leq p} \sup\limits_{x \in K_p} |D^\alpha \varphi_p(x)|$. Each $\|\varphi_p\|_p^{K_p}$ is finite. Hence **24**
there exists a sequence (ε_p) of $+ve$ numbers such that

$$\sum_{p=1}^{\infty} \varepsilon_p \|\varphi_p\|_p^{Kp} < \infty. \tag{4.3}$$

Let f_m be defined by

$$f_m(x) = \sum_{p=1}^{m} \varepsilon_p \varphi_p(x).$$

If K is any compact set in $\mathbb{R}^n$, $K \subset K_r$ for some r. (4.3) implies in particular that for integer $m > r$,

$$\sum_{p>m} \varepsilon_p \|\varphi_p\|_p^K \leq \sum_{p>m}^{\infty} \varepsilon_p \|\varphi_p\|_p^{K_p} < \infty.$$

Hence $\{f_m\}$ is a Cauchy sequence in C^∞, and by the completeness of C^∞ [Theorem 1, §1], f_m converges to a function φ, in C^∞. Clearly φ has the required properties. □

Theorem 2. *If $f\colon \Omega \to \mathbb{R}^m$ is a C^∞ map where $f = (f_1, f_2, \ldots, f_m)$, then $\{f_i\}_{1 \leq i \leq m}$, are functionally dependent on every compact subset of Ω if and only if rank $(df)(x) < m$ for $x \in \Omega$.*

Proof. If $\{f_i\}$ are functionally dependent on the compact set K, let $f = \mathbb{R}^m \to \mathbb{R}$ be a C^∞ map such that $g \circ f = 0$ and $g^{-1}(0)$ nowhere dense in $\mathbb{R}^m$. clearly $f(K) \subset g^{-1}(0)$is nowhere dense. If rank $(df)(x) = m$ for some $x \in \overset{\circ}{K}$, then rank $(df)(x) = m$ in an open neighbourhood $U \subset \overset{\circ}{K}$ of x and by the rank theorem $f|U$ is open, so that $f(U)$ cannot be nowhere dense. □

25 Conversely if rank $(df)(x) < m$ for $x \in \Omega$, then by Theorem 1, for any subset K of Ω, $f(K)$ has measure zero in $\mathbb{R}^m$. Hence $f(K)$ is nowhere dense in $\mathbb{R}^m$. Also K being compact, $f(K)$ is closed in $\mathbb{R}^m$. Hence by the above, lemma, there exists a C^∞ function g: $\mathbb{R}^m \to \mathbb{R}$ such that $g^{-1}(0) = f(K)$, so that $g \circ f = 0$ on K.

Only a somewhat weaker statement is true of analytic dependence.

Theorem 2′. *If f: $\Omega \to \mathbb{R}^m$ is an analytic map, and if rank $df(x) < m$ at every point of Ω, then there exists a nowhere dense closed set $E \subset \Omega$ such that for any $a \in \Omega - E$, there exists a neighbourhood U of a, $U \subset \Omega$, such that $f_i|U$ are analytically dependent.*

Proof. We may suppose that Ω is connected. Let $p =$ max. rank $(df)(x)$, and let $b \in \Omega$ be such that $p = rank(df)(b)$. This means that there exist $i_1, \ldots i_p$, and $j_1, j_2, \ldots, j_p$, such that if we set $h(x) = \det\left|\dfrac{\partial f_{i_r}}{\partial x_{j_s}}\right|$, we have $h(b) \neq 0$. Let $E = \{x \in \Omega | h(x) = 0\}$. Since h is analytic in Ω and $\not\equiv 0$, E can contain no open set, and so is nowhere dense. □

Now clearly rank (df) $(x) = p$ for $x \in \Omega - E$. By the rank theorem, given $a \in \Omega - E$, there exist neighbourhoods U of a, V of $f(a)$,cubes Q_1, in $\mathbb{R}^n$, Q_2 in $\mathbb{R}^m$ and analytic homeomorphisms u_1: $Q_1 \to U$, u_2: $V \to Q_2$ such that $u_2 \circ f \circ u_1$ is the map which sends $(y_1, y_2, \ldots, y_n)$ into the point $(y_1, \ldots, y_p, \ldots, 0)$. If $u_2 = (u^{(1)}, \ldots, u^{(m)})$, and we take

$$g = u^{(r)}, r > p,$$

then $g \circ f = 0$ on U.

26 **Example.** If $\varphi(z)$ is an entire function of the complex variable z, not a

polynomial, and real on the real axis, (e.g. $\varphi(z) = e^z$), consider the map $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$f(x_1, x_2) = (x_1, x_1x_2, x_1 f(x_2))$$

It can be shown that there does not exist any analytic function $g \not\equiv 0$ in a neighbourhood of $0 \in \mathbb{R}^3$ with $g \circ f = 0$ in a neighbourhood of $0 \in \mathbb{R}^2$.

5 E. Borel's theorem and approximation theorems

Notation. If $f \in C^\infty$, $T(f)$ will denote the formal power series $\sum\limits_{|\alpha|<\infty} \dfrac{f^\alpha(0)}{\alpha!} x^\alpha$ and $T^m(f)$ will denote the polynomial $\sum\limits_{|\alpha|\le m} \dfrac{f^\alpha(0)}{\alpha!} x^\alpha$.

Definitions. (1) If $f \in C^k(\Omega)$ and if E is a closed subset of Ω, then f is said to be $m-$flat on E, $(m \le k)$, if $D^\alpha f(x) = 0$ for $x \in E$ and $|\alpha| \le m$.

(2) If $f \in C^\infty(\omega)$, E is a closed subset of Ω and if f is $m-$flat on E for every positive integer m, then f is said to be flat on E.

Lemma 1. *If $f \varepsilon C^\infty(\mathbb{R}^n)$ and if f is m-flat at* 0, *given $\varepsilon > 0$, there exists $g \varepsilon C^\infty(\mathbb{R}^n)$ such that $g(x) = 0$ in a neighbourhood of* 0 *and $\|g - f\|_m^{\mathbb{R}^n} < \varepsilon$.*

Proof. By the corollary to Theorem 1, 2, there exists a C^∞ function $k\colon \mathbb{R}^n \to \mathbb{R}$, such that

$$k(x) \begin{cases} = 0 & \text{for } |x| \le \dfrac{1}{2} \\ = 1 & \text{for } |x| \ge 1 \end{cases}$$

and $$k(x) \ge 0.$$

Let $g_\delta(x) = k\left(\dfrac{x}{\delta}\right) f(x)$ for $\delta > 0$. It is enough to prove that for each **27**
α, $|\alpha| \le m$,

$$|(D^\alpha f_\delta)(x) - D^\alpha f(x)| \to 0 \text{ uniformly on } \mathbb{R}^n \text{ as } \delta \to 0.$$

Now we have

$$\sup_{x\varepsilon\mathbb{R}} |(D^\alpha g_\delta)(x) - (D^\alpha f)(x)| = \sup_{|x|\leq\delta} |(D^\alpha g_\delta)(x) - (D^\alpha f)(x)|$$

and since f is m-flat at 0.

$$\sup_{|x|\leq\delta} |(D^\alpha f)(x)| \to 0, \text{ as } \delta \to 0, \text{ for } |\alpha| \leq m.$$

By Leibniz' formula.

$$D^\alpha g_\delta(x) = \sum_{\mu+\nu=\alpha} \binom{\alpha}{\nu} \delta^{-|\nu|} (D^\nu.k)(\frac{x}{\delta})(D^\mu.f)(x).$$

For each ν, there exists a constant M_ν , such that $|(D^\nu k)(x)| \leq M_\nu$. Hence

$$\left|(D^\alpha g_\delta)(x)\right| \leq \sum_{\mu+\nu=\alpha} M_\nu \binom{\alpha}{\nu} \delta^{-|\nu|} |(D^\mu f)(x)|$$

now $(D^\mu f)(x)$ is $(m - |\mu|)$ flat at 0. Therefore,

$$|(D^\mu f)(x)| = \circ(|x|^{m-|\mu|}) \text{ as } x \to 0$$

so that $\sup_{|x|\leq\delta} |(D^\mu I)(x)| = \circ(\delta^{m-|\mu|})$ and

$$\begin{aligned} \delta^{-|\nu|}|D^\mu f(x)| &= \circ(\delta^{m-|\mu|-|\nu)}) \\ &= \circ(1). \end{aligned}$$

28 Hence for $|\alpha| \leq m$, $(D^\alpha g_\delta)(x) \to 0$ uniformly as $\delta \to 0$ i.e., $||g_\delta - f||_M \mathbb{R}^n \to 0$ as $\delta \to 0$ Q.E.D. □

Note that the function g in the above lemma is m in particular, flat at 0.

Theorem 1 (E. Borel)**.** *Given an arbitrarily family* $\{C_\alpha\}$ *of constants there exists* $f \varepsilon C^\infty(\mathbb{R}^n)$ *such that* $T(f) = \sum_{|\alpha|<\infty} C_\alpha x^\alpha$, *i.e,* $\dfrac{D^\alpha f(0)}{\alpha!} = C_\alpha$ *for all* α.

Proof. Let $\sum_{|\alpha|\leq m} C_\alpha x^\alpha = P_m(x)$.

By the lemma above there, exists $g_m \varepsilon C^\infty$, flat at 0, such that

$$\|P_{m+1} - P_m - g_m\| < 2^{-m}.$$

Clearly because of the completeness of C^∞

$$f = P_\circ + \sum_{m=0}^{\infty}(P_{m+1} - P_m - g_m)\varepsilon C^\infty,$$

and, for any k, $\sum_{m\geq k}(P_{m+1} - P_m - g_m)$ is k-flat at 0. Hence

$$T^k(f) = T^k\left(P_0 + \sum_0^{k-1}(P_{m+1} - P_m - g_m)\right) = P_k.$$

This theorem of Borel is a very special case important theorems of H. Whitney [46] on differentiable functions on closed sets. We state, without proof, his main theorem in this direction. A simplified version of his proof is container in the paper [12] of G. Glaeser. A systematic account of this circle of ideas will be found in a forthcoming book B. **29**
Malgrange [26] on ideals of differentiable functions. □

Extension theorem of Whitney

Part 1. Let k be an integer > 0, Ω open in $\mathbb{R}^n$ and E a closed subset of Ω. To every n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of nonnegative integers with $|\alpha| \leq k$, suppose given a continuous function f_α on E. Then there exists $f \varepsilon C^k(\Omega)$ with $D^\alpha f|E = f_\alpha$ for $|\alpha| \leq k$ if and only if for any α, $|\alpha| \leq k$, we have

$$f_\alpha(x) = \sum_{|\beta|\leq k-|\alpha|} \frac{f_\alpha + \beta(y)}{\beta!}(x-y)^\beta + \circ(|x-y|^{k-|\alpha|})$$

uniformly for x, y in any compact subset of E, as $|x-y| \to 0$.

Part 2. Given a continuous function f_α on E for all n- tuples α, there exists $f \varepsilon C^\infty(\Omega)$ with

$$D^\alpha f \Big| E = f_\alpha \text{ for } all \;\; \alpha,$$

if and only if we have for any integer $k > 0$ and any compact $K \subset E$,

$$f_\alpha(x) = \sum_{|\beta| \leq k} \frac{f_\alpha + \beta(y)}{\beta!}(x-y)^\beta + \circ(|x-y|^k)$$

uniformly as $|x - y| \to 0$, $x, y \varepsilon K$

Borel's theorem is the special case of this second part in which E reduces to a single point.

Theorem 2 (Weierstrass)**.** *If* $f \varepsilon C^k(\Omega)$, $0 \leq k < \infty$, *given a compact subset* K *of* Ω *and* $\varepsilon > 0$, *there exists a polynomial* $p(x_1, \ldots, x_n)$ *such that* $\| f - p \|_k^k < \varepsilon$.

30 *Proof.* Without loss of generality we may assume that f has compact support. □

For $\lambda > 0$, define $g_\lambda(x)$ by

$$g_\lambda(x) = c\lambda^{n/2} \int_{\mathbb{R}^n} f(y) e^{-\lambda \|x-y\|^2} dy, \tag{5.1}$$

where c is the constant given by

$$c \int_{\mathbb{R}^n} e^{-\|x\|^2} dx = 1.$$

Then obviously $c\lambda^{n/2} \int_{\mathbb{R}^n} e^{-\lambda \|x\|^2} dx = 1$. We shall show that $\|g_\lambda - f\|_k^K \to 0$ as $\lambda \to \infty$. By uniform convergence of the integral in (5.1) and by a suitable change of variable, we have,

$$D^\alpha g_\lambda(x) = c\lambda^{n/2} \int_{\mathbb{R}^n} (D^\alpha f)(y) e^{-\lambda \|x-y\|^2} dy.$$

Hence $D^\alpha g_\lambda(x) - D^\alpha f(x) = c\lambda^{n/2} \int\limits_{\mathbb{R}^n} [D^\alpha f(y) - D^\alpha f(x)] e^{-\lambda\|x-y\|^2}\, \mathrm{dy}.$

Given $\varepsilon > 0$, there exists $\delta > 0$ such that

(5.2) $$|(D^\alpha f)(y) - (D^\alpha f)(x)| < \varepsilon/2 \text{ for } \|x - y\| \leq \delta.$$

Since f has compact support and $f \varepsilon C^k$, there exists a constant M such that for any α, $|\alpha| \leq k$,

(5.3) $$|D^\alpha f(y)| < M.$$

By (5.2) and (5.3) 31

$$\begin{aligned}
&\left|(D^\alpha g_\lambda)(x) - (D^\alpha f)(x)\right| \\
&= \left| c\lambda^{n/2} \int\limits_{\|x-y\|<\delta} [D^\alpha f(y) - D^\alpha f(x)]\, e^{-\lambda\|x-y\|^2} dy + c\lambda^{n/2} \right. \\
&\qquad\qquad \left. \int\limits_{\|x-y\|\geq\delta} [D^\alpha f(y) - D^\alpha f(x)] e^{-\lambda\|x-y\|^2} dy \right| \\
&\leq \varepsilon/2 c\lambda^{n/2} \int\limits_{\mathbb{R}^n} e^{-\lambda\|x-y\|^2} dy + 2M.C.\lambda^{n/2} \int\limits_{\|x-y\|\geq\delta} e^{-\lambda\|x-y\|^2} dy \\
&\leq \varepsilon/2 + 2Mc\lambda^{n/2} e^{-\lambda\frac{\delta^2}{2}} \int\limits_{\|x-y\|\geq\delta} e^{-\frac{1}{2}\lambda\|x-y\|^2} dy.
\end{aligned}$$

The product $c\lambda^{n/2} \int\limits_{\mathbb{R}^n} e^{-\frac{\lambda}{2}\|x-y\|^2} dy = 2^n$ and $\lambda^{\frac{n}{2}} e^{-\lambda\frac{\delta^2}{2}} \to 0$ as $\lambda \to \infty$. Hence we have $|(D^\alpha g_\lambda)(x) - (D^\alpha f)(x)| \to 0$ uniformly as $\lambda \to \infty$ for $|\alpha| \leq k$; i.e.

$$\|g_\lambda - f\|_k^K \to 0 \text{ as } \lambda \to \infty.$$

Choose λ_0 such that

$$\|g_{\lambda_0} - f\|_k^K < \varepsilon/2.$$

Now, $$e^{-\lambda_0\|x-y\|^2} = \sum_{p=0}^{\infty}\frac{(-\lambda_0)^p}{p!}\|x-y\|^{2p}.$$

If we set $Q_N(x,y) = \sum\limits_{p=0}^{N}\frac{(-\lambda_0)^p}{p!}\|x-y\|^{2p}$, then $D^\alpha_x Q_N(x,y) \to D^\alpha_x$
32 $e^{-\lambda_0\|x-y\|^2}$ as $N \to \infty$, uniformly for x, y in a compact set. Hence, if

$$P_N(x) = c\lambda_0^{n/2}\int f(y)Q_N(x,y)dy,$$

then P_N is a polynomial and $\|f - P_N\|^K_k \to 0$ for any compact set K.

Corollary 1. *If Ω_i is open in $\mathbb{R}^{n_i}$, $i = 1, 2$ then the finite linear combinations $\sum\limits_{\mu,\nu}\varphi(x_i)\psi_\gamma(x_2)\left\{x_i \text{ denoting a general point in } \mathbb{R}^{n_i}\right\}$ where $\varphi_\mu(x_1)$ is C^∞ in Ω_1, $\psi_\gamma(x_2)$ in Ω_2, are dense in the space $C^k(\Omega_1\times\Omega_2)$.*

Since the topology on $C^k(\Omega_1\times\Omega_2)$ involves only approximation on compact sets, by multiplying φ_μ, ψ_ν by suitable functions with compact support we obtain

Corollary 2. *With the notation as in Cor. 1. the finite linear combinations $\sum\varphi_\mu(x_1)\psi(x_2)$, where the φ_μ, ψ_ν are C^∞ functions with compact support in Ω_1, Ω_2 respectively, are dense in $C^k(\Omega_1\times\Omega_2)$.*

Theorem 3 (Whitney)**.** *If Ω is an open set in $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a C^k map $(0 \le k \le \infty)$ then for any continuous function $\eta > 0$ on Ω, there exists an analytic function g in Ω such that for any $x\varepsilon\Omega$, we have $|D^\alpha f(x) - D^\alpha g(x)| < \eta(x)$ for $0 \le |\alpha| \le \min\left(k, \dfrac{1}{\eta(x)}\right)$.*

If K_p is any sequence of compact subsets of Ω, $K_p \subset K^\circ_{p+1}$, $\cup K_p = \Omega$ and if $\varepsilon_p > 0$, there exists a continuous function η on Ω with $\eta(x) < \varepsilon_p$ *on* $K_{p+1} - K_p$. Consequently, Theorem 3 is equivalent with the following

33 **Theorem 3′.** *If Ω is an open set in $\mathbb{R}^n$; $f : \Omega \to \mathbb{R}^1$ is C^k, $0, \le k \le \infty$, and if $\{K_p\}$ are compact subsets of Ω such that $\bigcup\limits_{p\ge1} K_p = \Omega$ $K_0 = \phi$ and $K_p \subset K^\circ_{p+1}$ then given a sequence $\{\varepsilon_p\}$ of positive numbers $\varepsilon_p \downarrow 0$, and*

a sequence $\{m_p\}$ *of non-negative integers with* $0 \leq m_p \leq k$, *there exists an analytic function* $g\colon \Omega \to \mathbb{R}$ *such that* $\| f - g\|_{m_p}^{K_{p+1}-K_p} > \varepsilon_p$ *for every* $p \geq 0$.

Proof. We may assume that $m_{p+1} \geq m_p$ for $p \geq 1$. Using Leibniz' formula we see at once that there is a sequence $\{C_p\}$ of numbers $C_p \geq 1$, such that for $\varphi, \psi \varepsilon C^{m_p}(\Omega)$ and say subset E of Ω, we have

$$\|\varphi\psi\|_{m_p}^E \leq C_p \|\phi\|_{m_p}^E \|\psi\|_{m_p}^E.$$

By Theorem 1, §2, there exist functions $\varphi_p \varepsilon C^\infty(\Omega)$, such that

$$\begin{aligned} \varphi_p \quad & \text{has compact support in } \Omega, \\ \varphi_p(x) &= 0 \text{ for } x \text{ in a neighbourhood of } K_{p-1} \\ &= 1 \text{ for } x \text{ is a neighbourhood of } (\overline{K_{p+1} - K_p}). \end{aligned}$$

Let $M_p = \|\varphi_p\|_{m_p} + 1$. Choose a sequence $\{\delta_p\}$ of positive numbers $\delta_p \downarrow 0$ such that

$$\sum_{q \geq p} C_m M_{q+1} \delta < \frac{1}{4} \varepsilon_p \text{ for all } p \geq 0. \tag{5.4}$$

For a continuous function f, $I_\lambda(f)$ will denote the function with $I_\lambda(f)(x) = c\lambda^{n/2} \int\limits_{\mathbb{R}^n} f(y) e^{-\|x-y\|^2}$ by where c is chosen so that $c \int\limits_{\mathbb{R}^n} e^{-\|x\|^2}$ $dx = 1$. By theorem 2, we may choose λ_0 such that, if $g_0 = I_{\lambda_0}(\varphi_0 f)$, **34**

$$\|g_0 - \varphi_0 f\|_{m_0}^{K_1} < \delta_0.$$

For $p \geq 1$, let

$$g_p = I_{\lambda_p}\left[\varphi_p\left(f - \sum_0^{p-1} g_i\right)\right]$$

where λ_p is so chosen that

$$\|g_p - \varphi_p\left(f - \sum_0^{p-1} g_i\right)\|_{m_p}^{K_{p+1}} < \delta_p. \tag{5.5}$$

Note that, for $p \geq 1$, λ_p can be chosen to be any number $>$ a constant l_p depending only on $\lambda_0, \ldots, \lambda_{p-1}$. The inequality (5.5) implies, in particular, that

$$\|g_p\|_{m_p}^{K_{p+1}} < \delta_p \tag{5.6}$$

and

$$\|f - \sum_{0}^{p} g_p\|_{m_p}^{K_{p+1}-K_p} < \delta_p \tag{5.7}$$

Consequently, (5.5), with p replaced by $p+1$, implies that

$$\begin{aligned}
\|g_{p+1}\|_{m_p}^{K_{p+1}-K_p} &\leq \|\varphi_{p+1}\left(f - \sum_{0}^{p}\right) g_q\|_{m_p}^{K_{p+1}-K_p} + \delta_{p+1} \\
&\leq C_p \|\varphi_{p+1}\|_{m_p} \|\left(f - \sum_{0}^{p} g_q\right)\|_{m_p}^{K_{p+1}-K_p} + \delta_{p+1} \\
&\leq C_p M_{p+1} \delta_p + \delta_{p+1} \leq 2\delta_p C_p M_{p+1};
\end{aligned}$$

35 also $\|g_{p+1}\|_{m_p}^{K_p} \leq \delta + p + 1$.

Hence

$$\|g_{p+1}\|_{m_p}^{K_{P+1}} \leq 2\delta_p C_p M_{p+1}$$

i.e.,

$$\|\sum_{p+1}^{\infty}\|_{m_p}^{K_{p+1}} \leq 2 \sum_{q>p} \delta_q C_p M_{q+1} < \frac{1}{2}\varepsilon_p.$$

Hence by the completeness of C^k,

$$g = \sum_{0}^{\infty} g_q \varepsilon C^{m_p}$$

and $\|f - g\|_{m_p}^{k_{p+1}-K_p} \leq \|f - \sum_{\circ}^{p} g_1\|_{m_p}^{K_{p+1}-K_p} + \|\sum_{p+1}^{\infty} g_i\|_{m_p}^{K_{p+1}-K_p} < \delta_p + \frac{1}{2}\varepsilon_p < \varepsilon_p.$

Now we shall prove that g is analytic if the λ_p are suitably chosen. By definition,

$$g_q(x) = c\lambda_q^{n/2} \int\limits_{\Omega_q} (y) \left[f(y) - \sum_0^{q-1} g_i(y) \right] e^{-\lambda_q \|x-y\|^2} dy$$

and φ_q has compact support. Hence g_q is analytic for such each q. Let $2\mu_p = d(K_p, \Omega - K_{p+1})$; clearly $\mu_p >$. There is an open set U_p in $\mathbb{C}^m$, $U_p \supset K_p$ such that if $z \varepsilon U_p$, $y \varepsilon \Omega - K_{p+1}$, then

$$Re\left[(z_1 - y_1)^2 + \cdots + (z_n - y_n)^2 \right] > \mu_p.$$

□

For any q, define 36

$$g_q(z) = c\lambda_q^{n/2} \int\limits_{\Omega} \phi_q(y) \left[f(y) - \sum_{r=0}^{q-1} g_r(y) \right] e^{-\lambda_q [(z_1-y_1)^2 + \cdots + (z_n-y_n)^2] dy}$$

Since ϕ_q has compact support, g_q is an entire function of $z_1, \ldots, z_n$. Further, for $q > p+1$, the integral defining g_q may be replaced by $\int\limits_{\Omega - K_{p+1}}$ since $\varphi_q = 0$ on K_{p+1}; hence

(5.8) $$\left| g_q(z) \right| \leq c\lambda_q^{n/2} H_q e^{-\lambda_q \mu_p}, \quad \text{for } q > p + 1, z \varepsilon U_p;$$

here H_q is a constant depending only on $\lambda_o, \ldots, \lambda_{q-1}$. We can choose, by induction, λ_q such that $\lambda_q > l_q$ (the constant depending on $\lambda_o, \ldots, \lambda_{q-1}$ which is involved in the validity of the inequality (5.5)) and such that the series.

$$\sum \lambda_q^{n/2} H_q e^{-\lambda_q \mu} < \infty \text{ for any } \mu > 0.$$

[It suffices, e.g. to choose λ_q such that $\lambda_q^{n/2} H_q(\lambda_0, \ldots \lambda_{q-1}) e^{\frac{-\lambda_q}{q}} < \frac{1}{q^2}$.]

For this choice of the sequence λ_q, the inequality (5.8) implies that the series $\sum g_q(z)$ converges uniformly for $z \varepsilon U_p$; hence the sum is holomorphic in U_p for any p. Since g is the restriction of this sum to Ω, g is real analytic in Ω.

We shall now consider analogues of these theorems for approxima-
37 tion by polynomials in complex variables. Clearly, since a uniform limit of holomorphic functions is holomorphic, we can at best hope to approximate *holomorphic* functions by polynomials. But there are geometric and analytic conditions on an open set U in the space $\mathbb{C}^n$ in order that any holomorphic function on U be approximable by polynomials.

Definition. An open set $U \subset \mathbb{C}^n$ is called a Runge domain if every holomorphic function f on U can be approximated by polynomials, uniformly on every compact subset of U.

The following theorem is contained in a general approximation theorem which we shall prove in Chap. *III*. For a simple direct proof based on Cauchy's integral formula (the original proof of Runge) see e.g [4].

Theorem (Runge). *An open connected set U in the complex plane is Runge domain if and only if U is simply connected.*

Let U be an open set in $\mathbb{C}^n$ and $\alpha\colon U \to \mathbb{R}$, a continuous function such that $\alpha(z) > 0$. Let dv denote Lebesgue measure in $\mathbb{C}^n$, and let $A(\alpha)$ denote the set of holomorphic functions f on U for which $\int |f(z)^2 \alpha(z) dv < \infty$.

Lemma 1. *For $f, g \varepsilon A(\alpha)$, set $(f, g) = \int f(z)\overline{g(z)}\alpha(z) dv$. Then $A(\alpha)$ is a Hilbert space with the inner product (f, g).*

Proof. In view of the completeness of the space $L^2(\alpha; dv)$ it suffices to prove that if $f_p \varepsilon A(\alpha)$ and

$$\int\limits_U |f_p(z) - f_q(z)|^2 \alpha(z) dv \to 0 \text{ as } p, q \to \infty,$$

38 then f_p converges uniformly on compact subsets of U. Since α is bounded below by a positive constant on any compact subset of U, this assertion follows from the following □

Lemma 2. *If $\{f_p\}$ is a sequence of holomorphic functions such that $\int\limits_U |f_p - f_q|^2 dv \to 0$ as $p, q \to \infty$, then f_p is uniformly convergent on every compact subset of U.*

Proof. If $g(z)$ is holomorphic in a neighbourhood of the closed disc $|z - a| \leq \rho$ in the plane it follows from Cauchy's integral formula that

$$g(a) = \frac{1}{\pi\rho^2} \int\limits_{|z-a|\leq\rho} g(a+z) dv.$$

□

Applying this n times, we find that if $h(z_1, \ldots, z_n)$ is holomorphic in a neighbourhood of the set $|z_1 - a_1| \leq \rho, \ldots, |z_n - a_n| \leq \rho$, then

$$h(a) = \frac{1}{(\pi\rho^2)^n} \int\limits_{|z-a|\leq\rho} h(a+z) dv.$$

Let K be a compact subset of U and let $\rho > 0$ be so small that the set $K_\rho = \{z\varepsilon\mathbb{C}^n \big| \exists a\varepsilon K \text{ with } |z-a| \leq \rho\}$ is compact in U. Then, for $a\varepsilon K$, if f is holomorphic in U,

$$\big|f(a)\big|^2 = \frac{1}{(\pi\rho^2)^n} \Big| \int\limits_{|z-a|\leq\rho} (f(a+z))^2 dv \Big|$$

so that

$$\sup_{a\varepsilon K} \big|f(a)|^2 \leq \frac{1}{(\pi\rho^2)^n} \int\limits_{K_\rho} |f(z)|^2 dv.$$

Lemma 2 follows if we apply this inequality to the differences $f_p - f_q$. 39

Let φ_ν be a complete orthonormal system in $A(\alpha)$.Then we have, for any $f\varepsilon A(\alpha)$, $f = \sum C_\nu \varphi_\nu$ where $C_\nu = (f, \varphi_\nu)$ and the series converges in the Hilbert space $A(\alpha)$. From Lemma 2 we deduce

Lemma 3. *If $\{\varphi_\nu\}$ is a complete orthonormal system in $A(\alpha)$, then any $f\varepsilon A(\alpha)$ can be approximated, uniformly on compact subsets of U, by finite (complex) linear combinations $\sum\limits_{\nu=1}^{p} C_\nu \rho_\nu$.*

Proposition. *If U is an open set in $\mathbb{C}^n$, V an open set in $\mathbb{C}^m$ and α: $U \to \mathbb{R}$, β: $V \to \mathbb{R}$ are positive continuous functions and if $\{\varphi_\nu\}$, $\{\psi_\mu\}$ are complete orthonormal systems in the Hilbert spaces $A(\alpha)$ and $A(\beta)$ respectively, then $\{\varphi_\nu \psi_\mu\}$ is a complete orthonormal system in $A(\alpha \times \beta)$ where $\alpha \times \beta$: $U \times V \to \mathbb{R}$ is defined by*

$$(\alpha \times \beta)(z, w) = \alpha(z)\beta(w).$$

Proof. We have only to show that $\{\varphi_\nu \psi_\mu\}$ form a complete system in $A(\alpha \times \beta)$. □

Let $f(z, w) \varepsilon A(\alpha)$ be such that

$$\int f(z, w)\alpha(z)\beta(w)\overline{\varphi_\nu(z)}\ \overline{\psi_\mu(w)} dv = 0$$

for each ν and μ, dv, Lebesgue measure in $\mathbb{C}^{n+m}$.

We have to show that $f(z, w) = 0$. Let dv_z, dv_w be the Lebesgue measures in $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively. If we show that for any μ, the
40 integral $g(z) = g^{(\mu)}(z) = \int f(z, w)\beta(w)\overline{\psi_\mu(w)} dv_w$, which exists for almost all z, defines a function in $A(\alpha)$, the proof follows immediately from the completeness of $\{\varphi_\nu\}$ and $\{\psi_\mu\}$. Let K_p be compact subsets of V such that $\bigcup K_p = V$ and $K_p \subset K_{p+1}$.

Define $g_p(z)$ by

$$g_p(z) = \int_{K_p} f(z, w)\beta(w)\overline{\psi_\mu(w)} dv_w.$$

Then g_p is holomorphic in U. We have for $q > p$,

$$g_q(z) - g_p(z) = \int_{K_q - K_p} f(z, w)\beta(w)\overline{\psi_\mu(w)} dv_w.$$

By Schwarz's inequality,

$$|g_q(z) - g_p(z)|^2 \leq \int_{K_q - K_p} |f(z, w)|^2 \beta(w) dv \int_{K_q - K_p} |\psi_\mu(w)|^2 \beta(w) dv_w$$

$$\leq \int_{K_q - K_p} |f(z,w)|^2 \beta(w) dv, \text{ since} ||\psi_\mu|| = 1 \text{ in } A(\beta)$$

Hence

$$\int_U |g_q(z) - g_p(z)|^2 \alpha(z) dv_z \leq \int_{U \times (K_q - K_p)} |f(z,w)|^2 \alpha(z) \beta(w) dv_w$$

and
$$\int_{U \times (K_q - K_p)} |f(z,w)|^2 \alpha(z) \beta(w) dv_w \to 0 \text{ as } p, q \to \infty$$

since $f \varepsilon A(\alpha \times \beta)$.

Hence $\int_K |g_q(z) - g_p(z)|^2 dv_z \to 0$ as p, $q \to \infty$ for any compact **41**
subset of U and, by Lemma 2, g_q converges uniformly to a holomorphic function $g(z)$. Further we clearly have
$\int_U |g_p(z)|^2 \alpha(z) dv_z \leq \int_{U \times V} |f(z,w)|^2 \alpha(z) \beta(w) dv$, so that $g \varepsilon A(\alpha)$, and proposition is proved.

Theorem 4. *If U is an open set in $\mathbb{C}^n$, V an open set in $\mathbb{C}^m$, the linear combinations $\sum \varphi_i(z) \psi_j(w)$, where φ_i and ψ_j are holomorphic functions on U and V respectively, are dense in the space of holomorphic functions on $U \times V$ (with the topology of uniform convergence on compact sets).*

Proof. Let $f(z,w)$ be a holomorphic function on $U \times V$. Since f is continuous on $U \times V$ there exists a positive continuous function η: $U \times V \to \mathbb{R}$ such that $f \varepsilon A(\eta)$, i.e. $\int_{U \times V} |f|^2 \eta dv$ is finite. Let K_p, L_q be compact subsets of U and V respectively such that $\bigcup_{p \geq 1} K_p = U$ and $\bigcup_{q \geq 1} L_q = V$ and $K_p \subset \mathring{K}_{p+1}$, $L_q \subset \mathring{L}_{q+1}$. Then

$$\bigcup_{p \geq 1} (K_p \times L_p) = U \times V.$$

□

There exist positive numbers ε_p such that $\eta(z,w) \geq \varepsilon_p > 0$ on $K_p \times L_p$ and $\varepsilon_p \leq 1$. There exist positive continuous functions α and β on U and V respectively such that

$$\alpha(z) \leq \varepsilon_p \text{ for } z \text{ in } (K_p - K_{p-1})$$

and $$\beta(w) \leq \varepsilon_p \text{ for } w \text{ in } (L_p - L_{p-1}).$$

42 This is easily deduced from Theorem 1. §2. Now

$$\{K_p \times L_p\} - \{K_{p-1} \times L_{p-1}\} = \{K_p \times (L_p - L_{p-1}\} \bigcup \{K_p - K_{p-1}) \times L_p\}.$$

It follows trivially that $\alpha(z)\beta(w) \leq \varepsilon_p \leq \eta(z,w)$ for $(z,w)\varepsilon(K_p \times L_p - K_{p-1} \times L_{p-1})$ for each p i.e. $\eta(z,w) \geq \alpha(z)\beta(w)$ for $(z,w)\varepsilon U \times V$. Hence $f \varepsilon A(\alpha \times \beta)$.

If $\{\varphi_\nu\}$ and $\{\psi_\mu\}$ form complete orthonormal systems of $A(\alpha)$ and $A(\beta)$ respectively, then by the last proposition, $\{\varphi_\nu \psi_\mu\}$ form a complete orthonormal system of $A(\alpha \times \beta)$; by Lemma 3 the finite linear combinations $\sum C_{\nu\mu}\varphi_\nu(z)\psi_\mu(w)$ approximate f uniformly on compact subsets of $U \times V$. q.e.d

Corollary. *If U is Runge in $\mathbb{C}^n$ and V is Runge in $\mathbb{C}^m$, then $U \times V$ is Runge in $\mathbb{C}^{n+m}$; in particular, if $U_1, \ldots, U_n$ are simply connected plane domains, then $U_1 \times \cdots \times U_n$ is Runge in $\mathbb{C}^n$.*

We shall deal with deeper properties of Runge domains in $\mathbb{C}^n$ later.

6 Ordinary differential equations

Lemma 1. *If I is an interval, containing 0, in $\mathbb{R}$ and $w \colon I \to \mathbb{R}$ is a continuous map such that $w(t) \geq 0$ and if $w(t) \leq M \int_0^t w(s)ds + \eta$, then $w(t) \leq \eta e^{Mt}$.*

43 *Proof.* We have, for $t \geq 0$,

$$e^{Mt}\frac{d}{dt}\left\{e^{-Mt}\int_0^t w(s)ds\right\} = w(t) - M\int_0^t w(s)ds \leq \eta.$$

hence $$\frac{d}{dt}\left\{e^{-Mt}\int\limits_0^t w(s)ds\right\} \leq \eta e^{-Mt}$$

i.e. $$\int\limits_0^t w(s)ds \leq \eta\frac{\{1-e^{-Mt}\}}{M}e^{Mt}.$$

□

Theorem 1. *Let Ω and Ω' be open sets in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, I an open interval in $\mathbb{R}^1$ with $0\varepsilon I$, $f : \Omega\times I\times\Omega' \to \mathbb{R}^n$ a continuous map. We denote a point in $\Omega\times I\times\Omega'$ by (x,t,α). If f is uniformly Lipschitz with respect to x on every subset $K\times I\times K'$ of $\Omega\times I\times\Omega'$, K, K' being compact subsets of Ω and Ω' respectively, then given $x_0\varepsilon\Omega$, there exists an interval $I_0 = \{t\big||t| < \varepsilon\}$, $\varepsilon > 0$ and a unique continuous map x: $I_\circ\times K' \to \Omega$ such that*

$$f(x(t,\alpha),t,\alpha) = \frac{\partial x}{\partial t}(t,\alpha) \tag{6.1}$$

and

$$x(0,\alpha) = x_0. \tag{6.2}$$

Further if the condition that f is Lipschitz is replaced by the (stronger) condition that $f\varepsilon C^k(\Omega\times I\times\Omega')$, $1\leq k\leq\infty$, then $x\varepsilon C^k(I_0\times K')$.

Proof. Let M be the Lipschitz constant, i.e.

$$||f(x,t,\alpha) - f(y,t,\alpha)|| \leq M||x-y|| \text{ for } x,y\varepsilon K \text{ and } \alpha\varepsilon K'.$$

□

Consider $\Omega_0 = \{x\big|||x-x_0|| \leq r\} \subset \Omega$ and let $\Omega_0 \subset K$. Clearly $|f|$ is bounded on $\Omega_0\times I\times K'$, say by C. Let $\varepsilon' > 0$ be such that **44**

$$\{t\big||t| < \varepsilon'\} \subset I$$

Let $I_0 = \left\{t \middle| |t| < \varepsilon, \varepsilon = \min(\varepsilon', \frac{r}{c}\right\}$. For $n \geq 0$, define functions x_n: $I_0 \times K' \to \Omega_0$ by $x_0(t, \alpha) = x_0$

$$x_n(t, \alpha) = x_0 + \int_0^t f(x_{n-1}(\tau, \alpha), \tau, \alpha) d\tau. \tag{6.3}$$

It is easily seen, by induction, that $x_n(t, \alpha) \varepsilon \Omega_0$ and that $||x_n - x_{n+1}|| \leq \dfrac{m^{n-1}|t^n|C}{n!}$. Hence as $n \to \infty$, $x_n(t, \alpha)$ converges uniformly to a function $x(t, \alpha)$. clearly $x(t, \alpha)$ is continuous and from (6.3), it follows that

$$x(t, \alpha) = x_0 + \int_0^t f(x(\tau, \alpha), \tau, \alpha) d\tau$$

so that $\dfrac{\partial x}{\partial t}(t, \alpha) = f(x(t, \alpha), t, \alpha)$ and $x(0, \alpha) = x_0$. If x and y are two continuous functions satisfying the differential equation (6.1) and the initial condition (6.2), let

$u(t, \alpha) = x(t, \alpha) - y(t, \alpha)$; then u is continuous and $||u(t, \alpha)|| \leq M \int_0^t ||u(\tau, \alpha)|| d\tau$ for $t \geq 0$. By Lemma 1 with $\eta = 0$, we conclude that
45 $u(t, \alpha) = 0$ for $t \geq 0$. Similar arguments apply to the range $t \leq 0$. This proves the uniqueness of the solution.

To prove the last part of the theorem, we shall first show that if $f \varepsilon C^1$, then $x \varepsilon C^1$. If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$, it is enough to prove that $\dfrac{\partial x}{\partial \alpha_i}$ exists and is continuous for each i, since (6.1) implies apply if $t < 0$.

Consider

$$A(t, \alpha) = (d_1 f)(x(t, \alpha), t, \alpha);$$
$$B(t, \alpha) = \frac{\partial f}{\partial \alpha_i}(x(t, \alpha), t, \alpha).$$

A is, for each t, α, a linear map of $\mathbb{R}^n$ into itself. Since f is C^1, $A(t, \alpha)$ is a continuous linear map and $B(t, \alpha)$ is continuous. Therefore the linear differential equation

$$\frac{dy}{dt} = A(t,\alpha)y + B(t,\alpha) \tag{6.4}$$

for $y\varepsilon\mathbb{R}^n$, has a solution $y(t,\alpha)$, which is continuous in t and α, and for which $y(0,\alpha) = 0$. If $(c_1, c_2, \ldots c_m)\varepsilon K'$, hereafter α will denote $(c_1, c_2, \ldots, \alpha_i, \ldots, c_m)$ and α^h the point, $(c_1, c_2, \ldots, \alpha_i + h, \ldots, c_n)$. Consider $\dfrac{x(t,\alpha^h) - x(t,\alpha)}{h} = \beta_h(t)$. Then since $f\varepsilon C^1$, by Taylor's formula,

$$\beta_h(t) = \int\limits_0^t \left[\{A(s,\alpha) + \varepsilon_1(h,\alpha,s)\}\beta_h(s) + \beta(s,\alpha) + \varepsilon_2(s,h)\right] ds$$

where, for fixed h, α and s, ϵ_1 is an endomorphism of $\mathbb{R}^n, \varepsilon_2\varepsilon\mathbb{R}^n$ and both tend uniformly to zero as $h \to 0$. Hence 46

$$\left|\beta_h(t)\right| \leq M_1 \int\limits_0^t |\beta_h| ds + M_2$$

for some M_1 and M_2 independent of h. Hence, by Lemma 1,

$\left|\beta_h(t)\right| \leq e^{M_1 t} M_2$ and β_h is bounded as $h \to 0$.

Let $\beta_h(t) - y(t,\alpha) = z_h(t)$; then

$$\left|z_h(t)\right| \leq \int\limits_0^t \left|A(s,\alpha)\right|. \left|z_h(s)\right| ds + \varepsilon_1^1 \int\limits_0^t \left|z_h(s)\right| ds + \varepsilon_2'$$

where ε_1' and $\varepsilon_2^1 \to 0$ as $h \to 0$. Also $\int\limits_0^t \left|\beta_h(s)\right| ds$ is bounded.

Hence $\left|z_h(t)\right| \leq \int\limits_0^t |A(s,\alpha)|\left|z_h(s)\right| ds + \varepsilon$ where $\varepsilon \to 0$ as $h \to 0$. By Lemma 1, this implies that

$$\left|z_h(t)\right| \to 0 \text{ as } h \to 0$$

i.e. $$x\varepsilon C^1 \text{ and } \frac{\partial x}{\partial \alpha_i} = y(t,\alpha).$$

If $f\varepsilon C^k$, assume, by induction, that the result is proved for functions in C^{k-1}. Then $x\varepsilon C^{k-1}$, so that $A(t,\alpha)$, $B(t,\alpha)\varepsilon C^{k-1}$; because of the differential equation

$$\frac{dy}{dt} = A(t,\alpha)y + B(t,\alpha),$$

47 and the induction hypothesis, $y\varepsilon C^{k-1}$. Since $\dfrac{\partial x}{\partial \alpha_i} = y(t,\alpha)$ and $\dfrac{\partial x}{\partial t} =$ $f(x,t,\alpha)$, if follows that $x\varepsilon C^k$.

Corollary. *If $f\colon \Omega \times I \times \Omega' \to \Omega'$ is in C^k, then the function $x(t,\alpha,x_o)$ for which*

$$\frac{dx}{dt} = f(x,t,\alpha), x(0,\alpha,x_0) = x_0$$

is C^k in $I \times \Omega' \times \Omega$.

We have only to consider the equation

$$\frac{dy}{dt} = g(y,t,\alpha,x_0), \tag{6.4}$$

where $g(y,t,\alpha,x_0) = f(x_0 + y,t,\alpha)$, on $\Omega \times I \times \Omega' \times \Omega$; we have

$$x(t,\alpha,x_0) = y(t,\alpha),$$

if $y(t,\alpha)$ is the solution of (6.4) with $y(0,\alpha) = x_0$.

Remark. If the function f in the above theorem is real analytic, then there exists a neighbourhood $U \times D \times U'$ of $\Omega \times I \times \Omega'$ in $\mathbb{C}^{n+1+m}$ such that f has holomorphic extension to $U \times D \times U'$. Then the equation

$$\frac{dx}{dt} = f(x,t,\alpha) \text{ for } (x,t,\alpha)\epsilon U \times D \times U'$$

has a holomorphic solution $x(t,\alpha)$ in $D_0 \times U'$. We set $x_0(t,\alpha) = x_0$,

$$x_k(t,\alpha) = x_0 + \int_0^t f(x_{k-1}(\tau,\alpha),\tau,\alpha)\, d\tau,$$

the integral being taken along the line joining 0 to t. Each x_k is holomor-
48 phic and hence so is $x(t, \alpha) = \lim_{k \to \infty} x_k(t, \alpha)$. Since by induction, each x_k is real for real t, so is x, so that by the uniqueness assertion, the restriction of x to $I_0 \times \Omega'$ is the solution of the differential equation in $I_0 \times \Omega'$. Hence this solution is real analytic.

{ For all this material and further developments, see Coddington and Levinson [8]. }

Chapter 2

Manifolds

1 Basic definitions

Definitions. (1) Let V be a hausdorff topological space. It is said to be a (C^0) manifold of dimension n if each $x \in V$ has an open neighbourhood U, which is homeomorphic to an open set in $\mathbb{R}^b$. 49

(2) If V is a topological space which is hausdorff, V is said to be a C^k manifold, $(0 \leq k \leq \infty)$, of dimension n, or a differentiable manifold of class C^k, if there is given a family of pairs (U_i, φ_i), U_i an open set in V and φ_i, a homeomorphism of U_i onto an open set in $\mathbb{R}^n$ such that

$$\cup U_i = V \text{and, if } U_i \cap U_j \neq \phi,$$
$$\varphi_j \circ \varphi_i^{-1} \big| \varphi_i(U_i \cap U_j) \text{ is a } C^k \text{ map of } \varphi_i(U_i \cap U_j) \text{into } \mathbb{R}^n.$$

(3) If V is a C^k manifold of dimension n, a C^k atlas on V is a maximal set $\{(U_i, \varphi_i)\}$ such that $\cup U_i = V$ and whenever $U_i \cap U_j \neq \phi$,

$$\varphi_j \circ \varphi_i^{-1} \big| \varphi_i(U_i \cap U_j) \text{is a } C^k \text{ map of } \varphi_i(U_i \cap U_j) \text{into } \mathbb{R}^n.$$

Remarks. 1. Any set of pairs as in (2) can be completed to a C^k atlas and conversely an atlas defines the structure of a C^k manifold.

2. The dimension of V is independent of the "coordinate systems" $\{U_i, \varphi_i\}$ according to a theorem of $L.E.J.$ Brouwer which asserts that if a non-empty open set in $\mathbb{R}^n$ is homeomorphic to one in $\mathbb{R}^m$, then $m = n$. We shall not prove this theorem here. For a proof, see eg [18].

50 3. A hausdorff topological space V is said to be a real analytic (complex analytic) manifold if there is given a family of pairs (U_i, φ_i), U_i an open set in V, φ_i, a homeomorphism of U_i onto an open set in $\mathbb{R}^n$ (an open set in $\mathbb{C}^n$), such that $\bigcup U_i = V$ and whenever $U_i \cap U_j \neq \phi$, $\varphi_j \circ \varphi_i^{-1} \big| \varphi_i(U_i \cap U_j)$ is real analytic (complex analytic = holomorphic).

4. If V is a C^k manifold and U an open set in V, a map $f : U \to \mathbb{R}$ is called C^r, $0 \leq r \leq k$ if for each coordinate neighbourhood (U_i, φ_i), with $U_i \cap U \neq \phi$, $f \circ \varphi_i^{-1} \big| \varphi_i(U_i \cap U)$ is C^r. We denote the set of C^r functions on V by $C^r(V)$, $0 \leq r \leq k$.

5. If V and V' are two C^k manifolds of dimensions n and m respectively, U, an open set in V, a map $f\colon U \to V'$ is called C^r, $0 \leq r \leq k$ if for coordinate neighbourhoods (U_i, φ_i) and (U'_j, φ'_j) of V and V' respectively, such that $U_i \cap U \neq \phi$ and $f(U_i \cap U) \subset U'_j$, the map $\varphi'_j \circ f \circ \varphi_i^{-1} \big| \varphi_i(U_i \cap U)$ is of class C^r.

We denote set of C^k maps of V into W by $C^k(V, W)$. If a C^k map $f\colon V \to W$ is a bijection and $f^{-1}\colon W \to V$ is also C^k, we say that f is a C^k-diffeomorphism, (or diffeomorphism or C^k-isomorphism) of V onto W. Real analytic and holomorphic mappings between real and complex analytic manifolds may be defined in the same way. We also introduce real and complex analytic isomorphisms between such manifolds just as we did diffeomorphisms.

Examples. 1. $S_1 = \{x \in \mathbb{R}^2 \big| \|x\| = 1\}$ is a C^∞ manifold of dimension 1.

2. If V is a C^k manifold and $\tilde{V}$ a hausdorff space, $p\colon \tilde{V} \to V$ local
51 homeomorphism, there is a unique structure of C^k manifold on $\tilde{V}$ such that for $\tilde{a} \in \tilde{V}$, $p(\tilde{a}) = a$, there exist neighbourhoods $\tilde{U}$ of $\tilde{a}$, U of a such that $p\colon \tilde{U} \to U$ is a C^k isomorphism.

A more interesting class of Examples (Grassmann manifolds) is described at the end of the section.

It is clear that a complex analytic manifold carries a natural real analytic structure; a real analytic manifold a C^∞ structure and a C^k manifold $(0 < k \leq \infty)$ a C^r structure $(0 \leq r < k)$. Conversely, it follows from results of H. Whitney [48] that any paracompact C^1 manifold carries a real analytic structure. Further, the imbedding theorem of H.Grauert [13] (see §9 for the statement) and the approximation theorem of Whitney (Chap.I, §5) imply that this structure is unique. However, a C^0 manifold may have no differentiable structure (M. Kervaire [20]) and even when it has,this is not unique. For example, the sphere S^7 can carry two differentiable structures such that there is *no* diffeomorphism of one onto the other (J. Milnor [28]). The problem of the existence and uniqueness of complex structures is a problem of quite a different nature, and had given rise to a vast literature (see in particular H. Hopf [16], K.Kodaria and D.C. Spencer [21]).

Let a be a point in a C^k manifold V. Consider all ordered pairs (f, U) where U is an open set containing a and f, a C^k map $U \to \mathbb{R}$. In the set of these ordered pairs we define an equivalence relation as follows. $(f, U) \sim (f', U')$ if there exists an open set Ω containing a such that $\Omega \subset U \cap U'$, and such that $f\big|\Omega = f'\big|\Omega$. The equivalence classes of these ordered pairs are called germs (of C^k functions) at a. We shall frequently identify a germ with a function defining it when there is no **52**
fear of confusion.

Definition. A germ f of a C^k functions, $k \geq 1$, at a is said to be stationary at a if there exists a coordinate neighbourhood (U, φ) with $a \in U$ such that all the first partial derivatives of $f \circ \varphi^{-1}$ vanish at a. Here (f, U) is a pair defining f. It is clear that the above definition depends only on the germ f.

Notation. C^k_a denotes the set of all C^k germs at a, $S^k_a = S_a$ denotes the set of all stationary C^k germs at a and $m^k_a = m_a$, the set of all C^k germs vanishing at a. C^k_a is a vector space over $\mathbb{R}$; S^k_a and m^k_a are subspaces..

Definition. (1) The quotient space C^k_a/S_a is called the space of differentials (or cotangent vectors or co vectors) and is denoted by $T^*_a(V)$.

The image of $f \in C_a^k$ in $T_a^*(V)$ is denoted by $(df)_a$.

(2) The dual space of $T_a^*(V)$, i.e., the space of all linear functionals X: $C_a^k \to \mathbb{R}$ with $X(f) = 0$ for $f \in S_a$, is called the tangent space at a and is denoted by $T_a(V)$. A point in $T_a(V)$ is called a tangent vector.

