Lectures On
Some Fixed Point Theorems
Of Functional Analysis

By

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Introduction

These lectures do not constitute a systematic account of fixed point theorems. I have said nothing about these theorems where the interest is essentially topological, and in particular have nowhere introduced the important concept of mapping degree. The lectures have been concerned with the application of a variety of methods to both non-linear (fixed point) problems and linear (eigenvalue) problems in infinite dimensional spaces. A wide choice of techniques is available for linear problems, and I have usually chosen to use those that give something more than existence theorems, or at least a promise of something more. That is, I have been interested not merely in existence theorems, but also in the construction of eigenvectors and eigenvalues. For this reason, I have chosen elementary rather than elegant methods.

I would like to draw special attention to the Appendix in which I give the solution due to B. V. Singbal of a problem that I raised in the course of the lectures.

I am grateful to Miss K. B. Vedak for preparing these notes and seeing to their publication.

Frank F. Bonsall

Contents

Chapter 1

The contraction mapping theorem

Given a mapping T of a set E into itself, an element u of E is called a 1
fixed point of the mapping T if $Tu = u$. Our problem is to find conditions on T and E sufficient to ensure the existence of a fixed point of T in E. We shall also be interested in uniqueness and in procedures for the calculation of fixed points.

Definition 1.1. Let E be a nonempty set. A real valued function d defined on $E \times E$ is called a distance function or metric in E if it satisfies the following conditions

i) $d(x, y) \geq 0,\ x, y \varepsilon E$

ii) $d(x, y) = 0 \Longleftrightarrow x = y$

iii) $d(x, y) = d(y, x)$

iv) $d(x, z) \leq d(x, y) + d(y, z)$

A nonempty set with a specified distance function is called a metric space.

Example. Let X be a set and E denote a set of bounded real valued functions defined on X. Let d be defined on $E \times E$ by

$$d(f, g) = \sup \left\{ |f(t) - g(t)| : t \varepsilon X \right\}, \; f, g \varepsilon E.$$

Then d is a metric on E called the uniform metric or uniform distance function.

2 **Definition 1.2.** A sequence $\{x_n\}$ in a metric space (E, d) is said to converge to an element x of E if

$$\lim_{n \to \infty} d(x_n, x) = 0$$

A sequence x_n of elements of a metric space (E, d) is called a Cauchy sequence if given $\epsilon > 0$, there exists N such that for $p, q \geq N, d(x_p, x_q) < \epsilon$.

A metric space (E, d) is said to be complete if every Cauchy sequence of its elements converges to an element of E. It is easily verified that each sequence in a metric space converges to at most one on point, and that every convergent sequence is a Cauchy sequence.

Example. The space $C_R[0, 1]$ of all continuous real valued functions on the closed interval $[0, 1]$ with the uniform distance is a complete metric space. It is not complete in the metric d' defined by

$$d'(f, g) = \int_0^1 |f(x) - g(x)| dx f, \quad g \varepsilon C_R[0, 1].$$

Definition 1.3. A mapping T of a metric space E into itself is said to satisfy a Lipschitz condition with Lipschitz constant K if

$$d(Tx, Ty) \leq K d(x, y) \; (x, y \varepsilon E)$$

If this conditions is satisfied with a Lipschitz constant K such that $0 \leq K < 1$ then T is called a contraction mapping.

Theorem 1.1 (The contraction mapping theorem)**.** Let T be a contraction mapping of a complete metric space E into itself. Then

3 i) T has a unique fixed point u in E

ii) If x_o is an arbitrary point of E, and (x_n) is defined inductively by

$$x_{n+1} = Tx_n \ (n = 0, 1, 2, \ldots),$$

then $\lim_{n\to\infty} x_n = u$ and

$$d(x_n, u) \le \frac{K^n}{1-K} d(x_1, x_o)$$

where K is a Lipschitz constant for T.

Proof. Let K be a Lipschitz constant for T with $0 \le K < 1$. Let $x_o \varepsilon E$ and let x_n be the sequence defined by

$$x_{n+1} = Tx_n \ (n = 0, 1, 2, \ldots)$$

We have

$$d(x_{r+1}, x_{s+1}) = d(Tx_r, Tx_s) \le Kd(x_r, x_s) \tag{1}$$

and so

$$d(x_{r+1}, x_r) \le K^r(x_1, x_o) \tag{2}$$

Given $p \ \ q$, we have by (1) and (2),

$$\begin{aligned} d(x_p, x_q) &\le K^q d(x_{p-q}, x_o) \\ &\le K^q \Big\{ d(x_{p-q}, x_{p-q-1}) + d(x_{p-q-1}, x_{p-q-2}) + \cdots + d(x_1, x_o) \Big\} \\ &\le K^q \Big\{ K^{p-q-1} + K^{p-q-2} + \cdots + K + 1 \Big\} d(x_1, x_o) \\ &\le \frac{K^q}{1-K} d(x_1, x_o) \qquad (3) \end{aligned}$$

since the right hand side tends to zero as $q \to \infty$, it follows that (x_n) is a 4
Cauchy sequence, and since E is complete, (x_n) converges to an element u of E. Since $d(x_{n+1}, Tu) \le Kd(x_n, u) \to 0$ as $n \to \infty$,

$$Tu = \lim_{n\to\infty} x_{n+1} = u.$$

$d(u, x_n) \leq d(u, x_p) + d(x_p, x_n) \leq d(u, x_p) + \dfrac{K^n}{1-K} d(x_1, x_o)$ for $n < p$, by (3). Letting $p \to \infty$, we obtain

$$d(u, x_n) \leq \frac{K^n}{1-K} d(x_1, x_o)$$

□

Example. As an example of the applications of the contraction mapping theorem, we prove Picard's theorem on the existence of solution of ordinary differential equation.

Let D denote an open set in R^2, $(x_o, y_o) \varepsilon D$. Let $f(x, y)$ be a real valued function defined and continuous in D, and let it satisfy the Lipschitz condition:

$$|f(x, y_1) - f(x, y_2)| \leq M|y_1 - y_2| \ ((x, y_1), (x, y_2) \varepsilon D)$$

Then there exists a $t > 0$, and a function $\phi(x)$ continuous and differentiable in $[x_o - t, x_o + t]$ such that $i) \phi(x_o) = y_o$, $(ii) y = \phi(x)$ satisfies the differential equation

$$\frac{dy}{dx} = f(x, y) \text{ for } x \varepsilon [x_o - t, x_o + t]$$

We show first that there exists an $\epsilon > 0$ and a function $\phi(x)$ contin-
5 uous in $[x_o - t, x_o + t]$ such that $iii) \phi(x) = y_o + \int_{x_o}^{x} f(t, \phi(t)) dt$ $(x_o - t \leq x \leq x_o + t)$, and $\epsilon(x, \phi(x)) \varepsilon D (x_o - \leq x \leq x_o + t)$. Then it follows from the continuity of $f(t, \phi(t))$ that $\phi(x)$ is in fact differentiable in $[x_o - t, x_o + t]$ and satisfies (i) and (ii).

Let $\mathcal{U}$ denote a closed disc of centre (x_o, y_o) with positive radius and contained in the open set D, and let m denote the least upper bound of the continuous function $|f|$ on the compact set $\mathcal{U}$. We now choose t, δ such that $0 < t < M^{-1}$, the rectangle $|x - x_o| \leq \delta$ is contained in $\mathcal{U}$, and $mt < \delta$. Let E denote the set of all continuous functions mapping $[x_o - t, x_o + t]$ into $[y_o - \delta, y_o + \delta]$. With respect to the uniform distance function E is a closed subset of the complete metric space $C_R[x_o - t, x_o + t]$ and is therefore complete. We define a mapping $T\phi = \psi$ for $\phi \in E$ by

$$\psi(x) = y_o + \int_{x_o}^{x} f(t, \phi(t)) dt$$

Clearly $\psi(x)$ is continuous in $[x_o - t, x_o + t]$. Also $|\psi(x) - y_o| \leq m|x - x_o| \leq mt < \delta$ whenever $|x - x_o| \leq t$. Thus T maps E into it self. Finally, T is a contraction mapping, for if $\phi_i \in E, \psi_i = T\phi)i(i = 1, 2)$, then

$$\begin{aligned}|\psi_1(x) - \psi_2(x)| = |&\int_{x_o}^{x} \left\{ f(t', \phi_1(t')) - f(t', \phi_2(t')) \right\} dt'| \\ &\leq |x - x_o| M \sup |\phi_1(t')| - \phi_2(t')| \; (x_o - t \leq t \leq x_o + t)| \\ &\leq tMd(\phi_1, \phi_2)\end{aligned}$$

Hence 6

$$d(\psi_1, \psi_2) \leq tMd(\phi_1, \phi_2)$$

As $tM < 1$, this proves that T is a contraction mapping. By the contraction mapping theorem, there exists $\phi \varepsilon E$ with $T\phi = \phi$ i.e., with

$$\phi(x) = y_o + \int_{x_o}^{x} f(t, \phi(t)) dt$$

This complete the proof of Picard's theorem.

A similar method may be applied to prove the existence of solutions of systems of ordinary differential equations of the form

$$\frac{dy_i}{dx} = f_i(x, y_1, \ldots, y_n)(i = 1, 2, \ldots, n)$$

with given initial conditions. Instead of considering real valued functions defined on $[x_o - t, x_o + t]$, one considers vector valued functions mapping $[x_o - t, x_o + t]$ into R^n.

In the following theorem we are concerned with the continuity of the fixed point.

Theorem 1.2. Let E be a complete metric space, and let T and $T_n(n = 1, 2, \ldots)$ be contraction mappings of E into itself with the same Lipschitz constant $K < 1$, and with fixed points u and u_n respectively. Suppose that $\lim\limits_{n \to \infty} T_n x = Tx$ for every $x \varepsilon E$. Then $\lim\limits_{n \to \infty} u_n = u$. By the inequality in Theorem 1.1, we have for each $r = 1, 2, \ldots$,

$$d(u_r, T_r^n x_o) \leq \frac{K^n}{1 - K} d(T_r x_o, x_o), \quad x_o \varepsilon E$$

setting $n = 0$ and $x_o = u$, we have 7

$$d(u_r, u) \leq \frac{1}{1-K} d(T_r u, u) = \frac{1}{-1} d(T_r u, Tu)$$

But $d(T_r u, Tu) \to 0$ as $r \to \infty$. Hence

$$\lim_{r\to\infty} d(u_r, u) = 0$$

Example. In the notation of the last example, suppose that y_n is a real sequence converging to y_o and let T_n be the mapping defined on E by

$$(T_n\phi)(x) = y_n + \int_{x_o}^{x} f(t, \phi(t))dt$$

Then $|(T_n\phi)(x) - y_o| \leq |y_n - y_o| + m\varepsilon < \delta$ for n sufficiently large i.e. T_n map E into itself for n sufficiently large. Also the mapping T_n, T have the same Lipschitz constant $\varepsilon M < 1$. Obviously for each $\phi\varepsilon E$, $\lim\limits_{n\to\infty} T_n\phi = T\phi$. Hence if ϕ_n is the unique fixed point of $T_n (n = 1, 2, \ldots)$ then $\lim\limits_{n\to\infty} \phi_n = \phi$. In other words, if ϕ_n is the solution of the differential equation

$$\frac{dy}{dx} = f(x, y)$$

in $[x_o - t, x_o + t]$ with the initial condition $\phi_n(x_o) = y_n$, then ϕ_n converges uniformly to the solution ϕ with $\phi(x_o) = y_o$.

Remark. The contraction mapping theorem is the simplest of the fixed point theorems that we shall consider. It is concerned with mappings of a
8 complete metric space into itself and in this respect is very general. The theorem is also satisfactory in that the fixed point is always unique and is obtained by an explicit calculation. Its disadvantage is that the condition that the mapping be a contraction is a somewhat severe restriction. In the rest of this chapter we shall obtain certain extension of the contraction mapping theorem in which the conclusion is obtained under modified conditions.

Definition 1.4. A mapping T of a metric space E into a metric space E' is said to be continuous if for every convergent sequence (x_n) of E,

$$\lim_{n\to\infty} T_{x_n} = T(\lim_{n\to\infty} x_n).$$

Theorem 1.3. Let T be a continuous mapping of a complete metric space E into itself such that T^k is a contraction mapping of E for some positive integer k. Then T has a unique fixed point in E.

Proof. T^k has a unique fixed point u in E and $u = \lim_{n\to\infty}(T^k)^n x_o$, $x_o \varepsilon E$ arbitrary. □

Also $\lim_{n\to\infty}(T^k)^n T x_o = u$. Hence

$$\begin{aligned} u &= \lim_{n\to\infty}(T^k)^n T x_o = \lim_{n\to\infty} T(T^k)^n x_o \\ &= T \lim_{n\to\infty}(T^k)^n x_o \text{ (by the continuity of) T} \\ &= Tu. \end{aligned}$$

The uniqueness of the fixed point of T is obvious, since each fixed point of T is also a fixed point of T^k.

Example. We consider the non-linear integral equation 9

$$f(x) = \lambda \int_a^x K(x, y, f(y))dy + g(x) \tag{1}$$

where g is continuous in $[a, b]$ and $K(x, y, z)$ is continuous in the region $[a, b] \times [a, b] \times R$ and satisfies the Lipschitz condition.

$$|K(x, y, z_1) - K(x, y, z_2)| \leq M|z_1 - z_2|.$$

(The classical Volterra equation is obtained by taking $K(x, y, z) = H(x, y).z$, with H continuous in $[a, b] \times [a, b]$). Let $E = C_K[a, b]$ and T be the mapping of E into itself given by

$$(Tf)(x) = \lambda \int_a^x K(x, y, f(u))dy + g(x) \ (f \varepsilon E, a \leq x \leq b).$$

Given f_1, f_2 in E it is easy to prove by induction on n that

$$|(T^n f_1)(x) - (T^n f_2)(x)| \leq \frac{1}{n!}|\lambda|^n M^n d(f_1, f_2)(x-a)^n, \ (a \leq x \leq b)$$

Then

$$d(T^n f_1, T^n f_2) \leq \frac{1}{n!}|\lambda|^n M^n (b-a)^n d(f_1, f_2)$$

This proves that all T^n and in particular T, are continuous and, for n sufficiently large $\frac{1}{n!}|\lambda|^n M^n (b-a)^n < 1$, so that T^n is a contraction mapping for n large. Applying the theorem, we have a unique $f \varepsilon E$ with $Tf = f$ which is the required unique solution of the equation (1).

10 **Definition 1.5.** Let (E, d) be a metric space and $\varepsilon > 0$. A finite sequence $x_o, x_1, \ldots, x_n$ of points of E is called an ε - chain joining x_0 and x_n if

$$d(x_{i-1}, x_i) < \varepsilon \ (i = 1, 2, \ldots, n)$$

The metric space (E, d) is said to be ε *-chainable* if for each pair (x, y) of its points there exists an ε- chain joining x and y.

Theorem 1.4 (Edelstein)**.** Let T be a mapping of a complete ε- chainable metric space (E, d) into itself, and suppose that there is a real number K with $0 \leq K < 1$ such that

$$d(x, y) < \varepsilon \Rightarrow d(Tx, Ty) \leq K d(x, y)$$

Then T has a unique fixed point u in E, and $u = \lim\limits_{n \to \infty} T^n x_o$ where x_o is an arbitrary element of E.

Proof. (E, d) being ε chainable we define for $x, y \varepsilon E$,

$$d_\varepsilon(x, y) = \inf \sum_{i=1}^{n} d(x_{i-1}, x_i)$$

where the infimum is taken over all ε - chains $x_0, \ldots, x_n$ joining $x_0 = x$ and $x_n = y$. Then d is a distance function on E satisfying

i) $d(x, y) \leq d_\varepsilon(x, y)$

ii) $d(x, y) = d_\varepsilon(x, y)$ for $d(x, y) < \varepsilon$

From (ii) it follows that a sequence (x_n), $x_n \varepsilon E$ is a Cauchy sequence
11 with respect to d_ε if and only if it is a Cauchy sequence with respect to d and is convergent with respect to d_ε if and only if it converges with respect with respect to d. Hence (E, d) being complete, (E, d_ε) is also a complete metric space. Moreover T is a contraction mapping with respect to d_ε. Given $x, y \varepsilon E$, and any ε-chain $x_o, \ldots, x_n$ with $x_o = x, x_n = y$, we have

$$d(x_{i-1}, x_i) < \varepsilon \ (i = 1, 2, \ldots, n),$$

so that

$$d(Tx_{i-1}, Tx_i) \leq Kd(x_{i-1}, x_i) < \varepsilon \ (i = 1, 2, \ldots, n)$$

Hence $Tx_o, \ldots, Tx_n$ is an ε- chain joining T_x and T_y and

$$d_\varepsilon(Tx, Ty) \leq \sum_{i=1}^{n} d(Tc_{i-1}, Tx_i) \leq K \sum_{i=1}^{n} d(x_{i-1}, x_i)$$

$x_o, \ldots, x_n$ being an arbitrary ε- chain, we have

$$d_\varepsilon(Tx, Ty) \leq Kd_\varepsilon(x, y)$$

and T has a unique fixed point $u \varepsilon E$ given by

$$\lim_{n \to \infty} d_\varepsilon(T^n x_0, u) = 0 \text{ for } x_o \varepsilon E \text{ arbitrary} \tag{1}$$

But in view of the observations made in the beginning of this proof, (1) implies that

$$\lim_{n \to \infty} d(T^n x_o, u) = 0$$

□

Example. Let E be a connected compact subset of a domain D in the
complex plane. Let f be a complex holomorphic function in D which 12
maps E into itself and satisfies $|f'(z)| < 1$ $(z, \varepsilon E)$. Then there is a unique point z in E with $f(z) = z$. Since f' is continuous in the compact set E, there is a construct K with $0 < K < 1$ such that $|f'(z)| < K$ $\varepsilon E)$. For each point $\omega \varepsilon E$ there exists $\rho_\omega > 0$ such that $f(x)$ is holomorphic in the disc $S(\omega, 2\rho_\omega)$ of center ω and radius $2\rho_\omega$ and satisfies $|f'(z)| < K$ there.

E being compact, we can choose $\omega_1, \ldots, \omega_n \varepsilon E$ such that E is covered by

$$S(\omega_1, 2\rho_{\omega_1}), \ldots, S(\omega_n, 2\rho_{\omega_n})$$

Let $\varepsilon = \min\{\rho\omega_i, i = 1, 2, \ldots, n\}$. If $z, z' \varepsilon E$ and $|z - z'| < \varepsilon$ then $z, z' \varepsilon S(\omega_i, 2\rho_{\omega_1})$ for some i and so

$$|f(z) - f(z')| = |\int_{z'}^{z} f'(\omega) d\omega| \leq K|z - z'|.$$

This proves that Theorem 1.4 is applicable to the mapping $z \to f(z)$ and we have a unique fixed point.

Definition 1.6. A mapping T of a metric space E into itself is said to be *contractive* if

$$d(Tx, Ty) < d(x, y) \; (x \neq y, x, y \varepsilon E)$$

and is said to be ε-contractive if

$$0 < d(x, y) < \varepsilon \implies d(Tx, Ty) < d(x, y)$$

Remark. A contractive mapping of a complete metric space into itself
13 need not have a fixed point. e.g. let $E = \{x/x \geq 1\}$ with the usual
distance $d(x, y) = |x - y|$,let $T : E \to E$ be given by $Tx = x + \dfrac{1}{x}$.

Theorem 1.5 (Edelsten)**.** Let T be an ε-contractive mapping of a metric space E into itself, and let x_o be a point of E such that the sequence $(T^n x_o)$ has a subsequence convergent to a point u of E. Then u is a periodic point of T, i.e. there is a positive integer k such that

$$T^k u = u$$

Proof. Let (n_i) be a strictly increasing sequence of positive integers such that $\lim\limits_{i \to \infty} T^{n_i} x_o = u$ and let $x_i = T^{n_i} x_o$. There exists N such that $d(x_i, u) < \varepsilon/4$ for $i \geq N$. Choose any $i \geq N$ and let $k = n_{i+1} - n_i$. Then

$$d(x_{i+1}, T^k u) = d(T^k x_i, T^k u) \leq d(x_i, u) < \varepsilon/4$$

and

$$d(T^k u, u) \leq d(T^k u, x_{i+1}) + d(x_{i+1}, u) < \varepsilon/2$$

$\square$

Suppose that $v = T^k u \neq u$. Then T being ε - contractive,

$$d(Tu, Tv) < d(u, v) \text{ or } \frac{d(Tu, Tv)}{d(u, v)} < 1.$$

The function $(x, y) \to \dfrac{Tx, Ty}{d(u, v)}$ is continuous at (u, v). So there exist $\delta, K > 0$ with $0 < K < 1$ such that $d(x, u) < \delta, d(y, v) < \delta$ implies that $d(Tx, Ty) < Kd(x, y)$. As $\lim\limits_{r\to\infty} T^K x_r = T^K u = v$, there exists $N' \geq N$ such that $d(x_r, u) < \delta, d(Tx_r, v) < \delta$ for $r \geq N'$ and so **14**

$$d(Tx_r, TT^K x_r) < Kd(x_r, T^K x_r) \tag{1}$$

$$\begin{aligned} d(x_r, T^K x_r) &\leq d(x_r, u) + d(u, T^K u) + d(T^K u, T^K x_r) \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon \text{ for } r \geq N' > N \end{aligned} \tag{2}$$

From (1) and (2)

$$d(Tx_r, TT^K x_r) < Kd(x_r, T^K x_r) < \varepsilon \text{ for } r \geq N'$$

and so T being ε-contractive,

$$d(T^p x_r, T^p T^k x_r) < Kd(x_r, T^K x_r) \text{ for } n \geq N', p > 0 \tag{3}$$

Setting $p = n_{r+1} - n_r$ in (3)

$$d(x_{r+1}, T^k x_{r+1}) < Kd(x_r, T^k x_r) \text{ for any } r \geq N'$$

Hence $d(x_s, T^k x_s) < K^{s-r} d(x_r, T^k x_r) < K^{s-r}\varepsilon$ and $d(u, v) < d(u, x_s) + d(x_s, T^k x_s) + d(T^k x_s, v) \to 0$ as $s \to \infty$ This contradicts the assumption that $d(u, v) > 0$. Thus $u = v = T^k u$.

Theorem 1.6 (Edelstein)**.** Let T be a contractive mapping of a metric space E into itself, and let x_o be a point of E such that the sequence $T^n x_o$ has a subsequence convergent to a point u of E. Then u is a fixed point of T and is unique.

Proof. By Theorem 1.5, there exists an integer $k > 0$ such that $T^k u =$ **15**
u. Suppose that $v = Tu \neq, u$. Then $T^k u = u, T^k v = v$ and $d(u, v) = d(T^k u, T^k v) < d(u, v)$, since T is contractive. As this is impossible, $u = v$ is a fixed point. The uniqueness is also immediate. □

Corollary . *If T is a contractive mapping of a metric space E into a compact subset of E, then T has a unique fixed point u in E and $u = \lim_{n\to\infty} T^n x_o$ where x_o is an arbitrary point of E.*

Chapter 2

Fixed point theorems in normed linear spaces

In Chapter 1, we proved fixed point theorems in metric spaces without 16
any algebraic structure. We now consider spaces with a linear structure but non-linear mappings in them. In this chapter we restrict our attention to normed spaces, but our main result will be extended to general locally convex spaces in Chapter 3

Definition 2.1. Let E be a vector space over. A mapping of E into R is called a norm on E if it satisfies the following axioms.

i) $p(x) \geq 0$ $(x \in E)$

ii) $p(x) = 0$ if and only if $x = 0$

iii) $p(x + y) \leq p(x) + p(y)$ $(x, y \in E)$.

A vector space E with a specified norm on it called a normed space. The norm of an element $x \in E$ will usually be denoted by $\|x\|$. A normed space is a metric space with the metric $d(x, y) = x - y$ $(x, y \varepsilon E)$ and the corresponding metric topology is called the normed topology. A normed linear space complete in the metric defined by the norm is called a Banach space. We now recall some definitions and well known properties of linear spaces. Two norms p_1 and p_2 on a vector space E are said to be equivalent if there exist positive constants k, k' such that

$$p_1(x) \leq kp_2(x),\ \ p_2(x) \leq k' p_1(x)\ (x \in E)$$

17 Two norms are equivalent if and only if they define the same topology.

Definition 2.2. A mapping f of a vector space E into R is called a linear functional on E if it satisfies

i) $f(x + y) = f(x) + f(y)\ (x, y \in E)$

ii) $f(\alpha x) = \alpha f(x)\ (x \in E, \alpha \in R)$.

A mapping $p : E \to R$ is called a sublinear functional if

i)$'$ $p(x + y) \leq p(x) + p(y)\ (x, y \in E)$

ii)$'$ $p(\alpha x) = \alpha p(x)\ (x \in E, \alpha \geq 0)$.

Hahn- Banach Theorem. Let E_0 be a subspace of a vector space E over R; let p be a sublinear functional on E and let f_o be a linear functional on E_o that satisfies

$$f_o(x) \leq p(x)\ (x \in E_0).$$

Then there exists a linear functional f on E that satisfies

i) $f(x) \leq p(x)\ (x \in E)$,

ii) $f(x) = f_o(x)\ (x \in E_0)$.

[For the proof refer to Dunford and Schwartz ([14], p. 62) or Day [13, p.9]].

Corollary. *Given a sublinear functional on E and $x_o \in E$, there exists a linear functional f such that*

$$f(x_0) = p(x_0),\ \ f(x) \leq p(x)\ (x \in E).$$

In particular, a norm being a sublinear functional, given a point x_0
18 of a normed space E, there exists a linear functional f on E such that

$$|f(x)| \leq \|x\| \ (x \in E) \text{ and } f(x_o) = \|x_o\|$$

Definition. *A norm p on a vector space E said to be strictly convex if $p(x + y) = p(x) + p(y)$ only when x and y are linearly dependent.*

Theorem 2.1 (Clarkson). If a normed space E has a countable everywhere dense subset, then there exists a strictly convex norm on E equivalent to the given norm.

Proof. Let S denote the surface of the unit ball in E,

$$S = \{x : \|x\| = 1\}$$

Then there exists a countable set (x_n) of points of S that is dense in S. For each n, there exists a linear functional f_n on E such that

$$f_n(x_n) = \|x_n\| = 1 \text{ and } |f_n(x)| \leq \|x\| \ (x \in E). \qquad \square$$

If $x \neq 0$, then $f_n(x) \neq 0$ for some n. For, by homogeneity, it is enough to consider x with $\|x\| = 1$, and for such x there exists n with $\|x - x_n\| < \frac{1}{2}$. But then

$$\begin{aligned} f_n(x) = f_n(x_n) + f_n(x - x_n) &\geq 1 - f_n(x - x_n) \\ &\geq 1 - \|x - x_n\| > \frac{1}{2} \end{aligned}$$

We now take $p(x) = \|x\| + \left\{ \sum_{n=1}^{\infty} 2^{-n}(f_n(x))^2 \right\}^{\frac{1}{2}}$. It is easily verified 19
that p is a norm on E and that

$$\|x\| \leq p(x) \leq 2\|x\|.$$

Finally p is strictly convex. To see this, suppose that

$$p(x + y) = p(x) + p(y),$$

and write $\xi_n = f_n(x), y_n = f_n(y)$. Then

$$\left\{\sum_{n=1}^{\infty} 2^{-n}(\xi_n + \eta_n)^2\right\}^{\frac{1}{2}} = \left\{\sum_{n=1}^{\infty} s^{-n}\xi_n^2\right\}^{\frac{1}{2}} + \left\{\sum_{n=1}^{\infty} 2^{-n}\eta_n^2\right\}^{\frac{1}{2}}$$

and we have the case of equality in Minkowsiki's inequality. It follows that the sequence (ξ_n) and (η_n) are linearly dependent. Thus there exist λ, ν, not both zero, such that

$$\lambda\xi_n + \mu\eta_n = 0 \ (n = 1, 2, \ldots)$$

But this implies that

$$f_n(\lambda x + \mu y) = 0 \ (n = 1, 2, \ldots),$$

and so $\lambda x + \mu y = 0$. This completes the proof.

