Lectures on
Exterior Differential Systems

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Introduction

To begin with, we shall roughly state the main problem that we shall be 1
considering in the following. Let D denote a domain in the n-dimensional Euclidean space R^n and let $\theta_1, \ldots, \theta_m$ be a system of homogeneous differential forms on D, which we shall denote by $\sum$. We adopt the convention that a function is a homogeneous differential form of degree zero. A submanifold M of D is called an integral submanifold or simply an integral of $\sum$ if the restrictions of $\theta_1, \ldots, \theta_m$ to M vanish. We will be concerned mainly with the following problem: given a system $\sum$ of homogeneous differential forms on D, to determine sufficient conditions for constructing all the integrals of $\sum$, and to obtain some information regarding the structure of the set of integrals of $\sum$. We shall discuss such conditions given by E.Cartan. He called systems satisfying his conditions "systems in involution". We shall also discuss the prolongations of differential systems, the main idea of which is also due to him.

The above mentioned problem is essentially a problem in the theory of partial differential equations. This fact is made clear by the following simple example.

Let $u(x, y)$ be a function of two independent real variables defined
in a certain domain D in R^2 and satisfy the system of partial differential 2
equations

$$F_\alpha\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 \quad (\alpha = 1, 2, \ldots, m)$$

u may be assumed to be once continuously differentiable. We will construct, introducing new variables p and q, a system $\sum$ of homogeneous differential forms in a suitable domain D_1 in R^5 of coordinate system (x, y, u, p, q)

$$(\sum)\begin{cases} F_\alpha(x, y, u, p, q), \\ du - pdx - qdy \end{cases}$$

Let M^2 be a two dimensional submanifold of D_1 expressed parametrically by $(x, y, u(x, y), p(x, y), q(x, y))$. It can be easily seen that M^2 is an integral of the system $\sum$ if and only if $u(x, y)$ is a solution of the system of differential equations

$$F_\alpha\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 \quad (\alpha = 1, \ldots, m)$$

together with $p(x, y) = \dfrac{\partial u(x, y)}{\partial x}, q(x, y) = \frac{\partial u(x,y)}{\partial y}$.

However, it seems that, in our approach, it is convenient to handle the system of homogeneous differential forms rather than solving the system of partial differential equations. Moreover, sometimes our approach is quite useful for certain geometric problems also.

We shall restrict our attention only to the case of systems of real
3 analytic differential forms. The extension of our results to the case of C^∞ forms (differentiable case) appears to be very much more complicated and remains unsolved. we shall also confine ourselves to the so called local problem.

Contents

Chapter 1

Parametrization of sets of integral submanifolds

1.1

In order to illustrate the problem with which we will be concerned in this 4
chapter, let us consider an ordinary differential equation, for instance,

$$\frac{du}{dx} = F(x)$$

where F is defined and real analytic in a neighbourhood of $x = 0$. Then there exists a unique function $u(x, w)$, real analytic in x, depending real analytically on a parameter w such that for sufficiently small fixed w, $u(x, w)$ is a solution of the differential equation and $u(o, w) = w$. Thus the solutions are parametrized by the parameter w.

More generally, in order to consider the situation independent of the coordinate systems, we shall use the following terminology. We say that a real analytic function $v(x, w_1, \ldots, w_h)$ is a parametrisation of solutions of the equation, when, for any fixed $(w_1^o, \ldots, w_h^o)$, with w_i^o small, $v(x, w_1^o, \ldots, w_h^o)$ is a solution of the differential equation and conversely any solution which is sufficiently small at the origin is obtained by choosing $(w_1^o, \ldots, w_h^o)$ suitably. Then, for any parametrization, the number of parameters is the same and is a constant determined by the equation (being equal to 1 in the above instance).

5 In the case of partial differential equations, the solutions are often parametrized by arbitrary functions. Take as a simple example, the partial differential equation

$$\frac{\partial u}{\partial x} = 0,$$

where u is an unknown function of the variables (x, y). Then, for any real analytic function $f(y), u = f(y)$ is a solution of the above differential equation and any real analytic solution is so obtained. In such a case the solutions of the partial differential equation are said to depend on one arbitrary function in two variables. However, no strict definition of this notion is known. As a consequence, the number of arbitrary functions on which the solutions of the equation depend may not be an invariant of the equation. For instance, we can give another parametrization of the solution of the above partial differential equation, in which the solutions depend on two arbitrary functions. Namely, for any two real analytic functions $f = \sum a_n y^n, g = \sum b^n y^n$, we associate a solution $u = \sum(a_n y^{2n} + b_n y^{2n+1})$. The main purpose of this chapter is to introduce a notion of parametrization of a set of submanifolds by arbitrary functions. This notion will be used to define systems of partial differential equations or an exterior differential system, the solutions of which depend on certain number of arbitrary functions. In this definition the number of arbitrary functions and the number of variables will be invariants of the system.

1.2

6 Let H_p denote the vector space of power series, in p variables $x_1, \ldots, x_p$, which are convergent on a neighbourhood of the origin and with coefficients in the field C of complex numbers. We set $H_o = C$. If $u > 0, v > 0$ are real numbers, let $H_p(u, v)$ denote the subset of H_p consisting of all power series ξ satisfying the following conditions: On a polydisc $\{x; |x_r| < u + \varepsilon\}, \xi$ converges and $|\xi(x)| < v - \varepsilon$, where $\varepsilon > 0$ depending on ξ. In particular $H_o(u, v) = \{z \in C; |z| < v\}$. Let H_p^s denote the direct

sum of s copies of the vector spaces H_p.

Definition . *By a system S of characters we mean an ordered set of non-negative integers $s_o, s_1, \ldots, s_p$. Denote by $H(S)$ the direct sum of $H_o^{s_o}, H_1^{s_1}, \ldots, H_p^{s_p}$. $\sum\limits_{q=o}^{p} (H_q(u,v))^{s_q}$ can be naturally identified with a subset of $H(S) = \oplus \sum\limits_{q=o}^{p} (H_q)^{s_q}$. We denote the subset by $H(S; u, v)$. It is clear that*

(1) $H(S; u, v) \subseteq H(S; u, v')$ *if* $v \leq v'$

(2) $H(S; u, v) \subseteq H(S; u', v)$ *if* $u \geq u'$

and therefore (3) $H(S; u, v) \subseteq H(S; u', v')$ *if* $v \leq v'$ *and* $u \geq u'$.

Let $K(a)$ denote the open disc in C of radius a about the origin.

Definition . *A mapping $\mathscr{C}$ of $K(a)$ into $H(S; u, v)$ is called a regular curve in $H(S; u, v)$ if each component $\mathscr{C}(z)_\lambda(x_1, \ldots, x_q)$ is an analytic function in $(z, x_1, \ldots, x_q)$ for $|z| < a, |x_i| > u$.*

Let $S' = (s'_o, s'_1, \ldots, s'_{p'})$ be another system of characters. (p' may be different from p).

Definition . *A mapping F of $H(S; u, v)$ into $H(S'; u', v')$ is said to be* 7
regular if

(i) $f(0) = 0 \in H(S')$,

(ii) *for any regular curve $\mathscr{C}$ in $H(S; u, v) f \circ \mathscr{C}$ is a regular curve in $H(S'; u', v')$.*

Proposition 1. *If F is a regular mapping of $H(S; u, v)$ into $H(S'; u', v')$ then for any real number b with $0 < b \leq 1$ and for any $\varepsilon > 0$*

$$F[H(S; u, bv)] \subseteq H(S', u', bv')$$

Proof. Take a ξ in $H(S;u,v)$. For sufficiently small ε', $z\xi$ is in $H(S;u,v)$ if $|z| \leq 1+\varepsilon'$. Hence $z \to z\xi$ is a regular curve in $H(S;u,v)$. F being regular, $F(z\xi)$ is a regular curve in $H(S';u',v')$. The component $F(z\xi)_\lambda(x_1,\ldots,x_q)$ is an analytic function $f(z)$ for any fixed $(x_1,\ldots,x_q)$ with $|x_r| < u'$ and $|z| \leq 1+\varepsilon'$. $f(z)$ satisfies the following two conditions:

$$f(0) = 0 \text{ since } F(0) = 0$$

and

$$|f(z)| \leq v' \text{ for } |z| \leq 1+\varepsilon'$$

Hence by Schwarz's lemma it follows that $|f(z)| \leq \dfrac{b}{1+\varepsilon'}v'$ for $|z| < b < 1$. Therefore the image of $H(S;u,bv)$ is contained in $H(S';u',bv')$. □

Proposition 2. *If F and G are two regular maps of $H(S;u,v)$ into*
8 *$H(S';u',v')$ and $H(S';u',v')$ into $H(S'';u'',v'')$ respectively, then GoF is a regular map of $H(S;u,v)$ into $H(S'';u'',v'')$.*

This follows immediately from the definition of regular maps.

Germs of regular maps. We remark that $H(S'';u'',v'')$ is contained in $H(S;u,v) \cap H(S';u',v')$ whenever $u'' > u,u'$ and $v'' < v,v'$. Let F_r be regular maps of $H(S;u_r,v_r)$ into $H(S';u'_r,v'_r)(r=1,2)$. We shall introduce an equivalence relation, denoted by $\sim$, in the set of all regular maps as follows. F_1 is said to be equivalent to F_2 (denoted by $F_1 \sim F_2$) if there exist $u > u_1,u_2$ and $v < v_1,v_2$ such that the restrictions of F_1 and F_2 to $H(S;u,v)$ are equal. Clearly $\sim$ is an equivalence relation.

Definition. *An equivalence class of regular maps under $\sim$ is called a* germ of regular maps *of $H(S)$ into $H(S')$.*

A germ of regular maps containing a representative F is denoted by $\mathscr{F}$.

Let us introduce the following notations. For any $\xi \in H_p$ and $z \in C$ we define $[\xi.z] \in H_p$ by setting $[\xi.z]\,(x_1,\ldots,x_p) = \xi(z\,x_1,\ldots,z\,x_p)$ and for any $\xi \in H(S)$ we define $[\xi.z]$ in $H(S)$ by setting $[\xi\,z]_\lambda = [\xi_\lambda\,z]$. Then clearly

(i) if $\xi \in H_p(u,v)$, $[\xi,z] \in H_P(|z|^{-1}\,u,v)$ and hence

(ii) if $\xi \in H(S; u, v), [\xi, z] \in H(S; |z|^{-1}u, v)$.