(3) A linear function L: $C_a^k \to \mathbb{R}$ is called a derivation if for $f, g \in C_a^k(V)$,

$$L(f.g) = L(f)\, g(a) + f(a)L(g).$$

Proposition 1. *Any tangent vector X in $T_a(V)$ is a derivation.*

Proof. For any $f, g \in C_a^k$ the function φ given by

$$\varphi = fg - f(a)g - f.g(a), \text{ is in } S_a.$$

□

53 Hence $X(\varphi) = 0$
i.e $X(fg) = f(a)X(g) + X(f).\, g(a)$.

Definition. If (U, φ) is a coordinate neighbourhood and for a point $x \in U$, $(x_1, \ldots, x_n)$ are the coordinates of $\varphi(x)$ in $\mathbb{R}^n$, for *a* C^1 function f: $U \to \mathbb{R}$, $a \in U$, we define

$$\left(\frac{\partial f}{\partial x_1}\right)_a, \left(\frac{\partial f}{\partial x_2}\right)_a, \ldots, \left(\frac{\partial f}{\partial x_n}\right)_a \text{ by}$$
$$\left(\frac{\partial f \circ \varphi^{-1}}{\partial x_1}\right)_{\varphi(a)}, \ldots, \left(\frac{\partial f \circ \varphi^{-1}}{\partial x_n}\right)_{\varphi(a)}$$

respectively. We define tangent vectors $\left(\frac{\partial}{\partial x_i}\right)_a$ at a by $\left(\frac{\partial}{\partial x_i}\right)_a f = \left(\frac{\partial f}{\partial x_i}\right)_a$.

Proposition 2. $\left(\frac{\partial}{\partial x_1}\right)_a, \ldots, \left(\frac{\partial}{\partial x_n}\right)_a$ *are linearly independent in $T_a(V)$ and span $T_a(V)$.*

Proof. If $f \in C_a^k$, g defined by

$$g(x) = f(x) - f(a) - \sum x_i \left(\frac{\partial f}{\partial x_i}\right)_a,$$

is in S_a. Hence for $X \in T_a(V)$, $X(g) = 0$

i.e. $$X(f) = \sum X(x_i)\left(\frac{\partial f}{\partial x_i}\right)_a$$

i.e. $$X = \sum X(x_i)\left(\frac{\partial}{\partial x_i}\right)_a.$$ □

Therefore $\left(\frac{\partial}{\partial x_1}\right)_a, \ldots, \left(\frac{\partial}{\partial x_n}\right)_a$ span $T_a(V)$. Further $\left(\frac{\partial x_j}{\partial x_i}\right)_a = \delta_{ij}$ hence $\left\{\left(\frac{\partial}{\partial x_i}\right)_a\right\}$, $1 \le i \le n$, are linearly independent.

Corollary. *$T_a(V)$ and $T_a^*(V)$ are n dimensional vector spaces.* **54**

It follows from this that any tangent vector defines a linear function on the germs of C^1 functions, which vanishes on stationary functions and is a derivation.

Proposition 3. *If X is a derivation of C_a^{k-1}, $k \ge 1$, X is in $T_a(V)$. [Note that there is a natural injection of C_a^k in C_a^{k-1}].*

Proof. If $f \in S_a$, we can assume without loss of generality that f is defined on an open set U containing a, (U, φ) a coordinate neighbourhood, and for some open set $U' \subset U$, with $a \in U'$ and, if $x \in U'$, $t\varphi(x) \in \varphi(U)$, $0 \le t \le 1$. Then for $x \in U'$,

$$\begin{aligned} f(x) &= \int_0^1 \frac{\partial f}{\partial t}[\varphi^{-1}(t\varphi(x))]dt \\ &= \sum x_i g_i(x) \end{aligned}$$

where $g_i(x) = \int_0^1 \frac{\partial f}{\partial x_i}[\varphi^{-1}(t\varphi(x)]dt.$ □

Clearly $g \in C_a^{k-1}$.

We may also assume that $\varphi(a) = 0$. Then

$$X(f) = \sum x_i(a) X(g_i) + \sum X(x_i) g_i(a)$$

but $x_i(a) = 0 = g_i(a)$ $1 \leq i \leq n$.

Hence $X(f) = 0$ i.e. X is a linear map $C_a^{k-1} \to \mathbb{R}$ which vanishes on S_a^k i.e. X is a tangent vector.

Corollary 1. *If V is a C^∞ manifold and $f \in m_a^\infty$, then f is stationary at a if and only if $f \in (m_a^\infty)^2$.*

55 *Proof.* The maps x_i and g_i in the above proof are in m_a^∞, which proves the necessity. The sufficiency is trivial. □

Corollary 2. *For a C^∞ manifold; we have $T_a^* = m_a/(m_a)^2$.*

One has also the following "geometric" definition of tangent vectors. Let $\gamma\colon I \to V$ be a C^k curve (i.e. a C^k map of a neighbourhood of the unit interval $I = [0, 1]$ on $\mathbb{R}$ into V). The *tangent to γ at $a = \gamma(0)$* is the tangent vector X at a defined by

$$X(f) = \frac{d}{dt} f \circ \gamma(t)\Big|_{t=0} \text{ for } f \in C_a^1.$$

[It is easily verified that this defines a tangent vector.] One has

Proposition 4. *Any tangent vector at $a \in V$ is the tangent at a to some curve γ with $\gamma(0) = a$.*

Proof. We may suppose that V is the open cube $|x_i| < 1$ in $\mathbb{R}^n$. Any tangent vector X at $x = 0$ is of the form

$$X = \sum a_i \left(\frac{\partial}{\partial x_i}\right)_0.$$

□

Let γ_i, $1 \leq i \leq n$, be C^k functions in a neighbourhood of I with $|\gamma_i| < 1$, $\gamma_i(t) = a_i t$ in a neighbourhood of $t = 0$. We may take for γ the curve given by $\gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t))$.

Unless otherwise stated, in what follows V denotes a C^k manifold of dimension n and W denotes a C^k manifold of dimension m. Let $F : V \to W$ be a C^k map. Then the maps

$$f_* : T_a(V) \to T_{f(a)}(W) \text{ and}$$
$$f^* : T^*_{f(a)}(W) \to T^*_a(V) \text{are defined by}$$
$$f_*(X)(g) = X(g \circ f) \text{and } (f^*(d\varphi))_{(a)} = [d(\varphi \circ f)]_a$$

when $g \in C^k_{f(a)}$, $(d\varphi) \in T^*_{f(a)}(W)$ and $X \in T_a(V)$. Note that if $g \in S_{f(a)}$, **56**
$g \circ f \in S_a$. It is easily verified that f_* and f^* are transposes of one another.

Remark that if f_1: $V_1 \to V_2$, $f_2 : V_2 \to V_3$ are C^k maps, then we have, for any $a \in V_1$, $(f_2 \circ f_1)^*_a = (f^*_2)_{f_1(a)} \circ (f_1)^*_a$. It follows that if $f : V \to W$ is a C^k isomorphism, then f^*_a is an isomorphism for any a. Hence $T_a(V)$ and $T_{f(a)}(W)$ have the same dimension; hence V, W have the same dimension. Thus the fact that the dimension of a C^k manifold, $k \geq 1$, is invariant of the (C^k) local coordinates chosen, is obvious. (Compare with Remark 2 after the definition of a manifold.) Let $T(V) = \bigcup\limits_{a \in V} T_a(V)$. We shall prove the following

Theorem 1. *If V is a C^k manifold , $k \geq 1$, $T(V)$ carries a natural structure of a C^{k-1} manifold dimension $2n$.*

Proof. It follows from Proposition 2 that, relative to a coordinate system (U_i, φ_i), with $a \in U_i$, a tangent vector X in $T_a(V)$ is completely determined by $\{\alpha_\nu = X(x_\nu)_a\}_{1 \leq \nu \leq n}$. Let (U_j, φ_j) be another coordinate neighbourhood with $a \in U_i$ and let the tangent vector be given by $\{\beta_\nu = X(y_\nu)_a\}_{1 \leq \nu \leq n}$ with respect to (U_j, φ_j). We denote by $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$, the local coordinates of $\varphi_i(x)$ and $\varphi_j(x)$ respectively. Then for any $g \in C^k_a$, □

$$X(g) = \sum_i \alpha_i \left(\frac{\partial g}{\partial x_i}\right)_a = \sum_j \beta_j \left(\frac{\partial g}{\partial y_j}\right)_a$$

$$= \sum_j \beta_j \left(\sum_\nu \left(\frac{\partial g}{\partial x_\nu} \right)_a \left(\frac{\partial x_\nu}{\partial y_j} \right)_a \right)$$

$$= \sum_\nu \left(\sum_j \beta_j \left(\frac{\partial x_\nu}{\partial y_j} \right)_a \right) \left(\frac{\partial g}{\partial x_\nu} \right)_a$$

57 Hence

$$\alpha_i = \sum_j \beta_j \left(\frac{\partial x_i}{\partial y_j} \right)_a \tag{1.1}$$

i.e.

$$(\alpha_1, \ldots, \alpha_n) = (\beta_1, \ldots, \beta_n)(m_{ij})_a$$

where $(m_{ij})_a$ is the matrix $(\dfrac{\partial x_j}{\partial y_i})_a$. Clearly $(m_{ij})_a$ is non-singular and $(m_{ij})_a (m_{ji})_a = I$. Now consider the topological union $E - \bigcup_i (U_i \times \mathbb{R}^n \times i)$ and define an equivalence relation, $\sim$, by $(x, v, i) \sum (x', v', j)$ if $x = x'$ and $v = v'(m_{ij})_x$, [where $(m_{ij})_x$ is the matrix defined above]. Clearly there is an obvious bijective map from $E/_\sim$ onto $T(V)$. It suffices to show that $E/_\sim$ carries a natural structure of C^{k-1} manifold.

It is clear that $\sim$ is an open equivalence relation. Let $\eta : E \to E/\sim$ denote the natural map, and let $p' : E \to V$ the continuous map $p((x, v, i)) = x$. Clearly p' maps equivalent points onto the same point in V, so that p' defines a continuous map p: $E/_\sim \to V$. Further $\eta_i = \eta \{U_i \times \mathbb{R}^n \times i\}$ is a homeomorphism onto $p^{-1}(U_i)$; in particular $p^{-1}(U_i)$
58 is hausdorff; we identity $U_i \times \mathbb{R}^n$ with $U_i \times \mathbb{R}^n \times i$. We assert that $E/_\sim$ is hausdorff: in fact if $e_1, e_2 \in E/\sim$ $e_1 \neq e_2$, then if $p(e_1) \neq p(e_2)$ and Ω_i is a neighbourhood of e_i, $\Omega_1 \cap \Omega_2 = \phi$, then $p^{-1}(\Omega_1)$, $p^{-1}(\Omega_2)$ are disjoint neighbourhoods of e_1, e_2 respectively. If $p(e_1) = p(e_2)$, then e_1, $e_2 \epsilon p^{-1}(U_i)$ for some i, since $p^{-1}(U_i)$ is open in $E/_\sim$ and is hausdorff, e_1, e_2 can be separated.

If φ_i is the given C^k homeomorphism of U_i onto an open set U'_i in $\mathbb{R}^n$, then $(\varphi_i \times id) \circ \eta_i^{-1} = \Phi_i$ is a homeomorphism of $p^{-1}(U_i)$ onto $U'_i \times \mathbb{R}^n$; that the mappings $\Phi_j \circ \Phi_i^{-1}$ are C^{k-1} follows at once from (1.1) [note that (1.1) involves derivatives of C^k functions].

We remark that the C^{k-1} structure of $T(V)$ so obtained does not depend on the system $\{U_i, \varphi_i\}$ used.

$T(V)$ is an example of a real vector bundle (see Chap.III, §1).

If $0 \leq p \leq n$, we consider the vector space $\Lambda^p T_a^*(V)$. An element of this space is called a *p-co vector* at the point a. If (U, φ) is a coordinate system at a, then the differentials $(dx_1)_a, \ldots, (dx_n)_a$ form a basis of $T_a^*(V)$. Hence a basis of $\Lambda^p T_a^*(V)$ is given by the elements $(dx_{i_1})_a \Lambda \cdots \Lambda (dx_{i_p})_a$, $i_1 < \cdots < i_p$. In exactly the same way as above, we prove the following

Theorem 2. *The set* $\overset{p}{\Lambda} T^*(V) = \bigcup\limits_{a \in V} \overset{p}{\Lambda} T_a^*(V)$ *carries a natural structure of* C^{k-1}*manifold [of dimension* $n + \binom{n}{p}$*].*

Grassmann manifolds.

Let $0 < r < n$, and let $G_{r,n}$ denote the set of r-dimensional linear subspaces of $\mathbb{R}^n$. We shall show that $G_{r,n}$ carries a natural structure of real analytic manifold.

Let $M(r, n)$ denote the space of $r \times n$ real matrices and $N = N(r, n)$ **59**
the subset of matrices of rank r. $M(r, n)$ is clearly homeomorphic to $\mathbb{R}^{rn}$ and $N(r, n)$ to an open subset. Let $G = GL(r, \mathbb{R})$ denote the group of nonsingular $r \times r$ matrices. We have natural map $G \times N \to N$ defined by $(A, B) \rightsquigarrow A.B$, where $A \in G$, $B \in N$.

We assert that there is a natural bijection $p : N/_G \to G_{r,n}$, where $N/_G$ is the quotient of N by the equivalence relation: $B_1 \sim B_2$ if there is $A \in G$ with $B_2 = A, B_1$.

Proof. If $B \in N$ we may look upon B as a column $\begin{pmatrix} v_1 \\ \vdots \\ v_r \end{pmatrix}$ where $v_\nu \in \mathbb{R}^n$; let $p(B)$ denote the subspace spanned by $v_1, \ldots, v_r$. If $B \in N$, this subspaces has dimension r. The assertion that p is a bijection is equivalent with the obvious assertion that the sets $(v_1, \ldots, v_r)$, $(w_1, \ldots, w_r)$ of points of $\mathbb{R}^n$ span the same r-dimensional subspace if and only if there is an $A \in G$

with

$$\begin{pmatrix} w_1 \\ \vdots \\ w_r \end{pmatrix} = A \begin{pmatrix} v_1 \\ \vdots \\ v_r \end{pmatrix}$$

□

We put on $G_{r,n}$ the quotient topology; clearly the equivalence relation defined above is open.

Let K denote the set of r-tuples $j_1 < \cdots < j_r$ of integers j_ν with $1 \leq j_\nu \leq n$. For $\alpha \in K$, let V be the subset of $M(r, n)$ consisting of matrices $B = (b_{ij})_{1 \leq i \leq r, 1 \leq j \leq m}$, for which

$$B^\alpha = (b_{ij_\nu})_{1 \leq i \leq r, 1 \leq \nu \leq r}, \alpha = (j_1, \ldots, j_r)$$

is non -singular. We have $\bigcup V_\alpha = N$. It is clear that if B_1, $B_2 \in N$ and
60 $A \in G$ satisfies $B_2 = AB_1$, and if $B_1 \in V_\alpha$, then $B_2 \in V_\alpha$ and we have $B_2^\alpha = AB_1^\alpha$.

For $B_1 \in V_\alpha$, we shall write symbolically, $B = (B^\alpha, C^\alpha)$, where B is the matrix defined above and C^α is the $r \times (n - r)$ matrix

$$C^\alpha = (b_{ij_\nu}) \text{ with } 1 \leq i \leq r, 1 \leq j_1 < \cdots < j_{n-r} \leq n, j_\nu \notin \alpha.$$

We shall identify $M(r, n - r)$ with $\mathbb{R}^{r(n-r)}$. Let $\psi_\alpha : V_\alpha \to \mathbb{R}^{r(n-r)}$ denote the mapping

$$\psi_\alpha(B) = (B^\alpha)^{-1} C^\alpha;$$

ψ_α is clearly continuous and open. Then, if U_α is the subset $p(V_\alpha)$ of $G_{r,n}$, there is a homeomorphism

$$\varphi_\alpha : U_\alpha \to \mathbb{R}^{r(n-r)}$$

such that, $\varphi_\alpha op = \psi_\alpha$. In fact it is easy to verify that $\psi_\alpha(B_1) = \psi_\alpha(B_2)$ if and only if $B_1 \sim B_2$, which gives us the existence of a bijection. This is continuous and open, since ψ_α is

We assert next that $G_{r,n}$ is Hausdorff. Since the equivalence relations is open, we have only to prove that its graph Γ consisting of pairs $(B_1, B_2) \in N \times N$ with $B_1 \sim B_2$ is closed in $N \times N$. Suppose that

$$((B_1)_\nu, (B_2)_\nu) \in \Gamma, (B_i)_\nu \to B_i \in N$$

and let $A_\nu \in G$ satisfy

$$(B_2)_\nu = A_\nu (B_1)_\nu.$$

Since $B_1 \in N$, $B_1 \in V_\alpha$ for some α. Then so does $(B_1)_\nu$ for sufficiently large ν and we have 61

$$(B_1)^\alpha_\nu \to B^\alpha_1 \text{ as } \nu \to \infty$$

Then we have $(B_2)\nu \in V_\alpha$ and

$$(B_2)^\alpha_\nu = A_\nu (B_1)^\alpha_\nu.$$

Since $(B_1)^\alpha_\nu \to B^\alpha_1 \in G$, and since $(B_2)^\alpha_\nu$ converges to a matrix $A^{(1)} \in M(r,r)$ (since, by assumption, $(B_2)_\nu \to B_2$ in N), the matrix A_ν converges to $A = (B^\alpha_1)^{-1} A^1$ as $\nu \to \infty$. Since

$$(B_2)_\nu = A_\nu (B_1)_\nu,$$

we deduce that $B_2 = AB_1$. However, since B_2 has rank r, A has rank $\geq r$; since $A \in M(r,r)$, $A \in G$ so that $B_1 \sim B_2$ and $(B_1, B_2) \in \Gamma$.

The covering $\{U_\alpha\}_{\alpha \in K}$ and the homeomorphisms φ_α: $U_\alpha \to \mathbb{R}^{r(n-r)}$ make of $G_{r,n}$ an $r(n-r)$ dimensional real analytic manifold. In fact the coordinate changes $\varphi_\alpha \circ \varphi^{-1}_\beta$ are easily seen to be *rational*functions.

Let $0(n)$ denote the orthogonal group of $\mathbb{R}^n$, i.e. the set of $n \times n$ matrices A for which

$$A \cdot {}^t A = I;$$

here I is the unit $n \times n$ matrix and ${}^t A$ is the transpose of A. $0(n)$ acts on $G_{r,n}$: if $B_1 \in N$, $0 \in 0(n)$, then $B_1 0 \in N$ and, if $B_1 \sim B_2$ we have 62 $B_1 0 \sim B_2 0$. It is easy to show that $0(n)$ is compact and that it acts transitively on $G_{r,n}$. We deduce the following

Proposition. *The Grassmannian $G_{r,n}$ is a compact, real analytic manifold of dimension $r(n-r)$.*

Remarks. 1. The manifold $G_{1,n}$ is called $(n-1)$ -dimensional projective space $\mathbb{P}^{n-1}(\mathbb{R})$.

2. It can be proved in the same way that the set $G_{r,n}(\mathbb{C})$ of complex r-dimensional subspaces of $\mathbb{C}^n$ is a compact complex manifold of complex dimension $r(n-r)$, $G_{1,n}(\mathbb{C})$ is the complex projective space $\mathbb{P}^{n-1}(\mathbb{C})$.

For much of the material contained in §§1, 2 see Schwartz [40].

2 Vector fields and differential forms

Let V be a C^k manifold and p: $T(V) \to V$ the projection given by $p(X) = a$ for $X \in T_a(V)$ for any $a \in V$.

Definition. *A C^r Vector field X, $0 \le r \le k-1$ is , by definition,* a *C^r map $X : V \to T(V)$ such that*

$$poX = \text{ identity on } V.$$

Clearly if X is a vector field, $X(a) \in T_a(V)$ for any $a \in V$. If (U, φ) is a coordinate neighbourhood, we may represent the vector field X by the formula

$$X_a = \sum \xi_i(a) \left(\frac{\partial}{\partial x_i}\right)_a.$$

63 Then X is of class C^r if and only if the $\xi_i(a)$ are C^r functions.

Definition. *A p differential form ω of class C^r is a C^r map ω: $V \to \overset{p}{\Lambda} T^*(V)$ such that $\omega(a) \in \overset{p}{\Lambda} T^*_a(V)$ for each $a \in V$.*

If (U, φ) is a coordinate neighbourhood ω has a representation

$$\omega_a = \sum_{i_1 < i_2 < \cdots < i_p} \xi_{i_1 \cdots i_p}(a)(dx_{ij})_a \Lambda (dx_{i_r})_a \Lambda \cdots \Lambda (dx_{i_p}).$$

again ω is of class C^r if and if the $\xi_{i_1 \cdots i_p}$ a C^r functions. Let $\mathscr{G}$ denote the module [over the ring $C^{k-1}(V)$ of C^{k-1} functions on V] of C^{k-1} vector fields on V. If ω is a p-form on V, it defines a p-linear map of $\mathscr{G}^p$ into $C^{k-1}(V)$; in fact we have only to set

$$\omega(X_1, \ldots, X_p)(a) = \omega_a((X_1)_a, \ldots, (X_p)_a).$$

[Note that $\overset{p}{\Lambda} T_a^*(V)$ is the dual of the space $\overset{p}{\Lambda} T_a(V)$.] This map his following two properties: (a) it is alternate; (b) it is multilinear over $C^{k-1}(V)$. Conversely, any alternate map φ of $\mathscr{G}^p$ into $C^{k-1}(V)$, which is multilinear over $C^{k-1}(V)$ defines a differential p-form ω; in fact, if $(X_1)_a, \ldots, (X_p)_a$ are vectors at $a \in V$, and if $X_1, \ldots, X_p$ are vector fields on V extending these vectors, we define the p-co vector ω_a by

$$\omega_a((X_1)_A, \ldots, (X_p)_a) = \varphi(X_1, \ldots, X_p).$$

It is easily verified, using the fact that φ is $C^{k-1}(V)$-linear that $\varphi(Y_1, \ldots, Y_p) = 0$ at a point b if $(Y_i)_b = 0$ for some i, so that the above **64**
definition is independent of the extension of the vectors $(X_i)_a$ to vector fields on V. If $f\colon V \to W$ is a C^k map and $a \in V$, $b = f(a)$, we have defined linear maps $f_*\colon T_a(V) \to T_b(W)$ and $f^*\colon T^*_{f(a)}(W) \to T^*_a(V)$. This defines a map, which denote f^*, of $\overset{p}{\Lambda} T^* f(a)(W) \to \overset{p}{\Lambda} T_a^*(V)$. f^* is clearly an algebra homomorphism of $\Lambda T^*_{f(a)}(W)$ into $\Lambda T^*_a(V)$.

Hence if ω is a p form on W of class C^r we may associate to any $a \in V$ the p co vector $f^*(\omega_{f(a)})$. It is easy to see that this defines a p-form $f^*(\omega)$ of class C^r on V. However, the map f_* does not in general, transform vector fields.

Definition. If $f : V \to W$ is a C^k map , $k \geq 1$, f is said to have rank r at $a \in V$, if

$$\text{rank } f_* : T_a(V) \to T_{f(a)}(W) \text{ is } r.$$

We can easily calculate the map f_* in terms of local coordinates (U, φ) at a and (U', φ') at $b = f(a)$. In terms if the bases

$$\left(\frac{\partial}{\partial x_1}\right)_a, \ldots, \left(\frac{\partial}{\partial x_n}\right)_a \quad \text{and} \quad \left(\frac{\partial}{\partial y_1}\right)_b, \ldots, \left(\frac{\partial}{\partial y_m}\right)_b$$

of $T_a(V)$ and $T_b(W)$, if $X = \sum a_i \left(\frac{\partial}{\partial x_i}\right)_a$, $g \in C^k_b$ then

$$\sum b_j \left(\frac{\partial g}{\partial y_j}\right)_b = f_*(X).(g) = X(gof)$$

$$= \sum_i a_i \sum_j \left(\frac{\partial g}{\partial y_j}\right)_b \left(\frac{\partial f_j}{\partial x_i}\right)_a$$

so that, $f_*(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$, with

$$b_j = \sum a_i \left(\frac{\partial f_j}{\partial x_i}\right)_a .$$

65 This was precisely the map $d(f)(a)$ defined in Chap. I §1 , if we look upon f as a map of an open set in $\mathbb{R}^n$ into $\mathbb{R}^m$. We obtain therefore the following theorems form the inverse function theorem, the rank theorem and Sard's theorem, proved in Chap. I.

Inverse function theorem. *If V and W are C^k (real analytic) manifolds of dimension n and $f : V \to W$ a C^k real analytic map, and if $f_* : T_a(V) \to T_{f(a)}(W)$ is an isomorphism for some $a \in V$, then there exist neighbourhoods Ω and Ω' of a and $f(a)$ receptively, such that $f|\Omega$ is a C^k (real analytic) isomorphism onto Ω'.*

Rank Theorem. *If V^n and W^m are C^k (real analytic) manifolds and $f\colon V \to W$, a C^k(real analytic) map such that rank f is a constant, r, for all points in V, then for every point $a \in V$, there exists coordinate neighbourhoods $(U, \varphi), (U', \varphi')$ of a and $f(a)$ respectively such that $\varphi' o f o \varphi^{-1}|_{\varphi\Omega}$ is given by*

$$\varphi'_1 o f o \varphi^{-1}(x_1, \ldots, x_n) = (x_1, x_2, \ldots, x_r, 0, \ldots, 0)$$

Definition. If V and W are C^1 manifolds of dimension n and m respectively, and $f\colon V \to W$ a C^1 map a point $a \in V$ is called critical if rank ${}_a f < m$.

Definition. If W is a C' manifolds of dimension m, countable at ∞, a set E in W is said to have measure zero in W if for any coordinate neighbourhood (U, φ), $\varphi(E \cap U)$ has measure zero in $\mathbb{R}^m$.

It is clear that the notion of a set being of measure zero is dependent of the coordinate neighbourhoods used in the definition.

Sard's theorem. If V and W are C^∞ manifolds of dimension n and m respectively which are countable at infinity,and $f : V \to W a C^\infty$ map and if A is the set of critical points of f in V, then $f(A)$ is of measure zero in W.

66 As in Chapter I, we can prove the existence of partitions of unity : we have only to use the fact that if (U, φ) is a coordinate neighbourhood and $K \subset U$ is compact, then there is a C^k function η on V with compact support $\subset U$ such that $\eta(x) > 0$ for $x \in K$. We formulate this as separate theorem.

Partition of unity. Given an open covering $\{U_i\}_{i\in I}$ of a C^k manifold $(0 \leq k \leq \infty)V$ which is countable at infinity, there exists a family $\{\varphi_i\}_{i\in I}$ of C^k functions, $\varphi_i \geq 0$, with supp. $\varphi_i \subset U_i$ such that the family $\{supp.\varphi_i\}$ is locality finite and $\sum \varphi_i(x) = 1$ for any $x \in V$.

Corollary. *If F is a closed subset of V and $U \supset F$ is open, there exists a C^k function φ on V with* $\varphi(x) = \begin{cases} 1 & \text{if } x \in F \\ 0 & \text{if } x \in V - U \end{cases}$.

Let V be a C^k manifold, $k < 2$. For any C^{k-1}vector field $X, a \in V$, let x given by $X = \sum_i \alpha_i \dfrac{\partial}{\partial x_i}$ in a neighbourhood of *****.

Then for $f \in C^k_a$, $X(f)$ can be considered as a function in C^{k-1}_a, given by

$$X(f)(y) = \sum_i \alpha_i(y) \left(\frac{\partial f}{\partial x_i}\right)(y) \text{ for } y \text{ in a neighbourhood of } a. \tag{1.2}$$

If Y be another C^{k-1}vector field given by $Y = \sum \beta_i \dfrac{\partial}{\partial x_i}$ in a neighbourhood of a. We define a C^{k-2} vector field $[X, Y]$ by

$$[X, Y]_a(f) = X_a[Y(f)] - Y_a[X(f)]$$

and by (1.2),

$$Y_a[X(f)] = \sum_j \beta_j(a) \left[\sum_i \left\{\left(\frac{\partial \alpha i}{\partial x_j}\right)_a \left(\frac{\partial f}{\partial x_i}\right)_a + \alpha_i(a)\left(\frac{\partial^2 f}{\partial x_j \partial x_i}\right)\right\}_a\right]$$

Hence 67

$$[X,Y]_a(f) = \sum_i \left[\sum_j \alpha_j(a) \left(\frac{\partial \beta_i}{\partial x_j} \right)_a - \beta_j(a) \left(\frac{\partial \alpha_i}{\partial x_j} \right)_a \right] \left(\frac{\partial f}{\partial x_i} \right)_a$$

It can be easily verified that for C^{k-1} vector fields X, Y, Z, $k \geq 3$, $[X, Y] = -[Y, X]$ and, if $k \geq 4$,

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

This is called the *Jacobi identity*.

Differential forms on the product of two manifolds

Let V and V' be C^k manifolds (countable at infinity), $W = V \times V'$, and π, π' the projections of W on V, V' respectively. Then any C^{k-1} form ω of degree p on V can be identified with the form $\pi^*(\omega)$on W; a similar remark applies to V'.

Let $A(V), \ldots$ denote the space of forms on $V, \ldots$ We topologise $A(V)$ as follows: a sequence $\{\omega_\nu\}$ of forms $\omega_\nu \in A(V)$ tends to zero if, for any coordinate neighbourhood U on V (coordinates $x_1, \ldots, x_n$) and any compact subset K of U, if ω^I_ν denotes the coefficient of $dx_{i_1} \Lambda \cdots \Lambda dx_{i_p} [I = (i_1, \ldots, i_p), i_1 < \cdots < i_p, p = 0, 1, \ldots, n]$, then for any I, ω^I_ν and all its partial derivatives of order $< k$ tend to zero as $\nu \to \infty$.

Using a partition of unity, we prove easily by applying Cor. 2 to Theorem 2 of Chap. I, §5, the following

68 **Proposition 1.** *Finite linear combinations of forms of the type* $\prod^*(\omega) \Lambda \prod^*(\omega')$*, where* ω *is a form on* V*,* ω' *one on* V'*, are dense in* $A(V \times V')$.

This implies of course that finite linear combinations of forms of the type $\prod^*(\omega) \Lambda \prod^*(\omega')$ where degree $\omega +$ degree $\omega' = p$ are dense in the space of p-forms on W; for $p = 0$ this means that functions on W can be approximated by finite linear combinations of products of functions on V, V' respectively.

Corresponding statement for holomorphic forms on the product of two complex manifolds are also true. If $\mathscr{H}(V)$ denotes the space of holomorphic forms on the complex manifold V, we topologies it by means of convergence of the coefficient on compact subsets of coordinate neighbourhood, just as we did above, (the convergence of the derivatives is here a consequence of the convergence of the coefficient since they are holomorphic functions). The density can be proved along the lines of Theorem 4 of Chap. I, §5; we have only to introduce the Hilbert space corresponding to the space $A(\alpha)$ introduced in Chap. I, §5.

Let $\{U_i\}$ be a locally finite covering of V by coordinate neighbourhoods, and α a positive continuous functions on V. Let ω be a homomorphic form on V, and let

$$\omega = \sum_i \omega_I^{(i)} dz_I^{(i)} \begin{cases} I = (i_1, \ldots, i_p), i_1 < \cdots < i_p, p = 0, \ldots, n \\ ds_I^{(i)} = dz_{i_1}^{(i)} \Lambda \cdots \Lambda dz_{i_p}^{(i)} \end{cases}$$

in U_i. Let $\mathscr{H}_V(\alpha)$ denote the set of forms ω for which

$$||\omega||^2 = \sum_i \sum_I \int\limits_{U_i} \left|\omega_I^{(i)}\right|^2 \alpha(z^{(i)}) dv_z(i) < \infty.$$

Define the Scalar products of $\omega, \omega' \in \mathscr{H}_V(\alpha)$ by **69**

$$(\omega, \omega') = \sum_i \sum_I \int\limits_{U_i} \omega_I^{(i)} \overline{\omega_I'^{(i)}} \alpha(z^{(i)}) dv_z(i).$$

It is follows from Lemma 2 of Chap. I, §5 that convergence in $\mathscr{H}_V(\alpha)$ implies verified that $\mathscr{H}_V(\alpha)$ is complete. We can now prove, exactly as Theorem 4 of Chap I, §5 the following

Proposition 2. *If $\{U_i\}, \{U'_j\}$ are locally finite coverings of V, V' and α, α' are positive continuous functions on V, V', if $\{\varphi'_\mu\}$, $\{\varphi'_\nu\}$ are orthonormal bases for $\mathscr{H}_V(\alpha)$, $\mathscr{H}_{V'}$, (α'), then $\prod^*(\varphi_\nu) A \prod^*(\varphi'_\mu)$ from an orthonormal basis for $\mathscr{H}_{V\times V'}(\alpha\times\alpha')$ with respect to the covering $U_i\times U'_j$. Further, finite linear combinations of forms of the type $\prod^*(\omega)\Lambda'^*(\omega')$, ω, ω' holomorphic forms on V, V' respectively, are dense in the space of holomorphic forms on $V \times V'$.*

3 Submanifolds

Definition. Let V be a C^k manifold, $k \geq 1$. *A* C^r submanifold of V, $0 < r \leq k$ is a C^r manifold W and an injection i: $W \to V$ such that i is a C^r map and the map i_*: $T_A(W) \to T_{i(a)}(V)$ is an injection for every $a \in W$.

We identity the submanifolds (W_1, i_1) and (W_2, i_2) if there exists a C^r isomorphism h: $W_1 \to W_2$ such that $i_2 oh = i_1$.

70 **Remarks.** 1 It follows immediately that

$$\dim .W \leq \dim V.$$

Further, if the dimension of $W = m$, from the rank theorem it follows that for $a \in W$, there exist coordinate neighbourhoods (U_1, φ_1) of a and (U_2, φ_2) of $i(a)$ such that,

$$\varphi_2 \circ i \circ \varphi_1^{-1} | \varphi_1(U_1) \text{ is given by}$$
$$\varphi_2 \circ i \circ \varphi_1^{-1}(x_1, \ldots, x_m) = (x_1, \ldots, x_m, 0, \ldots, 0).$$

Hence given a system of local coordinates at a, it can be "extended "(in an obvious sense) to a system at $i(a)$.

2 If W is a closed subset of V, V being a C^k manifold of dimension n, if for each $a \in W$, there exists a coordinate neighbourhood (U, φ), and if the local coordinate $(x_1, \ldots, x_n)$ in U can be so chosen that

$$W \cap U = \left\{ x \middle| x_{r+1} = x_{r+2} = \cdots = x_n = 0 \right\},$$

then W is a C^k manifold of dimension r, and is a submanifold of V.

Proof. With the manifold structure defined in the obvious way, W is a C^k manifold and the injection $i : W \to V$ is a C^k map. It is easily verified that i_*: $T_a(W) \to T_{i(a)}(V)$ is an injection. □

3 If V is a C^k manifold of dimension n and if $f_{r+1}, \ldots, f_n$ are C^k functions on V such that $df_{r+1}, df_{r+2}, \ldots, df_n$ are linearly independent at all points of $W = \left\{ x \in V \middle| f_{r+1}(x) = f_{r+2}(x) = \cdots = f_n(x) = 0 \right\}$, then W is a submanifold of V, of dimension r.

71 *Proof.* Since $df_{r+1}, \ldots, df_n$ are locally independent at any point $a \in W$, we can find C^k functions $f_1, \ldots, f_r$ such that $(df_i)_a, 1 \leq i \leq n$, are linearly independent at a; if $f = (f_1, \ldots, f_n)$ then, by the inverse function theorem, f is a C^k diffeomorphism in a neighbourhood of a, By the change of coordinates $(x_1, \ldots, x_n) \to (y_1, \ldots, y_n), y_i = f_i(x_1, \ldots, x_n)$, we have,

$$W \cap U = \left\{ x \middle| y_{r+1} = \cdots = y_n = 0 \right\}.$$

□

Hence by remark 2), W is a sub manifold of dimension r.

Remark. Similar definitions and results apply to real and complex analytic submanifolds.

Corollary. *In $\mathbb{R}^{n+1}$, the unit sphere given by*

$$S^n = \left\{ x \middle| x_0^2 + x_1^2 + \cdots + x_n^2 = 1 \right\},$$

is a real analytic submanifold of dimension n.

Proof. If f is the function $x_0^2 + \cdots + x_n^2 - 1$, df is $\neq 0$ at all points of $S^n = \left\{ x \in \mathbb{R}^{n+1} \middle| f(x) = 0 \right\}$. □

4 If V, V' are C^k (real, complex analytic) manifolds, $V \times V'$ carries a natural structure of C^k (real, complex analytic) manifold.

Definitions. 1) Let V and W be C^k manifolds. Then a continuous map $f : V \to W$ is called locally proper if for every $y \in f(V)$, there exists a compact neighbourhood U of y in W such that $f^{-1}(U)$ is compact .

2) If V and W are C^k manifolds then a continuous map $f\colon V \to W$ is proper if for every compact set K in W, $f^{-1}(K)$ is compact. 72

Remark. If V and W are C^k manifolds and $f\colon V \to W$ is locally proper, then f is proper if and only if $f(V)$ is closed in W.

Proposition 1. *If i: $W \to V$ is a submanifold of V^n, then the following statement are equivalent.*

1) i is a homeomorphism of W onto $i(W)$ with the induced topology from V.

2) The map i: $W \to V$ is locally proper.

Proof. If the topology on W is same as that on $i(W)$, for any $a \in W$ there exists a compact neighbourhood K in W for which $i(K)$ is a compact neighbourhood of $i(a)$ in $i(W)$. Hence $i(K) \supset U_1 \cap i(W)$, U_1 open in V. Let U_2 be a relatively compact neighbourhood of $i(a)$ in U_1; then $\bar{U}_2 \cap i(W) \subset \bar{U}_1 \cap i(W)$ and hence $i(K)$ is compact and hence closed, $\bar{U}_1 \cap i(W) \subseteq i(K)$,

i.e. $\bar{U}_2 \cap i(W) \subset i(K)$,

i.e. $i^{-1}(\bar{U}_2) \subset K$ and $i^{-1}(\bar{U}_2)$ is compact. □

Hence 1) implies 2).

If the map i is locally proper, for each $i(a)$ there exists a compact
73 neighbourhood U in V such that that i^{-1} is compact. Then $i^{-1}(U)$ is a compact neighbourhood of a such that $i|i^{-1}(U)$ is a homeomorphism onto $i(U)$ since a continuous bijective map form a compact space to a hausdorff space is a homeomorphism. Hence 2) implies 1).

Note that if 1) or 2) is satisfied, then $i(W)$ is locally closed in V. The converse is, however, false.

Definition. A submanifold W of V is called a closed submanifold if i: $W \to V$ is proper.

We shall give an example of a submanifold for which the injection i does not preserve the topology. For that we use the following

Theorem (Kronecker). *Let $\alpha_1, \ldots, \alpha_n$ be n real numbers which are linearly independent over the ring $\mathbb{Z}$ of integers, Let $T^n = S^1 \times \cdots \times S^1 = \{e^{i\theta_1}, \ldots, e^{i\theta_n})|\theta_i$ real $\}$, and let ω: $\mathbb{R} \to T^n$ denote the map $\omega(t) = (e^{i\alpha_1 t}, \ldots, e^{e\alpha_n t})$. Then the image $\omega(\mathbb{R})$ is dense in T^n.*

The best proof of this theorem is, without question, that given by H. Weyl [45].

Example. T^n defined above is a real analytic manifold of dimension n. Consider the map ω: $\mathbb{R} \to T^n$ defined above. ω is an injection for if $\omega(x_1) = \omega(x_r)$,

$$\alpha_i x_1 = 2\pi m_i + \alpha_i x_2, i = 1, \ldots, n, m_i \in \mathbb{Z}$$

and if $x_1 \neq x_2$, and $d_1, \ldots k_n$ are integers, not all zero, with $\sum k_i m_i = 0$, then $\alpha k_1 + \cdots + \alpha_n k_n = 0$, which contradicts the hypothesis that 74
$\alpha_1, \ldots, \alpha_n$ are linearly independent over $\mathbb{Z}$. Also rank (di) is maximal $= 1$ at all points of $\mathbb{R}$. Hence $\mathbb{R}$ is a submanifold of T^n. Let $\omega(\mathbb{R}) = D$. If ω preserves the topology, by the proposition proved above, D is locally closed and hence $D = \overline{D} \cap U$, U open in T^n. By Kronecker's theorem D is dense in T^n i. e. $D = T^n \cap U = U$. But D is not open in T^n if $n > 1$, and thus we arrive at a contradiction.

Proposition 2. *If i: $W \to V$ is a C^k submanifold of V, and M is a C^k manifold, then a continuous map f: $M \to W$ is C^k if and only if $i o f$: $M \to V$ is C^k.*

Proof. Let $a \in W$; choose coordinate neighbourhoods U of a in W and U' of $i(a)$ in V such that $i|U \to U'$ is the map $i(x_1, \ldots, x_m) = (x_1, \ldots, x_m, 0, \ldots, 0)$. We may restrict ourselves to the subset $N = f^{-1}(U)$ of M. The proposition is than obvious since if $f : N \to U$ has components given by $f(u) = (f_1(u), \ldots, f_m(u))$, then $i \circ f : N \to U'$ has components given by $i \circ f(u) = (f_1(u), \ldots, f_m(u), 0, \ldots, 0)$. ◻

Proposition 3. *If $i : W \to V$ is a C^k submanifold, then for a germ g_a of a continuous function at $a \in W$ to be C^k, it is necessary and sufficient that there is a C^k germ G_b at $b = i(a)$ such that $G_b o i = g_a$. Conversely. if i is a continuous injection of the C^k manifold W into V having this property, then $i : W \to V$ is a submanifold.*

Proof. Let i be a submanifold and choose coordinates at a, $(U; x_1, \ldots, x_m), (U'; x_1, \ldots, x_n)$ at $b = i(a)$ such that $i|U$ is the map $i(x_1, \ldots, x_m) = (x_1, \ldots, x_m, 0, \ldots, 0)$. If g is C^k on U, and G is the C^k function on U' defined by $G(x_1, \ldots, x_n) = g(x_1, \ldots, x_m)$, clearly $G \circ i = g$. ◻

Conversely, let $i : W \to V$ be an injection such that C^k germs g at a **75**
are precisely the germs G 0 i, G, a C^k germ on V at $b = i(a)$. Then i is C^k for if, in terms of local coordinates $(U'; x_1, \ldots, x_n)$ at b, $i_1 \ldots, i_n$ are the components of i, then $i_l = x_l \circ i$, and $x_l \in C^k$. We assert that there exists a germ of C^k map $p : V \to W$ at $b \in V$, $p(b) = a$, such that $p \circ i =$ identity near a in W. In fact, if $(U; x_1, \ldots, x_m)$ are local coordinates at $a \in W$, then, by hypothesis, there exist C^k germs $P_l, l = 1, \ldots, m$ at b such that $x_l - p_l$
circi; the p_l may be looked upon as the germ of a C^k mapping p: $V \to U$ for which $p(b) = a$, $poi =$ identity near a in U.

We then have

$$(p_*)_{i(a)} o (i_*)_a = \text{ identity on } T_a(W),$$

so that $(i_*)_a$ is injective.

Proposition 4. *If i: $W \to V$ is a closed submanifold, i. e. i is proper, then for any C^k function g on W, there exists a C^k function G on V such that Goi = g.*

Proof. We identify W with $i(W)$. Let U_a be a neighbourhood of a in V, G_a a C^k function in U_a with $G_a = g$ on $U_a \cap W$. Let $\{U_{a_\alpha}, V - W\}_{\alpha \in A}$ be a locally finite covering of V such that for each α, $U_{a_\alpha} \subset U_a$ for some $a \in W$. Let $(\varphi_\alpha, \varphi)$ be a C^k partition of unity relative to this covering and $h_\alpha = \varphi_\alpha . G_{a_\alpha}$ in U_{a_α}, 0 in $V - U_{a_\alpha}$. Clearly, if $G = \sum h_\alpha$, then G is C^k on V and, for $x \in W$, $G(x) = \sum\limits_{x \in U_{a_\alpha}} h_\alpha(x) = \sum\limits_{x \in U_{a_\alpha}} \varphi_\alpha(x)$. $G_{a_\alpha}(x) = g(x) \sum\limits_{x \in U_{a_\alpha}} \varphi_\alpha(x) = g(x)$. □

76 **Remark.** Propositions 2 and 3 and their proofs remain valid for real or complex analytic manifolds. Prop. 3 is true for real analytic manifolds, but is very difficult to prove; see H. Cartan [6] and H. Grauert [13], it is false for complex manifolds in general. A very important special case, due to K Oka, for which it is true will be dealt with later (§7).

4 Exterior differentiation

If V is a C^k manifold, $A^p_r(V)$ denotes the C^r differential forms of degree p, on V, $0 \leq r < k$ if $p > 0$, $0 \leq r \leq k$ if $p = 0$. In what follows V shall denote a C^k manifold which is countable at ∞ with $k \geq 2$.

Definition. An exterior differentiation d is a map d: $A^p_r(V) \to A^{p+1}_{r-1}(V)$ for each $p \geq 0$ and $1 \leq r < k$ if $p > \circ$, $1 \leq r \leq k$ if $p = 0$, satisfying the following.

1) *d is $\mathbb{R}$-linear, i. e.* $d\left[\alpha\omega_1 + \beta\omega_2\right] = \alpha d\omega + \beta d\omega_2$ *for* $\alpha, \beta \in \mathbb{R}$, ω_1, $\omega_2 \in A^p_r(V)$.

2) $d|A^0_k(V)$ *is given by* $(df)_a =$ *the image of* f *in* $T^*_a(V)$.

3) $d(df) = 0$ *for* $f \in C^k_a$.

4) *If* $\omega_1 \in A^p_r(V)$, $\omega_2 \in A^q_r(V)$, $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge d\omega_2$.

We deduce the following properties of an exterior differentiation form its definition.

I. *d is a local operator, i. e. if for an open set U we have $\omega \nmid U = 0$. then $d\omega \nmid_U = 0$.*

Proof. If U' is a coordinate neighbourhood $\subset U$, and U'' is a relatively **77**
compact subset of U, there exists a C^k function f on U' which is ***** on $U'' = 1$ in a neighbourhood of $\partial U'$; hence there exists $f \in C^k(U)$ such that

$$\begin{aligned} f(x) &= 0 \text{ for } x \in U'' \\ &= U \text{ for } x \in V - U. \end{aligned}$$

□

Hence if $\omega \nmid U = 0, \omega = f\omega$ so that $d\omega = (df) \wedge \omega + fd\omega$ Since $f = 0$, and by 2), $df = 0$ on U'', we deduce that $d\omega \nmid U'' = 0$. It follows that $d\omega$ vanishes in a neighbourhood of any point of U, so that $d\omega|U = 0$.

II. $d^2 = 0$ (if $k \geq 3$).

Proof. It is enough to prove this with V replaced by a coordinate neighbourhood. Let

$$\omega = \sum_{i_1<\cdots<i_p} f_{i_1\ldots i_p} dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_p} \epsilon A^p_{k-1}(V).$$

□

Then

$$d(d\omega) = \sum_{i_1<\cdots<i_p} d[df_{i_1\ldots i_p} \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_p} + f_{i_1\ldots i_p} d(dx_{1_1} \wedge \ldots \wedge dx_{i_p})].$$

Now, by 3) and 4), $d(dx_{i_1} \wedge \ldots dx_{i_p})$

$$= \sum_{r=1}^{p} (-1)^{r-1} (dx_{i_1} \wedge \ldots \wedge d^2 x_{i_r} \wedge \cdots \wedge dx_{i_p})$$
$$= 0.$$

Hence $d(d\omega) = \sum_{i_1<\cdots<i_p} \{df_{i_1\ldots i_p} \wedge d(dx_{i_1} \wedge \ldots dx_{i_p}) + d^2 f_{i_1\ldots i_p} \wedge dx_{i_1} \ldots \wedge dx_{i_p}\} = 0.$

78 We shall now prove *the existence and uniqueness of the exterior differentiation*. It suffices to prove the existence and uniqueness any coordinate neighbourhood.

Define d_1 by $d_1(\omega) = \sum_{i_1<\cdots<i_p} d(f_{i_1\ldots i_p}) \Lambda dx_i \Lambda \ldots \Lambda dx_{i_p}$ where $\omega \in A^p_r(V)$ is given by

$$\omega = \sum_{i_1<\cdots<i_p} f_{i_1\ldots i_p} dx_{i_1} \Lambda dx_{i_2} \Lambda \ldots \Lambda dx_{i_p}$$

It is easily seen that d_1 satisfies the conditions 1) and 2). As for 3)

$$d_1 f = \sum_i \frac{\partial f}{\partial x_i} dx_i$$

$$\begin{aligned}
\Rightarrow d_1^2 f &= \sum_i \left(\sum_j \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j \right) \Lambda dx_i \\
&= \sum_{i,j} \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j \Lambda dx_i = \sum_{j<i} \left(\frac{\partial^2 f}{\partial x_j \partial x_i} - \frac{\partial^2 f}{\partial x_i \partial x_j} \right) dx_j \Lambda dx_i \\
&= 0
\end{aligned}$$

We shall show that d_1 satisfies 4). It is enough to verify this for $\omega_1 = f_1 dx_{I_1}$, $\omega_2 = f_2 dx_{I_2}$, where $dx_{I_1} = dx_{i_1} \Lambda \ldots \Lambda dx_{i_p}$ and $dx_{I_2} = dx_{j_1} \Lambda \ldots \Lambda dx_{j_q}$.

Now, 79

$$\begin{aligned}
\omega_1 \Lambda \omega_2 &= f_1 f_2 dx_{I_1} \Lambda dx_{I_2} \\
d_1(\omega_1 \Lambda \omega_2) &= d_1(f_1 f_2) \Lambda dx_1 \Lambda dx_{I_2} \\
&= [(d_1 f_1). f_2 + f_1(d_1 f_2)] \Lambda dx_{I_1} \Lambda dx_{I_2} \\
&= d_1 f_1 \Lambda dx_{I_1} \Lambda f_2 dx_{I_2} + (-1)^p f_1 dx_{I_1} \Lambda d_1 f_2 \Lambda dx_{I_2} \\
&= (d_1 \omega_1) \Lambda \omega_2 + (-1)^p \omega_1 \Lambda (d_1 \omega_2).
\end{aligned}$$

We are using the obvious fact that the d in 2) of the definition of exterior differentiation satisfies $d(f_1 f_2) = f_1 d f_2 + f_2 d f_1$. Hence d_1 defined above is an exterior differentiation. If d_2 is another exterior differentiation, $\omega = f dx_{i_1} \Lambda \cdots \Lambda dx_{i_p}$, it follows from 4) that $d_2 \omega = d_2 f \Lambda dx_{i_1} \Lambda \cdots \Lambda dx_{i_p} + \sum_{r=1}^{p} (-1)^{r-1} f dx_{i_1} \Lambda \cdots \Lambda d_2(dx_{i_r}) \Lambda \cdots \Lambda dx_{i_p}$.

By 2) and 3) it follows that $d_2 f = d_1 f$ and $d_2(dx_{i_r}) = d_2(d_2 dx_{i_r}) = 0$. Hence $d_2 \omega = df \Lambda dx_{i_1} \Lambda \cdots \Lambda dx_{i_p}$, i. e. the exterior differentiation is unique.

We have already remarked that $T_a^*(V)$ is the dual of $T_a(V)$. Consider $\overset{p}{\Lambda} T_a^*(V)$ as the dual of $\overset{p}{\Lambda} T_a(V)$, i. e. for every p-form $\omega . \omega_a$ defines an alternate linear function of $\sum_{r=1}^{p} T_a(V)$ which determines ω_a uniquely. Hence ω gives rise to an alternate mapping of p-tuples of C^{k-1} vector fields into C^{k-1} functions.

Proposition 1. *If ω is a p-form, $X_1, \ldots X_{p+1}$, C^{k-1} vector fields, then for* 80 *any $a \in V$, $d\omega$ is the linear function given by*

$$(d\omega)(X_1, X_2, \ldots, X_{p+1}) = \sum_1^{p+1} (-1)^{i-1} X_i(\omega(X_1, \ldots X_i, \ldots, x_{p+1}))$$
$$+ \sum_{i<j} (-1)^{i-j} \omega([X_i, X_j]\ , X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+1}).$$

Here the Λ over a term means that this term is to be omitted.

We shall prove the proposition only when ω is a 1-form. The general case involves more complicated calculations. It is sufficient to prove the formula in a coordinate neighbourhood. By linearity, it is enough to prove it for forms of the type $\omega = f dg$. If $\omega = f dg$, f, g functions, then $d\omega = df \Lambda dg$.

Hence

$$D = (df \Lambda dg)(X_1, X_2)$$
$$= \det \begin{vmatrix} (df)(X_1) & (df)(X_2) \\ (dg)(X_1) & (dg)(X_2) \end{vmatrix}$$

where

$$(df)X_1 = X_1(f).$$

Hence

$$D = X_1(f)X_2(g) - X_2(f)X_1(g)$$
$$= X_1(fX_2(g) - X_2(f\ X_1(g))\ - fX_1(X_2(g))) + fX_2(X_1(g)).$$
$$= X_1(\omega(X_2)) - X_2(\omega(X_1))\omega([X_1, X_2])$$

which is the required formula.