Lemma 2.1. *Let K be a compact convex subset of a normed space E with a strictly convex norm. Then to each point x of E corresponds a unique point Px of K at K at minimum distance from x, i.e., with*

$$\|x - Px\| = \inf\{\|x - y\| : y \varepsilon K\}$$

and the mapping $x \to Px$ is continuous in E.

Proof. Let $x \in E$, and let the function f be defined on K by $f(y) = \|x - y\|$. Then f is a continuous mapping of the compact set K into $\mathbb{R}$ and therefore attains its minimum at a point z say of K

$$\|x - z\| = \inf\{\|x - y\| : y \in K\}. \qquad \square$$

Evidently for $x \in K, z = x$ is uniquely determined. If $x \notin K$, suppose that z' is such that

$$0 \neq \|x - z\| = \|x - z'\| \tag{1}$$

since K is convex, $y = \dfrac{1}{2}(z + z') \in K$ and therefore

$$\|x - y\| \geq \|x - z\| = \|x - z'\| = \frac{1}{2}\|x - z\| + \frac{1}{2}\|x - z'\|$$

But

$$(x-y) = \frac{1}{2}(x-z) + \frac{1}{2}(x-z'),$$

so that
$$\|x-y\| \leq \frac{1}{2}\|x-z'\| + \frac{1}{2}\|x-z'\|$$

Hence $\|x-y\| = \|\frac{1}{2}(x-z) + \frac{1}{2}(x-z')\| = \|\frac{1}{2}(x-z)\| + \|\frac{1}{2}(x-z')\|$

As the norm is strictly convex,

$$\lambda(x-z) + \mu(x-z') = 0$$

for λ, μ not both zero. By (1), $|\lambda| = |\mu|$ and so $x - z = \pm(x - z')$. If $x - z = -(x - z')$, then $x = \dfrac{z+z'}{2} \in K$, which is not true. Hence $x - z = x - z'$ or $z = z'$. This proves that the mapping $x \to Px = z$ is uniquely on E. Given $x, x' \in E$,

$$\|x - Px\| \leq \|x - Px'\| \leq \|x - x'\| + \|x' - Px'\|,$$

and similarly $\|x' - Px'\| \leq \|x - x'\| + \|x - Px\|$. So **21**

$$|\ \|x - Px\| - \|x' - Px'\|\ | \leq \|x - x'\| \tag{2}$$

Let $x_n \in E (n = 1, 2, \ldots)$ converge to $x \in E$. Then the sequence Px_n in the compact metric space K has a subsequence Px_{n_k} converging to $y \in K$. Then

$$\lim_{k\to\infty} \|x_{n_k} - Px_{n_k}\| = \|x - y\| \tag{3}$$

By (2)$|\ \|x_n - Px_n\| - \|x - Px\|\ | \leq \|x_n - x\| \to 0$ as $n \to \infty$, and so $\|x - y\| = \|x - Px\|$. Hence $Px = y i.e. \lim\limits_{k\to\infty} Px_{n_k} = Px$. Thus if (x_n) converges to $x, (Px_n)$ has a subsequence converging to Px ans so every subsequence of (Px_n) has a subsequence converging to Px. Therefore (Px_n) converges to Px and P is continuous. □

Definition 2.3. The mapping P of Lemma 2.1 is called the metric projection onto K.

Definition 2.4. A subset A of a normed space is said to be bounded if there exists a constant M such that $\|x\| \leq M$ $(x \in A)$.

We now state without proof three properties of finite dimensional normed spaces.

Lemma 2.2. *Every finite dimensional normed space is complete.*

Lemma 2.3. *Every bounded closed subset of a finite dimensional normed space is compact.*

Lemma 2.4 (Brouwer fixed point theorem)**.** *Let K be a non-empty com-*
22 *pact convex subset of a finite dimensional normed space, and let T be a continuous mapping of K into itself. Then T has a fixed point in K.*

The proofs of the first two of these Lemmas are elementary. (Refer to Dunford and Schwartz [14, p. 244-245].) The Brouwer fixed point theorem on the other hand is far from trivial. For a proof using some elements of algebraic topology refer to P. Alexandroff and H. Hopf ([1], p.376-378). A proof of a more analytical kind is given by Dunford ans Schwartz ([14], p.467).

Theorem 2.2 (Schauder)**.** Let K be a non-empty closed convex subset of a normed space. Let T be a continuous mapping of K into a cumpact subset of K. Then T has fixed point in K.

Proof. Let E denote the normed space and let $TK \subset A$, a compact subset of K. A is contained in a closed convex bounded subset of E.

$$T(B \cap K) \subset T(K) \subset A \subset B$$

so $T(B \cap K)$ is contained in a compact subset of B, K and there is no loss of generality in supposing that K is bounded. If A_o is a countable dense subset of the compact metric space A, then the set of all rational linear combinations of elements of A_o is a countable dense subset of the closed linear subspace E_o spanned by A_o and $A \subset E_0$. Then $T(K \cap E_0) \subset T(K) \subset A$, a compact subset of E_0, and $K \cap E_0$ is closed and convex. Hence without loss of generality we may assume that K is a bounded closed convex subset of a separable normed space E with a strictly convex norm (Theorem 2.1). $\square$

Given a positive integer n, there exists a $\frac{1}{n}$-net $Tx_1, \ldots, Tx_m$ say in TK, so that 23

$$\min_{1\le k\le n} \|Tx - Tx_k\| < \frac{1}{n} \ (x \in K) \qquad (1)$$

Let E_n denote the linear hull of $Tx_1, \ldots, Tx_m$. $K_n = K \cap E_n$ is a closed bounded subset of E_n and therefore compact (Lemma 2.3). Since the norm is strictly convex, the metric projection Pn of E onto the convex compact subset Kn exists. $T_n = P_n T$ is a continuous mapping of the non-empty convex compact subset Kn into itself, and therefore by the Brouwer fixed point theorem, it has a fixed point $u_n \varepsilon K_n$,

$$T_n u_n = u_n \qquad (2)$$

By (1), since $Tx_k \in K_n$ $(k = 1, 2, \ldots, m)$, we have

$$\|Tx - T_n x\| < \frac{1}{n} \qquad (3)$$

The sequence $\{Tu_n\}$ of TK has a subsequence Tu_{n_k} converging to a point $v \in K$. By (2) and (3), $\|u_{n_k} - v\| = \|T_{n_k} u_{n_k} - v\| \le \|T_{n_k} u_{n_k} - Tu_{n_k}\| + \|Tu_{n_k} - v\| < \frac{1}{n} + \|Tu_{n_k} - v\|$. Therefore, $\lim_{k\to\infty} u_{n_k} = u$, and by continuity of T, $\lim_{k\to\infty} Tu_{n_k} = Tv$ or $Tv = v$.

Example. Suppose that a function $f(x, y)$ of two real variables is continuous on a neighbourhood of (x_o, y_o). Then we can choose $\varepsilon > 0$ such that f is continuous in the rectangle

$$|x - x_o| \le \varepsilon, |y - y_o| \le m\varepsilon$$

and satisfies there the inequality 24

$$|f(x, y)| \le m.$$

Let E denote the Banach space $C_R[x_o - \varepsilon, x_o + \varepsilon]$, which is a Banach space with the uniform norm

$$\|\varphi\| = \sup\left\{|\Phi(t)| : |t - x_o| \le \varepsilon\right\}$$

Let K be the subset of E consisting of all continuous mappings of $[x_o - \varepsilon, x_o + \varepsilon]$ into $[y_o - m\varepsilon, y_o + m\varepsilon]$. Then K is a bounded closed convex subset of E. Let T be the mapping defined on K by

$$(T\phi)(x) = y_o + \int_{xo}^{x} f(t, \phi(t))dt \qquad (|x - x_o| \leq \varepsilon)$$

Then $TK \subset K$. Also since

$$\left|(T\phi)(x) - (T\phi)(x')\right| \leq \left| \int_{x'}^{x} f(t, (t))dt\right| \leq m|x - x'| \; (\phi \in K),$$

TK is an equicontinuous set. Since also TK is bounded, TK is contained in a compact set by the Ascoli - Arzela theorem. Therefore, by Theorem 2.2, T has a fixed point ϕ in K i.e.,

$$\phi(x) = y_o + \int_{x_o}^{x} f(t, \phi(t))\, dt \; (|x - x_o| \leq \varepsilon).$$

Then ϕ is differentiable in $[x_o - \varepsilon, x_o + \varepsilon]$ and provides a solution $y = \phi(x)$ there of the differential equation

$$\frac{dy}{dx} = f(x, y)$$

25 with $\phi(x_o) = y_o$. This is Peano's theorem. As a particular case of Schauder's theorem, we have

Theorem 2.3. Let K be a non-empty compact convex subset of a normed space, and let T be a continuous mapping of K into itself. Then T has a fixed point in K.

Remark. Theorem 2.2 and 2.3 are almost equivalent, in the sense that Theorem 2.2, with the additional hypothesis that K be complete, follows from Theorem 2.3. For, if K is a complete convex set and TK is contained in a compact subset A of K, then the closed convex hull of A is a compact convex subset K_o of K, and $TK_0 \subset K_0$.

Definition 2.5. A mapping T which is continuous and maps each bounded set into a compact set is said to be completely continuous.

Theorem 2.4. Let T be a completely continuous mapping of a normed space E into itself and let TE be bounded. Then T has a fixed point.

Proof. Let K be the closed convex hull of TE. Then K is bounded and so TK is contained in a compact subset of K. By Theorem 2.2, T has a fixed point in K. ◻

The Theorem 2.4 implies Theorem 2.3 is seen as follows. Let K be a compact convex set and let T be continuous mapping of K into itself. There is no loss of generality is supposing that the norm in E is strictly convex. Let P be the metric projection of E onto K, and let **26**
$\tilde{T} = TP$. Then $\tilde{T}$ satisfies the conditions of theorem 24, and so there exists u in E with $Tu = u$. Since T maps E into K, we have $u \in K$ and so $Pu = uTu = TRu = u$.

Lemma 2.5. *Let K be a non-empty complete convex subset of a normed space E, let A be a continuous mapping of K into a compact subset of E, and let F be a mapping of $K \times K$ into K such that*

(i) $\|F(x,y) - F(x,y)\| \leq k\|y - y'\|$ $(x, y, y' \in K)$, where k is a constant with $0 < k < 1$,

(ii) $\|F(x,y) - F(x',y)\| \leq \|Ax - Ax'\|$ $(x, x', y \in K)$. Then there exists a point u in K with
$$F(u,u) = u.$$

Proof. For each fixed x, the mapping $y \to F(x,y)$ is a contraction mapping of the complete metric space K into itself, and it therefore has a unique fixed point in K which we denote by Tx,

$$Tx = F(x, Tx) \qquad (x \in K).$$

$$\begin{aligned}
\text{We have } \|Tx - Tx'\| &= \|F(x, Tx) - F(x', Tx')\| \\
&\leq \|F(x, Tx) - F(x', Tx)\| \\
&\quad + \|F(x', Tx) - F(x', Tx')\| \\
&\leq \|Ax - Ax'\| + k\|Tx - Tx'\|
\end{aligned}$$ ◻

Therefore $\|Tx - Tx'\| \leq \dfrac{1}{1-k}\|Ax - Ax'\|$, (1) which shows that Tk is continuous and that $TK \subset K$ is precompact since AK is compact, since K is complete, $\overline{TK} \subset K$ is compact. By the Schander theorem, T has fixed point u in K,

$$Tu = u.$$

27 But then

$$F(u, u) = F(u, Tu) = Tu = u$$

□

Theorem 2.5 (Kranoselsku). Let K be a non-empty complete convex subset of a normed space E, let A be a continuous mapping of K into a compact subset of E, let B map K and satisfy a Lipschitz condition

$$\|Bx - Bx'\| \leq k\|x - x'\| \qquad (x, x' \in k)$$

with $0 < k < 1$ and let $Ax + By \in K$ for all x, y in K. Then there is a point $u \in K$ with

$$Au + Bu = u$$

Proof. Take $F(x, y) = Ax + By$ and apply Lemma 2.5. □

Corollary. *Let K be a non-empty complete convex subset of a normed space, let A be a continuous of K into a compact subset of K, let B map K into itself ans satisfy the Lipschitz condition*

$$\|Bx - Bx'\| \leq \|x - x'\| \quad (x, x' \varepsilon K),$$

and let $0 < \alpha < 1$. *Then there exists a point $u \in K$ with*

$$\alpha Au + (1 - \alpha)Bu = u$$

In general, under the condition of Schauder's theorem, we have no method for the calculation of a fixed point of a mapping. However there is a special case in which this can be done using a method due to Krasnoselsku.

Definition 2.6. A norm p is uniformly convex if it satisfies

$$p(x_n) = p(y_n) = 1 \ (n = 1, 2, \ldots), \lim_{n\to\infty} p(x_n + y_n) = 2 \Longrightarrow \lim_{n\to\infty} p(x_n - y_n) = 0.$$

28

Lemma . *Let p be a uniformly convex norm, and let εM be positive constants. Then there exists a constant δ with $0 < \delta < 1$ such that*

$$p(x) \leq M, p(y) \leq M, p(x - y) \geq \varepsilon \Rightarrow p(x + y) \leq 2\delta \max(p(x), p(y)).$$

Proof. For all x, y, we have

$$p\left(\frac{1}{2}(x + y)\right) \leq \frac{1}{2}p(x) + \frac{1}{2}p(y) \leq \max(p(x), p(y)). \tag{1}$$

If there is no constant δ with the stated properties, there exist sequences $(x_n), (y_n)$ with $p(x_n) \leq M$, $p(y_n) \leq M$,

$$p(x_n - y_n) \geq \varepsilon, \tag{2}$$

and

$$p\left(\frac{1}{2}(x_n + y_n)\right) > \left(1 - \frac{1}{n}\right) \max(p(x_n), p(y_n)). \tag{3}$$

Let $\alpha_n = p(x_n), \beta_n = p(y_n), \gamma_n = \max(\alpha_n, \beta_n)$. By (1) and (2),

$$\gamma_n \geq \frac{1}{2}, \tag{4}$$

and so, by (1) and (3)

$$\lim_{n\to\infty} \frac{1}{\gamma_n} p\left(\frac{1}{2}(x_n + y_n)\right) = 1. \tag{5}$$

It follows from (1) and (5), that

$$\lim_{n\to\infty} \frac{\alpha_n + \beta_n}{2\gamma_n} = 1. \tag{6}$$

Since (γ_n) is bounded, there exists a convergent sequence (γ_{n_k}), and by (4)

$$\lim_{k\to\infty} \gamma_{n_k} = \gamma \geq \varepsilon_{\gamma_2} \tag{7}$$

$$\lim_{k\to\infty} (\gamma_{n_k} - \alpha_{n_k}) + (\gamma_{n_k} - \beta_{n_k}) = 0,$$

29 and, since each bracket is non-negative, each tends to zero. Therefore

$$\lim_{k\to\infty} \alpha_{n_k} = \lim_{k\to\infty} \beta_{n_\alpha} = \gamma \tag{8}$$

By discarding some terms of the subsequence if necessary, we may suppose that $\alpha_{n_k} \geq 0$ and $\beta_{n_k} \geq 0$ for all k. Since

$$\begin{aligned} \left| p\left(\frac{1}{\alpha_{n_k}} x_{n_k} + \frac{1}{\beta_{n_k}} y_{n_k}\right) - p\left(\frac{1}{\gamma_{n_k}} x_{n_k} + \frac{1}{\gamma_{n_k}} y_{n_k}\right) \right| \\ \leq p\left(\frac{1}{\alpha_{n_k}} - \frac{1}{\gamma_{n_k}}\right) x_{n_k} + \left(\frac{1}{\beta_{n_k}} - \frac{1}{\gamma_{n_k}}\right) y_{n_k} \\ \leq M\left\{\frac{1}{\alpha_{n_k}} - \frac{1}{\gamma_{n_k}} + \frac{1}{\beta_{n_k}} - \frac{1}{\gamma_{n_k}}\right\}, \end{aligned}$$

it follows from (5), (7), (8), that

$$\lim_{k\to\infty} p\left(\frac{1}{\alpha_{n_k}} x_{n_k} + \frac{1}{\beta_{n_k}} y_{n_k}\right) = 2.$$

Therefore,

$$\lim_{k\to\infty} p\left(\frac{1}{\alpha_{n_k}} x_{n_k} - \frac{1}{\beta_{n_k}} y_{n_k}\right) = 0.$$

and so

$$\lim_{k\to\infty} \frac{1}{\gamma_{n_k}} p(x_{n_k} - y_{n_k}) = 0,$$

which contradicts (2). □

30 **Theorem 2.6** (Krashoselsku)**.** Let K be a bounded closed convex set in a Banach space E with a uniformly convex norm. Let T be a mapping of K into a compact subset of K that satisfies a Lipschitz condition with

Lipschitz constant 1, and let x_o be an arbitrary point of K. Then the sequence defined by

$$x_{n+1} = \frac{1}{2}(x_n + Tx_n) \qquad (n = 0, 1, 2, \ldots)$$

converges to a fixed point of T in K.

Proof. By Schauder's theorem, there is a nonempty set F of fixed points of T in K. We prove first that

$$\|x_{n+1} - y\| \leq \|x_n - y\| \quad (y \in F, n = 0, 1, 2, \ldots)$$

In fact if $y = T_y$, then

$$\begin{aligned}
\|x_{n+1} - y\| &= \|\frac{1}{2}(x_n + Tx_n) - \frac{1}{2}(y + Ty)\| \\
&= \|\frac{1}{2}(x_n - y) + \frac{1}{2}(Tx_n - Ty)\| \\
&\leq \frac{1}{2}\|x_n - y\| + \frac{1}{2}\|Tx_n - Ty\| \\
&\leq \|x_n - y\|
\end{aligned}$$

which is (1). □

Suppose that there exist an $\varepsilon > 0$ and N, such that

$$\|x_n - Tx_n\| \geq \varepsilon \text{ for all } n \geq N \qquad (2)$$

Then $\|x_n - y - (Tx_n - T_y)\| \geq \varepsilon$ for all $n \geq N, y \in F$.
Also $\|Tx_n - T_y\| \leq \|x_n - y\| \leq \|x_o - y\|$, by (1).
Since the norm is uniformly convex, this implies that there exists a constant $\delta, 0 < \delta < 1$, such that

$$\begin{aligned}
\|x_{n+1} - y\| &= \|\frac{1}{2}(x_n + Tx_n) - \frac{1}{2}(y + Ty)\| \\
&= \|\frac{1}{2}(x_n - y) + \frac{1}{2}(Tx_n - Ty)\| \\
&\leq \max\{\|x_n - y\|, \|Tx_n - Ty\|\}
\end{aligned}$$

$$\leq \|x_n - y\| \text{ for } n \geq N.$$

Therefore $\lim\limits_{n\to\infty} x_n = y$ where $T_y = y$. **31**

If there does not exist an $\varepsilon > 0$ for which (2) holds, there exists a sequence n_k of integers such that $\lim\limits_{k\to\infty}(x_{n_k} - Tx_{n_k}) = 0$, and such that (Tx_{n_k}) converges. But this implies that $\lim\limits_{k\to\infty} x_{n_k} = u = \lim\limits_{k\to\infty} Tx_{n_k}$ and so $Tu = u$.

Hence $\|x_{n+1} - u\| \leq \|x_n - u\|$, by (1). Since $\lim\limits_{k} \|x_{n_k} - u\| = 0$, we have $\lim\limits_{n\to\infty} \|x_n - u\| = 0$ and the theorem is proved.

The following theorem was proved by Altam by means of the concept of 'degree of a mapping', but we can easily deduce it from schander's theorem.

Theorem 2.7 (Altman)**.** Let E be a normed space, let Q be the closed ball of radius $r > 0$,

$$Q = \{x : \|x\| \leq r\}$$

and let T be a continuous mapping of Q into a compact subset of E such that

$$\|Tx - x\|^2 \geq \|Tx\|^2 - \|x\|^2 \quad (\|x\| = r)$$

Then T has fixed point in Q.

Proof. Suppose T has no fixed point in Q then

$$\|Tx - x\| + \|x\| > \|Tx\| \qquad (\|x\| = r) \tag{1}$$

For

$$(\|Tx - x\| + \|x\|)^2 - \|Tx\|^2 = \|Tx - x\|^2 + \|x\|^2 - \|Tx\|^2$$
$$+2\|x\|\,\|Tx - x\| \geq 2r\|Tx - x\| > 0$$

32 Let P be the mapping defined by

$$Px = \begin{cases} x & (x \in Q) \\ \frac{r}{\|x\|}x & (x \notin Q) \end{cases}$$

Plainly P is a continuous projection of E onto Q. □

Let $\tilde{T} = PT$

Then T maps Q continuously into a compact subset of Q. Hence, by the Schauder theorem, $\tilde{T}$ has a fixed point u in Q,

$$PTu = u$$

If $Tu \in Q$, then $PTu = Tu$, and

$$Tu = u$$

If $Tu \notin Q$, then $\|Tu\| > r$ and

$$u = PTu = \frac{r}{\|Tu\|} Tu$$

If follows that $\|u\| = r$, and we have

$$\begin{aligned} \|Tu - u\| + \|u\| &= \|\frac{\|Tu\|}{r} u - u\| + \|u\| \\ &= \left(\frac{\|Tu\|}{r} - 1 + 1 \right) \|u\| = \|Tu\| \end{aligned}$$

which contradicts (1)

supplementary results and exercises

(1) For further results connected with Theorem 2.6, see [19]

(2) Let A be a continuous mapping of a normed space E into itself which maps bounded sets into compact sets and satisfies

$$\lim_{\|x\| \to \infty} \frac{\|Ax\|}{\|x\|} = 0$$

Then given arbitrary real $\lambda > 0$ and y in E, the equation

$$x = \lambda Ax + y$$

has a solution x in E

Consider the mapping $Tx = \lambda Ax + y$

Clearly T has all the properties of A

Let $S_n = \{x : x \in E, \|x\| \leq n\} \quad (n = 1, 2, \ldots)$

Then

$TS_n \subset S_n$ for some n (1)

Otherwise $\|Tx_n\| > n$, for some $x_n \in S_n \ \ n = 1, 2, \ldots$ (2)

If $\{x_n\}$ were bounded, then $\{Tx_n\}$ will be contained in a compact set and therefore $\|Tx_n\|$ will be bounded which contradicts (2). Hence $\|x_n\| \to \infty$ as $n \to \infty$ But

$$\frac{\|Tx_n\|}{\|x_n\|} > 1 \text{ so } \lim_{\|x_n\|\to\infty} \frac{\|Tx_n\|}{\|x_n\|} \geq 1$$

As this is not true, T maps some S_n into its compact subset; Schauder's theorem then a gives a fixed point x which is the required solution.

(3) Let $\sum\limits_{k=1}^{\infty} a_k$ be a convergent series of non-negative real numbers and let (f_k) be a sequence of continuous mappings of the real line R into itself such that

$$|f_k(t)| \leq a_k \qquad (t \in R, k = 1, 2, \ldots)$$

Given $\alpha \in R$, there exists a convergent real sequence (ξ_k) such that

(i) $\xi = \alpha$

(ii) $\xi_{k+1} - \xi_k = f_k(\xi_k) \quad (k = 1, 2, \ldots)$

consider the mapping T of (c) into itself given by

$$(Tx)_1 = \alpha$$

$$(Tx)_{n+1} = \alpha + \sum_{k=1}^{n} f_k(\xi_k) \quad (n = 1, 2, \ldots)$$

 where $x = (\xi_k)$.

(4) Let E be a Banach space with a uniformly convex norm, and let K be a bounded closed convex subset of E. Then the metric projection $E \xrightarrow{\text{onto}} K$ exists and is uniformly continuous on each bounded subset of E.

(5) Brodsku and Milman [10], give conditions under which a convex set in a Banach space has a point invariant under all isometric self mappings. In this connection see also Dunford and Schwartz ([14], p.459).

(6) Browder (11) gives some generalization of the Schauder theorem which appear to lie rather deep. Perhaps the most striking of these results is the following generalization of theorem 2.4. Let T be a continuous mapping of a Banach space E into itself that maps bounded sets into compact sets. If, for some positive integer m, $T^m E$ is bounded, then T has a fixed point. For a generalization of the Schauder theorem of a different kind see Stepaneek (32).

(7) Aronszajn [2] gives general regularity condition on T sufficient to establish that the set of its fixed points is an R_δ i.e. is a homeomorphic image of the intersection of decreasing sequence of absolute retracts.

Chapter 3

The Schauder - Tychonoff theorem

It this chapter we are concerned with non-linear operators in general locally convex spaces. **35**

Definition 3.1. A vector space E over R which is also a topological space is called a *linear topological space (l.t.s)* if the mappings

$$(x, y) \to x + y$$
$$(\alpha, x) \to \alpha x$$

from $E \times E$ and $R \times E$ respectively into E are continuous. If also every open set in E is a union of convex open sets, then F is said to be *locally convex*.

We establish the elementary properties of a $l.t.s$ E. Since the mapping $(\alpha, x) \to \alpha x$ is continuous, the mapping $x \to \alpha x$, with fixed α, is continuous. Therefore, if α is a non-zero constant then the mapping $x \to \alpha x$ is a homeomorphism, and so

a) G open, $\alpha \neq 0 \Longrightarrow \alpha G$ open.

In particular

b) G open implies that - G is open.

c) Similarly, G open $\Longleftrightarrow y + G$ open, and so

d) V is neighbourhood of 0 if and only if $y + V$ is neighbourhood of y.

36 Let V be a neighbourhood of 0, and let $x \in E$. Since the mapping $\alpha \to \alpha x$ is continuous, and $0x = 0$, we have $\frac{1}{\lambda} x \in V$ for all sufficiently large λ, i.e.,

e) $x \in \lambda V$ for all sufficiently large λ.

We prove next that

f) The closure of a convex set is convex.

For $0 \leq \alpha \leq 1$, the mapping $f : E \times E \to E$ given by

$$(x, y) \to \alpha x + (1 - \alpha) y$$

is continuous and $f(K \times K) \subset K$. Therefore $f(\overline{K \times K}) \subset \bar{K}$, where $\bar{K}$ denotes the closure of K. But $\overline{K \times K} = \bar{K} \times \bar{K}$ and so $f(\bar{K} \times \bar{K}) \subset \bar{K}$ i.e., $\alpha a + (1 - \alpha) b \in \bar{K}$ for $a, b \in \bar{K}$.

g) The interior of a convex set is convex.

Let K_0 be the interior of a convex set K, let $a, b \in K_0$ and $0 < \alpha < 1$. By $(a), \alpha K_0, (1 - \alpha) K_0$ are open sets. By (c) $\alpha K_0 + (1 - \alpha) K_0$ is a union of open sets and is therefore open. Since

$$\alpha a + (1 - \alpha) b \in K_0 + (1 - \alpha) K_0 \subset K,$$

it follows that $\alpha a + (1 - \alpha) b \in K_0$. A subset A of a vector space E over R is said to be symmetric if $-A = A$.

h) Let U be a neighbourhood of 0 in a locally convex *l.t.s.* Then there exists a closed convex symmetric neighbourhood V of 0 with $V \subset U$.

Since 0 is an interior point of U and the space is locally convex,
37 there exists a convex open set G with $0 \in G \subset U$. Let $H = \frac{1}{2}(G \cap -G)$, and $V = \bar{H}$. By (b) and (f) V is a closed convex symmetric neighbourhood of 0. Finally $V \subset U$; for if $v \in V$, then $v + H$ is an

open set containing and therefore has nonempty intersection with H, ie there exists h, h' in H with $v + h = h'$. Since H is convex and symmetric,

$$v = h' - h \in 2H \subset G. \text{ Thus } V \subset G \subset U.$$

Definition. *Given an l.t.s. E over K, a subset A of E is said absorb points if for every x in E,*

$$x \in \lambda A$$

for all sufficiently large λ.

Definition. *Given a convex set K that absorbs points, the Minkowski functional p_K is defined by*

$$p_K(x) = \inf\{\lambda; \lambda > 0, \text{ and } x \in \lambda K\}$$

Definition. *A mapping p of E into R is called a seminorm on E if it satisfies the axioms.*

i) $p(x) \geq 0 \qquad (x \in E)$

ii) $p(\alpha x) = |\alpha| p(x) \qquad (x \in E, \alpha \in R)$

iii) $p(x + y) \leq p(x) + p(y)$

Given a seminorm p, the *seminorm topology* determined by p is the class of unions of open balls

$$S(x, \in) = \{y : p(y - x) < \varepsilon\} \ (\varepsilon > 0)$$

With this topology E is a locally convex *l.t.s.* which is not in general a Hausdorff space.