Further, for $w \in \mathbb{C}$ and for any $\xi \in H(S; u, v)$, we define $w\xi$ in $H(S; u, wv)$ by the usual multiplication by w.

Proposition 3. *If $F_r (r = 1, 2)$ are two regular maps of $H(S; u, v)$ into* 9
$H(S'; u'_r, v'_r)$ such that $F_1 \sim F_2$ then $F_1 = F_2$.

Proof. F_1 and F_2 being equivalent there exist $u^* \geq u$ and $v^* \leq v$ such that F_1 and F_2 coincide on $H(S; u^*, v^*)$. Take ξ in $H(S; u, v)$. There exists an $\varepsilon > 0$ such that ξ_λ is convergent in the polydisc of radius $(1 + 2\varepsilon)u$ and $(1 + 2\varepsilon)|\xi_\lambda(x_1, \ldots, x_q)| < v$ for $|x_i| < u(i = 1, \ldots, q)$. Let f be the mapping of $K(1 + \varepsilon)$ defined by $f(z) = z[\xi.z]$. f is a regular curve in $H(S; u, v)$ and $f(z)$ is in $H(S; u^*, v^*)$ for z sufficiently near the origin, say for example $|z| < \delta$. Then $F_1(f(z)) = F_2(f(z))$ for $|z| < \delta$. □

Hence by the theorem of uniqueness $F_1(f(1)) = F_2(f(1))$ which means $F_1(\xi) = F_2(\xi)$. A germ $\mathscr{F}$ of regular maps of $H(S)$ into $H(S')$ is said to be defined at an element ξ in $H(S)$ if there exists a representative F of $\mathscr{F}$ defined on $H(S; u, v)$ with $\xi \in H(S; u, v)$. The value $F(\xi)$ is said to be the value of the germ $\mathscr{F}$ at ξ. The Proposition 3 shows that the value of the germ $\mathscr{F}$ at an element ξ in $H(S)$ is uniquely defined and independent of the choice of the representatives.

Composition of regular maps. Let $\mathscr{F}$ and $\mathscr{G}$ be two germ of regular maps of $H(S)$ into $H(S')$ and $H(S')$ into $H(S'')$ respectively. We say that the composition of $\mathscr{F}$ and $\mathscr{G}$ are defined whenever there exist representatives F of $\mathscr{F}$ and G of $\mathscr{G}$ such that F is a regular map of $H(S; u, v)$ into $H(S'; u', v')$ and G is a regular map of $H(S'; u', v')$ into $H(S'', u'', v'')$.

It is clear that the composition of any two germs need not always 10
be defined. Whenever the composition of two germs $\mathscr{F}$ and $\mathscr{G}$ is defined, the germ of regular maps of $H(S)$ into $H(S'')$ containing GoF as a representative is called the composition of $\mathscr{F}$ and $\mathscr{G}$ and is denoted by $\mathscr{G}o\mathscr{F}$. It is clear that $\mathscr{G}o\mathscr{F}$ is uniquely defined.

Definition. *A mapping F of $H(S_1; u_1, v_1)$ into $H(S_2; u_2, v_2)$ is said to be* infinite analytic *if*

(i) *F is regular and*

(ii) *there exist strictly positive real numbers u_*, v, v', w and an integer k such that for every u with $0 < u < u_*$ there exists a regular map F_u of $H(S_1; u, u^k v)$ into $H(S_2; wu, v')$ which is equivalent to F.*

k is called a *degree* of the infinite analytic map F.

Remarks. *Let F be an infinite analytic map of $H(S_1; u_1, v_1)$ into $H(S_2; u_2, v_2)$ and let u_*, v, v', w and k satisfy the conditions of the above definition.*

(i) Any $w' \leq w$ also satisfies the requirement because obviously $H(S_2; wu, v')$ is contained in $H(S_2; w'u, v')$.

(ii) Any $k' \geq k$ also satisfies the requirement. Hence we can without loss of generality assume k to be non-negative.

Definition. *A germ of regular maps is called a germ of* infinite analytic maps *if every representative of it is an infinite analytic map.*

11 We shall give two simple examples of infinite analytic maps.

Example 1. The mapping F defined by $\xi(x) \to \dfrac{d\xi}{dx}(x)$ of $H_1(u, v_1)$ into $H_1\left(\dfrac{u}{2}, \dfrac{2v_1}{u}\right)$ is an infinite analytic map as can be seen using Cauchy's integral formula. Here we can take $k = 1, w = \dfrac{1}{2}, v = v_1$ and $v' = 2v_1$.

Example 2. The mapping F defined by $\xi(x) \to \int_0^x \xi(y)dy$ of $H_1(u, v_1)$ into $H_1(u, uv_1)$ is infinite analytic. Here we can take $k = -1, w = 1, v = v_1 = v'$.

Proposition 4. *If $\mathscr{F}$ and $\mathscr{G}$ are germs of infinite analytic maps of $H(S)$ into $H(S')$ and $H(S')$ into $H(S'')$ then the composition $\mathscr{G}$ o $\mathscr{F}$ is always defined and is a germ of infinite analytic maps of $H(S)$ into $H(S'')$.*

Proof. Let F (resp. G) be a representative of $\mathscr{F}$ (resp. $\mathscr{G}$). Then there exist (u_*, v, v', w, k) and $(\overline{u}_*, \overline{v}, \overline{v'}, \overline{w}, \overline{k})$ as in the definition of an infinite

analytic map and $F_a \in \mathscr{F}$ and $G_b \in \mathscr{G}$ for any $0 < a < u_*$ and $0 < b < \bar{u}_*$. According to remark 1, we assume $\underline{k} > 0, k' > 0$ and by remark 2, we can choose a w so small that $(u_* w)^{\underline{k}} \bar{v}(2v')^{-1} < 1$ and $wa < \bar{u}_*$ for any $0 < a < u_*$. Let $v_2 = w^{\underline{k}} \bar{v}\, v\, (2v')^{-1}$. For an a with $0 < a < u_*, F_a$ is a regular map of $H(S; a, a^k v)$ into $H(S'; wa, v')$. From Proposition 1 it follows that for a constant c with $0 < c < 1$ the image of $H(S; a, ba^k v)$ by F_a is contained in $H(S'; wa, bv')$. Taking $b = \dfrac{\underline{a}^k v_2}{\underline{v}}$ the image of $H(S; a, a^{k+\bar{k}} v_2)$ by F is contained in $H(S'; wa, (wa)^{\underline{k}} \bar{v}_1)$. 12
Now since $wa < \bar{u}_*$ we have that the image of $H(S'; wa, (wa)^{\underline{k}} \bar{v})$ by G_{wa} is contained in $H(S''; \bar{w} wa, \bar{v}^1)$. Thus the composite of G_{wa} and F_a is defined and this completes the proof. □

If $1 \geq 0$ is an integer let $H_p^{(1)}$ denote the vector subspace of H_p consisting of all $\xi \in H_p$ such that the first non-zero term, in the expansion of ξ in terms of homogeneous polynomials, is of degree ≥ 1. In other words ξ can be put in the form

$$\xi = \sum_{k \geq 1} A_k(x_1, \ldots, x_p)$$

where $$A_k(x_1, \ldots, x_p)$$

denotes a homogeneous polynomial in $x_1, \ldots, x_p$ of degree k. If $S = (s_o, \ldots, s_p)$ is a system of characters, we can define the vector subspace $H(S)^{(1)}$ of $H(S)$ by

$$H(S)^{(1)} = \{\xi \in H(S) : \xi_\lambda \in H_q^{(1)}\}$$

For any subset A of $H(S)$ we denote by $H(S; u, v)^{(1)}$ the set $H(S; u, v) \cap H(S)^{(1)}$.

Definition. *For an element ξ in $H_p(u, v)$ the norm of ξ in $H_p(u, v)$, denote by $|\xi|_u$, is defined by*

$$|\xi|_u = \sup_x \{|\xi(x)| : |x_r| < u (r = 1, \ldots, p)\}$$

and for any ξ in $H(S; u, v)$ the norm of ξ in $H(S; u, v)$, denoted again by $|\xi|_u$, is defined by

$$|\xi|_u = \max_\lambda |\xi_\lambda|_u$$

13 *In a phrase like "ξ is in $H(S;u,v)$ and $|\xi|_u$" we shall often omit "ξ is in $H(S;u,v)$ and" when there is no possible confusion.*

The following proposition is an immediate consequence of the above definitions.

Proposition 5. *An element ξ in $H(S)$ belongs to $H(S)^{(1)}$ if and only if there exists a constant $C > 0$ such that for sufficiently small u we have the inequality $|\xi|_u \leq c.u^1$.*

Proposition 6. *Let F be a regular map of $H(S;u,v)$ into $H(S';u',v')$. Then for any $\xi, \eta \in H(S;u,v/4)$,*

$$|F(\xi) - F(\eta)|_{u'} \leq K|\xi - \eta|_u$$

where K is a positive constant.

Proof. Suppose $\zeta = \xi - \eta$, the function $f(z) = \xi - z\zeta$ is in $H(S;u,v)$ for $|z| < R = \dfrac{3v}{4}|\zeta|_u^{-1}(1+\varepsilon)$. The function $g(z) = F[f(0)]_\lambda(x) - F[F(z)]_\lambda(x)$ is a holomorphic function for $|z| < R$ and for x in $|x_r| < u'.g(z)$ satisfies the conditions: $(i)|g(z)| < 2\ v'$ and $g(0) = 0$. By Schwarz's lemma $|g(z)| < \dfrac{2v'\ |z|}{R}$ and hence we have

$$|g(1)| = |F(\xi) - F(\eta)|_{u'} < K\frac{v'}{v}|\xi - \eta|_u$$

if we take $K \geq \dfrac{8}{3}$. □

The Propositions 1 and 5 together imply the following.

Proposition 7. *If $\mathscr{F}$ is a germ of infinite analytic maps (of degree k) of*
14 *$H(S)$ into $H(S')$ and if $\xi \in H(S)$ then there exists a positive real number a_1 depending on ξ such that for any $a < a_1$, ξ is defined at $a\xi$; moreover if ξ is in $H(S)^{(1+k)}$ (with $1 \geq 0, 1+k \geq 0$) then $\mathscr{F}(a\xi)$ is in $H(S')^{(1)}$.*

1.3 Regular linear maps

Let S, S' be two systems of characters.