81 **Proposition 2.** *If V and W are C^k manifolds and $f\colon V \to W$ a C^k map, we have, for any p form ω on W,*

(3.1) $$d_{(V)}(f^*(\omega)) = f^*(d_{(W)}(\omega)).$$

Proof. We may clearly suppose that W is an open set in $\mathbb{R}^m$. Since f^* is an algebra homomorphism of $\Lambda T^*(W)$ into $\Lambda T^*(V)$, it is enough to prove (3.1) for a system of generators of $\Lambda T^*(W)$ e.g. when ω is a function or is the exterior derivative of a function. If $\omega = \varphi$ is a function,

$$(d\varphi)_{f(a)} \in C^k_{f(a)}/S^k_{f(a)}$$

and $f^*[(d\varphi)_{f(a)}] = d(\varphi \circ f)_a$ by definition of f^*. □

If $\omega = d\varphi$ where φ is a function,

$$f^*(d(d\varphi)) = 0 \text{ and}$$
$$d[f^*(d\varphi)] = d[d(\varphi o f)] = 0.$$

q.e.d.

For a somewhat different approach to exterior differentiation set Koszul [22].

5 Orientation and Integration

Definition. On a C^k manifold V with $k \geq 1$, a continuous n-form ω which is nowhere zero on V is called an orientation on V and if there exists an orientation on V, V is called orientable.

Proposition 1. *A manifold V is orientable if and only if there exists* 82
a system of coordinates (U_i, φ_i), $\bigcup U_i = V$, such that the transformation $\varphi_i \circ \varphi_j^{-1}|\varphi_{j(U_i \cap U_j)}$ has positive jacobian det $|d(\varphi_i \circ \varphi_j^{-1})|$ whenever $U_i \bigcap U_j \neq \phi$.

Proof. If ω is an orientation of V, for any $a \in V$ there exists a connected coordinate neighbourhood (U_a, φ) of a such that in terms of local coordinates $\omega_x = f(x) dx_1 \Lambda \cdots \Lambda dx_n$, for $x \in U_a$. Further φ can be so chosen that $f(x) > 0$ for $x \in U_a$ (change x_1 to $-x_1$ if necessary). Consider a system of coordinate neighbourhoods (U_i, φ_i), such that for any $x \in U_i$, ω_x in terms of local coordinates can be written as $\omega_x = f_i(x) dx_1^{(i)} \Lambda \cdots \Lambda dx_n^{(i)}$, $f_i(x) > 0$. Then the jacobian of the transformation $\varphi_i \circ \varphi_j^{-1}$ is a quotient of the functions $f_i \circ \varphi_i^{-1}$ and so $> d$. □

Conversely if there exists a system of coordinate neighbourhoods (U_i, φ_i) with the above property, consider a partition of unity $\{\Psi_i\}$ subordinate to the covering $\{U_i\}$. Define ω_x in terms of local coordinates as $w_x = \sum_i \Psi_i(x) dx_1^i \Lambda \cdots \Lambda dx_n^i$.

Then ω_x is a continuous n form which is > 0 for every x and hence is an orientation of V.

Remark. It follows that on a C^k manifold, there is a $C^{k-1} n$ form which is nowhere zero.

Let

$$E = \{\xi \in \overset{n}{\Lambda} T^*(V) | \xi \neq 0\}$$

and

$$p(\xi) = a \text{ if } \xi \in \overset{n}{\Lambda} T_a^*(V).$$

83 Define an equivalence relation in E by

$$\xi_1 \sim \xi_2 \text{ if } x = p(\xi_1) = p(\xi_2), \text{ and there is } \lambda > 0 \text{ with } \xi_1 = \lambda \xi_2.$$

Let $\tilde{V} = E/_\sim$.

Proposition 2. *V is hausdorff and $\tilde{V} \to V$ is a covering.*

Proof. The equivalence relation is clearly open and it is easily seen that the graph of the equivalence relation in $E \times E$ is closed Hence V is hausdorff. Let (U, φ) be a coordinate neighbourhood of $a \in V$. Define ξ and η in terms of local coordinates as

$$\xi_x = dx_1 \Lambda \cdots \Lambda dx_n$$

and

$$\eta_x = -dx_1 \Lambda \cdots \Lambda dx_n.$$

□

Then $p^{-1}(U) = \left(\bigcup_{x \in U} \xi_x\right) \bigcup \left(\bigcup_{x \in U} \overline{\eta}_x\right)$ and $\tilde{V}$ is a covering.

Corollary 1. *If V is connected, V is orientable if and only if $\tilde{V}$ is not connected.*

Proof. If V is connected and orientable, let ω be an orientation. Then $\bigcup_{x \in V} \overline{(\omega_x)}$ is non- empty open and closed subset of $\tilde{V}$ and hence $\tilde{V}$ is not connected. □

If V is connected and $\tilde{V}$ not connected, consider $\overline{\xi_a} \in \tilde{V}$ and let U_a be the connected component of $\overline{\xi_a}$ in $\tilde{V}$. Then if $p|U_a = \pi$, $\pi\colon \to V$ is a covering and if $p^{-1}(x) = \pi^{-1}(x)$ for some $x \in V$, $p^{-1}(y) = \pi^{-1}(y)$ for every $y \in V$ and $U_a = \tilde{V}$ which is a contradiction. Hence for any 84
$x \in V$, $\pi^{-1}(x)$ contains exactly one point, so that π is a homeomorphism. It is easily verified that there is a continuous n form ω on V for which $\omega_x \in \pi^{-1}(x)$ for any x; hence V is orientable.

Corollary 2. *If V is a simply connected manifold it is orientable.*

Proof. If V is simply connected, clearly $\tilde{V}$ is not connected and the proof follows form Corollary 1. □

It can be be show easily that $\tilde{V}$ is *always* orientable; in fact if, for $a \in \tilde{V}$, $\omega_{p(a)}$ is an n-co vector at $p(a)$ with $\omega_{p(a)} \in a$, we see at once (partition of unity) that there is an n-form $\tilde{\omega}$ on $\tilde{V}$ with $\tilde{\omega}_a = \lambda_a p^*(\omega_{p(a)})$ with $\lambda_a > 0$.

Let $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1 \geq \circ\}$.

Definition. A hausdorff topological space V is said to be a C^k manifold with boundary and of dimension n if there exists a system of "coordinate neighborhoods" (U_i, φ_i), such that $\bigcup U_i = V$ and φ_i is a homeomorphism of U_i onto an open subset of $\mathbb{R}^n_+$ for which whenever $U_i \cap U_j \neq 0$, then map $\varphi_i \circ \varphi_j^{-1} \Big| \varphi_j(U_i \cap U_i)$ is a C^k map of $\varphi_j(U_i \cap U_j)$ as a subset of $\mathbb{R}^n_+$.

If f is a real valued function on $\mathbb{R}^n_+$, $\dfrac{\partial f}{\partial x_i}$, $i \geq 2$ are defined in the same way as for a function on $\mathbb{R}^n$ and

$$\frac{\partial f}{\partial x_1}\Big|_a = \lim_{h \to +0} \frac{f(a_1 + h,\ a_2, \ldots, a_n) - f(a_1, \ldots, a_n)}{h}.$$

For a C^k manifold V with boundary, $k \geq 1$, C^k functions, tangent vectors, $T_a(V)$, differential forms etc. are defined in the same way as for 85 a manifold. Orientation is also defined in an analogous way. Hereafter V will denote a C^k manifold with boundary, which is countable ∞.

Definition. A vector X in $T_a(V)$ is called a positive tangent vector or an inner normal if for any $f \in m_a^k$ with $f(x) \geq 0$ for x in a neighbourhood of a, we have $X(f) \geq 0$ and if there exists an $f \in m_a^k$, $f(x) \geq 0$ in a neighbourhood of a, for which $X(f) > 0$. (m_a^k is the set of C^k germs at a which vanish at a.) A tangent vector X is negative (or an outer normal) if $-X$ is positive.

Let a be a point in V. If there exists a coordinate neighbourhood (U, φ) of a, such that $\varphi(U)$ is an open set in $\mathbb{R}^n$, for any $f \in m_a^k$ consider f as a function of local coordinates. If $f(x) \geq 0$ for x in a neighbourhood U' of a, $U' \subset U$, f has a minimum at a and hence $\left.\dfrac{\partial f}{\partial x_i}\right|_a = 0$, $1 \leq i \leq n$. Hence for any $X \in T_a(V)$, $X(f) = \sum x(x_i)\dfrac{\partial f}{\partial x_i}|_a = 0$, i. e. there does not exist a positive tangent vector in $T_a(V)$.

Proposition 3. *Let $a \in V$ and suppose that there exists a coordinate neighbourhood (U, φ) of a such that $\varphi(U)$ is an open set of $\mathbb{R}_+^n$ and $\varphi(a) \in \{x \in \mathbb{R}_+^n \big| x_1 = 0\}$, then a tangent vector $X_i \in T_a(V)$ given by $X = \sum \alpha_i \dfrac{\partial}{\partial x_i}$ is a positive tangent vector if and only if $\alpha_1 > 0$.*

Proof. We may suppose that $\varphi(a) = 0$. If $f \in m_a^k$ and $f(x) \geq 0$ in a neighbourhood U' of a, consider $f(a_1, x_2, \ldots, x_n)$ as a function of $(x_2, \ldots, x_n)$, in terms of local coordinates. The set $U_1 = \{(x_2, \ldots, x_n)$ $(x_1, \ldots, x_n) \in \varphi(U)\}$, is open in $\mathbb{R}^{n-1}$ and by the same argument as above $\left.\dfrac{\partial f}{\partial x_i}\right|_a = 0$, $2 \leq i \leq n$. Now, if $f \in m_a^k$, $f \geq 0$, we have

$$\left.\frac{\partial f}{\partial x_1}\right|_a = \lim_{h \to +0} \frac{f(h, 0, \ldots, 0) - f(0, \ldots, 0)}{h}$$
$$= \lim_{h \to +0} \frac{f(h, 0, \ldots, 0)}{h} \geq 0$$

86 and by choosing $f(x) = x_1$, we see that

$$\left.\frac{\partial f}{\partial x_1}\right|_a = 1, \ f \in m_a^k, \ \text{ and } \ f(x) \geq 0 \text{ for } x \text{ in } U.$$

□

Then $$X(f) = \alpha_1 \left(\left.\frac{\partial f}{\partial x_1}\right|_a \right) = \alpha_1.$$

Hence if $\alpha_1 > 0$, X is a positive tangent vector and conversely if X is a positive tangent vector $\alpha_1 > 0$.

Definition. An element $\omega \in T_a^*(V)$ is called positive (negative) if $\omega(X) > 0 (< 0)$ for any positive $X \in T_a(V)$.

In terms of local coordinates (U, φ), $\varphi(U) \subset \mathbb{R}_+^n$, $\varphi(a) = 0$, an element $\omega = \sum \alpha_i dx_i$ is > 0 if and only if $\alpha_1 > 0$, $\alpha_2 = \cdots = \alpha_n = 0$.

Definition. The set $\partial V = \{x \in V \big|$ there exists a coordinate neighbourhood (U, φ) of x with $\varphi(x) = 0\}$ is called the boundary of V.

Remark. It is clear from the above discussion that $x \in \partial V$ if and only if there exists a positive tangent vector in $T_x(V)$. V is said to have no boundary if $\partial V = \phi$.

Proposition 4. *∂V is a C^k manifold of dimension $n - 1$.*

Proof. If $a \in \partial V$, there exists a coordinate neighbourhood (U, φ) of a such that $\varphi(a) = 0$. □

Let $U' = \left\{x \in U \middle| \varphi(x) = \circ\right\}$. Clearly $U' = \partial V \cap U$. For $x \in U'$, **87**
define φ' by

$$\varphi'(x) = (x_i)_{2 \leq i \leq n}$$

φ' is obviously a homeomorphism of U' onto open set in $\mathbb{R}^{n-1}$. If (U_1, φ_1), (U_2, φ_2) are coordinates in V inducing coordinates (U_1', φ_1'), (U_2', φ_2') on ∂V, the map $\varphi_1' \circ (\varphi_2')^{-1}$ is the restriction of $\varphi_1 \circ \varphi_2^{-1}$ to a submanifold of $\varphi_2(U_2)$, and so is C^k and so is C^k. Thus (U_i', φ_i') is a system of coordinate neighbourhood for ∂V and ∂V is a C^k manifold of dimension $n - 1$. It is obvious that ∂V has no boundary.

Further ∂V is clearly a C^k submanifold of V which is in fact a closed submanifold. We shall therefore identify, for $a \in \partial V$, the tangent space $T_a(\partial V)$ with a subspace of $T_a(V)$.

Proposition 5. *If $a \in \partial V$ and X_1, X_2 are positive tangent vectors in $T_a(V)$, then there exists $\alpha > 0$ such that $X_1 - \alpha X_2 \in T_a(\partial V)$.*

Proof. In terms of a local coordinate system (U, φ) at a, let

$$X_1 = \sum \alpha_i \frac{\partial}{\partial x_i}, X_2 = \sum \alpha_i \frac{\partial}{\partial x_i}.$$

□

Then $\alpha_1 > 0$, $\beta_1 > 0$. Let $\alpha = \dfrac{\alpha_1}{\beta_1}$, then

$$X_1 - \alpha X_2 = \sum_{i=2}^{n} (\alpha_i - \alpha\beta_i) \frac{\partial}{\partial x_i} \in T_a(\partial V).$$

Definition. $\xi \in \overset{n-1}{\Lambda} T_a(V)$, is called positive if for any outer normal $e \in T_a(V)$, $e \Lambda \xi$, as an element of $\overset{n}{\Lambda} T_a(V)$, is positive.

88 It is clear that we may look upon $\overset{n-1}{\Lambda} T_a(\partial V)$ as a subspace of $\overset{n-1}{\Lambda} T_a(V)$; similar remarks apply to $\overset{n-1}{\Lambda} T_a^*(\partial V)$ and $\overset{n-1}{\Lambda} T_a^*(V)$.

Proposition. *If V is oriented, so is ∂V.*

Proof. As in the above definition, we say that $\omega_1 \in \overset{n-1}{\Lambda} T_a^*(\partial V)$ is positive if for any

$$\omega_2 \in T_a^*(V),\ \omega_2 < 0,\ \text{we have}\ \ \omega_2 \Lambda \omega_1 > 0.$$

□

Let ω be the orientation of V and in terms of ”positive” local coordinates suppose that

$$\omega_x = f(x) dx_1 \Lambda \cdots \Lambda dx_n,\ f(x) > 0\ \text{ for each } x.$$

then $\omega'_x = -dx_2 \Lambda \cdots \Lambda dx_n$ is in $\overset{n-1}{\Lambda} T_a^*(\partial V)$ and is positive for each x in ∂V. The condition of the positivity of the Jacobians is trivially verified.

Remark. If V is a C^k manifold and D an open set such that for any $a \in (\bar{D} - D)$, there exists a neighbourhood U in V and a C^k function g in U with $(dg)_a \neq 0$, $D \cap U = \{x \in U \,\big|\, g(x) > 0\}$, then $\bar{D}$ is a manifold with boundary and ∂D coincides with the topological boundary $(\bar{D} - D)$ of D.

Theorem 1 (Formula for change of variable)**.** *Let Ω, Ω' be open sets in $\mathbb{R}^n$ and $h\colon \Omega' \to \Omega$ a C^1 homeomorphism (so that h is bijective and the jacobian det $dh(y) \neq 0$ for $y \in \Omega'$). Then, if f is a continuous function with compact support in Ω, we have*

$$\int_{\Omega} f(x) dx_1 \ldots dx_n = \int_{\Omega'} (f \circ h)(y).|det\ dh(y)| dy_1 \ldots dy_n. \tag{5.1}$$

Proof. We first prove the formula when h is a linear transformation. 89
Let A denote the matrix of h(with respect to the canonical basis of $\mathbb{R}^n$). By the elementary divisors theorem, A can be written as a product of finitely many matrices A_i each of which is either a diagonal matrix or an elementary matrix viz. the matrix corresponding to one of the linear transformations

(a) $h(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_k, x_{i+1}, \ldots, x_{k-1}, x_i, x_{k+1}, \ldots, x_n)$

(b) $h(x_1, \ldots, x_n) = (x_1 + x_2, x_2, \ldots, x_n)$.

□

It is clearly sufficient to prove (1) for matrices of these special kinds. For diagonal matrices of type (a), the formula (1) is a trivial consequence of Fubini's theorem. For transformations h of type (b) we have, by Fubini's theorem

$$\int_{\Omega'} (f \circ h)(y) |\det dh(y)| dy_1 \cdots dy_n$$
$$= \int_{\mathbb{R}^n} f(x_1 + x_2, x_2, \ldots, x_n) dx_1 \cdots dx_n$$

$$= \int_{\mathbb{R}^{n-1}} dx_2 \cdots dx_n \int_{\mathbb{R}^n} f(x_1 + x_2, \ldots x_n) dx_1$$

$$= \int_{\mathbb{R}^{n-1}} dx_2 \cdots dx_n \int_{\mathbb{R}} f(x_1, \ldots, x_n) dx_1$$

(since Lebesque measure on $\mathbb{R}$ is translation invariant)

$$= \int_{\mathbb{R}^n} f(x) dx_1 \cdots dx_n = \int_{\Omega} f(x) dx_1 \cdots dx_n.$$

To prove (5.1) in general, we remark that it suffices to prove the inequality

$$\int_{\Omega} f(x) dx_1 \cdots dx_n \leq \int_{\Omega'} (f \circ h)(y) |\det dh(y)| dy_1 \cdots dy_n \tag{5.2}$$

90 for all non-negative f with compact support. (Apply the inequality to h^{-1} to obtain equality.) Moreover, by the definition of the Riemann integral, it is sufficient to prove the following statement (90) If Q is a closed cube with equal sides contained in Ω', we have

$$\mu(h(Q)) \leq \int_{Q} |\det dh(y)| dy_1 \cdots dy_n$$

here μ denotes Lebesgue measure in $\mathbb{R}^n$.

Proof of (5.3). Let K denote any closed cube, with equal sides say, δ, contained in Ω'. For an $n \times n$ matrix $A = (a_{ij})$, set $||A|| = \max_i \sum_j |a_{ij}|$. Note that if I is the unit matrix, we have $||I|| = 1$.

Let $h = (h_1, \ldots, h_n)$. Taylor's formula shows that if $x, y \in K$

$$h_i(x) - h_i(y) = \sum_j \frac{\partial h_i}{\partial x_j}(\theta_i)(x_j - y_j), \theta_i \in K,$$

so that
$$|h_i(x) - h_i(y)| \leq \sup_{a \in K} ||dh(a)||.\delta.$$

Consequently, $h(K)$ is contained in a cube of side $\delta. \sup_{a\in K} \|dh(a)\|$ so that

(5.4) $$\mu(h(K)) \leq \{\sup_{a\in K} \|dh(a)\|\}^n \mu(K)$$

If we apply (5.4) to the transformation $g = A.h$, where A is the inverse of the linear transformation $(dh)(a)$ for a fixed $a \in K$, and observe **91**
that, by (5.1) applied to the linear transformation A we have

$$\mu(g(K)) = |\det dh(a)|^{-1} \mu(h(K)),$$

we obtain

(5.5) $$\mu(h(k)) \leq |\det dh(a)| \{\sup_{b\in K} \|(dh(a))^{-1} dh(b)\|\}^n \mu(K).$$

We observe that as the sides of K tend to zero, $(dh(a))^{-1} dh(b) \to I$, uniformly for b in any compact subset of Ω' (90) is now easy to prove. Divide Q into ε^{-n} cubes K_i of side (ε. side of Q), and let $a_i \in K_i$. Then

$$\sup_{b\in K_i} \|(dh(a_i))^{-1} dh(b)\| \leq 1 + \alpha(\varepsilon), \text{ where } \alpha(\varepsilon) \to 0 \text{ as } \varepsilon \to 0.$$

The inequality (5) now gives

$$\mu(h(Q)) \leq \sum_i \mu(h(K_i)) \leq (1 + \alpha(\varepsilon))^n \sum_i |\det |dh(a_i)| \mu(K_i).$$

As $\varepsilon \to 0$, by definition, the sum on the right converges to $\int\limits_Q |\det dh(y)| dy_1 \cdots dy_n$, so that, since $\alpha(\varepsilon) \to 0$, we obtain (5).

Integration.

Let V be an oriented n dimensional C^k manifold (with or without boundary) countable at infinity ($k \geq 1$). Let ω be a continuous n form on V with compact support. We shall define the integral

$$\int_V \omega$$

as follows.

Let $\{U_i, \varphi_i\}$ be a locally finite family of coordinate systems such that 92
φ_i induces the given orientation on U_i from that of $\mathbb{R}^n$; the Jacobians det $|d(\varphi_i \circ \varphi_j^{-1}|$ are then all positive. Let $\{\alpha_i\}$ be a partition of unity relative to the covering $\{U_i\}$; let $\Omega_i = \varphi_i(U_i)$, and let $x_1^{(i)}, \ldots, x_n^{(i)}$ denote the running coordinates in Ω_i. Let $(\varphi_i^{-1})^*(\alpha_i \omega) = g_i(x^{(i)}) dx_1^{(i)} \wedge \cdots \wedge dx_n^{(i)}$. We set

$$\int_V \omega = \sum_i \int_{\Omega i} g_i(x^{(i)}) dx_1^{(i)}, \ldots, dx_n^{(1)}$$

(the latter integral being an ordinary Riemann or Lebesgue integral); the sum is finite since ω has compact support. The integral so defined is linear: we have $\int_V (\omega_1 + \omega_2) = \int_V (\omega_1 + \omega_2) = \int_V \omega_1 + \omega_2$ and $\int_V \lambda\omega = \lambda \int_V \omega$ for $\lambda \in \mathbb{R}$. It is, however, necessary in applications to know that the definition above is independent of the covering U_i, and the functions α_i used in the definition. We shall denote the integral defined above temporarily by $I(\omega)$. Since I is linear, its invariance of the $\{U_i, \alpha_i\}$ results at once from the following.

Lemma 1. *Let (U, φ) be any coordinate system such that* det $|d(\varphi \circ \varphi_i^{-1}|$ *is positive on $\varphi_i(U_i \cap U)$ for each i. Let ω be an n form with support in U, and , in terms of the local coordinates in $\varphi(U) = \Omega$; let*

$$\omega = f(x) dx_i \wedge \cdots \wedge dx_n.$$

Then we have

$$\int_\Omega f(x) dx_1 \cdots dx_n = I(\omega).$$

93 *Proof.* It is enough to prove that if

$$\alpha_i \omega = f_i(x) dx_1 \wedge \cdots \wedge dx_n = g_i(x^{(i)}) dx_1^{(i)} \wedge \cdots \wedge dx_n^{(i)}$$

then

$$\int_\Omega f_i(x) dx_1, \ldots, dx_n = \int_{\Omega_i} g_i(x^{(i)}) dx_1^{(i)}, \ldots, dx_n^{(i)}.$$

□

The integrals are respectively $\int\limits_{\varphi(U_i\cap U)}$ and $\int\limits_{\varphi_i(U_i\cap U)}$; let h_i: $\varphi(U \cap U_i) \to \varphi(U \cap U_i)$ be the mapping $\varphi \circ \varphi_i^{-1}$; since $\alpha_i\omega = f_i(x)dx_i \wedge \cdots \wedge dx_n = g_i(x_i^{(i)})dx_1^{(i)} \wedge \cdots \wedge dx_n^{(i)}$, we have

$$f_i \circ h(x^{(i)}), \det(dh_i)(x^{(i)}) = g_i(x^{(i)});$$

however, by hypothesis, det $(dh_i)(x^{(i)}) > 0$, and the assertion follows from the formula for change of variable.

Theorem 2 (Stokes' theorem)**.** *If V is an oriented manifold of dimension n, V an (n - 1) form of class C^1, having compact support, we gave*

$$\int\limits_{\partial V} \omega = \int\limits_{V} d\omega.$$

In particular, the above formula holds for all C^1 forms ω if V is compact

Proof. If (U_i, φ_i) is a locally finite system of coordinate neighbourhoods, (η_i) a partition of unity subordinate to $\{U_i\}$, it is enough to prove that

$$\int\limits_{\partial V} \eta_i\omega = \int\limits_{V} d(\eta_i\omega)$$

□

Case I. *If $\varphi_i(U_i)$ is open in $\mathbb{R}^n$,* **94**

$$\int\limits_{\partial V} \eta_i\omega = 0$$

Further, if $\eta_i\omega = \sum\limits_{j=1}^{n} f_j dx_1 \wedge \cdots \wedge d\hat{x}_j \wedge \cdots \wedge dx_n$, we gave $d(\eta_i\omega) = \sum \dfrac{\partial f_j}{dx_j}(-1)^{j-1} dx_1 \wedge \cdots \wedge dx_n$

$$\text{and} \int\limits_{V} d(\eta_i\omega) = \int\limits_{\varphi_i(U_i)} \sum(-1)^{j-1}\frac{\partial f_j}{\partial x_j} dx_1 \wedge \cdots \wedge dx_n$$

$$= \int_{\mathbb{R}^n} \left(\sum (-1)^{j-1} \frac{\partial f_j}{\partial x_j}\right) dx_1 dx_2 \cdots dx_n$$

since f_j has compact support for each j, $\int_{\mathbb{R}} \frac{\partial f_j}{\partial x_j} dx_j = 0$. Hence it follows from Fubini's theorem that

$$\int_V d(\eta_i \omega) = 0 = \int_{\partial V} \eta_i \omega.$$

Case II. *If $\varphi_i(U_i)$ is not an open set in $\mathbb{R}^n$*

$$\int_V d(\eta_i \omega) = \int_{\mathbb{R}^n_+} \left(\sum (-1)^{j-1} \frac{\partial f_j}{\partial x_j}\right) dx_1 \cdots dx_n.$$

Now

$$\int_{\mathbb{R}^n_+} \frac{\partial f_j}{\partial x_j} dx_1 \cdots dx_n = 0 \text{ if } j = 1 \text{ as in case I :}$$

further, if $j \neq 1$, $f_j dx_1 \wedge \cdots \wedge d\hat{x}_j \wedge \cdots \wedge dx_n | \partial V = 0$. Also

$$\int_{\mathbb{R}^n_+} \frac{\partial f_1}{\partial x_1} dx_1 \cdots dx_n = \int_{\mathbb{R}} dx_n \int_{\mathbb{R}} dx_{n-1} \cdots \int_{x_1 \geq 0} \frac{\partial f_1}{\partial x_1} dx_1.$$

95 Hence

$$\int_{\mathbb{R}^n_+} \frac{\partial f_1}{\partial x_1} dx_1 \cdots dx_n = -\int_{\mathbb{R}^{n-1}} f_1(0, x_2, \ldots, x_n) dx_2 \cdots dx_n$$

and

$$\int_V d(\eta_i \omega) = -\int_{\mathbb{R}^{n-1}} f_1(0, x_2, \ldots, x_n) dx_2 \cdots dx_n$$

$$= \int_V \eta_i \omega$$

6 One parameter groups and the theorem of Frobenius

In what follows V denotes a C^k manifold countable at ∞ with $k \geq 3$.

Definition. A C^r map φ: $\mathbb{R} \times V \to V$, $0 < r \leq k$, is called a one parameter group of C^r transformations of it satisfies the following conditions:

(1) for every $t \in \mathbb{R}$, $\varphi(t, x) = \varphi_t(x)$ is a C^r diffeomorphism of V onto itself;

(2) $\varphi_{t+s}(x) = \varphi_t \circ \varphi_s(x)$ for $s, t \in \mathbb{R}$ and $x \in V$.

Definition. If U is an open subset of V, a local one parameter group of C^r transformations of U into V is a C^r map φ: $I_\varepsilon \times U \to V$, $I_\varepsilon = \{t \in \mathbb{R} \big| |t| < \varepsilon\}$, $\varepsilon > 0$, which satisfies the following conditions:

1. for any $t \in I_\varepsilon$, $\varphi(t, x) = \varphi_t(x)$ is a C^r diffeomorphism of U into V **96**
   (i.e. onto an open subset of V);

2. if $s, t, s + t \in I_\varepsilon$ and $x, \varphi_t(x) \in U$, then $\varphi_{s+t}(x) = \varphi_S \circ \varphi_t(x)$

Given a one parameter group φ: $\mathbb{R} \times V \to V$ we can associate to it a vector field X_φ defined by $(X_\varphi)_a(f) = \dfrac{\partial(f \circ \varphi_t)}{\partial t}\Big|(0, a)$ for $f \in C^k_a$; i.e. $(X_\varphi)_a$ is precisely the tangent to the curve $t \to \varphi_t(a)$ at a, X_φ is called the vector field induced by φ. A local one parameter group of transformations of U into V induces a vector field on U in the same way.

Proposition 1. *Given a C^{k-1} vector field X, there exists, for every $a \in V$, a neighbourhood U of a and a local one parameter group of C^{k-1} transformations of U, $\varphi : I_\varepsilon \times U$ which induces X on U, i.e. we have*

$$X_b(f) = \frac{\partial(f \circ \varphi)}{\partial t}(0, b) \text{ for } b \in U \text{ and } f \in C^k_b.$$

Proof. Let the vector field X be given by

$$X = \sum a_i(x) \frac{\partial}{\partial x_i}$$

in terms of local coordinates in an open set $U' \ni a$. We have then to solve the differential equation

$$\sum \frac{\partial \varphi_i}{\partial t} \frac{\partial}{\partial x_i} = \sum a_i \frac{\partial}{\partial x_i}$$

i.e.
$$\frac{\partial \varphi}{\partial t} = a(\varphi(t_1, x))$$

with the initial condition $\varphi(0, x) = x$; [here φ stands for an n-tuple of functions]. □

97 Since X is a C^{k-1} vector field, $(k \geq 3)$, $a_i \in C^{k-1}$ and by Chapter I, §6, there exists $\varepsilon > 0$, a neighbourhood U of a and a unique C^{k-1} map φ: $I_\varepsilon \times U \to V$ satisfying the differential equation

$$\frac{\partial \varphi}{\partial t} = a(\varphi(t, x)), \varphi(0, x) = x.$$

For s, t, $s+t \epsilon I_\varepsilon$ and x, $\varphi_t(x) \epsilon U$, it can be easily verified that $\varphi_s \circ \varphi_t(x)$ and $\varphi_{s+t}(x)$ are both solutions of the differential equation

$$\frac{\partial \psi}{\partial s} = a(\psi(s, x)), \psi_i(0, x) = \varphi_i(t, x).$$

Hence by the uniqueness of the solution of equations of this form, we have

$$\varphi_s \circ \varphi_t(x) = \varphi_{s+t}(x) \text{ for } x, \varphi_t(x) \in U.$$

It now remains to show that for $t \in I_\varepsilon$, $\varphi_t(x)$ is a diffeomorphism of U into V. Since

$$(d_2\varphi)(0, x) = \text{ identity, and } \varphi \in C^{k-1}, \text{ it}$$

follows that for sufficiently small ε, $t \in I_\varepsilon$ implies $(d_2\varphi)(t, x)$ is non-singular and hence, by the rank theorem, $\varphi_t(x)$ is a diffeomorphism of U into V if U is chosen small enough (see also proof of the following corollary).

Corollary. *Given a C^{k-1} vector field X on V and a relatively compact open set U, there exists a local one parameter group φ_t of C^{k-1} transformations of U into V which induces X on U.*

98 *Proof.* Let U' be an open set with $U \subset\subset U' \subset\subset V$. (We write $A \subset\subset B$ to mean that A is relatively compact in B.) For any a $a \in \overline{U'}$ there is a neighbourhood U_a in V and a local one parameter group $\varphi_t^{(a)}$: $U_a \to V$, $|t| < \varepsilon(a)$, which induces X on U_a. Suppose $a_1, \ldots, a_k$ so chosen that $\bigcup U_{a_i} \supset U'$. Let $\varepsilon' = \min \varepsilon(a_i)$. If $U_{a_i} \bigcap U_{a_j} \neq \phi$, then $\varphi_t^{(a_i)}$, $\varphi_t^{(a_j)}$ induce X on $U_{a_i} \cap U_{a_j}$, and hence coincide there. Define $\varphi_t(x) = \varphi_t^{(a_i)}(x)$ for $x \in U_{a_i}$. Let $\varepsilon < \varepsilon'$ be so small that $\varphi_t(U) \subset U'$ for $|t| < \varepsilon$. Since each φ_t is a 1-parameter group, we have only to show that each φ_t is injective on U. But this is obvious since $\varphi_{-t}(\varphi_t(x)) = x$ for $x \in U$. (Note that $\varphi_t(x) \epsilon U'$ and φ_{-t} is defined on U'). ◻

Remark. If V is compact, X gives rise to a global 1- parameter group ψ_s. In fact, as is easily deduced from the above corollary, there is $\varepsilon > 0$ such that $\varphi_t : V \to V$ is a diffeomorphism (onto) for $|t| < \varepsilon$. Given $s \in \mathbb{R}$, we set $\psi_s = (\varphi_{s/k})^k$ where k is an integer so chosen that $|s/k| < \varepsilon$ and $(\varphi_{s/k})^k$ denotes the composite of $(\varphi_{s/k})$ with itself k times. (ψ_s is independent of the k chosen).

We have remarked earlier that a differentiable map does not transfer vector fields into vector fields. However, let σ be a C^r diffeomorphism of an open set $U \subset V$, into V and X, a C^{r-1} vector field on U, let $U' = \sigma(U)$. The assignment to $a \in U'$ of the vector $\overset{\sigma}{*}(X_{\sigma-1(a)})$ at a, is clearly a C^{r-1} vector field on U', denoted by$\overset{\sigma}{*}(X)$ or $\overset{\sigma}{*}X$. If f is a C^k function on U', we have,

$$\sigma_*(X)(f) = X(f \circ \sigma) \circ \sigma^{-1}.$$

If X, Y are two vector fields on U, we have 99

$$\begin{aligned}
[\overset{\sigma}{*}X, \sigma_* Y](f) &= \overset{\sigma}{*}(X)[Y(f \circ \sigma) \circ \sigma^{-1}] - \overset{\sigma}{*}(Y)[X(f \circ \sigma) \circ \sigma^{-1}] \\
&= [X(Y(f \circ \sigma)) - Y(X(f \circ \sigma))] \circ \sigma^{-1} \\
&= \overset{\sigma}{*}([X, Y])(f),
\end{aligned}$$

$$\text{i.e. } [\overset{\sigma}{*}X, \overset{\sigma}{*}Y] = \overset{\sigma}{*}[X, Y].$$

Proposition 2. *If σ is a diffeomorphism $U \to U'$ and if a local one parameter group of transformations φ: $(U \cup U') \to V$ induces the vector* *field X, then $\overset{\sigma}{*}X$ is induced by the local one parameter group $\sigma\circ\varphi\circ\sigma^{-1}$: $U' \to U$.*

Proof.

$$\begin{aligned}\overset{\sigma}{*}(X)(f) &= X(f\circ\sigma)\circ\sigma^{-1}\\ &= \frac{\partial}{\partial t}(f\circ\sigma\circ\varphi_t)|_{t=0}\circ\sigma^{-1}\\ &= \frac{\partial}{\partial t}(f\circ\sigma\circ\varphi_t 0\sigma^{-1}|_{t=0}\end{aligned}$$

□

Corollary 1. *σ commutes with φ_t for every t if and only if $\sigma_*(X) = X$.*

Definition. A local one parameter group φ is said to leave a vector field X invariant if $(\varphi_{t_*})(X) = X$ for every t.

100 **Remark.** If φ induces the vector field X_φ, X_φ is invariant under φ.

Definition. If φ is a local one parameter group $U \to V$, of C^2 transformations, and Y, a vector field on V, and if $(\varphi_t)_*Y = Y_t$, we define the vector field $\dfrac{dY_t}{dt}$ by

$$\left(\frac{dY_t}{dt}\right)(f) = \frac{d}{dt}[Y_t(f)].$$

Proposition 3. *If Y is a C^{k-1} vector field on V, $k \geq 3$ and if a one parameter group φ induces the $k-1$ vector field X on U we gave*

$$\left.\frac{dY_t}{dt}\right|_{t_0} = [Y_{t_0}, X] \text{ on } U.$$

Proof. We shall first prove the result for $t_0 = 0$. We have □

$$\begin{aligned}\left.\frac{dY_t}{dt}\right|_0(f) &= \lim_{t\to 0}\frac{1}{t}[Y_t - Y](f)\\ &= \lim_{t\to 0}\frac{1}{t}[Y[f\circ\varphi_t]\circ\varphi_{-t} - Y(f)]\end{aligned}$$

$$= \lim_{t\to 0} \frac{1}{t}[Y(f\circ\varphi_t) - Y(f)\circ\varphi_t]\circ\varphi_{-t}$$
$$= \lim_{t\to 0} \frac{1}{t}[Y(f\circ\varphi_t) - Y(f) - \circ Y(f)\circ\varphi_t + Y(f)]\circ\varphi_{-t}$$
$$= \lim_{t\to 0} \frac{Y(f\circ\varphi_t - f)}{t} - \lim_{t\to 0}\frac{Y(f)\circ\varphi_t - Y(f)}{t},$$

since $\lim_{t\to 0}\varphi_{-t}$ = identity. Now

$$\lim_{t\to 0}\frac{Y(f)\circ\varphi_t - Y(f)}{t} = X(Y(f))$$

by definition of X. Consider $h(t,x) = f\circ\varphi_t(x)$. Clearly $h \in C^2$ since **101**
$h \in C^{k-1}$ and

$$\frac{h(t,x) - h(0,x)}{t} = \frac{f\circ\varphi_t - f}{t} \in C^1.$$

Hence
$$\lim_{t\to 0}\frac{Y[f\circ\varphi_t - f]}{t} = Y\left[\lim_{t\to 0}\frac{f\circ\varphi_t - f}{t}\right] = Y(X(f)).$$

Hence
$$\left.\frac{dY_t}{dt}\right|_{t=0}(f) = Y[X(f)] - X[Y(f)] = [Y_0, X](f).$$

i.e.
$$\left.\frac{dY_t}{dt}\right|_{t=0} = [Y_0, X].$$

For any t_0 in the interval of definition.

$$(\varphi_{t_0})_*\left(\frac{dY_t}{dt}\right)_{t=0} = \left(\frac{dY_t}{dt}\right)_{t=t_0}$$

and
$$(\varphi_{t_0})_*[Y_0, X] = [(\varphi_{t_0})_* Y_0, (\varphi_{t_0})_* X] = [Y_{t_0}, X].$$

Hence $\frac{dY_t}{dt}|_{t=t_0} = [Y_{t_0}, X]$.

Corollary. *If X, Y are vector fields on V which give rise to local one parameter groups φ and ψ: $U \to V$ respectively, then for all t, s, φ_t and ψ_s commute (i.e. $\varphi_t\circ\psi_s = \psi_s\circ\varphi_t$ on the common of definition) if and only if* $[X, Y] = 0$.

102 *Proof.* If φ_t and ψ_s commute for sufficiently small t and s, φ_t leaves Y invariant.

Hence $\frac{dY_t}{dt} = [Y_t, X] = [Y, X] = 0.$ □

Conversely if $[Y, X] = 0$

$$\left.\frac{dY_t}{dt}\right|_{t=t_0} = [Y_{t_0}, X] = [Y_{t_0}, X_{t_0}]$$
$$= (\varphi_{t_0})_*[Y, X] = 0.$$

Hence φ_t leaves Y invariant, which, with the corollary to Prop.2, completes the proof.

In what follows we consider a C^k manifold V. The vector fields will be C^{k-1} and differentiable functions, mappings will be C^k.

Definition. 1. A distribution (or differential system) $\mathscr{D}$ of rank p, on (a C^k manifold) V is an assignment to each point $a \epsilon V$ of a subspace $\mathscr{D}(a)$ of $T_a(V)$, of dimension p.

2. A distribution $\mathscr{D}$ is called differentiable if for every $a \in V$ there exists a neighbourhood U of a and differentiable vector fields X_1, $X_2, \ldots, X_p$ such that X_{1_b}, $X_{2_b}, \ldots X_{p_b}$ form a basis of $\mathscr{D}(b)$ for every $b \in U$.

3. A submanifold i: $W \to V$ of V (more generally, a C^k mapping i: $W \to V$) is called an integral of $\mathscr{D}$ if for $a \in W, i_*(T_a(W)) \subset \mathscr{D}(i(a))$.

4. A distribution $\mathscr{D}$ is said to be completely integrable if for every $a \in V$, there exists a neighbourhood U_a and a system of local
103 coordinates $(x_1, \ldots, x_n)$, such that for sufficiently small c_i, $p+1 \leq i \leq n$, the submanifolds given by $U_c = \{x \epsilon U | x_i = c_i, i \geq p+1\}$ are integrals of $\mathscr{D}$.

Remark. Any submanifold of an integral is itself an integral.

Lemma 1. *If $\mathscr{D}$ is a completely integrable differentiable distribution and if $W \subset U$ is a connected integral of $\mathscr{D}$, then $W \subset U_c$ for some $c = (c_i)_{p+1 \leq i \leq n}$, [where U carries a coordinate system as in* (4) *above].*

Proof. We have $i_*(T_a(W)) \subset \mathscr{D}(i(a))$. Now for any $c, T_{i(a)}(U_c)$ has dimension p and hence $T_{i(a)}(U_c) = \mathscr{D}(i(a))$

Hence $$i_*(T_a(W)) \subset T_{i(a)}(U_c).$$ □

Now $T_{i(a)}(U_c)$ is the subspace of $T_{i(a)}(V)$, orthogonal to the 1- forms $\{dx_i\}_{i>p}$. Hence $\{dx_i\}_{i>p}$ are orthogonal to $i_*(T_a(W))$, i.e. $dx_i|W = 0, i > p$ and hence $x_i = c_i$ for some constant $c_i, i > p$, since W is connected.

Definition. A differentiable distribution $\mathscr{D}$ is called involutive (or complete) if for any $a \in V$, there is a neighbourhood U and vector fields $X_1, \ldots, X_p$ generating $\mathscr{D}$ in U such that, we gave, for $b \in U$

$$[X_i, X_j]_b \in \mathscr{D}(b) \text{ for } i, j \leq p.$$

Note that there then exist differentiable functions a^k_{ij} in U such $[x_i, x_j] = \sum\limits_{k=1}^{p} a^k_{ij} X_k$.

Remark. The above definition is independent of the basics $X_1, \ldots, X_p$.

Lemma 2. *If a differential system $\mathscr{D}$ is involutive, for any $a \in V$ there exists a neighbourhood U of a and a basis $X_1, \ldots, X_p$ of $\mathscr{D}$ in U such that $[X_i, X_j] = 0$ in U.* **104**

Proof. Let $(Y_i)_{1 \leq i \leq p}$ be a basis of $\mathscr{D}\big|U$. □

In terms of local coordinates, let

$$Y_i = \sum_{r=1}^{n} a_{ir} \frac{\partial}{\partial x_r}.$$

We may assume without loss of generality that the matrix $(a_{ir}(x)) = A(x)$, $1 \leq i \leq p$, $1 \leq r \leq p$ is of rank p at the point $x = a$. If U is small enough, $A(x)$ has rank p for $x \in U$. If $B(x) = (b_{ir})(x) = [A(x)]^{-1}$, then the b_{ir} are differentiable. Let

$$X_i = \sum_{k=1}^{p} b_{ik} Y_k.$$

$$\text{Then } X_i = \frac{\partial}{\partial x_i} + \sum_{r>p} C_{ir} \frac{\partial}{\partial x_r} \quad \text{and} \qquad (X_i)_{1 \le i \le p}$$

form a basis of $\mathscr{D}\big|U$. Since $\mathscr{D}$ is involutive, we have

$$[X_i, X_j] = \sum_{r=1}^{p} \lambda_r X_r.$$

But $\left(\frac{\partial}{\partial x_i}\right)_{1 \le i \le p}$ commute with each other and, if $[X_i, X_j] = \sum_{r=1}^{n} \mu_r \left(\frac{\partial}{\partial x_r}\right)$, then $\mu_r = 0$ for $r \le p$. Clearly we therefore have

$$\lambda_r = \mu_r = 0 \text{ for } r \le p.$$

105 **Proposition 4.** *Let $X_1, \ldots, X_p$ be vector fields on V which are linearly independent at every point of V and such that $[X_i, X_j] = 0$, then for any $a \in V$ there exists a neighbourhood U and coordinates t_1, $t_2, \ldots, t_p$, $x_{p+1}, \ldots, x_n$ in U such that $X_i = \frac{\partial}{\partial t_i}$ for $i \le p$.*

Proof. We can assume that $X_1, \ldots, X_p$ are induced by local one parameter groups of transformations, $\varphi^{(1)}, \varphi^{(2)}, \ldots, \varphi^{(p)}$ in a neighbourhood U of a. We suppose that $\varphi_t^{(i)}$ are defined for $|t| <$. After a linear change of coordinates on U we may suppose that the vectors

$$(X_1)_a, \ldots, (X_p)_a, \left(\frac{\partial}{\partial x_{p+1}}\right)_a, \ldots, \left(\frac{\partial}{\partial x_n}\right)_a$$

are linearly independent. We suppose further that the coordinates of a are zero. Let $U' \subset \mathbb{R}^{n-p}$ be the set of $x' = (x_{p+1}, \ldots, x_n)$ with $(0, x') \in U$, $Q \subset \mathbb{R}^p$, the set $|t_i| < \delta$ and let h: $Q \times U' \to U$ be the mapping

$$h(t_1, \ldots, t_p, x_{p+1}, \ldots, x_n) = \varphi_{t_1}^{(1)} \circ \ldots \varphi_{t_p}^{(p)}(0, x'),$$

ε being chosen so small that the composites are all defined. □

For any C^k function f on U, we have $\dfrac{\partial}{\partial t_1}[foh]_{t=0} = (X_1)_a(f)$, by definition of $\varphi_t^{(1)}$, and since the $\varphi_{t_i}^{(i)}$ commute, (because $[X_i, X_j] = 0$), we have

$$h_*\left[\left(\frac{\partial}{\partial t_i}\right)_o\right] = (X_i)_a, 1 \leq i \leq p.$$

It is obvious that

$$h_*\left(\frac{\partial}{\partial x_i}\right)_0 = \left(\frac{\partial}{\partial x_i}\right)_0, i > p.$$

This, however, implies that h_* has the maximum rank $= n$ and is a diffeomorphism in a neighbourhood of 0. Hence $t_1, \ldots, t_p, x_{p+1}, \ldots, x_n$ may be considered as local coordinates in U if U is small enough. Further, exactly as above, we show that $h_*(\frac{\partial}{\partial t_i}) = X_i$, $i \leq p$, which gives the proposition. **106**

Theorem 1 (Frobenius)**.** *A differential system on V is involutive if and only it is completely integrable.*

Proof. If $\mathscr{D}$ is a completely integrable system for $a \in V$ there exists a neighbourhood U of a such that for all sufficiently small $(C_i)_{p+1\leq i\leq n}$, $U_c = \{x \in U | x_i = s_i, i > p\}$ are integrals of $\mathscr{D}$ Hence $\left(\dfrac{\partial}{\partial x_i}\right)_{1\leq i\leq p}$ form a basis of $\mathscr{D}|U$ and $\mathscr{D}$ is involutive. This together with Lemma 2 and Proposition 4 above proves the theorem. □

Remark. We have proved the theorem of Frobenius for C^2 distributions, i.e. distributions having a basis of C^2 vector fields [We have used the condition essentially in the proof of Prop. 3.] However the theorem is valid also for C^1 vector fields. We have only to prove Prop. 3 for C^1 vector fields. This can be by approximating the fields by C^2 fields and using the results of Chap I, §6, to conclude that the local 1-parameter group associated to a vector field X depends continuously on X.

Let $\omega_{p+1}, \ldots, \omega_n$ be 1-forms on V which are linearly independent at every point. We can define a distribution $\mathscr{D}$ by setting

$$\mathscr{D}(a) = \{X \in T_a(V) \Big| (\omega_i)_a(X) = 0 \text{ for } i = p+1, \ldots, n\}.$$

If the ω_i are differentiable then so is $\mathscr{D}$. In fact, considering suitable
107 linear combinations of the ω_i with differentiable coefficients, we may suppose that, in a neighbourhood of any given point a of V, we have

$$\omega_i = dx_i + \sum_{r \leq p} a_{ir} dx_r, i > p.$$

Then $\mathscr{D}$ is the distribution spanned by the vector fields

$$X_r = \frac{\partial}{\partial x_r} - \sum_{j>p} a_{jr} \frac{\partial}{\partial x_j}, 1 \leq r \leq p :$$

(it is obvious that the X_r are orthogonal to the ω_i and they are clearly linearly independent).

For distributions given in this form, the theorem of Frobenius is as follows.

Theorem 2. *Let $\omega_{p+1}, \ldots, \omega_n$ be 1-forms which are linearly independent at every point. Then, in order that the distribution $\mathscr{D}$ defined by them be completely integrable, it is necessary and sufficient that every point $a \in V$ has a neighbourhood in which there exists 1-forms $\alpha^r_{j_n}$ such that, for $j > p$,*

$$(6.1) \qquad d\omega_j = \sum_{k=p+1}^{n} \omega_k \wedge \alpha^r_j;$$

i.e. $d\omega_j$ belongs to the ideal generated by the ω_k.

[Note that the condition (6.1) is invariant under 'change of basis', i.e. if η_j are 1-forms which span the same subspace of $T^*_a(V)$ for any a, then the condition (6.1) is satisfied if and only if the corresponding condition on the η_j is.]

108 *Proof.* If $\mathscr{D}$ is completely integrable, and $a \in V$, choose coordinates at a such that the "planes" $x_{p+1} = c_{p+1}, \ldots, x_n = c_n$ are integrals of $\mathscr{D}$. Then $\mathscr{D}(b)$ is the space orthogonal to $(dx_{p+1})_b, \ldots, (dx_n)_b$. Hence $dx_{p+1}, \ldots, dx_n$ span the same subspace of $T^*_b(V)$ as $\omega_{p+1}, \ldots, \omega_n$ for $b \in U$. The equation (6.1) for the dx_j is trivial. ◻

Suppose conversely that there exist α_j^r satisfying (6.1). Let $X_1, \ldots, X_p$ be vector fields in a neighbourhood of a generating $\mathscr{D}$. We have

$$(d\omega_k)(X_i, X_j) = X_i\omega_r(X_j) - X_j\omega_r(X_i) - \omega_r([X_i, X_j]).$$

Because of (6.1) $(d\omega_k)(X_i, X_j) = 0$; and by definition, $\omega_r(X_i) = \omega_r(X_j) = 0$. Hence $\omega_r([X_i, X_j]) = 0$ so that $[X_i, X_j]_b$ is orthogonal to $(\omega_r)_b$ for all r, so that $[X_i, X_j]_b \in \mathscr{D}(b)$. This proves that $\mathscr{D}$ is involutive, hence completely integrable

One can prove that through any point passes a maximal integral. More precisely, we have

Theorem 3′. *If $\mathscr{D}$ is completely integrable, then for any $a \in V$, there exists a connected integral i: $W \to V$ of $\mathscr{D}$ such that if j: $W' \to V$ is any connected integral of $\mathscr{D}$ with $j(a') = a$ then W' is a submanifold of W.*

Proof. Let W be the set of points x of V with the following property: there exists a chain of differentiable mappings γ_i: $I \to V$, $0 \leq i \leq N$ (I being the closed unit interval) with $\gamma_\circ(0) = a$, $\gamma_N(1) = x$, $\gamma_{i+1}(0) = \gamma_i(1)(0 \leq i < N)$, such that each γ_i is an integral of $\mathscr{D}$ (in the obvious sense). We topologize W as follows. Let $x_0 \in W$, and U an open set about x_0 carrying coordinates $x_1, \ldots, x_n, x_i(x_0) = 0$, such that all the **109**
"planes" $U_c = \{x_{p+1}, = c_{p+1}, \ldots, x_n = c_n\}$ are integrals of $\mathscr{D}$. We may suppose that U is a "cube", so that these planes are connected. Clearly every point of U_0 belongs to W. The sets $W_\varepsilon(x_0) = U_0 \cap \{x \in U \big| |x| < \varepsilon\}$ will, by definition, form a fundamental system of neighbourhoods of x_0 in W. [Note that by Lemma 1 the *sets* U_c are completely determined by $\mathscr{D}$] Also if $\gamma_0, \ldots, \gamma_N$ is a chain as in the definition of W, $\gamma_N(I) \subset U_0 \subset \overline{W}$. It is clear that this topology is Hausdorff. We make W into a C^{k-1} manifold by requiring that the obvious mappings $W_\varepsilon(x_0) \to \{(x_1, \ldots, x_p) \in \mathbb{R}^p \big| |x_i| < \varepsilon\}$ determine coordinates on W. It is then clear that W is a connected integral of $\mathscr{D}$. □

If $j : W' \to V$ is any connected integral with $j(a') = a$, let, for $w' \in W', \gamma_0', \gamma_1', \ldots, \gamma_N'$ be diffeomorphism of I into W' such that $\gamma_0'(0) =$

$a', \gamma'_{i+1}(0) = \gamma'_i(1) = w'$. Let $w = j(w')$. Then $\gamma_i = j(\gamma'_i)$ is a chain as in the definition of W joining a to w. Hence $w \in W$. Thus there is a mapping $\eta : W' \to W$ with $io\eta = j$. Clearly η makes of W' a submanifold of W.