The Minkowski functional of a convex set K that absorbs points is **38**
sublinear, and if K is also symmetric, then it is a seminorm. Also if $x \in \lambda K$ and $\mu > \lambda$, then $x \in \mu K$, for $0 \in K$ since K absorbs points and

$$\frac{1}{\mu} x = \frac{\lambda}{\mu} \left(\frac{1}{\lambda} x \right) + \left(1 - \frac{\lambda}{\mu} \right) \cdot 0 \in K$$

i) If K is a closed convex symmetric neighbourhood of 0 in a *l.t.s*, the p Minkowski functional p_K is a continuous semi-norm in E, and

$$K = \{x; p_K(x) \leq 1\}$$

Conversely, if p is a continuous seminorm in E, then $\{x : p(x) \leq 1\}$ is a closed convex symmetric neighbourhood K of 0, and $p_K = p$.

Proof. Let K be a closed convex symmetric neighbourhood of 0. ◻

Then p_K is a seminorm on E, and so

$$|p_K(x') - p_K(x)| \leq p_K(x' - x) \qquad (x', x \in E)$$

Given $\varepsilon > 0$,

$$\begin{aligned} x'\varepsilon, x + \varepsilon K &\Rightarrow x' - x\varepsilon K \\ &\Rightarrow p_K(x' - x) \leq \varepsilon \\ &\Rightarrow |p_K(x') - p_K(x)| \leq \varepsilon \end{aligned}$$

since $x + \varepsilon K$ is a neighbourhood of x, this shows that p_K is continuous. If $x\varepsilon K$, then $p_K(x) \leq 1$, by the definition of p_k. On the other hand, if $p_K(x) \leq 1$, then $x \; \varepsilon \; \lambda K(\lambda > 1)$,

$$\frac{1}{\lambda} x \in K(\lambda > 1),$$

and, since K is closed, $x \; \in K$.

39 Thus

$$K = \{x : p_k(x) \leq 1\}.$$

Conversely, let p be a continuous semi-norm on E, and let $K = \{x : p(x) \leq 1\}$. That K is a closed convex symmetric neighbourhood of the origin is evident. We have

$$p(x) \leq 1 \Leftrightarrow x \in K \Leftrightarrow p_k(x) \leq 1,$$

and, since p and p_k are both positive-homogeneous, it follows that $p = p_k$.

(j) Let x be a nonzero point of a Hausdorff locally convex $l.t.s.E$. Then there exists a continuous semi-norm p on E with $p(x) > 0$.

Proof. Since $x \neq 0$ and E is a Hausdorff space, there exists a neighbourhood U of 0 such that $x \notin U$. By (h) there exists a closed convex symmetric neighbourhood U of 0 with $V \subset U$. By (i), there exists a continuous semi-norm p on E such that

$$V = \{y : p(y) \leq 1\}$$

Hence $p(x) > 1$. □

(k) Let E be a vector space over K. Let p be a semi-norm on E, and let $N = \{x : p(x) = 0\}$. Then N is a subspace of E, and the functional q defined on the quotient space $\dfrac{E}{N}$ by

$$q(\tilde{x}) = p(x) \qquad (x \in \tilde{x}, \tilde{x}\ \frac{E}{N})$$

is a norm on E/N

Proof. If $x, y \in N$, then 40

$$0 \leq p(x + y) \leq p(x) + p(y) = 0,$$

and so $x + y \in N$. Also $p(x) = 0$ implies $p(\lambda x) = 0$, and so N is a linear subspace of E. The definition of $q(\tilde{x})$ is in fact free from ambiguity, for if $x, x' \in \tilde{x}$, then $x - x' \in N$, and so

$$|px - p(x')| \leq p(x - x') = 0,$$
$$p(x) = p(x').$$

Finally that q satisfies the axioms of a norm is entirely straightforward. □

Lastly, among these preliminary results, we need a proposition which is a special case of a general theorem on uniform spaces. However, it is more convenient for our purposes to prove the special case than to invoke the general theory.

(b) Let E, F be linear topological spaces, let K be a compact subset of E and let T be a continuous mapping of K into F. Given a neighbourhood U of 0 in F, there exists a neighbourhood V of 0 in E such that

$$x, x' \in K,\ x - x' \in V \Rightarrow Tx - Tx' \in U.$$

Proof. Let H be an open set containing 0 such that

$$H - H \subset U$$

41 Given $x \epsilon K$, there exists a neighbourhood $G(x)$ of 0 such that $x' \in K \cap (X + G(x)) \Rightarrow Tx' \in Tx + H$.

Let $V(x)$ be an open neighbourhood of 0 in E such that

$$V(x) + V(x) \subset G(x),$$

since K is compact and is covered by open sets $x + V(x)$, it has a finite covering

$$x_1 + V(x_1), \ldots, x_n + V(x_n).$$

Let $V = \bigcap_{i=1}^{n} V(x_i)$.

Then V is a neighbourhood of 0 in E. Suppose $x, x' \in K$ and $x - x' \in V$. Then there exists j with

$$x' \in x_j + V(x_j) \subset x_j + G(x_j)$$

$$x - x_j = x - x' + x' - x_j \in V + V(X_j) \subset V(x_j) + V(x_j) \subset G(x_j)$$

since $$x, x' \in x_j + G(x_j),$$

we have $$Tx \in Tx_j + H, Tx' \in Tx_j + H,$$

and so $$Tx - Tx' \in H - H \subset U.$$

We are now ready to prove the main theorem by which we are able to deduce properties of operators in a locally convex linear topological space

from the corresponding properties of operators in normed spaces. The
42 main idea of this theorem was derived from the proof of the Schauder-Tychnoff theorem in Dunford and Schwartz [14] p.454. □

Theorem 3.1. Let K be a compact subset of a locally convex l.t.s E, T a continuous mapping of K into itself, $p_\circ$ a continuous semi-norm on E.

Then there exists a semi-norm q on the linear $L(K)$ of K such that

i) $q(x) \geq p_\circ(x) \quad (x \in L(K))$;

ii) q is continuous on $K - K$;

iii) K is compact with respect to the semi-norm topology given by q;

iv) T is uniformly continuous in K with respect to q i.e., given $\epsilon > 0$, there exists $\delta > 0$, such that

$$x, x' \in K,\ q(x - x') < \delta \Rightarrow q(Tx - Tx') < \epsilon$$

Remark. It would be better if one could prove the existence of a continuous semi-norm q on E satisfying (i) and (iv).

Proof. Since $p_\circ$ is bounded on K there is no real loss of generality in supposing that

$$p_\circ(x) \leq 1 \quad (x \in K).$$

It is convenient to introduce the following definition. We say that a set Γ of continuous semi-norm *dominates* a set Γ of continuous semi-norms if the following two conditions are satisfied. □

a) $p'(x) \leq 1 \quad (x \in K, p' \in \Gamma')$ 43

b) given $p \in \Gamma$ and $\epsilon > 0$, there exists $p' \in \Gamma'$ and $\delta > 0$ such that $x, x' \in K, p'(x - x') < \delta \Rightarrow p(Tx - Tx') < \epsilon$.

We construct a countable self-dominating set containing $p_\circ$. Given a continuous semi-norm p, and a positive integer n, the set

$$\left\{x : p(x) < \frac{1}{n}\right\}$$

is a neighbourhood of 0. Therefore, by proposition (2), there exists a neighbourhood V of 0 in E such that

$$x, x' \in K, x - x' \in V \Rightarrow p(Tx - Tx') < \frac{1}{n}.$$

By (h), we may suppose that V is a closed convex symmetric neighbourhood of 0, and then by (i), p_V is a continuous semi-norm and

$$V = \{x : p_V(x) \leq 1\}.$$

Multiplying p by an appropriate positive constant δ_n, we obtain a continuous semi-norm q_n such that

$$q_n(x) \leq 1 \quad (x \in K),$$

and such that $$x, x' \in K, q_n(x - x') < \delta_n \Rightarrow p(Tx - Tx') < \frac{1}{n}.$$

44 Plainly the set of semi-norms q_n is a countable set dominating the set (p).

It follows that given a countable set Γ of continuous semi-norms, there exists a countable set Γ' that dominates Γ. Now the set $(p_\circ)$ is dominated by a countable set Γ_1, Γ_1 is dominated by a countable set Γ_2, and so on. Finally, we take

$$\Gamma = (p_\circ) \cup \bigcup_{n=1}^{\infty} \Gamma_n.$$

Then Γ is a countable self-dominating set. Let $(p_n)_\circ^\infty$ be an enumeration of Γ and take

$$q(x) = \sum_{n=0}^{\infty} 2^{-n} p_n(x) \tag{1}$$

since

$$p_n(x) \leq 2 \quad (x \in K - K),$$

the series (1) converges uniformly on $K - K$, and so q is continuous on $K - K$. Also the series converges on $L(K)$ (linear hull of K) and q is a semi-norm there satisfying (i). Given $x \epsilon K$, let

$$S(x, \rho) = \{x'; x' \in K \text{ and } q(x - x') < \rho\}$$

since q is continuous on $K - K$, $S(x,\rho)$ is an open subset of K in the topology τ on $L(K)$ induced from the initial topology on E. Hence each open subset of K in the topology induced by τ_q (topology on $L(K)$ de- 45
fined by q) is also open in the topology induced by τ. (iii) is now an immediate consequence of the τ-compact-ness of K.

Given $\epsilon > 0$, we choose N with $2^{-N} < \dfrac{\epsilon}{4}$ since we have

$$\sum_{n=N+1}^{\infty} \frac{1}{2^n} p_n(x - x') \leq \sum_{n=N+1}^{\infty} \frac{1}{2^{n+1}} < \frac{\epsilon}{2} \quad (x, x' \in K),$$

and so

$$q(x - x') \subset \sum_{n=0}^{N} \frac{1}{2^n} p_n(x - x') + \frac{\epsilon}{2} \quad (x, x' \in K) \tag{2}$$

since T maps K into itself, (2) gives

$$q(Tx - Tx') < \sum_{n=0}^{N} \frac{1}{2^n} p_n(Tx - Tx') + \frac{\epsilon}{2} \quad (x, x' \in K) \tag{3}$$

since Γ is self-dominated, for each n, there exists k_n and $\delta_n > 0$ such that

$$p_{kn}(x - x') < \delta_n \Rightarrow p_n(Tx - Tx') < \frac{\epsilon}{4}(x, x' \in K) \tag{4}$$

Let $N' = \max(k_\circ, \ldots, k_N)$, and

$$= 2^{-N'} \min(\delta_\circ, \ldots, \delta_N).$$

Then since $p_n \leq 2^{N'} q$ for $n \leq N'$, we have

$$q(x - x') < \delta \Rightarrow_{p_{k_n}} (x - x') < \delta_n \quad (n \leq N)$$

and so, by (4)

$$x, x' \epsilon k, q(x - x'), < \delta \Rightarrow p_n(Tx - Tx') < \frac{\epsilon}{4}\ (n = 0, 1, \ldots, N).$$

Therefore, by (3), 46

$$x, x' \in K, q(x - x') < \delta \Rightarrow q(Tx - Tx') < \varepsilon.$$

Theorem 3.2 (Schauder-Tychonoff)**.** Let K be a non- empty compact convex subset of a locally convex Hausdorff $l.t.sE$, and let T be a continuous mapping of K into itself. Then T has a fixed point in K.

Proof. There is no loss of generality in supposing that $L(K) = E$. Suppose that T has no fixed point in K. Then $Tx - Tx \neq 0\,(x \in K)$. ◻

It follows by proposition (j) that for each point x of K there exists a continuous semi-norm p_x such that

$$p_x(Tx - x) > 0$$

By continuity of T and p_x, there exists a neighbourhood U_x of x such that

$$p_x(T_y - y) > 0 \quad (y \in U_x)$$

Since K is compact, there is a finite covering of K by such neighbourhood say

$$U_{x_1}, \ldots, U_{x_m}.$$

$$\textit{Let} \qquad p = p_{x_1} + p_{x_2} + \cdots + p_{x_m}.$$

47 Then p is a continuous semi-norm and

$$p(Tx - x) > 0 \qquad (x \in K) \tag{1}$$

Let q be the semi-norm constructed as in Theorem 3.1 with $p_0 = p$. Then q is defined on $L(K) = E, q \geq p, K$ is compact in the semi-norm topology τ_q, and given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x, x' \epsilon K, q(x - x') < \delta \Rightarrow q(Tx - Tx') < \varepsilon \tag{2}$$

Let $N = \{x : q(x) = 0\}$. By lemma 3.4, E/N is a normed space with the norm given by

$$\|\tilde{x}\| = q(x)$$

where $\tilde{x}$ is the coset of x. Let $\tilde{K} = \{x : x \in K\}$. Then since mapping $x \to \tilde{x}$ is a continuous homomorphism from E with the topology τ_q to

E/N with the norm topology, K is a compact convex set in E/N. Also by (2),

$$x, x' \in K, q(x - x') = 0 \Rightarrow q(Tx - Tx') = 0 \tag{3}$$

For each $\tilde{x}$ in $\tilde{K}$ there exists a point x in $\tilde{x} \cap K$, and we define $\tilde{T}\tilde{x}$ by taking

$$\tilde{T}\tilde{x} = \widetilde{Tx}$$

By (3), this definition is unambiguous, and $\tilde{T}$ maps $\tilde{K}$ into itself. Also, by (2), given $\varepsilon > 0$, there exists $\delta > 0$ such that $\tilde{x}, \tilde{x'} \in K, \|\tilde{x} - \tilde{x}'\| < \delta \Rightarrow \|\tilde{T}\tilde{x} - \tilde{T}\tilde{x}'\| < \epsilon$. For given $\tilde{x}$, $\tilde{x}' \epsilon K$, there exist $x \in K \cap \tilde{x}$ and 48
$x' \epsilon K \cap x'$ and $q(x - x') = \|\tilde{x} - \tilde{x}'\|$. Hence $\tilde{T}$ is a continuous mapping of the compact convex subset $\tilde{K}$ of the normed space E/N. Applying the Schauder fixed point theorem, $\tilde{T}$ has a fixed point $\tilde{u}$ say

$$\tilde{T}\tilde{u} = \tilde{u}.$$

Since $\tilde{u} \epsilon \tilde{K}$, there exists $u \in K \cap \tilde{u}$, and we have $\tilde{T}\tilde{u} = \widetilde{Tu}$. Thus

$$Tu - u \in N,$$
$$i.e., \qquad q(Tu - u) = 0$$

It follows that $p(Tu - u) = 0$, which contradicts (1) since $u \in K$.

Problem. It will be noticed that Theorem 3.2 generalizes theorem 2.3 rather than the full force of the Schauder theorem (2.2). It is not known whether the following proposition is true.

Q. Let K be a closed convex subset of a locally convex Hausdorff *l.t.s.* E, and let T be a continuous mapping of K into a compact subset of K. Then T has a fixed point in K.

It is obvious that if T maps K into a *compact convex* subset H of K, then T has a fixed point. For

$$TH \subset TK \subset H$$

and we can apply theorem 3.2 to H instead of K. In particular, Q will 49
hold if every compact subset of K is contained in compact convex subset of K. By an elementary theorem of Bourbaki (Espaces Vectoriels

Topologiques, $Ch.II$, $p.80$) the convex hull of a precompact subset of a locally convex Hausdorff $l.t.s$ is precompact. Thus we can obtain a true theorem from Q by supposing that K be complete instead of closed or E quasi-complete. However, this is certainly unnecessarily restrictive. By the Krein-Smulian Theorem [21], if E is a branch space with the weak topology as the specified topology, then the closed convex hull of each compact subset of E is compact, and so the proposition Q holds, even though K need not be completes (in the weak topology).

Example. Let E be a reflexive Banach space, K a closed convex subset of E, T a weakly continuous mapping of K into a bounded subset of K. Then T has a fixed point in K.

For since K is norm closed and convex it is also weakly closed. Also since E is reflexive, each bounded weakly closed subset of E is weakly compact. Hence the weakly closed convex hull of TK is weakly compact.

Chapter 4

Nonlinear mappings in cones

The theorems in this chapter are mainly due to Krein and Rutman [20] **50**
and to Schaefer [28]. They may be regarded as a further step in the transition from nonlinear to linear problems. We will be content with considering normed spaces only, through theorems of the kind studied here have been proved for general locally convex spaces by H. Schaeffer.

Definition 4.1. A subset C of a vector space E over R is called a *positive cone* if it satisfies

(i) $x, y \in C \Rightarrow x + y \in C$

(ii) $x \in C, \alpha \geq 0 \Rightarrow \alpha \in C$

(iii) $x, -x \in C \Rightarrow x = 0$

(iv) C contains non-zero vectors.

A vector space E over R with a specified positive cone is called a *partially ordered vector space* , and we write $x \leq y$(or $y \geq x$) to denote that $y - x \in C$. It is easily verified that this relation $\leq$ is a relation of partial order in the usual sense i.e.,

(v) $x \leq x$ $(x \in E)$,

(vi) $x \leq y, y \leq z \Rightarrow x \leq z$,

(vii) $x \leq y, y \leq x \Rightarrow x = y$.

Also the partial ordering and the linear structure are related by the prop-
51 erties:

(viii) $x_i \leq y_i$ $(i = 1, 2) \Rightarrow x_1 + x_2 \leq y_1 + y_2$,

(ix) $x \leq y, 0 \leq \alpha \leq \beta \Rightarrow \alpha x \leq \beta y$.

Conversely, given a non-trivial relation $\leq$ in E satisfying $(v), \ldots,$ (xi), the set $\{x : 0 \leq x\}$ is a positive cone in E to which the given relation corresponds in the above manner.

Definition 4.2. Let C be a positive cone in a normed space E. A mapping T of a subset D of C into C is said to be *strictly positive* in D if

$$x_n \in D, \lim_{n\to\infty} Tx_n = 0 \Rightarrow \lim_{n\to\infty} x_n = 0$$

A mapping T defined on C is said to be *completely* continuous in C if it is continuous in C and maps each bounded subset of C into a compact set.

Theorem 4.1 (Morgenstern [23])**.** Let C be a closed positive cone in a normed vector space E such that the norm is additive on $C i.e.$

$$\|x + y\| = \|x\| + \|y\| \quad (x, y \in C)$$

Let $c > 0$, let $K = \{x : x \in C, \|x\| = c\}$, and let T be a continuous and positive mapping on K and strictly positive on K and map K into a compact subset of C. Then there exists u in K and $\lambda > 0$ such that $Tu = \lambda_u$.

Proof. Since the norm is additive on C, K is a convex set. Since T is strictly positive on K,

$$\inf\{\|Tx\|, x \in K\} > 0$$

and therefore the mapping A defined on K by

$$Ax = c\|Tx\|^{-1}Tx$$

52 is continuous and maps K into a compact subset A itself. By the Schauder theorem, there exists $u \in K$ with $Au = u$. $\square$

Corollary. *Let C be a closed positive cone in a normed vector space E such that the norm is additive on C. Let T be a strictly positive and completely continuous mapping of C into itself. Then for each $c > 0$, there exists u_c in C and $\lambda_c > 0$ such that $Tu_c = \lambda_c u_c$ and $\|u_c\| = c$.*

Definition 4.3. A positive cone C in a normed vector space is said be normal if there exists a positive constant γ such that

$$\|x + y\| \geq \gamma\|x\| \ (x, y \in C).$$

Theorem 4.2 (Schaefer)**.** Let C be a closed normal positive cone in a normed space. Let $c > 0$ and let $K = \left\{x|^{x\in C}_{\|x\|\leq c}\right\}$. Let T be a continuous and strictly positive on K and map K into a compact set. Then there exists $u \in C$, and $\lambda > 0$, such that $Tu = \lambda_u$ and $\|u\| = c$.

Proof. Since TK is contained in a compact set we can choose $\mu > 0$, such that $\mu Tk \subset K$. Let $A = \mu T$, let y be a point of K with $y = c$, and let B be the mapping defined on K by $Bx = c^{-1}xAx + c^{-1}(c - x)y \ (x \in k)$, since K is convex, we have $BK \subset K$. Also B is continuous in K, and maps K into a compact set. Since T is strictly positive on K, there exists $c > 0$ such that

$$x \in K, \|x\| \geq \frac{1}{2}c \Rightarrow \|Ax\| \geq \varepsilon$$

since C is a normal cone, it follows that 53

$$x \in K, \|x\| \geq \frac{1}{2}c \Rightarrow \|Bx\| \geq \gamma c^{-1}\frac{1}{2}c\varepsilon = \frac{1}{2}\gamma\varepsilon$$

On the other hand,

$$x \in K, \|x\| \leq \frac{1}{2}c \Rightarrow c^{-1}(c - \|x\|)\|y\| \geq \frac{1}{2}c \Rightarrow \|Bx\|\frac{1}{2}\gamma c$$

Therefore $\|Bx\| \geq \frac{1}{2}\gamma \min(\varepsilon, c) > 0 \ (x \in K)$.

It follows that the mapping

$$x \to c\|Bx\|^{-1}Bx$$

is a continuous mapping of K into a compact subset of itself. Therefore, there exists $u \in K$ with

$$u = c\|Bu\|^{-1}Bu$$

Plainly $\|u\| = c$, and so $Bu = Au$, and we have

$$Au = \lambda, \text{ with } \lambda = c^{-1}\|Au\| > 0.$$

The following theorem due to Krein and Rutman [20, Theorem 9.1] marks a further transition towards a linear problem. □

Definition 4.4. Let E be a partially ordered vector space with positive cone C, let T be a mapping of C into itself and let c be be positive real number. T is said to be

54 (i) positive-homogeneous of

$$T(\alpha x) = \alpha T x \ (\alpha \geq 0, x \in C)$$

(ii) monotonic increasing if

$$x, y \in C, x \leq y \Rightarrow Tx \leq Ty$$

(iii) c-dominant if there exists a nonzero vector u in C with $Tu \geq cu$.

Lemma 4.1. *Let C be a closed positive cone in a normed space E, and let u be a point that does not belong to $-C$. Then there exists a continuous linear functional f on E such that*

(i) $f(u) = d(u, -C) > 0,$

(ii) $f(x) \geq 0 \ (x \in C),$

(iii) $\|f\| \leq 1$

Proof. Let

$$p(x) = d(x, -C) = \inf\{\|x + y\| : y \in C\}$$

Then p is a sublinear functional on E, with the properties

(a) $p(u) = d(u, -C) > 0$,

(b) $p(x) = 0(x \in -c)$,

(c) $p(x) \leq \|x\|$ $(x \in E)$.

□

By the Hahn-Banach theorem there exists a linear functional f on E with $f(u) = p(u)$ and with

$$f(x) \leq p(x) \quad (x \in E).$$

Plainly f has the required properties. 55

Theorem 4.3 (Krein and Rutman)**.** Let E be a partially ordered normed vector space with a closed positive cone C. Let T be a completely continuous mapping of G in to itself which is positive-homogeneous, monotonic increasing, and c-dominant for some $c > 0$. Then there exists a nonzero vector v in C and a real number $\lambda \geq c$ such that $Tv = \lambda v$.

Proof. Since T is positive-homogeneous and c-dominant, there exists a vector u in C with $\|u\| = 1$ and

$$Tu \geq cu \tag{1}$$

since $u \notin -C$, Lemma 4.1 establishes the existence of a continuous linear functional f on E with

$$f(u) > 0,\ f(x) \geq 0\ (x \in C) \tag{2}$$

and
$$\|f\| = 1 \tag{3}$$

We now prove that

$$x \in C, \alpha > 0, \beta > 0, Tx = \alpha x - \beta u \Rightarrow \alpha > c. \tag{4}$$

Let Γ denote the set of positive real numbers t with $x \geq tu$. Since $x = \dfrac{\beta}{\alpha}u + \dfrac{1}{\alpha}Tx \geq \dfrac{\beta}{\alpha}u$, we have $\dfrac{\beta}{\alpha} \in \Gamma$. Also Γ is bounded above, for otherwise

$$\frac{1}{n}x \geq u\ (n = 1, 2, \ldots),$$

and since C is closed, this gives $0 \geq u, u = 0$ with is not true.

Let m denote the least upper bound of Γ. Using again the fact that C is closed, we have 56

$$x \geq mu$$

and therefore

$$Tx \geq T(mu) = mTu \geq mcu$$

Since $Tx = \alpha x - \beta u$, this gives

$$x \geq \frac{\beta + mc}{\alpha} u,$$

and therefore

$$\frac{\beta + mc}{\alpha} \leq m,$$
$$m(\alpha - c) \geq \beta > 0,$$
$$\alpha > c$$

In the rest of the proof ε will denote a real number with

$$0 < \varepsilon < \frac{1}{2} \tag{5}$$

Let $K_\varepsilon = \{x : x \in E, \| x \| \leq 1, xgeq\varepsilon \| x \| u, f(x) \geq \varepsilon f(u)$ Clearly K is a closed, convex, bounded subset of E. Next we note that, for some $\delta > 0$,

$$\| Tx \| \geq \delta \in f(u) 0 (x \in K_\epsilon) \tag{6}$$

For $x \in K_\varepsilon, x \geq \varepsilon \| x \| u$ gives

$$Tx \geq \varepsilon \| x \| Tu$$

57 since T is positive homogeneous and monotonic increasing .

By (3), $\| Tx \| \geq f(Tx)$

and by (2) $f(Tx) \geq f(\varepsilon c \| x \| u) = \varepsilon c \| x \| f(u) \; (x \epsilon K_\epsilon)$

i.e., $\| Tx \| \geq \varepsilon c \| x \| f(u) \geq \varepsilon e f(x) f(u)$

$$\geq \delta\varepsilon f(u) \quad (x \in K_\varepsilon)$$

with $\delta = \varepsilon c f(u) > 0$.

Let V_ϵ be the mapping defined by taking

$$V_\epsilon(0) = 0,$$
$$V_\varepsilon(x) = \| x \| \,.\, \| x + 2\varepsilon \| x \| u \|^{-1} (x + 2\varepsilon \| x \| u), x \neq 0$$

V is well defined since

$$\begin{aligned} \| x + 2\epsilon \| x \| u \| &\geq \| x \| - 2\varepsilon \| x \| \quad \| u \| \\ &= \| x \| (1 - 2\varepsilon) > 0 \text{ if } \| x \| \neq 0. \end{aligned}$$

Plainly V_ϵ is continuous in E and

$$\| V_\epsilon x \| = \| x \| \tag{7}$$

Also

$$x \in C, \| x \| = 1 \; V_\varepsilon x \in K_\varepsilon \tag{8}$$

For $f(V_\varepsilon x) = \|x\| \|x + 2\varepsilon \|x\| u\|^{-1} \{ f(x) + 2\varepsilon \|x\| f(u) \}$

$$\leq \frac{2\epsilon}{1 + 2\epsilon} f(u) \geq \varepsilon f(u)$$

Let A_ε be the mapping defined on K_ϵ by 58

$$A_\varepsilon = V_\varepsilon LT$$

when $$Lx = \frac{x}{\|x\|} \quad x \neq 0$$

Then by (6) and (8)

$$A_\varepsilon K_\varepsilon \subset K_\varepsilon$$

By (6), $V_\varepsilon L$ is continuous in $\overline{TK_\varepsilon}$, and A_ε continuously into a compact subset of K_ε. Applying the Schauder theorem, we see that there exists a point x_ε in K_ε such that

$$A_\varepsilon x_\varepsilon = x_\varepsilon,$$

$$\text{i.e.,} \quad V_\varepsilon \left\{ \frac{Tx}{\|Tx_\varepsilon\|} \right\} = x_\varepsilon \tag{9}$$

i.e. $\|Tx_\varepsilon\|^{-1}Tx_\varepsilon + 2\varepsilon U = \| \, \|Tx_\varepsilon\|^{-1}Tx_\varepsilon + 2\varepsilon u\| x_\varepsilon$ This can be written in the form

$$Tx_\varepsilon = \alpha_\varepsilon x_\varepsilon - \beta_\varepsilon u, \tag{10}$$

where $\quad c < \alpha_\varepsilon < (1+2\varepsilon)\|Tx_\varepsilon\|$.