Definition. *A regular map F of $H(S; u, v)$ into $H(S'; u', v')$ is said to be linear if the following condition is satisfied: for every ξ, η in $H(S; u, v)$ such that $\alpha\xi, \beta\eta$ and $\alpha\xi + \beta\eta$ are in $H(S; u, v)$ where $\alpha, \beta \in C$*

$$F(\alpha\xi + \beta\eta) = \alpha F(\xi) + \beta F(\eta).$$

For a strictly positive real number u let $H(S; u) = \bigcup_{v>0} H(S; u, v)$. Clearly $H(S; u)$ is a subvector space of $H(S)$. Let F be a regular linear map of $H(S; u, v)$ into $H(S'; u', v')$. Then, for any $\xi \in H(S; u)$ and $\alpha \in C$ such that $\alpha\xi \in H(S; u, v), \alpha^{-1} F(\alpha\xi)$ does not depend on the choice of such an α.

In fact, if $\beta \in C$ such that $\beta\xi$ is also in $H(S; u, v)$, we see that $F(\alpha\xi) = \beta^{-1}\alpha F(\beta\alpha^{-1}\alpha\xi) = \beta^{-1}\alpha F(\beta\xi)$ by the linearity of F.

Setting $F'(\xi) = \alpha^{-1} F(\alpha\xi)$ with $\alpha \in C$ such that $\alpha\xi$ is in $H(S; u, v)$ we obtain a map F' of $H(S; u)$ into $H(S'; u')$. The restriction of F' to $H(S; u, v)$ is equal to F because of the linearity of F. Also the restriction of F' to $H(S; u, v')$ for any $v_1 > 0$ is a regular map and is equivalent to F. Further F' is a linear map of the vector space $H(S; u)$ into the vector **15**
space $H(S'; u')$.

In fact let $\xi, \eta \in H(S; u)$. Then $F'(\alpha\xi + \beta\eta) = \gamma_1^{-1} F(\gamma_1(\alpha\xi + \beta\eta))$ for $\alpha, \beta \in C$ and any $\gamma_1 \in C(\gamma_1 \neq 0)$ such that $\gamma_1(\alpha\xi + \beta\eta)$ is in $H(S; u, v)$. Let $\gamma_2, \gamma_3 \in C(\gamma_2, \gamma_3 \neq 0)$ such that $\gamma_2\alpha\xi, \gamma_3\beta\eta \in H(S; u, v)$. If $\gamma \in C$ is such that $|\gamma| = \min(|\gamma_1|, |\gamma_2|, |\gamma_3|)$ it follows by linearity of F that

$$\begin{aligned} F'(\alpha\xi + \beta\eta) = \gamma^{-1} F(\gamma(\alpha\xi + \beta\eta)) &= \gamma^{-1}\{F(\gamma\alpha\xi) + F(\gamma\beta\eta)\\ &= \alpha(\gamma\alpha)^{-1} F(\gamma\alpha\xi) + \beta(\gamma\beta)^{-1} F(\gamma\beta\eta)\\ &= \alpha F'(\xi) + \beta F'(\eta). \end{aligned}$$

Definition. *A germ $\mathscr{F}$ of regular maps of $H(S)$ into $H(S')$ is said to be linear if $\mathscr{F}$ contains a representative F which is linear.*

Proposition 8. *If $\mathscr{F}$ is a germ of linear infinite analytic maps then $\mathscr{F}$ is defined everywhere and linear mapping of the vector space $H(S)$ into $H(S')$.*

Proof. Every ξ in $H(S)$ is an element of some $H(S;u)$ for sufficiently small $u > 0$. There exists a representative F of $\mathscr{F}$ defined at $\xi \in H(S;u,v_1)$ for sufficiently small $v_1 > 0$ and this F can be extended into $H(S;u)$. The linearity of the germ $\mathscr{F}$ is clear from the above remark. ◻

16 Now we introduce the following terminology: if $S = (s_o, \ldots, s_p)$ with $s_p \neq 0$ is a system of characters then p is called the degree of $H(S)$ and s_p is called the multiplicity of $H(S)$.

Definition. *Two vector spaces $H(S)$ and $H(S')$ are said to be isomorphic if there exist germs of linear infinite analytic maps $\mathscr{F}$ and $\mathscr{G}$ of $H(S)$ into $H(S')$ and $H(S')$ into $H(S)$ respectively such that $\mathscr{G}$ o $\mathscr{F}$ and $\mathscr{F}$ o $\mathscr{G}$ are germs of identity maps on $H(S)$ and $H(S')$ respectively.*

This is denoted by $H(S) \cong H(S')$.

Proposition 9. *Two vector spaces $H(S)$ and $H(S')$ are isomorphic if and only if $H(S)$ and $H(S')$ have the same degree and multiplicity.*

Proof. We shall first prove that $H(S) \cong H(S')$ implies that their degrees and multiplicities are the same. ◻

We observe that for any integer $1 > 0$ the dimension of the quotient space $\dfrac{H_r}{H_r^{(1)}}$ is $\binom{r+1-1}{r} = \frac{1}{r!}\, 1^r +$ (lower powers of 1) $= f_r(1)$ where $f_r(X)$ denotes the polynomial $\dfrac{X^r}{r!} + \ * \ X^{r-1} + \cdots$ of degree r. Then

$$\dim\left(\frac{H(S)}{H(S)^{(1)}}\right) = \sum_{r=0}^{p} s_r \ f_r\ (1)$$

and

$$\dim\left(\frac{H(S')}{H(S')^{(1)}}\right) = \sum_{r=0}^{q} t_r \ f_r\ (1)$$

where $S = (s_o, \ldots, s_p)$ and $S' = (t_o, \ldots, t_q)$ with $s_p \neq 0, t_q \neq 0$. Let $\mathscr{F}$ and $\mathscr{G}$ be the germs of linear infinite analytic maps defining the isomorphism. We can, without loss of generality assume that $\mathscr{F}$ has

a degree $k > 0$. The map $\mathscr{F}$ of $H(S)$ into $H(S')$ is surjective and
17 $\mathscr{F}[H(S)^{(1+k)}] \subseteq H(S')^{(1)}$ (Proposition 5). Hence there exists an induced surjective map of $\dfrac{H(S)}{H(S)^{(1+k)}}$ onto $\dfrac{H(S')}{H(S')^{(1)}}$ and therefore

$$\sum_{r=0}^{p} s_r \; f_r\,(1) \geq \sum_{r=0}^{q} t_r \; f_r(1).$$

For large 1, the comparison of dominant factors on either side of the inequality shows that $p \geq q$. Similarly, we show that $q \leq p$ and hence $p = q$. Then, by the same reasoning we show that $s_p = s_q$.

The converse is proved in two steps.

Case 1. *Consider the particular case where $t_\nu = s_\nu + s_{\nu+1} + \cdots + s_p$ for $\nu = 0, \ldots, p$. Let ξ_r be the component of $\xi \in H(S)$ in $H_r^{s_r}$. Then ξ_r has s_r components which are functions of $(x_1, \ldots, x_r)$ and we denote them by $\xi_r^1, \ldots, \xi_r^{s_r}$ when $s_r \neq 0$. Similarly η_r^σ (for $\sigma = 1, \ldots, s_r + \cdots + s_p$) denote the components of an element η in $H(S')$.*

Now we define a linear map F of $H(S)$ into $H(S')$ as follows: for any $\xi \in H(S)$ set

$$F(\xi)_o^{s_o+\cdots+s_{q-1}+\sigma} = \xi_q^\sigma(0) \text{ for } q = 0, \ldots, p; \sigma = 1, \ldots, s_q \text{ if } s_q \neq 0$$

where $s_o + \cdots + s_{q-1} + \sigma$ means σ when $q = 0$ and

$$F(\xi)_r^{s_r+\cdots+s_{q-1}+\sigma} = \frac{\partial\, \xi_q^\sigma}{\partial\, x_r} \text{ for } {\scriptstyle r=1,\ldots,p;q=r,\ldots,p \atop \sigma=1,\ldots,s_q \text{ if } s_q\neq 0}$$

where $s_r + \cdots + s_{q-i} + \sigma$ means σ when $q = r$. Clearly F is infinite analytic (see ex.1 of §1.2) and linear because of the linearity of partial 18
derivation. The inverse map G of $H(S')$ into $H(S)$ is defined as follows: for any $\eta \in H(S')$ set

$$G(\eta)_r^\lambda = \eta_o^{s_o+\cdots+s_{r-1}+\lambda} + \int_o^{x_1} \eta_1^{s_1+\cdots+s_{r-1}+\lambda} dx_1 + \int_o^{x_2} \eta_2^{s_2+\cdots+s_{r-1}+\lambda} + \cdots + \int_o^{x_r} \eta_r^\lambda \, d\, x_r$$

for $r = 1, \ldots, p; \lambda = 1, \ldots, s_r$ if $s_r \neq 0$. Again G is infinite analytic (ex.2 of §1.2) and linear because of the linearity of integration. Obviously F and G are surjective. It is easy to verify that F and G define germs of infinite analytic maps $\mathscr{F}$ and $\mathscr{G}$, and that $\mathscr{G}\ o\ \mathscr{F}$ (resp. $\mathscr{F}\ o\ \mathscr{G}$) is identity on $H(S)$ (resp. $H(S')$). The assertion follows in this special case.