Finally, we give the Frobinius theorem in another form. In this form, it may be looked upon as a direct generalisation of the existence theorem for ordinary differential equations proved in Chap. I, §6.

Theorem 4. *Let Ω be an open set in $\mathbb{R}^n$ with coordinates $(x_1, \ldots, x_n)$, Ω' an open set in $\mathbb{R}^m$ with coordinates $(t_1, \ldots, t_m)$. Let $f_i : \Omega \times \Omega' \to \mathbb{R}^n$ be C^k functions, $i = 1, \ldots, m (k \geq 2)$. In order that to every $t_0 \in \Omega'$ and $x_0 \in \Omega$ there is a neighbourhood U of t_0 and a unique C^k map $x : U \to \Omega$ such that*

$$\frac{\partial x(t)}{\partial t_i} = f_i(x(t), t), \ i = 1, \ldots, m, x(t_0) = x_0, \tag{6.2}$$

110 *it is necessary and sufficient that we have*

$$\begin{aligned} &\frac{\partial f_i}{\partial t_j}(x,t) + (d_1 f_i)(x,t). f_j(x,t) \\ = &\frac{\partial f_j}{\partial t_i}(x,t) + (d_1 f_j)(x,t). f_i(x,t) \end{aligned}$$

for $1 \leq i, j \leq m$, $(x,t) \in \Omega \times \Omega$

[Note that $d_1 f_i$ is a linear mapping of $\mathbb{R}^n$ into itself.]

Proof. The uniqueness of the solution, if it exists, follows from the uniqueness theorem for solutions of ordinary differential equations proved in Cha. I, §6. If the equations (6.2) are solvable, the equations (6.3) hold; in fact the two sides of the equality at the point (x_0, t_0) are then simply

$$\left.\frac{\partial^2 x(t)}{\partial t_j \partial t_i}\right|_{t=t_0}.$$

□

To prove the converse, we proceed as follows. The equations (6.2) can be written

$$\frac{\partial x_r}{\partial t_i} = f_{ir}(x,t),\ f_i = (f_{i1},\dots,f_{in}), r = 1,\dots,n. \tag{6.4}$$

Consider the differential forms

$$dx_r - \sum_{i=1}^{m} f_{ir}(x,t)dt_i, r = 1,\dots,n \tag{6.5}$$

on $\Omega \times \Omega'$, and let $\mathscr{D}$ be the form differential system of rank m defined
by them. If $\mathscr{D}$ has an integral manifold of the form $x-\varphi(t) = 0$, where φ **111**
is a C^k map of a neighbourhood of t_0 into Ω with $\varphi(t_0) = x_0$, then $x = \varphi$ is a solution of (6.2). Suppose now that $u_1,\dots,u_n$ are C^k functions near (x_0,t_0) such that $(du_1)(x_0,t_0),\dots,(du_n)(x_0,t_0)$ are linearly independent. Then if the manifold $W = \{u_1 = \dots = u_n = 0\}$ [in a neighbourhood of (x_0,t_0) this is a manifold by the rank theorem] is an integral of $\mathscr{D}$, it is clear that the forms (6.5) and the forms $du_1,\dots,du_n$ generate the same subspace of $T^*_{(x_0,t_o)}(\Omega\times\Omega')$. Hence $(d_1u_1)(x_0,t_0),\dots,(d_1u_1)(x_0,t_0)$ are linearly independent. Hence by the implicit function theorem, W is given by equations $x-\varphi(t) = 0$, and by our remark above, the equations (6.2) are solvable. Thus, if $\mathscr{D}$ is completely integrable, then the equations (6.2) are solvable.

Now, as we have seen before, $\mathscr{D}$ has a basis given by the vector fields

$$X_i = \frac{\partial}{\partial t_i} + \sum_{r=1}^{n} f_{ir}(x,t)\frac{\partial}{\partial x_r}; \tag{6.6}$$

further, we have seen in the proof of Lemma 2 that $\mathscr{D}$ is completely integrable if and only if

$$[X_i, X_j] = 0 \text{ for } i, j \leq m.$$

It is easily verified that these latter conditions are precisely the condition (6.3). Thus, if conditions (6.3) are satisfied, $\mathscr{D}$ is completely integrable, and in particular the equations (6.2) are solvable.

Remark. Theorem 4 is true also for C^1 functions f_i; *a* proof of this statement can be obtained by using the remark made after the proof of Frobenius' theorem. 112

We remark that if, in Theorem 4, we take $n = 1$ and the f_i to be functions independent of x, we obtain the following result.

In order that there exist a C^k function $x(t_1, \ldots, t_m)$ for which, in a neighbourhood of t_0, we have

$$\frac{\partial x}{\partial t_i} = f_i(t),$$

it is necessary and sufficient that

$$\frac{\partial f_i}{\partial t_i} = \frac{\partial f_j}{\partial t_i}$$

This can be formulated as follows. Consider the 1-form $\omega = \sum_{i=1}^{m} f_i(t)$ dt_i. Then there is a function f with $df = \omega$ in a neighbourhood of any point if and only if $d\omega = 0$.

This result is a special case of Poincare's lemma, which we shall prove later.

For the material concerning 1-parameter groups, see Nomizu [33]. A different treatment of the Frobenius theorem (in the first form given here) will be found in Chevalley [7].

7 Poincare's lemma, the type decomposition of complex co vectors, and Grothendieck's lemma

Definition. If V is a C^k manifold of dimension n a differential form ω,
113 of degree p, is said to be closed if $d\omega = 0$ and is said to be exact if there exists a form ω_1 of degree $p - 1$, such that $d\omega_{11} = \omega$.

Since $d^2 = 0$, an exact form is closed. We denote the set of closed p-differential forms by $Z^p(V)$ and the set of exact p-differential forms by $B^p(V)$. The quotient $H^p(V) = Z^p/B^p(V)$ is called the p^{th} de Rham group of V. A basic theorem of de Rham, which we shall not prove here, implies that the $H^p(V)$ are topological invariants. i.e., if V, V' are homeomorphic, then $H^p(V) \approx H^p(V')$. For a proof, see e.g. A.Weil [44].

Poincare's lemma. If D is a convex open set $\mathbb{R}^n$, every closed form of degree ≥ 1 on D is exact, i,e. $H^p(D) = 0$ for $p \geq 1$.

Proof. We may suppose without loss of generality that $0 \in D$. Let $I = (0, 1)$, be the open unit interval. Consider the map $h : D \times I \to D$ given by $h(x, t) = t.x$. □

If ω is a closed p form on D, $p \geq 1$, let $\omega = \sum\limits_I a_I(x).\ dx_I$ in terms of the coordinates of $\mathbb{R}^n$. Then $h^*(\omega)$ is a form on $D \times I$ given by

$$\begin{aligned} h^*(\omega) &= \sum_I a_I(tx) d(tx_I), I = (i_1, \ldots, i_p), i_1 < i_2 \ldots < i_p \\ &= \sum_I a_I(tx) t^p dx_I + t^{p-1} \sum_I a_I(tx) (\sum_j (-1)^{j-1} x_j dt \wedge dx_{I,j}) \end{aligned}$$

where

$$\begin{aligned} dx_{I,j} &= dx_i \wedge \ldots \wedge d\hat{x}_j \wedge \ldots \wedge dx_{i_n} \text{ if } j \in I \\ &= 0 \text{ otherwise.} \end{aligned}$$

Hence $h^*(\omega) = \sum a_I(tx) t^p d(x_I) + dt \wedge \omega'$ where ω' is a $(p-1)$ form on $D \times I$. We have 114

$$0 = h^*(d\omega) = d(h^*(\omega)),$$

so that $\displaystyle \sum_I \frac{\partial}{\partial t}(t^p a_I(tx)) dt \wedge dx_I + t^p \sum_{j,I} \frac{\partial}{\partial x_j}(a_I(tx)) dx_j \wedge dx_I - dt \wedge d\omega' = 0.$

This implies that

$$\sum \frac{\partial}{\partial x_j}(a_I(tx)) dx_j \wedge dx_I = 0$$

and that

$$\sum \frac{\partial}{\partial t}(t^p a_I(tx)) dt \wedge dx_I$$

$$= dt \wedge d\omega'.$$

Since dx_I does not contain dt, this implies that

$$\frac{\partial}{\partial t}\Big(\sum t^p a_I(tx) dx_I\Big) = d_x\omega'$$

where $$d_x\omega' = \sum dx_i \wedge \frac{\partial \omega'}{\partial x_i}$$

Hence

$$\int_0^1 \frac{\partial}{\partial t}\Big(\sum a_I(tx) t^p dx_I\Big) dt = \omega \text{ (since } p \geq 1)$$

$$= \int_0^1 d_x\omega' dt.$$

$$= d_x\Big[\int_0^1 \omega' dt\Big].$$

115 i.e. $\omega = d\omega_1$ where $\omega_1 = \int_0^1 \omega' dt$.

Compare this proof with the one given in A. Weil [44].

We introduce on $\mathbb{R}^{2n}$ the structure of a vector space over $\mathbb{C}$ by means of the $\mathbb{R}$ isomorphism of $\mathbb{R}^{2n}$ isomorphism of $\mathbb{R}^{2n}$ onto $\mathbb{C}^n$ given by

$$(x_1, \ldots, x_{2n}) \leftrightarrow (z_1, \ldots, z_n)$$

where $z_j = x_{2j-1} + ix_{2j}$.

If E is a vector space over $\mathbb{C}$, of dimension n, consider the complex vector space $\mathscr{E}^* = \operatorname{Hom}_{\mathbb{R}}(E, \mathbb{C})$, of $\mathbb{R}$-linear mappings of E into $\mathbb{C}$.

Let $F = \{f | f$ an $\mathbb{R}$ linear form $: E \to \mathbb{C}$ such that $f(iv) = if(v)\}$.

$\overline{F} = \{f | f$ an $\mathbb{R}$ linear form $: E \to \mathbb{C}$ such that $f(iv) = -if(v)\}$.

Then $\mathscr{E}^* = F \oplus \overline{F}$.

[For if $g \in \mathscr{E}^*$, consider f' and f'' defined by

$$f'(v) = \frac{1}{2}\{g(v) - ig(iv)\}$$
$$f''(v) = \frac{1}{2}\{g(v) + ig(iv)\}.$$

Then $g = f' + f''$ and $f' \in F$, $f'' \in \bar{F}$]

We denote F by $E(1,0)$ and $\bar{F}$ by $\underline{E}^{(0,1)}$. Conjugation $z \to \bar{z}$ in $\mathbb{C}$ defines an $\mathbb{R}$-isomorphism of F onto $\bar{F}$. Let $(\underline{e}_1, e_2, \ldots, e_n)$ form a $\mathbb{C}$ basis of F. Then $(\bar{e}_1, \ldots, \bar{e}_n)$ forms a $\mathbb{C}$ basis of $\bar{F}$.

We shall have to consider the vector space $\wedge^r \mathscr{E}^*$. For fixed p, q with $p + q = r$, let $\mathscr{E}^*_{p,q}$ denote the complex subspace of $\wedge^r \mathscr{E}^*$ generated by 116
the elements of the form

$$e_I \wedge \bar{e}_J = e_{i_1} \wedge \ldots \wedge_{e_{i_p}} \wedge \bar{e}_{j_1} \ldots \wedge \bar{e}_{j_q}$$

where $i_1 < \ldots < i_p, j_1 < \ldots < j_q$ (but there is no relation between the i and the j). Then the elements $e_I \wedge \bar{e}_J$ are linearly independent and span $\wedge^r \mathscr{E}^*$ if I, J run over all increasing sequences of p and q integers respectively, so that $\wedge^r \mathscr{E}^* = \sum\limits_{p+q=r} \mathscr{E}^*_{p,q}$.

In what follows, V is a complex analytic manifold of complex dimension $n, (x_1, y_1, \ldots, x_n, y_n)$ denotes the real local coordinates and $(z_1, \ldots, z_n), z_j = x_j + iy_j$, complex coordinates. Let $T_a = T_a(V)$ be the tangent space to V at a considered as a C^∞ manifold of dimension $2n$ over $\mathbb{R}$.

Let $\mathscr{T}^*_a = \mathrm{Hom}_{\mathbb{R}}(T_a, \mathbb{C})$.

Clearly $\mathscr{T}^*_a$, as a vector space over $\mathbb{C}$ has dimension $2n$. Since $(dx_j)_a, (dy_j)_a \in \mathrm{Hom}_{\mathbb{R}}(T_a, \mathbb{R}) \subset \mathrm{Hom}_{\mathbb{R}}(T_a, \mathbb{C})$, the expressions $(dz_j)_a = (dx_j)_a + i(dy_j)_a, (d\bar{z}_j)_a = (dx_j)_a - i(dy_j)_a$ are well defined elements of $\mathbb{J}^*_a$; it is clear that they form a $\mathbb{C}$ basis of $\mathscr{T}^*_a$. Note that for any complex valued C^∞ function g on V, the differential $(dg)_a \in \mathscr{T}^*_a$.

We note the mapping $T_a(V) \to \mathbb{R}^{2n}$ defined by $X \to (dx_1(X), dy_1(X), \ldots, dx_n(X), dy_n(X))$ is an $\mathbb{R}$-isomorphism. Hence the map $x \to (dz_1(X), \ldots, dz_n(X)$ is an $\mathbb{R}$- isomorphism of $T_a(V)$ onto $\mathbb{C}^n$. This isomorphism defines the structure of complex vector space on $T_a(V)$. This structure is independent of the complex coordinate system used. It is 117 seen at once that it is uniquely characterised by the following property.

If f is a germ of holomorphic function at $a \in V$, we have

$$(df)_a((\alpha + i\beta)X) = (\alpha + i\beta)(df)_a(X), \alpha, \beta \in \mathbb{R}, X \in T_a(V).$$

We may also consider the space $\mathscr{T}_a(V) = T_a(V) \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Hom}(\mathscr{T}_a^*, \mathbb{C})$. This is called the space of complex tangent vectors at a. $\mathscr{T}_a(V)$ has a basis. dual to the basis $dz_1, \ldots, dz_n$, $d\overline{z}_1, \ldots d\overline{z}_n$ of $\mathscr{T}_a^*$; this basis is denoted by $\dfrac{\partial}{\partial z_1}, \ldots, \dfrac{\partial}{\partial z_n}, \dfrac{\partial}{\partial \overline{z}_1}, \ldots, \dfrac{\partial}{\partial \overline{z}_n}$. It is easily verified that, in terms of the tangent vectors $\dfrac{\partial}{\partial x_j}, \dfrac{\partial}{\partial y_j}$ (which also form a basis of $\mathscr{T}_a(V)$ we have,

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

$\mathscr{T}_a^*$ is the complexification of T_a^* and elements of $\mathscr{T}_a^*$ are called complex co vectors at a. $\wedge^p \mathscr{T}^*(V)$ is a C^∞ manifold of real dimension $2n + \binom{4n}{p}$. Hereafter, by a p differential form ω, we mean a complex p differential form, i.e., a C^∞ map $\omega : V \to \wedge^p \mathscr{T}^*(V)$ such that $\omega(a) \in \wedge^p \mathscr{T}_a^*(V)$.

We return now to our remarks on $\wedge^r \mathscr{E}^*$ for a complex vector space E, where $\mathscr{E}^* = \mathrm{Hom}_{\mathbb{R}}(R, \mathbb{C})$. We take for E, the space $T_a = T_a(V)$ with the complex structure introduced above. It is immediate that $\mathscr{E}^*_{1,0}$ is the space spanned by $dz_1, \ldots, dz_n, \mathscr{E}^*_{0,1}$, that spanned by $d\overline{z}_1, \ldots, d\overline{z}_n$. Hence $\mathscr{E}^*_{p,q}$ is spanned by the convectors $dz_I \wedge d\overline{z}_J = dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge$
118 $\cdots \wedge d\overline{z}_{j_q}$

A differential form ω is said to be of type (p, q) if for each $a \in V$, $\omega \in \mathscr{E}^*_{p,q}$ that is to say,

$$\omega_a = \sum_{\substack{i_1 < \ldots < i_p \\ j_1 < \ldots < j_p}}^{a} \omega_{IJ} dz_I \wedge d\overline{z}_J, I = (i_1, \ldots, i_p), J = (j_1, \ldots, j_p).$$

The operator d of exterior differentiation defined on real valued forms extends obviously to a $\mathbb{C}$ linear map from $C^\infty p$ forms to $C^\infty (p+1)$ form, with properties similar to those proved before.

If f is a complex valued function, we have a decomposition

$$df = \partial f + \overline{\partial} f$$

where ∂f is of type $(1,0)$ and $\overline{\partial} f$ of type $(0,1)$, since the space $\mathscr{E}^* = \mathscr{E}_{1,0} \oplus \varepsilon^*_{1,0}$. In terms of local coordinates, we have

$$\partial f = \sum \frac{\partial f}{\partial z_k} dz_k, \overline{\partial} f = \sum \frac{\partial f}{\partial \overline{z}_k} d\overline{z}_k.$$

If ω is a form of type (p,q) say,

$$\begin{aligned} \omega &= \sum \omega_{IJ} dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \ldots \wedge d\overline{z}_{j_q} \\ &= \sum \omega_{IJ} dz_I \wedge d\overline{z}_J \text{ say }, \\ \text{then} \quad d\omega &= \sum d\omega_{IJ} \wedge dz_I \wedge d\overline{z}_J \\ &= \sum (\partial \omega_{IJ} + \partial \omega_{IJ}) \wedge dz_I \wedge d\overline{z}_J \end{aligned}$$

so that $d\omega = \partial\omega + \overline{\partial}\omega$, where $\partial\omega$ is of type $(p+1,q)$ and $\overline{\partial}\omega$ of type $(p,q+1)$. From the fact that the decomposition $\wedge^r \mathscr{E}^* = \sum \mathscr{E}^*_{p,q}$ is direct we see at once that the fact that $d^2 = 0$ is equivalent with the three conditions

$$\partial^2 = 0, \partial\overline{\partial} + \overline{\partial}\partial = 0, \overline{\partial}^2 = 0.$$

Note further that we have $\overline{\partial} f = \overline{\partial}(\overline{f})$ [The operation $\overline{\partial} f$ is the con- **119**
jugation $\mathscr{E}^*_{1,0} \to \mathscr{E}^*_{0,1}$ defined earlier].

Definition. A differential form ω is holomorphic if ω is of type $(p,0)$ and $\overline{\partial}\omega = 0$.

Remark. If f is a $0-$ form, it is holomorphic if and only if f, as a function of $(z_1,\ldots,z_n)$, the complex local coordinates is holomorphic. Further if ω is of type $(p,0)$, if $\omega = \sum f_I(z) dz_I$ in local coordinates, ω is holomorphic if and only if $f_I(z)$ is holomorphic for each I.

We make two further remarks.

1. Any complex manifold is orientable. In fact the jacobian determinant of a holomorphic map $f : \Omega \to \mathbb{C}^n$, Ω open in $\mathbb{C}^n$, considered as a C^∞ map of an open set in $\mathbb{R}^{2n}$ into $\mathbb{R}^{2n}$ (in terms of the identification of $\mathbb{R}^{2n}$ and $\mathbb{C}^n$ made earlier) is equal to $|D|^2$, where $D = \det\left(\dfrac{\partial f_i}{\partial x_j}\right)$.

2. Let V, V' be complex manifolds, $f : V \to V'$ a holomorphic map. f induces a $\mathbb{C}$ linear map $T_a(V) \to T_{f(a)}(V')$ since $\varphi \circ f$ is holomorphic for any holomorphic φ. Hence, the map $f^* : \mathscr{T}^*_{f(a)} \to \mathscr{T}^*_a$ maps $(\mathscr{E}^*_{1,0})_{f(a)}$ into $(\mathscr{E}^*_{1,0})_a$. Hence $f^*(\omega')$ is of type (p, q) if ω' is of type (p, q). Since moreover, for any form ω' of type (p, q) on V', we have,

$$\begin{aligned}\partial(f^*(\omega')) + \bar{\partial}(f^*(\omega')) = d f^*(\omega') &= f^*(d\omega') \\ &= f^*(\partial\omega') + f^*(\bar{\partial}\omega'),\end{aligned}$$

120 and f^* preserves the type, we deduce that

$$\partial f^*(\omega') = f^*(\partial\omega'), \bar{\partial} f^*(\omega') = f^*(\bar{\partial}\omega').$$

[Note that this is not true for any C^∞ map f.] As in the case of a C^k manifold, we set $Z^{p,q}(V)$ = set of C^∞ forms ω of type (p, q) with $\bar{\partial}\omega = 0$ and $B^{p,q}(V)$ = set of C^∞ forms ω of type (p, q), for which there is a C^∞ form ω' of type $(p, q-1)$ with $\bar{\partial}\omega = \omega$. Then, since $\bar{\partial}^2 = 0$, we have $B^{p,q}(V) \subset Z^{p,q}(V)$. We set $H^{p,q}(V) = Z^{p,q}(V)/B^{p,q}(V)$. These groups are called the Dolbeault groups of V.

These groups are not topological invariants of V. They depend essentially on the holomorphic structure of V.

We now look for an analogue of Poincare's lemma, i.e. for a class of domains D in $\mathbb{C}^n$ for which $H^{p,q}(D) = 0$ for $q * * * * * 1$. We begin with the following lemma.

Lemma. *Let K, L and L' be compact sets in $\mathbb{C}, \mathbb{C}^r$ and $\mathbb{R}^n$, respectively. We denote a point in $K \times L \times L'$ by (z, w, t). If g is a C^∞ function defined in a neighbourhood of $K \times L \times L'$ and if g is holomorphic in w for each fixed z and t, then there exists a C^∞ function f in a neighbourhood of $K \times L \times L'$ which is holomorphic in w for fixed z and t such that $\dfrac{\partial f}{\partial \bar{z}} = g$ in a neighbourhood of $K \times L \times L'$.*

Proof. We may assume that g has compact support in $\mathbb{C}$ for any fixed w and t [Multiply g if necessary, by $\varphi(z)$ where φ has compact support and

= 1 in a neighbourhood of K]. Define

$$f(z,w,t) = -\frac{1}{\pi}\int\limits_{\mathbb{C}} \frac{g(\zeta,w,t)}{\zeta - z} d\xi \wedge d\eta$$

□

Then 121

$$\begin{aligned}
\frac{\partial f}{\partial \overline{z}} &= -\frac{1}{\pi}\int\limits_{\mathbb{C}} \frac{\partial g(\zeta+z,w,t)}{\partial \overline{z}} - \frac{1}{\zeta} d\xi \wedge d\eta \\
&= \lim_{\mathcal{E}\to 0} -\frac{1}{\pi}\int\limits_{|\zeta|\geq\mathcal{E}} \frac{\partial g(\zeta+z,w,t)}{\partial \overline{\zeta}} - \frac{1}{\zeta} d\xi \wedge d\eta \\
&= \lim_{\mathcal{E}\to 0} -\frac{1}{2\pi i}\int\limits_{|\zeta|\geq\mathcal{E}} \frac{\partial g(\zeta+z,w,t)}{\partial \overline{\zeta}} - \frac{1}{\zeta} d\overline{\xi} \wedge d\eta \\
&= \lim_{\mathcal{E}\to 0} -\frac{1}{2\pi i}\int\limits_{|\zeta|\geq\mathcal{E}} d\left(\frac{g(\zeta+z,w,t)d\zeta}{\zeta}\right).
\end{aligned}$$

Now by Stoke's theorem,

$$\begin{aligned}
&-\lim_{\mathcal{E}\to 0} \frac{1}{2\pi i}\int\limits_{|\zeta|\geq\mathcal{E}} d\left(\frac{g(\zeta+z,w,t)d\zeta}{\zeta}\right) \\
&= \lim_{\mathcal{E}\to 0} \frac{1}{2\pi i}\int\limits_{|\zeta|=\mathcal{E}} \frac{d(\zeta+z,w,t)d\zeta}{\zeta} \\
&= g(z).
\end{aligned}$$

Clearly f is a C^∞ function and holomorphic in w for fixed z and t Grothendieck's lemma (or Poincare's lemma for $\overline{\partial}$). If $D = D_1\times\cdots\times D_n$, where D_i is a domain in $\mathbb{C}$, $1 \leq i \leq n$, then if $\overset{(p,q)}{\omega}$ is a C^∞ differential form on D with $\overline{\partial}\omega = 0$, $q \geq 1$, then there exists a C^∞ differential form ω' on D such that $\overline{\partial}\omega = \omega$; in other words $H^{p,q}(D) = 0$ for $q \geq 1$.

Proof. To make clear the basic idea we shall first prove the lemma for $(0,1)$ forms on $K = K_1 \times \cdots \times K_n$ where K_i are compact sets in. □

[i.e. for forms in some neighbourhood of K, the above equations holding in some neighbourhood of K (not necessarily the same)].

122 Let $\omega = a_1 d\overline{z}_1 + \cdots + a_n d\overline{z}_n$. By the lemma, there exists a C^∞ function b_n in a neighbourhood of $K_1 \times \cdots \times K_n$ such that

$$\frac{\partial b_n}{\partial \overline{z}_n} = a_n.$$

Let $\omega_{n-1} = (\omega - \overline{\partial} b_n) = a'_1 d\overline{z}_n + \cdots + a_{n-1} d\overline{z}_{n-1}$.

Then $\overline{\partial}\omega = 0 \Rightarrow \overline{\partial}\omega_{n-1} = 0$.

Hence

$$d\overline{z}_n \wedge \sum_{i \leq n-1} \frac{\partial a'_i}{\partial \overline{z}_n} d\overline{z}_i = 0$$

i.e. a'_i are holomorphic in z_n.

Hence by the lemma there exists a C^∞ function b_{n-1} in a neighbourhood of $K_1 \times \cdots \times K_n$ which is holomorphic in z_n and for which $\dfrac{\partial b_{n-1}}{\partial \overline{z}_{n-1}} = a'_{n-1}$.

Let $\omega_{n-2} = \omega - \overline{\partial} b_n - \overline{\partial} b_{n-1}$.

Then $\omega_{n-2} = a''_1 d\overline{z}_1 + \cdots a''_{n-2} d\overline{z}_{n-2}$ with a''_1 holomorphic in z_{n-1}, z_{n-2}. We continue the process and obtain

$$\omega_1 = \omega - \overline{\partial} b_n - \overline{\partial} b_{n-1} \cdots - \overline{\partial} b_1 = 0$$

$$\text{i.e.} \qquad \omega = \overline{\partial}(b_n + b_{n-1} \cdots + b_n).$$

We shall now prove by induction the lemma for forms on

$$K = K_1 \times \cdots \times K_n, K_i \text{ compact in } \mathbb{C}.$$

Let $\mathscr{O}_k$ = the set of differential forms of type (p, q) not containing $d\overline{z}_k, \ldots d\overline{z}_n$ in their expressions in local coordinates.

123 Assume that the lemma is proved for differential forms in $\mathscr{O}_i$, $i \leq k$. (The lemma is trivial for $\mathscr{O}_1$). Let ω be a differential form in $\mathscr{O}_{k+1}$. Then

$$\omega = d\overline{z}_k \wedge \omega_1 + \omega_2 \text{ where}$$
$$\omega_1^{p,q-1}, \omega_2^{p,q} \in \mathscr{O}_k.$$

If $\bar{\partial}\omega = 0$, $d\bar{z}_k \wedge \bar{\partial}\omega_1 + \bar{\partial}\omega_2 = 0$ hence $\dfrac{\partial\omega_1}{\partial\bar{z}_j} = 0$ for $j > k$.

Since, by assumption, $\dfrac{\partial\omega_2}{\partial\bar{z}_j} = 0$ for $j \geq k$. By the lemma, there exists $\overset{p,q-1}{\Phi}$ in a neighbourhood of K, holomorphic in z_j, $j > k$ such that $\dfrac{\partial\Phi}{\partial\bar{z}_k} = \omega_1$. Then $\omega - \bar{\partial}\Phi \in \mathscr{O}_k$, $\bar{\partial}(\omega - \bar{\partial}\Phi) = 0$ and by the induction hypothesis there exists ψ such that

$$\bar{\partial}\psi = \omega - \bar{\partial}\Phi$$
$$\text{i.e.} \quad \omega = \bar{\partial}(\Phi + \psi).$$

We shall now prove the lemma for $D = D_1 \times \cdots \times D_n$. Let K_i^ν be a sequence of compact sets, $K_i^\nu \uparrow D_i$ as $\nu \to \infty$ and let $K^\nu = K_1^\nu \times \cdots \times K_n^\nu$. By what we have proved above, there exist differential forms ω^ν of type $(p, q-1)$ in neighbourhood of K^ν such that

$$\bar{\partial}\omega^\nu = \omega \text{ in a neighbourhood of } K^\nu.$$

We shall consider two different cases

(i) $q \geq 2$ and (ii) q = 1

(i) If $q > 1, \bar{\partial}(\omega^{\nu+1} - \omega^\nu) = 0$ in a neighbourhood of K^ν.

Since $\omega^{\nu+1} - \omega^\nu$ is of type of $(p, q-1)$ and $q - 1 \geq 1$, there exists a **124**
differential form $\varphi^{\nu+1}$ of type $(p, q-1)$ in D such that $\bar{\partial}\varphi^{\nu+1} = \omega^{\nu+1} - \omega^\nu$ on K.

Let $\psi^{\nu+1} = \omega^{\nu+1} - \bar{\partial}\varphi^{\nu+1} - \cdots - \bar{\partial}\varphi^1$.

Then
$$\psi^{\nu+1} - \psi^\nu = \omega^{\nu+1} - \omega^\nu - \bar{\partial}\varphi^{\nu+1}$$
$$= 0 \text{ in a neighbourhood of } K^\nu.$$

Hence the form $\psi = \psi^\nu$ in K^ν, $\nu \geq 1$, is well defined, and $\bar{\partial}\psi = \omega$.

We suppose that K_i^ν have the property that any holomorphic function in a neighbourhood of K_i^ν can be approximated, uniformly on K_i^ν by holomorphic functions in D_i [It is a classical theorem that any domain in $\mathbb{C}$ can be approximated by such compact sets: this result is a consequence of the Runge theorem proved in Chap. III.] From Chap I, §5, it follows that any holomorphic function on K^ν can be approximated on K^ν by holomorphic functions in D and in view of the remark following the definition of holomorphic forms, there exist holomorphic forms $\varphi^{\nu+1}$ of type $(p,0)$ on D such that $\|\varphi^{\nu+1} - (\omega^{\nu+1} - \omega^\nu)\| < \dfrac{1}{2^\nu}$ on K^ν. [the inequality holding for all coefficients]. Hence $\sum\limits_1^\infty \{\varphi^{\nu+1} - (\omega^{\nu+1} - \omega^\nu)\}$ is uniformly convergent on any compact subset of D.

Let $\omega' = \sum\limits_0^\infty \{\omega^{\nu+1} - \omega^\nu - \varphi^{\nu+1}\}$ where $\omega^0 = 0$; we have

$$\omega' = \omega^r - \varphi^r - \varphi^1 + \sum_r^\infty (\omega^{\nu+1} - \omega^\nu - \varphi^{\nu+1})$$

on K^r; since the φ^ν and $\sum\limits_r^\infty (\omega^{\nu+1} - \omega^\nu - \varphi^{\nu+1})$ are holomorphic on K^r we conclude that $\bar\partial\omega' = \omega$ in D, and that ω' is C^∞.

125 The proof of Grothendieck's lemma on compact sets given above follows essentially the exposition by Serre [42] of the original proof of Grothendieck. It is to be remarked that also the proof of Poincare's lemma (for cubes instead of arbitrary convex sets) can be given on the same lines as that of the Grothendieck lemma. This is essentially the proof given by E. Cartan [5]; this proof of *E*. Cartan was in fact the origin of the proof of the Grothendieck lemma.

8 Applications to complex analysis. Hartogs' continuation theorem and the Oka-Weil theorem

Proposition 1. *Let Ω be a convex open set in $\mathbb{C}^n$ and φ a real valued C^∞ function on Ω. In order the there exist a holomorphic function f on Ω such that Re $f = \varphi$, it is necessary and sufficient that*

$$\frac{\partial^2\varphi}{\partial z_i \partial \bar{z}_j} = 0 \text{ for } 1 \le i, j \le n.$$

Proof. If $\varphi = Re f = \frac{1}{2}(f + \overline{f})$, then $\frac{\partial^2\varphi}{\partial z_i \partial \overline{z}_j} = 0$ since $\frac{\partial f}{\partial \overline{z}_j} = 0, \frac{\partial \overline{f}}{\partial z_i} = \overline{\frac{\partial f}{\partial \overline{z}_i}} = 0$. Suppose conversely that these equations are satisfied. We see at once that the form of type (1.1)

$$\overline{\partial}\partial\varphi = 0.$$

Since $d = \partial + \overline{\partial}$ and $\overline{\partial}^2 = 0$, this can be written $d\partial\varphi = 0$. By Poincare's lemma, there is a complex valued function g on Ω with

$$dg = \partial\varphi.$$

Since $\partial\varphi$ is of type $(1, 0)$, we have $\partial g = \partial\varphi, \overline{\partial} g = 0$, so that g is holomorphic. Further 126

$$d(g + \overline{g}) = dg + \overline{dg} = \partial\varphi + \overline{\partial\varphi} = d\varphi,$$

so that $g + \overline{g} - \varphi$ is constant, and the proposition follows.

This implies the following

Proposition 1′. *Let φ be a C^∞ real valued function on the complex manifold V. In order that φ be locally the real part of a holomorphic function, it is necessary and sufficient that $\overline{\partial}\partial\varphi = 0$.*

Lemma 1. *If $D = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \| z_i| < R_i\}$, $n \geq 2$, U is a neighbourhood of ∂D in $\mathbb{C}^n$ and if f is a holomorphic function in $U \cap D$, there exists a neighbourhood V of ∂D and a holomorphic function F in D such that $F|V \cap D = f$.*

Proof. Let $\mathscr{E}_1, \mathscr{E}_2$ be two positive numbers such that if $U_1 = \{(z_1, \ldots, z_n)$ $|R_1 - \mathscr{E}_1 < |z| < R_1, |z_2| < R_2, \ldots, |z_n| < R_n\}$ and $U_2 = \{(z_1, \ldots, z_n)| |z_1| < R_1, R_2 - \mathscr{E}_2 < |z_2| < R_2, \ldots, |z_n| < R_n\}$, then $U_1 \cup U_2 \subset U$. □

For any holomorphic function f on U_1 there exist holomorphic functions a_r in $\{|z_2| < R_2, \ldots, |z_n| < R_n\}$ such that $f(z) = \sum_{-\infty}^{\infty} a_r(z')z_1^r$ where $z' = (z_2, \ldots, z_n)$. Let $z' = (z_2, \ldots, z_n)$ be any point with

$$R_2 - \varepsilon_2 < |z_2| < R_2, \ldots, |z_n| < R_n.$$

Then $f(z_1, z')$ is holomorphic for $|z_1| < R_1$ since f is holomorphic in U_2. Hence there can be no terms containing negative powers of z_1 in 127
the Laurent as expansion of f: thus $a_r(z') = 0$ for $r < 0$, if $R_2 - \varepsilon_2 < |z_2| < R_2$. By the principle of analytic continuation, this implies that $a_r(z') = 0$, for $r < 0$, $|z_2| < R_2, \ldots |z_n| < R_n$, and

$$f(z) = \sum_0^\infty a_r(z') z_1^r \text{ in } U_1 \cup U_2.$$

By Abel's lemma, $\sum\limits_0^\infty a_r(z') z_1^r$ is uniformly convergent on compact subsets of D and hence

$$F(z) = \sum_0^\infty a_r(z') z_1^r$$

is a holomorphic extension of $f|U_1 \cup U_2$ to D. Hence $F = f$ in the connected component Ω of $U \cap D$ containing $U_1 \cup U_2$; since ∂D is connected, $\Omega = V \cap D$where V is a neighbourhood of D.

Lemma 2. *If ω is a differential form of type* $(0, 1)$ *with compact support in $\mathbb{C}^n$, $n \geq 2$, and if $\bar{\partial}\omega = 0$, there exists a C^∞ function φ on $\mathbb{C}^n$, with compact support, such that $\bar{\partial}\varphi = \omega$.*

Proof. Choose $R > 0$ such that if

$$\square \qquad D\left\{(z_1, \ldots, z_n)\Big||z_i| < R\right\}, \text{ then supp. } \omega \subset D.$$

By Poincare's lemma for $\bar{\partial}$, there exists a C^∞ function f on $\mathbb{C}^n$ such that

$$\bar{\partial} f = \omega.$$

Now we have $\omega = \bar{\partial} f = 0$ in a neighbourhood of ∂D, i.e. f is holomorphic in a neighbourhood of ∂D. Hence by Lemma 1 there exists
128 a function F, holomorphic on D such that $F(z) = f(z)$ for x in a certain neighbourhood of ∂D. Consider

$$\varphi(z) = \begin{cases} f(z) - F(z) & \text{for z} \in D \\ 0 & \text{for z} \notin D. \end{cases}$$

Then clearly φ is C^∞ function with compact support and $\bar\partial\varphi = \omega$.

We shall now prove the following important theorem of Hartogs.

Theorem 1 (Hartogs). *Let D be a bounded open connected subset of $\mathbb{C}^n$, $n \geq 2$, such that $\mathbb{C}^n - D$ is connected, and U, a neighbourhood of ∂D. If f is a holomorphic function on U, then there exists a neighbourhood V of ∂D and a holomorphic function F on D such that $F|V \cap D = f$.*

Proof. We can assume without loss of generality that $f \in C^\infty$ in D. [If not, multiply f by a C^∞ function α with compact support in U such that $\alpha(z) = 1$ for z in a neighbourhood of ∂D.] Let $\omega = \bar\partial f$ in D; since f is holomorphic near $\partial D, \omega$ has compact support in D. We extend it to $\mathbb{C}^n$ by setting $\omega = 0$ outside D. □

Then ω is of type $(0, 1)$ and has compact support and $\bar\partial\omega = 0$. Hence by Lemma 2, there exists a C^∞ function φ, in $\mathbb{C}^n$, with compact support such that $\bar\partial\varphi = \omega$.

In particular φ is holomorphic on each open set on which ω vanishes and hence φ is holomorphic in a neighbourhood of $\mathbb{C}^n - D$. Also φ has compact support and $\mathbb{C}^n - D$ is connected. Hence, by the principle of analytic continuation $\varphi = 0$ in a connected neighbourhood of $\mathbb{C}^n - D$ and hence $\varphi = 0$ in a neighbourhood V of ∂D. Consider $F = f - \varphi$; we have

$$\bar\partial F = 0 \text{ in } D, F = f \text{ near } \partial D.$$

Hence F is a holomorphic function with the required properties. **129**

Definition. A domain D in $\mathbb{C}^n$ is said to be a Cousin domain if given a differential form ω of type (p, q), $q \geq 1$, $p \geq 0$, such that $\bar\partial\omega = 0$, there exists a differential form ω' of type $(p, q-1)$ such that $\bar\partial\omega' = \omega$; (in this case we shall also any that D is Cousin).

Theorem 2 (Oka). *Let $B = \{z \in \mathbb{C} \big| |z| < 1\}$. If D is a domain in $\mathbb{C}^n$ such that $D \times B$ is Cousin and if f is a holomorphic function on D, $D_f = \{z \in D \big| |f(z)| < 1\}$, then D_f is Cousin. Further given a differential form $\overset{(p,q)}{\omega}$, $q \geq 0$ on D_f, such that $\bar\partial\omega = 0$, there exists a form Ω of type (p, q) on $D \times B$ with $\bar\partial\Omega = 0$ such that if $i : D_f \to D \times B$ is the map given by $i(z) = (z, f(z))$, we have $i^*(\Omega) = \omega$.*

Proof. We begin with the remark that i: $D_f \to D \times B$ is injective and proper; further i_* is injective at every point, so that $i(D_f)$ is a closed complex analytic submanifold of $D \times B$. Let π: $D \times B \to D$ be the projection

$$\pi(z, z') = z, (z, z') \in D \times B.$$

□

Let $\pi^{-1}(D_f) = D_f \times B = V$. Then V is a neighbourhood of $i(D_f)$ in $D \times B$.

Let V' be a neighbourhood of $i(D_f)$ in V such that $\overline{V'} \subset V$. Then there exists a C^∞ function α on $D \times B$ such that

$$\begin{aligned} \alpha(z, z') &= 1 \text{ if } (z, z') \text{ is in a neighbourhood of } i(Df) \\ &= 0 \text{ if } (z, z') \notin V'. \end{aligned}$$

Let $\varphi = \pi^*(\omega)$ on V; since π is holomorphic, φ is of type (p, q).
130 Further, since $\pi \circ j$ = identity on D_f, we have $i^*(\varphi) = \omega$. Then if φ' is defined on $D \times B$ as $\varphi' = \alpha\varphi$ on $V = 0$ outside V, φ' is a C^∞ form of type (p, q) on $D \times B$, and since $\varphi' = \varphi$ near $i(D_f)$ we have $i^*(\varphi') = \omega$. Let ω_1 be the form defined on $D \times B$ by

$$\begin{aligned} \omega_1 = &\text{ in a neighbourhood of } i(D_f) \\ = &\frac{1}{z' - f(z)} \bar\partial(\varphi') \text{ in } D \times B - i(D_f). \end{aligned}$$

Then $\bar\partial\omega_1 = 0$ and ω_1 is of type $(p, q+1), q \geq o$. [ω_1 is C^∞ since $\bar\partial(\varphi') = \bar\partial(\varphi) - 0$ in a neighbourhood of $i(D_f)$.] Hence there exists ψ of type (p, q) such that $\bar\partial\psi = \omega_1$.

Consider $\bar\partial(\varphi' + (z' - f(z))\psi)$

$$= \bar\partial(\varphi') - (z' - f(z))\bar\partial\psi.$$

Clearly $\bar\partial[\varphi' + (z' - f(z))\psi] = 0$ on $D \times B$ and $i^*[\varphi' + (z' - f(z))\psi] = i^*(\varphi') = \omega$.

Hence given a differential form $\overset{(p,q)}{\omega}$, $q \geq 0$ on D_f with $\bar\partial\omega = 0$ there exists a form $\Omega\{= \varphi' + (z - f(z))\psi\}$, on $D \times B$ such that $i^*(\Omega) = \omega$ and $\bar\partial\Omega = 0$. Since $D \times B$ is Cousin it follows immediately that D_f is Cousin.

Corollary. *With the same notation as in the theorem, if $D \times B^r$ is Cousin for every positive integer r, so is $D_f \times B^r$ for every positive integer r.*

Proof. Consider $D \times B^{r+1}$, a point in $D \times B^{r+1}$ being denoted by $(z, w_1, \ldots, w_{r+1})$. Then, if $D' = (D \times B^r)$, $D'_f = D_r \times B^r$ and by applying the lemma to D' the corollary is proved. □

Theorem 3 (Oka)**.** *If $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and if $f_i(z)_{1 \leq i \leq r}$ are holomorphic functions in $\mathbb{C}^n$ and if $U = \left\{ z \in \mathbb{C}^n \middle| |f_i(z)| < 1i, 1 \leq i \leq r \right\}$ the map $U \to \mathbb{C}^n \times B^r$ given by $i(z) = (z, f_1(z), \ldots, f_r(z))$, then given a holomorphic function g on U, there exists a holomorphic function F on $\mathbb{C}^n \times B^r$ such that $G \circ i = g$.* 131

Proof. Let

$$D_0 = \mathbb{C}^n, D_{k+1} = \{z \in D_k \Big| |f_{k+1}(z)| < 1\}; 0 \leq k < r.$$

Clearly $D_r = U$. □

Let i_r be the map $D_r \to D_{r-1} \times B$ defined by $i_r(z) = (z, f_r(z))$, i_{r-1}: $D_{r-1} \times B \to D_{r-2} \times B^2$ the map defined by $i_{r-1}(z, w_1) = (z, w_1, f_{r-1}(z))$, and so on. Then we have $i = i_1 \circ i_2 \circ \cdots \circ i_r$. Further, since $\mathbb{C}^n \times B^m$ is Cousin for every m (by Poincare's lemma for $\bar\partial$), it follows by the corollary to Theorem 2 that $D_k \times B^m$ is Cousin for $0 \leq k \leq r$, and all $m \geq 0$, so that, by Theorem 2, for any form ω_k of type (p, q) on $D_k \times B^{r-k}$ with $\bar\partial\omega_k = 0$, there is a form ω_{k+1} of type (p, q) on $D_{k-1} \times B^{r-k+1}$ with $\bar\partial\omega_{k+1} = 0$ and $i_k^*(\omega_{k+1}) = \omega_k$. Hence, by induction for any $\bar\partial$ closed form ω of type (p, q) on $U = D_r$, there is a $\bar\partial$ closed form Ω of type (p, q) on $\mathbb{C}^n \times B^r$ with $i^*(\Omega) = \omega$. Theorem 3 is the special case of this for which $p = 0$, $q = 0$.

Theorem 4 (Oka - Weil approximation theorem)**.** *If $\{f_i(z)\}_{1 \leq i \leq r}$ are entire functions in $z_1, z_2, \ldots, z_n$, and if $U = \left\{ z \in \mathbb{C}^n \middle| |f_i(z)| < 1, 1 \leq i \leq r \right\}$, then U is a Runge domain.*

Proof. With the notation of Theorem 3, given a holomorphic function 132
G on $\mathbb{C}^n \times B^r$ such that $G \circ i = g$. Let a point in $\mathbb{C}^n \times B^r$ be denoted by

(z, w). Then G can be expanded in a uniformly convergent Taylor series, $G(z, w) = \sum a_{\alpha\beta} z^\alpha w^\beta$. ◻

Hence $G(z, w) = \lim_{k\to\infty} g_k(z, w)$, uniformly on compact sets where $g_k(z, w)$ are polynomials in $z_1, \ldots, z_n, w_1, \ldots, w_r$.

Hence
$$g = G \circ i = \lim_{k\to\infty} g_k \circ i$$
$$= \lim_{k\to\infty} \sum_{\alpha+\beta\leq k} a_{\alpha\beta} z^\alpha f^\beta, \text{ uniformly on compact}$$

subsets of U, where $f = (f_1(z), \ldots, f_r(z))$. Thus g can be approximated by polynomials in $z_1, \ldots, z_n$, $f_1, \ldots, f_1$. Since, the f_i being entire, the f_i can be approximated by polynomials in $z_1, \ldots, z_n$, so can g, and U is a Runge domain.

Remark. We have used Grothendieck's lemma for a domain of the form $D = D_1 \times \cdots \times D_n$; however, for the proof of the Oka-weil theorem, it would suffice to use it for *compact* sets $K = K_1 \times \cdots \times K_n$. However, the extension theorem of Oka (Theorem 3) is very important, so that we have given the proof for open, rather than compact, sets.

As a corollary to the Oka-Weil theorem we have the following

Proposition 2. *A convex open set in $\mathbb{C}^n$ is a Runge domain.*

Proof. It is enough to prove that a bounded convex set in Runge. Consider U as a convex set in $\mathbb{R}^{2n}$. Then for any point z_0 on the boundary,
133 there exists a linear function $l, l(z) = \sum_1^n a_i x_i + \sum_1^n b_i y_i + c$ such that $U \subset \left\{z \middle| l(z) < 0\right\}$ and $l(z_0) = 0$. Let $L(z)$ be a linear function, $L(z) = \sum_i^n d_i z_i + e$, d_i, $e \in \mathbb{C}$ such that $1(z) = Re[L(z)]$. Hence $U \subset \left\{z \middle| ReL(z) < 0\right\}$, while $ReL(z_0) = 0$. Let K be any compact subset of U. If $z_0 \in \partial U$, we may therefore find a linear function L with $ReL(z) < 0$ for $z \in K$, $ReL(z_0) > 0$ (replace the L constructed above by $L + \delta$ where $\delta > 0$ is sufficiently small).Then Re $L(z) > 0$ for z in a neighbourhood of z_0. Since ∂U is compact, there exist finitely many linear functions $L_1, \ldots, L_r$ such that ◻

Re $L_i(z) < 0$ for $z \in K$, $ReL_j(z) > 0$ for at least one j if $z \in \partial U$. Hence the set

$$\Omega_U = \left\{ z \in U \middle| ReL_i(z) < 0, i = 1, \ldots, r \right\}$$

contains K and is relatively compact in U. Since the set. $\Omega = \{z \in \mathbb{C}^n \big| ReL_i(z) < 0, i = 1, \ldots, r\}$ is convex, hence connected and $\Omega \cap U = \Omega_U$ is relatively compact in U, it follows that $\Omega \subset U$. Now

$$\Omega_U = \left\{ z \in \mathbb{C}^n \middle| |f_i(z)| < 0, i = 1, \ldots, r \right\}$$

where $f_i(z) = e^{L_i(z)}$, so that Ω is Runge by theorem 4. Hence any holomorphic function on $U(\supset \Omega)$can be approximated, uniformly on K, by polynomials. Since K is an arbitrary compact subset of U, the proposition is proved.

The proof of Hartogs' theorem given here is suggested by the proof **134**
of the Runge theorem of Malgrange-Lax (see Chap. III §10; also Malgrange [27]). That of the Oka-well theorem is merely a translation of Oka's own proof [34] into the language of differential forms.

9 Immersions and imbeddings: the theorems of Whitney

In what follows V, V' are C^k manifolds, $1 \leq k \leq \infty$ countable at infinity.

Definitions. (1) A C^k map $f\colon V \to V'$ is called an immersion if for every $a \in V$, $f_*\colon T_a(V) \to T_{f(a)}(W)$ is injective . If $f_*\colon T_a(V) \to T_{f(a)}(W)$ is injective for every a in a subset E of V, we say that f is regular on E.

(2) A C^k map $f\colon V \to V'$ is called an imbedding if f is an immersion and f is injective.

(3) An imbedding (immersion) $f\colon V \to V'$ is called a closed imbedding (immersion) if f is proper.

[Note that the set of points where f is regular is open.]

Let $\{U_i\}$ be a locally finite covering of V, U_i being relatively compact coordinate neighbourhoods. Then there exist compact sets $K_i \subset U_i$ with $\cup K_i = V$. Let η be a continuous function on V, $\eta(x) > 0$ for all x, and N, a non- negative integer $\leq k$. Given a C^k function f on V, another C^k function g is said to approximate f within η upto N^{th} order (with respect to the covering $\{U_i\}$), if

$$|D^\alpha f(x) - D^\alpha g(x)| < \eta(x) \text{ for } |\alpha| \leq N \text{ and } x \in K_i$$

and we denote this fact by g approximates f with respect to $(U_i, \eta, N)''$.

135 If $\{U_i\}$ is given, we say that g approximates f within η upto order N.

Remark. If$\{U_i\}$, $\{U'_j\}$ are two locally finite coverings of V, $K_i \subset U_i$, $K_j \subset U_j$, K_i, K'_j compact sets of V such that $\cup K_i = \cup K'_j = V$ then there exists a positive continuous function δ such that if g approximates f with respect to (U_i, η, N) then g approximates f with respect to $(U'_j, \delta\eta, N)$.

Proof. Since $\{U'_j\}$ is locally finite, it suffices to prove that if $\{y^j_1, \ldots, y^j_n\}$, $\{x^i_1, \ldots, x^i_n\}$ are coordinate in U'_j, U_i respectively, then for any C^k function h on V, we have

$$\left|D^\alpha_{y^j} h(y)\right| \leq C_j \sup_{K'_j \cap U_i \neq 0} \sum_{|\beta| \leq N} \left|D^\beta_{x^i} h(x^{(i)})\right|$$

for y in K'_j and some constant C_j independent of h. This is, however, obvious. □

This remark implies that if $\{U_i\}$, N are such that f can be approximated by functions g in a given class $\mathscr{C}$ with respect to (η, N) for any η then the same is true if $\{V)i\}$ is replaced by any other locally finite covering $\{U'_j\}$ consisting of relatively compact coordinate neighbourhoods.