We now choose a sequence (ε_n) such that $\lim\limits_{n\to\infty} \varepsilon_n = 0$, and such that the sequences (Tx_{ε_n}) and $(\alpha\varepsilon_n)$ converges. Let $v = \lim\limits_{n\to\infty} Tx_{\varepsilon_n}$ and
59 $\lambda = \lim\limits_{n\to\infty} \alpha_{\epsilon_n}$. Then $\lambda \geq c$, and since $\lim n \to \infty \beta_{\varepsilon_n} = 0$, (10) gives

$$\lim_{n\to\infty} x_{\varepsilon_n} = \frac{1}{\lambda} v$$

By continuity and positive homogeneity of T,

$$\frac{1}{\lambda} Tv = T\left(\frac{1}{\lambda} v\right) = \lim_{n\to\infty} Tx_{\epsilon_n} = v$$

Finally by (7) and (9), $\|x_\epsilon\| = 1$ and so $v \neq 0$. $\square$

Remark. The theorems in this chapter are unsatisfactory in that each of them involves an adhoc condition (strict positivity and c- domi-nance). It turns out that for linear mappings such an ad hoc condition can be avoided, and I think that there is still scope for proving a better theorem on non-linear mappings also.

Chapter 5

Linear mapping in cones

If A is a linear operator in $R^{(n)}$ with a matrix (a_{ij}) with non-negative 60
elements, $a_{ij} \geq 0$, then, by a famous theorem of Perron and Frobenius (see for example Gantmacher, The theory of matrices), there exists an eigen vector of A with non-negative coordinates and with eigenvalue ρ, such that all other eigenvalues satisfy $|\lambda| \leq \rho$.

If we take $E = R^{(n)}$ and C to be the set of all vectors in E with non-negative coordinates, then C is a positive cone in E, and $n \times n$ matrices with non-negative elements correspond to linear operators in E that map C into itself. Then theorems of the present chapter may be regarded as generalizations of the Perron-Frobenius Theorem. A great many such generalizations with various methods of proof have been published during recent years, and our list of references is far from complete.

The idea of the method of proof adopted here is the use of the simple concept of 'topological divisor of zero'.

Let $\mathcal{U}$ be a Banach algebra with a unit element e, and let a be a frontier point of the set of invertible elements. Then a is a topological divisor of zero, i.e., there exists a sequence (x_n) with $\|x_n\| = 1 (n = 1, 2, \ldots)$ such that

$$\lim_{n\to\infty} ax_n = 0$$

61

Proof. There exists a sequence (a_n) of invertible elements such that

$$\lim_{n\to\infty} a_n = a$$

Then the sequence $(\|a_n^{-1}\|)$ is unbounded. For otherwise

$$\lim_{n\to\infty} (a_n - a)a_n^{-1} = 0$$

i.e.,

$$\lim_{n\to\infty} (e - aa_n^{-1}) = 0$$

But this implies that aa_n^{-1} is invertible for some n, and therefore a has a right inverse. Similarly a has a left inverse, and a is invertible, which is absurd since the set of invertible elements is open. □

We may therefore suppose that $\|a_n^{-1}\| \to \infty$, and take $x_n = \|a_n^{-1}\|^{-1}$ a_n^{-1}. Then $\|x_n\| = 1$, and

$$\lim_{n\to\infty} ax_n = \lim_{n\to\infty} (a - a_n)x_n + a_n x_n = 0.$$

In particular, if λ is a frontier point of the spectrum of an element b, then $\lambda e - b$ is a frontier point of the set of invertible elements of $\mathcal{U}$ and so there exists (x_n), with $\|x_n\| = 1$ and

$$\lim_{n\to\infty} (\lambda e - b)x_n = 0$$

62 If further we know that $\lambda \neq 0$, and $bx_{n_k} \to u$ for some subsequence (x_{n_k}). Then

$$\lambda x_{n_k} \to u,$$
$$\lambda b x_{n_k} \to bu,$$

so that $bu = \lambda u$ and $u \neq 0$, since $\|u\| = \lambda$.

Actually, our method is not quite so simple as this, for our Banach algebra is a Banach algebra of operation on a Banach space X, and we have to replace the sequence (x_n) of operators by a sequence of elements of X.

Until we reach the statement of our main theorem (Theorem 5.1) we shall use the following notation.

E will denote a normed and partially ordered vector space with norm $||x||$ and positive cone C. We suppose that C is complete with respect to $||x||$, and that

$$E = C - C$$

We do not suppose that E is complete with respect to $||x||$. We denote by B the intersection of C and the closed unit of E, i.e.,

$$B = \{x : x \in C \text{ and } ||x|| \leq 1\}$$

We denote by B^0 the convex symmetric hull of B, i..e.

$$B^0 = \{\alpha x - \beta y : x, y \in B, \alpha \geq 0, \beta \geq 0, \alpha + \beta = 1\},$$

and by $||x||_c$ the Mindowski functional of B^0, i.e., **63**

$$||x||_c = \inf\{\lambda : \lambda > 0, x \in \lambda B^0\}$$

Lemma 5.1. (α) $||x||_c$ *is a norm on E and satisfies*

$$||x||_c = ||x|| \ (x \in C), ||x||_c \geq ||x|| \qquad (x \in E)$$

(β) *E is complete and C is a closed subset of E with respect to* $||x||_c$.

Proof. (α) since $E = C - C, B^0$ is an absorbing set for E, and so the Mindowski functional $||x||_c$ is defined on E. Since B^0 is convex and symmetric, $||x||_c$ is a seminorm on E. If $z \in E$ and $||z||_c < 1$, then $z \in B^0$ i.e. $z = \alpha x - \beta y$ with $x, y \in B$ and $\alpha \geq 0, \beta \geq 0$, $\alpha + \beta = 1$. Therefore

$$||z|| = ||\alpha x - \beta y|| \leq \alpha ||x|| + \beta ||y|| \leq \alpha + \beta = 1.$$

This proves that

$$||x|| \leq ||x||_c \qquad (x \in E)$$

and completes the proof that $||x||_c$ is a norm. Since $B \subset B^0$, we have

$$||x||_c \leq 1 \qquad (x \in B),$$

and therefore

$$||x||_c \leq ||x|| \qquad (x \in C)$$

This completes the proof of (α). 64

(β) Let (z_n) be a Cauchy sequence in E with respect to x_c. Then there exists a strictly increasing sequence (n_k) of positive integers such that

$$p, q \geq n_k \Rightarrow \|z_p - z_q\|_c < 2^{-k}$$

Let $w_k = z_{n_k}$ $(k = 1, 2, \ldots)$. Then, in particular,

$$\|w_{k+1} - w_k\|_c < 2^{-k} \ (k = 1, 2, \ldots)$$

Therefore

$$w_{k+1} - w_k \in 2^{-k} B^0,$$

and so $w_{k+1} - w_k = \alpha_k x_k - \beta_k y_k$, with

$$\alpha_k \geq 0, \beta \geq 0, \alpha_k + \beta_k = 1, x_k, y_k \in 2^{-k} B$$

Let

$$s_n = \sum_{k=1}^{n} \alpha_k x_k, \ \ t_n = \sum_{k=1}^{n} \beta_k y_k.$$

□

Then $p > q$ gives

$$\|s_p - s_q\| \leq \sum_{q+1}^{p} \alpha_k \|x_k\| \leq \sum_{k=q+1}^{p} 2^{-k} < 2^{-q},$$

and similarly

$$\|t_p - t_q\| < 2^{-q}.$$

65 Since C is complete, there exist s, t in C such that

$$\lim_{n\to\infty} \|s - s_n\| = 0, \quad \lim_{n\to\infty} \|t - t_n\| = 0$$

Also, $\lim\limits_{p\to\infty} s_p - s_q = s - s_q$, and $s_p - s_q \in C$ whenever $p > q$. Hence $s - s_n \in C$ $(n = 1, 2, \ldots)$. Therefore

$$\|s - s_n\|_c = \|s - s_n\| \quad (n = 1, 2, \ldots),$$

and so $$\lim_{n\to\infty} \|s - s_n\|_c = 0.$$

Similarly, $\lim_{n\to\infty} \|t - t_n\|_c = 0$, and so (w_n) converges with respect to $\|x\|_c$ to $w_1 + s - t$. It is now easily seen that (z_n) converges with respect to $\|x\|_c$, and so E is complete with respect to $\|x\|_c$. That C is a closed subset of E with respect to $\|x\|_c$ is a simple consequence of the inequality

$$\|x\| \leq \|x\|_c \qquad (x \in E)$$

and the closeness of C with respect to $\|x\|$, (in fact a larger norm gives a stronger topology).

Definition 5.1. A linear operator in E is said to be *positive* if it maps C into C and to be *partially bounded* if it maps B into a bounded set. The *partial bound* $p(T)$ of a partially bounded linear operator T is defined by

$$p(T) = \sup\{\|T\| : x \in B\}$$

Given partially bounded positive linear operators S and T, we have **66**

$$p(ST) \leq p(S)p(T),$$

and therefore the limit

$$\lim_{n\to\infty} \{p(T)^n\}^{\frac{1}{n}}$$

exists. It is called the *partial spectral radius* of T.

Lemma 5.2. *A positive linear operator T is partially bounded if and only if it is a bounded linear operator in the Banach space $(E, \|x\|_c)$. For such an operator T,*

$$p(T) = \|T\|_c = \sup\{\|Tx\|_c : x \in E \text{ and } \|x\|_c \leq 1\},$$

and the partial spectral radius μ of T is equal to its spectral radius as an operator in $(E, \|x\|_c)$, i.e.

$$\mu = \lim_{n\to\infty} \{\|T^n\|_c\}^{\frac{1}{n}}$$

Finally, if $\lambda > \mu$, there exists a partially bounded positive linear operator R_λ, such that

$$(\lambda I - T)R_\lambda = R_\lambda(\lambda I - T) = I,$$

where I is the identity operator in E.

Proof. Let T be a partially bounded positive linear operator in E and let $x \in E$ with $\|x\|_c < 1$. Then $x \in B^0$, and so

$$x = \alpha y - \beta z$$

67 with $\alpha \geq 0, \beta \geq 0$, $\alpha + \beta = 1$, $y, z \in B$. Then

$$Tx = \alpha Ty - \beta Tz,$$

and so

$$\begin{aligned}\|Tx\|_c &\leq \alpha\|Ty\|_c + \beta\|Tz\|_c \\ &= \alpha\|Ty\| + \beta\|Tz\| \\ &\leq (\alpha + \beta)p(T) = p(T)\end{aligned}$$

Thus T is a bounded linear operator in $(E, \|x\|_c)$ and

$$\|T\|_c \leq p(T)$$

For the converse and the reversed inequality it is enough to note that

$$\begin{aligned}\|T\|_c \geq &\sup\{\|Tx\|_c : x \in C \text{ and } \|x\|_c \leq 1\} \\ &\sup\{\|Tx\| : x \in B\} = p(T).\end{aligned}$$

□

That $\mu = \lim_{n\to\infty} \{\|T^n\|_c\}^{\frac{1}{n}}$ is an obvious consequence of the fact that $\|T\|_c = p(T)$ for each partially bounded linear operator.

If $\lambda > \mu$, the series

$$\frac{1}{\lambda}I + \frac{1}{\lambda 2}T + \frac{1}{\lambda^3}T^2 + \cdots$$

68 converges with respect to the operator norm for bounded linear operators in the Banach space $(E, \|x\|_c)$ to a bounded linear operator R_λ, and

$$(\lambda I - T)R_\lambda = R_\lambda(\lambda I - T) = I.$$

Since C is closed with respect to $\|x\|_c$, and the partial sums of the series are obviously positive operators, it follows that R_λ is a positive operator.

Definition 5.2. A positive linear operator T is said to be a *normalising operator* if it satisfies the following condition:

$$\alpha_n \geq 0, y_n \in B, \alpha_n x \geq T y_n, \ \lim_{n\to\infty} \alpha_n = 0 \qquad \lim_{n\to\infty} \|T y_n\| = 0$$

If C is a normal cone, then every positive linear operator is obviously a normalising operator. We shall see later that a certain compactness condition on T suffices to make T a normalising operator (without restriction on C).

Lemma 5.3. *Let T be a normalizing partially bounded positive linear operator with partial spectral radius μ. Then*

$$\lim_{\lambda\to\mu+0} p(R_\lambda) = \infty$$

Corollary. *For each such operator T, the partial spectral radius μ is in the spectrum of T regarded as an operator in the Banach space $(E, \|x\|_c)$.*

Suppose that the conditions of the lemma are satisfied but that $p(R_\lambda)$ does not tend to infinity as λ decreases to μ. Then there exists a positive 69
constant M such that $p(R_\nu) \leq M$ for some ν gather than and arbitrarily close to μ.

The case $\mu = 0$ is easily settled. For if $\mu = 0$, then, from the formula $(\lambda I - T)R_\lambda = I$, it follows that

$$\lambda R - \lambda x \geq x \qquad (\lambda > 0, x \in C) \tag{1}$$

If we let λ tend to zero through values for which $p(R_\lambda) \leq M$, the left hand side of (1) tends to zero, and, since C is closed, we obtain

$$-x \in C \qquad (x \in C).$$

But this implies that $C = (0)$ which was excluded by our axioms on C.

Suppose now that $\mu > 0$. Then we may choose λ, ν with

$$0 < \lambda < \mu < \nu < \lambda + M^{-1},$$

and with $p(R_\nu) \leq M$. Since $\|R\|_c = p(R_\nu)$, it follows that the series

$$R_\nu + (\nu - \lambda)R_\nu^2(\nu - \lambda)^2 R_\nu^3 + \cdots$$

converges with respect to the operator norm $\| \ \|_c$ to a partially bounded positive linear operator S which is easily seen to satisfy

$$S(\lambda I - T) = (\lambda I - T)S = I$$

70 This gives

$$Sx = \lambda^{-1}x + \lambda^{-1}TSx \qquad (x \in C),$$

from which it follows by induction that

$$Sx \geq \lambda^{-(n+1)}T^n x \qquad (x \in C, n = 0, 1, 2, \ldots) \tag{2}$$

since $\lambda < \mu$ and $\lim\limits_{n\to\infty} \|T^n\|_c^{\frac{1}{n}} = \mu$, we have

$$\lim_{n\to\infty} \|\lambda^{-(n+1)}T^n\|_c = \infty$$

By the principle of uniform boundedness, there exists a point x in E for which the sequence $\|\lambda^{-(n+1)}T^n x\|_c$ is unbounded. Since $E = C - C$, it follows that there exists a point w in C for which the sequence $\|\lambda^{-(n+1)}T^n w\|$ is unbounded. But given any unbounded sequence $\{a_n\}$ of non-negative real numbers, there exists a subsequence (a_{n_k}) such that

$$a_{n_k} > k \qquad (k = 1, 2, \ldots) \tag{3}$$

$$a_{n_k} > a_j \qquad (j < n_k, k = 1, 2, \ldots) \tag{4}$$

71 This is easily proved by induction. For if $n_1, \ldots, n_{r-1}$ have been chosen so that (3) and (4) are satisfied for $k = 1, 2, \ldots, r-1$, we take n_r to be the smallest positive integer s for which

$$a_s > a_{n_{r-1}} + r$$

Hence we see that there exists a strictly increasing sequence (n_k) of positive integers for which

$$\lim_{k\to\infty} \lambda^{-(n_k+1)}T^{n_k} = \infty \tag{5}$$

and

$$\|\lambda^{-(n_k+1)}T^{n_k}\| \geq \|\lambda^{-n_k}T^{n_k-1}w\| \tag{6}$$

since

$$\|T^{n_k}w\| \leq p(T)\|T^{n_k-1}w\|,$$

we also have

$$\lim_{k\to\infty} \|\lambda^{-n_k}T^{n_k-1}w\| = \infty \tag{7}$$

By (7), there is no loss of generality in supposing that $T^{n_k-1} \neq 0$ for all k, and we may take

$$y_k = \|T^{n_k-1}w\|^{-1}T^{n_k-1}w \tag{8}$$

Then, by (2),

$$\lambda^{n_k}\|T^{n_k-1}w\|^{-1}Sw \geq \lambda^{-1}Ty_k \tag{9}$$

Since $y_k \in B$, and T is a normalizing operator, it follows from (7) 72
and (9) that

$$\lim_{k\to\infty} \lambda^{-1}\|Ty_k\| = 0 \tag{10}$$

But, by (6),

$$\lambda^{-1}\|T^{n_k}w\| \geq \|T^{n_k-1}w\|,$$

which obviously contradicts (10). This contradiction proves the lemma.

To deduce the corollary, it is enough to appeal to the continuity of the resolvent operator on the resolvent set.

Definition 5.3. Let τ_N denote the given norm topology in E, τ a second linear topology in E, and A a subset of the positive cone C. We say that τ is sequentially *stronger than* τ_N *at* 0 *relative to* A if 0 is a (τ_N)-cluster point of each sequence of point of A of which 0 is a τ-cluster point.

We recall that to say that 0 is a τ-cluster point of a sequence (a_n) means that each τ-neighbourhood of 0 contains points a_n which *arbitrarily large n*.

Theorem 5.1. Let E be a normed and partially ordered vector space with norm topology τ_N, positive cone C complete with respect to the norm, and let $B = \{x : x \in C, \|x\| \leq 1\}$. Let T be a partially bounded positive linear operator in E with partial spectral radius μ, and let τ be a linear
73 topology in E with respect to which C is closed and T is continuous.

Let A be a subset of C that is contained in a countably τ-compact subset of C, and let τ be sequentially stronger than τ_N at 0 relative to A. If either

(i) $A = TB$ and $\mu > 0$,

or

(ii) $A = B$,

then there exists a non-zero vector u in C with $Tu = \mu u$.

Proof. Since we can restrict our consideration to $C - C$, we shall suppose that in fact $E = C - C$. □

Since $TB \subset p(T)B$, both (i) and (ii) imply

(iii) TB is contained in a subset of C that is countably compact with respect to τ, and τ is sequentially stronger than τ_N at 0 relative to TB.

We prove that under condition (iii), T is a normalizing operator. Let

$$\alpha_n x = Ty_n + z_n$$

with $\alpha_n \geq 0$, $\lim_{n\to\infty} \alpha_n = 0, y_n \in B$ and $z_n \in C$. If $\|Ty_n\|$ does not converge to zero, we may select a subsequence (Ty_{n_k}) such that

$$\|Ty_{n_k}\| \geq \varepsilon > 0 \qquad (k = 1, 2, \ldots) \tag{1}$$

We have

$$Ty_{n_k} + z_{n_k} \to 0 \qquad (\tau),$$

and (Ty_{n_k}) has a τ-cluster point, v say, in C. It follows that $-v$ is a τ-cluster point of (z_{n_k}), and, since C is τ-closed, $-v \in C$, $v = 0$. This now implies that 0 is a τ-cluster point of (Ty_{n_k}) and therefore is a τ_N-cluster point of (Ty_{n_k}), which contradicts (1). **74**

Thus, by Lemma 5.3, we have

$$\lim_{\lambda\to\mu+0} p(R_\lambda) = \infty,$$

i.e.,

$$\lim_{\lambda\to\mu+0} \|R_\lambda\|_c = \infty.$$

Applying the principle of uniform boundedness, we see that there exists a sequence (λ_n) converging decreasingly to μ and a vector w in C with $\|w\| = 1$ such that

$$\lim_{n\to\infty} R_{\lambda_n} w = \infty,$$

and we may suppose that $\|R_{\lambda_n} w\| \neq 0$ $(n = 1, 2, \ldots)$. Let $\alpha_n = \|R_{\lambda_n} w\|^{-1}$ and $u_n = \alpha_n R_{\lambda_n} w$. Then $u_n \in B, \|u_n\| = 1, \lim\limits_{n\to\infty} \alpha_n = 0$, and $\mu u_n - Tu_n = (\mu - \lambda_n)u_n + (\lambda_n I - T)u_n = (\mu - \lambda_n)u_n + \alpha_n w$.

Suppose that condition (ii) in the statement of the theorem is satisfied. Since B is τ-countably compact and $u_n \in B$, it follows from (2) that

$$\lim_{n\to\infty} \mu u_n - Tu_n = 0 \qquad (\tau) \tag{3}$$

75

Also (u_n) has a τ-cluster point u in C, and (3) shows that $\mu u - Tu = 0$.

We have $u \neq 0$, for otherwise a is 0 τ_N-cluster point of (u_n), which contradicts $\|u_n\| = 1$.

Finally suppose that the condition (i) is satisfied. Then by (2).

$$(\mu - I - T)Tu_n = T(\mu I - T)u_n = (\mu - \lambda_n)Tu_n + \alpha_n Tw$$

Since TB is contained in a τ-countably-compact subset of C, this shows that

$$\lim_{n\to\infty} (\mu I - T)Tu_n = 0 \qquad (\tau)$$

and that (Tu_n) has a τ-cluster point v in K. By the τ-continuity of $(\mu I - T)$, we have therefore

$$(\mu I - T)v = 0$$

If $v = 0$, then 0 is a τ_N-cluster point of (Tu_n). But, by (2),

$$\lim_{n\to\infty} \mu u_n - Tu_n = 0 \qquad (\tau_N)$$

and therefore 0 is a τ_N-cluster point of (μu_n). But since $\mu > 0$, and $\|u_n\| = 1$, this is absurd.

The statement of Theorem 5.1 is somewhat complicated in that it
76 seeks to combine generally and precision. A number of less complicated but also less general theorems are easily deduced from it.

Theorem 5.2. Let C be a complete positive cone in a normed space E, let $B = \{x : x \in C \text{ and } \|x\| \leq 1\}$, and let T be a positive linear operator which is continuous in C, and maps B into a compact set and has a positive partial spectral radius μ. Then there exists a non-zero vector u in C with

$$Tu = \mu u.$$

Proof. In Theorem 5.1, take $\tau = \tau_N$. □

Example 1. Let $E = C_R[0, 1]$ with the uniform norm, and let C be the positive cone in E consisting of those functions f belonging to E that are increasing, convex in $[0, 1]$ and satisfy $f(0) = 0$.

Let $0 < k < 1$, and let T denote the operator in E defined by

$$(Tf)(x) = f(kx) \qquad (f \in E, 0 \leq x \leq 1)$$

Plainly T is a bounded linear operator in E and maps C into itself.

For f in C we have $\|f\| = f(1)$, and also since $x = (1 - x)0 + x1$,

$$f(x) \leq (1 - x)f(0) + xf(1) = xf(1) \qquad (0 \leq x \leq 1)$$

since $T^n f(x) = f(K^n x)$, it follows that

$$\|T^n f\| = f(k^n) \leq k^n f(1) = k^n \|f\| \qquad (f \in C)$$

77 and equality is attained with the function $f(x) = x$. Thus

$$p(T^n) = k^n,$$

and the partial spectral radius of T is $k > 0$. Also T maps B into a compact set, for given a convex function f and $0 < x_1 < x_2 \leq y_1 < y_2 \leq 1$, we have

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(y_2) - f(y_1)}{y_2 - y_1}$$

Given $f \in B$, and $0 \leq s < t \leq 1$, we therefore have

$$0 \leq \frac{f(kt) - f(ks)}{kt - ks} \leq \frac{f(1) - f(k)}{1 - k} \leq \frac{1}{1 - k},$$

i.e.,
$$0 \leq (Tf)(t) - (Tf)(s) \leq \frac{k}{1 - k}(1 - s)$$

This proves that TB is an equicontinuous set of functions, and, since TB is also bounded, it is contained in a compact set.

For this particular operator T the conclusion of Theorem 5.2 is of course trivial since the function u given by $u(x) = x(0 \leq x \leq 1)$ plainly satisfies $Tu = ku$.

The example is however of interest in that it provides a simple example of a bounded linear operator completely continuous in a cone that is not a compact linear operator in any subspace of E that contains C. In fact let g_n, h_n, f_n be defined by

$$g_n(x) = k^{-n}x \quad (0 \leq x \leq 1),$$

$$h_n(x) = \begin{cases} 0 & (0 < x < k^n) \\ k^{-n}(x - k^n) & (k^n \leq x \leq 1) \end{cases}$$

and $\quad f_n = g_n - h_n.$

Then $g_n, h_n \in C$, $f_n \in C - C$ and since 78

$$f_n(x) = \begin{cases} k^{-n}x & (0 < x < k^n) \\ 1 & (k^n \leq x \leq 1) \end{cases}$$

we have
$$\|f_n\| = 1 \qquad (n = 1, 2, \ldots).$$

Also,

$$T f_n = f_{n-1} \qquad (n = 1, 2, \ldots),$$

and for $r > s$,

$$\|f_r - f_s\| \geq f_r(k^r) - f_s(k^r) = 1 - k^{r-s} > 1 - k.$$

It follows that no subsequence of $(T f_n)$ converges.

Example 2. A slight modification of the last example yields a less trivial application of Theorem 5.2. Let C denote the class of continuous, increasing, convex functions f on $[0, 1]$ with $f(0) = 0$, and let ϕ be an element of C that satisfies

$$\phi(1) < 1, \;\; \phi^1_+(0) > 0.$$

Then there exists an element g of C such that $g \circ \phi = \phi^1_+(0)g$. $(f \circ g$
79 denotes the composition $(f \circ g)(x) = f(g(x)))$. As before we take $E = C_R[0, 1]$, and consider the linear operator T given by

$$T f = f \circ \phi \qquad (f \in E)$$

That Theorem 5.2 is applicable is proved as in the last example, except for showing that the partial spectral radius μ is given by

$$\mu = \phi'_+(0)$$

To prove this we consider the sequence (ϕ_n) of functions defined by

$$\phi_n = T^n \phi \qquad (n = 1, 2, \ldots).$$

Let $k = \phi\,(1)$. Then, for f in C,

$$\|T f(x)\| = (T f)(1) = f(\phi(1)) \leq \phi(1) f(1),$$

and so $p(T) \leq k < 1$.

$$\lim_{n\to\infty} \phi_n(1) = 0.$$

Hence

$$\lim_{n\to\infty} \frac{\|t^n \phi\|}{\|T^{n-1}\phi\|} = \lim_{n\to\infty} \frac{\phi_n(1)}{\phi_{n-1}(1)} = \lim_{n\to\infty} \frac{\phi(\phi_{n-1}(1))}{\phi_{n-1}(1)} = \phi'_+(0),$$

and therefore $\lim_{n\to\infty} \|T^n\phi\|^{\frac{1}{n}} = \phi'_+(0)$,

$$\mu \geq \phi'_+(0).$$

Finally $(T^n f)(x) = f(\phi_n(x))$, so that

$$p(T^n) \leq \phi_n(1),$$

and so $\mu \leq \phi'_+(0)$. This completes the proof that Theorem 5.2 is appli- **80**
cable.

In this particular example, we can calculate an eigenvector g by an iterative process. In fact, if we take g_n defined by

$$g_n(x) = \frac{\phi_n(x)}{\phi_n(1)} \qquad (0 \leq x \leq 1, \qquad n = 1, 2, \ldots)$$

Then the sequence (g_n) converges decreasingly to a function g with the required properties.

Example 3. Let E, C and ϕ be defined as in the last example, and let $K(x, y)$ be a function continuous on the square

$$[0, 1] \times [0, 1]$$

which belongs to C as a function of x for each fixed y in $[0, 1]$. Let T be the operator defined on E by

$$(Tf)(x) = f(\phi(x)) + \int_0^1 K(x, y) f(y) dy \qquad (0 \leq x \leq 1).$$

Then T is a bounded linear operator mapping C into itself. T maps B into a compact set, and its spectral radius μ satisfies

$$\mu \geq \phi'_+(0) > 0.$$

Thus Theorem 5.2 is again applicable.

Example 4. A variant on Example 3 is given by

$$(Tf)(x) = f(\phi(x)) + \int_0^x k(y)f(y)dy$$

where k is increasing non-negative and continuous in $[0,1]$. Again Theorem 5.2 is applicable and so there exists a non-zero function f in C with **81**

$$\mu f(x) - f(\phi(x)) = \int_0^x k(y)f(y)dy \qquad (0 \leq x \leq 1),$$

where μ is the partial spectral radius of T. From this we see that $\mu f(x) - f(\phi(x))$ is differentiable and we have a solution of the functional equation

$$\frac{d}{dx}\{\mu f(x) - f(\phi(x))\} = k(x)f(x)$$

Example 5. That the conclusion of Theorem 5.2 need not hold if $\mu = 0$ is seen by taking the following example. Let $K(x,y)$ be continuous and non-negative in the square $[0,1] \times [0,1]$, and suppose that

$$K(1,y) > 0 \quad (0 \leq y \leq 1).$$

Let $E = C_R[0,1]$, let C consist of all $f \in E$ with

$$f(x) \geq 0 \quad (0 \leq x \leq 1),$$

and let T be the Volterra operator defined by

$$(Tf)(x) = \int_0^x K(x,y)f(y)dy.$$

Then T satisfies the condition of Theorem 5.2 except that its spectral radius is zero (and hence its partial spectral radius is zero).