Case 2. *To prove the assertion in the general case we shall write $H(S)$ explicitly as $H(s_o, \ldots, s_p)$. It is sufficient to prove that $H(S) \cong H_p^{s_p}$. In view of the Case 1 proved above we can without any loss of generality assume that $s_o \neq 0, \ldots, s_p \neq 0$ because $H(S)$ is isomorphic to $H(s_o + \cdots + s_p, \ldots, s_{p-1} + s_p, s_p)$. Then it can be seen without much difficulty that our assertion is a consequence of the following statement: if $s_r > 0, \ldots, s_p > 0, H(0, \ldots, 0, s_r, s_{r+1}, \ldots, s_p) \cong H(0, \ldots, 0, s_r - 1, s_{r+1}, \ldots, s_p)$. Now since $H_r \cong H_o \oplus H_1 \oplus \cdots \oplus H_r$ we can write*

$$\begin{aligned}
&H(0, \ldots, 0, s_r, s_{r+1}, \ldots, s_p)\\
&\quad \cong H(0, \ldots, 0, s_r - 1, s_{r+1}, \ldots, s_p) \oplus H_r\\
&\quad \cong H(0, \ldots, 0, s_r - 1, s_{r+1}, \ldots, s_p) \oplus (H_o \oplus \cdots \oplus H_r)\\
&\quad \cong H(0, \ldots, 0, s_r - 1, s_{r+1} - 1, \ldots, s_p) \oplus H_{r+1} \oplus (H_o \oplus \cdots \oplus H_r)\\
&\quad \cong H(0, \ldots, 0, s_r - 1, s_{r+1} - 1, \ldots, s_p) \oplus H_{r+1}\\
&\quad \cong H(0, \ldots, 0, s_r - 1, s_{r+1}, \ldots, s_p).
\end{aligned}$$

19 This is obtained by successive usage of Case 1.

1.4

Hitherto we had restricted our attention to the case of convergent power series with coefficients in the field of complex numbers. We can, without much difficulty, extend all the notions and results proved in the previous sections to the case of convergent power series with real coefficients (which we call, hereafter, as the real analytic case). Let $S = (s_o, \ldots, s_p)$ be a system of characters. Let $H^R(S)$ denote the (real) subvector space of $H(S)$ consisting of ξ in $H(S)$ with all its components ξ_r^λ to be real analytic. Set $H^R(S; u, v) = H^R(S) \cap H(S; u, v)$. A mapping F_R of $H^R(S; u, v)$

into $H^R(S'; u', v')$ is said to be a (real) regular map if there exists a regular map F_C of $H(S; u, v)$ into $H(S'; u', v')$ such that F_R is the restriction of F_C to $H^R(S; u, v).F_C$ is called the complexification of F_R and by the theorem of identity it can be shown that such an F_C is unique. Two (real) regular maps are said to be equivalent when their complexifications are equivalent. Thus we can define germs of (real) regular maps of $H^R(S)$ into $H^R(S')$. Similarly we can define the notion of germs of infinite analytic maps of $H^R(S)$ into $H^R(S')$ when there exist germs of infinite analytic map of $H(S)$ into $H(S')$ mapping $H^R(S)$ into $H^R(S^1)$. **20**

1.5 Examples

(1) Consider a system of functions

$$A_\lambda(y_1, \ldots, y_s, x_{p+1}, \ldots, x_{p+1}) \quad (\lambda = 1, \ldots, s')$$

defined and analytic in the domain $|y_\sigma| \leq v, |x_{p+\mu}| \leq u^*$ $(\sigma = 1, \ldots, s; \mu = 1, \ldots, 1)$ satisfying the following conditions:

(i) $A_\lambda(0, \ldots, 0) = 0$

(ii) $|A_\lambda(y_1, \ldots, y_s; x_{p+1}, \ldots, x_\mu)| < v' - \varepsilon$ for small $\varepsilon > 0$ in the above domain.

Then we define a regular map F_u of $H^s_p(u, v)$ into $H^{s'}_{p+1}(u, v')$ for every $0 < u < u^*$ by setting

$$\begin{aligned}&[F_u\ (\xi)]_\lambda(x_1, \ldots, x_{p+1})\\ &\qquad = A_\lambda(\xi_1(x_1, \ldots, x_p), \ldots, \xi_s(x_1, \ldots, x_p), x_{p+1}, \ldots, x_{p+1})\end{aligned}$$

for every λ with $1 \leq \lambda \leq s'$. The germ of F_u can be verified to be a germ of infinite analytic maps.

(2) Consider a system of functions

$$A_\lambda(x_1, \ldots, x_{p+1}, y_1, \ldots, y_s, \ldots, y^r_\mu, \ldots) \quad (\lambda = 1, \ldots, s)$$

defined and analytic in the domain $|x_i| < v^*, |y_\mu| < v^*, |y^r_\mu| < v^*$ $(i = 1, \ldots, p+1; r = 1, \ldots, p; \mu = 1, \ldots, s)$. Consider the system of partial differential equations

$$(\mathscr{U})\frac{\partial y_\lambda}{\partial x_{p+1}} = A_\lambda\left(x_1, \ldots, x_{p+1}, y_1, \ldots, y_s, \ldots, \frac{\partial\, y_\mu}{\partial\, x_r}, \ldots\right).$$

21 Such a system of partial differential equations is called a system of partial differential equations of Cauchy-Kowalewski type.

Now we make the following definitions.

Definition. *If* $\xi \in H^s_p$ *such that* $|\xi_\lambda(0)| < v^*$ *and* $\left|\frac{\partial \xi_\lambda}{\partial x_r}(0)\right| < v^*$ *for* $\lambda = 1, \ldots, s$ *and* $r = 1, \ldots, p$ *then an element* $\eta \in H^s_{p+1}$ *is called a* solution *of Cauchy-Kowalewski system* $(\mathscr{U})$ *with the initial condition* ξ *if*

(i) $\eta_\lambda(x_1, \ldots, x_p, 0) = \xi_\lambda(x_1, \ldots, x_p) \quad (\lambda = 1, \ldots, s)$

(ii) $y_\lambda = \eta_\lambda$ *is a solution of* $(\mathscr{U})$.

Let u, v, u', v' be strictly positive real numbers.

Definition. *A mapping* F *of* $H^s_p(u, v)$ *into* $H^s_{p+1}(u', v')$ *(where* $v, u^{-1}v < v^*$*) is called a solution mapping of the Cauchy-Kowalewski system* $(\mathscr{U})$ *if, for every* $\xi \in H^s_p(u, v)$ $F(\xi)$ *is a solution of* $(\mathscr{U})$ *with the initial condition* ξ.

We remark that there is no ambiguity in this definition since the conditions $u, u^{-1}v < v^*$ imply that $|\xi_\lambda(0)| < v^*$ and $\left|\frac{\partial\, \xi_\lambda}{\partial\, x_r}(0)\right| < v^*$.

Now the classical theorem of Cauchy-Kowalewski on the existence and uniqueness of solutions of Cauchy problems can be generalised as follows.

Theorem 1.

(i) *Given an element* ξ *in* H^s_p *with* $\left|\xi_\lambda(0)\right| < v^*$ *and* $\left|\frac{\partial\, \xi_\lambda}{\partial\, x_r}(0)\right| < v^*$*, the solution of the system* $(\mathscr{U})$ *with the initial condition* ξ *is (locally) unique if it exists;*

22 (ii) *there exists a solution mapping; and*

(iii) *when $A_\lambda(x, 0) = 0$ for all $\lambda = 1, \ldots, s$, any solution mapping is a regular map and all solution mappings are equivalent to each other.*

Assume $A_\lambda(x, 0) = 0$ for all $\lambda = 1, \ldots, s$. Then the solution mappings of the system $(\mathscr{U})$ define a germ of regular maps called the *solution germ* of the system $(\mathscr{U})$.

Theorem 2. *The solution germ of $(\mathscr{U})$ is a germ of infinite analytic maps of H^s_p into H^s_{p+1}.*

The proofs of these two theorems will be given in sections 1.7 and 1.9 respectively. Section 1.6 deals with some preliminaries required in the following sections.

1.6

We will adopt the following notations in the course of the proofs of the above two theorems and the preliminary propositions:

$$x = (x_1, \ldots, s_{p+1}), y = (y_1, \ldots, y_s).$$

Let $(r, r', \ldots, r_i, \ldots)$ denote integers running independently between 1 and p and $(\lambda, \lambda', \ldots, \lambda_i, \ldots)$ and $(\mu, \mu', \ldots, \mu_i, \ldots)$ denote integers running independently between 1 and s. β denotes an ordered set of integers $(r_1, \ldots, r_h)$ where, by the above convention $1 \leq r_i \leq p$ and $h = |\beta|$ denotes the length of β. k or k_i denote integers $0, 1, 2, \ldots$. When we consider a finite set of β's we often use the notation $\beta(1), \ldots, \beta(h)$. We include the case where β is the empty set $(\beta = \phi)$ and in this case $|\beta| = 0$. For a function $g = g(x_1, \ldots, x_{p+1})$, sufficiently differentiable, we set 23

$$\partial^\beta g = \frac{\partial^{|\beta|} g}{\partial x_{r_1} \cdots \partial x_{r_h}} \qquad \text{if } \beta = (r_1, \ldots, r_h) \neq \phi$$
$$= g \qquad \text{if } \beta = \phi.$$

Let us introduce a set of indeterminates $w_k^{\beta k_1\cdots k_s}$ and $w_k^{\beta k_1\cdots k_s;r}\lambda\mu$. Let P denote the ring of polynomials in these indeterminates over the field of rational numbers. A polynomial Φ in P is said to be positive whenever all its coefficients are non-negative. For any system of functions

$$C = (\cdots, C_\lambda(x,y), \ldots, C^r_{\lambda\mu}(x,y), \ldots),$$

sufficiently differentiable and for any Φ in P we denote by ΦC the function, in (x, y), obtained in substituting

$$w_k^{\beta k_1\cdots k_s;r}\lambda = \frac{\partial^{k_1}}{\partial y_1^{k_1}}\cdots\frac{\partial^{k_s}}{\partial y_s^{k_s}}\,\frac{\partial^k}{\partial\, x_{p+1}^k}\,(\partial^\beta C_\lambda),$$
$$w_k^{\beta k_1\cdots k_s;r}\lambda\mu = \frac{\partial^{k_1}}{\partial y_1^{k_1}}\cdots\frac{\partial^{k_s}}{\partial y_s^{k_s}}\,\frac{\partial^k}{\partial\, x_{p+1}^k}\,(\partial^\beta C^r_\lambda\mu) \text{ in } \Phi.$$

First we consider the following special case of the system of equations $(\mathscr{U})$ where $A_\lambda = B_\lambda(x,y) + B^r_{\lambda\mu}(x,y)\dfrac{\partial\, y_\mu}{\partial\, x_r}$. Consider the system of equations

$$(\mathscr{L}) \qquad \frac{\partial\, y_\lambda}{\partial\, x_{p+1}} = B_\lambda B(x,y) + B^r_{\lambda\mu}(x,y)\frac{\partial\, y_\mu}{\partial\, x_r}$$

24 where the usual tensor summation convention is used. Let $B = (\cdots, B_\lambda$ $(x,y), \ldots, B^r_{\lambda\mu}(x,y), \ldots)$. The proof of the theorem is given in the following lemmas, in this special case. Let 1 denote an integer ≥ 1.