Proposition 1. *If $f : V^n \to \mathbb{R}^p$ is a C^1 map which is an immersion, given any locally finite $\{U_i\}$ as above, there exists a positive continuous function η on V^n such that if g approximates f with respect to $(U_i, \eta, 1)$, then g is an immersion.*

Proof. The rank $(df)(x) = n = \dim V$ for any $x \in V$. Hence there exists a locally finite covering $\{U_i\}$, compact sets $K_i \subset U_i$, $\cup K_i = V$, and positive numbers $\delta_i < 1$ such that if $|D^\alpha f(x) - D^\alpha g(x)| < \delta_i$ for x **136**
in $U_i, |\alpha| \leq 1$, then rank $(dg)(x) = n$. Let $\{\alpha_i\}$ be a partition of unity subordinate to $\{U_i\}$ and $\delta'_1 = \inf .\{\delta_{i_1}, \delta_{1p}\}$, then infimum being over those i_k for which $K_i \cap U_{i_k} \neq \phi$. We may then take $\eta = \sum \delta'_i \alpha_i$. ◻

Lemma 1. *If K is a compact set in V, L a neighbourhood of K and f: $V \to \mathbb{R}^p$ is an imbedding, there exists a positive number δ such that for any C^1 map g: $V \to \mathbb{R}^p$ such that $\| f - g \|^L_1 < \delta$, $g|K$ is injective.*

Proof. Since rank $(df)(x) = n$, for any x in V, the rank theorem implies that for any $x \epsilon V$ there is a relatively compact neighbourhood U and a positive number δ' such that $|f(x') - f(x'')| \geq \delta' |x' - x''|$ for x', $x'' \in U$. Let $0 < \varepsilon < \delta'$ and $\| g - f \|^U_1$ is sufficiently small, and $h = g - f$, we have

$$|h(x') - h(x'')| \leq \varepsilon \| x' - x'' \| \text{ for } x', x'' \in U.$$

◻

Then $|g(x') - g(x'')| \geq (\delta' - \varepsilon)|x' - x''|$, i.e. $g|U$ is injective. Since K compact, there exists a finite number of points $x_1, \ldots x_n$ and neighbourhood $U_1, \ldots U_n$, $L \supset \cup U_i \supset K$, such that if $\| g - f \|^{U_i}_1$ is sufficiently small, $g|U_i$ is injective. Hence there exists a neighbourhood Ω of the diagonal Δ in $K \times K$ and a positive number δ_1 such that if $\| g - f \|^L_1 < \delta_1$, we have $g(x) \neq g(y)$ for any $(x, y) \in \Omega - \Delta$. Again there exists $\delta_2 > 0$ such that for $(x, y) \in K \times K - \Omega$, $|f(x) - f(y)| \geq \delta_2$. Let $\delta = \min(\delta_1, \dfrac{\delta_2}{4})$. Then if $\| g - f \|^L_1 < \delta$, and $(x, y) \epsilon K \times K - \Omega$, $\| g(x) - g(y) \| \geq \dfrac{\delta_2}{2}$ and clearly $g|K$ is injective.

We shall not need the next proposition, but have included it because **137**
it is of interest and is useful in many questions.

Proposition 2. *If f: $V^n \to \mathbb{R}^p$ is an imbedding and f is locally proper, there exists continuous function η on V such that if g approximates f within η upto 1^{st} order, then g is an imbedding.*

Proof. It follows from Proposition 1 that there exists a continuous function η_1, such that if g approximates f within η_1, upto 1^{st} order, g is an immersion. Now for g satisfying this condition, we shall find a positive continuous function η_2 such that if g approximates f within η_2 upto 1^{st} order g is an imbedding. Let K_m be compact sets such that $K_m \subset \overset{\circ}{K}_{m+1}$ and $\cup K_m = V$. Define $L_m = \overline{K_{m+1} - K_m}$. Then since f is locally proper, (therefore proper into and open set Ω in $\mathbb{R}^p$), there exist open sets U_m in $\mathbb{R}^p$ such that $f(L_m) \subset U_m$ and $U_m \cap U_{m'} = \phi$ if $m' \geq m + 2$. [This is because $\{f(L_m)\}$ is a locally finite system of compact sets in Ω such that $f(L_m) \cap f(L_{m'}) = \phi$ if $m' \geq m + 2$]. Now choose $\delta_m > 0$ such that

$$\| f - g \|_1^{L_m} < \delta_m \text{ for all } m \Rightarrow g(L_m) \subset U_m$$

and $g|L_m \cup L_{m+1}$ is injective. Then if $\eta_2(x) < \delta_m$ for x in L_m and g approximates f within η_2 upto 1^{st} order, g is injective. For if $g(x) = g(y)$, $x \in L_m$, and $x \neq y$, since $g|L_m \cup L_{m+1}$ is injective $y \in L_{m'}$, where $m' \geq m+2$ or $m' \leq m-2$. But $g(L_m) \subset U_m$ for every m and $U_m \cap U_{m'} = \phi$ if $m' \geq m + 2$ or $m' \leq m - 2$. Hence we have a contradiction i.e. g is injective. ◻

138 The proposition is false if we drop the assumption that f is locally proper. Further even on compact subsets, an approximation to an injective map (which is not regular) need not be injective.

Lemma 2. *If Ω is bounded open set in $\mathbb{R}^n$, f a C^k map: $\Omega \to \mathbb{R}^p$, $p \geq 2n$, then for any $\varepsilon > 0$ there exists a C^kmap g: $\Omega \to \mathbb{R}^p$ such that $\| g - f \|_1^{\Omega} < \varepsilon$ and $(\dfrac{\delta g}{\delta x_i})_{1 \leq i \leq n}$ are linearly independent at any point of Ω.*

Proof. We may suppose that $f \in C^2$ because of Whitney's approximation theorem (Chap. 1 §5). Let $f_0 = f$. If $f_1, \ldots, f_r$ are C^k maps such that $\| f_s - f \|_1^{\Omega} < \varepsilon$ and $\dfrac{\delta f_s}{\delta x_1}, \ldots \dfrac{\delta f_s}{\delta x_s}$ are linearly independent on Ω, for $0 \leq s \leq r < n$ we shall define f_{r+1} such that

$$\| f_{r+1} - f \|_1^{\Omega} < \varepsilon \text{ and } \frac{\partial f_{r+1}}{\partial x_1}, \ldots, \frac{\partial f_{r+1}}{\partial x_{r+1}}$$

are linearly independent on Ω. ◻

Let $v_i(x) = \dfrac{\partial f_r}{\partial x_1}$ $1 \leq i \leq n$.
Define

$$\varphi : \mathbb{R}^r \times \Omega \to \mathbb{R}^p \text{ by}$$
$$\varphi(\lambda_1, \ldots, \lambda_r, x) = \sum_1^r \lambda_i \frac{\partial f_r}{\partial x_i} - v_{r+1}(x).$$

Now we have $\dim \mathbb{R}^r \times \Omega < p$ and $\varphi \in C^1$. Hence the image of $\mathbb{R}^r \times \Omega$ by φ has measure zero in $\mathbb{R}^p$. Hence given any $\delta > 0$, there exists $a \in \mathbb{R}^p$ such that $\| a \| < \delta$ and $a \notin \varphi(\mathbb{R}^r \times \Omega)$. For sufficiently small δ, if we define $f_{r+1}(x) = f_r(x) + a.x_{r+1}$, $a \in \mathbb{R}^p$ having the above property, 139
we have $\dfrac{\partial f_{r+1}}{\partial x_i} = \dfrac{\partial f_r}{\partial x_i}$ for $i \leq r$ and $\dfrac{\partial f_{r+1}}{\partial x_{r+1}} = v_{r+1}(x) + a$ which is linearly independent of $\dfrac{\partial f_r}{\partial x_i}$, $1 \leq i \leq r$ since $a \notin \varphi(\mathbb{R}^r \times \Omega)$. The lemma is proved with $g = f_n$.

Note that in the above lemma, $g|\Omega$ is an immersion.

Theorem 1. *If $p \geq 2n$, f: $V^n \to \mathbb{R}^p$ is a C^k map if η is positive continuous function on V and $\{U_i\}$ any locally finite covering of V by relatively compact coordinate neighbourhoods, then there exists an immersion g: $V^n \to \mathbb{R}^p$ such that g approximates f with respect to $(U_i, \eta, 1)$.*

Proof. Because of the remark made at the beginning, we may replace $\{U_i\}$ by any other similar covering. We may therefore suppose that $\{U_i\}$ is a locally finite covering of V by relatively compact coordinate neighbourhoods such that U_i are diffeomorphic to bounded open sets in $\mathbb{R}^n$. Let K_i be compact sets with $K_i \subset U_i$ and $\cup K_i = V$. Let $f_0 = f$. Assume that $f_1, \ldots, f_m$ are defined and have the following properties

(i) f_m approximates f with respect to $(U_i, \eta, 1)$,

(ii) f_m is regular on $\bigcup\limits_{i \leq m} K_i$,

(iii) Supp $.(f_{m+1} - f_m) \subset U_{m+1}$.

□

Let α_m be a C^∞ function: $V \to \mathbb{R}$, having compact support in U_{m+1}, while $\alpha_m(x) = 1$ for x in a neighbourhood of K_{m+1}. By the lemma proved above, $f_m|U_{m+1}$ has approximation h_m within δ_m upto 1^{st} order
140 such that h_m is regular on U_{m+1}; let η' be a positive continuous function, $\eta' < \eta$ such that, if g approximates f_m within η' upto 1^{st} order, then g is regular on $\bigcup_{i \leq m} K_i$

Define

$$f_{m+1} = f_m + \alpha_m(h_m - f_m).$$

Then clearly if δ_m is small enough,

i) f_{m+1} approximates f within η upto the 1^{st} order,

ii) f_{m+1} is regular on $\bigcup_{i \leq m} K_i$ (since it approximates f_m within η') and $f_{m+1} = h_m$ in neighbourhood of K_{m+1} and so regular on $\bigcup_{i \leq m} K_i$,

iii) $\operatorname{Supp}(f_{m+1} - f_m) \subset U_{m+1}$.

Hence by induction we have functions $\{f_m\}_{m \geq 1}$ satisfying (i), (ii) and (iii) above. We now define $g = \lim_{m \to \infty} f_m$. Since $\{U_i\}$ is locally finite and $\operatorname{Supp}.(f_{m+1} - f_m) \subset U_{m+1}$, g is well defined and it is easily verified that g satisfies the conditions stated in the theorem.

Theorem 2. *Let $f \colon V^n \to \mathbb{R}^p$ be an immersion, $p \geq 2n+1$, $\{U_i\}$ a locally finite covering of V by relatively compact coordinate neighbourhoods, K_i compact sets, $K_i \subset U_i$, $\cup K_i = V$, such that $f|U_i$ is injective and let η be a positive continuous function on V. Then there exists an imbedding g, approximating f within η upto 1^{st} order.*

Proof. We shall define, by induction, regular maps $f_m \colon V \to \mathbb{R}$, $m \geq 1$,

(i) $f_m|U_i$ is injective for each i,

(ii) f_m is injective on $\bigcup_{i \leq m} K_i$,

141 (iii) f_m approximates f within η upto 1^{st} order and $\operatorname{Supp}.(f_{m+1} - f_m) \subset U_{m+1}$.

Let f_0 and assume that $f_1, \ldots, f_n$ are define. Let α_m be a C^k function $\alpha_m \colon V \to \mathbb{R}$ with compact support in U_{m+1} such that $\alpha_m(x) = 1$ in a neighbourhood of K_{m+1}. Let Ω be the open subset of $V \times V$ defined by $\Omega = \{(x, y) | \alpha_m(m) \neq \alpha_m(y)\}$. Then Ω is a C^k manifold of dimension $2n$; define $\varphi \colon \Omega \to \mathbb{R}^p$ by

$$\varphi(x, y) = \frac{f_m(y) - f_m(x)}{\alpha_m(x) - \alpha_m(y)}.$$

Since $p \geq 2n+1$ and $\varphi \in C^1$, $\varphi(\Omega)$ has measure zero in $\mathbb{R}^p$. Hence we can choose $a \in \mathbb{R}^p$, arbitrarily near 0, such that a $\notin \varphi(\Omega)$ and if $f_{m+1}(x) = f_m(x) + a\alpha_m(x)$, f_{m+1} approximates f_m within a suitable positive function η' so that f_{m+1} is regular and f_{m+1} approximates f within η. We shall now prove that f_{m+1} thus defined satisfies (i), (ii) and (iii). If $f_{m+1}(x) = f_{m+1}(y)$, then

$$a\{\alpha_m(x) - \alpha_m(y)\} = f_m(y) - f_m(x) \tag{9.1}$$

and it follows from the choice of a that

$$\alpha_m(x) - \alpha_m(y) = 0 \text{ i.e. } f_m(x) = f_m(y).$$

□

Hence $f_{m+1}|U_i$ is injective for each if and $f_{m+1}|\bigcup_{i \leq m} K_i$ is injective. Moreover if $x \in K_{m+1}$ and $f_{m+1}(x) = f_{m+1}(y)$ for $y \in \bigcup_{i \leq m+1} K_i$ then $y \in U_{m+1}$, [for otherwise $\alpha_m(x) = 1$ and $\alpha_m(y) = 0$ which contradicts the 142
choice of because of (9.1)] and since $f_{m+1}|U_{m+1}$ is injective $x = y$ i.e. $f_{m+1}|\bigcup_{i \leq m+1} K_i$ is injective. Hence we have, by induction, a family $\{f_m\}$ satisfying (i) (ii) and (iii) and if $g \lim_{m \to \infty} f_m$, g is seen to have the required properties.

Lemma 3. *If $f \colon V^n \to \mathbb{R}$ is continuous proper map and $g \colon V^n \to \mathbb{R}$ is a continuous map which satisfies $|f(x) - g(x)| < 1$ then g is proper.*

Proof. Clearly $\{x \in V \big| |g(x)| \leq C\} \subset \{x \in V \big| |f(x)| \leq C + 1\}$ and so is compact for every C. □

Theorem 3 (Whitney). *If V is a C^k manifold, $k \geq 1$, of dimension n, then there exists a closed immersion of V into $\mathbb{R}^{2n}$ and there exists a closed imbedding of V into $\mathbb{R}^{2n+1}$.*

Proof. Let $\{U_i\}$ be a locally finite covering of V as before, and let $\{K_i\}$ be compact sets, $K_i \subset U_i$ and $\cup K_i = V$. Let $\{\alpha_i\}$ be C^k functions, $supp.\alpha_i \subset U_i$ and $\alpha_i(x) = 1$ for x in a neighbourhood of K_i. Define

$$\varphi : V \to \mathbb{R} \text{ by } \varphi(x) = \sum_{i \geq 1} i\alpha_i(x).$$

clearly φ is C^k. Moreover if $x \in K_m$, we have $\varphi(x) \geq m\alpha_m(x) = m$. Hence $\varphi^{-1}[0, m] \subset \bigcup_{i \leq m+1} K_i$ and so is compact. Hence φ is proper. define φ': $V \to \mathbb{R}^{2n}$ by $\varphi'(x) = (\varphi(x), 0, \ldots, 0)$. Choose η_1, a positive continuous function, with $0 < \eta_1(x) < 1$. By the lemma above if f approximates φ' within η_1, it is proper. Then by Theorem 1, there exists
143 an immersion f which approximates φ' within $\eta_1/2$ and this proves the first part of the theorem. □

Let f: $V \to \mathbb{R}^{2n}$ be a proper immersion. Choose a locally finite covering $\{U_i\}$ of V such that $f|U_i$ is injective and there exists compact sets $\{K_i\}$, $K_i \subset U_i$, $\cup K_i = V$. Define F: $V \to \mathbb{R}^{2n+1}$ by $F(x) = (f_1(x), ., f_{2n}(x), 0)$. Then by Theorem 2, there exists an imbedding g, approximating F within $\eta_1/2$ upto 1^{st} order. Hence g approximates φ' within η_1 and hence is proper i.e. g: $V \to \mathbb{R}^{2n+1}$ is a closed imbedding.

We add a note about th embedding of real analytic manifolds. Let V be real analytic, and suppose that V admits a proper real analytic imbedding i in $\mathbb{R}^p$ for some p. Then if f is C^∞ on V, there exists $F \in C^\infty$ on $\mathbb{R}^p$ with $F \circ i = f$. If follows easily from this and Whitney's approximation theorem (Chap. I, §5) that for any locally finite $\{U_i\}$, $\eta > 0$ and $N > 0$, and C^∞ function f can be approximated a real analytic function g with respect to $\{U_i, \eta, N\}$ (we have only to approximate F by G and set $g = G \circ i$). Hence it follows, from Whitney's Theorem 3 and Proposition 2 that such a manifold has a closed immersion in $\mathbb{R}^{2n}$, and a closed imbedding in $\mathbb{R}^{2n+1}$. These results immersion in $\mathbb{R}^{2n}$, and a closed imbedding in $\mathbb{R}^{2n+1}$. These results have been completed by H.Grauert

[13] by showing that any real analytic manifold countable at ∞ can be analytically imbedded in $\mathbb{R}^p$ for some p. It follows from our remarks above that we have the following theorem.

Theorem. *Any real analytic manifold of dimension n which is countable at ∞ admits a real analytic closed immersion in $\mathbb{R}^{2n}$, and a real analytic closed imbedding in $\mathbb{R}^{2n+1}$.*

The problem of holomorphic imbeddings of complex manifolds is of a different nature. Only so called *Stein manifolds* (see Chap. III for definition) can be imbedded as closed submanifolds of $\mathbb{C}^p$. (See R. Narasimha [31] and E. Bishop [3]). **144**

Whitney [50] has proved that if V is a C^k manifold ($k \geq 1$) of dimension $n(\geq 2)$ and $g\colon V \to \mathbb{R}^{2n-1}$ is any continuous map, then there is a C^k immersion $f\colon V \to \mathbb{R}^{2n-1}$ approximating f. From our remarks above it follows that any real *differentiable manifold* (C^k or analytic) *admits a* a closed immersion in $\mathbb{R}^{2n-1}$. (This is obviously false for $n = 1$; the circle cannot be immersed in the line.) He has further proved [49] that any C^k manifold of dimension n can be imbedded in $\mathbb{R}^{2n}$. In particular, compact have *closed* imbeddings in $\mathbb{R}^{2n}$. These results have been completed by M.W Hirsch [15] by proving that *a non-compact manifold of dimension n has an imbedding in $\mathbb{R}^{2n}$ (hence, a closed imbedding in $\mathbb{R}^{2n-1}$).*

These results are best possible.

Note. The proof of the imbedding theorem (Theorem 3 above) given here is essentially that of Whitney [51].

Chapter 3

1 Vector bundles

Definition. Let X and E be hausdorff spaces and p: $E \to X$ a continuous map. 1 Theotriple (E, pX) is called a continuous complex (real) vector bundle of rank q if the following conditions are satisfied. 145

(i) For $x \in X$, $E_x = p^{-1}(x)$ is a vector space of dimension q over $\mathbb{C}(\mathbb{R})$.

(ii) If $x \in X$, there is a neighbourhood U of x and a homeomorphism h of $E_U = P^{-1}(U)$ onto $U \times \mathbb{C}^q (U \times \mathbb{R}^q)$ such that if π is the projection of $U \times \mathbb{C}^q (U \times \mathbb{R}^q)$ onto U, we have $\pi(h(y)) = x$ if $y \in E_x$ and $h|E_x$ is a $\mathbb{C}(\mathbb{R})$-isomorphism of E_x onto $\{x\} \times \mathbb{C}^q (\{x\} \times \mathbb{R}^q)$.

If E and X are C^k manifolds ($1 \leq k \leq \infty$), if p is a C^k map and if the isomorphisms h_U can be chosen to be C^k diffeomorphisms, p: $E \to X$ is called a C^k bundle (or differentiable bundle of class C^k).

If X is a real (complex) analytic manifold real (complex) analytic vector bundle can be defined in the same way. Complex analytic bundles are also called holomorphic vector bundles. A vector bundle of rank 1 is called a line bundle.

It follows from the definition that if $p : E \to X$ is a complex vector bundle of rank q there exists an open covering $\{U_i\}$ of X and homeomorphisms φ_i: $E_{U_i} \to U_i \mathbb{C}^q$ such that if $U_{ij} = U_i \cap U_j$, then $\varphi_j \circ \varphi_i^{-1}$: $U_{ij} \times \mathbb{C}^q \to U_{ij} \times \mathbb{C}^q$ is a homeomorphism and $\varphi_j \circ \varphi_i^{-1}(x, y) = (x, g_{ij}(x)v)$ where, for each x, $g_{ij}(x)$ is in $GL(q, \mathbb{C})$ and $g_{ij}(x) g_{jk}(x) = g_{ik}(x)$ for $x \in U_{ijk} = U_i \cap U_j \cap U_k$. Clearly $g_{ij} : U_{ij} \to GL(q, \mathbb{C})$ is continuous. 146
The g_{ij} are called transition map (or transition functions) of the bundle.

If the bundle is C^k (or real or complex analytic), the transition maps $g_{ij} : U_{ij} \to GL(q, \mathbb{C})$ are C^k (or real or complex analytic).

Conversely let X be a hausdorff topological space, $\{U_i\}_{i\in I}$ an open covering of X and $g_{ij} : U_{ij} \to GL(q, \mathbb{C})$ continuous map satisfying $g_{ij}(x)g_{jk}(x) = g_{ik}(x)$ for $x \in U_{ijk}(= U_i \cap U_j \cap U_k)$. Then let S be the topological sum $\bigcup_i \{U_i \times \mathbb{C}^q \times \times i\}$. Define an equivalence relation $\sim$ on S by $(x, v, i) \sim (x', v', j)$ if $x = x'$ and $v' = g_{ij}(x).v$. It is easily verified that the equivalence relation is open and that the graph is closed. Hence $E = S/\sim$ is hausdorff. Let $p' : S \to X$ be defined by $p'(x, v, i) = x$. Clearly equivalent points have the same image in X so that p' defines a map $p : E \to X$. Then $p^{-1}(U_i) = \{\overline{(x, v, i)} | x \in U_i, v \in \mathbb{C}^q\}$ and hence $p^{-1}(U_i)$ is "isomorphic" to $U_i \times \mathbb{C}^q$ and $p : E \to X$ is a complex vector bundle of rank q. Thus a vector bundle $p : E \to X$ is characterised by an open covering $\{U_i\}$ of X such that $p^{-1}(U_i)$ is isomorphic to $U_i \times \mathbb{C}^q$, and the transition maps $g_{ij} : U_{ij} \to GL(q, \mathbb{C})$. If X is a C^k manifold, and the g_{ij} are C^k, maps, then the vector bundle constructed above is also C^k. A similar remark applies to real and complex analytic vector bundles.

(Compare the above construction with the introduction of the topology on the tangent bundle as given in Chap. II §1).

147 **Definition.** Let $p : E \to X$, $p' : E' \to X$ be two complex vector bundles on X. A bundle map or a homomorphism $h : E \to E'$ is a continuous map $h : E \to E'$ such that for any $x \in X$, $h|E_x[= p^{-1}(x)]$ is a $\mathbb{C}$-linear map into $E'_x[= p^{-1}(x)]$. If in addition h is a homeomorphism (so that $h|E_x$ is an isomorphism onto E'_x), h is called an isomorphism. E and E' are isomorphic if there is an isomorphic of E onto E'.

Similar definitions apply to C^k, real analytic and holomorphic bundle maps and isomorphisms.

Remark. Let a vector bundle $p : E \to X$ be given by the open covering $\{U_i\}_{i\in I}$ and the transition map $g_{ij} : U_{ij} \to GL(q, \mathbb{C})$. Let $\{V_\alpha\}_{\alpha\in A}$ be a refinement of $\{U_i\}$ and $\tau : A \to I$ a map such that $V_\alpha \subset U_{\tau(\alpha)}$. Consider the vector bundle $p' : E' \to X$ where $E' = S'/\sim$, $S' = \{(x, v, \alpha) | x \in V_\alpha, v \in \mathbb{C}^q\}$ constructed with the transition maps $g'_{\alpha\beta} = \beta_{\tau(\alpha)\tau(\beta)} | V_\alpha \cap V_\beta$. The map τ defines a continuous map $h' : S' \to S$, viz. $h'(x, v, \alpha) =$

$(x, v, \tau(\alpha))$. It is easily verified that h' maps equivalent points into equivalent points, and so define a continuous map $h : E' \to E$. This map is easily seen to be an isomorphism of the vector bundles E and E'.

Proposition 1. *Let $p : E \to X$ and $p' : E' \to X$ be two vector bundles given by the open coverings $\{U_i\}_{i\in I}, \{V_\alpha\}_{\alpha\in A}$ and transition map (g_{ij}), $(g'_{\alpha\beta})$. Then a necessary and sufficient condition that the two vector bundles are isomorphic is the following: there exists a common refinement $\{W_k\}_{k\in K}$ of $\{U_i\}$ and $\{V_\alpha\}$, refinement maps $\tau_1 : K \to I$, $\tau_A : K \to A$, [i.e. $W_k \subset U_{\tau_I}(k) \cap V_{\tau_A}(k)$] and continuous maps $h : W_k \to GL(q, \mathbb{C})$* **148**
such that if g_{kl}, g'_{kl} denote the restrictions to W_{kl} of $g_{\tau_I(k),\tau_I(l)}$, $g'_{\tau A(k),\tau_A(l)}$ respectively, we have

$$h_l g_{kl} h_k^{-1} = g'_{kl} \text{ on } W_{kl}.$$

Proof. Let $p : E \to X$ and $p' : E' \to X$ be isomorphic and $h : E \to E'$ and isomorphic between then. Let $\{W_k\}$ be a common refinement of $\{U_i\}$ and $\{V_\alpha\}$. In view of the remark made above, we may suppose that E and E' are constructed using the covering $\{W_k\}$ and the transition maps g_{kl}, g'_{kl} respectively. Let $\varphi_k : E_{W_k} \to W_k \times \mathbb{C}^q$ and $\varphi'_k : E'_{W_k} \to W_k \times \mathbb{C}^q$ be the isomorphisms corresponding to E and E' respectively. Let $h'_k = \varphi'_k o h o \varphi_k^{-1} : W_k \times \mathbb{C}^q \to W_k \times \mathbb{C}^q$. Define $h_k : W_k \to GL(q, \mathbb{C})$ by the formula

$$h'_k(x, v) = (x, h_k(x), v).$$

Then since $h'_k \circ \varphi_k \circ \varphi_1^{-1} \circ h_1'^{-1} = \varphi'_k \circ \varphi_l'^{-1}$ and $(x, g_{kl}(x)v) = \varphi_k \circ \varphi_1^{-1}(x, v)$, we obtain at once the relation $h_k g_{lk} h_1^{-1} = g'_{lk}$. For the converse, suppose that $p : E \to X$ and $p' : E' \to X$ are two vector bundles and let $\{W_k\}$ be a common refinement of the covering $\{U_i\}$, $\{V_\alpha\}$ corresponding to E and E' respectively. If there exists map $h_k : W_k \to GL(q, \mathbb{C})$, satisfying $h_l g_{kl} h_k^{-1}$, let $\varphi_k : E_{W_k} \to W_k \times \mathbb{C}^q$ and $\varphi'_k : E'_{W_k} \to W_k \times \mathbb{C}^q$ be the isomorphisms corresponding to E and E' respectively. Then $h : E \to E'$ is defined as follows: let $h'_k : W_k \times \mathbb{C}^q \to W_k \times \mathbb{C}^q$ be the isomorphism defined by

$$h'_k(x, v) = (x, h_k(x)v);$$

set $h^{(k)} = \varphi_k'^{-1} \circ h'_k \circ \varphi_k$ on E_{W_k}. We have $h^{(k)} = h^{(l)}$ on $E_{W_{kl}}$ because of **149**
the formula $h_l g_{kl} h_k^{-1} = g'_{kl}$. □

Examples. (1) Let $I_q = X \times \mathbb{C}^q$ and $p : I_q \to X$ be the projection p, $(x, v) = x$. Then $p : I_q \to X$ is a complex vector bundle of rank q and is called the trivial vector bundle of rank q. A bundle of rank q is trivial if it is isomorphic to I_q. Since given a vector bundle $p : E \to X$, every point $x \in X$ has a neighbourhood U such that E_U is isomorphic to $U \times \mathbb{C}^q$, every vector bundle is locally trivial.

(2) Let $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ be two vector bundles of rank q_1 and q_2 respectively. Set $F = \bigcup_{x \in X} (E_{1x} \otimes E_{2x})$ and define the map $p : F \to X$ by $(E_{1x} \otimes E_{2x}) = x$. For any $x \in X$, there exists a neighbourhood U such that E_{1U} and E_{2U} are isomorphic to $U \times \mathbb{C}^{q_1}$ and $U \times \mathbb{C}^{q_2}$ respectively. Let $\varphi_1 : E_{1U} \to U \times \mathbb{C}^{q_1}$ and $\varphi_2 : E_{2U} \to U \times \mathbb{C}^{q_2}$ be such isomorphisms. Define $\varphi : F_U \to \times \varphi_1 : E_{1U} \to U \times \mathbb{C}^{q1+q2}$ by $\varphi(e_{1x} \otimes e_{2x}) = (x, \bar{\varphi_1}(e_{1x} \bar{\varphi_2}(e_{2x}))$, where $e_{ix} \in E_{ix}$ and $\bar{\varphi_i}(e_i)$ is the projection on $\mathbb{R}^{q_i}$ of $\varphi_i(e_i)$; here $e_i \in E_{iU}$. Clearly there exists a unique topology on F such that above maps are homeomorphisms and $p : F \to X$ is a vector bundle. The transition maps g_{ij} of F are given by $g_{ij} = g^1_{ij} \oplus g^2_{ij}$ and g^2_{ij} are transition maps of E_1 and E_2 respectively. F is called the direct (or Whitney) sum of E_1 and E_2 and we write $F = E_1 \oplus E_2$.

If $p : E \to X$ and $p' : E' \to X$ are complex vector bundles, then
150 $\bigcup_{x \in X} E_x \oplus E_{x'} \bigcup_{x \in X} Hom(E_x, E'_x)$ and $\bigcup_{x \in X} \wedge^p E_x$ can be given, in the same way, suitable topologies so as to make them vector bundles. They are denoted by $E \otimes E'$, Hom (E, E'), $\overset{p}{\wedge} E$ respectively. When E' is a trivial bundle of rank 1, $\mathrm{Hom}(E_x, E'_x) = E^*_x$ is the dual of E_x and we wrote E^* for the corresponding bundle. $E \otimes E'$ is called the tensor product of E and E', $\wedge^p E$, the p-fold exterior products of E.

We remark explicitly that if g_{ij}, g'_{ij} are transition maps of E, E' relative to a covering $\{U_i\}$, those of $E \otimes E'$ are $g_{ij} \otimes g'_{ij}$ (Kronecker or tensor product of matrices), those of E^* are $(t_{g_{ij}})^{-1}$, t_A denoting the transpose of the matrix A. In particular if E' is a line bundle, $E \otimes E'$ has transition maps $g'_{ij}.g_{ij}$. If we apply this to a line bundle E its dual $E^* = E'$, we see that *if $\underline{E}$ is a line bundle, $E \otimes E^*$ is trivial*.

This isomorphism is intrinsically defined as follows: for $x \in X$, we have a bilinear map $E_x \oplus E_x^* \to \mathbb{C}$, viz. $(e \oplus e^*) \to e^*(e)$. This defines a linear map $E_x \otimes E_x^* \to \mathbb{C}$, and so a map $h : E \otimes E^* \to 1_1$; h is an isomorphism.

Other examples of vector bundles are the following. If V is a C^k manifold of dimension n, $T(V) = \bigcup_{a \in V} T_a(V)$ is a vector bundle of class C^{k-1} and rank n; it is called the tangent bundle of V. The bundle of p-forms on V is the space $\wedge^p T^*(V) = \bigcup_{a \in V} \wedge^p T_a^*(V)$. [Note that $\wedge^p T^*(V)$ is in fact the p-fold exterior product of $T^*(V)$.]

Let E, F, E', F' be vector bundles on X, $h : E \to F$, $h' : E' \to F'$ bundle maps. For any x, we have a linear map $h_x \otimes h'_x : E_x \otimes E'_x \to F_x \otimes F'_x$; this defines a bundle map $h \otimes h' : E \otimes E' \to F \otimes F'$. In the same way, we have a transpose bundle map $h^* : F^* \to E^*$ and a map $\wedge^p h : \wedge^p E \to \wedge^p F$. [If h, h' are C^k, analytic, holomorphic, so are $h \otimes h', h^*, \wedge^p h$.]

Definitions. (1) Let V be a C^k manifold $p : E \to V$ a C^k vector bundle **151**
and U an open set in V. Then a C^k section s of E on U is a C^k map $s : U \to E$ such that $p\ o\ s$ = identity on U.

$C^k(U, E)$ denotes the set of all C^k sections of E over U. Analytic (holomorphic) sections of analytic (holomorphic) bundles are similarly defined.

(2) The support of a section s of E over U is defined to be the closure U of $\{x | x \epsilon U, s(x) \neq 0\}$ [0 stands for the zero of the vector space E_x]. The set of C^k sections on U saving compact support in U is denoted by $C_0^k(U, E)$.

Note that if $E = 1_q$, $C^k(U, 1_q)$ can be canonically identified with the space of C^k maps $U \to \mathbb{C}^q(\mathbb{R}^q)$. Let E be a vector bundle, $\{U_i\}$ a covering of $X, \varphi_i : E_{U_i} \to U_i \times \mathbb{C}^q$ isomorphisms and $g_{ij} : U_{ij} \to GL(q, \mathbb{C})$ the corresponding transition maps. If $s : X \to E$ is a section, we have elements $s_i \epsilon C^k(U_i, 1_q)$, $viz. s_i = \varphi_i \circ s$ and hence mappings $\sigma_i : U_i \to \mathbb{C}^q$; since $\varphi_j \circ \varphi_i^{-1} \circ s_i = s_j$ on U_{ij}, we have $\sigma_j = g_{ij}\sigma_i$ on U_{ij}. Conversely, mapping $\sigma_i : U_i \to \mathbb{C}^q$ with $\sigma_j = g_{ij}\sigma_i$ on U_{ij} define a section $s : X \to E$. This section is C^k, analytic, holomorphic, according as the σ_i are C^k, analytic, holomorphic.

We denote the set of C^k maps $U \to \mathbb{C}^q($ or $\mathbb{R}^q)$ by $C^{k,q}(U)$; those with compact support by $C_0^{k,q}(U)$.

2 Linear differential operators: the theorem of Peetre

In what follows, V is a C^∞ manifold and all vector bundles over V are
152 C^∞, real vector bundles. $C_0^\infty(V,E)$ denotes the set of C^∞ sections of E over V having compact support.

Definition. Given a C^∞ manifold V and vector bundles $p_1 : E \to V$ and $p_2 : F \to V$, a differential operator L from E to F (written $L : E \to F$) is an $\mathbb{R}$ linear map $L : C_0^\infty(V,E) \to C^0(V,F)$ such that supp. $(Ls) \subset$ supp(s) for every $s \epsilon C_0^\infty(V,E)$. These are also called operators (or sheaf maps). Note that L does not define a bundle map $E \to F$.

Remarks. A differential operator gives rise to an $\mathbb{R}$ linear map $L : C^\infty(V,E) \to C^0(V,F)$ as follows. For $x \epsilon V$, let U be a relatively compact neighbourhood of x. Let φ be a C^∞ function $\varphi : V \to \mathbb{R}$ such that for y in a nighbourhood of x, $\varphi(y) = 1$ and $\varphi(y) = 0$ for $y \notin U$. Then for any $s \epsilon C^\infty(V,E)$, we set

$(Ls)(x) = L(\varphi s)(x)$; since φ has compact support, $L(\varphi s)$ is well defined. $(Ls)(x)$ is independent of the φ chosen since L does not increase supports.

If E, E' are C^∞ vector bundles of rank q, q' respectively, and if, U is a coordinate neighbourhood of V such that E_U and E'_U are trivial then $C_0^\infty(U,E)$ can be identified with $C_0^{\infty,q}(U)$, the set of q tuples of C^∞ functions with compact support in U. A linear differential operator defines then an $\mathbb{R}$ linear map $L : C_0^{\infty,q}(U) \to C^{0,q'}(U)$.

Lemma 1. *Let V be a C^∞ manifold and U,a coordinate neighbourhood on V, (coordinate system $x_1, \ldots, x_n$). Let L be a defferential operator $C_0^{\infty,q}(U) \to C^{0,p}(U)$ [i.e. an operator from $1_q \to 1_p$ on U]. Then for any point $a \epsilon U$, there exists a neighbourhood U' of a, a positive integer m and a constant $C > 0$, such that*

$$\|Lf\|_0 \leq C\|f\|_m \text{ for any } f \epsilon C_0^{\infty,q}(U' - \{a\}).$$

We recall that the norms on $C^{k,q}(U)$ are defined by 153

$$\|f\|_m = \sum_{|\alpha|\leq m}\sum_{i=1}^{q}\sup|D^\alpha f_i(x)| \text{ if } f=(f_1,\ldots,f_q).$$

Proof. Let $a\epsilon U$ and suppose that the lemma does not hold. Let U_0 be a neighbourhood of a, relatively compact in U. Then there exists an open set $U_1 \subset\subset (U_0 - \{a\})$ and $f_1 \epsilon C_0^{\infty,q}(U_1)$ such that

$$\|Lf_1\|_0 > 2^2\|f_1\|_1.$$

□

Now consider the open neighbourhood $(U_0 - \bar{U}_1)$ of a; by our assumption there exists an open set $U_2, U_2 \subset\subset (U_0 - \bar{U}_1 - \{a\})$, and $f_2 \epsilon C_0^{\infty,q}(U_2)$ such that

$$\|Lf_2\|_0 > 2^{2.2}\|f_2\|_2.$$

By induction we have a sequence of open sets $\{U_k\}$ with $\bar{U}_k \subset \{U_0 - a\}$ and $\bar{U}_k \bigcap \bar{U}_1 = \phi$ if $k \neq 1$ and $f_k \epsilon C_0^{\infty,q}(U_k)$ with $\|Lf_k\|_0 > 2^{2k}\|f_k\|_k$ Let $f = \sum\limits_{k=1}^{\infty}\dfrac{2^{-k}f_k}{\|f_k\|_k}$. Since $\sum\dfrac{2^{-k}f_k}{\|f_k\|}$ is convergent in the C^∞ topology, $f\epsilon C_0^{\infty,q}(U_0)$ and $f|U_k = \dfrac{2^{-k}f_k}{\|f_k\|_k}$ so that $L(f)|U_k = 2^{-k}L(f_k)|U_k/\|f_k\|_k$. Since $\|Lf_k\|_0 > 2^{2k}\|f_k\|_k$, we have a sequence (x_k), $x_k\epsilon U_k$ such that

$$\left|Lf_k(x_k)\right| \rangle 2^{2k}\|f_k\|_k.$$

Hence $|Lf(x_k)| \geq 2^k$. But Lf is continuous in U, while $(Lf)(x_k)$ is 154
unbounded and $\{x_k\}$ lie in the relatively compact subset U_0 of U. This is contradiction, so that the lemma is established.

Theorem (Peetre). *Let V be a C^∞ manifold and E, F, C^∞ vector bundles of rank q and p respectively. Let Lbe a differential operator $C_0^\infty(V, E) \to C^0(V, F)$ and let U be a coordinate neighbourhood such that E_U and F_U are trivial. We identify $C^k(U, E)$ with $C^{k,q}(U)$. Then for any* *relatively compact open subset Ω of U there exists a positive integer m and continuous functions a_α on $\Omega, |\alpha| \leq m$ [with values in the space of linear maps from $\mathbb{R}^q$ to $\mathbb{R}^p$, i.e $p \times q$ matrices] such that for $f \epsilon C^{\infty,q}(U)$ and $x \epsilon \Omega$ we have*

$$(2.1) \qquad (Lf)(x) = \sum_{|\alpha \leq m|} a_\alpha(x)(D^\alpha f)(x).$$

Proof. Let Ω' be an open subset of Ω. We shall first prove equation (2.1) on Ω' with the following additional assumption: there exists a constant $C > 0$ and an integer m such that if $f \epsilon C_0^{\infty,q}(\Omega')$, we have

$$(2.2) \qquad ||Lf||_0 \leq C||f||_m.$$

□

First we remark that if $\varphi \epsilon C_0^{\infty,q}(\Omega')$ and if φ is m-flat $a \epsilon \Omega'$ then $(L\varphi)(a) = 0$. In fact by §5 Chapter I, there exists a sequence $\{f_\nu\}$ of functions in $C_0^{\infty,q}(\Omega')$ such that $f_\nu(x) = 0$ for x in a neighbourhood of a and $||\varphi - f_\nu||_n^{\Omega'} \to 0$. Since $\text{supp}(Lf_\nu) \subset \text{supp}\, f_\nu$ we have $Lf_\nu(a) = 0$ and because of the inequality (2.2), $(L\varphi)(a) = \lim_{\nu\to\infty}(Lf_\nu)(a) = 0$. For $a \epsilon \Omega'$
155 and $f \epsilon C^{\infty,q}(\Omega')$, Taylor's formula gives us the following:

$$f(x) = \sum_{|\alpha| \leq m} \frac{(x-a)^\alpha}{\alpha!} . D^\alpha f(a) + g(x),$$

where g is m-flat at a. Hence by the remark above,

$$(Lg)(a) = 0$$

$$\text{i.e} \qquad (Lf)(a) = \sum_{|\alpha| \leq m} \frac{L[(x-a)^\alpha D^\alpha f(a)](a)}{\alpha!}.$$

In what follows we write elements of $C^{\infty,q}$ as columns. Let $f = \begin{bmatrix} f_1 \\ \vdots \\ f_q \end{bmatrix}$, f_i being C^∞ functions and let $e_k = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}$, where 1 occurs in the k^{th} place.

Then $\dfrac{(x-a)^\alpha D^\alpha f(a)}{\alpha!} = \sum_{1\le k\le q} \dfrac{D^\alpha f_k(a).(x-a)^\alpha e_k}{\alpha!}$. Hence

$$(Lf)(a) = \sum_{|\alpha|\le m} \sum_{1\le k\le q} \frac{D^\alpha f_k(a)}{\alpha!} L[(x-a)^\alpha e_k](a).$$

[Recall our remark that L can be applied to C^∞ functions which are not compactly supported.] Now

$$(x-a)^\alpha e_k = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} x^\beta (-a)^{\alpha-\beta} e_k,$$

and, by definition, $L(x^\beta e_k)$ is continuous on U (and not just on Ω'). Hence $(L(x-a)^\alpha e_k)(a)$ is a continuous function of a in U and can be identified with a p-tuple

$$[L(x-a)^\alpha e_k](a) = \begin{bmatrix} a_\alpha^{1k}(a) \\ \vdots \\ a_\alpha^{pk}(a) \end{bmatrix}.$$

Thus, if Ω' is an open subset of U and if there exists m, C such that **156**

$$\|Lf\|_0 \le C\|f\|_m \text{ for } f \in C_0^{\infty,q}(\Omega'),$$

there exist continuous functions a_α <u>on Ω</u> such that

$$(Lf)(x) = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha \text{ for } f \epsilon C^{\infty,q}(\Omega'), x\epsilon\Omega'.$$

Moreover if $Lf = \sum a_\alpha D^\alpha f$ for all $f\epsilon C^{\infty,q}(W)$, where W is an open subset of Ω, the a_α are uniquely determined on W by L. Consequently, if suffices to prove that every a $\epsilon\Omega$ has a neighbourhood W such that (2.1) holds for all $f\epsilon C^{\infty,q}(W)$. Now, by the remark above and Lemma 1 there is a W such that

$$(Lf)(x) = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha f(x), x\epsilon W - \{a\}, f\epsilon C^{\infty,q}(W - \{a\})$$

where the a_α are continuous in W. Since, for $f \epsilon C^{\infty,q}(W)$, both sides of this equation are continuous in W, the result is proved.

Note. A result even somewhat more general than the one proved here is due to J. Peetre [35].

3 The Cauchy Kovalevski Theorem

Lemma 1. *Let $D = \left\{w \epsilon \mathbb{C} \Big| |w| < R\right\}$ and h, be a holomorphic function on D. If, for $w \epsilon D$, $|h'(w)| \leq A|w|^r$ and $h(0) = 0$, then*

$$|h(w)| \leq A \frac{|w|^{r+1}}{r+1}.$$

157 **Lemma 2.** *Let $D = \left\{w \epsilon \mathbb{C} \Big| |w| < R\right\}$ and h be a holomorphic function on D. If*

$$h(0) = 0, \left|h'(w)\right| < \frac{A}{(R - |w|)^{r+1}} \text{ for } w \epsilon D,$$

$$\text{then} \qquad |h(w)| < \frac{A}{r(R - |w|)^r}.$$

The proof of the above lemmas follows at once from the equation $\int_0^w h'(z)dz = h(w)$.

Lemma 3. *If $D = \left\{w \epsilon \mathbb{C} \Big| |w| < R\right\}$ and h s is a holomorphic function on D, and if*

$$|h(w)| < \frac{A}{(R - |w|)^r}, \text{ for } w \epsilon D,$$

$$\text{then} \qquad \left|h'(w)\right| < \frac{3A(r+1)}{(R - |w|)^{r+1}}.$$

Proof. Let $w_0 \epsilon D$ and $0 < \epsilon < R - |w_0|$. Then by Cauchy's inequality we have

$$\left|h'(w_0)\right| \leq \frac{1}{\varepsilon} \sup_{|w - w_0| = \varepsilon} |h(w)|.$$

Hence $|h'(w_0)| \leq \dfrac{A}{\varepsilon\{R - |w_0| - \varepsilon\}^r}$ for any ε with $o < \varepsilon < R - |w_0|$.

Take $\varepsilon = \dfrac{R - |w_0|}{r+1}$.

Then $|h'(w_0)| \leq \dfrac{A(r+1)}{(R-|w_0|)^{r+1}} . \left(\dfrac{r+1}{r}\right)^r$.

Hence $|h'(w_0)| < \dfrac{3A(r+1)}{(R-|w_0|)^{r+1}}$. □ **158**

Theorem (Cauchy -Kovalevski). *Let* $\Omega = \left\{(z_1, \ldots, z_n) \epsilon \mathbb{C}^n \middle| |z_i| < r_i\right\}$. *Let* g, φ*:* $\Omega \to \mathbb{C}^q$ *be holomorphic functions on* Ω *and let* $\beta = (\beta_1, \ldots, \beta_n)$*,* $\beta_i \epsilon \mathbb{Z}^+$ *with* $\beta_n > 0$*. Let* α *run over the multiindices with* $|\alpha| \leq |\beta|, \alpha_n < \beta_n$*, and suppose that for each* α *is given a holomorphic map* a_α *of* Ω *into the space of* $q \times q$ *matrices. Then exists a neighbourhood* U *of* 0 *and a unique holomorphic functions* f *on* U*,* $f : U \to \mathbb{C}^q$ *such that*

(3.1) $$D^\beta f(z) = \sum_{\substack{|\alpha| \leq |\beta| \\ \alpha_n < \beta_n}} a_\alpha(z) . D^\alpha f(z) + g(z),$$

and

(3.2) $$\left(\frac{\partial}{\partial z_i}\right)^l (f - \varphi) = 0 \text{ for } z_i = 0 \text{ and } 0 \leq l < \beta_i.$$

Proof. We may suppose without loss of generality that $r_i \leq 1$ and that $\varphi = 0$, since if $h = f - \varphi$, the problem then would be to solve the equation

$$D^\beta h = \sum_{\substack{|\alpha| \leq |\beta| \\ \alpha_n < \beta_n}} a_\alpha D^\alpha h + g',$$

with $(\dfrac{\partial}{\partial z_i})^l (h) = 0$, for $z_i = 0$ and $0 \leq l < \beta_i$, where g' is holomorphic on Ω. We may further suppose that the a_α are bounded in Ω. □ **159**

We first remark that for a holomorphic function h on Ω, there exists a unique holomorphic function u on Ω such that

$$D^\beta u = h$$

$$\text{and} \quad \left(\frac{\partial}{\partial z_i}\right)^l u = 0 \text{ for } z_i = 0 \text{ and } 0 \leq l < \beta_i.$$

To prove this it is enough to show that there exists a unique holomorphic function u on Ω such that $\dfrac{\partial u}{\partial z_i} = h$ and $u = 0$ on $z_1 = 0$. But this is immediate; we must set $u(z) = \int_0^z h(\zeta)d\zeta$. We define holomorphic functions $f_k : \Omega \to \mathbb{C}^q$ as follows: $f_0(z) = 0$ for $z\epsilon\Omega$, and $f_k(z)$, for $k \geq 1$, is defined, by induction, as the unique holomorphic solution of

$$D^\beta f_k = \sum_{\substack{|\alpha|\leq|\beta|\\ \alpha_n<\beta_n}} a_\alpha D^\alpha f_{k-1} + g$$

with $$\left(\frac{\partial}{\partial z_i}\right)^l f_k = 0 \text{ for } z_i = 0 \text{ and } 0 \leq l < \beta_i$$

It is clear from the remark made above that $\{f_k\}_{k\geq 0}$ are defined on Ω. Let $u_0(z) = 0$, $u_k(z) = f_k(z) - f_{k-1}(z)$. Then u_k satisfies

$$D^\beta u_{k+1} = \sum_{\substack{|\alpha|\leq|\beta|\\ \alpha_n<\beta_n}} a_\alpha D^\alpha u_k \tag{3.3}$$

with $\left(\frac{\partial}{\partial z_i}\right)^l u_k = 0$ for $z_i = 0, 0 \leq l < \beta_i$.

Let $\rho(z) = (r_1 - |z_1|)\cdots(r_{n-1} - |z_{n-1}|)$ and let $|\beta| = m$. We shall now
160 prove that there exists a constant A such that the relations (3.3) imply the estimates

$$\left|D^\beta u_r(z)\right| \leq \frac{A^r|z_n|^r}{\{\rho(z)\}^{mr+1}}. \text{ for } z\epsilon\Omega. \tag{3.4}$$

Assume that (3.4) holds for $r = k$. Then

$$\left|D^\beta u_k(z)\right| \leq \frac{A^k|z_n|^k}{\{\rho(z)\}^{mk+1}}.$$

Applying Lemma 1 with respect to z_n, $(\beta_n - \alpha_n)$ times and Lemma 2 with respect to z_i, β_i times for $1 \leq i \leq n-1$, we have, since $r_i \leq 1$,

$$\left|\left(\frac{\partial}{\partial z_n}\right)^{\alpha_n} u_k(z)\right| \leq$$

$$\frac{A^k|z_n|^{k+\beta_n-\alpha_n}}{(k+1),\ldots,(k+\beta_n-\alpha_n)\prod_q^{n-1}(r_i-|z_i|)^{mk+1}(mk),\ldots,(mk+1\beta_i)}$$

hence, since $\alpha_n < \beta_n$, $\left|\left(\dfrac{\partial}{\partial z_n}\right)^{\alpha_n} u_k(z)\right| \leq \dfrac{A^k|z_n|^{k+1}}{[\rho(z)]^{mk+1}}.\ k^{-(m-\alpha_n)}$

Now using Lemma 3 with respect to z_i, α_i times, for $1 \leq i \leq n-1$, we obtain

$$\left|D^\alpha u_k(z)\right| \leq \frac{3^m A^k|z_n|^{k+1}}{[\rho(z)]^{m(k+1)+1}}.\left[\frac{m(k+1)+1}{k}\right]^{m-\alpha_n}.$$

Hence by equation (3.3), since the a_α are bounded,

$$\left|D^\beta u_{k+1}(z)\right| \leq \frac{A^k|z_n|^{k+1}}{[\rho(z)]^{m(k+1)+1}}.3^m.M\left[\frac{m(k+1)+1}{k}\right]^{m-\alpha_n}$$

for some constant M (independent of α and k).

Hence if $A = \sup_k.\ 3^m.\ M\left[\dfrac{m(k+1)+1}{k}\right]^m$, the inequality (3.4) is **161**
proved. Consequently, if z satisfies $|z_n| < [b\rho(z)]^m$, $\sum_k \left|D^\beta u_k(z)\right|$ is convergent. Hence there exists a neighbourhood U of 0 such that $\sum_k |D^\beta u_k|$ is uniformly on U. This clearly implies that f_k is uniformly convergent on U and if $f(z) = \lim_{k\to\infty} f_k(z)$, $f(z)$ is a holomorphic function which satisfies equation (3.1) with the initial conditions (3.2). Again if f and f' are two holomorphic solutions of (3.1) satisfying (3.2) let $f(z) - f'(z) = u(z)$. Then if $u_k(z) = u(z)$, $k \geq 1$, we have

$$D^\beta u_{k+1}(z) = \sum a_\alpha D^\alpha u_k$$

and

$$\left(\frac{\partial}{\partial z_i}\right)^l u_k = 0 \text{ for } z_i = 0 \text{ and } 0 \leq l \leq \beta_i,$$

i.e (u_k) satisfies equations (3.3) and by the discussion above, there exists a neighbourhood U' of 0 such that $\sum |u_k|$ is uniformly convergent on U'. But this implies that $u(z) = u_k(z) = 0$, which proves the uniqueness of the solution.