82 Also, if $f \in C$ and $Tf = 0$, we have

$$\int_0^1 K(1,y)f(y)dy = 0,$$

and so $f = 0$.

Definition 5.4. Given a normed and partially ordered vector space X with positive cone K, a non zero linear functional f such that $f(x) \geq 0$ $(x \in K)$ is called a *positive continuous linear functional.*

The following theorem is quite easily deduced from Theorem 5.1.

Theorem 5.3. Let X be a normed and partially ordered vector space with a closed positive cone K, and suppose that there exists a subset H of K with the properties:

(i) Given $x \varepsilon X$ with $||x|| \leq 1$, there exists hH with $-hxh$,

(ii) H is contained in a compact set.

Let T be a partially bounded positive linear operator in X with partial spectral radius μ.

Then there exists a positive continuous linear functional f and a real number μ^* with $0 \leq \mu^* \leq \mu$ such that

$$f(Tx) = \mu^* f(x) \quad (x \in X).$$

If also K is a normal cone, then $\mu = \mu^*$

Proof. Let X^* be the dual space of X, and let C be the dual cone K^* consisting of all f in X^* that satisfy

$$f(x) \geq 0 \quad (x \in K)$$

We have seen in Lemma 4.1 (chapter 4) that since K is closed, $K^* \neq 0$. **83**
By condition (i) in the theorem, $X = K - K$, and therefore $K^* \cap (-K^*) = (0)$. This proves that $C = K^*$ is indeed a positive cone. It is clearly a closed, and therefore complete, subset of the Banach space X^*. We take $E = C - C$. □

Let $M = \sup\{||h|| : h \in H\}$, and as usual, let $B = \{f : f \in C$ and $||f|| \leq 1\}$. Given $f \in B$ and $x \in X$ with $||x|| \leq 1$, there exists $h \in H$ with

$$-h \leq x \leq h,$$

and therefore $-Th \leq Tx \leq Th$,

$$-f(Th) \leq f(Tx) \leq f(Th),$$
$$|f(Tx)| \leq f(Th) \leq \|Th\| \leq p(T)\|h\| \leq p(T).M,$$

where $p(T)$ denotes the partial bound of T. Thus for each $f \in E$ we have an element $T^* f$ of E given by

$$(T^* f)(x) = f(Tx) \quad (x \in X),$$

and we obtain a partially bounded positive linear operator T^* in E with partial bound $p(T^*)$ satisfying

$$p(T^*) \leq Mp(T).$$

84 similarly

$$p(T^{*n}) \leq Mp(T^n),$$

and therefore

$$\mu^* \leq \mu$$

where μ^* denotes the partial spectral radius of T^*. We take τ to be the weak topology in E. Plainly C is τ-closed, B is τ-compact and T^* is τ-continuous. In order to apply Theorem 5.1 it only remains to prove that τ is sequentially stronger than τ_N at 0 relative to B. To prove this, suppose that $f_n \in B(n = 1, 2 \ldots)$ and that 0 is a τ-cluster point of the sequence (f_n).

Since H is contained in a (norm) compact set, given $\varepsilon > 0$. there exists $h_1, \ldots, h_r$ in H such that for each $h \in H$ there is some $k(1 \leq k \leq r)$ with

$$\|h - h_k\| < \frac{\varepsilon}{2} \tag{1}$$

Since 0 is a weak $*$-cluster point of (f_n), there exists an infinite set Λ of positive integers such that

$$|f_n(h_k)| < \frac{\varepsilon}{2} \quad (k = 1, 2, \ldots, r; n \in \Lambda) \tag{2}$$

Therefore by (1) and (2) and the fact that $\|f_n\| \leq 1$,

$$|f_n(h)| < \varepsilon \quad (h \in H, n \in \Lambda) \tag{3}$$

Given $x \in X$ with $\|x\| < 1$, there exists $h \in H$ with $-h \leq x \leq h$, and
85 so, by (3),

$$|f_n(x)| \leq |f_n(h)| < \varepsilon \quad (n \in \Lambda)$$

Therefore

$$\|f_n\| \leq \varepsilon \quad (n \in \lambda),$$

and 0 is a τ_N-cluster point of the sequence (f_n).

Suppose now that K is a normal cone, *i.e.* for some $\gamma > 0$,

$$\|x + y\| \geq \gamma \|x\| \quad (x,\ y \in K).$$

Then, for each point x in K,

$$d(x, -K) = \inf_{y \in K} \|x + y\| > \gamma \|x\|$$

Therefore, for each x in K, there exists $f \in K^*$ with $\|f\| \leq 1$ and

$$f(x) \geq \gamma \|x\|$$

In particular, given $\varepsilon > 0$, there exists $x_0 \in K$ with $\|x_0\| \leq 1$, and

$$\|Tx_0\| > p(T) -$$

Then there exits $f_0 \in B$ with

$$f_0(Tx_0) \geq \gamma \|Tx_0\| > \gamma(p(T) - \varepsilon).$$

Therefore

$$\|T^* f_0\| > \gamma(p(T) - \varepsilon),$$

and

$$p(T^*) \geq \gamma p(T).$$

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Since, similarly,

$$p(T^{*n}) \geq \gamma p(T^n),$$

we have

$$\mu^* \geq \mu,$$

and the proof is complete.

As a special case of Theorem 5.3, we have

Theorem 5.4 (Krein and Rutman)**.** Let X be a normed and partially ordered over space with a closed normal positive cone K with non-empty interior. Let T be a positive linear operator in X. Then

(i) T is a bounded linear operator in X.

(ii) There exists a positive continuous linear functional f such that

$$f(Tx) = \rho f(x) \quad (x \in X),$$

where ρ is the spectral radius of T.

Proof. Since K has non-empty interior, there exists a point e of K such that

$$\|x\| \leq 1 \Rightarrow e + x \in K$$

87 since $\| - x\| \leq \|x\|$, we have

$$\|x\| \leq 1 \quad e \pm x \in K \Rightarrow -e \leq x \leq e \tag{1}$$

Thus conditions (i) and (ii) of Theorem 5.3 are satisfied with $H = (e)$. □

Give a positive linear operator T and $\|x\| \leq 1$, we have $-e \leq x \leq e$, and so $-Te \leq Tx \leq Te$. Since K is a normal cone, there exists a positive constant γ with

$$\|y + z\| \geq \gamma\|y\| \quad (y, z \in K) \tag{2}$$

We have $Te \pm Tx \varepsilon K$, and so

$$2\|Te\| = \|(Te + Tx) + (Te - Tx)\| \geq \gamma\|Te - Tx\| \geq \{\|Tx\| - \|Te\|\},$$

and so
$$\|Tx\| \leq \frac{2+\gamma}{\gamma}\|Te\|,$$

which proves that T is bounded, and also gives

$$\|T\| \leq \frac{2+\gamma}{\gamma}\|Te\| \tag{3}$$

It follows from (3) that

$$\|T\| \leq \frac{2+\gamma}{\gamma}\|e\| \; p(T),$$

and similarly, for any positive integer *n*,

$$\|T^n\| \leq \frac{2+\gamma}{\gamma}\|e\|\ p(T^n),$$

Therefore **88**

$$\rho = \lim_{n\to\infty} \|T^n\|^{1/n} \leq \lim_{n\to\infty} p(T^n)^{1/n} = \mu$$

On the other hand it is obvious that $\mu \leq \rho$, and so the theorem now follows from Theorem 5.3.

As a special case of Theorem 5.2 we have the following theorem.

Theorem 5.5 (Krein and Rutman theorem 6.1)**.** Let X be a partially ordered Banach space with a positive cone K such that X is the closed linear hull of K. Let T be a compact linear operator in X that maps K into itself and has a positive spectral radius ρ. Then there exists a non-zero vector u in K and a positive continuous linear functional f such that

$$Tu = \rho u, T^* f = \rho f.$$

The proof depends on the following lemma concerning compact liner operators.

Lemma 5.4. *Let T be compact linear operator in a normed space X, and let T have a positive spectral radius. Then there exists a vector x in X such that*

$$\limsup_{n\to\infty} \|T^n\|^{-1}\|T^n x\| > 0.$$

Proof. Let $\varepsilon > 0$, and suppose that the lemma is false. Then,for each x in X there exists a positive integer N_x such that

$$n \geq N_x \Rightarrow \|T^n x\| < \frac{\varepsilon}{2}\|T^n\|$$

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Also

$$\|x' - x\| < \frac{\varepsilon}{2} \Rightarrow \|T^n x' - T^n x\| < \frac{\varepsilon}{2}\|T^n\|,$$

and so $$\|T^n x'\| < \varepsilon\|T^n\| \; (n \geq N_x, \|x' - x\| < \frac{\varepsilon}{2})$$

Let S denote the closed unit ball in X.Then $\overline{TS}$ is compact and so has a finite covering by open balls of radius $\frac{\varepsilon}{2}$ and centers $x_1, \ldots x_m$ say. Let

$$N \max(N_{x_1}, \ldots, N_{x_m}).$$

Then

$$\|T^n x\| < \varepsilon\|T^n\| \; (n \geq N, x \in TS),$$
$$\|T^{n+1} x\| \leq \varepsilon\|T^n\| \; (n \geq N, x \in S),$$

and so
$$\|T^{n+1}\| \leq \varepsilon\|T^n\| \; (n \geq N),$$
from which it follows that $\lim\limits_{n\to\infty} \|T^n\|^{1/n} \leq \varepsilon$. □

Proof of Theorem 5.5. By Lemma 5.4 and the fact that $X = \overline{K - K}$, there exists a vector x in C with

$$\lim_{n\to\infty} \sup \|T^n\|^{-1}\|T^n x\| > 0.$$

90 It easily follows from this that

$$\mu \lim_{n\to\infty} p(T^n)^{1/n} = \lim_{n\to\infty} \|T^n\|^{1/n} = \rho$$

Applying Theorem 5.2 with $E = X$ and $C = X$ we see that there exists a non-zero vector u in K with

$$Tu = \rho u$$

As in the beginning of the proof of Theorem 5.3, the set K^* of continuous linear functions f with

$$f(x) \geq 0 \quad (x \in K)$$

is a positive cone in the dual space X^*. Also, T^* is a compact linear operator in X^* and maps K^* into itself. Applying Theorem 5.2 with

$E = X^*$ and $C = K^*$, we conclude that there exists a positive continuous linear functional f with

$$T^* f = \mu^* f,$$

where μ^* is the partial spectral radius of T^*. Since the spectral radius of T^* is equal to that of T, we have $\mu^* \leq \rho$. It only remains to prove that $\mu^* \geq \rho$. There exists $u \in K$ with $||u|| = 1$ and $Tu = \rho u$. We have

$$T_u^n = \rho^n u \quad (n = 1, 2, \ldots).$$

Since $\delta = d(u, -K) > 0$, there exists ϕ in K^* with $||\phi|| \leq |$ and $\phi(u) = \delta$. Then 91

$$\phi(T^n u) = \phi(\rho^n u) = \rho^n \delta,$$

and so we have in turn $(T^{*^n}\phi)(u) = \rho^n \delta$,

$$||T^{*^n}\phi|| \geq \rho^n \delta,$$

$$p(T^{*^n}) \geq \rho^n \delta, \qquad (n = 1, 2, \ldots)$$

$$\mu^* \geq \rho$$

Remark. It is in fact enough in Theorem 5.5 that K be a complete cone in a normed space (rather than a closed cone in a complete space).

In our next theorem we give a formula for the calculation of positive eigenvectors corresponding to ρ valid under the conditions of Theorem 5.5. For its proof we shall need some result from the classical Riesz Schauder theory of compact operators in Banach spaces. We shall state these results without proof (see Dunford and Schwartz [14]).

Let T be a compact linear operator in a complex Banach space X. The spectrum $\sigma(T)$ is the set of all complex numbers λ for which $\lambda I - T$ is not a one-to-one mapping of X onto itself. Then $\sigma(T)$ is a countable set contained in the disc $|\zeta|| \leq \rho$ where $\rho = \lim_{n\to\infty} ||T^n||^{1/n}$. Also each element of $\sigma(T)$ other than 0 is an eigenvalue, and is an isolated point of $\sigma(T)$, *i.e.* has a neighbourhood containing no other point of $\sigma(T)$. 92

Let λ be a non-zero point of $\sigma(T)$. Then there is a positive integer $\gamma = \gamma(\lambda)$ called the *index* of λ which is the smallest integer n with the property that

$$(T - \lambda I)^{n+1} x = 0 \Rightarrow (T - \lambda I)^n x = 0.$$

[The null space of $(T-\lambda I)^n$ increases with n, but eventually we come to an integer after which it remains constant.] Let

$$N = \{x : (T - \lambda I)^\gamma x = 0\},$$

and let

$$M = (T - \lambda I)^\gamma X.$$

Then N and M are closed subspaces of X and

$$X = N \oplus M \tag{1}$$

(i.e., each vector is x has a unique expression $x = y + z$ with $y \in N$ and $z \in M$.)

Also M and N are invariant subspaces for T,

$$T(M) \subset M, \quad TN \subset N.$$

There exist bounded linear projections P and Q orthogonal to each other and with ranges M and N respectively.

$$\begin{aligned} &I = P + Q, PQ = QP = 0, P^2 = P, Q^2 = Q \\ &PX = M, QX = N. \end{aligned} \tag{2}$$

93 Let T_M denote the restriction of T to M. Then T_M is a compact linear operator in M and

$$\alpha \neq \lambda, \alpha \in \sigma(T) \Longleftrightarrow \alpha \in \sigma(T_M) \tag{3}$$

Theorem 5.6. Let X, K, T, ρ satisfy the conditions of Theorem 5.5 and let γ be the index of ρ. Let Q be the projection onto the null space of $(T-\rho I)^\gamma$ which is orthogonal to the range of $(T-\rho I)$. Then

(i) $\lim\limits_{n\to\infty} (1+\rho)^{-n} \binom{n}{\gamma-1}^{-1} (I+T)^n = (1+\rho)^{1-\nu}(T-\rho I)^{-1} Q,$

(ii) $\lim\limits_{n\to\infty} \|(I+T)^n\|^{-1}(I+T)^n = \|(T-\rho I)^{\nu-1} Q\|^{-1}(T-\rho I)^{-1} Q,$

the convergence being with respect to the operator norm.

Proof. Since there is a natural isometry between bounded linear operator in a real Banach space and their complexifications, and since this isometry preserves compactness of operators, there is no loss of generality in supposing that X is a complex Banach space. □

Taking $\lambda = \rho$ in the above considerations, we have continuous linear projections P, Q on to the range M and null space N of A^{γ}, γ being the index of ρ, where $A = T - \rho I$, and (1), (2), (3) hold.

Suppose α is a non-zero point in the spectrum of $P + TP$. We prove that $\alpha - 1$ is in the spectrum of T_M (the restriction of T to the subspace M.) In fact

$$S = \alpha I - (P + TP)$$

is not a $(1 - 1)$ mapping of X onto itself. Either **94**

i) the mapping S is not $(1 - 1)$,

or

ii) the range of the mapping S in not the whole of X.

In case (i) there exists a non-zero vector x in X with

$$(P + TP)x = \alpha x.$$

It follows that $x \in PX = M$, $x = Px$, and so

$$(I + T)x = \alpha x, \;\; Tx = (\alpha - 1)x, \;\; \alpha - 1 \in \sigma(T_M).$$

In case (*ii*), since $SM \subset M, SN \subset N$ and $X = M+N$, either $SM \neq M$ or $SN \neq N$. But as P is zero on N, and $\alpha \neq 0$, we have $SN = N$. Hence $SM \neq M$. But

$$Sx = \{(\alpha - 1)I - T\}x, \quad (x \in M)$$

and so $\alpha - 1 \in \sigma(T_M)$. It follows from this and (3) that if α is a non-zero point of the spectrum of $P + TP$, then

$$\alpha - 1 \in \sigma(T)\ (\rho).$$

Therefore

$$\lim_{n\to\infty} \|(P + TP)^n\|^{1/n} \leq \sup\{|1 + \zeta| | \zeta \in \sigma(T), \zeta \neq \rho\}$$

Since all points ζ of $\sigma(T)$ satisfy $|\zeta| \leq \rho$, and since ρ is as isolated point **95**
of $\sigma(T)$, it follows that

$$\lim_{n\to\infty} \|(P + TP)^n\|^{1/n} = k < 1 + \rho$$

since $TM \subset M$, we have $PTP = TP$, and so

$$\lim_{n\to\infty} \|(I + T)^n P\|^{1/n} = k.$$

We choose η with $k < \eta < 1 + \rho$. Then there exists n_0 with

$$\|(I + T)^n P\| < \eta^n \quad (n \geq n_0) \tag{1}$$

We have

$$I + T = (1 + \rho)I + A$$

and

$$A^n Q = 0 \ (n \geq \nu).$$

Hence

$$(I + T)^n Q = \left\{(1 + \rho)^n I + \binom{n}{1}(1 + \rho)^{n-1} A + \cdots + \binom{n}{-1}(1 + \rho)^{n-\nu+1} A^{\nu-1}\right\} Q$$

It follows that

$$\lim_{n\to\infty} (1 + \rho)^{-n} \binom{n}{\nu - 1}^{-1} (I + T)^n Q = (1 + \rho)^{1-\nu} A^{\nu-1} Q \tag{2}$$

Also (1) gives,

$$\lim_{n\to\infty} (1 + \rho)^{-n} \binom{n}{\nu - 1}^{-1} (I + T)^n P = 0 \tag{3}$$

96 and, since $(I + T)^n = (I + T)^n P + (I + T)^n Q$, (2) and (3) give

$$\lim_{n\to\infty} (1 + \rho)^{-n} \binom{n}{\nu - 1}^{-1} (I + T)^n = (1 + \rho)^{1-\nu} A^{\nu-1} Q \tag{4}$$

Taking norms, we have

$$\lim_{n\to\infty} (1 + \rho)^{-n} \binom{n}{\nu - 1}^{-1} \|(I + T)^n\| = (1 + \rho)^{1-\nu} \|A^{\nu-1} Q\| \tag{5}$$

By definition of ν, the null space of $A^{\nu-1}$ is not the whole of N, and so $A^{\gamma-1}Q \neq 0$. Thus (4) and (5) combine to give (ii).

Further results connected with Theorem 5.5 are given by Krein and Rutman [20]. In particular very precise results are proved (Theorem 6.3 [20]) for an operator T which satisfies the conditions of Theorem 5.5 and also maps each non-zero point of K into the interior of K. In this case $\nu = 1$, and the result of Theorem 5.6 takes the specially simple from

$$\lim_{n\to\infty} (1 + \rho)^{-n}(I + T)^n = Q.$$

Here Q is a dimensional operator

$$Qx = f(x)u,$$

where u is a positive eigenvector of T, f a positive eigenvector of T^* normalized by taking $f(u) = 1$.

Chapter 6

Self-adjoint linear operator in a Hilbert space

It would be foolish of me to attempt to give in these lectures an account **97**
of the many methods that have been developed for the study of the spectral resolution of a self adjoint operator. I shall limit myself to giving an account of a certain explicit formula for the projections belonging to the spectral resolution. In general this is a theorem about projections; in the case of a compact operator it becomes a theorem about eigenvectors.

Definition 6.1. A complex (real) Hilbert space is a vector space over $C(R)$ with a mapping

$$H \times H \to C(R)$$

called the scalar product and denoted by (x, y) which satisfies the following axioms

i) $(x, y) = \overline{(y, x)}$

ii) $(x_1 + x_2, y) = (x_1, y) + (x_2, y)$ $(x_1, x_2, y \in H)$

iii) $(\alpha x, y) = \alpha(x, y)$ $(x, y \in H, \alpha \in C(R))$

iv) $(x, y) > 0$ for $x \neq 0$; $(x, x) = 0$ for $x = 0$ $\quad (x \in H)$

v) H is a Banach space with the norm $\|x\| = (x, x)^{\frac{1}{2}}$.

Let H be a real or complex Hilbert space, and let $\mathscr{S}$ denote the class of all *bounded symmetric* operators in H, i.e., bounded linear mappings of T into itself such that

$$(Tx, y) = (x, Ty) \quad (x, y \in H)$$

98 $\mathscr{S}$ is the class bounded self adjoint operators, $T = T^*$. A relation $\leq$ is introduced into $\mathscr{S}$ by writing $A \leq B$ or $B \geq A$ to denote that

$$(Ax, x) \leq (Bx, x) \quad (x \in H).$$

In particular, the operators T belonging to $\mathscr{S}$ that satisfy

$$T \geq 0$$

are called *positive operators* .

Note that this definition of 'positive' has nothing to do with the property of mapping a cone into itself that we have considered in earlier chapters.

We establish a few elementary properties of the relation. First we recall a few obvious properties of commutants.

Let B denote the class of all bounded linear operators in H, and given $E \subset B$, let

$$E' = \{T : T \in B \text{ and } AT = TA(A \in E)\}$$

E' is called the *commutant* of E.

(i) $E_1 \subset E_2 \Rightarrow E_2' \subset E_1'$ (obvious)

(ii) E' is strongly closed.

Let $T_n \in E'$ converge strongly towards $T \in B$. Then $T_nA = AT_n \quad (n = 1, 2, \ldots)$

$$TA_x = \lim_{n\to\infty} T_nAx = \lim_{n\to\infty} AT_nx = AT_x \quad (x \in H)$$

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(iii) E self-adjoint $\Rightarrow E'$ self-adjoint.

$$\begin{aligned} T \subset E' \Rightarrow AT &= TA \quad (A \in E) \\ A^*T &= TA^* \quad (A \in E) \\ T^*A &= AT^* \quad (A \in E) \end{aligned}$$

(iv) E' is a complex linear algebra (obvious)

(v) $E = (T) \Rightarrow E''$ is a strongly closed commutative algebra containing T.

$$(T) \subset (T)' \Rightarrow (T)'' \subset (T)'$$
$$A_1, A_2 \in (T)' \Rightarrow A_1 \in (T)'', \quad A_2 \in (T)' \quad A_1A_2 = A_2A_1.$$

We next establish some well known elementary propositions concerning positive operators.

(a) For any positive operator T, the *generalized Schwartz inequality* holds i.e.,

$$|(Tx, y)|^2 \leq (Tx, x)(Ty, y)$$

Proof. $B(x, y) = (Tx, y)$ is a positive semi-definite symmetric bilinear form and so the generalized Schwarz inequality for this form. □

(b) If T is a positive operator, then

$$\|T\| = \sup\{(Tx, x) : \|x\| \leq 1\}$$

Proof. Let T be a positive operator and let **100**

$$M = \sup\{(Tx, x) : \|x\| \leq 1\}$$

By the Schwarz inequality

$$|(Tx, x)| \leq \|Tx\| \, \|x\|,$$

and so $M \leq \|T\|$. On the other hand putting $y = Tx$ in the generalized Schwarz inequality, we have

$$\|Tx\|^4 = (Tx, Tx)^2 = |(Tx, y)|^2 \leq (Tx, x)(Ty, y) \leq M^2\|x\|^2\|Tx\|^2,$$

and so $\|T\| \leq M$. □

(c) The set of positive operators is a positive cone in $\mathscr{S}$.

Proof. If T and $-T$ are both positive, we have

$$(Tx, n) = 0 \quad (x \in H)$$

By (b), this gives $T = 0$. The other properties of the cone are obvious. □

(d) Let (T_n) be a bounded increasing sequence of elements of $\mathscr{S}$, i.e.,

$$T_n \leq T_{n+1} \leq M.I \quad (n = 1, 2, \ldots)$$

Then (T_n) converges strongly to an element T of s i.e.

$$\lim_{n\to\infty} T_n x = Tx \quad (x \in H)$$

101 *Proof.* For $m < n$, let $A_{m,n} = T_n - T_m$. By the generalized Schwartz inequality (a) with $T = A_{m,n}$ and $y = A_{m,n}x$, we have $\|A_{m,n}x\|^4 = (A_{m,n}x,\ A_{m,n}x)^2 = |(A_{m,n}x, y)|^2 \leq (A_{m,n}x, x)(A_{m,n}y, y)$. Since $0 \leq A_{m,n} \leq MI$, we have $(A_{m,n}y, y) \leq M^3\|x\|^2$. Hence $\|T_nx - T_mx\|^4 \leq M^3\|x\|^2 \{T_nx, x) - T_mx, x)\}$. Since the sequence $\{(T_nx, x)\}$ is a bounded increasing sequence of real numbers, it follows that (T_nx) is a Cauchy sequence which converges to an element $Tx \in H$ in view of the axiom (v) in definition 6.1.

(e) $T \geq 0 \Rightarrow T^n \geq 0$ $(n = 1, 2, \ldots)$.

□

Proof. $(T^{2K}x, x) = (T^K x, T^K x) \geq 0$

$$(T^{2k+1}x, x) = \ (T.T^x, T^k x) \geq 0.$$

(f) Each positive operator T is the square of a positive operator $T^{\frac{1}{2}}$, and $T^{\frac{1}{2}}$ belongs to the second commutant $(T)''$ of T.

□

Proof. Suppose that $0 \leq A \leq I$, and let $B = I - A$, so that also

$$0 \leq B \leq I.$$

□

Let Y_n be the sequence defined inductively by

$$Y_0 = 0,\ Y_{n+1} = \frac{1}{2}(B + Y_n^2)\ (n = 0, 1, 2, \ldots).$$

By induction we have $\|Y_n\| \leq 1$, and so

$$0 \leq Y_n \leq I.$$

Also, since **102**

$$Y_{n+1} - Y_n = \frac{1}{2}(Y_n^2 - Y_{n-1}^2) = \frac{1}{2}(Y_n + Y_{n-1})(Y_n - Y_{n-1}),$$

we see by induction that $Y_{n+1} - Y_n$ is a polynomial in B with non-negative real coefficients. Since $B^n \geq 0$, for every n, it follows that (Y_n) is an increasing sequence. Hence (Y_n) converges strongly to a positive operator Y, and we have

$$0 \leq Y \leq I,\ Y = \frac{1}{2}(B + Y^2)$$

Let $X = I - Y$. Then X is a positive operator and

$$X^2 = A.$$

If a bounded linear operator commutes with A, it commutes with each polynomial in A, hence it commutes with Y_n, and therefore with X.

If $0 \leq T \leq MI$, $A = \dfrac{1}{M}.T$ satisfies $0 \leq A \leq I$ and so the proposition holds for T.

The positive square root $T^{\frac{1}{2}}$ is in fact unique but we do not need this fact.

(g) $A \geq 0,\ B \geq 0,\ AB = BA \Rightarrow AB \geq 0.$

Proof. Since $A \in (B')$, we have $B^{\frac{1}{2}} \in (A)'$ and so

$$AB = AB^{\frac{1}{2}}B^{\frac{1}{2}} = B^{\frac{1}{2}}AB^{\frac{1}{2}}.$$

□

Finally, $(B^{\frac{1}{2}}AB^{\frac{1}{2}}x, x) = (AB^{\frac{1}{2}}x, B^{\frac{1}{2}}x) \geq 0$.
(h) $T \geq 0$ $I + T$ is invertible, $(I + T^{-1}) \geq 0$, and

$$(I + T)^{-1} \in (T)''.$$

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Proof. We have

$$I \leq I + T \leq (1 + M)I,$$
$$\frac{1}{1+M} \leq A \leq I,$$

where $A = \dfrac{1}{1+M}(I + T)$. Therefore

$$\|I - A\| \leq \|\left(1 - \frac{1}{1+M}\right)I\| = \frac{M}{1+M} < 1.$$

□

Therefore the Neumann series

$$I + (I - A) + (I - A)^2 + \cdots$$

converges in operator norm and since $(I - A)^k \geq 0$ its sum is a positive operator B. We have

$$I = B(I - A) = I + (I - A)B = B,$$

and so $AB = BA = I$.