Definition. *Given the equation $(\mathscr{L})$, the property $\{1,\beta\}$ is said to hold if the following condition is satisfied: there exist positive polynomials ${}^1\Phi^{\beta\mu_1\cdots\mu_h}_{\lambda\beta(1)\cdots\beta(h)}$ in P depending only on p, s and the indicated indices (for any $|\beta(1)| + \cdots + |\beta(h)| \leq |\beta| + 1$) such that the following equation holds:*

$$(\mathscr{L})^{1,\beta}\frac{\partial^1(\partial^\beta y_\lambda)}{\partial\, x^1_{p+1}} = \left({}^1\Phi^{\beta\mu_1\cdots\mu_h}_{\beta(1)\cdots\beta(h)}\, B\right)\left(\partial^{\beta(1)}y_{\mu_1}\right)\cdots\left(\partial^{\beta(h)}y_\mu\right)$$

where the usual summation convention is used. ($h = 0$ is also included).

Lemma 1. *Given the equation $(\mathscr{L})$, the property $\{1,\beta\}$ holds. Moreover ${}^1\Phi^{\beta\mu_1\cdots\mu_k}_{\beta(1)\cdots\beta(k)}$ depends only on p and s.*

Proof. The proof follows by induction argument in two steps.

(i) As for $\{1, \phi\}$ is concerned ${}^1\Phi^{\phi\mu_1\cdots\mu_h}_{\beta(1)\cdots\beta(h)} = 0$ unless $h = 0, 1$. Also ${}^1\Phi^{\phi}_{\lambda} = w_o^{\beta 0\cdots 0;\lambda}$ and when $\beta = \{r\}, {}^1\Phi^{\phi\mu}_{\lambda\beta} = w_o^{\beta o,\ldots,o;r}\lambda\mu$. Therefore $(\mathscr{L})^{1,\phi}$ is nothing but the system $(\mathscr{L})$. Then we proceed by induction on $|\beta|$ to prove $\{1, \beta\}$. by formally differentiating both sides of $(\mathscr{L})^{1,\beta'}$ with respect to x_r where β' is chosen such that $\dfrac{\partial}{\partial\, x_r}(\partial^{\beta'} y) = \partial^{\beta}\, y$.

(ii) Assuming the assertion of the lemma to have been proved for $\{1', \beta\}$ for any β and all $1' < 1$ we can show that $\{1, \beta\}$ holds on 25
the same lines as above.

□

Lemma 2. *Whenever there exists a solution of the system of equations $(\mathscr{L})$ exists, the solution is unique for a given initial condition.*

Proof. Let $\xi \in H^s_p$ and $\eta = (\eta_\lambda)$ be a solution of $(\mathscr{L})$ with $\eta_\lambda(x_1, \ldots, x_p, 0) = \xi_\lambda(x_1, \ldots, x_p)$. By Lemma 1

$$\{1, \phi\}\frac{\partial^1\, y_\lambda}{\partial\, x^1_{p+1}} = \left({}^1\Phi^{\phi\mu_1\cdots\mu_h}_{\beta(1)\cdots\beta(h)}\, B\right)\,\left(\partial^{\beta(1)}\, y_{\mu_1}\right)\cdots$$
$$\cdots(\partial^{\beta(h)} y_{\mu_h})\ (|\beta(1)| + \cdots + |\beta(h)| \leq 1)$$

is a consequence of the system of equations $(\mathscr{L})$. On the other hand $\partial^{\beta(k)}$ $(k = 1, \ldots, h)$ contain only partial derivatives $\dfrac{\partial}{\partial x_r}$ $(r = 1, \ldots, p)$. Hence if $\xi = (\xi_1(x), \ldots, \xi_s(x))$ is the given initial condition $(\xi \in H^s_p)$, the solution $\eta = (\eta_\lambda)$ must be of the form

$$\eta_\lambda = \xi_\lambda(x) + \sum_{1=1}^{\infty} \frac{(x_{p+1})^1}{1\,!}\,\left({}^1B^{\mu_1\cdots\mu_h}_{\lambda\beta(1)\cdots\beta(h)}\,(x, \xi)\right)$$
$$(\partial^{\beta(1)}\,\xi_{\mu_1})\cdots(\partial^{\beta(h)}\xi_{\mu_h})\ (|\beta(1)| + \cdots + |\beta(h)| \leq 1)$$

where $\quad {}^1B^{\mu_1\cdots\mu_h}_{\lambda\beta(1)\cdots\beta(h)}\,(x, y) = \left({}^1\Phi^{\phi\mu_1\cdots\mu_h}_{\lambda\beta(1)\cdots\beta(h)}.B\right)(x_1, \ldots, x_p, 0, y_1, \ldots, y_s)$

□

The right hand side of the above expansion being unique for a given
26 $\xi \in H^s_p$, it follows that the solution is unique whenever there exists one. *q.e.d.*

Let us denote the formal power series representing η_λ by $T^{\mathscr{L}}_\lambda(\xi)$. It only remains to prove that $T^{\mathscr{L}}_\lambda(\xi)$ is convergent in a neighbourhood of $x_{p+1} = 0$. For this purpose we introduce the following notation.

Let $f(v_1, \dots, v_h)$ and $g(v_1, \dots, v_h)$ be formal power series in $(v_1, \dots, v_h)$. A power series g is said to be positive if each coefficient of g is ≥ 0. A power series f is said to be majorized by a positive power series g if the absolute value of each coefficient of f is majorized by the corresponding coefficient of g. This we denote by $f \alpha g$.

Now consider the system of linear differential equations

$$(\mathscr{L}) \qquad \frac{\partial y_\lambda}{\partial x_{p+1}} = C_\lambda\,(x,y) + \sum_{\mu,r} C^r_{\lambda\mu}\,(x,y) \frac{\partial y_\mu}{\partial x_r}$$

where $C_\lambda\ (x,y), C_{\lambda,\mu}^{\ r}\,(x,y)$ are positive. Assume that $B_\lambda \alpha C_\lambda$ and $B^r_{\lambda\mu}$ $\alpha C^r_{\lambda\mu}$. Let $\xi, \zeta \in H^s_p$ with ζ_λ positive for all $\lambda = 1, \dots, s$. In $\{\ell, \phi\}$,$^\ell$ $\Phi^{\phi\mu_1\cdots\mu_h}_{\lambda\beta(1)\cdots\beta(h)}$ are all positive and are independent of $\beta_\mu, B^r_{\lambda\mu}, C_\mu, C^r_{\lambda\mu}$. Hence it follows from lemmas (1) and (2) that, if $\xi_\lambda \alpha \zeta_\lambda$ $(\lambda = 1, \dots, s)$, $T^{\mathscr{L}}_\lambda(\zeta)$ is positive and $T^{\mathscr{L}}_\lambda(\xi) \alpha T^{\mathscr{L}}_\lambda(\zeta)$. Therefore it is sufficient to prove the convergence of $T^{\mathscr{L}}_\lambda(\zeta)$.

Lemma 3. *There exists a solution of the system* ($\mathscr{L}$) *when the initial condition* $\xi \in H^s_p$ *is given.*

27 *Proof.* First we shall construct a system $\mathscr{L}$ satisfying the above requirements. Assume that

$$B_\lambda, B^r_{\lambda\mu} \alpha \left[b \Big/ 1 - \frac{1}{a} \left(\frac{^x p + 1}{z} + x_1 + \cdots + x_p + y_1 + \cdots + y_s \right) \right]$$

for any $0 < z < 1$. we prove that the system ($\mathscr{L}(a, b, z)$)

$$\frac{\partial y_\lambda}{\partial x_{p+1}} = \left[b \Big/ 1 - \frac{1}{a} \left(\frac{^x p + 1}{z} + x_1 + \cdots + x_p + y_1 + \cdots + y_s \right) \right]$$

$$\left(1 + \sum_{\mu,r} \frac{\partial y_\mu}{\partial x_r} \right)$$

has a solution $f_{a,b,z}(x)$ with positive coefficients. Then it follows that $\left[T_\lambda^{\mathscr{L}}\right](x)\alpha f_{a,b,z}(x)$. Now we prove that $f_{a,b,z}(x)$ exists for $\dfrac{1}{z} >$ bsp as follows: Consider a function $h(t)$ of the variable t such that $y_\lambda = h(x_1 + \cdots + x_p + \frac{x_{p+1}}{z})$ is a solution of $(\mathscr{L}(a,b,z))$. This is possible if and only if

$$\frac{1}{z}\left(\frac{dh}{dt}\right) = \left[b\Big/1 - \frac{1}{a}(t+sh)\right]\left[1 + sp\frac{dh}{dt}\right]$$

or equivalently if and only if

$$\left(\frac{1}{z} - \frac{bsp}{1 - \frac{t+sh}{a}}\right)\frac{dh}{dt} - \frac{b}{1 - \frac{t+sh}{a}}.$$

□

This equation takes the form $\dfrac{dh}{dt} = F(h,t)$ where $F(h,t)$ is a positive convergent power series in h,t, if $\dfrac{1}{z} > bsp$. But it is known that there exists a positive solution $h(t)$ and so $(\mathscr{L}(a,b,z))$ has a positive solution if $\dfrac{1}{z} > bsp$. This shows that $(\mathscr{L})$ has a solution with the initial condition 0. Now, η is a solution of $(\mathscr{L})$ with the initial condition ξ if and only if $(w_\lambda) = (\eta_\lambda - \xi_\lambda)$ is a solution of the system

$$(\mathscr{L})^\xi \qquad \frac{\partial w_\lambda}{\partial x_{p+1}} = B_\lambda^\xi(x,w) + B_{\lambda\mu}^{\xi r}(x,w)\frac{\partial w_\lambda}{\partial x_r}$$

with the initial condition 0, where 28

$$B_\lambda^\xi(x,w) = B_\lambda(x, w+\xi(x)) + B_{\lambda\mu}^r(x, w+\xi(x))\frac{\partial \xi_\mu}{\partial x_r},$$
$$B_{\lambda\mu}^{\xi r}(x,w) = B_{\lambda\mu}^r(x, w+\xi(x))$$

Therefore, $(\mathscr{L})$ has a solution with the initial condition ξ.