4 Fourier transforms, Plancherel's theorem

Definitions. (1) *If $f\epsilon L'(\mathbb{R}^n)$, the fourier transform of f, denoted by $\hat{f}$, is defined by*

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} f(x) e^{-ix\xi} dx.$$

162 (2) Let $\mathscr{S}$ be the set of C^∞ functions f on $\mathbb{R}^n$ such that for any polynomial P on $\mathbb{R}^n$ and any α, we have, $\sup\limits_{x\epsilon\mathbb{R}^n} |P(x)D^\alpha f(x)| < \infty$. The space $\mathscr{S}$ is called Schwartz space.

Remarks. (1) For every $p \geq 1$, $\mathscr{S} \subset L^p$ and $\mathscr{S}$ is dense in L^p (in L^p norm) if $p < \infty$.

(2) If $f\epsilon\mathscr{S}$, $D^\alpha f\epsilon\mathscr{S}$ for every α.

(3) Any function in $\mathscr{S}$ is bounded.

(4) If $f\epsilon\mathscr{S}$, it is verified by integration by parts that $(i)(D^\alpha f)(\hat{\xi}) = i^{|\alpha|}\xi^\alpha \hat{f}(\xi)$ and $(ii) D^\alpha \hat{f}(\xi) = \{(-i\xi)^\alpha f(\xi)\}$.

(5) For any $f\epsilon L'$.

$$\sup_{\xi\epsilon\mathbb{R}^n} \left|\hat{f}(\xi)\right| \leq \sup_{\xi\epsilon\mathbb{R}^n} \int \left|e^{-ix\xi} f(x)\right| dx.$$

(6) It follows from remarks (4) and (5) that if $f\epsilon\mathscr{S}$, so is $\hat{f}$.

Proposition 1 (Inversion formula)**.** *If $f\epsilon\mathscr{S}$, we have,*

$$f(y) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} \hat{f}(\xi) e^{iy\xi} d\xi.$$

Proof. Let $\varphi\epsilon\mathscr{S}$. Consider $\int\limits_{\mathbb{R}^n} \varphi(\xi)\hat{f}(\xi)e^{iy\xi}d\xi$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \varphi(\xi) e^{iy\xi} (\int_{\mathbb{R}^n} f(x) e^{-ix\xi} dx) d\xi.$$

□

By Fubini's theorem, we have

$$\int \varphi(\xi)\hat{f}(\xi)e^{iy\xi}d\xi = \frac{1}{(2\pi)^{\frac{n}{2}}}\int f(x)dx \int \varphi(\xi)e^{-i(x-y)\xi}d\xi$$
$$= \frac{1}{(2\pi)^{\frac{n}{2}}}\int f(y+t)e^{it\xi}\varphi(\xi)d\xi dt \; [x-y=t]$$

(4.1) $$= \int f(y+t))\hat{\varphi}(t)dt.$$

Now, set $\varphi(\xi) = \psi(\epsilon\xi)$, where $\psi\epsilon\mathscr{S}$. Then, as is easily verified, we have 163

$$\hat{\varphi}(t) = \varepsilon^{-n}\hat{\psi}(\frac{t}{\varepsilon}).$$

Hence

$$\int \varphi(\xi)\hat{f}(\xi)e^{iy\xi}d\xi = \int \psi(\varepsilon\xi)\hat{f}(\xi)e^{iy\xi}d\xi$$
$$= \int f(y+t)\varepsilon^{-n}\hat{\psi}\left(\frac{t}{\varepsilon}\right)dt$$

i.e $$\int_{\mathbb{R}^n} \psi(\varepsilon\xi)\hat{f}(\xi)e^{iy\xi}d\xi = \int_{\mathbb{R}^n} f(y+\varepsilon t)\hat{\psi}(t)dt.$$

Since f and $\psi\epsilon\mathscr{S}$, we can take the limits as $\varepsilon \to 0$ under the integrals, so that

$$\psi(0)\int_{\mathbb{R}^n} \hat{f}(\xi)e^{iy\xi}d\xi = f(y)\int_{\mathbb{R}^n} \hat{\psi}(t)dt.$$

If we set $\psi(t) = e^{-\frac{t^2}{2}}$, it is easily verified that $\psi(0) = (2\pi)^{\frac{n}{2}} \int_{\mathbb{R}^n} \hat{\psi}(t)dt$, so that

$$(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} \hat{f}(\xi)e^{iy\xi}d\xi = f(y).$$

Corollary. *For f and $\varphi\epsilon\mathscr{S}$, we have,* 164

$$\int_{\mathbb{R}^n} \varphi(\xi)\hat{f}(\xi)d\xi = \int_{\mathbb{R}^n} f(\xi)\hat{\varphi}(\xi).dt$$

This follows from equation (4.1) on putting $y = 0$.

Remark. As is evident from the proof, (4.1) holds whenever $f \epsilon L^1(\mathbb{R}^n)$ and $\varphi \epsilon \mathscr{S}$.

Lemma. *If* $f \epsilon \mathscr{S}$, *we have* $\|f\|_{L^2} = \|\hat{f}\|_{L^2}$

$$[\text{ For } g \epsilon L^p, \|g\|_{L^p} = \text{ norm of } g \text{ in } L^p$$
$$= \left(\int |g(\xi)|^p d\xi\right)^{\frac{1}{p}} \text{ when } 1 \leq p < \infty,$$
$$\|g\|_{L^\infty} = ess.\sup |g(x)|.]$$

Proof. It follows from the inversion formula that for $f \epsilon \mathscr{S}\Big|$, $\hat{\hat{f}}(-y) = f(y)$. Define ψ: $\mathbb{R}^n \to \mathbb{C}$ by

$$\psi(t) = \overline{\left\{\hat{f}(t)\right\}};\ \text{we have}$$
$$\hat{f}(t) = \int f(\xi)e^{-it\xi}d\xi$$
$$= \overline{\left(\int \bar{f}(\xi)e^{it\xi}d\xi\right)}.$$

Hence

$$\hat{\bar{\hat{f}}}(t) = \bar{\hat{\hat{f}}}(-t) \tag{4.2}$$

i.e

$$\hat{\psi}(t) = \bar{f}(t).$$

165 By the corollary to the inversion formula,

$$\int f(t)\hat{\psi}(t)dt = \int \psi(t)\hat{f}(t)dt.$$

Now $\psi(t) = \bar{\hat{f}}$ and $\hat{\psi}(t) = \bar{f}$, from which it follows that

$$\|f\|_{L^2} = \|\hat{f}\|_{L^2}.$$

□

Definition. If $f \epsilon L^p$ for some $p \geq 1$, $\hat{f}$ is defined as the linear functional on $\mathscr{S}$, for which $\hat{f}(\psi) = \int f(t)\hat{\psi}(t)dt, \psi \epsilon \mathscr{S}$. If there exists a function $g \epsilon L^p$ (for some p with $1 \leq p \leq \infty$), such $\int g(t)\psi(t)dt = \int f(t)\hat{\psi}(t)dt$ for all $\psi \epsilon \mathscr{S}$, we shall identify f with the function g. Note that if $f \epsilon L^1$, this compatible with the definition given at the beginning (as follows the remark after the inversion formula).

Theorem (Plancherel). *If $f \epsilon L^2$, then there exists $g \epsilon L^2$ such that the linear map $\hat{f}$ is given by*

$$\hat{f}(\psi) = \int \psi(t) g(t) dt$$

and

$$\|f\|_{L^2} = \|g\|_{L^2}.$$

In other words, $\hat{f} \epsilon L^2$ *and* $\|f\|_{L^2} = \|\hat{f}\|_{L^2}$.

Proof. Since $\mathscr{S}$ is dense in L^2, there is a sequence $\{f_\nu\}$ of function in $\mathscr{S}$ such that $\|f_\nu - f\|_{L^2} \to 0$. It follows from the lemma above that

$$\|f_\nu - f_\mu\|_{L^2} = \|\hat{f}_\nu - \hat{f}_\mu\|_{L^2}$$

so that $\|\hat{f}_\nu - \hat{f}_\mu\|_{L^2} \to$ as $\nu, \mu \to \infty$. Hence there exists $g \epsilon L^2$ such that **166**

$$\|\hat{f}_\nu - g\|_{L^2} \to 0.$$

□

Clearly $\|f\|_{L^2} = \|g\|_{L^2}$.

Now for any $\psi \epsilon \mathscr{S}$, we have

$$\int f_k(t)\hat{\psi}(t)dt = \int \hat{f}_k(t)\psi(t)dt.$$

Since $\|f_k - f\|_{L^2} \to 0$ and $\|\hat{f}_k - g\|_{L^2} \to 0$, we have, taking limits as $k \to \infty$,

$$\int f(t)\hat{\psi}(t)dt = \int g(t)\psi(t)dt.$$

Remark. The inversion formula can be written

$$\hat{\hat{f}}(-y) = f(y), \text{ for } f \epsilon \mathscr{S}.$$

It is an immediate consequence of Plancherel's theorem that this relationship holds if $f \epsilon L^2$. Further, if $f \epsilon L^1$, then as in the proof of the inversion formula, we have, for $\psi \epsilon \mathscr{S}$,

$$\int \psi(\varepsilon\xi)\hat{f}(\xi)e^{iy\xi}d\xi = \int f(y+\varepsilon t)\hat{\psi}(t)dt,$$

so that if we suppose that we have also $\hat{f} \epsilon L^1$ we may take limits as $\varepsilon \to 0$, [the term on the right converges to $f \int \hat{\psi}(t)dt$ in L^1 norm].

167 From this we conclude that $\hat{f}(-y) = (-y)$. [This implies in particular that f is then bounded and continuous.]

Proposition 2. *If $f, g \epsilon L^1$, then $\int\limits_{\mathbb{R}^n} |f(x-y)g(y)|dy < \infty$ for almost all x and if*

$$(f * g)(x) = \int f(x-y)g(y)dy,$$
$$||f * g||_{L^1} \leq ||f||_{L^1}.||g||_{L^1}.$$

Proof. It is enough to prove the proposition for $f \geq 0$ and $g \geq 0$. We have by Fubini's theorem

$$\int dx \int f(x-y)g(y)dy = \int g(y)dy \int f(x-y)dx.$$
$$= \left(\int f(x)dx\right)\left(\int g(y)dy\right) < \infty$$

and the proposition follows. ◻

Proposition 3. *If $f, g \epsilon \mathscr{S}$, $f * g \epsilon \mathscr{S}$ and*

$$\widehat{(f * g)} = (2\pi)^{\frac{n}{2}}\hat{f}\hat{g}.$$

Proof. It is clear that $f * g\epsilon\mathscr{S}$. Now,

$$\begin{aligned}(f * g\hat{)}(x) &= (2\pi)^{-\frac{n}{2}} \int e^{-ixt} dt \int f(t-y)g(y)dy \\ &= (2\pi)^{-\frac{n}{2}} \int g(y)dy \int f(t-y)e^{-ixt} dt \\ &= (2\pi)^{-\frac{n}{2}} \int g(y)e^{-ixy} dy \int f(t)e^{-ixt} dt \\ &= (2\pi)^{\frac{n}{2}} \hat{f}(x).\hat{g}(x).\end{aligned}$$

□

Corollary. *For $f, g\varepsilon\mathscr{S}$, we have,* 168

$$(fg\hat{)} = (2\pi)^{-\frac{n}{2}} \hat{f} * \hat{g}.$$

This follows from the above proposition and the inversion formula.

Remark. In fact, the above result is true for $f\epsilon L^i$, $i = 1, 2$ and $g\epsilon\mathscr{S}$.

Proof. Let $\{f_\nu\}$ be a sequence in $\mathscr{S}$ such that

$$\|f_\nu - f\|_{L^i} \to 0$$

Then $$(f_\nu.g)^\wedge = (2\pi)^{-\dfrac{n}{2}} \hat{f}_\nu * \hat{g}.$$

If $$f \in L^2, = \pi^2 \hat{f}_\nu * \hat{g}(t) - \hat{f} * \hat{g}(t) = \int (\hat{f}_\nu - \hat{f})(t-y)\hat{g}(y)dy,$$

and using Schwarz's inequality,

$$\lim_{\nu\to\infty} \hat{f}_\nu * \hat{g}(t) = \hat{f} * \hat{g}(t).$$

□

If $f \epsilon L^1$ and $\|f_\nu - f\|_{L^1} \to 0$. then $\hat{f}_\nu \to \hat{f}$ uniformly, so that $\hat{f}_\nu * \hat{g}(t) \to \hat{f} * \hat{g}(t)$ uniformly and hence $\lim\limits_{\nu\to\infty} \hat{f}_\nu * \hat{g}(t) = \hat{f} * \hat{g}(t)$. Further, since g is bounded, $f_\nu g \to fg$ in L^i, so that $(f_\nu g)\hat{}(t) \to (fg)\hat{}(t)$ [pointwise for $i = 1$, in L^2 for $i = 2$].

Hence, if $f \epsilon L^1$ or $f \epsilon L^2$, we have,

$$\begin{aligned}(fg)\hat{}(t) = \lim_{\nu\to\infty}(f_\nu g)\hat{}(t) &= \lim_{\nu\to\infty}(2\pi)^{-n/2}\hat{f}_\nu * \hat{g}(t)\\ &= (2\pi)^{-n/2}\hat{f} * \hat{g}(t).\end{aligned}$$

5 The Sobolev spaces $H_{m,p}$

169 In this section we have given proofs of the most important results in L^p; however since we shall need only the L^2 statements, we have included simple proofs in this special case (based on Plancherel's theorem).

Let Ω be an open set in $\mathbb{R}^n$, p, a real number, $p \geq 1, q, m$ integers, $q > 0, m \geq 0$. Let $f = (f_1, \ldots, f_q) : \Omega \to \mathbb{C}^q$ be a C^∞ map. Consider the space$\{f : \Omega \to \mathbb{C}^q,\ f \in C^\infty(\Omega) | \sum\limits_{\substack{|\alpha|\leq m\\ 1\leq i\leq q}} \int |D^\alpha f_i(x)|^p dx < \infty\}$.

Define a norm $|f|_{m,p}$ on this space by

$$|f|^p_{m,p} = \sum_{|\alpha|\leq m}\sum_{1\leq i\leq q}\int |D^\alpha f_i|^p dx.$$

We shall write $|f|^\Omega_{m,p}$ for this norm when its dependence on Ω is relevant. The completion of the above space is called the Sobolev space $H_{m,p}(\Omega)$. If a sequence $\{f_\nu\}$ of C^∞ functions converges in $H_{m,p}(\Omega)$, the sequence $D^\alpha f_\nu$ is convergent in L^p, to f^α, say. The limit of f_ν in $H_{m,p}(\Omega)$ is denoted by f and f^α is called the derivative of order α of f, and we write $D^\alpha f = f^\alpha$. [We shall see below that f^α is independent of the sequence $\{f_\nu\}$]. We shall denote $H_{m,2}(\Omega)$ by $H_m(\Omega)$. For a mapping $f = (f_1, \ldots, f_q)$: $\Omega \to \mathbb{C}^q$, we write $f \in L^p$ if $f_i \epsilon L^p$ for each i: for $f \epsilon L^p$, we define $\|f\|_{L^p}$ by.

$$\|f\|^p_{L^p} = \sum_{i=1}^q \|f_i\|^p_{L^p}.$$

170 Let $C_o^{\infty,q}$ be the subspace of $H_{m,p}(\Omega)$, of c^∞ functions $g : Omega \to \mathbb{C}^q$, with compact support. Then the closure of $C_o^{\infty,q}$ in $H_{m,p}(\Omega)$ is denoted by $H^0_{m,p}(\Omega)$.

For vectors $v_1, v_2 \in \mathbb{C}^q$ (or $\mathbb{R}^q$) we shall denote by (v_1, v_2), the usual scalar product, i.e. if $v_i = (v_i^1, \ldots, v_i^q)$, then $(v_1, v_2) = \sum_{k=1}^q v_1^k \overline{v_2^k}$; similarly, for mappings $f = (f_1, \ldots, f_q), g = (g_1, \ldots, g_q) : \Omega \to \mathbb{C}^q$, we write

$$(f, g) = \sum_{i=1}^q \int_\Omega f_i(x)\overline{g_i(x)}dx.$$

Definitions. (1) If $f \in L^p$ and if $f \in H_{m,p}(\Omega')$ for every relatively compact subset Ω' of Ω, then f is said to be strongly differentiable, upto order m, in L^p. If $p = 2$, we speak simply of strong differentiability.

(2) If $f \in L^p$ and if there exist functions h^α in $L^p, |\alpha| \leq m$, such that for any $g \epsilon C_\circ^{\infty,q}$,

$$\int_\Omega (f(x), D^\alpha g(x))dx = (-1)^{|\alpha|} \int_\Omega (h^\alpha(x), g(x))dx,$$

then f is said to have weak derivatives upto order m in L^p and the h^α are called the weak derivatives of f.

Remark. (1) If $\int_\Omega (h^\alpha(x), g(x))dx = \int_\Omega (h'^\alpha(x), g(x))dx$ for all functions $g \epsilon C_0^\infty$, clearly $h^\alpha(x) = h'^\alpha(x)^\Omega$ almost everywhere and hence the weak derivatives of f, if they exist, are uniquely determined.

(2) If a function in L^p has strong derivatives upto order m they are weak derivative of f. This follows at once from Holder's inequality. In particular, if $f_\nu \epsilon C^{\infty,q}$ and $f_\nu \to f$ in $H_{m,p}$, the limits $\lim\limits_{\nu\to\infty} D^\alpha f_\nu$ in L^p are independent of the sequence $\{f_\nu\}$, being weak derivatives of f. 171

(3) Let $0 \leq m' \leq m$ and $f \epsilon H_{m,p}$. Then there exists a sequence $\{f_\nu\}$ of C^∞ functions such that $f_\nu \to f$ in $H_{m,p}$. But this implies that $f_\nu \to f$ in $H_{m',p}$ and if $D^\alpha f_\nu \to f^\alpha$ in L^p, $f^o = f$ almost everywhere. Hence there exists a map i: $H_{m',p}(\Omega) \to H_{m,p}(\Omega)$ with $i(f)$ = Limit in $H_{m',p}(\Omega)$ of $\{f_\nu\}$. Further if $i(f) = 0$ in $H_{m',p}(\Omega)$, then $f^o = 0$ in L^p.

Now , for $g\epsilon C_o^{\infty,q,}$ we have

$$\int_\Omega (D^\alpha f_\nu(x), g(x))dx = (-1)^{|\alpha|} \int_\Omega (f_\nu(x), D^\alpha g(x))dx \, (|\alpha| \leq m)$$

and by Holder's inequality,

$$\int_\Omega (D^\alpha f(x), g(x))dx = (-1)^{|\alpha|} \int (f^o(x), D^\alpha g(x))dx = o \text{ for any } g \in C_o^{\infty,q}.$$

Hence $D^\alpha f = 0$, for $|\alpha| \leq m$ i.e the map $i : H_{m,p}(\Omega) \to H_{m',p}(\Omega)$ is an injection. Of course i maps $\overset{\circ}{H}_{m',p}$ into $\overset{\circ}{H}_{m',p}$.

(4) If $f\epsilon H_{m,p}(\Omega)$ and $\varphi\epsilon C_o^{\infty,1}$, then $\varphi f \epsilon H_{m,p}\Omega$ and $D^\alpha(\varphi f) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} D^\beta \varphi D^{\alpha-\beta} f$.

Proof. If $\{f_\nu\}$ is a sequence of C^∞ functions converging to f in $H_{m,p}, \varphi f_\nu$
172 $\to \varphi f$ in $H_{m,p}$, i.e.

$$D^\alpha(\varphi f_\nu) \to D^\alpha(\varphi f) \text{ in } L^p.$$

Hence $$D^\alpha(\varphi f) = \lim_{\nu\to\infty} D^\alpha(\varphi f_\nu) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} D^\beta \varphi D^{\alpha-\beta} f.$$ □

(5) If $f \in \overset{\circ}{H}_m(\Omega)$, there exists a sequence $\{f_\nu\}$ of C^∞ functions with compact support $\subset \Omega$, such that $f_\nu \to f$ in $H_m(\Omega)$. If we extend f_ν to functions on $\mathbb{R}^n$ by setting $f_\nu(x) = 0$ for $x \notin \Omega$, then $\{f_\nu\}$ is convergent in $H_m(\mathbb{R}^n)$, to f' say. We define $i' : \overset{\circ}{H}_m(\Omega) \to \overset{\circ}{H}_m(\mathbb{R}^n)$, by $i'(f) = f'$. Then i' is injective and preserves norms.

(6) If Ω is bounded, we have, for any $f \in C_o^{\infty,q}(\Omega)$, $f(x) = \int_{-M}^{x_1} \frac{\partial f}{\partial x_1}(t, x_2, \ldots, x_n)$ dt, for large M, so that

$$\|f\|_{L^p} \leq C(\Omega) \|\frac{\partial f}{\partial x_1}\|_{L^p}.$$

It follows that for any $f \in \overset{\circ}{H}_{m,p}(\Omega)$, we have,

$$|f|_{m,p} \leq C_m(\Omega) \sum_{|\alpha|=m} |D^\alpha f|_{o,p}.$$

(This is sometimes called Poincare's inequality.)

Lemma 1. *Let* $\varphi \geq 0$ *be a* C^∞ *function with* $\operatorname{supp}\varphi \subset \{x \big| |x| < 1\}$ *and* $\int \varphi dx = 1$. *Let* $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(\dfrac{x}{\varepsilon})$. *Then if* Ω *is an open set in* $\mathbb{R}^n$ *and, for* $x \in \mathbb{R}^n$, *we set* $\varphi_\varepsilon * f(x) = \int_\Omega \varphi_\varepsilon(x-y) f(y) dy$, *then*

(i) for any $f \in L^p(\Omega)$, $\varphi_\varepsilon * f \to f$ *in* $L^p(\Omega)$ *and (ii) for any* $f \in \overset{\circ}{H}_{m,p}(\Omega)$

$$D^\alpha(\varphi_\varepsilon * f) = \varphi_\varepsilon * D^\alpha f.$$

Proof. If we extend f to $\mathbb{R}^n$ by setting $f(x) = 0$ for $x \notin \Omega$, we have **173**

$$\begin{aligned}(\varphi_\varepsilon * f - f)(x) &= \int \varphi_\varepsilon(x-y)[f(y) - f(x)]dy \\ &= \int_{|y|\leq\varepsilon} \varphi_\varepsilon(y)[f(y+x) - f(x)]dy,\end{aligned}$$

so that, by Holder's inequality, if $p'^{-1} = 1 - p^{-1}$,

$$\begin{aligned}\|\varphi_\varepsilon * f - f\|_{L^p} &\leq \{\int_{|y|\leq\varepsilon} [\varphi_\varepsilon(y)]^{p'} dy\}^{\frac{1}{p'}} \{\int_{|y|\leq\varepsilon} dy \int |f(x+y) - f(x)|^p dx\}^{\frac{1}{p}} \\ &\leq C.\|\varphi\|_{L^{p'}}.\sup_{|y|\leq\varepsilon}\{\int |f(x+y) - f(x)|^p dx\}^{\frac{1}{p}} \to 0 \text{ as } \varepsilon \to 0.\end{aligned}$$

□

[Note that $\{\int_{|y|\leq\varepsilon} [\varphi_\varepsilon(y)]^{p'} dy\}^{\frac{1}{p'}} = \varepsilon^{-\frac{n}{p}}.\|\varphi\|_{L^{p'}}$; that the last term tends to zero is trivial if f is continuous with compact support and follows for general $f \in L^p$ since continuous functions with compact supports are dense in L^p].

If $f \in \overset{\circ}{H}_{m,p}(\Omega)$, let $\{f_\nu\}$ be a sequence of C^∞ functions with compact support converging to f in $_{m,p}(\Omega)$. Then

$$\begin{aligned} D^\alpha(\varphi_\varepsilon * f)(x) &= \int_\Omega D^\alpha \varphi_\varepsilon(x-y) f(y) dy \\ &= \lim_{\nu \infty} \int_{\mathbb{R}^n} D^\alpha \varphi_\varepsilon(x-y) f_\nu(y) dy \\ &= \lim_{\nu \to \infty} \int_{\mathbb{R}^n} \varphi_\varepsilon(x-y) D^\alpha f_\nu(y) dy \\ &= \int_\Omega \varphi_\varepsilon(x-y) D^\alpha f(y) dy \\ &= \varphi_\varepsilon * D^\alpha f(x). \end{aligned}$$

174 **Remark.** This proposition, when $p = 2$, follows immediately from Plancherel's theorem. In fact, if we extend f to $\mathbb{R}^n$ by setting it $= 0$ outside Ω, we have,

$$\begin{aligned} (\varphi_\varepsilon * f)^\wedge(\xi) &= (2\pi)^{\frac{n}{2}} \hat{\varphi}_\varepsilon(\xi). \hat{f}(\xi) = (2\pi)^{\frac{n}{2}} \hat{\varphi}(\varepsilon\xi) \hat{f}(\xi) \\ &\to (2\pi)^{\frac{n}{2}} \hat{\varphi}(0) \hat{f}(\xi) = \hat{f}(\xi) \text{ in } L^2, \text{as } \varepsilon \to 0. \end{aligned}$$

Proposition 1. *If $f \in H_{m,p}(\Omega)$ and if the $D^\alpha f$, for $|\alpha| \leq m$, are strongly differentiable upto order m' in L^p, then f is strongly differentiable upto order $m + m'$ in L^p.*

Proof. It is enough to prove the proposition for a function f with compact support $\subset \Omega', \Omega'$ being a relatively compact open subset of Ω. If φ_ε is defined as in the lemma above, then $\varphi_\varepsilon * f(x) = \int_\Omega \varphi_\varepsilon(x-y) f(y) dy$ is a C^∞ function of x and for $|\alpha| \leq m$, we have by (*ii*) in the lemma above,

$$D^\alpha(\varphi_\varepsilon * f) = \varphi_\varepsilon * D^\alpha f.$$

□

Again since $D^\alpha f \in \overset{\circ}{H}_{m',p}(\Omega)$, we have, for $|\alpha| \leq m, |\beta| \leq m'$

$$D^{\alpha+\beta}(\varphi_\varepsilon * f) = D^\beta[D^\alpha(\varphi_\varepsilon * f)] = D^\beta(\varphi_\varepsilon * D^\alpha f) = \varphi_\varepsilon * D^\beta(D^\alpha f)$$

(the last two equations hold because of Lemma 1).

If $f_\nu(x) = \varphi_{1/\nu} * f(x)$, then by Lemma 1 (i), $f_\nu \to f$ in $H_{m+m',p}(\Omega')$ and hence the proposition is proved.

Proposition 2. *Let Ω be a bounded open set in $\mathbb{R}^n$. If φ_ε is defined as in the lemma above, then for any $f \in \overset{\circ}{H}_{m,p}(\Omega)$, we have* 175

$$|\varphi_\varepsilon * f - f|_{m-1,p} \leq A\varepsilon \|\varphi\|_{L^{p'}} |f|m, p, where \frac{1}{p} + \frac{1}{p'} = 1$$

and A is a constant depending on Ω.

Proof. We shall first prove that if Ω' is a bounded open set with $\Omega \Subset \Omega'$, and ε is small enough, then for $f \in \overset{\circ}{H}_{m,p}(\Omega)$,

$$|\varphi_\varepsilon * f - f|^{\Omega'}_{0,p'} \leq A\varepsilon \|\varphi\|_{L^{p'}} |f|_{1,p}.$$

□

Since $C_0^{\infty,q}(\Omega)$ is dense in $\overset{\circ}{H}_{1,p}(\Omega)$, it is enough to prove this inequality for $f \in C_0^{\infty,q}(\Omega)$. We have

$$f(x+y) - f(x) = \sum_{i=1}^{n} y_i \int_0^1 \frac{\partial f}{\partial x_i}(x+ty)\, dt$$

so that
$$|f(x+y) - f(x)|^p \leq n^p \sum_{i=1}^{n} |y_i|^p \int_0^1 \left|\frac{\partial f}{\partial x_i}(x+ty)\right|^p dt.$$

Hence, if $g_y(x) = f(x+y) - f(x)$, we have

$$\begin{aligned} \|g_y\|^p_{L^p} &\leq n^p \sum_{i=1}^{n} |y_i|^p \int_0^1 dt \int_{\mathbb{R}^n} \left|\frac{\partial f}{\partial x_i}(x+ty)\right|^p dx \\ &\leq \left(n^p \sum_1^n |y_i|^p\right) |f|^p_{1,p}, \end{aligned}$$

so that

(5.1) $$\|g_y\|^p_{L^p} \le n^{p+1}\varepsilon^p |f|^p_{1,p} \text{ if } |y| \le \varepsilon.$$

Now 176

$$\varphi_\varepsilon * f(x) - f(x) = \int\limits_{\mathbb{R}^n} \varphi_\varepsilon(x-y)[f(y) - f(x)]dy$$
$$= \int\limits_{\mathbb{R}^n} \varphi_\varepsilon(y)[f(x+y) - f(x)]dy.$$

Since supp $.\varphi_\varepsilon \quad \subset \{x \big| |x| < \varepsilon\}$, this gives

$$\varphi_\varepsilon * f(x) - f(x) = \int\limits_{|y|<\varepsilon} \varphi_\varepsilon(y)[f(x+y) - f(x)]dy.$$

If $p > 1$, we use Holder's inequality and obtain

$$\left|\varphi_\varepsilon * f(x) - f(x)\right| \le \left(\int |\varphi_\varepsilon(y)|^{p'} dy\right)^{\frac{1}{p'}} \left(\int\limits_{|y|<\varepsilon} |f(x+y) - f(x)|^p dy\right)^{\frac{1}{p}}.$$

Since, as is easily verified,

$$\left(\int |\varphi_\varepsilon(y)|^{p'} dy\right)^{\frac{1}{p'}} = \varepsilon^{-\frac{n}{p}} \|\varphi\|_{L^{p'}}$$

this gives $$|\varphi_\varepsilon * f(x) - f(x)|^p \le \varepsilon^{-n} \|\varphi\|^p_{L^{p'}} \int |f(x+y) - f(x)|^p dy.$$

This inequality clearly holds also if $p = 1$, if we replace $\|\varphi\|_{L^{p'}}$ by $v.\|\varphi\|_{L^\infty} = v. \sup\limits_x |\varphi(x)|$, where $v = \int\limits_{|y|<1} dy$. Hence

$$\int\limits_{\Omega'} \left|\varphi_\varepsilon * f(x) - f(x)\right|^p dx \le v.\varepsilon^{-n} \|\varphi\|^p_{L^{p'}} \int\limits_{\Omega'} dx \int\limits_{|y|<\varepsilon} |f(x+y) - f(x)|^p dy$$

$$= v.\varepsilon^{-n}||\varphi||^p_{L^{p'}} \int\limits_{|y|<\varepsilon} ||g_y||^p_{L^p} dy$$
$$\leq A^p \varepsilon^p ||\varphi||^p_{L^{p'}} |f|^p_{1,p} \text{ (because of (5.1))}$$
$$\left[A^p = v^2.n^{p+1}\right].$$

Hence $|\varphi_\varepsilon * f - f|_{0,p} \leq A\varepsilon ||\varphi||_{L^{p'}} .|f|_{1,p}$ for some constant A. **177**

Now, for $|\alpha| \leq m-1, D^\alpha f \in \overset{\circ}{H}_{1,p}$ and

$$|\varphi_\varepsilon * D^\alpha f - D^\alpha f|^{\Omega'}_{0,p} \leq A\varepsilon .||\varphi||_{L^{p'}} |D^\alpha f|_{1,p},$$

which prove the proposition.

Lemma 2. *Let Ω be a bounded open set in $\mathbb{R}^n$ and k, a continuous function with compact support. Then for $f \epsilon L^p(\Omega)$, the function*

$$(Kf)(x) = \int_\Omega k(x-y) f(y) dy \in L^p(\mathbb{R}^n)$$

and the operator $K : L^p(\Omega) \to L^p(\mathbb{R}^n)$ is completely continuous.

Proof. The first part is obvious since Kf is clearly continuous and with compact support, with support $\subset \{a+b | a \in \Omega,\ b \in \text{supp}\,.k\}$, which is relatively compact. Further, by Holder's inequality, Kf is uniformly bounded on the set $||f||_{L^p} \leq 1$. By Ascoli's theorem, it suffices to prove that the family Kf, $||f||_{L^p} \leq 1$, is equicontinuous. If $\eta(\varepsilon) = \sup\limits_{|a-b|\leq\varepsilon} |k(a) - k(b)|$, we have

$$|(Kf)(x) - (Kf)(x')| \leq \eta(|x-x'|)||f||_{L^1} \leq A_p \eta(|x-x'|)||f||_{L^p}$$

(since Ω is bounded), which proves the lemma. □ **178**

Theorem 1 (Rellich)**.** *Let Ω be a bounded open set in $\mathbb{R}^n$ and $0 \leq m' < m$. Then the natural map $i : \overset{\circ}{H}_{m,p}(\Omega) \to \overset{\circ}{H}_{m',p}(\Omega)$ is completely continuous.*

Proof. Let Ω' be a bounded open set, $\overline{\Omega} \subset \Omega'$. We have only to prove that the natural map $j : \overset{\circ}{H}_{m',p}(\Omega) \to \overset{\circ}{H}_{m',p}(\Omega')$ composite of i and the isometry $\overset{\circ}{H}_{m',p}(\Omega) \to \overset{\circ}{H}_{m',p}(\Omega')$ is completely continuous. □

For any operator T between these two spaces, we set

$$\|T\| = \sup_{f\neq 0} \cdot \frac{|Tf|_{m',p}}{|f|_{m,p}}.$$

Let T_ε be the operator $T_\varepsilon(f)(x) = \varphi_\varepsilon * f(x)$. If ε is sufficiently small, $T_\varepsilon(f) \in \overset{\circ}{H}_{m',p}(\Omega')$, and because of Prop. 2, we have

$$\|T_\varepsilon - j\| \to 0 \quad \text{as } \varepsilon \to 0.$$

Since the uniform limit of completely continuous operators is completely continuous, the theorem follows at once from Lemma 2.

Proposition 3. *There exist positive constants C_1 and C_2 such that for any $f \in \overset{\circ}{H}_m(\mathbb{R}^n)$,*

$$C_1 \int (1+|\xi|^2)^m |\hat{f}(\xi)|^2 d\xi \leq |f|_m^2 \leq C_2 \int (1+|\xi|^2)^m |\hat{f}(\xi)|^2 d\xi$$

($|\hat{f}|$ denotes the norm in $\mathbb{C}^q$).

179 *Proof.* Since functions with compact support are dense in $H_m^0(\mathbb{R}^n)$, it is enough to prove the proposition for f with compact support. Now,

$$\begin{aligned} |f|_m^2 &= \sum_{|\alpha|\leq m} \sum_{i\leq q} \int |D^\alpha f_i(x)|^2 dx \\ &= \sum_{|\alpha|\leq m} \sum_{i\leq q} |D^{\hat{\alpha}} f_i|_0^2, \text{ by Plancherel's theorem, since } D^\alpha f \in |L^2. \end{aligned}$$

□

Hence $|f|_m^2 = \sum_{|\alpha|\leq m} \sum_{i\leq q} |\xi^\alpha|^2 |\hat{f}_i(\xi)|^2 d\xi$.

Now there exist constants C_1 and C_2 such that

$$C_1(1+|\xi|^2)^m \leq \sum_{|\alpha|\leq m} |\xi^\alpha|^2 \leq C_2(1+|\xi|^2)^m \text{ for } \xi \in \mathbb{R}^n$$

and hence the proposition.

Remark. When $p = 2$, Theorem 1 can be proved very simply, using Plancherel's theorem. In fact, if $f \in \overset{\circ}{H}_m(\Omega)$ and $|f|_m \leq 1$, clearly $\hat{f}$ is bounded and so is $\dfrac{\partial \hat{f}}{\partial \xi_k} = (2\pi)^{-n/2} \int i x_k e^{i\xi x} f(x) dx$ since Ω is bounded, so that the $\{\hat{f}\}$ form an equicontinuous family. Hence given a sequence $\{f_\nu\}, |f_\nu|_m \leq 1$, we may select a subsequence $\{f_{\nu_k}\}$ such that $\{\hat{f}_{\nu_k}\}$ converges uniformly on compact sets of $\mathbb{R}^n$. Now,

$$|f_{\nu_p} - f_{\nu_q}|^2_{m-1} \leq C_2 \int_{\mathbb{R}} (1 + |\xi|^2)^{m-1} |\hat{f}_{\nu_p} - \hat{f}_{\nu_q}|^2 d\xi. \text{ Given } \varepsilon > 0,$$

we may choose A that $1 + |\xi|^2 > \dfrac{1}{\varepsilon}$ for $|\xi| > A$, so that

$$\int_{|\xi|>A} (1 + |\xi|^2)^{m-1} |\hat{f}_{\nu_p} - \hat{f}_{\nu_q}|^2 d\xi \leq C_3 \varepsilon |f_{\nu_p} - f_{\nu_q}|^2_m < 2C_3 \varepsilon.$$

while, if p, q are large, $\int_{|\xi|\leq A} (1 + |\xi|^2)^{m-1} |\hat{f}_{\nu_p} - \hat{f}_{\nu_q}|^2 d\xi < \varepsilon$ since $\{\hat{f}_{\nu_p}\}$ **180**
converges uniformly on compact sets. This shows that $\{f_{\nu_p}\}$ converges in $H_{m-1,p}$.

Proposition 4. *We have*

$$H^0_m(\mathbb{R}^n) = H_m(\mathbb{R}^n) = \left\{ f | f \in L^2, \int_{\mathbb{R}^n} (1 + |\xi|^2)^m |\hat{f}(\xi)|^2 d\xi < \infty \right\}.$$

Proof. Let $f \in H_m(\mathbb{R}^n)$ and φ, a be C^∞ function with compact support, $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $\varphi(x) = 1$ for $|x| \leq 1$ and $0 \leq \varphi(x) \leq 1$. □

Let $\varphi_\nu(x) = \varphi\left(\dfrac{x}{\nu}\right)$. Then $\varphi_\nu(x) \to 1$ and each φ_ν has compact support. By remark (3) above,

$$D^\alpha \varphi_\nu f = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} D^\beta \varphi_\nu D^{\alpha-\beta} f.$$

Since $D^\beta\varphi_\nu$ are bounded and tend to zero (for $|\beta| \geq 1$) everywhere, it follows from Lebesgue's theorem on bounded convergence that $D^\beta\varphi_\nu . D^{\alpha-\beta} f \to 0$ in L^2 for $|\beta| \geq 1$, so that

$$D^\alpha(\varphi_\nu f) \to D^\alpha f \text{ in } L^2, \text{ for } |\alpha| \leq m.$$

Hence $\varphi_\nu f \to f$ in $H_m(\mathbb{R}^n)$ and since $\{\varphi_\nu\}$ is a sequence of functions with compact support, it follows that $f \in \overset{\circ}{H}_m(\mathbb{R}^n)$. This proves that $H_m(\mathbb{R}^n) = \overset{\circ}{H}_m(\mathbb{R}^n)$. It is clear from Proposition 3 that

$$\overset{\circ}{H}_m(\mathbb{R}^n) \subset \left\{ f \middle| f \in L^2, \int\limits_{\mathbb{R}^n} (1 + |\xi|^2)^m |\hat{f}(\xi)|^2 d\xi < \infty \right\}.$$

181 Conversely, if $f \in L^2$ and $\int\limits_{\mathbb{R}^n} (1+|\xi|^2)^m |\hat{f}(\xi)|^2 d\xi < \infty$, $(1+|\xi|^2)^{\frac{m}{2}} \hat{f}(\xi) \in L^2$. Hence there exists a sequence $\hat{g}_\nu$ in $\mathscr{S}$ such that $\hat{g}_\nu(\xi) \to (1 + |\xi|^2)^{\frac{m}{2}} \hat{f}(\xi)$ in L^2. Let $h_\nu \in \mathscr{S}$ be such that its Fourier transform $\hat{h}_\nu = \hat{g}_\nu/(1 + |\xi|^2)^{m/2}$ [which exists by the inversion theorem]. Then $h_\nu \in H_m$ and $\int\limits_{\mathbb{R}^n} (1 + |\xi|^2)^m |\hat{h}_\nu(\xi) - \hat{h}_\mu(\xi)|^2 d\xi \to 0$ as $\mu, \nu \to \infty$, i.e., by Proposition 3, h_ν is convergent in H_m. It is clear that $h_\nu \to f$ in L^2. Hence $f \in H_m(\mathbb{R}^n)$, which proves the proposition.

Lemma 3. *Let η be the map $\eta : \mathbb{R}^+ \times S^{n-1} \to \mathbb{R}^n - \{0\}$, given by $\eta(t, x) = tx = y$. Then there exists an $(n-1)$ form ω on S^{n-1} such that $\eta^*(dy_1 \wedge \cdots \wedge dy_n) = t^{n-1} dt \wedge \omega$. [A point in $\mathbb{R} - \{0\}$ is denoted by $y = (y_1, \ldots, y_n)$.]*

Proof. In fact if $x_1, \ldots, x_n$ are the restrictions to S^{n-1} of the coordinate functions in $\mathbb{R}^n$, we may take $\omega = \sum\limits_{k=1}^{n} x_k dx_1 \wedge \cdots \wedge d\hat{x}_k \wedge \cdots \wedge dx_n$. (The hat over a term means that the term is omitted.) □

Remark. Since $\int\limits_U dy_1 \wedge \cdots \wedge dy_n$, over any non-empty open set $U \subset \mathbb{R}^n - 0$ is positive, we have $\int\limits_{S^{n-1}} \omega \neq 0$.

Theorem 2 (Sobolev's lemma)**.** *Let Ω be an open set in $\mathbb{R}^n$ and $m > \dfrac{n}{p}$. Then for any compact set $K \subset \Omega$, there exists a constant C_K such that for any C^∞ function $f : \Omega \to \mathbb{C}^q$ with supp: $f \subset K$, we have*

$$\sup_{x\in K} |f(x)| \leq C_{K,m}|f|_{m,p}.$$

Proof. We may suppose that $\Omega = \mathbb{R}^n$. Further, we can choose a compact set K' such that for any $x \in K, g(y) = f(x + y)$ is a C^∞ function with supp $.g \subset K'$. Hence it is enough to prove that there exists a constant C such that for C^∞ f with supp. $f \subset K$, we have, 182

$$|f(0)| \leq C|f|_{m,p}.$$

□

Let $\eta : \mathbb{R}^+ \times S^{n-1} \to \mathbb{R}^n - \{0\}$ be the map as defined in the lemma above.

Let $f(y) = g(tx)$, where $y = tx$ for $y \neq 0$, $t \in \mathbb{R}^+, x \in S^{n-1}$ and $g(0) = f(0)$.

Then $f_i(0) = C_1 \int\limits_0^M \dfrac{\partial g_i(tx)}{\partial t^m} t^{m-1} dt$ for some constants $M = M_K$ and $C_1 = C_1(m)$.

Multiplying by ω and integrating over S^{n-1}, we have

$$\begin{aligned} f_i(0) \int\limits_{S^{n-1}} \omega &= C_1 \int\limits_{S^{n-1}} \int\limits_0^M \frac{\partial^m g_i(tx)}{\partial t^m} t^{m-1} dt \wedge \omega \\ &= C_1 \int\limits_{S^{n-1}} \int\limits_0^M t^{m-n} \frac{\partial^m g(tx)}{\partial t^m} t^{n-1} dt \wedge \omega. \end{aligned}$$

Since $\int\limits_{S^{n-1}} \omega \neq 0$, this gives

$$f_i(0) = C_2 \int\limits_{|y|<M} t^{m-n} \frac{\partial^m g_i(tx)}{\partial t^m} dy \tag{5.2}$$

for some constant C_2 and $t = |y|$.

183 Now for $p > 1$, using Holder's inequality,

$$|f_i(0)| \leq C_2 \left(\int\limits_{|y|<M} t^{(m-n)p'} dy \right)^{\frac{1}{p'}} \left(\int\limits_{|y|<M} \left|\frac{\partial^m g_i(tx)}{\partial t^m}\right|^p dy \right)^{\frac{1}{p}}$$

where $\dfrac{1}{p'} + \dfrac{1}{p} = 1$. Hence

$$|f_i(0)| \leq C_2 . \left(\int\limits_{S^{n-1}} \int\limits_0^M t^{(m-n)p'} t^{n-1} dt \wedge \omega \right)^{\frac{1}{p'}} \left(\int\limits_{|y|<M} \left|\frac{\partial^m g_i(tx)}{\partial t^m}\right|^p dy \right)^{\frac{1}{p}} .$$

Since $m > \dfrac{n}{p}$, we have $(m-n)p' + n - 1 > -1$ and hence $\int\limits_{S^{n-1}} \int\limits_0^M$ $t^{(m-n)p'} t^{n-1} dt \wedge \omega < \infty$. Now

$$\frac{\partial^m g_i(tx)}{\partial t^m} = \sum_{|\alpha| \leq m} q_\alpha(y) D^\alpha f_i(y),$$

where $q_\alpha(y)$ are bounded functions of y and hence there exists a constant C_3 such that

$$\int\limits_{|y|<M} \left|\frac{\partial^m g_i(tx)}{\partial t^m}\right|^p dy \leq C_3 (|f|_{m,p})^p.$$

Hence $|f(0)| \leq C_4 |f|_{m,p}$, for some constant C_4 depending on K. This proves the theorem for $p > 1$.

If $p = 1, m \geq n$, it follows immediately from that (5.2) $|f(0)| \leq C_K |f|_{m,1}$ for a constant C_K

Corollary 1. *If Ω is an open set in $\mathbb{R}^n$, and K is a compact subset of Ω, then, for any $f \in C^{\infty,q}(\Omega)$, we have*

$$\sup_{x \in K} |f(x)| \leq C_{K,\Omega,m} |f|_{m,p} \text{ for } m > n/p.$$

184 *Proof.* Apply Theorem 2 to ηf, where η is a fixed function with compact support in Ω, which is $= 1$ on K. □

Corollary 2. *If Ω is an open subset of $\mathbb{R}^n$ and $m > \frac{n}{p}$, then if $f \in H_{m,p}(\Omega)$, there exists a function $g \in H_{m,p}(\Omega)$ such that $f = g$ almost everywhere and g has continuous derivatives of all orders*

$$\leq\ m - \left[\frac{n}{p}\right] - 1.$$

Proof. By multiplying f by a suitable function, we may suppose that $f \in \overset{\circ}{H}_{m,p}(\Omega)$; moreover, we may suppose, that Ω is bounded. Let $\{f_\nu\}$ be a sequence of C^∞ functions, with compact support in Ω, converging to f in $H_{m,p}(\Omega)$. Clearly $D^\alpha(f_\nu - f_\mu) \in H_{m-|\alpha|,p}$, for $0 \leq |\alpha| \leq m$, and if $m - |\alpha| > \frac{n}{p}$, and $K \subset \Omega$ is compact, we have, by Sobolev's lemma,

$$\text{supp}_{x \in K} |D^\alpha f_\nu(x) - D^\alpha f_\mu(x)| \leq C_K |f_\nu - f_\mu| m, p.$$

□

Hence $D^\alpha f_\nu$ is uniformly convergent on K, for $|\alpha| < m - \frac{n}{p}$; if $g = \lim f_\nu$, this implies that g has continuous derivatives upto order $\leq m - \left[\frac{n}{p}\right] - 1$.

Remark. The proof of Sobolev's lemma, for $p = 1$ or 2, simplifies as follows If $p = 1$, $f_i(x) = \int\limits_{-M}^{x} \cdots \int\limits_{-M}^{x_n} \frac{\partial^n f_i(t_1, \ldots, t_n)}{\partial x_1 \cdots \partial x_n} dt_1 \cdots dt_n$ for a constant M depending on K.

Hence $|f(x)| \leq A|f|_{n,1} \leq A|f|_{m,1}$, for $m \geq n$ and a constant A. 185
Further, by Holder's inequality applied to this formula, we get

$$|f(x)| \leq C_{K,p}|f|_{m,p} \text{ if } m \geq n, \text{ and } p \geq 1.$$

Thus, the statement that any $f \in H_{m,p}$ has continuous derivatives of order $\leq m - n$ is trivial. If $p = 2$, by the remark following the inversion formula in §4,

$$f_i(x) = \frac{1}{(2\pi)^{n/2}} \int e^{ix\xi} \hat{f}_i(\xi) d\xi$$
$$= \frac{1}{(2\pi)^{n/2}} \int e^{ix\xi} (1 + |\xi|^2)^{\frac{m}{2}} \frac{\hat{f}_i(\xi)}{(1 + |\xi|^2)^{m/2}} d\xi,$$

and by Schwarz's inequality

$$|f_i(x)| \leq A \left(\int (1 + |\xi|^2)^{-m} d\xi \right)^{\frac{1}{2}} \left(\int |\hat{f}_i(\xi)|^2 (1 + |\xi|^2)^m d\xi \right)^{\frac{1}{2}}$$

for some constant A.

Now, $\int\limits_{\mathbb{R}^n} (1 + |\xi|^2)^{-m} d\xi < \infty$ if $m > \frac{n}{2}$.

Hence it follows from Prop. 2, that for $m > \frac{n}{2}, |f(x)| \leq B|f|_{m,2}$, for some constant B. This latter proof applies to a such larger class of functions than functions with support in a fixed compact set.

Rellich's lemma remains true if we replace $\overset{\circ}{H}_{m,p}(\Omega)$ by $H_{m,p}(\Omega)$ if the boundary of Ω is sufficiently smooth (see Rellich [37]).

Several proofs of Sobolev's lemma have been given; Sobolev [43] obtained several very precise inequalities. However most of these proofs are more complicated than the one given here.

6 Elliptic differential operators: the inequalities of Gårding and Friedrichs

186 In what follows, Ω is an open set in $\mathbb{R}^n$ and L is a linear differential operator, $L : C_0^{\infty,q}(\Omega) \to C_0^{\infty,p}(\Omega)$.

Definition. (1) If L can be written as $Lf = \sum\limits_{|\alpha| \leq m} a_\alpha . D^\alpha f$, with continuous mappings a_α of Ω into the space of $p \times q$ complex matrices, and if there exists α such that $|\alpha| = m$ and $a_\alpha \not\equiv 0$ on Ω, then L is said to have order m on Ω.

(2) If L is a differential operator of order m on Ω, for $\xi \in \mathbb{R}^n$, the characteristic polynomial of L is defined by $p(x,\xi) = \sum\limits_{|\alpha|=m} \xi^\alpha a_\alpha(x)$; it is a mapping of $\Omega \times \mathbb{R}^n$ into the space of $p \times q$ matrices.

(3) If $p(x,\xi)$ is the characteristic polynomial of L and if for any $\xi \in \mathbb{R}^n$, $\xi \neq 0$ and $x \in \Omega$, the map $p(x,\xi) : \mathbb{C}^q \to \mathbb{C}^p$ is injective, then L is said to be elliptic.

(4) If $p = q$ and $p(x,\xi)$ is the characteristic polynomial of L and if for any $\xi \neq 0, \xi \in \mathbb{R}^n, x \in \Omega$ and $v \in \mathbb{C}^q, v \neq 0$, we have $Re(p(x,\xi)v, v) \neq 0$, then L is said to be strongly elliptic.

(5) If $p = q$, L is of order m, and $p(x,\xi)$ is its characteristic polynomial, and if there exists a constant $c > 0$ such that for any $\xi \in \mathbb{R}^n, x \in \Omega$ and $v \in \mathbb{C}^q, Re(p(x,\xi)v, v) \geq c|\xi|^m|v|^2$, then L is said to be uniformly strongly elliptic.

If $n > 1$, then a strongly elliptic operator (or its negative) is uniformly strongly elliptic an any connected subset $\Omega' \subset\subset \Omega$.

In fact, since S^{n-1} is connected, Re $(p(x,\xi)v, v)$ has constant sing on 187
$\Omega' \times S^{n-1}$.

Further, if $n > 1$, then any strongly elliptic operator is of even order. In fact, for fixed x and $v \neq 0$, $Q(\xi) = Re(p(x,\xi)v, v)$ is a homogeneous polynomial of degree m = order L. It is clear that for almost all values of $a, b \in \mathbb{R}^n$, the polynomial $Q(a + \lambda b)$ of the real variable λ has degree m, hence has a real zero if m is odd. If $n > 1$, we may choose a, b such that $a + \lambda b \neq 0$ for all real λ and Q would then have a real, non-trivial root.