Finally $(1 + M)^{-1}B$ is the required positive inverse of $I + T$. By a *projection* we mean an operator P belonging to $\mathscr{S}$ with $P^2 = P$.

(i) Each projection P satisfies $0 \leq P \leq I$.

Proof. Since $P \in \mathscr{S}$ and $P = P^2$, we have $P \geq 0$. $\square$

Since $I - P \in \mathscr{S}$ and $(I - P)^2 = I - P$ we have $I - P \geq 0$.

(j) For projections P_1, P_2

$$P_1 \geq P_2 \iff P_2 = P_2 P_1 \iff P_2 = P_1 P_2$$

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Proof. $P_1 P_2 = P_2 \Rightarrow P_2 = P_2^* = (P_2 P_1)^* = P_1^* P_2^* = P_1 P_2$

$$P_1 P_2 = P_2 \; P_2 = P_2^* = (P_2 P_1)^* = P_2^* P_1^* = P_2 P_1.$$

$\square$

Thus, if $P_2 = P_2 P_1$, we also have $P_1 P_2 = P_2$, and so

$$(P_1 - P_2)^2 = P_1^2 - P_1 P_2 - P_2 P_1 + P_2^2 = P_1 - P_2,$$

and therefore $\quad P_1 \geq P_2$.

Finally suppose that $P_1 \geq P_2$. If $P_1 x = 0$, then $P_2 x = 0$, for

$$(P_2 x,\ P_2 x) = (P_2^2 x, x) = (P_2 x, x) \leq (P_1 x, x) = 0.$$

Since $P_1(I - P_1) = 0$,

$$P_2(I - P_1) = 0,$$

i.e.,
$$P_2 = P_2 P_1.$$

Lemma 6.1. *Let $A \geq 0$, and let $B = 2A^2(I + A^2)^{-1}$. Then*

i) $B \in A''$,

ii) $0 \leq B \leq A$,

iii) $I - B = (I - A)(I + A)(I + A^2)^{-1}$,

iv) *if P is a projective permutable with A and $P \leq A$, then $P \leq B$.*

Proof. Proposition (h) implies (i). □

105 That $B \geq 0$ is clear since A^2 and $(I + A^2)^{-1}$ are permutable. Also

$$(I + A^2)(A - B) = A + A^3 - 2A^2 = A(I - A)^2 \geq 0,$$

and so, using the permutability of the operators,

$$A - B = (I + A^2)^{-1}(I + A^2)(A - B) \geq 0$$

This proves (ii), and (iii) is straight forward.

Let P be a projection such that $P \in A'$ and $P \leq A$. We have

$$P = P^2 \leq PA \leq A^2,$$

and therefore

$$P = P^2 \leq A^2 P.$$

Therefore

$$(I + A^2)(B - P) = 2A^2 - (I + A^2)P \geq 2A^2 - 2A^2 P$$
$$= 2A^2(I - P) \geq 0$$

Finally, since $(I + A^2)^{-1}$ is permutable with all the operators concerned,

$$B - P \geq (I + A^2)^{-1} 2A^2 (I - P) \geq 0.$$

Theorem 6.1. Let A be a positive operator, and let the sequence (A_n) be defined inductively by

$$A_1 = A,\ A_{n+1} = 2A_n^2(I + A_n^2)^{-1} \quad (n = 1, 2, \ldots)$$

106 Then

i) $0 \leq A_{n+1} \leq A_n \quad (n = 1, 2, \ldots)$,

ii) the sequence (A_n) converges strongly to a projection Q belonging to A''.

iii) $Q \leq A$,

iv) $(I - A)(I - Q) \geq 0$,

v) Q is maximal in the sense if P is a projection permutable with A and satisfying $P \leq A$, then $P \leq Q$.

Proof. (i) This follows at once from Lemma 1. (ii) and (iii). It follows from (i) and Proposition (d) that (A_n) converges strongly to a positive operator Q with $Q \leq A$, and that $Q \in (A)''$. It remains to prove that Q is a projection. □

Since $0 \leq A_n \leq A$, we have

$$\|A_n\| \leq \|A\| \ (n = 1, 2, \ldots);$$

and therefore, since

$$\lim_{n\to\infty} A_n x = Qx \ (x \in H),$$

we have in turn,

$$\lim_{n\to\infty} A_n^2 x = Q^2 x \ (x \in H),$$

$$\lim_{n\to\infty} A_{n+1}\{(I + Q^2)x - (I + A_n^2)x\} = 0 \quad (x \in H),$$

$$\lim_{n\to\infty} A_{n+1}(I + Q^2)x = \lim_{n\to\infty} 2A_n^2 x = 2Q^2 x \quad (x \in H).$$

But 107

$$\lim_{n\to\infty} A_{n+1}(I + Q^2)x = Q(I + Q^2)x \quad (x \in H),$$

and so

$$Q(I + Q^2) = 2Q^2.$$

Therefore

$$(Q - Q^2)^2 = 0.$$

But since $Q - Q^2$ is symmetric, this gives

$$(Q - Q)^2 = 0,$$

i.e., Q is a projection.

(iv) By Lemma 1 (iii),

$$\begin{aligned}(I - A_n) &= (I - A_{n-1})(I + A_{n-1})(I + A_{n-1}^2)^{-1},\\ &= (I - A)\prod_{k=1}^{n-1}(I + A_k)(I + A_k^2)^{-1};\end{aligned}$$

and so $$(I - A)(I - A_n) \geq 0 \quad (n = 1, 2, \ldots),$$
which gives $$(I - A)(I - Q) \geq 0.$$

(v) Let P be a projection permutable with A such that $P \leq A$.

108 By repeated application of Lemma 1 (iv), P is permutable with A_n and $P \leq A_n$. In the limit we have $P \leq Q$.

Theorem 6.2. Let T be a bounded symmetric operator with

$$mI \leq T \leq MI.$$

Let $E_M = I$, and for $\lambda < M$ let E_λ be the projection Q of Theorem 1 corresponding to A given by

$$A = \frac{1}{M - \lambda}(MI - T)$$

Then

i) the projections E belongs to $(T)''$;

ii) $E_\lambda = 0$ $(\lambda < m)$, $E_M = I$;

iii) $E_\lambda \leq E_\mu$ $(\lambda \leq \mu)$;

iv) $\lambda(E_\mu - E_\lambda) \leq T(E_\mu - E_\lambda) \leq \mu(E_\mu - E_\lambda)$ $(\lambda \leq \mu)$;

v) the maily (E_λ) is strongly continuous on the right.

Proof. i) follows from Theorem 1 (*ii*).

ii) If $E_\lambda \neq 0$, there exists a non-zero x_0 with $E_\lambda x_0 = x_0$.

□

By Theorem 1 (iii), we have

$$E_\lambda \leq \frac{1}{M-\lambda}(MI - T),$$

and so

$$(x_0, x_0) = (E_\lambda x_0, x_0) \leq \frac{1}{M-\lambda}((MI - T)x_0, x_0).$$

Since $MI - T \ \leq (M - m)I$, this gives

$$(x_0, x_0) \leq \frac{M-m}{M-\lambda}(x_0, x_0),$$

and so $\lambda \geq m$. This proves (*ii*). **109**

(iii) This is obvious except when $\lambda < \mu < M$. In this case, since E_λ is a projection permutable with $\frac{1}{M-\mu}(MI - T)$, and

$$E_\lambda \leq \frac{1}{M-\lambda}(MI - T) \leq \frac{1}{M-\mu}(MI - T),$$

Then 1 (v) shows that

$$E_\lambda \ \leq \ E_\mu$$

(iv) Suppose that $\lambda < \mu \leq M$. By Theorem 1 (*iv*),

$$\left(I - \frac{1}{M-\lambda}(MI - T)\right)(I - E_\lambda) \geq 0,$$

i.e.,

$$(I - \lambda I)(I - E_\lambda) \geq 0.$$

Since $E_\lambda \leq E_\mu$, we have $E_\lambda E_\mu = E_\lambda$, and so this gives

$$(T - \lambda I)(E_\mu - E_\lambda) \geq 0,$$

which is the left hand inequality in (*iv*). The right hand inequality is obvious if $\mu = M$ (since $T \leq MI$); and if $\mu < M$ we have

$$E_\mu \leq \frac{1}{M-\mu}(MI - T),$$

i.e.,
$$T \leq MI - (M-\mu)E_\mu.$$

Since $E_\mu(E_\mu - E_\lambda) = E_\mu - E_\lambda$, this gives.

$$T(E_\mu - E_\lambda) \leq \mu(E_\mu - E_\lambda),$$

110 and (iv) is proved.

(v) Suppose $\mu < M$. If (E_λ) is not strongly continuous on the right at μ, there exists a sequence (λ_n) convergent decreasing to μ but such that E_{λ_n} does not converge strongly to E_μ.

Since (E_{λ_n}) is a decreasing sequence of operators, with $E_{\lambda_n} \geq E_\mu$, there exists a positive operator J such that $J \geq E_\mu$ and (E_{λ_n}) converges strongly to J. Then J is a projection, permutable with T, and

$$E_\mu \leq J \leq E_{\lambda_n} \leq \frac{1}{M-\lambda_n}(MI - T) \quad (n = 1, 2, \ldots)$$

It follows that

$$E_\mu \leq J \leq \frac{1}{M-\mu}(MI - T),$$

and so by the maximal property of E_μ,

$$J \leq E_\mu.$$

This completes the proof of the theorem.

Corollary. (The spectral theorem).

$$T = \int_{M-\varepsilon}^{M} \lambda dE_\lambda \quad (\varepsilon > 0)$$

The integral being the uniform limit of its Riemann-Stieltjes sums. In fact let

$$m - \varepsilon = \lambda_0 < \lambda_1 < \cdots < \lambda_n = M.$$

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Then

$$\lambda_{k-1}(E_{\lambda_k} - E_{\lambda_{k-1}}) \leq T(E_{\lambda_k} - E_{\lambda_{k-1}}) \leq \lambda_k(E_{\lambda_k} - E_{\lambda_{k-1}})$$

and so since $E_{\lambda_n} = I$ and $E_{\lambda_0} = 0$,

$$\sum_{k=1}^{n} \lambda_{k-1}(E_{\lambda_k} - E_{\lambda_{k-1}}) \leq T \leq \sum_{k=1}^{n} \lambda_k(E_{\lambda_k} - E_{\lambda_{k-1}}),$$

$$0 \leq T - \sum_{k=1}^{n} \lambda_{k-1}(E_{\lambda_k} - E_{\lambda_{k-1}}) \leq \sum_{k=1}^{n} (\lambda_k - \lambda_{k-1})E_{\lambda_k} - E_{\lambda_{k-1}})$$
$$\leq \max(\lambda_k - \lambda_{k-1})I$$

Hence

$$\|T - \sum_{k=1}^{n} \lambda_{k-1}(E_{\lambda_k} - E_{\lambda_{k-1}})\| \to 0 \text{ as } \max(\lambda_k - \lambda_{k-1}) \to 0$$

Moreover

$$T^r = \int_{m-\varepsilon}^{M} \lambda^r dE_\lambda \quad (r = 0, 1, 2, \ldots)$$

To see this we rewrite (*iv*) in the form

$$(M - \mu)(E_\mu - E_\lambda) \leq (MI - T)(E_\mu - E_\lambda) \leq (M - \lambda)(E_\mu - E_\lambda)$$

since $MI - T \geq 0$, $M - \lambda \geq 0$, and $M - \mu \geq 0$, it follows that

$$(M - \mu)^r(E_\mu - E_\lambda) \leq (MI - T)^r(E_\mu - E_\lambda) \leq (M - \lambda)^r(E_\mu - E_\lambda)$$

Therefore, as in the preceeding argument, **112**

$$(MI - T)^r = \int_{m-\varepsilon}^{M} (M - \lambda)^r \, dE_\lambda \quad (r = 0, 1, 2, \ldots)$$

and the required result follows by induction.

In the next theorem we consider the special simplification which occurs when the operator is also compact. We need a simple lemma.

Lemma 6.2. *Let A be a positive operator, and let (A_n) be the sequence constructed as in Theorem 7. Then $A_n = A^2 B_n$ $(n = 2, 3, \ldots)$, where each B_n belongs to $(A)''$ and*

$$0 \le B_{n+1} \le B_n \ \ (n = 2, 3, \ldots)$$

Proof. We have

$$A_2 = 2A^2(I + A^2)^{-1} = A^2 B_2,$$

with $B_2 = 2(I + A^2)^{-1}$. If $A_n = A^2 B_n$ with $B_n \ge 0$ and $B_n \in (A)''$, then

$$\begin{aligned} A_{n+1} &= 2A^4 B_n^2 (I + A^4 B_n^2)^{-1} \\ &= A^2 B_{n+1}, \end{aligned}$$

with $B_{n+1} = 2A^2 B_n^2 (I + A^4 B_n^2)^{-1}$. Then

$$\begin{aligned} B_n - B_{n+1} &= (I + A^4 B_n^2)^{-1} \{B_n (I + A^4 B_n^2) - 2A^2 B_n^2\} \\ &= (I + A^4 B_n^2)^{-1} \ B_n \ (I - A^2 B_n)^2, \end{aligned}$$

113 so that $\quad 0 \le B_{n+1} \le B_n.$ □

Theorem 6.3. Let A be a compact positive operator, and let (A_n) and Q be the corresponding and projection defined as in Theorem 1. Then

(i) (A_n) converges uniformly to Q;

(ii) Q has finite rank;

(iii) the range of Q is spanned by eigenvectors of A corresponding to eigenvalues λ with $\lambda \ge 1$, and all such eigenvectors lie in the range of Q.

Proof. Let B_n be the sequence defined in Lemma 2. Then (B_n) converges strongly to an operator B in $(A)''$. Since $A_n = A^2 B_n$ and A_n converges strongly to Q, we have

$$Q = A^2 B.$$

□

Let K be the unit ball in the Hilbert space H, and let $E = \overline{(AK)}$. Then E is a norm compact set. Since $\lim\limits_{n\to\infty} B_n x = Bx$ $(x \in H)$, we have

$$\lim_{n\to\infty} (B_n x, x) = (Bx, x) \ \ (x \in E).$$

With respect to the norm topology in E, the functions $(B_n x, x)$ are continuous real functions converging decreasingly to the continuous real 114
function (Bx, x). Therefore, by Dini's theorem, the convergence is uniform on E. Since

$$(A_n x, x) = (A^2 B_n x, x) = (B_n Ax, Ax)$$

It follows that $(A_n x, x)$ converges uniformly on K to $(BAx, Ax) = (A^2\ Bx, x) = (Qx, x)$. Therefore $A_n \to Q$ uniformly (i.e., with respect to the operator norm).

Since $Q = A^2 B$ and A is compact, we know that Q is compact, and therefore has finite rank, i.e., its range H_Q is finite dimensional; for Q is the identity operator in the Banach space H_Q and a Banach space in which a ball is compact is finite -dimensional.

The rang H_Q of Q is a finite dimensional Hilbert space and A maps Q into itself (since $QA = AQ$). By the elementary theory of symmetric matrices, H_Q is spanned by eigenvectors $u_1, \ldots, u_r$ with real eigenvalues $\lambda_1, \ldots, \lambda_r$,

$$A_i\ u_i = \lambda_i\ u_i$$

Since $Qu_i = u_i$, and $A \geq Q$, we have

$$\lambda_i (u_i, u_i) = (Au_i,\ u_i) \geq (Qu_i,\ u_i) = (u_i,\ u_i),$$

and so $\lambda_i \geq 1$.

Conversely, let u be an eigenvector of A with $Au = \lambda u, \lambda \geq 1$. We may suppose that $\|u\| = 1$ and then define a projection P by taking 115

$$Px = (x, u)u.$$

$APx = (x, u)Au = \lambda(x, u)u = (x, \lambda u)u = (x, Au)u = (Ax, u) = PAx$. Also $P \leq A$, for given $x \in H$, we have $x = H$, we have $x = \xi u + v$ with $(u, v) = 0$. Then $Ax = \ \lambda\xi u + Av$, and

$$(u, Av) = (Au, v) = \lambda(u, v) = 0. \text{ So}$$

$$(Ax, x) = \lambda|\xi|^2 (u, u) + (Av, v)^2$$

$$\geq \lambda|\xi|^2(u, u) = \lambda(Px, x) \geq (Px, x).$$

By the maximal property of Q, $P \leq Q$. Hence $P = QP$, $u \in H_Q$.

The iterative method, given here can also be applied to construct the projections belonging to the spectral family of an unbounded self-adjoint operator, and details of this may be found in my paper.

Chapter 7

Simultaneous fixed points

In this brief chapter we are concerned with the existence of a simultaneous fixed point of a family $\mathscr{F}$ of mappings **116**

$$Tu = u \ (T \in \mathscr{F}).$$

We first state without proof two well known theorems on this question proofs of which will be found in Dunford and Schwartz [14] pp.456-457. We then prove a theorem on families of mappings of a ∞*ne* into itself. Some further results in the present context will appear in the next chapter in the theory of a special class of semialgebras.

Theorem 7.1 (Markov- Kankutani)**.** Let K be a compact convex subset of a Hausdorff linear topological space. Let $\mathscr{F}$ be a commuting family of continuous affine mappings of K into itself. Then there exists a point **117**
u in K with

$$Tu = u \ (T \in \mathscr{F})$$

A mapping T is said to be *affine* if

$$T(\alpha x + (1-\alpha)y) = \alpha Tx + (1-\alpha)Ty \quad (x, y \in K, \ 0 \leq \alpha \leq 1)$$

Theorem 7.2 (Kakutani)**.** Let K be a compact convex subset of a Hausdorff locally convex space and let G be a group of linear mappings which is equi-continuous on K and satisfies $GK \subset K$. Then there exists u in K with

$$Tu = u \quad (T \in G)$$

G is said to be equi-continuous on K if given a neighbourhood V of 0, there exists a neighbourhood U of 0 such that $k_1, k_2 \in K$, $k_1 - k_2 \in U$ implies that $Tk_1 - Tk_2 \in V (T \in G)$.

We have chosen to include a theorem on mappings of cones into themselves in this chapter because of the opportunity it gives to employ a method of proof quite different from the methods used in the rest of these lectures. The method depends on the concept of *ideal* in a partially ordered vector space.

Definition 7.1. Let E be a partially ordered vector space with positive cone C. A subset J of E is called an ideal if

(i) J is a linear subspace

(ii) $j \in J \Rightarrow [0, j] \subset J$,

Here $[a, b] = \{x : a \leq x \leq b\}$; and conditions (i) and (ii) are equivalent to (i) and

(ii)′ $j \in J \Rightarrow [-j, j] \subset J$

For let $j \in J$ and $-j \leq x \leq j$. Then $0 \leq x + j \leq 2j$. By (i), $2j \in J$ and by (ii), $x + j \in J$. Again by (i), $x \in J$.

Example. If $E_\circ$ is a linear subspace of E with $E_\circ\, C = (0)$. Then $E_\circ$ is an ideal. (For then [0, j] for $j \in J$ is empty unless $j = 0$).

118 **Definition 7.2.** An element e of C is called an *order unit* if $[-e, e]$ absorbs all points of E, i.e., if for every x in E

$$-\lambda e \leq x \leq \lambda e$$

for all sufficiently large λ.

By definition of a positive cone, C contains non -zero elements, and so certainly

$$e > 0.$$

We establish a few elementary properties of ideals. An ideal J is *proper* if $0 \neq J \neq E$.

a) An element of C is an order unit of E if and only if it is not contained in any proper ideal of E.

Proof. Let J be an ideal containing an order unit e. Then $\lambda e \in J$ and

$$E \subset \bigcup_{\lambda>0} [-\lambda e, \lambda e] \subset J$$

Hence $E = J$ is not a proper ideal. □

Conversely, if $e \in C$ is not contained in any proper ideal, then the ideal $\bigcup_{\lambda>0} [-\lambda e, \lambda e]$ which contains e is the whole of E i.e., e is an order unit.

(b) If E has an order unit, then each proper ideal of E is contained in a maximal proper ideal of E.

Proof is immediate using (a) and Zorn's lemma.

(c) If E has an order unit e, and J is a proper ideal of E, then E/J is a partially ordered vector space with order unit. **119**

Proof. Since J is a linear subspace, E/J is a vector space whose elements are the cosets $\tilde{x} = x + J,\ x \in E$. The set $\tilde{C} = \{\tilde{x} : x \in C\}$ is a positive cone in E/J. It contains the non-zero element $\tilde{e}$, and $\tilde{x},\ -\tilde{x} \in \tilde{C}$ implies that there exist $j,\ j' \in J$ such that

$$x + j,\ \ -x + j' \in C.$$

This gives

$$0 \leq x + j \leq j' + j$$

and so

$$x + j \in [0, j + j] \subset J,$$

$$x \in J,\ \tilde{x} = 0.$$

Finally $\tilde{e}$ is an order unit. For

$$-\lambda e \leq x \leq \lambda e$$

implies that $$-\lambda\tilde{e} \leq \tilde{x} \leq \lambda\tilde{e}$$

which proves that $\tilde{e}$ is an order unit since $x \to \tilde{x}$ is an 'onto' mapping. □

120 (d) If E has an order unit, and M is a maximal proper ideal of E, then E/M has no proper ideals.

Proof. If $\tilde{J}$ is an ideal of E/M, then

$$J = \{x : \tilde{x} \in \tilde{J}\}$$

is an ideal of E containing M. Hence $J = M$ or $J = E$ *i.e.* $\tilde{J} = (0)$ or $\tilde{J} = E/M$. □

(e) If E has no proper ideals then $E \simeq R$.

Proof. C contains a non-zero element e. Since there are no proper ideals, e is an order unit. □

Let
$$p(x) = \inf[\xi : x \leq \xi e] \quad (x \in E) \tag{1}$$

Let $y = p(x)e - x$.

If $x \in E$, then either $x \in C$ or $-x \in C$, for otherwise (x) is an ideal. Hence $y \in C$ or $-y \in C$ i.e. $y \geq 0$ or $y < 0$. If $y < 0$ for some x, then $-y \geq \varepsilon e$ for some $\varepsilon > 0$, and $x \geq (p(x) + \varepsilon)e$ which contradicts (1). If $y > 0$, for some $x \in E$, then y is an order unit and so

$$y \geq \varepsilon e \text{ for some } \varepsilon > 0$$

But then $(p(x) - \varepsilon)e \geq x$ which also contradicts (1). Hence

$$y = p(x)e - x = 0 \quad x \in E)$$

121 i.e. $e \neq 0$ spans the whole space E and so E is isomorphic to R.

(f) Let E have an order unit e, and let M be a maximal proper ideal. Then there exists a linear functional f on E with

(i) $f(x) \geq 0 \qquad (x \in C)$,

(ii) $f(e) = 1$,

(iii) $f(x) = 0, \qquad (x \in M)$.

(i.e., M is the null space of a normalised positive linear functional).

Proof. E/M has no proper ideals, and $\tilde{e} > 0$ where $x \to \tilde{x}$ is the canonical mapping $E \to E/M$. By (e), for each $\tilde{x}$, there exists a real number ξ with

$$\tilde{x} = \xi\tilde{e}$$

Let $f(x) = \xi$. Then it is easily verified that f is a linear functional with the required properties. □

(g) Let E have an order unit and have dimension greater than one, and let T be a positive linear mapping of E into itself.

Then there exists a proper T-invariant ideal, i.e. a proper ideal J with $TJ \subset J$.

Proof. Let e be the order unit, let p be the Minkowski functional of $[-e, e]$, and let

$$N = \{x : p(x) = 0\}.$$

The set N is an ideal in E. For if $j \in N$ then **122**

$$-\varepsilon \leq j \leq \varepsilon e \text{ for all } \varepsilon > 0$$

Hence, if $0 \leq x \leq j$, then

$$-\varepsilon e \leq x \leq \varepsilon e \quad (\varepsilon > 0)$$

and so $p(x) = 0$ and $x \in N$. $N \neq E$ since $p(e) = 1$. Also $TN \subset N$. For $x \in N$, we have

$$-\varepsilon e \leq x \leq \varepsilon e \qquad (\varepsilon > 0)$$

and so $$-\varepsilon Te \leq Tx \leq \varepsilon Te \quad (\varepsilon > 0)$$

But

$$-\alpha e \leq Te \leq \alpha e \quad \text{for some} \quad \alpha > 0$$

so that $$-\varepsilon\alpha e \leq Tx \leq \varepsilon\alpha e \text{ for some } \alpha > 0 \text{ and all } \varepsilon > 0$$

Hence

$$\alpha p(Tx) = 0 \text{ i.e. } Tx \in N.$$

$\square$

Thus if $N \neq (0)$, it is a T-invariant proper ideal. Suppose $N = (0)$, so that p is a norm. Let H denote the set of all positive linear functionals f with $f(e) = 1$. As E is of dimension greater than one, by $(e), (b)$ and $(f), H$ is nonempty. If for some $f \in H$, we have $f(Te) = 0$, then $f(Tx) = 0$ for all $x \in E$ and so the null-space of f is a proper T-invariant ideal. Suppose then that $f(Te) \neq 0$ for all f in H. Clearly H is a convex weak * closed subset of the dual space E^* of the normed space (E, p). Also H is contained in the unit ball of E^*.

123 For we have

$$-p(x)e \leq x \leq p(x)e \qquad (x \in E)$$

so that $$-p(x)f(e) \leq f(x) \leq p(x)f(e) \qquad (x \in E, f \in H)$$

i.e., $$-p(x) \leq f(x) \leq p(x) \qquad (x \in E, f \in H)$$

or $$|f(x)| \leq p(x)$$

Hence H is weakly compact.

As

$$-p(x)Te \leq T(x) \leq p(x)T(e) \qquad (x \in E),$$

T is a bounded linear transformation of (E, p) into itself with $\|T\| \leq T(e)$. Therefore its transpose T^* is a weak* continuous mapping of E^* into itself. Thus the mapping S defined by

$$Sf = \frac{1}{f(Te)} T^* f$$

is a weak* continuous mapping of the convex, weak* compact subset H of E^* into itself. By the Schander-Tychnoff fixed point theorem S has a fixed point f_0 in H, and the null space of f_0 is a proper T-invariant ideal.

(h) Under the condition of (g) there exists a maximal proper ideal M and a non-negative real number such that

$$Tx - \mu x \in M \qquad (x \in E).$$

Proof. By (g) and Zorn's lemma, there exists a maximal proper T-invariant ideal M. In fact M is a maximal proper ideal, for otherwise **124**
E/M has dimension greater than 1 and so there is a proper $\tilde{T}$-invariant ideal $\tilde{M}_1$ in E/M, where $\tilde{T}$ is the mapping on E/M given by

$$\tilde{T}\tilde{x} = \tilde{T}x,$$

$x \to \tilde{x}$ being the canonical mapping $E \to E/M$. Then the inverse image M_1 of $\tilde{M}_1$ by this mapping is a proper T-invariant ideal containing M strictly which contradicts the definition of M.

The maximal proper ideal M is the null space of a normalised positive linear functional f.

Since $f(e) = 1$, we have

$$x - f(x)e \in M \qquad (x \in E),$$

and so

$$Tx- = f(x)Te \in M \qquad (x \in E)$$

$$f\{Tx - f(x)Te\} = 0 \qquad (x \in E)$$

Writing $\mu = f(Te)$, we have

$$f(Tx - \mu x) = 0 \qquad (x \in E),$$

and so $Tx - \mu x \in M$ $(x \in E)$. □

Theorem 7.3. Let E be a partially ordered vector space with an order unit e and with dimension greater than one. $\mathscr{F}$, be a commuting family of positive linear mappings of E. Then there exists a maximal proper **125**
ideal which is T-invariant for all T in.

Proof. We prove that there is a proper ideal that is $\mathscr{F}$ invariant (i.e. T-invariant for every $T \in \mathscr{F}$), and then the proof is completed by applying Zorn's lemma as in (h). ◻

If every $T \in \mathscr{F}$ is a constant multiple of the identity mapping, then this assertion is obvious, for every proper ideal is then $\mathscr{F}$-invariant. Suppose then that $T_0 \in \mathscr{F}$ is not a constant multiple of the identity. Then by (h), there exists a maximal proper ideal M and a constant real number such that

$$T_0 x - \mu x \in M \qquad (x \in E).$$

Let $E_0 = \{T_0 x - \mu x : x \in E\}$. Then E_0 is a proper subspace of E and since

$$T(T_0 x - \mu x) = T_0(Tx) - \mu(Tx) \qquad (T \in \mathscr{F}),$$

E_0 is $\mathscr{F}$-invariant. If $E_0 \cap C = (0)$ then E_0 is the required proper $\mathscr{F}$-invariant ideal.