Lemma 4. *There exists a solution mapping for the system of equations $(\mathscr{L})$.*

Proof. Let $B_\lambda(x, y), B^r_{\lambda\mu}(x, y)$ be defined for $|x_k|, |x_{p+1}|, |y_\sigma| < v^*$ ($k = 1, \ldots, p; \sigma = 1, \ldots, s$) and let $|B_\lambda|, |B^r_{\lambda\mu}| < v' - \varepsilon$ on this domain. Then, if $\xi \in H^s_p(v^*, v^*/2)$, we have that $B^\xi_\lambda(x, y), B^{\xi r}_{\lambda\mu}(x, y)$ are defined for $|x_k|$, $|x_{p+1}|$, $|y_\sigma| < v^*/2$ and on this domain $|B^\xi_\lambda|, |B^{\xi r}_{\lambda\mu}| < (1 + sp)v' - \varepsilon$ (since $\frac{\partial\ \xi_\lambda}{\partial\ x_r} \in H_p(\frac{v^*}{2}, 1)$). Hence for large $b > 0$ and small $a > 0$

$$B^\xi_\lambda, B^{\xi\ r}_{\lambda\mu} \alpha b \Big/ \Big[1 - \frac{1}{a}(2\ spb\ x_{p+1} + x_1 + \cdots + x_p + y_1 + \cdots + y_s)\Big]$$

whenever $\xi \in H^s_p(v^*, v^*/2)$. Hence, by Lemma 3, $T^{(\mathscr{L})}_\lambda(\xi)$ converges for any $\xi \in H^s_p(v^*, v^*/2)$ and is majorized by $f_{a,b,}\dfrac{1}{2spb}$. Thus there exists a solution mapping T of $H^s_p(v^*, v^*/2)$ into $H^s_{p+1}(u', v')$ for suitable u', v'. □

If T is any solution mapping of $H^s_p(u, v)$ into $H^s_{p+1}(u', v')$ for the system $(\mathscr{L})$ then $T_\lambda(\xi)$ must have the expression $T^{\mathscr{L}}_\lambda(\xi)$ in Lemma 2 and
29 hence T must be regular. It is also clear that the solution mappings are equivalent to each other. Thus the Lemmas 1, 2, 3, 4 together complete the proof of Theorem 1 in our special case of the system $(\mathscr{L})$.

1.7

To simplify the expressions we adopt the following notation. If in $\beta = (r_1 \cdots r_h)$ the integers $1, \ldots, p$ occur $a_1, \ldots, a_p$ times respectively we shall denote by C^β the constant $\dfrac{1}{a'_1 \cdots a'_p}$. Then, if $\xi \in H_p(u, v)$, we have $|C^\beta\ \partial^\beta\ \xi(0)| < \left(\dfrac{u}{2}\right)^{-|\beta|} v$. We rewrite the expansion of $T_\lambda(\xi)$ in the form:

$$T_\lambda(\xi) = \sum_{n=o}^{\infty} x^n_{p+1}\ {}^nB^{\mu_1\cdots\mu_h}_{\lambda\beta(1)\cdots\beta(h)}(x, \xi)\left(c^{\beta(1)}\partial^{\beta(1)}\xi_{\mu_1}\right)$$
$$\cdots\left(c^{\beta(h)}\ \xi_{\mu_h}\right)\ (|\beta(1)| + \cdots + |\beta(h)| \leq n).$$

The new ${}^nB^{\mu_1\cdots\mu_h}_{\lambda\beta(1)\cdots\beta(h)}$ differs from the previous one by a constant factor.

Lemma 5. *The solution mapping T of $H_p^s(u,v)$ into $H_{p+1}^s(u',v')$, for the system $(\mathscr{L})$, defined by $\xi \rightarrow (T_1(\xi),\dots,T_s(\xi))$ is an infinite analytic map, provided $T_\lambda(0)=0$.*

Proof. (i) Given $u,v>0$ there exist $\tilde{u},v_1>0,u_1\geq 1$ with the following property: For any $|x_r^o|<\tilde{u}$ and $|y_\lambda^\beta|<v_1(u_1)^{-|\beta|}$. There is a ξ in $H_p^s(u,v)$ such that $(c^\beta\partial^\beta\xi_\lambda)(x^o)=y_\lambda^\beta$. For, given x_r^o,y_λ^β, set $\xi_\lambda=\sum\limits_\beta y_\lambda^\beta(x_{r_1}-x_{r_1}^o)\cdots(x_{r_h}-x_{r_h}^o),(\beta=(r_1,\dots,r_h))$. Then

$$|\xi_\lambda|_u\leq\sum v_1\frac{1}{u_1^{|\beta|}}(u+\tilde{u})^{|\beta|}=v_1\sum_\beta t^{|\beta|}$$

where $t=\dfrac{u+\tilde{u}}{u_1}$. Choose $\tilde{u},v_1$ sufficiently small and u_1 sufficiently 30
large so that $v_1\sum\limits_\beta t^{|\beta|}<v$. Then $\xi\in H_p(u,v)$ and $(c^\beta\partial^\beta\xi_\lambda)(x^o)=y_\lambda^\beta$.

(ii) For any $\xi\in H_p^s(u,v)$ we have, by Cauchy's formula, that

$$\left|{}^nB_{\lambda\beta(1)\cdots\beta(h)}{}^{\mu_1\cdots\mu_h}(x,\xi)(c^{\beta(1)}\partial^{\beta(1)}\xi_{\mu_1})\cdots(c^{\beta(h)}\,\partial^{\beta(h)}\xi_{\mu_h})\right|<(\frac{u'}{2})^{-n}\,v'.$$

Therefore, by (i), $\left|{}^nB^{\mu_1\cdots\mu_h}_{\lambda\beta(1)\cdots\beta(h)}(x,y)y_{\mu_1}^{\beta(1)}\cdots y_{\mu_h}^{\beta(h)}\right|<(\dfrac{u'}{2})^{-n}\,v'$ whenever $|x_r|<\tilde{u},|y_\mu^\beta|<\dfrac{v_1}{u_1|y_\mu^\beta|}$ $(y_\mu^\phi=y_\mu)$.

Hence, for $|x_r|<\tilde{u},|y_\mu^\beta|<v_1'/\sigma|\beta|$ for any v_1' $\sigma>0$ we have

$$\left|{}^nB^{\mu_1\cdots\mu_h}_{\lambda\beta(1)\cdots\beta(h)}(x,y)\prod_{j=1}^h\left(\frac{v_1}{v_1'}\left(\frac{\sigma}{u_1}\right)^{|\beta(j)|}y_{\mu_j}^{\beta(j)}\right)\right|<\left(\frac{u'}{2}\right)^{-n}\,v'.$$

More explicitly,

$$\Big|\sum_{k=0}^n\sum_{|\beta(1)|+\cdots+|\beta(h)|=k}\left(\frac{\sigma}{u_1}\right)^k\left(\frac{v_1}{v_1'}\right)^h\,{}^nB^{\mu_1\cdots\mu_h}_{\lambda\beta(1)\cdots\beta(h)}$$

$$(x,y)y_{\mu_1}^{\beta(1)}\cdots y_{\mu_h}^{\beta(h)}\Big| < \left(\frac{u'}{2}\right)^{-n} v'$$

That is

$$\left|\sum_{|\beta(1)|+\cdots+|\beta(h)|=k} {}^{n}B^{\mu_1\cdots\mu_h}_{\lambda\beta(1)\cdots\beta(h)}(x,y)\, y_{\mu_1}^{\beta(1)}\cdots y_{\mu_h}^{\beta(h)}\right| < \left(\frac{u'}{2}\right)^{-n} v' \left(\frac{u_1}{\sigma}\right)^{k}\left(\frac{v_1'}{v_1}\right)^{h}.$$

(iii) Let $0 < t < 1$ be fixed and let $0 < a < \tilde{u}$; then if $\xi \in H_p^s(a, v_1')$, $|c^\beta \partial^\beta \xi_\mu| < (ta)^{-|\beta|} v_1'$ in a polydisc of radius $(1-t)a$, by Cauchy's formula. Therefore

$$\begin{aligned} |T_\lambda(\xi)|_{(1-t)a} &\leq \sum_{n=0}^{\infty}(1-t)^n a^n \sum_{k=o}^{n}\sum_{n}\left(\frac{u_1}{ta}\right)^{k}\left(\frac{u'}{2}\right)^{-n} v' \left(\frac{v_1'}{v_1}\right)^{h} \\ &\leq v' \sum_{n=o}^{\infty}(n+1)\left(\frac{1-t}{t}\,\frac{u_1}{u'}.2\right)^{n}\sum_{h}\left(\frac{v_1'}{v_1}\right)^{h} \text{ if } a<1 \end{aligned}$$

31 Choose v_1' such that $\frac{v_1'}{v_1} < 1$ and t such that $\frac{1-t}{t}\,\frac{u_1}{u'} < \frac{1}{4}$. Then we obtain the majorization $|T_\lambda(\xi)|_{(1-t)a} < k\Big/\left(1-\frac{v_1'}{v_1}\right)$ where K is a constant > 0.

Thus the proof of Theorem 2 is completed in the special case of the system of equations ($\mathscr{L}$). □

1.8

We shall now give the proof of the Theorems 1 and 2 in the general case of the system of equations ($\mathscr{U}$).