Let L_1 and L_2 be differential operators, $L_1 : C_0^{\infty,q}(\Omega) \to C^{0,p}(\Omega)$ and $L_2 : C_0^{\infty,p}(\Omega) \to C^{0,r}(\Omega)$, then if L_1 can be written as $L_1 f = \sum\limits_{|\alpha|\leq m} a_\alpha D^\alpha f, a_\alpha$ being C^∞ functions with values in $p \times q$ matrices, then we define $L_2 \circ L_1 : C_0^{\infty,q}(\Omega) \to C^{0,r}(\Omega)$ by

$$(L_2 \circ L_1)(f)(x) = (L_2(L_1 f))(x).$$

We also write $L_2.L_1$ for $L_2 \circ L_1$.

Let L_2 be given by

$$(L_2 f)(x) = \sum_{|\beta| \leq m'} b_\beta(x) D^\beta f(x), \text{for } f \in C^{\infty,p}(\Omega).$$

Then $L_2 \circ L_1$ is given by

$$(L_2 \circ L_1)(f)(x) = \sum_{|\gamma| \leq m+m'} c_\gamma(x) D^\gamma f(x)$$

where $$c_\gamma(x) = \sum_{\alpha+\beta=\gamma} b_\beta(x) a_\alpha(x) \text{ for } |\gamma| \leq m + m'.$$

188 Hence $L_2 \circ L_1$ has order $\leq m + m'$ and if $p_1(x,\xi), p_2(x,\xi)$ are the characteristic polynomials of L_1 and L_2 respectively, the characteristic polynomial $p(x,\xi)$ of $L_2 \circ L_1$ is given by

$$p(x,\xi) = p_2(x,\xi).p_1(x,\xi), \text{ unless } p_2(x,\xi).p_1(x,\xi) = 0 \text{ for all } x \text{ and } \xi.$$

If L_1 and L_2 are elliptic differential operators and $L_2 \circ L_1$ is defined as above, then $L_2 \circ L_1$ is elliptic. This obvious since if $p_1(x,\xi), p_2(x,\xi)$ are injective, $p(x,\xi)$ is injective.

Let L be a differential operator of order $m, L : C_0^{\infty,q} \to C^{o,p}$ and $Lf = \sum_{|\alpha| \leq m} a_\alpha D^\alpha f$, where a_α are C^∞ functions on Ω. Then we define the (formal) adjoint operator $L^* : C_0^{\infty,p} \to C^{0,q}$ by

$$(Lf,\varphi) = (f, L^*\varphi) \text{ for any } f \in C_0^{\infty,q}(\Omega) \text{and } \varphi \in C_0^{\infty,p}(\Omega).$$

We shall show that the operator L^* exists and is unique.

If for $\varphi_1, \varphi_2 \in C_0^{\infty,q}(\Omega), (\varphi_1, f) = (\varphi_2, f)$ for every $f \in C_0^{\infty,q}(\Omega)$, then clearly $\varphi_1 = \varphi_2$. Hence $L^*\varphi$, if it exists, is unique.

Since φ and f are C^∞ functions with compact supports,

$$\begin{aligned}(Lf,\varphi) &= \sum_\alpha \sum_{i=1}^{p} \sum_{j=1}^{q} a_\alpha^{ij}(x) D^\alpha f_j(x) \overline{\varphi_i(x)} dx \\ &= \sum_\alpha (-1)^{|\alpha|} \sum_{i=1}^{p} \sum_{j=1}^{q} \int f_j(x) \overline{D^\alpha(\overline{a_\alpha^{ij}}(x).\varphi_i(x))} dx\end{aligned}$$

$$= \sum_{\alpha} (-1)^{|\alpha|} (f(x), D^{\alpha}(\overline{t_{a_\alpha}(x)}.\varphi(x))),$$

where t_a is the transpose of the matrix a. Hence if we define $L^*\varphi$ by **189**

$$L^*\varphi = \sum_{\alpha} (-1)^{|\alpha|} D^{\alpha}[t_{\overline{a_\alpha}} \cdot \varphi],$$

we have, $(f, L^*\varphi) = (Lf, \varphi)$, for $f \in C_0^{\infty,q}(\Omega)$ and $\varphi \in C_0^{\infty,p}(\Omega)$.

This prove the existence and uniqueness of the adjoint operator L^*: $C_0^{\infty,p}(\Omega) \to C_0^{\infty,q}(\Omega)$. Moreover order of L^* = order of L. Further, for $|\alpha| = m$,

$$D^{\alpha}({}^t\overline{a}_\alpha\varphi) = t_{\overline{a}_\alpha}.D^{\alpha}\varphi + \sum_{|\beta|<m} b_\beta D^\beta \varphi \ , b_\beta$$

being functions on Ω with values in $q \times p$ matrices.

Hence if $p^*(x,\xi)$ is the characteristic polynomial of L^*,

$$p^*(x,\xi) = (-1)^m \sum_{|\alpha|=m} \xi^{\alpha t_{\overline{a}_\alpha}}(x) = (-1)^m t_{\overline{p(x,\xi)}}.$$

Remark. If L is an elliptic operator of order m, L: $C_0^{\infty,q} \to C_o^{\infty,p}$ and if L^*: $C_0^{\infty,p} \to C_0^{\infty,q}$ is the adjoint of L, then the operator $(-1)^m L^*. \ L$ is strongly elliptic.

Proof. If $A = (-1)^m L^*. \ L$, $p(x,\xi)$, $p^*(x,\xi)$ and $p'(x,\xi)$ are the characteristic polynomials of L, L^* and A respectively, and if $\xi \in \mathbb{R}^n$, $x \in \Omega$, $v \in \mathbb{C}^q$, $\xi \neq 0$, $v \neq 0$, have,

$$\begin{aligned} Re(p'(x,\xi)v, v) &= Re((-1)^m p^*(x,\xi).p(x,\xi)v, v) \\ &= Re(p(x,\xi)v, p(x,\xi)v) > 0. \end{aligned}$$

□

Corollary. *If Ω' is relatively compact in Ω and L: $C_0^{\infty,q}(\Omega) \to C_0^{\infty,p}(\Omega)$ is an elliptic operator, of order, m, $(-1)^m L^* \circ L$ is uniformly strongly elliptic on Ω', of even order, namely $2m$.* **190**

We remark further that if L is an elliptic operator L: $C_0^{\infty,q} \to C_0^{\infty,q}$ (i.e. if $q = p$), then L^* is also elliptic. In fact, for $\xi \neq 0$, $p(x,\xi)$ is an automorphism of $\mathbb{C}^q$ and hence so is $t_{\overline{p(x,\xi)}} = (-1)^m p^*(x,\xi)$.

Proposition 1. *Let Ω be an open set in $\mathbb{R}^n$. Then for $\varepsilon > 0$, there exists a constant $C(\varepsilon)$ such that for any $f \in \overset{\circ}{H}_m(\Omega)$, $(m > 0)$, we have*

$$|f|^2_{m-1} \leq \varepsilon |f|^2_m + C(\varepsilon)|f|^2_0.$$

Proof. It is enough to prove the inequality for C^∞ functions f with compact support $\subset \Omega$. By proposition 3, §5, there exists a constant C_2 such that

$$|f|^2_{m-1} \leq C_2 \int\limits_{\mathbb{R}^n} (1 + |\xi|^2)^{m-1} |\hat{f}(\xi)|^2 d\xi.$$

□

Now given ε, there exists $C'(\varepsilon)$ such that

$$\left(1 + |\xi|^2\right)^{m-1} \leq \frac{\varepsilon}{C_2}\left(1 + |\xi|^2\right)^m + C'(\varepsilon) \text{ for } \xi \in \mathbb{R}^n.$$

Hence $|f|^2_{m-1} \leq \varepsilon \int\limits_{\mathbb{R}^n} (1+|\xi|^2)^m |\hat{f}(\xi)|^2 d\xi + C(\varepsilon).\ |f|^2_0$, which by Proposition 3, §5, proves the required inequality.

Theorem 1 (Garding's inequality)**.** *Let L be a uniformly strongly elliptic*
191 *differential operators of even order $2m$ on Ω, Ω being an open set in $\mathbb{R}^n$. Then for any relatively compact open subset Ω' of Ω, there exist constants $C > 0$ and $B > 0$ such that for any C^∞ function $f \colon \Omega \to \mathbb{C}^q$ with* supp . $f \subset \Omega'$, *we have*

$$Re(-1)^m (Lf, f) \leq C|f|^2_m - B|f|^2_0.$$

Proof. We shall prove the theorem in three steps. □

Step I. *Let L be given by*

$$Lf = \sum_{|\alpha| \leq 2m} a_\alpha D^\alpha f,$$

where the a_α are constant matrices. Then we have, by Plancherel's theorem,

$$(Lf, f) = (\hat{Lf}, \hat{f}).$$

Now

$$\hat{Lf}(\xi) = \sum_{|\alpha|\le 2m} a_\alpha \hat{D^\alpha f}(\xi)$$
$$= (-1)^m \sum_{|\alpha|=2m} a_\alpha.\xi^\alpha \hat{f}(\xi) + \sum_{|\alpha|\le 2m-1} a_\alpha i^{|\alpha|}.\xi^\alpha \hat{f}(\xi).$$

Clearly the characteristic polynomial is independent of x; we denote it by $p(\xi)$. Then

$$(Lf, f) = (-1)^m \int_{\mathbb{R}^n} (p(\xi)\hat{f}(\xi), \hat{f}(\xi))d\xi + \sum_{|\alpha|<2m} \int_{\mathbb{R}^n} (i^{|\alpha|} a_\alpha \xi^\alpha \hat{f}(\xi), \hat{f}(\xi))d.\xi$$

Since L is uniformly strongly elliptic on Ω, there exists, by definition, a constant C_1 such that

$$Re(p(\xi)v, v) \ge C_1|\xi|^{2m}|v|^2 \text{ for } \xi \in \mathbb{R}^n \text{ and } v \in \mathbb{C}^q.$$

Hence **192**

$$Re(-1)^m(\hat{Lf}, \hat{f} \ge C_1 \int_{\mathbb{R}^n} |\xi|^{2m}|\hat{f}(\xi)|^2 d\xi - M_1 \int_{\mathbb{R}^n} (1+|\xi|)^{2m-1}|\hat{f}(\xi)|^2 d\xi$$

where M_1 is a constant, depending only on the matrices a_α, $|\alpha| \le 2m-1$. Let A be a constant such that

$$C_1|\xi|^{2m} - M_1(1+|\xi|)^{2m-1} \ge C_2(1+|\xi|^2)^m$$

for $|\xi| \ge A$ and a suitable constant $C_2 > 0$. Then

$$Re(-1)^m(Lf, f) \ge C_2 \int_{|\xi|>A} (1+|\xi|^2)^m|\hat{f}(\xi)|^2 d\xi$$
$$- M \int_{|\xi|\le A} (1+|\xi|)^{2m-1}|\hat{f}(\xi)|^2 d\xi$$
$$\ge C_2 \int_{\mathbb{R}^n} (1+|\xi|^2)^m|\hat{f}(\xi)|^2 d\xi$$

$$- C_2 \int\limits_{|\xi|\leq A} (1 + |\xi|^2)^m |\hat{f}(\xi)|^2 d\xi$$
$$- M \int\limits_{|\xi|\leq A} (1 + |\xi|)^{2m-1} |\hat{f}(\xi)|^2 d\xi.$$

Let B be constant such that

$$C_2(1 + |\xi|^2)^m + M(1 + |\xi|)^{2m-1} < B \text{ for } |\xi| \leq A.$$

Then

$$Re(-1)^m(Lf, f) \geq C_2 \int\limits_{\mathbb{R}^n} (1 + |\xi|^2)^m |\hat{f}(\xi)|^2 d\xi - B \int\limits_{\mathbb{R}^n} |\hat{f}(\xi)|^2 d\xi.$$

By Proposition 3, §5 there exists a constant C such that

$$C_2 \int\limits_{\mathbb{R}^n} (1 + |\xi|^2)^m |\hat{f}(\xi)|^2 d\xi \geq C|f|_m^2$$

193 i.e. the inequality is proved when L has constant coefficients.

Step II. *We shall now prove that for any $x_0 \in \Omega$, there exist a relatively compact neighbourhood U of x_0, $U \subset \Omega$, and constants C, B such that for any C^∞ $f \colon \Omega \to \mathbb{C}^q$ with* supp . $f \subset U$, *we have*

$$\text{Re } (-1)^m (Lf, f) \geqq C\,|f|_m^2 - B\,|f|_0^2.$$

We may write L as $L = \sum\limits_{i=1}^{k} (B_i)^* A_i$ for some k, where A_i and B_i are differential operators of orders $\leq m$. For any $x_0 \in \Omega$, A_i and B_i can be written as $A_i = A_i^0 + A_i'$, $B_i = B_i^0 + B_1'$, where B_i^0, A_i^0 are differential operators with constant coefficients and A_i', B_i' are differential operators whose coefficients vanish at x_0. Then

$$(Lf, f) =$$
$$\sum_i (A_i^0\ f, B_i^0\ f) + \sum_i (A_i'\ f, B_i^0\ f) + \sum_i (A_i^0\ f, B_i'\ f) + \sum_i (A_i' f, B_i'\ F)$$

Let L^0 be the differential operator with constant coefficients, defined by

$$L^0 = \sum (B_i^0)^* A_i^0.$$

Since the coefficients of A_i' and B_i' vanish at x_0 and are continuous functions on Ω, given $\varepsilon > 0$ there exists a relatively compact neighbourhood U of x_0, $U \subset \Omega$ such that for f with supp. $f \subset U$,

$$|B_i' f|_0^U + |A_i' f|_0^U \leq \varepsilon |f|_m.$$

Then 194

$$\text{Re } (-1)^m (Lf, f) \geqq \text{Re}(-1)^m (L^0 f, f) - \varepsilon M |f|_m^2$$

where M is a constant depending on A_i, B_i. Now by the result in Step I, there exist constants C', B' such that for $C^\infty f$ with supp $f \subset U$,

$$\text{Re}(-1)^m (L^o f, f) \geq C'|f|_m^2 - B'|f|_0^2.$$

Hence Re $(-1)^m(Lf, f) \geqq (C' - \varepsilon M)|f|_m^2 - B'|f|_0^2$; since $\varepsilon \to 0$ as U shrinks to x_0, our assertion is proved.

Step III. *This is the general case. By step II above, for any relatively compact open subset Ω' of Ω, there exist points x_i, $1 \leq i \leq N$, and neighbourhoods U_i of x_i, $\cup U_i \supset \overline{\Omega'}$ and constants C, B such that for a C^∞ f with* supp . $f \subset U_i$, $\text{Re}(-1)^m(Lf, f) \geqq C|f|_m^2 - B|f|_0^2$. *We write, as in II, $L = \sum_{i=1}^{k} (B_i)^* A_i$, where A_i, B_i are differential operators of orders $\leq m$. Let η_k be C^∞ functions, $\eta_k \colon \Omega \to \mathbb{R}$, with* supp .$\eta_k \subset U_k$, $0 \leq \eta_k(x) \leq 1$, *and $\sum \eta_k^2(x) = 1$ for $x \in \Omega'$. (The η_k exist : see Chap. I, §2.) We first remark that if φ is a C^∞ function with compact support and $\triangle$ is a differential operator of order m, then*

$$\triangle(\varphi f) - \varphi \triangle f = \sum_{|\alpha| < m} a_\alpha D^\alpha f,$$

where the a_α are continuous function with compact supports (depending on φ). Now for C^∞ f with supp . $f \subset \Omega'$,

$$|\eta_k f|_m^2 \leq \frac{1}{C} (-1)^m \text{ Re} . \sum (A_i \eta_k f, B_i \eta_k f) + \frac{B}{C} |\eta_k f|_0^2.$$

By the remark made above, there exists a constant C_1 depending on 195
A_i, B_i, such that

$$(-1)^m \operatorname{Re}. \sum_i (A_i\, \eta_k\, f, B_i\, \eta_k\, f) \leq (-1)^m$$
$$\operatorname{Re} \sum_i (\eta_k\, A_i\, f, \eta_k\, B_i\, f) + C_1 |f|_m . |f|_{m-1}.$$

Hence

$$|\eta_k f|^2_m \leq \frac{1}{C}(-1)^m \operatorname{Re}. \sum_i (A_i f, \eta^2_k\, B_i f) + \frac{C_1}{C}|f|_m |f|_{m-1} + \frac{B}{C}|\eta_k\, f|^2_0.$$

Since $D^\alpha\, \eta_k\, f = \eta_k\, D^\alpha\, f + \sum\limits_{\beta<\alpha} \binom{\alpha}{\beta} D^\beta\, f\, D^{\alpha-\beta}\, \eta_k$, we have

$$|\sum_k |\, \eta_k\, f|^2_m - |f|^2_m| \leq C_2 |f|_m . |f|_{m-1}$$

for some constant C_2.

Hence summing over k,

$$|f|^2_m \leq C_3(-1)^m \operatorname{Re}.(Lf, f) + C_4\, |f|_m\, |f|_{m-1} + C_5 |f|^2_0$$

for some constants C_3, C_4, C_5. Now

$$|f|_m\, |f|_{m-1} \leq \frac{1}{2}.(\varepsilon|f|^2_m + \frac{1}{\varepsilon}|f|^2_{m-1}), 0 < \varepsilon < \frac{1}{2}.$$

Hence

$$|f|^2_m - \frac{\varepsilon}{2}|f|^2_m \leq C_3(-1)^m \operatorname{Re}(Lf, f) + \frac{C_4}{2\varepsilon}|f|^2_{m-1} + C_5|f|^2_0.$$

By Proposition 1, there exists a constant C_6 such that

$$|f|^2_{m-1} \leq \varepsilon^2 |f|^2_m + C_6 |f|^2_0.$$

196 Hence $(1-\varepsilon)$. $|f|^2_m \leq C_3(-1)^m$ Re. $(Lf, f) + C_7 |f|^2_0$ which proves the theorem.

Remark. If L is a uniformly strongly elliptic differential operator of order $2m$ which is homogeneous and has constant coefficients, i.e. $L = \sum_{|\alpha|=2m} a_\alpha D^\alpha$, then the above inequality holds in a stronger form, i.e. for $\Omega' \subset\subset \Omega$, there exists a constant C such that for C^∞ f with supp $. f \subset \Omega'$,

$$(-1)^m \operatorname{Re} .(Lf, f) \geq C.|f|_m^2.$$

Proof. We have, as in Step I above,

$$(Lf, f) = (\hat{Lf}, \hat{f}) = (-1)^m \int (p(\xi)\hat{f}(\xi), \hat{f}(\xi))\, d\xi,$$

and there exists constant C such that

$$\operatorname{Re} .(-1)^m (Lf, f) \geq C. \int_{\mathbb{R}^n} |\xi|^{2m}\, |\hat{f}(\xi)|^2\, d\xi.$$

□

By Plancherel's theorem

$$|D^\alpha f|_0^2 = |D^{\hat{\alpha}} f|_0^2 = \int_{\mathbb{R}^n} |\xi^{2\alpha}||\hat{f}(\xi)|^2\, d\xi.$$

Hence $\operatorname{Re} .(-1)^m (Lf, f) \geq C' \sum_{|\alpha|=m} |D^\alpha f|_0^2$ and since f is a C^∞ function with compact support $\subset \Omega'$ we have $\sum_{|\alpha|=m} |D^\alpha f|_0^2 \geq C''|f|_m^2$ for some constant $C'' > 0$ (Poincare's inequality; see Remark (6) after the Definitions in §5), which proves the required inequality.

Proposition 2. *If Ω is a bounded open subset of $\mathbb{R}^n$ and m is an integer > 0, then for any $A > 0$, there exists a constant C such that for $f \in H_m^0(\Omega)$,* 197

$$\int_{\mathbb{R}^n} (1 + |\xi|^2)^m |\hat{f}(\xi)|^2\, d\xi \leq C \int_{|\xi|>A} (1 + |\xi|^2)^m |\hat{f}(\xi)|^2\, d\xi.$$

[This Proposition may be looked upon as a stronger version, in the case $p = 2$, of Poincare's inequality (Remark (6) at the beginning of §5)].

Proof. If the proposition is false, there exists a sequence $\{f_\nu\}_{\nu \geq 1}$ of C^∞ functions with compact support $\subset \Omega$, such that $|f_\nu|_m = 1$ and

$$\int_{|\xi|>A} (1+|\xi|^2)^m |\hat{f}_\nu(\xi)|^2 d\xi \to 0 \text{ as } \nu \to \infty. \tag{6.1}$$

□

By Rellich's lemma the map i: $\overset{o}{H}_m(\Omega) \to \overset{o}{H}_0(\Omega)$ is completely continuous. Hence we may assume that $\{f_\nu\}$ converges in L^2 to f any. Now $\hat{f}_\nu(\xi)$ is an analytic function of ξ, $\nu \geq 1$, and since $f \in \overset{o}{H}_0(\Omega)$ and Ω is relatively compact, $f \in L^1$; clearly

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_\Omega e^{-i\,\xi\,x} f(x)\,dx$$

is analytic in ξ since Ω is bounded; moreover $\hat{f}_\nu$ converges uniformly to $\hat{f}$ on compact subsets of $\mathbb{R}^n$.

Now, because of assumption (6. 11), for every compact set K with $K \subset \{\xi \| \xi| > A\}$, we have $\int_K |\hat{f}_\nu(\xi)|^2\, d\xi \to 0$. Hence $\int_K |\hat{f}(\xi)|^2\, d\xi = 0$,
198 so that $\hat{f}(\xi) = 0$ for $\xi \in K$ and hence $f = 0$ since we may choose K such that $\overset{o}{K} \neq \phi$. Hence $\hat{f}_\nu$ converges to zero uniformly on compact sets so that $\int_{|\xi|\leq A} (1+|\xi|^2)^m |\hat{f}_\nu(\xi)|^2\, d\xi \to 0$ and by assumption $\int_{|\xi|>A} (1+|\xi|^2)^m |\hat{f}_\nu(\xi)|^2\, d\xi \to 0$.

But $|f|_m = 1$ and thus we have a contradiction. This proves the proposition.

Lemma 1. *Let Ω, Ω', Ω'' be open sets in $\mathbb{R}^n$, $\Omega'' \subset\subset \Omega' \subset\subset \Omega$. Let φ be in $C^\infty(\Omega)$ such that $\varphi(x) = 1$ for x in a neighbourhood of $\overline{\Omega''}$ and*

$\varphi(x) = 0$ for $x \notin \Omega'$. Then for any $\varepsilon > 0$, there exists a constant $C(\varepsilon)$, such that for $k \geq 1$ and for $f \in C^\infty(\Omega)$,

$$\sum_{|\beta|=k} |\varphi^k D^\beta f|_0^2 \leq \varepsilon \sum_{|\beta|=k+1} |\varphi^{k+1} D^\beta f|_0^2 + C(\varepsilon) \sum_{|\beta|=k-1} |\varphi^{k-1} D^\beta f|_0^2$$

[φ^0 stands for 1 on Ω', 0 outside].

Proof. It is enough to prove that for $k \geq 1$, and $|\beta| = k$ we have

$$|\varphi^k D^\beta f|_0^2 \leq \varepsilon \sum_{|\alpha|=k+1} |\varphi^{k+1} D^\alpha f|_0^2 + C(\varepsilon) \sum_{|\alpha|=k-1} |\varphi^{k-1} D^\alpha f|_0^2.$$

□

Now

$$(\varphi^k D^\beta f, \varphi^k D^\beta f) = (D^\beta f, \varphi^{2k} D^\beta f)$$

Let $\beta = \gamma + e$, where $|e| = 1$. Then

$$\begin{aligned}(\varphi^k D^\beta f, \varphi^k D^\beta f) &= -(D^\gamma f, D^e.\varphi^{2k} D^\beta f)\\ &= -(D^\gamma f, 2k\,\varphi^{2k-1} D^e\varphi.D^\beta f) - (D^\gamma f, \varphi^{2k} D^{\beta+e} f).\\ &= -(\varphi^{k-1} D^\gamma f, 2k\,\varphi^k D^e\varphi.D^\beta f)\\ &\qquad - (\varphi^{k-1} D^\gamma f, \varphi^{k+1} D^{\beta+e} f).\end{aligned}$$

By Schwarz's inequality, this gives, **199**

$$|\varphi^k D^\beta f|_0^2 \leq |\varphi^{k-1} D^\gamma f|_0.C_1|\varphi^k D^\beta f|_0 + |\varphi^{k-1} D^\gamma f|_0.|\varphi^{k+1} D^{\beta+e} f|_0$$

for a constant C_1 depending on φ. Now

$$|\varphi^{k-1} D^\gamma f|_0.|\varphi^k D^\beta f|_0 \leq \frac{1}{2}\left\{\frac{\varepsilon}{C_1}|\varphi^k D^\beta f|_0^2 + \frac{C_1}{\varepsilon}|\varphi^{k-1} D^\gamma f|_0^2\right\}$$

and $|\varphi^{k-1} D^\gamma f|_0.|\varphi^{k+1} D^{\beta+e} f|_0 \leq \frac{1}{2}\left\{\varepsilon|\varphi^{k+1} D^{\beta+e} f|_0^2 + \frac{1}{\varepsilon}|\varphi^{k-1} D^\gamma f|_0^2\right\}.$

Hence

$$(1-\varepsilon)|\varphi^k D^\beta f|_0^2 \leq \varepsilon|\varphi^{k+1} D^{\beta+e} f|_0^2 + C(\varepsilon).|\varphi^{k-1} D^\gamma f|_0^2.$$

This proves our assertion.

Theorem 2 (Friedrichs' inequality)**.** *Let Ω be a bounded open set in $\mathbb{R}^n$ and L, an elliptic differential operator on Ω of order m, given by $L = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$. Let r be an integer, $r \geq 0$.*

200 (I) If a'_α s are constant, there exists a constant C such that for $f \in C^\infty(\Omega)$,

$$|f|_{m+r} \leq C\, |Lf|_r.$$

(II) For any $x_0 \in \Omega$, there exists a neighbourhood U of x_0 and a constant C_1 such that for any C^∞ f with $\operatorname{supp} . f \subset U$, we have

$$|f|_{m+r} \leq C_1\, |Lf|_r.$$

(III) There exists a constant C_2 such that for any $f \in C^\infty(\Omega), \operatorname{supp} . f \subset \Omega$,

$$|f|_{m+r} \leq C_2\{|Lf|_r + |f|_0\}.$$

(IV) If Ω'', Ω' are open subsets of Ω, $\Omega'' \subset\subset \Omega' \subset\subset \Omega$, then there exists a constant C_3 such that for $f \in C^\infty(\Omega)$,

$$|f|^{\Omega''}_{m+r} \leq C_3\{|Lf|^{\Omega'}_r + |f|^{\Omega'}_0\}.$$

The proofs of this theorem are completely parallel to those of Gårding's inequality, but in this form do not follow at once from Theorem 1. In the case $r = 0$ (III) and (I) and (II) with the inequality $|f|_m \leq C|Lf|_0$ replaced by $|f|_m \leq C\{|Lf|_0 + |f|_0\}$ follows at once from Gårding's inequality applied to $\triangle = (-1)^m L^* L$.

Proof. (I) Since L is elliptic, there exists a constant $B_1 > 0$ such that

$$|p(\xi).v| \geq B_1|\xi|^m|v| \quad \text{for } \xi \in \mathbb{R}^n \text{ and } v \in \mathbb{C}^q. \tag{6.2}$$

□

201 Let $L_1 = \sum_{|\alpha|=m} a_\alpha D^\alpha$ and $L_2 = \sum_{|\alpha|<m} a_\alpha D^\alpha$. Then there exists a constant M depending on L_2 such that

$$|\widehat{L_2 f}(\xi)|^2 \leq M.(1 + |\xi|^2)^{m-1}\, |\hat{f}(\xi)|^2. \tag{6.3}$$

Also there exists a constant A such that
$\frac{B_1^2}{2}|\xi|^{2m} - M(1+|\xi|^2)^{m-1} \geq B_2\,(1+|\xi|^2)^m$ for $|\xi| > A$ where B_2 is a suitable constant > 0. By §5, Proposition 3, we have

$$\begin{aligned} |Lf|_r^2 &\geq c' \int_{\mathbb{R}^n} (1+|\xi|^2)^r |\widehat{Lf}(\xi)|^2 d\,\xi \\ &\geq c' \int_{|\xi|>A} (1+|\xi|^2)^r |\widehat{L_1 f}(\xi) + \widehat{L_2 f}(\xi)|^2 d\,\xi \\ &\geq c' \int_{|\xi|>A} (1+|\xi|^2)^r \{|\frac{1}{2} p(\xi)\hat{f}(\xi)|^2 - M(1+|\xi|^2)^{m-1}|\hat{f}(\xi)|^2\}\, d\,\xi \\ &\qquad (\text{ since } |a+b|^2 \geq |\frac{1}{2}|a|^2 - |b|^2 \text{ for } a,b \in \mathbb{C}) \\ &\geq c' \int_{|\xi|>A} (1+|\xi|^2)^r \{\frac{1}{2} B_1 |\xi|^{2m} - M(1+|\xi|^2)^{m-1}\}|\hat{f}|(\xi)|^2\, d\,\xi \end{aligned}$$

(by (6.2) and — (6.3)).

Now by the choice of A,

$$|Lf|_r^2 \geq B_2 c' \int_{|\xi>A} (1+|\xi|^2)^{m+r} |\hat{f}(\xi)|^2 d\xi$$

and hence by Proposition 2, there exists a constant c such that

$$|Lf|_r \geq c|f|_{m+r}.$$

II Let $L = L_0 + L_1$ where L_0 and L_1 are differential operators of orders $\leq m$ such that L_0 has constant coefficients of L_1 vanish at x_0. Since the coefficients of L_1 are continuous functions on Ω, there exists a neighbourhood U of x_0 such that for any C^∞ f with $\operatorname{supp} . f \subset U$, $|L_1 f|_r^2 \leq \frac{\varepsilon}{2}|f|_{m+r}^2$, ε depending on U and tending to zero as $U \to \{x_0\}$[1]. Then **202**

$$|L_0 f + L_1 f|_r^2 \geq \frac{1}{2}|L_0 f|_r^2 - |L_1\ f|_r^2.$$

[1]If $r > 0$, this involves integration by parts.

By (I) there exists a constant B such that

$$|L_0\ f|_r^2 \geq B|f|_{m+r}^2.$$

Hence

$$|Lf|_r^2 \geq (\frac{B}{2} - \varepsilon)|f|_{m+r}^2.$$

III Because of (II), there is a finite covering $\{U_1, \ldots, U_h\}$ of $\bar{\Omega}'$ such that if supp $f \subset U_i$ for some i, then

$$|Lf|_r \geq C|f|_{m+r}.$$

Let supp $f \subset \Omega'$ and $\varphi_1, \ldots, \varphi_h$ be C^∞ functions, $0 \leq \varphi_i \leq 1$, supp $\varphi_i \subset U_i$, $\sum \varphi_i^2 = 1$ on Ω'. Now, since supp $f \subset \Omega'$, we have $\Big| |f|_{m+r}^2 - \sum_i |\varphi_i\ f|_{m+r}^2 \Big| \leq C'|f|_{m+r-1}$, and $\Big| |Lf|_r^2 - \sum\limits_i |L(\varphi_i\ f)|_r^2 \Big| \leq C'|f|_{m+r-1}$, so that, since

$$|L(\varphi_i\ f)|_r \geq C|\varphi_i\ f|_{m+r}, \text{ and } |f|_{m+r-1} \leq \varepsilon|f|_{m+r} + C(\varepsilon)|f|_0,$$

203 we obtain the required inequality.

IV Define a C^∞ function φ on Ω such that $\varphi(x) = 1$ for x in a neighbourhood of $\bar{\Omega}''$ and $\varphi(x) = 0$ for $x \notin \Omega'$. Then if $f \in C^\infty(\Omega)$, $m' = m + r$, we have

$$D^\alpha(\varphi^{m'}\ f) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (D^\beta\ \varphi^{m'})\ D^{\alpha-\beta}\ f.$$

Now, $D^\beta\ \varphi^{m'}(x) = C_\beta(x)\varphi^{m'-|\beta|}(x)$, where $C_\beta(x)$ is a C^∞ function and $|C_\beta(x)| \leq A_1$ for some constant A_1. Then

$$D^\alpha(\varphi^{m+r}\ f) = \varphi^{m+r}\ D^\alpha\ f + \sum_{\substack{\beta \leq \alpha \\ \beta \neq 0}} C'_\beta\ \varphi^{m+r-|\beta|}\ D^{\alpha-\beta}\ f$$

where $C'_\beta = \binom{\alpha}{\beta} C_\beta$.

Now squaring both sides, using Schwarz's inequality and summing over α, $|\alpha| \leq m + r$, we obtain,

$$(6.4) \qquad \Big| |\varphi^{m+r} f|^2_{m+r} - \sum_{|\alpha|\leq m+r} |\varphi^{m+r} D^{\alpha} f|_0^2 \Big| \leq A_2 \sum_{|\beta|\leq m+r} |\varphi^{|\beta|}.D^{\beta} f|_0^2$$

for a suitable constant A_2.

By Part III above, we have, since supp $.\varphi^{m'} f \subset \Omega'$,

$$(6.5) \qquad |\varphi^{m+r}|^2_{m+r} \leq C\{|L(\varphi^{m+r} f)|_r^2 + (|f|_0^{\Omega'})^2\}.$$

By a repeated application of Lemma 1 to $|\varphi^{|\beta|} D^{\beta} f|_0^2$ for $|\beta| < m + r$, we have

$$(6.6) \qquad \sum_{|\beta|<m+r} |\varphi^{|\beta|} D^{\beta} f|_0^2 \leq \varepsilon \sum_{|\beta|=m+r} |\varphi^{m+r} D^{\beta} f|_0^2 + C(\varepsilon)(|f|_0^{\Omega'})^2.$$

It follows from (6.4), (6.5) and (6.6) that **204**

$$\sum_{|\alpha|\leq m+r} |\varphi^{m+r} D^{\alpha} f|_0^2 \leq C\left\{|\varphi^{m+r} Lf|_r^2 + \left(|f|_0^{\Omega'}\right)^2\right\}$$
$$+\varepsilon \sum_{|\alpha|\leq m+r} |\varphi^{m+r} D^{\alpha} f|_0^2 + C(\varepsilon)\left(|f|_0^{\Omega'}\right)^2$$

so that
$$\sum_{|\alpha|\leq m+r} |\varphi^{m+r} D^{\alpha} f|_0^2 \leq C_2\left\{\varphi^{m+r} Lf|_r^2 + \left(|f|_0^{\Omega'}\right)^2\right\}$$

for a suitable constant C_2.

Since $\varphi(x) = 1$ for $x \in \Omega''$ and supp $.\varphi \subset \bar{\Omega}'$, the theorem follows.

Remark. As in the remark following Gårding's inequality, parts (I) and (II) of Theorem 2 can be proved for homogeneous elliptic operators L, without appealing to Proposition 2; the reasoning is the same.

The proofs given in this section are essentially those of Garding [11] and Friedrichs [10].

7 Elliptic operators with C^∞ coefficients: the regularity theorem

Let Ω be an open set in $\mathbb{R}^n$.

Definition. If L is an elliptic differential operator $L : C_0^{\infty,q}, (\Omega) \to$
205 $C_0^{\infty,p}(\Omega)$ and $f \in H_0(\Omega)$, we define (Lf) as a linear functional on $C_0^{\infty,p}$ (Ω), by

$$(Lf)(\varphi) = (f, L^*\varphi) \text{ for } \varphi \in C_0^{\infty,p}(\Omega).$$

Further, (Lf) is said to be in $H_0(\Omega)$ (or $H_m(\Omega)$ or to be strongly differentiable) if there exists $g \in H_0(\Omega)$ (or $H_m(\Omega)$ or which is strongly differentiable) such that $(Lf)(\varphi) = (g, \varphi) = (f, L^*\varphi)$ for any $\varphi \in C_0^{\infty,p}(\Omega)$.

In what follows upto the regularity theorem, L denotes a uniformly strongly elliptic operator of order $2m$ with C^∞ coefficients, L: $C_0^{\infty,q}$ $(\Omega) \to C_0^{\infty,q}(\Omega)$ and $L = \sum\limits_{i=1}^{r} B_i^* A_i$, A_i, B_i being differential operators of orders $\leq m$. For φ, $\psi \in H_m(\Omega)$, we define $Q(\varphi, \psi)$ by $Q(\varphi, \psi) = \sum\limits_{i=1}^{r} (A_i\varphi, B_i\psi)$. Let $\Omega' \subset\subset \Omega$ and $h \in \mathbb{R}, h \neq 0$ be so small that $(x_1, \ldots, x_n) \in \Omega'$ implies $(x_1 + h, x_2, \ldots, x_n) \in \Omega$. We write $(x + h)$ for $(x_1 + h, x_2, \ldots, x_n)$. For $g \in H_m(\Omega)$, we define g^h: $\Omega' \to \mathbb{C}^q$ by $g^h(x) = \dfrac{g(x+h) - g(x)}{h}$.

Lemma 1. *(a) If $\eta \in C_0^{\infty,1}(\Omega)$, there exists a constant C such that for any $f \in H_0(\Omega)$, and h small enough, we have*

$$|(\eta\ f)^h - \eta\ f^h|_0 \leq C|f|_0.$$

(b) For $f \in H_m^0(\Omega)$, there is a constant $C > 0$ such that $|f^h|_{m-1} \leq C|f|_m$.

Proof.

$$(\eta f)^h(x) - (\eta\ f^h)(x) = \frac{\eta(x+h) - \eta(x)}{h} f(x+h)$$
$$= \eta^h(x)\ f(x+h).$$

□

206 This proves (a). The proof of (b), for $f \in C_0^{\infty,q}$, follows at once from

$$f^h(x) = \int_0^1 \frac{\partial f}{\partial x_1}(x_1 + th, x_2, \ldots, x_n)\, dt,$$

and, for $f \in \overset{o}{H}_m(\Omega)$, by passage to the closure.

Proposition 1. *If $f \in \overset{o}{H}_m(\Omega)$ and has compact support, and there exists a constant C such that for any $\varphi \in C_0^{\infty,q}(\Omega)$, $|Q(f,\varphi)| \leq C|\varphi|_{m-1}$, then $f \in H_{m+1}(\Omega)$.*

Proof. We have

$$Q(f^h, \varphi) = \sum_i (A_i\ f^h, B_i\ \varphi).$$

Since $D^\alpha\ f^h = (D^\alpha\ f)^h$, it follows from Lemma 1 (a) that $|A_i\ f^h - (A_i\ f)^h|_0 \leq C_1|f|_m$ for a constant C_1 depending on L. Hence

$$Q(f^h, \varphi) = \sum ((A_i f)^h, B_i\varphi) + 0(|\varphi|_m.|f|_m).$$

□

Now

$$\begin{aligned}((A_i\ f)^h, B_i\varphi) &= -(A_i f, (B_i\varphi)^{-h}) \\ &= -(A_i f, B_i\varphi^{-h}) + 0(|\varphi|_m) \text{ (Lemma 1 (a))}.\end{aligned}$$

Hence

$$Q(f^h, \varphi) = -Q(f, \varphi^{-h}) + 0(|\varphi|_m);$$

by hypothesis,

$$|Q(f, \varphi^{-h}) \leq C|\varphi^{-h}|_{m-1} \leq C'|\varphi|_m \text{(Lemma 1 (b))}.$$

Hence there exists a constant C_2 such that 207

$$|Q(f^h, \varphi)| \leq C_2|\varphi|_m.$$

This holds for any $\varphi \in C_0^{\infty,q}(\Omega)$. Since f^h has compact support $\subset \Omega$, choose a sequence $\{\varphi_\nu\}$ of functions in $C_0^{\infty,q}(\Omega)$, $\varphi_\nu \to f^h$ in H_m. Then, we have,

$$|Q(f^h, \varphi_\nu)| \leq C_2 |\varphi_\nu|_m$$

and passing to the limit,

$$|Q(f^h, f^h)| \leq C_2 |f^h|_m. \tag{7.1}$$

Now by Garding's inequality, there exists a constant B such that

$$|\varphi_\nu|_m^2 \leq B(-1)^m \operatorname{Re}(L\varphi_\nu, \varphi_\nu)| + |\varphi_\nu|_0^2,$$

hence

$$|\varphi_\nu|_m^2 \leq B|Q(\varphi_\nu, \varphi_\nu)| + |\varphi_\nu|_0^2;$$

taking limits as $\nu \to \infty$ and using (7.1), this gives

$$|f^h|_m^2 \leq B\, C_2 |f^h|_m + |f|_0^2.$$

Hence there exists a constant M such that

$$|f^h|_m \leq M.$$

Consider f^h for sufficiently small h. This is a bounded set in the Hilbert space $H_m(\Omega)$; hence there is a sequence $\{h_\nu\}$, $h_\nu \to 0$ such that f_ν^h is weakly convergent to a function g in $H_m(\Omega)$. Also $f^h \to \dfrac{\partial f}{\partial x_1}$ in
208 $H_0(\Omega)$. This implies that $\dfrac{\partial f}{\partial x_1} \in H_m(\Omega)$. Similarly we can show that $\dfrac{\partial f}{\partial x_i}$, $i \geq 2$ are in $H_m(\Omega)$. Hence it follows from Proposition 1, §5, that f is $(m+1)$ times strongly differentiable, since f has compact support, $f \in H_{m+1}(\Omega)$.

Proposition 2. *Suppose $f \in H_m(\Omega)$ and for a given r, $0 < r \leq m$, there exists a constant C such that $|Q(f, \varphi)| \leq C|\varphi|_{m-r}$ for any $\varphi \in C_0^{\infty,q}(\Omega)$; then f is $(m+r)$ times strongly differentiable.*

Proof. We shall prove the proposition by induction. □

Case $r = 1$**.** Suppose that

$$|Q(f,\varphi)| \leq C|\varphi|_{m-1}.$$

We assert that for any $\eta \in C_0^{\infty,1}$, there is a constant C' such that

$$|Q(\eta f,\varphi)| \leq C'|\varphi|_{m-1};$$

the case $r = 1$ of Proposition 2 then follows from Proposition 1. To prove the existence of C', we note that

$$(A_i\ \eta\ f, B_i\varphi) = (\eta\ A_i\ f, B_i\varphi) + (A' f, B_i\varphi)$$

[where A' has order $\leq m-1$ and has coefficients with compact support]

$$= (A_i f, B_i\eta\varphi) + (A' f, B_i\varphi) + (A_i f, B'\varphi)$$

where B' has order $\leq m-1$. Clearly

$$|(A_i f, B'\varphi)| \leq C''|\varphi|_{m-1}.$$

Now we can write $B_i = \sum\limits_k D_k\ B_k''$ where B_k'' have order $\leq m-1$, D_k has ordered.

Since $f \in H_m(\Omega)$, we then have, 209

$$|A' f, B_i\varphi)| = |\sum(D_k^* A' f, B_k''\varphi)| \leq C''|\varphi|_{m-1}.$$

Hence

$$Q(\eta f,\varphi) = Q(f,\eta\varphi) + 0(|\varphi|_{m-1})$$

and the result follows.

Let us now suppose that the result is proved for $r = k-1 > 0$; then f is $(m+k-1)$ times strongly differentiable; by restricting ourselves to $\Omega' \subset\subset \Omega$, we may then suppose that $f \in H_{m+k-1}(\Omega)$. Let $|\beta| = 1$. Now since $f \in H_{m+k-1}(\Omega)$, we have

$$\begin{aligned} Q(D^\beta f,\varphi) &= \sum(A_i D^\beta f, B_i\varphi) \\ &= \sum(D^\beta A_i\ f, B_i\varphi) + \sum(A_i'\ f, B_i\varphi) \end{aligned}$$

where the A_i's are differential operators of order $\leq m$. Hence

$$\begin{aligned} Q(D^\beta f, \varphi) &= -\sum (A_i\ f, D^\beta B_i \varphi) + \sum (A_i' f, B_i \varphi) \\ &= -\sum (A_i f, B_i D^\beta \varphi) + \sum (A_i'' f, B_i'' \varphi) \end{aligned}$$

where A_i'' and B_i'' are differential operators such that ord. $A_i'' \leq m+k-1$ ord. $B_i'' \leq m-k+1$ [for this last equality, write $B_i\, D^\beta - D^\beta\, B_i$ as a linear combination $\sum L_j\, L_j'$ where ord. $L_j \leq k-1$, ord. $L_j' \leq m-k+1$ and use the fact that $f \in H_{m+k-1}(\Omega)$, to shift L_j to A_i] . Since

$$|Q(f, \varphi)| \leq C|\varphi|_{m-k},$$

210 this gives

$$\left|\sum (A_i D^\beta f, B_i \varphi)\right| \leq C|D^\beta \varphi|_{m-k} + C_1|\varphi|_{m-k+1}$$

for some constant C_1.

Hence

$$|Q(D^\beta\ f, \varphi)| \leq C_2|\varphi|_{m-k+1}$$

and by the induction hypothesis, $D^\beta\ f$ is $(m+k-1)$ times strongly differentiable. By Proposition 1 of § 5 this implies that f is $(m+k)$ times strongly differentiable.

Proposition 3. *If $f \in H_m(\Omega)$ and Lf is r times strongly differentiable, them f is $(2m+r)$ times strongly differentiable.*

Proof. We shall prove the proposition by induction; by restricting ourselves to $\Omega' \subset\subset \Omega$, we may suppose that $Lf \in H_r(\Omega)$. Let $Lf = g \in H_0(\Omega)$; then for $\varphi \in C_0^{\infty, q}(\Omega)$, by definition,

$$Q(f, \varphi) = (f, L^* \varphi) = (g, \varphi);$$

so that $|Q(f, \varphi)| \leq C|\varphi|_0$ for some constant $C > 0$.

□

Now using Proposition 2 with $r = m$, we conclude that f is $2m$ times strongly differentiable. Let us suppose that proposition is true for $r = k$.

Then if $Lf \in H_{k+1}(\Omega)$, by the induction hypothesis, f is $(2m+k)$ times strongly differentiable. For $|\beta| \leq k+1$,

$L(D^\beta) - D^\beta L = \triangle_\beta$ is a differential operator of order $\leq 2m+k$, and since f is $(2m+k)$ times strongly differentiable, we have

$$LD^\beta f = D^\beta(Lf) + \triangle_\beta f \text{ and } D^\beta(Lf) + \triangle_\beta f \in H_0(\Omega').$$

Therefore by what we have proved above $(D^\beta f)$ is $2m$ times strongly differentiable. This, together with Proposition 1, §5 implies that f is $(2m+k+1)$ times strongly differentiable. 211

Proposition 4. *Let* $\triangle$ *denote the operator* $\sum\limits_{i=1}^{n} \dfrac{\partial^2}{\partial x_i^2}$ *acting on q-tuples of* C^∞ *functions. If* $\varphi \in H_0(\mathbb{R}^n)$ *and* $r \geq 1$ *is an integer, there exists* $\varphi' \in H_{2r}(\mathbb{R}^n)$ *such that* $(I-\triangle)^r \varphi' = \varphi$*, I being the identity.*

Proof. By Plancherel's theorem $\hat{\varphi} \in H_0(\mathbb{R}^n)$. Define φ' by

$$\hat{\varphi}'(\xi) = \frac{\hat{\varphi}(\xi)}{(1+\xi_1^2+\cdots+\xi_n^2)^r} \in H_0(\mathbb{R}^n).$$

□

Then

$$\int\limits_{\mathbb{R}^n} (1+|\xi|^2)^{2r} |\hat{\varphi}'(\xi)|^2 d\xi = |\hat{\varphi}|_0 < \infty.$$

Hence by Proposition 4, §5, $\varphi' \in H_{2r}$. Moreover using the fact that $D^{\hat{\alpha}} f(\xi) = i^{|\alpha|}\xi^\alpha \hat{f}(\xi)$ and the inversion formula we see immediately that

$$(I-\triangle)^r \varphi'(\xi) = \varphi(\xi).$$

In the next theorem, L need no longer have the properties stated at the beginning.

Theorem (The regularity theorem). *If L is an elliptic differential operator of order m with* C^∞ *coefficients, L:* $C_0^{\infty,q}(\Omega) \to C_0^{\infty,p}(\Omega)$ *and for an* $f \in H_0(\Omega)$*,* $Lf = g$ *is in* $H_0(\Omega)$*, and if* $g \in C^\infty$*, so is* f*.*

Proof. Let $A = (-1)^m L^* \circ L$; by restricting ourselves to $\Omega' \subset\subset \Omega$, we **212**
may suppose that A is uniformly strongly elliptic and A: $C_0^{\infty,q} \to C_0^{\infty,q}$. Then we shall prove that for any $f \in H_0(\Omega)$, if $Af \in C^\infty$, then $f \in C^\infty$. Since $(-1)^m L^* oLf = (-1)^m L^*(Lf) \in C^\infty$, this will imply the theorem. We may extend f to $\mathbb{R}^n$ by $f(x) = 0$ for $x \notin \Omega$. Let r be a positive integer, $r \geq m$. Then by Proposition 4, there exists a q-tuple $f^{(r)} \in H_{2r}(\mathbb{R}^n)$ such that if $\triangle = \sum_{i=1}^{n} \dfrac{\partial^2}{\partial x_i^2}$, $(I - \triangle)\ f^{(r)} = f$. Consider $B = (-1)^r A(I - \triangle)^r$. Then B is uniformly strongly elliptic and is of order $2(m + r)$; further

$$B\ f^{(r)} = (-1)^r A\ f \in C^\infty.$$

□

Since $r \geq m$, $f^{(r)} \in H_{m+r}$ and hence, by Proposition 3, $f^{(r)}$ is $2m + 2r + s$ strongly differentiable for any $s > 0$. Hence f is $2m + s$ times strongly differentiable for any $s > 0$.

It follows from the corollary to Sobolev's lemma that f has continuous derivatives of order $\leq 2m + s - n$ for any s, and hence $f \in C^\infty$.

Remark. We have in fact proved the following proposition in the case when L is strongly elliptic of even order.

Proposition 5. *Let L be an elliptic operator of order m, and* $f \in H_0(\Omega)$*. If* Lf *is r times strongly differentiable, r being an integer* ≥ 0*, then* f *is* $r + m$ *times strongly differentiable.*

The above proposition, for arbitrary L can be reduced, to the case of strongly elliptic operators of even order by considering $\triangle_1 = L^* L$, if $r \geq m$ (= order of L). The general case requires the use of the space
213 $H_{-k}(\Omega)$ which is the dual of $\overset{o}{H}_k(\Omega), k > 0$. We do not enter into the details.

The proof of the regularity theorem given here is a somewhat simplified version of that of Nirenberg [32]. There are now several other proofs available. The oldest, which operates with "fundamental solutions" was proposed by L. Schwartz [39]; very strong theorems that can be obtained by this method will be found in Hörmander [17]. The first proof using only 'a priori' estimates is due to Friedrichs [10] (who proves, however, only a slightly weaker assertion). Other proofs are due to F. John [19] and P. Lax [24] ; that of Lax is both brief and elegant. Schwartz has recently given another very elegant and very general proof, which operates, however, with singular integral operators; see [41]. There is a vast literature that has sprung up around this theorem and its generalizations (particularly the so called "regularity at the boundary"). References may be found in [1].

8 Elliptic operators with analytic coefficients

Lemma 1. *If K is a compact set in $\mathbb{R}^n$ and if $K_\varepsilon = \{x|d(x,K) < \varepsilon\}$, there exists $\varphi_\varepsilon \in C_0^{\infty,1}(\mathbb{R}^n)$ such that $0 \leq \varphi_\varepsilon \leq 1$, $\varphi_\varepsilon(x) = 1$ for $x \epsilon K$, $\operatorname{supp}.\varphi_\varepsilon \subset K_{2\varepsilon}$ and $|D^\alpha \varphi_\varepsilon| \leq \dfrac{C_\alpha}{\varepsilon^{|\alpha|}}$ for some constants C_α independent of ε and K.*

Proof. Let φ be a C^∞ function such that $\int\limits_{\mathbb{R}^n} \varphi(x)dx = 1$ and $\varphi \geq 0$, $\operatorname{supp}.\varphi \subset \{x|\|x\| < 1\}$. Let $\chi_\varepsilon(x) = 1$ for $x \in K_\varepsilon$ and $\chi_\varepsilon(x) = 0$ for $x \notin K_\varepsilon$. ◻

Let $\varphi_\varepsilon(x) = \varepsilon^{-n} \int \varphi\left(\dfrac{x-y}{\varepsilon}\right). \chi_\varepsilon(y)\, dy$.

Then clearly $\varphi(x) = 1$ for $x \in K$ and $\operatorname{supp}.\varphi_\varepsilon \subset K_{2\varepsilon}$. Also **214**

$$D^\alpha \varphi_\varepsilon(x) = \varepsilon^{-|\alpha|}\varepsilon^{-n} \int D^\alpha \varphi\left(\frac{x-y}{\varepsilon}\right)\chi_\varepsilon(y)dy.$$

Hence $|D^\alpha \varphi_\varepsilon(x)| \leq \varepsilon^{-|\alpha|} \int |D^\alpha \varphi(y)|dy$, which proves the lemma.