Otherwise, let

$$J = \bigcup_{y \in E_0} [-y, y]$$

Then $(0) \neq J \subset M$, and J is an $\mathscr{F}$ invariant ideal.

Corollary. *There exists a normalized positive linear functional f such that*

$$f(Tx) = f(Te) f(x) \qquad (T \in \mathscr{F}, x \in E).$$

Chapter 8

A class of abstract semi-algebras

The present chapter is somewhat of an intruder in this course of lectures. **126**
It has some incidental bearing on the Perron-Frobenius theorem, but our main purpose is to establish some algebraic properties of a certain class of semi-algebras.

Definition 8.1. A real Banach algebra is a linear associative algebra over $\mathbb{R}$ together with a norm under which it is a Banach space and which satisfies

$$\|xy\| \leq \|x\| \, \|y\| \qquad (x, y \in B)$$

Definition 8.2. A non-empty subset A of a real Banach algebra B is called a semi-algebra if

(i) $x, y \in A, \alpha \geq 0 \Rightarrow x + y, \alpha x \in A$ and $xy \in A$.

A semi-algebra A is called a *locally compact* semi-algebra if it satisfies the additional axioms

(ii) A contains non zero vectors;

(iii) the set of elements x of A with $\|x\| \leq 1$ is a compact subset of B.

It is easily seen that if A is a locally compact semi-algebra, then the intersection of A and each closed ball in B with its center at

the zero vector is compact, and hence that A is a closed subset
127 of B, and that each closed bounded subset of A is compact. It is easily seen that axioms (i), (ii) and (iii) are equivalent to (i), (ii) and

$(iii)'$ A with the relative topology induced from the norm topology in B is a locally compact space.

This is our justification for the use of the term locally compact in the present sense. Axiom (ii) of course merely excludes trivial exceptional cases.

If the Banach algebra B has finite dimensions, then its closed unit ball is compact, and therefore every nontrivial closed semi-algebra in B is locally compact. In particular, each closed semi-algebra of $n \times n$ real matrices is of this kind. However, the axioms do not imply that every locally compact semi-algebra is contained in a finite dimensional algebra, as the following example shows.

Example. Let E be the subset of the closed unit interval $[0, 1]$ consisting of the closed interval $[0, \frac{1}{2}]$ together with the point 1, and let E be given the topology induced form the usual topology in $[0, 1]$, so that E is a compact Hausdorff space.

Let A' denote the class of all functions belonging to $C_R[0, 1]$ that are non-negative, increasing, and convex in $[0, 1]$; and let A denote the class of all functions on E that are restrictions to E of functions belonging to A'.

It is obvious that A' is a semi-algebra in $C_R[0, 1]$. We prove that A
128 is a closed subset of $C_R(E)$. Each element f of A has a unique extension $f' \in A'$ which is linear in $[\frac{1}{2}, 1]$ defined by

$$f'(x) = f(x), 0 \leq x \leq \frac{1}{2}$$
$$f'(x) = \alpha f\left(\frac{1}{2}\right) + (1-\alpha) f(1) \text{ for } x = \alpha.\frac{1}{2} + (1-\alpha)^1, 0 \leq \alpha \leq 1$$

Let f_n be a sequence of elements of A that converges in norm to an element f of $C_R(E)$. Then the sequence $\{f'_n\}$, where $f'_n \in A'$ is the extension of f_n to $[0, 1]$ which is linear in $[\frac{1}{2}, 1]$, converges uniformly in $[0, 1]$ and since A' is a closed subset of $C_R[0, 1]$, the limit function f' belongs A'. But f is the restriction of f' to E, and so $f \in A$. Hence A is a closed semi-algebra in $C_R(E)$. To prove that it is locally compact it is enough to prove that A intersects the unit ball of $C_R(E)$ in an equicontinuous set.

If $f \in A$, and $||f|| \leq 1$, then

$$0 \leq f\left(\frac{1}{2}\right) \leq f(1) \leq 1;$$

and so, for any pair of points x_1, x_2 with

$$0 \leq x_1 < x_2 \leq \frac{1}{2},$$

we have, by the ∞nvexity of f,

$$0 \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(1) - f\left(\frac{1}{2}\right)}{1 - \frac{1}{2}} \leq 2$$

Thus the set of all such f is equi-continuous, and A is a locally compact semi-algebra. Finally it is obvious that A is not contained in any finite dimensional subspace of $C_R(E)$. **129**

Our principal results are concerned with the existence and properties of idempotents in a locally compact semi-algebra, and may be regarded as analogues of classical theorems of Wedderburn. As biproducts we obtain an abstract characterization of the semi-algebra of all $n \times n$ matrices with non-negative entries, and some results related to the Perron-Frobenius theorems.

Throughout this chapter we will denote by S_A the intersection of A with the surface of the unit ball in B, i.e.,

$$S_A = \{x : x \in A \text{ and } ||x|| \leq 1\}.$$

Obviously the set S_A is compact.

Definition 8.3. Given a subset E of a semi-algebra A, E_r will denote the *right annihilator* of E in A, i.e.

$$E_r = \{x : x \in A \text{ and } ux = 0\ (u \in E)\}.$$

In particular $(a)_r$ denotes the right annihilator of the set consisting of a only; and the left annihilators E_l and $(a)_l$ are similarly defined.

Definition 8.4. Given subsets P and Q of A, PQ will denote the set of all finite sums

$$p_1q_1 + \cdots + p_nq_n$$

130 with $p_i \in P$ and $q_i \in Q$; and P^2 will denote PP.

Definition 8.5. A semi-algebra J contained in A is called a *right ideal* of A if

$$a \in A, j \in J \Rightarrow ja \in J$$

Left ideals are similarly defined with ja replaced by aj, and semi-algebra contained in A is called a *two sided ideal* if it is both a left ideal and right ideal.

A closed right ideal J is called a *minimal closed right ideal* if $J \neq (0)$ and if the only closed right ideals contained in J are (0) and J. Similar definitions apply for minimal closed left and two-sided ideals. (Closed ideal means an ideal which is a closed subset of A in relative topology.)

Theorem 8.1. Each non-zero closed right ideal of a locally compact semi-algebra contains a minimal closed right ideal.

A similar statement holds for left and two-sided ideals.

Proof. Given a non-zero closed right ideal J, there exists, by Zorn's lemma, a maximal family Δ of non zero closed right ideals contained in J and totally ordered by the relation of set theoretic inclusion. The sets $I \cap S_A$ with $I \in \Delta$ are compact and have the finite intersection property. Hence their intersection is non-empty and therefore the intersection I_0

of the ideals I in Δ is non-zero. Clearly I_0 is a minimal closed right ideal. ◻

It is clear that similar results hold for left and two sided closed ideals. 131

Lemma 8.1. *Let E be a closed subset of a locally compact semi-algebra A such that $\alpha x \in E$ whenever $\alpha \geq 0$ and $x \in E$. Let a be an element of A such that $(a)_r \cap E = (0)$. Then aE is closed.*

Proof. Let $y = \lim\limits_{n\to\infty} ax_n$, $(x_n \in E)$. ◻

If $y = 0$ there is nothing to prove since $0 \in E$ and so $0 \in aE$. Let $y \neq 0$. Then we can assume that $x_n \neq 0$ $(n = 1, 2, \ldots)$. The sequence $s_n = \dfrac{x_n}{\|x_n\|}$ in the compact set $E \cap S_A$ has a subsequence (s_{n_i}) that converges to an element $s \in E \cap S_A$. We have

$$as_{n_i} \neq 0 \text{ and } \lim_{i\to\infty} as_{n_i} = as \neq 0,$$

since $$s_{n_i} \neq 0,\ s \neq 0 \text{ and } (a)_r \cap E = (0).$$

Hence

$$\|\,\mathrm{as}_{n_i}\| > m > 0 \qquad (i = 1, 2, \ldots).$$

Also since

$$\lim_{i\to\infty} \|x_{n_i}\| as_{n_i} = \lim_{i\to\infty} ax_{n_i} = y,$$

$\|x_n\| as_{n_i}$ is a bounded sequence. It follows that the sequence $(\|x_{n_i}\|)$ is bounded, and therefore has a subsequence convergent to $\lambda > 0$ say. Then

$$y = \lambda \text{ as } = a(\lambda s) \in aE$$

and the lemma is proved.

Theorem 8.2. Let M be a minimal closed right ideal of a locally compact semi-algebra A with $M^2 \neq (0)$. Then M contains an idempotent e and $M = eA$. 132

Proof. The proof begins on familiar algebraic lines. As $M^2 \neq (0)$, there exists $a \in M$ with $aM \neq (0)$. Hence

$$M \cap (a)_r \neq M$$

Since $M \cap (a)_r$ is a closed right ideal contained in M, the minimal property of M implies that

$$M \cap (a)_r = (0) \tag{1}$$

□

Hence, by Lemma 1, aM is a closed right ideal. We have

$$0 \neq aM \subset M$$

and therefore

$$aM = M.$$

In particular, there exists an element $e \in M$ with

$$ae = a \tag{2}$$

The complication of the rest of the argument is forced on us by the fact that we cannot assert at this point that $e^2 - e$ belongs to A. Our next is to prove that

$$\lim_{n\to\infty} \|e^n\|^{1/n} = 1 \tag{3}$$

133 By (2), we have

$$ae^n = a \qquad (n = 1, 2, \ldots)$$
$$\|a\|\|e^n\| \geq \|ae^n\| = \|a\|,$$
$$\|e^n\| \leq 1 \qquad (n = 1, 2, \ldots)$$

so that

$$\lim_{n\to\infty} \|e^n\|^{1/n} \geq 1$$

In order to prove that $\lim\limits_{n\to\infty} \|e^n\|^{1/n} \leq 1$, it suffices to show that $(\|e^n\|)$ is bounded.

Let $K = \inf\{\|am\| : m \in M \cap S_A\}$.

Since $M \cap S_A$ is compact, this infimum is attained; there exists $m_0 \in M \cap S_A$ with $\|am_0\| = K$. Therefore, by (1), $K \neq 0$. Thus $K > 0$, and we have

$$\|ax\| \geq K\|x\| \qquad (x \in M)$$

In particular $\|e^n\| \leq \frac{1}{2}\|ae^n\| = \frac{1}{2}\|a\|$ $(n = 1, 2, \ldots)$. This completes the proof of (3).

Suppose that $\lambda > 1$, and let

$$b_\lambda = \frac{1}{\lambda}e + \frac{1}{\lambda^2}e^2 + \ldots$$

The convergence of the series is established by (3), and we have $b_\lambda \in M$. Also,

$$\lambda b_\lambda - eb_\lambda = e \in M,$$

and therefore $b_\lambda \neq 0$. Let $\lambda_n > 1 (n = 1, 2, \ldots)$ and $\lim\limits_{n\to\infty} \lambda_n = 1$. By what 134
we have just proved, there exists for each n, an element m_n of $M \cap S_A$ such that

$$\lambda_n m_n - em_n \in M$$

Therefore, by the compactness of $M \cap S_A$, there exists an element m of $M \cap S_A$ such that

$$m - em \in M$$

Let

$$J = \{x : x \in M, x - ex \in M\}.$$

We have $J \neq (0)$ since $m \in J$. Also, J is a closed right ideal contained in M, and therefore $J = M$ i.e.,

$$x - ex \in M \qquad (x \in M)$$

But, by (2)

$$a(x - ex) = 0 \qquad (x \in M),$$

and so

$$x - ex \in M \cap (a)_r \qquad (x \in M)$$

Therefore, by (1),

$$x - ex = 0 \qquad (x \in M)$$

In particular $e = e^2$, and also 135

$$M = eM = eA.$$

Definition 8.6. An idempotent e in a semi-algebra A for which eA is a minimal closed right ideal is called *minimal idempotent.*

A semi-algebra A is called a *division semi-algebra* if it contains a unit element different from zero and if every non-zero element of A has an inverse in A.

Theorem 8.3. Let e be a minimal idempotent in a locally compact semi-algebra A. Then eAe is a closed division semi-algebra.

Proof. Let $A_0 = eAe$. Then A_0 is a semi-algebra with unit element e, and is closed since

$$A_0 = \{x : x \in A \text{ and } x = ex = ex\}.$$

□

Let eae e a non-zero element of A_0. Then

$$e \notin (eae)_r \cap eA, \quad e \in eA.$$

Since eA is a minimal closed right ideal and $(eae)_r \cap eA$ is a closed right ideal properly contained in it, we have

$$(eae)_r \cap eA = (0).$$

It follows, by Lemma 1, that $(eae)_r \cap eA$ is a closed right ideal. Since it contained eae and is contained in eA it coincides with eA, and therefore

$$(aea)A_0 = A_0.$$

136 This proves that every non-zero element of A_0 has a right inverse,

and a routine argument now completes the proof.

Given $x \neq A_0$, with $x \neq 0$, there exists $y \in A_0$ with $xy = e$. It follows that $y \neq 0$, and so there exists $z \in A_0$ with $yz = e$. But then

$$x = x(yz) = (xy)z,$$

and so $yx = e$ and x has an inverse y.

Definition 8.7. A semi-algebra A is said to be *strict* if $x, y \in A, x + y = 0 \Rightarrow x = 0$.

Theorem 8.4. Let A be a closed strict division semi-algebra. Then

$$A = R^+ e,$$

where e is the unit element of A and R^+ is the set of all non-negative real numbers.

Proof. We prove first that if $x, y \in A, y \neq 0$, and $\| x \|$ is sufficiently small then $y - x \in A$. □

Since $y \neq 0$, it has an inverse y^{-1} in A and for sufficiently small $\| x \|$, we have $\| z \| < 1$, where $z = y^{-1}x$. Since A is a closed semi-algebra in a Banach algebra, the series

$$e + z + z^2 + \cdots$$

converges to an element a of A, and $(e - z) = e$. This shows that $a \neq 0$, and it therefore has an inverse b in A therefore

$$e - z = (e - z)ab = b \in A.$$

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Finally

$$y - x = y(e - y^{-1}x) = y(e - z) \in A.$$

Suppose now that $u \in A, u \neq 0$, and let

$$\mu = \sup\{\lambda : e - \lambda u \in A\}.$$

By what we have just proved, we have $\mu > 0$. Also, the strictness of A implies that μ is finite, for otherwise we have

$$\frac{1}{n}e - u \in A \qquad (n = 1, 2, \ldots),$$

and so $-u \in A$, since A is closed; and then $u = 0$ (as A is strict) which is not true.

Let $y = e - \mu u$. Since A is closed we have $y \in A$. If $y \neq 0$, then, for sufficiently small $\lambda > 0$, we have

$$(e - \mu u) - \lambda u \in A,$$
$$\text{i.e.,} \qquad e - (\mu + \lambda)u \in A,$$

which is absurd. Therefore $e - \mu u = 0$,

$$A = R^{+}e.$$

Remark. It is of interest to consider what other division semi-algebras there are besides R^{+}. In any semi-algebra A, $A \cap (-A)$ is an ideal. Hence if A is a division semi-algebra *either* $A \cap (-A) = (0)$ and A is strict, or
138 $A \cap (-A) = A$ and A is a division algebra. Thus the only non-strict division semi-algebras are the familiar division algebras. On the other hand there are many strict (nonclosed) semi-algebras. For example, let E be a compact Hausdorff space and let A be the subset of $C_R(E)$ consisting of those functions $f \in C_R(E)$ such that either

$$f(t) = 0 \ (t \in E),$$

or $f(t) > 0$ $(t \in E)$.

It is easily seen that each such A is a strict division semi-algebra.

Definition 8.8. A semi-algebra A is said to be *semi-simple* if the zero ideal is the only closed two-sides ideal J with $J^2 = (0)$.

Lemma 8.2. *Let A be a semi-simple semi-algebra, and let I be an ideal (left, right, or two-sided) of A such that $I^n = (0)$ for some positive integer n. Then $I = (0)$.*

Proof. We first show that if J is any left ideal with $J^2 = (0)$ then

$$J = (0) \tag{1}$$

Let $H = (JA)$. Then H is a closed two-sides ideal, and since

$$(JA)(JA) = J(AJ)A \subset J^2A = (0),$$

we have $H^2 = (0)$ and so $H = (0), JA = (0)$. This gives $J \subset A_1$, and **139**
since A_1 is a closed two-sides ideal with

$$A_1^2 \subset A_1A = (0),$$

we have$A_1 = (0), J = (0)$

If now I is a left ideal and n is the least positive integer with $I^n = (0)$, then $I^{n-1}I = 0$, and so $n > 1$ would give $(I^{n-1})^2 = (0)$, and so by (1)

$$I^{n-1} = 0$$

Hence $n = 1, \qquad I = (0)$.

A similar argument applies to right ideals. □

Theorem 8.5. Let A be a semi-simple locally compact semi-algebra, and let e be an idempotent in A. Then e is a minimal idempotent if and only if eAe is a division semi-algebra.

Corollary . *eA is a minimal closed right ideal if and only if Ae is a minimal closed left ideal.*

Proof. That eAE is a division-algebra if e is a minimal idempotent was proved in Theorem 8.3. To prove the converse suppose that eAe is a division semi-algebra. Since

$$eA = \{x : x \in A \text{ and } x = ex\},$$

eA is a closed right ideal. Since it contains e it is non-zero, and it there-
fore contains a minimal closed right ideal M. Since A is semi-simple, **140**

Lemma 2 shows that $M^2 \neq (0)$; and therefore, by Theorem 8.2, M contains an idempotent f with $M = fA$. Since

$$(fA)^2 \subset (fA)(eA),$$

and A is semi-simple, we have

$$fAe \neq 0$$

Let a be an element of A with $fae \neq 0$. Then fae is non-zero element of eAe and therefore has an inverse b in eAe,

$$faeb = e.$$

It now follows that $eA \subset fA$; and so by the minimal properly of fA, eA is a minimal closed right ideal. □

The Corollary is evident from the symmetry of the conditions on A and eAe.

Theorem 8.6. Let A be a semi-simple locally compact semi-algebra, and let $\mathscr{I}$ be the set of all minimal idempotents in A. If $e \in \mathscr{I}$ and $a \in A$, then there exists $f \in \mathscr{I}$ and $b \in A$ with $ae = fb$.

Corollary. *$\mathscr{I}A$ is a two-sides ideal.*

Proof. Let $e \in \mathscr{I}$ and $a \in A$. If $ae = 0$, we take $f = e$ and $b = 0$. □

Suppose $ae \neq 0$. Then $e \notin (ea)_r$, and therefore the closed right ideal

$$(ea)_r \cap eA$$

141 is a proper subset of eA. Therefore, by the minimal property of eA,

$$(ae)_r \cap eA = (0)$$

By Lemma 8.1, it follows that

$$aeA = (ae)(eA)$$

is closed right ideal. It is a minimal closed right ideal, for if J is non-zero closed ideal properly contained in aeA, then $\{ex : aex \in J\}$ is a non-zero closed right ideal properly contained in eA. Since A is semi-simple and locally compact, there exists $f \in \mathscr{I}$ with $aeA = fA$. In particular

$$ae = fb \text{ for some } b \in A.$$

The corollary is obvious.

Theorem 8.7. Let A be a semi-simple locally compact semi-algebra. Then the set of minimal closed two-sides ideals of A is finite and non-empty.

Proof. By Theorem 8.1 and the fact that A is a non-zero closed two-sided ideal of itself, A has at least one minimal closed two-sided ideal. □

Suppose that A has an infinite set $\{M_\alpha : \alpha \in \Delta\}$ of minimal closed two-sides ideals. Then

$$M_\alpha \cap M_\beta = (0)\ (\alpha \neq \beta),$$

and so

$$M_\alpha M_\beta (0)\ (\alpha \neq \beta).$$

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For each $\alpha \in \Delta$, choose $m_\alpha \in M_\alpha \cap S_A$. By the compactness of S_A, there exists a sequence (α_n), of distinct elements of Δ such that (m_{α_n}) converges to an element m say of S_A. Given $\alpha \in \Delta$, we have

$$M_\alpha m_{\alpha_n} = (0) \text{ for all } n \text{ such that } \alpha \neq \alpha_n$$

and therefore

$$M_\alpha m = (0)\ (\alpha \in \Delta)$$

Let $$J = \bigcap_{\alpha \in \Delta} (M_\alpha)_r$$

Since M_α is a two-sided ideal, $(M_\alpha)_r$ is a closed two-sided ideal and so J is a closed two-sided ideal and is non-zero since $m \in J$. By Theorem 8.1, J contains a minimal closed two-sided ideal M_β, say. But $M_\beta^2 = (0)$, contradicting the semi-simplicity of A.

Theorem 8.8. Let A be a semi-simple locally compact semi-algebra and let its minimal closed two-sided ideals be denoted by $M_1, M_2, \ldots, M_n$. Let $\mathscr{I}$ be the set of all minimal idempotent in A and let $\mathscr{I}_k = \mathscr{I} \cap M_k$ $(k = 1, 2, \ldots, n)$. Then

i) the sets $\mathscr{I}_k$ are disjoint and their union is $\mathscr{I}$,

ii) For each k, $\mathscr{I}_k A$ is a two-sided ideal,

$$\mathscr{I}_k A = A\mathscr{I}_k = \mathscr{I}_k A \mathscr{I}_k,$$

143 iii) $M_k = d(\mathscr{I}_k A)$.

Proof. Given $e \in \mathscr{I}$, either $eA \cap M_k = eA$ or $eA \cap M_k = (0)$. In the first case $e \in M_k$, $e \in \mathscr{I}_k$. In the second case, since

$$eAM_k \subset eA \cap M_k,$$

we have $eAM_k = (0), e \in (M_k)_l$. Thus if $e \in \mathscr{I}$ but $e \notin \bigcup_{k=1}^{n} \mathscr{I}_k$, then

$$e \in \bigcap_{k=1}^{n} (M_k)_l = J.$$

Since J is a non-zero closed two-sided ideal, it contains one of the minimal closed two-sided, M_j say. But this lead to $M_j^2 = (0)$, which is impossible as A is semi-simple. Thus

$$\mathscr{I} = \bigcup_{k=1}^{n} \mathscr{I}_k.$$

The disjointness of the $\mathscr{I}_k$ follows from the fact that $M_j \cap M_k = (0)$ $(j \neq k)$. ◻

Given $e \in \mathscr{I}_k$, and $a \in A$, we have $ae = fb$ with $f \in \mathscr{I}$ and $b \in A$. If $ae = 0$, we can take $f = e \in \mathscr{I}_k$. Since $ae \in M_k$, we have $fb \in M_k$, and so if $ae \neq 0$,

$$fA \cap M_k \neq (0).$$

But this gives $fA \cap M_k = fA$, and so $f \in M_k$. This proves that $\mathscr{I}_k A$ is a two-sided ideal, and it is plainly the smallest two-sided ideal containing $\mathscr{I}_k$. Similarly $A\mathscr{I}_k$ and $\mathscr{I}_k A \mathscr{I}_k$ are both this smallest two-sides ideal, and so (*ii*) holds.

Finally, each $\mathscr{I}_k$ is non-empty, for M_k being a non-zero closed right **144**
ideal contains a minimal idempotent. Therefore $d(\mathscr{I}_k A)$ is non-zero closed two-sided ideal containing M_k; and the minimal property of M_k gives (iii).

Theorem 8.9. Let e be a minimal idempotent in a semi-simple locally compact semi-algebra A. Then eA is a minimal right ideal and Ae is a minimal left ideal.

Remark. Of course, we know that eA and Ae are minimal closed right and left ideals. But, a priori, they might contain smaller non-closed ideals.

Proof. With the notation of Theorem 8.8, $e \in \mathscr{I}_k$ for some k. Let J be a non-zero right ideal contained in eA, and choose $u \in J$ with $u \neq 0$. □

If $\mathscr{I}_k \subset (u)_r$, then $M_k \subset (u)_r$ by Theorem 8.8, and so $u \in (M_k)_l$. But since $u \in eA \subset M_k$, this implies that

$$M_k \cap (M_k)_l \neq (0),$$

and therefore $M_k^2 = (0)$, which impossible. Therefore there exists $f \in \mathscr{I}_k$ with $uf \neq 0$. Since fA is a minimal closed right ideal, it follows that

$$(u)_r \cap fA = (0)$$

Therefore, by Lemma 8.1, ufA is a closed right ideal. It is non-zero since it contains $uf^2 = uf$, and is containing in eA since $u \in eA$. There- **145**
fore

$$eA = ufA \subset J,$$
$$J = eA$$

Lemma 8.3. *Let e be a minimal idempotent in a strict locally compact semi-algebra A. Then*
$$eAe = R^+e.$$

Proof. This is an immediate consequence of Theorem 8.3 and 8.4 $\square$

Lemma 8.4. *Let A be a semi-simple strict, locally compact semi-algebra, and let e, f be minimal idempotent for which $fAe \neq (0)$. Then there exists an element ω of eAf such that*

$$eAf = R^+\omega,$$

and either $\omega^2 = \omega$ or $\omega^2 = 0$.

Proof. Choose a non-zero element v of fAe, say $v = fae$. since $ve = v \neq 0$, we have $e \notin (v)_r$. Using Lemma 1 and the fact that eA is a minimal closed ideal, we deduce that veA is closed right ideal. Since it contains v and is contained in fA, we have

$$veA = fA,$$
$$veAf = fAf.$$

Thus there exists $u \in eAf$ for which $vu = f$. Given $x \in A$, Lemma 8.3, gives
$$exfv = exfae = \lambda e$$

146 for some $\lambda \in R^+$. Therefore

$$exf = exfvu = \lambda eu = \lambda u$$

If $u^2 = 0$, we $\omega = u$. If $u^2 \neq 0$, then $u^2 = \alpha u$ with $\alpha > 0$, and we take $\omega = \dfrac{1}{\alpha}u$. $\square$

Lemma 8.5. *Let e, f be minimal idempotents in a semi-simple locally compact semi-algebra A. Then $fAe \neq (0)$ if and only if e and f belong to the same closed two-sided ideal.*

Proof. Suppose that $e \in M_i$ and $f \in M_j$ with M_i and M_j minimal closed two-sides ideals. Then

$$fA \subset M_j$$

and so, if $$M_i \neq M_j,$$

$$fAe \subset M_j M_i = (0)$$

On the other hand, suppose $fAe = (0)$. Then $f \in (Ae)_l$ since $(Ae)_l$ is closed two-sided ideal and its intersection with M_j contains f, it follows that

$$M_j \subset (Ae)_l$$

Therefore $M_j e = (0)$, and so $e \notin M_j$. □

Theorem 8.10. Let A be a semi-simple, strict, locally compact semi-simple, and let M_k and $\mathscr{I}_k$ be defined as in Theorem 8.8. For each pair e, f of minimal idempotent belonging to $\mathscr{I}_k$, there exists an element $\omega_{e,f}$ of eAf such that $eAf = R^+\omega_{e,f}$ and either $\omega_{e,f}^2 = \omega_{e,f}$ or $\omega_{e,f}^2 = 0$. Also **147**

$$J = \sum_{e,f \in \mathscr{I}_k} R^+\omega_{e,f}$$

is a two-sided ideal contained in M_k, and $M_k = clJ$. Finally, for all idempotents e, f, g, h in $\mathscr{I}_k$,

$$\omega_{e,f}\omega_{g,h} = \lambda\omega_{e,h}, \text{ for some } \lambda \in R^+.$$

Proof. Let e, f be idempotent belonging to $\mathscr{I}_k$. By Lemma 8.5, $fAe \neq 0$, and so Lemma 8.4, there exists $\omega_{e,f}$ in eAf such that

$$eAf = R^+\omega_{e,f},$$

and $\omega_{e,f}^2 = \omega_{e,f}$, or $\omega_{e,f} = 0$. That J is a two-sided ideal the closures of which is M_k, now follows from Theorem 8.8. Finally

$$\omega_{e,f}\omega_{g,h} = eafghk \in eAh = R^+\omega_{e,h}$$

□

We now consider the question: when does a semi-algebra contain exactly one minimal closed two-sided ideal?