Proof of Theorem 1. The system of equations ($\mathscr{U}$) being given we consider the following system of equations ($\mathscr{U}'$) with the unknown functions $y_1,\ldots,y_s,\ldots,y_\mu^r,\ldots$;

$$(\mathscr{U}')\begin{cases}\frac{\partial y_\lambda}{\partial x_{p+1}} = A_\lambda(x, y, \ldots, y^r_\mu, \ldots) \quad (\lambda = 1, \ldots, s)\\ \frac{\partial y^r_\lambda}{\partial x_{p+1}} = \frac{\partial A_\lambda}{\partial x_r}(x, y, \ldots, y^r_\mu, \ldots) + \frac{\partial A_\lambda}{\partial y_\mu}(x, y, \ldots, y^r_\mu, \ldots)\frac{\partial y_\mu}{\partial x_r}\end{cases}$$
$$+ \frac{\partial A_\lambda}{\partial y^{r'}_\mu}(x, y, \ldots, y^r_\mu, \ldots)\frac{\partial y^{r'}_\mu}{\partial x_r}\begin{pmatrix}\lambda = 1, \ldots, s;\\ r = 1, \ldots, p\end{pmatrix}$$

where A_λ are analytic functions of all their arguments in the domain $|x_k|, |y_\mu|, |y^r_\mu| < v^*(k = 1, \ldots p+1; \mu = 1, \ldots, s; r = 1, \ldots, p)$ and the second system is obtained by formally differentiating both the members of the system of equations $(\mathscr{U})$ with respect to the variables $x_r (r = 1, \ldots, p)$. Hence we easily see that $y_\lambda = \eta_\lambda(x)$ is a solution of $(\mathscr{U})$ **32**
with the initial condition ξ if and only if $y_\lambda = \eta_\lambda(x), y^r_\lambda = \dfrac{\partial \eta_\lambda}{\partial x_r}$ is a solution of $(\mathscr{U}')$ with the initial condition $\left(\xi, \ldots, \dfrac{\partial \xi_\mu}{\partial x_r}, \ldots\right)$. Because the system $(\mathscr{U}')$ is of the special type of system $(\mathscr{L})$ considered, it follows therefore that there exists a unique solution of the system of equations $(\mathscr{U})$ with a given initial condition ξ from lemmas 2 and 3.

Further, if ξ is in $H^s_p(u, v)$ then clearly $\left(\xi, \ldots, \dfrac{\partial \xi_\mu}{\partial x_r}, \ldots\right)$ is in $H^{s+ps}_p(u/2, 2v/u)$. There exists, by lemma 4, a solution mapping T^* of $H^{s+ps}_p(u/2, 2v/u)$ into $H^{s+ps}_{p+1}(u', v')$ for the system of equations $(\mathscr{U}')^p$. On the other hand the mapping F of $H^s_p(u, v)$ into $H^{s+ps}_p(u/2, 2v/u)$ defined by $F(\xi) = \left(\xi, \ldots, \dfrac{\partial \xi_\mu}{\partial x_r}, \ldots\right)$ is clearly infinite analytic. (See Ex. 1 of § 1.2). Similarly the projection mapping P defined by $p(\eta, \ldots, \eta^r_\mu, \ldots) = \eta$ of $H^{s+ps}_{p+1}(u', v')$ onto $H^s_{p+1}(u', v')$ is also infinite analytic. Therefore the composite mapping $T = p \circ T^* \circ F$ is an infinite analytic map since T^* is so by Lemma 5. Moreover T is a solution mapping because of the remark made in the beginning of this section. This complete the proof of Theorem 1 and 2.

1.9

In this section we consider the discussion of the case of a system of differential equations involving certain parameters. We prove in this case that the solution depends infinite analytically on the initial condition and the parameters.

Here in addition to the usual notations, $z = (z_1, \ldots, z_q)$ denotes the set of parameters. Let $A_\lambda(x, y, \ldots, y^r_\mu, \ldots, z)$ be a system of functions de-
33 fined and analytic (in all their arguments) in the domain $|x_k|, |y_\lambda|, |y^r_\lambda|, |z_\sigma| < v^*_1$ $(k = 1, \ldots p+1; r = 1, \ldots, p; \lambda = 1, \ldots s; \sigma = 1, \ldots, q)$. Given a $\zeta \in H^q_{p+1}$ with $|\zeta(0)| < v^*_1$ consider the system of partial differential equations

$$(\mathscr{U}_\zeta) \qquad \frac{\partial y_\lambda}{\partial x_{p+1}} = A_\lambda\left(x, y, \ldots, \frac{y_\mu}{\partial x_r}, \ldots \zeta(x)\right).$$

Definition. *A mapping T of the direct sum $(H^s_p + H^q_{p+1})(u, v)$ into H^{q+s}_{p+1} (u', v') (with $u^{-1}v, v < v^*_1$) is said to be a solution mapping of the system of partial differential equations $(\mathscr{U}_\zeta)$ with parameters if $y_\lambda = T_\lambda(\xi, \zeta)$ is a solution of $(\mathscr{U}_\zeta)$ with the initial condition ξ, for any (ξ, ξ) in $(H^s_p + H^q_{p+1})(u, v)$.*

Theorem 3. *For any given system of partial differential equations $(\mathscr{U}_\zeta)$ with parameters of Cauchy-Kowalewski type there exists a solution mapping. If moreover $A_\lambda(x, 0, 0) = 0$ then the solution mappings are regular and equivalent to each other.*

The additional condition $A_\lambda(x, 0, 0) = 0$ is imposed to ensure that the image by any solution mapping of 0 in $(H^s_p + H^q_{p+1})(u, v)$ is the 0 in $H^{q+s}_{p+1}(u', v')$. The germ of solution mappings in this case is called the solution germ of the system with parameters.

Theorem 4. *The solution germ of a system of partial differential equations of Cauchy-Kowalewski type with parameters is germ of infinite analytic maps.*

Proof of Theorem 2 and 3 Given the system of equations $(\mathscr{U}_\zeta)$ we construct a new system of equations $(\mathscr{U}'')$ by introducing another new variable t, with the unknown functions $y_1, \ldots, y_s, z_1, \ldots, z_q$.

$$(\mathscr{U}'') \begin{cases} \frac{\partial y_\lambda}{\partial x_{p+1}} = A_\lambda(x_1, \ldots, x_p, t, t_{p+1}, y_1, \ldots, y_s, \ldots, \frac{\partial y_\mu}{\partial x_r}, \ldots, z_1, \ldots, z_q) \\ \frac{\partial z_\sigma}{\partial x_{p+1}} = \frac{\partial z_\sigma}{\partial t} \qquad (\lambda = 1, \ldots, s) \\ \qquad\qquad (\sigma = 1, \ldots, q) \end{cases}$$

34

where $A_\lambda \left(x_1, \ldots, x_p, t, x_{p+1}, y_1, \ldots, y_s, \ldots, \frac{\partial y_\mu}{\partial x_r}, \ldots, z_1, \ldots, z_q\right)$

$$= A_\lambda(x_1, \ldots, x_p, x_{p+1}, y_1, \ldots, y_s, \ldots, \frac{\partial y_\mu}{\partial x_r}, \ldots, z_1, \ldots, z_q)$$

$(\mathscr{U}'')$ is a system of Cauchy-Kowalewski type treated in Theorems 1 and 2. Now if $y_\lambda = \eta_\lambda(x_1, \ldots, x_p, t, x_{p+1}), z_\sigma = \mu_\sigma(x_1, \ldots, x_p, t, x_{p+1})$ is a solution of the system $(\mathscr{U}'')$ then $\mu_\sigma(x_1, \ldots, x_p, t, x_{p+1})$ must be equal to $\zeta_\sigma(x_1, \ldots, x_p, t + x_{p+1})$ where $\zeta_\sigma(x_1, \ldots, x_p, t) = \mu_p(x_1, \ldots, x_p, t, 0)$, because $\dfrac{\partial z_\sigma}{\partial x_{p+1}} = \dfrac{\partial z_\sigma}{\partial t}$ is of Cauchy-Kowalewski type and so the solution must be unique for the initial condition. Therefore $y_\lambda = \eta_\lambda(x_1, \ldots, x_p, 0, x_{p+1})$ is a solution of the system $(\mathscr{U}_\zeta)$. Then the proof can be carried out by an argument similar to the one given when we deduced the theorems 1 and 2 from the special case $(\mathscr{L})$.

Remark. *Let F be a solution mapping of $(\mathscr{U}'')$. Then the mapping $\xi \to F(\xi) - F(0)$ is infinite analytic. This follows from the fact that $F - F(0)$ is a solution mapping of a system of partial differential equations of the same type as $(\mathscr{U}'')$.*

Next we shall briefly mention the case of a system of partial differential equations in which the derivatives of the parameters also occur. Given a $\zeta \in H^q_{p+1}(u, v)$ consider the system of equations

$$(\mathscr{U}'_\zeta) \qquad \frac{\partial y_\lambda}{\partial x_{p+1}} = A_\lambda \left(x, y, \ldots, \frac{\partial y_\mu}{\partial x_r}, \ldots, z_1, \ldots, z_q, \ldots, \frac{\partial z_\sigma}{\partial x_r}, \ldots\right)$$
$$(\lambda = 1, 2, \ldots, s),$$

where $A_\lambda(x, y, \ldots, y^r_\mu, \ldots, z_1, \ldots, z_q, \ldots, z^r_\sigma, \ldots)$ are functions defined **35**
and analytic in $|x_k|, |y_\mu|, |z_\sigma|, |y^r_\mu|, |z^r_\sigma| < v^*_1$ ($k = 1, \ldots, p+1, r = 1, \ldots, p$, $\mu = 1, \ldots, s, \sigma = 1, \ldots, q$). In this case also it is not difficult to give definitions of a solution mapping and a solution germ.

Definition. *A mapping T of the direct sum $(H^s_p + H^q_{p+1})(u, v)$ (with $u^{-1}v$, $v < v^*_1$) into $H^{q+s}_{p+1}(u', v')$ is called a solution mapping of the system $(\mathscr{U}'_\zeta)$ if $y_\lambda = T_\lambda(\xi, \zeta)$ is a solution of $(\mathscr{U}'_\zeta)$, for every (ξ, ζ) in $(H^s_p + H^q_{p+1})(u, v)$, with the initial condition ξ.*

Theorem 5. *For any system $(\mathscr{U}'_\zeta)$ of Cauchy-Kowalewski type there exists a solution mapping. Moreover if $A_\lambda(x, 0, 0) = 0$, then the solution mappings are regular and equivalent to each other.*

The germ of solution mappings is called the solution germ of the given system $(\mathscr{U}'_\zeta)$.