Notation. In what follows, R, ρ are real numbers, $0 < \rho < \min\{1, R\}$, and $M_\rho(f)$ is given by

$$[M_\rho(f)]^2 = \int\limits_{|x|<R-\rho} |f(x)|^2\, dx.$$

Proposition 1. *Let L be an elliptic operator of order m, with C^∞ coefficients, on $\{x \big| |x| < R+\delta\}$. Then there exists a constant C (independent of f, ρ, ρ_1) such that for $\rho, \rho_1 > 0$ and $f \in C^{\infty,q}$, we have*

$$\rho^m M_{\rho+\rho_1}(D^\alpha f) \leq C\left\{\rho^m M\rho_1(Lf) + \sum_{|\beta|<m} \rho^{|\beta|} M\rho_1(D^\beta f)\right\}.$$

for $|\alpha| = m$.

Proof. By Lemma 1 above there exists a C^∞ function φ on $\mathbb{R}^n$ such that $\varphi(x) = 1$ for $|x| < R-\rho-\rho_1$, $0 \leq \varphi(x) \leq 1$, supp $.\varphi \subset \{x \big| |x| < R-\rho_1\}$ and $|D^\alpha\varphi| \leq \dfrac{C_\alpha}{\rho^{|\alpha|}}$ where C_α are constants independent of ρ and ρ_1.

By Friedrichs' inequality, (Part III), there exists a constant C_1, independent of f and ρ_1 such that

$$|D^\alpha \varphi f|_0 \leq C_1\{|L(\varphi f)|_0 + |\varphi f|_0\}.$$

□

215 Let

$$L = \sum_{|\lambda|\leq m} a_\lambda D^\lambda.$$

Then

$$L(\varphi f) = \phi L f + \sum_{\substack{\beta<\lambda\\ |\lambda|\leq m}} a_\lambda \binom{\lambda}{\beta} D^{\lambda-\beta}(\varphi) D^\beta f.$$

Since a_λ are C^∞ in $|x| < R+\delta$ and $|D^\alpha\varphi| \leq \dfrac{C_\alpha}{\rho|\alpha|}$, there exist constants $C_{\lambda,\beta}$, independent of ρ such that

$$|a_\lambda \binom{\lambda}{\beta} D^{\lambda-\beta}\varphi| \leq \frac{C_{\lambda,\beta}}{\rho^{|\lambda-\beta|}} \text{ for } |x| \leq R.$$

Hence

$$|D^\alpha\varphi f|_0 \leq C_2\left\{M\rho_1(Lf) + \sum_{|\beta|<m} \beta^{-m+|\beta|} M\rho_1(D^\beta f)\right\}$$

for a constant C_2 and this proves the proposition.

Proposition 2. *Let Ω be an open set in $\mathbb{R}^n$, $0 \in \Omega$, and let L: $C^{\infty,q}(\Omega) \to C^{\infty,q}(\Omega)$ be an elliptic operator of order m with coefficients which are analytic in Ω (note that $L = \sum\limits_{|\lambda| \leq m} a_\lambda D^\lambda$ where a_λ are $q \times q$ matrices of analytic functions). Then if $R > 0$ and $R_1 > R$ are sufficiently small there exists a constant $A > 0$ such that for all ρ, $0 < \rho < \min(1, R)$, and $f \in C^{\infty,q}(\Omega)$, we have, for $|\alpha| \leq mr$, $r = 1, 2, \ldots,$*

$$\rho^{|\alpha|} M_{|\alpha|\rho}(D^\alpha f) \leq A^{|\alpha|+1} \left\{ \sum_{s=1}^{r} |L^s f|_0^{R_1} \rho^{(s-1)m} + |f|_0^{R_1} \right\}; \tag{8.1}$$

here $|f|_0^{R_1} = \int\limits_{|x|<R_1} |f(x)|^2 dx$; L^s denotes the iterate of L, s times.

Proof. We choose R_1 so small that the a_λ have holomorphic extensions to the polycylinder $|z| \leq R_1$. Let $C_1 = \sum_{|\lambda| \leq m} \sup\limits_{|z| \leq R_1} .|a_\lambda(z)|$; then we have (Cauchy's inequality) 216

$$\sum_{|\lambda| \leq m} |D^\alpha a_\lambda(x)| \leq C_1 \alpha! \rho^{-|\alpha|} \text{ for } |x| \leq R - \rho. \tag{8.2}$$

□

Let

$$\sum_{s=1}^{r} |L^s f|_0^{R_1} \rho^{(s-1)m} + |f|_0^{R_1} = S_r(f).$$

We first remark that

$$\rho^m S_r(Lf) \leq S_{r+1}(f). \tag{8.3}$$

We shall prove the proposition by induction. For $r = 1$, i. e. for $|\alpha| \leq m$, we apply Friedrichs' inequality, Part IV. There-exists a constants C_2 such that

$$M_0(D^\alpha f) \leq C_2 \{|Lf|_0^{R_1} + |f|_0^{R_1}\} \text{ for } |\alpha| \leq m.$$

Hence

$$\rho^{|\alpha|} M_{|\alpha|\rho}(D^\alpha f) \leq C_2 \{|Lf|_0^{R_1} + |f|_0^{R_1}\},$$

so that (8.1) is true for $r = 1$ if $A \geq C_2$. Now let $mr < |\alpha| \leq m(r+1)$, $r \geq 1$, and assume that (8.1) is already proved for all β with $|\beta| < |\alpha|$.

Let $\alpha = \alpha_0 + \alpha'$ where $|\alpha_0| = m$. Then we have by Proposition 1 with $\rho_1 = (|\alpha| - 1)\rho$, (and α_0 in place of α)

$$\rho^{|\alpha|} M_{|\alpha|\rho}(D^\alpha f) \leq$$

$$C\left\{\rho^{|\alpha|} M_{(|\alpha|-1)\rho}(LD^{\alpha'} f) + \sum_{|\beta|<m} \rho^{|\beta|+|\alpha'|} M_{(|\alpha|-1)_\rho}\left(D^{\beta+\alpha'} f\right)\right\} \tag{8.4}$$

217 Further,

$$D^{\alpha'} Lf = LD^{\alpha'} f + \sum_{|\lambda|\leq m} \sum_{\gamma<\alpha'} \binom{\alpha'}{\gamma} D^{\alpha-\gamma} a_\lambda D^{\gamma+\lambda} f.$$

Now, for $|x| \leq R - mr\rho$, we have

$$|D^{\alpha'-\gamma} a_\lambda(x) \leq C_1(\alpha' - \gamma)!(\rho mr)^{-|\alpha'-\gamma|}.$$

and $\binom{\alpha'}{\gamma}\dfrac{(\alpha'-\gamma)!}{|\alpha'-\gamma|} \leq \left(\dfrac{|\alpha'|}{mr}\right)^{|\alpha'-\gamma|} \leq 1$ since $|\alpha'| = |\alpha| - m \leq mr$. Hence for $|x| \leq R - mr\rho$, a fortiori for $|x| \leq R - (|\alpha| - 1)\rho$, we have,

$$|D^{\alpha'} Lf - LD^{\alpha'} f| \leq C_1 \sum_{|\lambda|\leq m} \sum_{\gamma<\alpha'} \rho^{-|\alpha'-\gamma|} |D^{\gamma+\lambda} f|. \tag{8.5}$$

Hence, by (8.4) and (8.5), we have for $mr < |\alpha| \leq m(r+1)$,

(8.6)

$$\rho^{|\alpha|} M_{|\alpha|\rho}(D^\alpha f) \leq C\left\{\rho^{|x|} M_{|\alpha'|\rho}(D^{\alpha'} Lf) + \sum_{|\beta|<m} \rho^{|\beta|+\alpha'} M_{|\beta+\alpha'|\rho}(D^{\beta+\alpha'} f)\right.$$
$$\left. + C_1 \sum_{|\lambda|\leq m} \sum_{\gamma<\alpha'} \rho^{m+|\gamma|} M_{(m+|\gamma|)\rho}(D^{\gamma+\lambda} f)\right\}.$$

We can apply our induction hypothesis to each of the three terms in brackets on the right.

The first term is $\leq \rho^m A^{|\alpha'|+1} S_r(Lf) \leq A^{\alpha'|+1}(f)$.

The second $\leq \sum_{|\beta|<m} A^{|\beta+\alpha'|/+1} S_{r+1}(f)$ and similarly for the third. This gives

$$\rho^{|\alpha|} M_{|\alpha|\rho}(D^{\alpha} f) \leq A^{|\alpha|+1} S_{r+1}(f) \left\{ \frac{C}{A^m} + C \sum_{|\beta| \leq m} \frac{1}{A} + C_1' \sum_{\gamma<\alpha'} A^{-|\alpha'-\gamma|} \right\}.$$

Now 218

$$\sum_{\gamma<\alpha'} A^{-|\alpha'-\gamma|} = \sum_{0<\beta\leq\alpha'} A^{-|\beta|} \leq A^{-1} \sum_{|\beta|\geq 0} A^{-|\beta|}$$

which clearly $\to 0$ as $A \to \infty$. Hence we can choose $A \geq C_2$ so large that

$$\frac{C}{A^m} + C \sum_{|\beta|\leq m} \frac{1}{A} + C_1' \sum_{\gamma<\alpha'} A^{-|\alpha'-\gamma|} < 1,$$

which gives us (8.1).

Theorem 1 (T. Kotake -M.S. Narasimhan)**.** *Let L: $C^{\infty,q}(\Omega) \to C^{\infty,q}(\Omega)$ be an elliptic operator of order m with analytic coefficients. If $f \in C^{\infty,q}(\Omega)$ and for any $\Omega' \subset\subset \Omega$, there exists a constant $M > 0$, such that*

$$|L^r f|_0^{\Omega'} \leq M^{r+1}(rm)!,$$

then f is analytic in Ω.

Proof. We may suppose that $0 \in \Omega$; it suffices moreover to show that f is analytic in a neighbourhood of 0. We choose R_1 such that Proposition 1 is true; we the have

$$|L^r f|_\circ^{R_1} \leq M^{r+1}(rm)!$$

so that

$$S_r(f) \leq \sum_{s=1}^{r} \rho^{(s-1)m} M^{s+1}(sm)! + M.$$

If $(r-1)m < |\alpha| \leq$ rm, we choose $\rho = \dfrac{c}{|\alpha|}$, where c is small. Since, 219
then $(sm)!\rho^{(s-1)m} \leq (rm)^{2m}$ for $s \leq r$, we conclude that $S_r(f) \leq B_1^{r+1}$ for a suitable constant B_1. By Proposition 2, this implies that $|f|_k^{R-c} \leq$

$B_2^{k+1}k^k$; if K is a compact subset of the set $|x| < R - c$, it follows from (the weak form of) Sobolev's lemma that

$$\sup_{x \in K} |D^\alpha f(x)| \leq B_3 B_2^{k+n+1}(k+n)^{k+n} \text{ if } |\alpha| = k.$$

□

Stirling's formula shows then that

$$\sup_{x \in K} |D^\alpha f(x)| \leq B_4^{k+1} k! \text{ if } |\alpha| = k.$$

so that f is analytic in $|x| < R - c$ by Chap I, §1.

Lemma 2. *Let L be any differential operator of order m, with coefficients which are holomorphic $q \times q$ matrices on $D = \{z \in \mathbb{C}^n \big| |z_i| < r_i \leq 1\}$ and let f be a bounded holomorphic map $D \to \mathbb{C}^q$. Then there exists a constant A such that*

$$|L^r f(z)| \leq \frac{(3A)^{r+1}(mr)!}{\prod(r_i - |z_i|)^{mr}} \textit{ for } z \in D.$$

Proof. We shall prove the lemma by induction. For $r = 0$, the lemma is trivial. Assume that it is true for $r = k - 1$. Then

$$|L^{k-1} f(z)| \leq \frac{(3A)^k \{m(k-1)\}!}{\prod(r_i - |z_i|)^{m(k-1)}} \text{ for } z \in D.$$

□

Let $\sum_{|\alpha| \leq m} |a_\alpha(z)| \leq A$, where $L = \sum a_\alpha D^\alpha$. We have, by Lemma 3, §3,

$$|D^\alpha L^{k-1} f(z)| \leq \frac{3(3A)^k (mk)!}{\Pi(r_i - |z_i|)^{m(k-1)+|\alpha|}}$$

220 and since $\sum_{|\alpha| \leq m} |a_\alpha(z)| \leq A$ on D, this implies that

$$|L^k f(z)| \leq \frac{(3A)^{k+1}(mk)!}{\Pi(r_i - |z_i|)^{mk}}.$$

Theorem 2 (Petrovsky)**.** *If L is an elliptic operator of order m, with analytic coefficients on Ω, and if Lf is analytic, then f is analytic.*

Proof. By replacing L by L^*L if necessary, we may suppose that L is an operator $C^{\infty,q}(\Omega) \to C^{\infty,q}(\Omega)$ with analytic coefficients. Let $L = \sum\limits_{|\alpha|\le m} a_\alpha D^\alpha$. We may assume that $\Omega = \{x \big| |x_i| < r_i\}$ and that a_λ and Lf extend to holomorphic functions on $D = \{z \in \mathbb{C} \big| |z_i| < r_i\}$. Let $g = Lf$. Then it follows from Lemma 2 that for any compact subset K of Ω, there exists a constant M such that

$$|L^r f|_0^K \le M^{r+1}(mr)!$$

Theorem 2 then follows from Theorem 1. ◻

Note: We indicate briefly how the proof of Theorem 1 simplifies in the special case needed for Theorem 2. We use inequalities (8.4) and (8.5). But now since $g = Lf$ is analytic, we apply the Cauchy inequalities to a holomorphic extension of g and conclude that

$$|D^\alpha Lf| \le \frac{C_3 \alpha!}{(mr\rho)^{|\alpha|}} \text{ in } |x| \le R - mr\rho$$

so that
$$\rho^{|\alpha'|} M_{(|\alpha|-1)\rho}(D^{\alpha'} Lf) \le C_4;$$

this leads easily to the estimate 221

$\rho^{|\alpha|} M_{|\alpha|\rho}(D^\alpha f) \le A^{|\alpha|+1}$ for all α, (A now depends on f) and the proof is completed as before. The point is that one does not need the somewhat complicated Part *IV* of Friedrichs' inequality. The main theorem of this section (Theorem 2) is a special case of results of Petrovsky [36] who considered also non-linear systems of differential equations. His proof is however very difficult. The main idea in the proof given here is contained in the paper of Morrey-Nirenberg [29]. The proof by Koteke-Narasimhan [23] of Theorem 1 involves more careful analysis although it is also based on the idea of Morrey-Nirenberg.

9 The finiteness theorem

Let V be an oriented C^∞ manifold, E, F, C^∞ vector bundles of ranks q, p respectively over V. Let L be a differential operator $L : C_0^\infty(V, E) \to$

$C^\infty(V, F)$. All coordinates systems considered will be assumed to be *positive*.

Definition. (1) The order of L at a point $a \in V$ is the largest integer m such that $L(F^m s)(a) \neq 0$ for some $f \in m_a^\infty$ and some section $s \in C_0^\infty(V, E)$.

(2) The order of L on V is defined to be $\max\limits_{a \in V}$ order of L at a.

(3) A differential operator L of order m is said to be elliptic if, for $a \in V$,
222 and every (real valued) $f \in m_a^\infty$ such that $(df)(a) \neq 0$, we have $L(f^m s)(a) \neq 0$ for every $s \in C_0^\infty(V, E)$ for which $s(a) \neq 0$.

Note that if $f \in m_0^\infty$ and $s(a) = 0$, then $L(f^m s)(a) = 0$. Further if $(df)(a) = 0$, $L(f^m s)(a) = 0$ for any s. Hence $L(f^m s)(a)$ defines a map (not linear) from $E_a \otimes T_a^*(V) \to F_a$; this gives rise to a C^∞ map $\sigma(L)$: $E \otimes T^*(V) \to F$ (which preserves fibres). This map is called the *symbol of L* (and replaces the characteristic polynomial which we considered earlier).

Remarks 1. (1) We shall prove that the definition (2) above is consistent with the definition (1) of §6. Let E, F be trivial and for $a \in V$, let U_a be a coordinate neighbourhood of a and let L be given by $L = \sum\limits_{|\alpha| \leq m_1} a_\alpha D^\alpha$, $a_{\alpha'} \not\equiv 0$ for some $|\alpha'| = m_1$. Then it is enough to show that the order of L on $U_a = m_1$. But this follows at once from

(i) $(D^\alpha f^m)(a) = 0$ for $|\alpha| < m$, if $f \in m_a^\infty$ and

(ii) $(D^\alpha f^m)(a) = (m!)(\dfrac{\partial f}{\partial x_1})^{\alpha_1}(a) \cdots (\dfrac{\partial f}{\partial x_n})^{\alpha_n}(a)$ for $|\alpha| = m$, if $f \in m_a^\infty$.

(2) If L has order m on V, with the same notation as in the remark (1),

$$L(f^m s)(a) = m! \sum_{|\alpha| = m} \xi^\alpha a_\alpha(a) S(a), \xi = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$$

Hence it follows that the definition (3) above and the definition (3) of §6 are consistent.

Examples. (1) Let V be a C^∞ manifold of dimension n, and let $\overset{p}{A}$ denote **223**
the set of p differential forms on V. Then $\overset{p}{A} = C^\infty(V, E^p)$, where E^p is a vector bundle of rank $\binom{n}{p}$ over V for which the fibre at $a \in V$ is $\overset{p}{\wedge} T_a^*(V)$. The exterior differentiation $d : \overset{p}{A} \to \overset{p}{A}p+1$ is a differential operator of degree 1. If $p = 0$, for $f \in m_a^\infty$ and $g \in \overset{\circ}{A}$, we have

$$d(fg)(a) = (df)(a)g(a) + f(a)(dg)(a).$$

Hence if $(df)(a) \neq 0, d(fg)(a) \neq 0$ whenever $g(a) \neq 0$ i. e. $d : \overset{\circ}{A} \to \overset{1}{A}$ is elliptic.

(2) Let V be a complex manifold of complex dimension n and let $\varepsilon^{p,q}$ denote the set of all differential forms of type (p, q). Then $\varepsilon^{p,q} = C^\infty(V, E^{p,q})$, where $E^{p,q}$ is a vector bundle of rank $\binom{n}{p}\binom{n}{q}$ over V; $[E^{p,q}$ is a bundle whose fibre over $a \in V$ is the space $\varepsilon_a^{p,q}$ of complex convectors of type (p, q) at $a]$.

Clearly $\bar{\partial}$: $\varepsilon^{p,q} \to \varepsilon^{p,q+1}$ is a differential operator of order 1. Let $q = 0$, $f \in m_a^\infty$ and $(df)(a) \neq 0$. Since f is real valued, we have

$$(df)(a) = \overline{(\bar{\partial} f)}(a) + (\bar{\partial} f)(a)$$

and hence $(df)(a) = \phi$ implies $(\bar{\partial} f)(a) \neq 0$. If $g \in \varepsilon^{p,0}$ and $g(a) \neq 0$ $\bar{\partial}(fg)(a) = (\bar{\partial} f)(a)g(a) \neq 0$, since $g(a)$ is of type $(p, 0)$. i.e. $\bar{\partial}$: $\varepsilon^{p,0} \to \varepsilon^{p,1}$ is elliptic.

In what follows, $\overset{n}{A}(V)$ is the (complex) line bundle $\wedge^n \mathcal{J}^*(V)$, where $\mathcal{J}^*(V)$ is the bundle of complex covectors on V i. e. for $a \in V$, $\mathcal{J}_a^*(V) =$ **224**
$T_a^*(V) \otimes_{\mathbb{R}} \mathbb{C}$ and E' is the vector bundle on V, given by

$$E' = E^* \otimes A^n(V).$$

Since $E' = E_a^* \otimes A^n(V)$, we have a map η: $E_a \times E'_a \to A_a^n(V)$, given by

$$\eta(x, y^* \otimes \omega_a) = (x, y^*)\omega_a \in A_a^n(V).$$

Now for any open subset U of V, η defines a map (which we again denote by η)

$$\eta : \Gamma(U, E) \times \Gamma(U, E') \to \Gamma(U, A^n(V))$$

given by $$\eta(s, s')(a) = \eta(s(a), s'(a)).$$

If one of s and s' has compact support, we define $\langle s, s'\rangle$ by

$$\langle s, s'\rangle = \int_V \eta(s, s').$$

Remarks. (1) Since for all line bundle D, $D \otimes D^*$ is (canonically) trivial, it follows that

$$\begin{aligned}(E')' &= E'^* \otimes \overset{n}{A}(V)\\ &= E \otimes (\overset{n}{A}(V))^* \otimes \overset{n}{A}(V) \simeq E.\end{aligned}$$

(2) If τ: $E \to V \times \mathbb{C}^q$ is an isomorphism of E with the trivial bundle and ${}^t\tau^{-1}$: $E^* \to V \times \mathbb{C}^q$, the associated isomorphism of the duals and if

$$\tau(x) = a \times (x_1, \ldots, x_q), {}^t\tau^{-1}(y^*) = a \times (y_1, \ldots, y_q),$$

225 then $y^*(x) = \sum x_i y_i$. We shall also write $\tau(x)$ for the projection $(x_1, \ldots, x_q)$ of $\tau(x)$ on $\mathbb{C}^q$.

Lemma 1. *If L is differential operator $C_0^\infty(V, E) \to C^\infty(V, F)$ then there exists a unique differential operator*

$$L' : C_0^\infty(V, F') \to C^\infty(V, E'), \text{ such that}$$

(9.1) $$\langle s, L'\sigma\rangle = \langle Ls, \sigma\rangle \ \text{ if } \ s \in C^\infty(V, E) \sigma \in C_0^\infty(V, F').$$

Proof. It is clear that an operator L', if it satisfies (9.1), is local (i. e. $\operatorname{supp} .L'\sigma \subset \operatorname{supp} .\sigma$) and is uniquely determined. We have therefore only to prove the existence locally. Let U be a positive coordinate neighbourhood with coordinates $(x_1, \ldots, x_n)$. We remark that any $\sigma \in F'_a = F^*_a \otimes \overset{n}{A}_a(V)$ can be uniquely written as

$$\sigma = g \otimes (dx_1 \wedge \cdots \wedge dx_n)_a, g \in F^*_a.$$

□

Suppose now that $\tau_E\colon E_U \to U\times\mathbb{C}^q$, $\tau_F\colon F_U \to U\times\mathbb{C}^q$ are isomorphisms and $\tau^*_E\colon E^*_U \to U\times\mathbb{C}^q$ is the transpose inverse. We suppose that, in terms of the isomorphism τ_E, τ_F, L is written

$$L = \sum_{|\alpha|\le m} a_\alpha D^\alpha \text{ on } U.$$

We define $L'\sigma$ by $L'\sigma = \lambda_\sigma dx_1\wedge\cdots\wedge dx_n$ where $\tau^*_E(\lambda_\sigma) = \bar{L}^*(\tau^*_F(g))$, if $\sigma = g\otimes dx_1\wedge\cdots\wedge dx_n$. (For any operator $A = \sum C_\alpha D^\alpha$, we denote by $\bar{A}$ the operator $\sum \bar{C}_\alpha D^\alpha$.) We have, if $\sigma\in C^\infty_0(U,F')$, $s\in C^\infty(V,E)$,

$$\begin{aligned}\langle Ls,\sigma\rangle &= \int_U (L\tau_E(s),\tau^*_F(g))dx_1\wedge\cdots\wedge dx_n\\ &= \int_U (\tau_E(s), L^*\tau^*_F(g))dx_1\wedge\cdots\wedge dx_n\\ &= \langle s, L'\sigma\rangle.\end{aligned}$$

Definition. The L' defined by Lemma 1 is called the transpose of the operator L. 226

Remarks. If rank E = rank F and L is elliptic, then L' is elliptic.

If $p\colon E\to V$ is a vector bundle, in what follows a section $s\colon V\to E$, is a map $V\to E$ (not necessarily continuous) such that

$$p\circ s = \text{ identity on } V.$$

Definition. A section $s\colon V\to E$ is said to be locally in H_m is every point $a\in V$ has a coordinate neighbourhood U such that there is an isomorphism $\tau\colon E_U\to U\times\mathbb{C}^q$ for which $\tau\circ s$ is in $H_m(U)$.

[We may speak of locally measurable, integrable sections in the same way.] The theorems proved in §7, §8, extend to differential operators between vector bundles. We state those results that we need. The proofs are immediate, and the details will be omitted.

If $L : C_0^0(V,E) \to C^\infty(V,F)$ is an elliptic differential operator, then for any locally (square) integrable section s of E on the open set $U \subset V$, Ls denotes the linear functional on $C_0^\infty(U,F')$ defined by

$$(Ls)(s') = \langle s, L's'\rangle \text{ for } s' \in C_0^\infty(U,F').$$

227 If there exists σ which is a locally square integrable (or $C^\infty,\ldots$) section of F on U for which

$$(Ls)(s') = \langle \sigma, s'\rangle \text{ for } s' \in C_0^\infty(U,F'),$$

we say that Ls is locally square integrable (or $C^\infty,\ldots$).

Regularity theorem. *If L is an elliptic operator, L: $C_0^\infty(V, E) \to C^\infty(V,F)$ and s is a locally square integrable section of E such that Ls is C^∞, then s is itself C^∞(i. e. equal almost everywhere to a C^∞ section).*

Analyticity theorem *Let V be an analytic manifold, E,F analytic vector bundles on V and L an elliptic operator from E to F with analytic coefficients (i. e. for any analytic section s: $U \to E$, Ls is an analytic section $U \to F$). Then if s is a locally square integrable section such that Ls is analytic, then s is itself analytic.*

Let K be a compact set in V. Then $H_m(K,E)$ denotes the set of sections $s : V \to E$, which are locally in H_m for which supp $.s \subset K$. Let $\mathscr{U} = \{U_1;\ldots,U_h\}$ be a finite covering of K, U_i being coordinate neighbourhoods such that E restricted to a neighbourhood U'_i of $\bar{U}_i$ is trivial. Let τ_i: $E_{U'_i} \to U'_i \times \mathbb{C}^q$ be isomorphisms. Let φ_i be C^∞ functions with supp $.\varphi_i \subset U_i$ and $\sum \varphi_i = 1$ in a neighbourhood of K. Then for
228 $s \in H_m(K,E)$, $\tau_i(\varphi_i s) \in H_m(U_i)$ and $|\tau_i(\varphi_i s)|_m^2 < \infty$. We define the norm $|s|_{m,\mathscr{U}}$ by $|s|^2_{m,\mathscr{U}} = \sum\limits_{i=1}^{h} \tau_i(\varphi_i s)|^2_m$. Then $H_m(K,E)$ is a complete normed linear space and in fact a Hilbert space.

Let $\mathscr{H}$ denote the Hilbert space $\oplus^h H_m(U_i)$ and η: $H_m(K,E) \to \mathscr{H}$ the map given by $\eta(s) = \oplus\tau_i(\varphi_i s)$. Clearly η is an isometry of $H_m(K,E)$ onto a closed subspace of $\mathscr{H}$.

We also have $\tau_i(s|U_i) \in H_m(U_i)$ and if $\|s\|_{m,\mathscr{U}}$ denotes

$$(\sum |\tau_i(s|U_i)|^2_m)^{\frac{1}{2}}, H_m(K,E)$$

is a complete normed linear space with the norm $\| \|_{m,\mathscr{U}}$.

Clearly $|s|_{m,\mathscr{U}} \leq c\|s\|_{m,\mathscr{U}}$ and

$$\begin{aligned}|\tau_i(s|U_i)|_m &\leq \sum_{j=1}^{h} |\tau_i(\varphi_j s|U_i)|_m\\ &\leq C\sum_{j=1}^{h} |\tau_j(\varphi_j s|U_j)|_m\end{aligned}$$

where C is a constant depending on the isomorphisms τ_i.

Hence the two norms are equivalent. It is easy to see that if $\mathscr{U}_1$ and $\mathscr{U}_2$ are two finite coverings having the same properties as $\mathscr{U}$, $| |_{m,\mathscr{U}_1}$ and $| |_{m,\mathscr{U}_2}$ are equivalent.

Rellich's lemma. The natural injection i: $H_m(K,E) \to H_{m-1}(K,E)$ is completely continuous.

This follows at once from the result of §5 and the definition of the norms on $H_r(K,E)$.

Proposition 1. *For any continuous linear functional l on $H_0(K,E)$, there exists a unique $s' \in H_0(K,E')$ such that $l(s) = \langle s, s'\rangle$ for any $s \in H_o(K,E)$.* 229

Proof. It is clear that if there exists $s' \in H_o(K,E')$ such that $l(s) = \langle s, s'\rangle$ for any $s \in H_o(K,E')$ then s' is unique. Let U be a coordinate neighbourhood such that F restricted to a neighbourhood U' of $\bar{U}$ is trivial, then it is enough to show that there exists $s' \in H_o(U,E)$ such that

$$l(s) = \langle s, s'\rangle \text{ for any } s \in H_o(U,E).$$

□

If $\tau : E_{U'} \to U' \times \mathbb{C}^q$ is an isomorphism, let $\tau^* : E^*_{U'} \to U' \times \mathbb{C}^q$ be the corresponding isomorphism of E^*_U. Let $s \in H_o(U,E)$ correspond to $\tau(s) = (s_1,\ldots,s_o)$; then $(s_1,\ldots,s_q) \in L^2(U)$. Then by the theorem of Riesz since $L^2(U)$ is a Hilbert space, there exists $t = (t_1,\ldots,t_q) \in L^2(U)$ such that

$$l(s) = (s,t) = \int\limits_U \sum_{i=1}^{q} s_i\bar{t_i}dx_1 \wedge \cdots \wedge dx_n.$$

Let $s' \in H_o(U, E')$ be given by

$$s' = (\tau')^{-1}(\overline{t_1}, \ldots, \overline{t_q}) \otimes dx_1 \wedge \cdots \wedge dx_n.$$

Then clearly

$$l(s) = \int_U \sum s_i \overline{t_i} dx_1 \wedge \cdots \wedge dx_n = \langle s, s' \rangle.$$

230 **Remark.** If L is a differential operator of order m, L: $C_o^\infty(V, E) \to C^\infty(V, F)$ and if K is a compact subset of V, L gives rise to a map $L_K : H_m(K, E) \to H_o(K, F)$.

We shall need the following result

Theorem 1. *If H_1, H_2 are Hilbert spaces and if A, B are continuous linear maps, $H_1 \to H_2$ such that A is injective and $A(H_1)$ is closed while B is completely continuous, then $(A + B)(H_1)$ is closed and the kernel of $(A + B)$ is of finite dimension.*

Proof. It follows from the closed graph theorem that A^{-1}: $A(H_1) \to H_1$ is continuous. Let $A + B = T$. If the kernel of T is of infinite dimension, there exists an orthonormal sequence (x_n) in H_1 such that $Tx_n = 0$. By the complete continuity of B, there exists a subsequence (x_{n_k}) of (x_n) such that Bx_{n_k} is convergent. Hence A. $x_{n_k} \to Ax_0$. It follows from the continuity of A^{-1} that $x_{n_k} \to x_o$ which contradicts the hypothesis that (x_n) are orthonormal. This proves that the kernel of T is of finite dimension. □

Let N be the kernel of T and let M be the orthogonal complement of N in H_1 and let $\widetilde{T}$ be the restriction of T to M. Clearly $\widetilde{T}$ is continuous and injective. It is enough to prove that $\widetilde{T}^{-1}$ defined on $T(H_1)$ is continuous. Let $y_n \in T(H_1), y_n \to 0$ and $y_n = \widetilde{T}x_n$, $x_n \in M$. If $x_n \not\to 0$, we may assume $\|x_n\| \geq \rho > 0$ for some positive number ρ. Put $z_n = \dfrac{x_n}{\|x_n\|}$; then $\widetilde{T}z_n \to 0$. Let (z_{n_k}) be a subsequence of z_n such that Bz_{n_k} is convergent. Then Az_{n_k} is convergent and let $Az_{n_k} \to Az_o$. It follows that $z_{n_k} \to z_o$ and obviously $\|z_o\| = 1$. But $\widetilde{T}z_n \to 0$, i.e. $\widetilde{T}z_o = 0$ and this is a contradiction.

231 **Theorem 2.** *Let V be an oriented C^∞ manifold and E, F, C^∞ vector bundles on V of rank q and p respectively. Let L be an elliptic differential operator of order m, L: $C_o^\infty(V,E) \to C^\infty(V,F)$ and K, a compact subset of V. Then L_K: $H_m(K,E) \to H_o(K,F)$ has a closed image and the kernel of L_K has finite dimension.*

Proof. Let $A : H_m(K,E) \to H_o(K,F) \oplus H_{m-1}(K,E)$ be the map $Au = (Lu) \oplus i(u)$, where i: $H_m(K,E) \to H_{m-1}(K,E)$ is the natural injection. By Friedrichs' inequality (Part III), for any $a \in V$, there exists a neighbourhood U_i and a constant C such that $|\varphi_i f|_m \leq C(|L\varphi_i f|_0 + |\varphi_i f|_0)$ for $C^\infty \varphi_i$ with supp $.\varphi_i \subset U_i$. ◻

It follows that we have an inequality of the form

$$|f|_{m,\mathscr{U}} \leq C\{|Lf|_{o,\mathscr{U}} + \|f\|_{m-1,\mathscr{U}}\}.$$

(with respect to a suitable covering $\mathcal{U}$ of K). Since $\|_{r,\mathcal{U}}, \|\|\|_{r,\mathcal{U}}$ are equivalent, this leads to an inequality

$$|f|_m \leq C_1\{|Lf|_o + |f|_{m-1}\}.$$

Now, since i is an injection, so is A. Further, because of the above inequality, $A(H_m(K,E))$ is closed. Let B: $H_m(K,E) \to H_o(K,F) \oplus H_{m-1}$ (K,E) be the map $Bu = 0 \oplus i(u)$. By Rellich's lemma, B is completely continuous. Hence, by Theorem 1, $A - B = L_K \oplus 0$ has a closed image and a finite dimensional kernel. The theorem clearly follows from this.

Proposition 2. *Let V be a compact oriented C^∞ manifold, E, F, C^∞ vector bundles of rank q, p respectively on V. Let L be an elliptic differential operator, L: $C^\infty(V,E) \to C^\infty(V,F)$. Let $L(H_m(V,E)) = M$. Then $M = \{s \in H_o(V,F) | < s, s' >= 0$ for every $s' \in H_o(V,F')$ such that $L's = 0\}$.* 232

Proof. Let $N = \{s \in H_o(V,F) | < s, s' >= 0$ for $s' \in H_o(V,F')$, $L's' = 0\}$ [the equation $L's' = 0$ means, of course, that $< Lu, s' >= 0$ for all $u \in C^\infty(V,E)$]. By definition of the equation $L's' = 0$ we have $M \subset N$. Suppose that $M \neq N$, then since M is closed by Theorem 2, there is a continuous linear functional l on $H_0(V,F)$ such that $l(M) = 0$, but

$l(N) \neq 0$. Now, there is $s' \in H_o(V, F')$ such that $l(s) =< s, s' >$ for $s \in H_o(V, F)$. Since $l(m) = 0$, we have $< Lu, s' >= 0$ if $u \in C^\infty(V, E)$. But this means precisely that $L's' = 0$ and by definition of N, we have $l(N) = 0$, a contradiction. □

The same reasoning gives the following

Proposition 2′. *Let V be an oriented C^∞ manifold, E, F C^∞ vector bundles on V and L: $C_0^\infty(V, F) \to C^\infty(V, F)$ an elliptic operator. Let K be a compact subset of V and $s \in H_o(K, F)$ be such that $< s, s' >= 0$ for any $s' \in H_o(K, F')$ with $L's' = 0$ on $\mathring{K}$. Then there is $\sigma \in H_m(K, E)$ with $L\sigma = s$.*

Proposition 3. *If V is a compact C^∞ manifold , E, F, C^∞ vector bundles of the same rank on V, L is an elliptic differential operator L: $C^\infty(V, E) \to C^\infty(V, F)$, then the image of Lis of finite codimension*

233 *Proof.* Consider the operator L_V: $H_m(V, E) \to H_0(V, F)$ and let L_V $[H_m(V, E)] = M$. By Proposition 2, $M = \{s \in H_o(V, F)| < s, s' >= 0$ for every $s' \in H_0(V, F')$ such that $L's' = 0\}$. Hence it follows that cokernel $L_V \simeq$ kernel $L'_V(L'_V : H_m(V, F') \to H_0(V, E'))$. Since rank $E =$ rank F, L' is also elliptic, so that by Theorem 2, kernal L'_V has finite dimension. Now, if $s' \in H_0(V, F')$ and $L's' = 0$, we have $s' \in C^\infty$. Hence $M \cap C^\infty(V, F) = L(C^\infty(V, E))$. Since M has finite codimension in $H_0(V, F), L(C^\infty(V, E))$ has finite codimension in $C^\infty(V, F)$ □

Remark. It can actually be shown that we have

$$C^\infty(V, F)/L[C^\infty(V, E)] \simeq \text{ kernel } L'_V.$$

Definition. If V is a compact oriented C^∞ manifold E, F are C^∞ bundles of the same rank, $L : C^\infty(V, E) \to C^\infty(V, F)$ an elliptic operator, the integer dim. (kernal L) - dim. (cokernel L) is called the index of L.

The study of the index of elliptic operators has recently become very important and has led to beautiful relationships between topology and analysis.

The results provided in the section are due mainly to L. Schwartz.

10 The approximation theorem and its application to open Riemann surfaces

Definition. Let V be a manifold, and S a subset of V. $\hat{S}$ denotes the union of S with the relatively compact connected components of $V - S$.

Remarks. (1) If K is compact, $\hat{K}$ is compact. For if U is a relatively open set containing, K, let $\Omega_1, \ldots, \Omega_h$ be open connected sets that $\cup\Omega_i \supset \partial U, \Omega_i \cap K = \phi$. Then there exists at most h relatively compact connected components of $V - K$ which are not contained in U; hence $\hat{K}$ is relatively compact and since $V - \hat{K} = \cup\{$ unbounded components of $V - K\}$, $V - \hat{K}$ is open i.e. $\hat{K}$ is compact. **234**

(2) If K is a compact subset of an open set Ω and if $V - \Omega$ has no compact components then $\hat{K} \subset \Omega$. For if U_α is a bounded component of $V - K$, not contained in Ω, let $a \in U_\alpha$ and $a \notin \Omega$. If V_a is the connected component of $V - \Omega$ containing a, we have $V_a \subset U_\alpha$, hence V_a is relatively compact and thus we have a contradiction.

(3) If $S_1 \subset S_2$, it is easy to see that $\hat{S}_1 \subset \hat{S}_2$.

(4) If U is open set then $\hat{U}$ is also open; this fact is not so trivial and since we shall not need it, we omit the proof; the same applies to

(5) If K is a compact set and $K = -\hat{K}$, then K has a fundamental system of open (compact) neighbourhoods $U(L)$ such that $U = \hat{U}(L = \hat{L})$.

Lemma 1. *Let V be an oriented C^∞ manifold , E, F C^∞ vector bundles and $L\colon E \to F$, an elliptic differential operator with C^∞ coefficients. If Ω is an open set on V and if $Lf = 0$ on V and $Lf_\nu = 0$ on V, the following are equivalent.*

(i) $f_\nu \to f$ *in* L^2 *locally on* Ω.

(ii) $f_\nu \to f$ *uniformly on compact subsets of* Ω. **235**

(iii) f_ν *and* $D^\alpha f_\nu$, *for every* α, *converge to* f *and* $D^\alpha f$ *respectively, uniformly on compact subsets of* Ω. *(Note that because of the regularity theorem* f, f_ν *are* C^∞.*)*

Proof. We may suppose that E, F are trivial and that V is an open aet in $\mathbb{R}^n$. Let $K \subset U \subset\subset U' \subset\subset \Omega$, K being compact and U, U' open. Let $r > 0$. Then by Friedrichs' inequality (Part IV) there is a constant C such that for any $g \in C^\infty(\Omega)$,

$$|g|^U_{m+r} \leq C\{|Lg|^{U'}_r + |g|^{U'}_o\}.$$

□

If $f_\nu \to f$ in $L^2(U')$ and $Lf_\nu = 0$ on V and $Lf = 0$ on Ω this gives

$$|f_\nu - f'|^U_{m+r} \leq C|f_\nu - f|^{U'}_o \to \text{ for every } r \geq 0.$$

By Sobolev's lemma, there exists a constant C_K such that

$$||f_\nu - f||^K_r \geq C_K|f_\nu - f|^U_{m+r+n},$$

and hence (i) implies (iii). Since trivially (iii) implies (ii) and (ii) implies (ii), the lemma is proved.

Theorem 1 (Malgrange-Lax)**.** *Let V be an oriented real analytic manifold, E, F analytic vector bundles of the same rank and L: $E \to F$, an elliptic operator of order m, with analytic coefficients. Then if Ω is an open set in V and if $V - \Omega$ has no compact connected components, then*
236 *any $f \in C^\infty(\Omega, E)$ with $Lf = 0$ on Ω can be approximated uniformly on compact subsets of ω, by solutions $s \in C^\infty(V, E)$ of the equation $Ls = 0$.*

Proof. Let K be a compact set in Ω. Then by the remarks (1) and (2) above, $\hat{K}$ is compact and $\hat{K} \subset \Omega$. □

Let K' be a compact set in V such that $\hat{K} \subset \overset{\circ}{K'}$. Let

$$A(K') = \{f \in H_o(K', E) | Lf = 0 \text{ on } \overset{\circ}{K'}\}$$

and $S(K) = \{f | Lf = 0$ in a neighbourhood of $\hat{K}\}$. Consider the map η: $H_o(V, E) \to H_o(\hat{K}, E)$, given by

$$\eta(s) = \begin{cases} s & \text{on } \hat{K} \\ 0 & \text{outside } \hat{K}. \end{cases}$$

If $\eta(A(K')) = M$, we shall prove that M is dense in $\eta(S(K))\{$ clearly $M \subset \eta(S(K))\}$. Let l be a continuous linear functional on $H_o(\hat{K}, E)$ such that $l(s) = 0$ for $s \in M$. By §9, proposition 1, there exists $u \in H_o(\hat{K}, E')$ such that $l(s) =< s, u >$ for $s \in H_o(K, E)$. Then $< s, u >= 0$ for every s with $Ls = 0$ on $\overset{\circ}{K}'$. Hence by §9. Proposition 2′, there exists $v \in H_m(K', F')$ such that $L'v = u$.

Now supp $u \subset \hat{K}$ i.e. $L'v = 0$ on $V - \hat{K}$. Hence by the analyticity theorem, v is analytic on $V - \hat{K}$. Hence by the analyticity theorem, v is analytic on $V - \hat{K}$. But supp . $v \subset K'$ and $V - \hat{K}$ has no relatively compact connected components; hence $v = 0$ on $V - \hat{K}$, i,e, $v \in H_m(\hat{K}, F')$. For any $s \in S(K)$, let U be a neighbourhood of $\hat{K}$ so that s is defined and $Ls = 0$ on U. Then $< \eta(s), u >=< s, u >=< s, L'v >_U =< Ls, v >_U = 0$ (since supp . $v \subset \hat{K}$), i.e. $l(s) = 0$ for any $s \in \eta(S(K))$. By the Hahn- Banach Theorem, this implies that M is dense in $\eta(S(K))$. Thus if $Lf = 0$ in a neighbourhood of $\hat{K}$, there exists a sequence of functions **237**
$\{f_\nu\}$ in $A(K')$ such that $f_\nu \to f$ in $H_o(\hat{K}, E)$; by Lemma 1, $f_\nu \to f$ uniformly on compact sets in $(\hat{K})^0$.

Let $\{K_r\}$ be a sequence of compact sets such that $\cup K_r = V$ and $\hat{K} \subset K^o_1$, $\hat{K}_1 \subset \Omega$, $\hat{K}_r \subset K^o_{r+1}$ for $r \geq 1$. Then if $Lf = 0$ in a neighbourhood of $\hat{K}_1$ there exists $f_1 \in A(K_2)$ such that

$$||f - f_1||^{\hat{K}} < \varepsilon/2$$

By induction, we have a sequence $f_\nu \in A(K_{\nu+1})$ such that $||f_\nu - f_{\nu+1}||^{\hat{K}_\nu} < \dfrac{\varepsilon}{2^\nu}$; of course, $f_\nu \in C^\infty(K^o_{\nu+1})$.

Define g on V by $g = g_r \equiv f_r + \sum\limits_{s=r+1}^{\infty} (f_s - f_{s-1})(= \lim\limits_{s\to\infty} f_s)$ on K_r; clearly the series converges uniformly on compact sets of V, and we have $g_r = g_{r+1}$ on K_r. Moreover $Lg = 0$; in fact, for any section $u \in C^\infty_o(V, F')$ we have $(Lg)(u) =< g, L'u >= \lim\limits_{s\to\infty} < f_s, L'u >= \lim\limits_{s\to\infty} < Lf_s, u >= 0$ [We have $< f_s, L'u >=< Lf_s u >$ if supp. $u \subset K_s$.] It is clear that $||f - g||^{\hat{K}} < \varepsilon$.

(The fact that $Lg = 0$ also follows from Lemma 1.)

Remarks. (1) It follows from Theorem 2 and remark (5) after the definition of $\hat{S}$ that the following proposition holds.

Proposition 4. *If K is a compact set such that $K = \hat{K}$, then any solution of the equation $Ls = 0$ in a neighbourhood of K can be approximated, uniformly on K, by solutions of the equation on V.*

(2) It can be proved that the condition that $V - \Omega$ have no compact
238 component is also necessary for every solution on Ω to be approximable by solutions on V. The proof depends on the existence theory for equations $Ls = f$, f being given, which we have not treated. See Malgrange [27].

(3) Let V be a complex manifold of complex dimension n and let $\varepsilon^{p,q}$ denote the set of differential forms of type (p, q). Consider $\bar{\partial}$: $\varepsilon^{p,0} \to \varepsilon^{p,1}$, discussed in example (2) of §9. In particular, if $p = 0$ and if the rank of $(0, 0)$ forms = rank of $(0, 1)$ forms, i.e. if $n = 1$, we may apply Theorem 1 to $\bar{\partial}$: $\varepsilon^{0,0} \to \varepsilon^{0,1}$ and obtain the following result.

Theorem 2 (Runge theorem for open Riemann surfaces: H. Behnke -K.Stein)**.** *If V is an open Riemann surface, (i.e. a connected, non compact complex manifold of complex dimension 1) and if Ω is an open subset of V such that $V - \Omega$ has no complete connected components, then if f is a holomorphic function on Ω, for any compact subset K of Ω, f is the uniform limit on K of a sequence of holomorphic functions on V.*

Note that when V is an open set in $\mathbb{C}$, the condition on Ω is also necessary. in fact it is seen easily that if $\{f_\nu\}$ is a sequence of holomorphic functions on V, converging uniformly on compact subsets of Ω, then $\{f_\nu\}$ converging uniformly on compact subsets of $\hat{\Omega}$. It follows that any holomorphic function on Ω which can approximated by holomorphic functions on V, admits a holomorphic extension to $\hat{\Omega}$. If $\Omega \neq \hat{\Omega}$, this is not the case for at least one holomorphic function on Ωe.g. $\dfrac{1}{z-a}$ where $a \in \hat{\Omega} - \Omega$. One can further use the Runge theorem to prove this latter
239 statement also when V is an arbitrary open Riemann surface, so that the condition is necessary for any open Riemann surface.

Definition. Let V be a complex manifold and let $\mathscr{H} = \mathscr{H}(V)$ denote the set of all holomorphic function on V. V is said to be a Stein manifold if the following three condition are satisfied.

(i) $\mathscr{H}$ separates points.

(ii) For any point $a \in V$, there exists functions in $\mathscr{H}$, which form a system of local coordinates in a neighbourhood of a.

(iii) For any compact subset K of V, the set $\hat{K}_{\mathscr{H}} = \{x \in V | f(x) \leq \sup_{y \in K}.$ $|f(y)|$, for every $f \in \mathscr{H}\}$, is compact.

Theorem 3 (Behnke-Stein)**.** *Every open Riemann surface V is a Stein manifold*

Proof. For $a, b \in V$, $a \neq b$, let (U_1, φ_1), (U_2, φ_2) be coordinate neighbourhood of a and b such that $U_1 \cap U_2 = \phi$ and

$$\varphi_1(U_1) = \{z \in \mathbb{C}|z| < 1\} = \varphi_2(U_2).$$

□

If $U'_1 = \{x \in U_1 \Big| |\varphi_2(x)| < r < 1\}$ and $U'_2 = \{x \in U_2 \Big| |\varphi_2(x)| < r < 1\}$, then $V - U'_1$ and $V - U'_2$ are connected and so is $V - U'_1 - U'_2$. Hence if $\Omega = U'_1 \cup U'_2$, $V - \Omega$ has no compact connected and by Theorem 2, any holomorphic function on Ω, can be approximated uniformly on compact subsets of Ω, by functions in $\mathscr{H}$.

Let f be given by **240**

$$f(x) = 0 \text{ for } x \in U'_1$$
$$\text{and } = 1 \text{for} x \in U'_2.$$

Then f is holomorphic on Ω and hence there exists $g \epsilon \mathscr{H}$ such that

$$||f - g||^{U'_1}, ||f - g||^{U'_2} < \frac{1}{2},$$

i.e. $|g(a)| < \frac{1}{2}$ and $|g(b)| > \frac{1}{2}$ and hence $\mathscr{H}$ separates points. For $a \in V$, let f be a holomorphic function in a neighbourhood W of a with (df) $(a) \neq 0$. Then f gives local coordinates at a. Let (U, φ) be a coordinate neighbourhood, $U \subset W$ such that $\varphi(U) = \{z \in \mathbb{C} \Big| |z| < 1\}$. Then,

with the same notation as above, $V - U'$ is connected and since f is holomorphic on U', there exists $g \in \mathscr{H}$ such that $\|f - g\|^{U'} < \varepsilon$. Since uniform convergence of holomorphic functions implies the uniform convergence of their derivatives, if ε is small enough, we have $(dg)(a) \neq 0$, so that g gives local coordinates at a. We shall now prove that for a compact set K in V, $\hat{K} = \hat{K}_{\mathscr{H}}$. Let $a \notin \hat{K}$; then $a \in U_\alpha$, U_α being component of $V - K$ which is not relatively compact. Let (U, φ) be a coordinate neighbourhood of a such that $\varphi(U) = \{z \in \mathbb{C} \big| |z| < 1\}$, $\bar{U} \subset U_\alpha$. Let S be a discrete unbounded set, contained in U_α and let $S' = S \cup \bar{U}$. Then S' is a closed set, $S' \subset U_\alpha$. Hence there exists a closed connected set A such that $S' \subset A \subset U_\alpha$. Let L be a compact neighbourhood of $\hat{K}$ such that
241 $L \cap A = \phi$. Then clearly $A \cap \hat{L} = \phi$ and $\hat{L}$ is a neighbourhood of $\hat{K}$, $V - \hat{L}$ has no relatively compact connected component. Clearly $V - \{\hat{L} \cap \bar{U}\}$ has no relatively compact connected component. Let f be defined on a neighbourhood of $\hat{L} \cup \bar{U}$ by $f(x) = 0$ for x near $\hat{L}$ $f(x) = 1$ for x near $\bar{U}$. Then f is holomorphic in a neighbourhood of $\hat{L} \cup \bar{U}$. According to the proof of Theorem 1 (for the operator $\bar{\partial}$: $\varepsilon^{0,0} \to \varepsilon^{0,1}$) f is the limit of holomorphic functions on V in $H_o(\hat{L} \cup \bar{U}, \varepsilon^{0,0})$. Since $K' = \hat{K} \cup \{a\}$is contained in the interior of $\hat{L} \cup \bar{U}$, and L^2convergence implies uniform convergence on compact subsets of the interior, f is the uniform limit, on K', of holomorphic functions on V. Hence there exists $g \in \mathscr{H}$ such that

$$|g(x)| < \frac{1}{2} \text{ for } x \in \hat{K}$$

and

$$|g(a)| > \frac{1}{2},$$

so that $a \notin \hat{K}_{\mathscr{H}}$. Hence $\hat{K}_{\mathscr{H}} \subset \hat{K}$. It follows from the theorem of maximum modulus that $\hat{K} \subset \hat{K}_{\mathscr{H}}$.

The main Theorem 1 is due to Malgrange [27] and Lax [25]. The application to open Riemann surfaces is essentially as in Malgrange [27]. The original treatment of Behnke - Stein [2] is quite different, and rather more difficult, but enables one to solve also the so called First and Second Problems of Cousin" on arbitrary open Riemann surfaces with little extra effort.

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