Definition 8.9. We say that a semi-algebra A is *prime* if $IJ \neq (0)$ whenever I and J are closed non-zero two-sided ideals.

A prime semi-algebra is obviously semi-simple. It is also clear that in a prime semi-algebra, if J is a non-zero left ideal, then $J_l = (0)$, and
148 if J is a non-zero right ideal, then $J_r = (0)$.

Theorem 8.11. Let A be a locally compact semi-algebra. If A is prime, then A has exactly one minimal closed two-sided ideal. Conversely, if A is semi-simple and has exactly one minimal closed two-sided ideal, then A is prime.

Proof. Minimal closed two-sided ideals annihilate each other, and therefore if A is prime there is exactly one such ideal. ◻

Suppose on the other hand that A is semi-simple and has exactly one minimal closed two-sided ideal, and let H, J be closed two-sided ideals with $HJ = (0)$. Then $(H \cap J)^2 = (0)$ and so, by semi-simplicity of A,

$$H \cap J = (0)$$

Then either $H = (0)$ or $J = (0)$, for otherwise by Theorem 8.1, they contain minimal closed two-sided ideals which are distinct since they have zero intersection.

Our next theorem is concerned with an abstract characterization of the semi-algebra of all $n \times n$ matrices with non-negative real entries.

Definition 8.10. We say that a semi-algebra A is *simple* if it has no two-sided ideals other than (0) and A.

Theorem 8.12. A simple, strict, locally compact semi-algebra with a unit element is isomorphic to the semi-algebra $M_n(R^+)$ of all $n \times n$ ma-
149 trices with non-negative real entries where n is some positive integer. Conversely, for each positive integer n, the semi-algebra $M_n(R^+)$ is simple, strict, locally compact and has a unit element.

Proof. Since A has a unit element, $A^2 \neq (0)$. But A is the only non-zero two-sided ideal, and therefore A is semi-simple and indeed prime. Let $\mathscr{I}$ denote the class of all minimal idempotents, and let 1 be the unit element. Then $\mathscr{I}A$ is a non-zero, two-sided ideal in A, and so

$$A = \mathscr{I}A.$$

□

In particular there exist $e_1, e_2, \ldots, e_n \in \mathscr{I}$ and $a_1, a_2, \ldots, a_n \in A$ such that

$$1 = e_1a_1 + e_2a_2 + \cdots + e_na_n, \tag{1}$$

and we may suppose that the expression (1) has been chosen so that n is as small as possible. From (1), we obtain

$$e_1 = e_1a_1e_1 + e_2a_2e_1 + \cdots + e_na_ne_1 \tag{2}$$

By Lemma 8.3, $e_1a_1e_1 = \lambda e_1$ with $\lambda \in R^+$. We have $\lambda \geq 1$; for if $\lambda < 1$, then (2) gives

$$(1-\lambda)e_1 = e_2a_2e_1 + \cdots + e_na_ne_1,$$

and we could rewrite (1) in the form,

$$1 = e_2b_2 + \cdots + e_nb_n,$$

contradicting our hypothesis that n was as small as possible. Therefore **150**
$\lambda \geq 1$; and rewriting (2) in the form

$$(\lambda - 1)e_1 + e_2a_2e_1 + \cdots + e_na_ne_1 = 0$$

and using the strictness of A, we obtain

$$\lambda = 1, e_ja_je_1 = 0 (j \neq 1).$$

By applying a similar argument with e_i in place of e_1, we obtain the formula

$$e_ia_ie_i = e_i, \quad e_ja_je_i = 0 \quad (i \neq j) \tag{3}$$

We take $u_i = e_i a_i$ $(i = 1, 2, \ldots, n)$. Then (3) gives

$$u_i^2 = u_i \ (i = 1, \ldots, n), \quad u_i u_j = 0 \quad (i \neq j), \tag{4}$$

and we also have

$$1 = u_1 + \cdots + u_n \tag{5}$$

Since $u_i A = e_i A$ each u_i is a minimal idempotent. And for each i, j, $u_i A u_j$ is non-zero and is of the form

$$u_i A u_j = R^+ e_{ij}$$

for some element e_{ij} of $u_i A u_j$. We choose the elements e_{ij} in such a way that

$$e_{ii} = u_i \quad (i = 1, 2, \ldots, n) \tag{6}$$

$$e_{ij} e_{jk} = e_{ik} \quad (i, j, k = 1, \ldots, n), \tag{7}$$

$$e_{ij} e_{kl} = 0 \quad (j \neq k) \tag{8}$$

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In the first place we have $u_i A u_i = R^+ u_i$, and so we can take $e_{ii} = u_i$ $(i = 1, 2, \ldots, n)$. Next, for $j = 2, \ldots, n$ we take e_{ij} to be an arbitrary non-zero element of $u_1 A u_j$. Then we have

$$u_1 A u_j = R^+ e_{ij} \quad (j = 1, \ldots, n).$$

Since $e_{ij} \neq (0), (Au_k)_l = (0)$, we have

$$e_{ij} u_j A u_k = e_{1j} A u_k \neq (0),$$

and so $$e_{1j} u_j A u_k = u_1 A u_k = R^+ e_{ik} \quad (j, k = 1, \ldots, n).$$

Therefore, for $j = 2, \ldots, n$ and $k = 1, \ldots, n$, we can select e_{jk} such that

$$e_{1j} e_{jk} = e_{1k}$$

Since $e_{11} = u_1$, this holds also for $j = 1$ i.e.

$$e_{1j} e_{jk} = e_{1k} \qquad (j, k = 1, \ldots, n) \tag{9}$$

We have now chosen e_{jk} for all j, k with $u_i\, Au_j = R^+\, e_{jk}$, and with (6) and (9) holding. To prove (7), we note that

$$e_{ij}e_{jk} \in u_i A u_k$$

and so $$e_{ij}e_{jk} = \lambda e_{ik} \text{ with } \lambda \geq 0.$$

multiplying by e_{1i}, we obtain by (9), 152

$$e_{1k} = e_{1j}e_{jk} = e_{1i}e_{ij}e_{jk} = \quad e_{1i}e_{ik} = \lambda e_{1k}$$

Therefore $\lambda = 1$ and (7) is proved. Finally (8) is obvious since $u_j u_k = 0$ $(j \neq k)$.

Given $x \in A$, we have

$$\begin{aligned} x = 1.x.1 &= \sum_{i,j=1}^{n} u_i x u_j \\ &= \sum_{i,j=1}^{n} \xi_{ij} e_{ij}, \end{aligned}$$

with $\xi_{ij} \geq 0$. Similarly, for $y \in A$,

$$y = \sum_{i,j}^{n} \eta_{ij} e_{ij}$$

and $$xy = \sum_{i,l=1}^{n} \left\{ \sum_{j=1}^{n} \xi_{ij}\eta_{jl} \right\} e_{il}.$$

It is now easily seen that the mapping $x \to (\xi_{ij})$ gives an isomorphism between the semi-algebra A and $M_n(R^+)$.

Conversely, with a given positive integer n, let $A = M_n(R^+)$. A is a closed semi-algebra in the Banach algebra $M = M_n(R)$ of all $n \times n$ real matrices (with an arbitrary Banach algebra norm), and is locally compact since B has finite dimension. That A is strict and has unit element is obvious.

Let u^{rs} be the matrix with (i, j) the element $u^{rs}_{ij} = \delta^r_i \delta^s_j$, where 153

$$\delta^\alpha_\beta = \begin{cases} 1 \text{ if} & \alpha = \beta \\ 0 \text{ if} & \alpha \neq \beta \end{cases}$$

The matrix u^{rs} belongs to A, and if $a = (\alpha_{ij})$ is an arbitrary element of A, we have

$$\begin{aligned}(u^{rs}au^{lm})_{ij} &= \sum \delta^r_i \delta^s_\mu \alpha_{\mu\gamma} \delta^l_\gamma \delta^m_j \\ &= \delta^r_i \alpha_{sl} \delta^m_j \\ &= \alpha_{sl} u^{rm}_{ij},\end{aligned}$$

so that

$$u^{rs}au^{lm} = \alpha_{sl}u^{r^m}.$$

It follows that every non-zero two-sided ideal of A contains all the matrices u^{rs}, and so is the whole of A. Thus A is simple and the proof is complete.

In the case when A is commutative our theorems on idempotents take a particularly simple form. In the first place we can determine semi-simple and prime commutative semi-algebras by annihilation properties of individual elements.

Lemma 8.6. *Let A be a closed commutative semi-algebra. If A is semi-simple, then*

$$a \in A, a \neq 0 \Rightarrow a^n \neq 0 \quad (n = 2, 3, \ldots)$$

154 *Conversely if*

$$a \in A,\ a^2 = 0 \Rightarrow a = 0,$$

Then A is semi-simple.

Also, A is prime if and only if it has no divisors of zero *ie*. $a, b \in A,\ ab = 0 \Rightarrow a = 0, b = 0$.

Proof. Entirely straight-forward using Lemma 8.2. In fact we can prove a stronger result than Lemma 8.6, that is analogous to a well-known fact about semi-simple commutative Banach algebras. □

Theorem 8.13. Let a be a non-zero element of a semi-simple commutative locally compact semi-algebra. Then

$$\lim_{n\to\infty} \|a^n\|^{1/n} > 0$$

Proof. Given a non-zero closed ideal J, let

$$K_J = \inf\left\{\|ax\| : x \in J \cap S_A\right\}.$$

□

By the compactnes of $J \cap S_A$, this infimum is attained, and so

$$K_J = 0 \Rightarrow J \cap (a)_r \neq (0)$$

For some J, we have $K_J > 0$. For otherwise every non-zero closed ideal J satisfies $J \cap (a)_r \neq (0)$, and therefore

$$M \cap (a)_r = M.$$

for every minimal closed ideal M i.e., $aM = (0)$ for every such M. Let 155
$I = cl(aA)$. Then I is a non-zero closed ideal and $IM = (Q)$. for every minimal closed ideal M. Since I contains a minimal closed ideal, this would contradict the semi-simplicity of A.

Let J be a closed ideal with $K_J > 0$. Then

$$\|ax\| \geq K_J\|x\| \qquad (x \in J),$$

and, since $ax^{n-1} \in J$,

$$\|a^n x\| \geq K_J\|a^{n-1}x\| \qquad (x \in J, n = 1, 2, \ldots)$$

Therefore

$$\|a^n\| \ \|x\| \geq K_J^n\|x\| \qquad (x \in J),$$

and so
$$\lim_{n\to\infty} \|a^n\|^{1/n} \geq K_J > 0.$$

Theorem 8.14. Let A be a semi-simple, commutative locally compact semi-algebra. Then the set $\mathscr{I}$ of minimal idempotents of A is a finite non-empty set $e_1, \ldots, e_n$. Each ideal $e_k A$ is a closed division semi-algebra with unit element e_k, and $e_k e_j = 0$ $(k \neq j)$. If also A is strict, then

$$e_k A = R^+ e_k \qquad (k = 1, 2, \ldots, n)$$

Remark. The elements e_k are simultaneous eignvectors for A,

$$ae_k = \lambda_a e_k \qquad (a \in A)$$

156 *Proof.* All ideals of A are two-sided, and so, by Theorem 8.7 A has only finitely many minimal closed ideals $M_1, \ldots, M_n$ and we have

$$M_i M_j = (0) \quad i \neq j.$$

□

Since M_i is a minimal closed right ideal, it is of the form $e_i A$ with $e_i \in \mathscr{I}$. Since $e_i A = e_i e_i A = e_i A e_i$, ($A$ being commutative), $e_i A$ is a closed division semi-algebra with unit element e_i.

Suppose now that e is a minimal idempotent. Then eA is a minimal closed ideal, and so $eA = M_i = eA_i$ for some i. Then e and e_i are both unit elements for M_i, and so $e = e_i$.

Theorem 8.15. Let A be a semi-simple, strict, commutative, locally compact semi-algebra, and let $e_1, \ldots, e_n$ be the minimal idempotents of A. Then, for each element a of A there exists non-negative real numbers $\lambda_1, \ldots, \lambda_n$ such that

$$ae_i = \lambda_i e_i \quad (i = 1, 2, \ldots, n),$$

and
$$\max\{\lambda_i : 1 \leq i \leq n\} = \rho_a$$

where
$$\rho_a = \lim_{k \to \infty} \|a^k\|^{1/k}, \text{ and } \rho_a > 0 \quad (a \neq 0)$$

Corollary. *If also A is prime then there exists exactly one minimal idempotent e and*

$$ae = \rho_a e \quad (a \in A).$$

157 *Proof.* We have already proved in Theorem 8.13 and 8.14 everything except that

$$\max \lambda_i = \rho_a.$$

□

Let $e = e_1 + \cdots e_n$. Since A is strict, we have

$$(e)_l = \bigcap_{i=1}^{n} (e_i)_l$$

Also $\bigcap_{i=1}^{n} (e_i)_l$ is a closed ideal, and so if it were non-zero it would contain one of the minimal idempotents e_i, which is absurd. Therefore $(e)_l = (0)$.

Let $a \in A$,

$$K = \inf\{\|xe\| : x \in S_A\}, \lambda = \max(\lambda_1, \ldots, \lambda_n),$$
$$\mu = \|e_1\| + \ldots + \|e_n\|$$

Since K is attained and $(e)_l = (0)$, we have $K > 0$, and so

$$\|x\| \leq K^{-1} \|xe\| \quad (x \in A)$$

For every positive integer k, we have

$$a^k e = \lambda_1^k e_1 + \cdots + \lambda_n^k e_n,$$

and so $$\|a^k e\| \leq \lambda^k \mu.$$

Therefore

$$\|a^k\| \leq K^{-1} \lambda^k \quad (k = 1, 2, \ldots)$$

On the other hand, for some i we have $\lambda = \lambda_i$, 158

$$a^k e_i = \lambda^k e_i,$$

and therefore $$\|a^k\| \geq \lambda^k$$

Therefore $\lambda = \rho_a$, and the proof is complete.

We now consider some concrete semi-algebras. First some semi-algebras of matrices.

Let n be a positive integer, $M_n(R)$ the Banach algebra of all $n \times n$ real matrices, $M_n(R^+)$ the semi-algebra of all matrices belonging to $M_n(R)$ with all their entries non-negative. Let $X = R^{(n)}$ and let C be the positive cone in $R^{(n)}$ consisting of all $x = (\xi_1, \ldots, \xi_n)$ such that $\xi_i \geq 0$ ($i = 1, 2, \ldots, n$). Let u_i be the vector $(0, 0, \ldots, 1, 0, \ldots, 0)$ with 1 in this i^{th} place and 0 elsewhere.

Given a subset Δ of $(1, 2, \ldots, n)$, let C_Δ be the set of all vectors

$$x = (\xi_1, \ldots, \xi_n) = \xi_1 u_1 + \cdots + \xi_n u_n,$$

with $\xi_i \geq 0$ $(i = 1, 2, \ldots, n), \xi_i = 0 (i \notin \Delta)$. We call each such cone C_Δ a *basic cone*, and call C a *proper basis cone* if Δ is a non-empty proper subset of $(1, 2, \ldots, n)$.

Each matrix $a \in M_n(R^+)$ may be regarded as a linear operator in X
159 that maps C into itself. Such a matrix is said to be *reducible* if there exists a proper basic cone C with

$$aC_\Delta \subset C_\Delta.$$

In term of the entries α_{ij} in the matrix a, this is equivalent to

$$i \notin \Delta, j \in \Delta \quad \alpha_{ij} = 0,$$

for it is equivalent to

$$au_j \in C \quad (j \in \Delta),$$

and au_j is the vector $(\alpha_{ij}, \alpha_{2j}, \ldots, \alpha_{nj})$. For example, if $\Delta = (1, 2, \ldots, r)$, then (α_{ij}) is of the form

$$\begin{pmatrix} b & c \\ o & d \end{pmatrix}$$

where b is an $r \times r$ matrix.

A matrix $a \in M_n(R^+)$ is said to be irreducible if it is not reducible.

Given a subset E of $M_n(R^+)$, let

$$N(E) = \{x : x \in C \quad \text{and} \quad E_x = (0)\}.$$

It follows that if a is an irreducible matrix and A is a semi-algebra with

$$a \in A \subset M_n(R^+),$$

then A is prime. **160**

For, let I, J be non-zero two-sided ideals of A with

$$IJ = (0)$$

Since $J \subset N(I)$, we have $N(I) \neq (0)$. Since $I \neq (0)$, we have $N(I) \neq C$. Therefore $N(I)$ is a proper basic cone. But since $Ia \subset I$, we have $aN(I)\Big\{ N(I)$, so that this would imply that a is reducible. $\Big\}$

Theorem 8.16. Let a_0 be an irreducible matrix belonging to $M_n(R^+)$, and let $A(a_0)$ denote the smallest closed semi-algebra in $M_n(R^+)$ that contains a_0. Then there exists an idempotent e of rank 1 in $A(a_0)$, such that

$$ae = \rho_a e,$$

with $\rho_a = \lim\limits_{n\to\infty} \|a^n\|^{1/n}$, for every element a of $M_n(R^+)$ that is permutable with a_0.

Proof. Let A be a closed commutative semi-algebra with

$$a_0 \in A \subset M_n(R^+).$$

□

Then A is a strict, prime, commutative, locally compact semi-algebra, and therefore there exists a unique minimal idempotent e_A in A and

$$ae_A = \rho_a e_A \quad (a \in A)$$

In particular

$$a_0 e_A = \rho_{a_0} e_A.$$

By a theorem of Function (Gantmacher, Theory of matrices, Vol. 2), **161**
ρ_{a_0} is a simple eigenvalue of the irreducible matrix a_0. Therefore

$$X = (u) + Y$$

where (u) is the one-dimensional null-space of $a_0 - \rho_{a_0}.1$, and Y is the range of this matrix. By a straight-forward argument, using the fact that the restriction of $a_0 - \rho_{a_0}.1$ to Y is nonsingular, we have

$$e_A u = u, e_A Y = (0)$$

In fact

$$(a_o - \rho_{a_o}.1)e_A = 0$$

and so
$$(a_0 - \rho_{a_0}.1)e_A X = (0)$$
$$e_A X \subset (u)$$

Since $e_A \neq 0$, this gives $e_A X = (u), (u)$ being one dimensional. Also, since

$$e_A(a_0 - \rho_{a_o}.1) = 0, \text{ and } (a_0 - \rho_{a_0}.1)Y = Y,$$

we have
$$e_A Y = e_A(a_0 - \rho_{a_0}.1)Y = (0).$$

162 This proves that e_A is the unique projection with $e_A u = u$ and $e_A Y = (0)$.

Therefore e_A is independent of the choice of the commutative semi-algebra A containing a_o, and so

$$e_A \in A(a_0).$$

Finally given any matrix a in $M_n(R^+)$ with $aa_0 = a_0 a$, there exists a closed commutative semi-algebra A in $M_n(R^+)$ that contains a and a_0.

Example. For $n = 2$, if

$$a_0 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

then $e = \dfrac{1}{2}a_0$ is a minimal idempotent in $A(a_0), A(a_0) = R^+ a_0$.

$a = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$ commutes with a_0 if and only if $\alpha_{11} = \alpha_{22}$ and $\alpha_{12} = \alpha_{12}$, i.e., if and only if

$$a = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$$

Then $ae = (\alpha + \beta)e$. (Note all such a are in $A(a_0)$).

Examples of semi-algebras of functions.

Let E denote a topological space, $B(E)$ the Banach algebra of all bounded continuous real functions on E, and let A be a locally compact semi-algebra in $B(E)$. At the beginning of this chapter we saw an example of such a semi-algebra A that had infinite dimension. In this particular example E was not connected and A was a type 2 semi-algebra. (We **163**
say that a semi-algebra A in $B(E)$ is of type n if $f \in A \Rightarrow \dfrac{f^n}{1+f} \in A$.)

In both these respects our example was as simple as possible. Such an example cannot have E connected and cannot be of type 1. In fact we can prove that the following propositions hold for any locally compact semi-algebra A in $B(E)$.

a) If A contains a non-constant function, then E is not connected.

b) If A is of type 1, then each element of A is constant on each connected subset of E, A contains a finite set $\chi_1, \ldots, \chi_n$ of characteristic functions of subsets of E, and each element f of A is a linear combination of these characteristic functions with non-negative coefficients.

$$f = \sum_{i=1}^{n} \lambda_i \chi_i, \quad (i = 1, 2, \ldots, n).$$

It is obvious that any semi-algebra in $B(E)$ is semi-simple and commutative. Let A contain a non-constant function. Since A is semi-simple it contains a minimal idempotent χ, and is the characteristic function of a set that is both open and closed. We have $\chi \neq 0$, and so if $\chi \neq 1$, then E is not connected.

Suppose that $\chi = 1$ and that E is connected. Then

$$A = A\chi$$

is a division semi-algebra. If $f \in A$ and f is not constant, then f has **164**
an inverse in A, and so, for every t, $f(t) \neq 0$. Since E is connected, we have either $f(t) > 0$, for all t or $f(t) < 0$ for all t. The second possibility cannot occur since $f + \lambda\chi \in A$ $(\lambda \geq 0)$. Therefore all non-constant

functions in A are contained in $B^+(E)$. Since A contains a non-constant function, it follows that in fact $A \subset B^+(E)$, for if A contains a negative constant function it also contains a non-constant function that is not in $B^+(E)$.

Since $A \subset B^+(E)$, A is strict, and so

$$A = A\chi = R^+\chi,$$

i.e., A contains only constant functions and (a) is proved.

b) Suppose A is of type 1. Given $f \in A$, and $\alpha > 0$, let

$$E(f, \alpha) = \{x : f(x) \geq \alpha\},$$

and let $\chi_{f,\alpha}$ denote the characteristic function f of $E(f, \alpha)$.

We prove that $\chi_{f,\alpha} \in A$. Let $g_1 = \dfrac{1}{\alpha} f$, and

$$g_{n+1} = \frac{2g_n^2}{1 + g_n^2} \qquad (n = 1, 2, \ldots)$$

Then
$$\lim_{n\to\infty} g_n(t) = \chi_{f,\alpha}(t) \quad (t \in E)$$

Since (g_n) is a bounded sequence of elements of A, it has a uniformly convergent subsequence, and so $\chi_{f,\alpha} \in A$.

165 Let E_0 be a subset of E, and f an element of A that is not constant on E_0, then we can choose points s, t in E_0 and a real number α with

$$f(s) < \alpha < f(t).$$

We have $\alpha > 0$, and since $\chi_{f,\alpha} \in A$, $E(f, \alpha)$ is both open and closed. But $t \in E(f, \alpha)$ and $s \notin E(f, \alpha)$, and so E_0 is not connected.

Let $\triangle$ denote the set of all characteristic functions that belongs to A. Then $\triangle$ is a finite set, for if (χ_n) were an infinite sequence of distinct elements of $\triangle$, it would have a uniformly convergent subsequence which is absurd, since

$$\|\chi_p - \chi_q\| = 1 \ (p \neq q)$$

We show that each element of A is a non- negative linear combination of the elements of

Give $f \in A$, we have $\chi_{f,\alpha} \in \triangle$ for every $\alpha > 0$, and so f takes only a finite set of different values $\alpha_1, \ldots, \alpha_n$ say with $0 < \alpha_1 < \cdots < \alpha_n$, and perhaps also the value 0. Let

$$h_i = \chi_{f,\alpha_i} \ (i = 1, \ldots, n),$$

and consider the function

$$h = \alpha_1 h_1 \sum_{i=2}^{n} (\alpha_i - \alpha_{i-1}) h_i$$

If $f(t) = \alpha_k$, then **166**

$$h_i(t) = \begin{cases} 0 & (i > k) \\ 1 & (i \leq k) \end{cases}$$

and so $$h(t) = \alpha_1 + \sum_{i=2}^{k} (\alpha_i - \alpha_{i-1}) = \alpha_k.$$

Also, if $f(t) = 0$, then $h_i(t) = 0$ $(i = 1, 2, \ldots,)$, and so $h(t) = 0$. Thus $f = h$, and we have proved (b)

Appendix The Schauder theorem for locally convex spaces

In Chapter 3, I asked whether the Schauder fixed point in its full gen- **167**
erality (Theorem 2.2) is true for locally convex spaces, and pointed out that this question did not seem to be answered in the literature. I am very much indebted to $B.V.$ Singbal who showed that this question could be settled affirmatively by using a technique due to Nagumo. The resulting proof of the general Schauder fixed point theorem is in my view the simplest proof even for the special case of normed spaces.

Lemma. (Nagumo [24]). *Let A be a compact subset of a locally convex l.t.s E, and V be a neighbourhood of 0 in E. Then there exists a finite set $a_1, \dots, a_m$ of points of A and continuous mapping S of A into the convex hull of $a_1, \dots, a_m$ such that*

$$Sx - x \in V \ (x \in A).$$

Proof. Let W be an open convex symmetric neighbourhood of 0 with $W \subset V$. Since A is compact, there exists a finite set $a_1, \dots, a_m$ in A such that

$$A \subset \bigcup_{i=1}^{n} (a_i + W). \tag{1}$$

□

Let P_W denote the Minkowski functional of W. Since W is open, 168

$$x \in W \Leftrightarrow p_W(x) < 1 \tag{2}$$

By (1) and (2), for each $x \in A$, there exists some i with

$$P_W(x - a_i) < 1.$$

Let S be the mapping of A into E defined by

$$Sx = \left[\sum_{i=1}^{m} q_i(x)\right]^{-1} \sum_{i=1}^{m} q_i(x) a_i \quad (x \in A),$$

where
$$q_i(x) = \max\{1 - p_w(x - a_i), 0\}.$$

For each x in A, there is at least one i with $q_i(x) > 0$. Since also $q_i(x) \geq 0$ for all i, it follows that S is defined and continuous on A and that it maps A into the convex hull of $a_1, \ldots, a_m$. For any i with $p_w(a_i - x) \geq 1$, we have $q_i(x) = 0$, and therefore

$$p_W(Sx - x) < 1 \ (x \in A),$$

i.e.,
$$Sx - x \in W \subset V \ (x \in A).$$

The mapping S constructed in the above Lemma serves essentially the purpose for which we used the metric projection in Chapter 2.

169 **Theorem.** (Singbal). *Let E be a locally convex Hausdorff l.t.s., K a non empty closed convex subset of E, T a continuous mapping of K into a compact subset of K. Then T has a fixed point in K.*

Proof. Let T map K into $A \subset K$, with A compact. For each neighbourhood V of 0, there exists a convex hull K_V of a finite subset of A and a continuous mapping S_V of A into K_V such that

$$S_V x - x \in V \ (x \in A). \tag{1}$$

□

Since K is convex and $A \subset K$, we have $K_V \subset K$. Let T_V be the mapping

$$T_V = S_V o T$$

of K_V into itself. By the Brower fixed point theorem T_V has a fixed point x_V in K_V,

$$T_V x_V = x_V \tag{2}$$

Since $x_V \in K_V \subset K$, we have $Tx_V \in A$. Since A is compact, there exists a point u of A such that every neighbourhood of u contains points x_V corresponding to *arbitrarily small* V, i.e. given neighbourhoods G, H of 0, there exists a neighbourhood V of 0 such that

$$(i) V \subset G, (ii) x_V \in u + H \tag{3}$$

Since $u \in K$ and T is continuous in K, given an arbitrary neighbourhood G of 0, there exists a neighbourhood H of 0 such that **170**

$$x \in (u + H) \cap K \Rightarrow Tx \in Tu + G, \tag{4}$$

and by (3) there exists a neighbourhood V of 0 such that

$$V \subset G, x_V \in u + (H \cap G). \tag{5}$$

Since $x_V \in K_V \subset K$, it follows from (4) that we then have, for such V,

$$Tx_V \in Tu + G. \tag{6}$$

Since $TK \subset A$, (1) gives

$$S_V Tx - Tx \in V \; (x \in K),$$

and, in particular,

$$x_V - Tx_V = T_V x_V - Tx_V = S_V Tx_V - Tx_V \in V \subset G \tag{7}$$

Since

$$u - Tu = (u - x_V) + (x_V - Tx_V) + (Tx_V - Tu),$$

(5), (6) and (7) give

$$u - Tu \in -G + G + G.$$

Since G is an arbitrary neighbourhood of 0, it follows that $u - Tu = 0$.

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