Theorem 6. *The solution germ of $(\mathscr{U}'_\zeta)$ is infinite analytic.*

Proof. First, consider (z_σ, z^r_σ) as independent parameters. Then restrict the parameter to $z^r_\sigma = \partial z_\sigma / \partial x_r$. □

1.10 Differentials of regular maps

Let S and S' be two systems of characters. Then, if F is a regular map of $H(S; u, v)$ into $H(S', u', v')$, for any ξ, η in $H(S; u, v/2)$ the mapping of $K(1 + \varepsilon)$ into $H(S; u, v)$ defined by $t \to \xi + t\eta$ is a regular curve in
36 $H(S; u, v)$. We pose the following definition.

Definition. *The differential dF of a regular map F of $H(S; u, v)$ into $H(S'; u', v')$ is defined to be the mapping of the direct sum $H\left(S; u, \dfrac{v}{2}\right) \oplus H\left(S; u, \dfrac{v}{2}\right)$ into $H(S'; u', v')$ by the formula*

$$dF_\lambda(\xi, \eta) = \left[\frac{\partial}{\partial t} F_\lambda(\xi + t\eta)\right]_{t=0}$$

Remarks. *Let $S = (s_0 \dots, s_p)$ be a system of characters. Set $S'' = (2s_0, \dots, 2s_p)$; then dF can be identified with a mapping of $H(S''; u, v)$ into $H(S'; u', v')$. It can be seen by direct verification that the following are immediate consequences of the above definition.*

(1) *For any regular map F, its differential is also a regular map.*

(2) *For any two regular maps F_1 and F_2 with $F_1 \sim F_2, dF_1 \sim dF_2$. This implies that one can define the differential of a germ $\mathscr{F}$ of regular maps of $H(S)$ into $H(S')$ to be germ containing the differential of any representative of $\mathscr{F}$. The differential of a germ $\mathscr{F}$ of regular maps is denoted by $d\mathscr{F}$. This definition is unambiguous.*

(3) *The differential dF of any infinite analytic map F of $H(S)$ into $H(S')$ is itself infinite analytic.*

The differential $d\mathscr{F}$ of any infinite analytic germ $\mathscr{F}$ is an infinite analytic germ.

Let F be any regular mapping of $H(S; u, v)$ into $H(S'; u', v')$, then 37
$dF_\lambda\,(0, \xi) = \left[\dfrac{\partial}{\partial t} F_\lambda\,((t, \xi)\right]_{t=0}$ and this is denoted by $(dF)_{o,\lambda}(\xi)$. Define $(dF)_0 = ((dF)_{o,\lambda}); (dF)_o$ is called the differential of F at the origin.

For any regular map F of $H(S; u, v)$ into $H(S'; u', v')$, it is easy to see that $(dF)_0$ is a linear map of $H(S; u, v)$ into $H(S'; u', v')$.

Now if F is a regular map of $H(S; u, v)$ into $H(S'; u', v')$, since the mapping of $K(1+\varepsilon)$ into $H(S; u, v)$ defined by $t \to t\,\xi$ is a regular curve, the mapping of $K(1+\varepsilon)$ into $H(S'; u', v')$ defined by $t \to F(t\,\xi)$ is a regular curve in $H(S'; u', v')$. Hence $F_\lambda(t\,\xi)$ has an expansion in powers of t in a neighbourhood of $t = 0$ as follows:

$$\begin{aligned} F_\lambda(t\xi) &= \left[\frac{\partial}{\partial t} F_\lambda\,(t\xi)\right]_{t=0} t + \text{ higher powers of } t \\ &= (dF)_{o,\lambda}\,(\xi)\, t + \text{ higher powers of } t. \end{aligned}$$

Now supposing G is a regular map of $H(S'; u', v')$ into $H(S''; u'', v'')$ then we have

$$(GoF)_\lambda\,(t\,\xi) = d(GoF)_{o,\lambda}\,(\xi)\, t + \text{ higher powers of } t.$$

But on the other hand

$$
\begin{aligned}
(GoF)_\lambda\ (t\,\xi) &= G_\lambda\Big[t\ (dF)_{o,\mu}\ (\xi) + \text{ higher powers of } t\Big]\\
&= G_\lambda\Big[t\ (dF)_{o,\mu}\ (\xi)\Big] + G_\lambda(\text{ higher powers of } t)\\
&= (dG)_{o,\lambda}\ o\ (dF)_{o,\lambda}\ (\xi)\ t + \text{ higher powers of } t.
\end{aligned}
$$

38 Comparing the coefficients of t in the right members of the two equalities we obtain the formula

$$d(GoF)_{o,\lambda} = (dG)_{o,\lambda}\ o\ (dF)_o$$

and hence we can write $(d(GoF))_0 = ((dG)_{o,\lambda}\ o\ (dF)_o)$.

Proposition 10. *If $\mathscr{F}$ and $\mathscr{G}$ are two germs of infinite analytic maps of $H(S)$ into $H(S')$ and $H(S')$ into $H(S)$ respectively such that $\mathscr{G}\ o\ \mathscr{F}$ and $\mathscr{F}\ o\ \mathscr{G}$ are germs of identity maps on $H(S)$ and $H(S')$ respectively then $H(S)$ and $H(S')$ have same degree and multiplicity.*

Proof. The germs $(d\mathscr{F})_o, (d\mathscr{G})_o$ are linear infinite analytic and are such that $(d(\mathscr{G}_o\ \mathscr{F}))_o = (d\mathscr{G})_o\ o\ (d\mathscr{F})_o$ and $(d(\mathscr{F}\ o\ \mathscr{G}))_o = (d\ \mathscr{F}_o\ o\ (d\mathscr{G})_o$ are germs of identity maps on $H(S)$ and $H(S')$ respectively. Hence, by Proposition 9, $H(S)$ and $H(S')$ have same degree and multiplicity. □

All our considerations above are in the case of complex analytic maps. The whole discussion can, without much difficulty, be carried out to the case of real analytic maps.

1.11 Germs of submanifolds of a manifold

Let M denote a real analytic manifold and z a point in M. Let (N, z) denote a real locally closed analytic submanifold of M passing through the point z. (N, z) is called a real analytic submanifold with center. Two real analytic submanifolds with centers (N, z) and (N, z') passing through z and z' respectively are said to be equivalent if

39 (i) $z = z'$ and

(ii) there exists a neighbourhood U of z such that $U \cap N = U \cap N'$.

This is denoted by $(N, z) \sim (N', z')$. Clearly $\sim$ is an equivalence relation.

Definition. *An equivalence class of real analytic submanifolds with centers under $\sim$ is called a germ of real analytic submanifolds.*

A germ of real analytic submanifolds of M is denoted by ϑ. An element $(N, z) \in \vartheta$ is called a representative of ϑ, z is called the origin of the germ ϑ, and ϑ is called a germ at z.

Let M and M' be two real analytic manifolds and ϖ be a real analytic mapping of M onto M'. ϖ is said to be locally trivial when, for any $z \in M, d\ \varpi$ maps the tangent vector space to M at z onto the tangent vector space to M' at $\varpi(z)$. A triple consisting of $M.M'$ and a locally trivial real analytic map ϖ of M onto M' is called of fibred manifold and is denoted (M, M', ϖ). For any point z in M, a coordinate system $(x_1, \ldots, x_n, y_1, \ldots, y_m)$ at z in M is called a coordinate system at z in the fibred manifold (M, M', ϖ) when there exists a coordinate system $(x'_1, \ldots, x'_n)$ at $\varpi(z)$ in M' such that $x_i = x'_i\ o\ \varpi$. It is clear that there exist coordinate systems in (M, M', ϖ) at any point of M.

Definition. *A germ ϑ of real analytic submanifolds at z, of M is called a germ of cross - sections of (M, M', ϖ) at z if ϑ contains a representative (N, z) such that ϖ induces a real analytic homeomorphism of N onto an* **40**
open neighbourhood of $\varpi(z)$.

ϑ is also called a germ of cross - sections of (M, M', ϖ) over $\varpi(z)$.

Again all these notions can be carried over without any change to the complex analytic case.

Let (M, M', ϖ) be a fibred manifold. Then for any fixed point $z' \in M'$, let $T(\varpi, z')$ denote the set of all germs ϑ of cross-sections of (M, M', ϖ) over z'. Let $\vartheta^0 \in T(\varpi, z')$.

A coordinate system $(x_1, \ldots, x_n, y_1, \ldots, y_m)$ around the origin of ϑ^o in (M, M', ϖ) is called a coordinate system with centre ϑ^0 if the germ ϑ^0 can be expressed by $(x_1, \ldots, x_n, 0, \ldots, 0)$. Let us fix an germ ϑ^o in $T(\varpi, z')$ and a coordinate system (x, y) in (M, M', ϖ) with centre ϑ^0. Denote by U the open set of M where (x, y) is defined and by $T(\varpi, U, z')$ the

set of germs ϑ' in $T(\varpi, z')$ with their origins in U. If ϑ is in $T(\varpi, U, z')$, ϑ can be expressed parametrically by $y_\lambda = \xi_\lambda(x_1, \ldots, x_n)$ $(\lambda = 1, \ldots, m)$, where $\xi_\lambda(x_1, \ldots, x_n)$ is a real analytic function defined in a neighbourhood of the point $x_1 = \cdots = x_n = 0$. Thus $\xi_\lambda(x_1, \ldots, x_n)$ can be regarded as a convergent power series in $x_1, \ldots, x_n$ and hence as an element of H_n. We set $\tau_\lambda(\vartheta) = \xi_\lambda \in H_n$, and $\tau(\vartheta) = (\tau_1(\vartheta), \ldots, \tau_m(\vartheta)) \in H_n^m$. The mapping τ defined by this is an injective mapping of $T(\omega, U, z')$ into H_0^m. For strictly positive real numbers u, v, $T(\omega, \vartheta^0, u, v)$ denotes the set
41 of germs ϑ in $T(\omega, U, z')$ such that $\tau(\vartheta)$ is in $H_n^m(u, v)$. For sufficiently small v, τ induces a bijective mapping of $T(\varpi, \vartheta^0, u, v)$ into $(H_n^m)^R(u, v)$. Evidently $T(\varpi, \vartheta^0, u, v)$ depends on the choice of the coordinate system chosen with centre ϑ^0. The idea behind this definition is to follow the analogy of coordinate systems in point sets for sets of germs of cross sections. Namely, the coordinate neighbourhoods and coordinate functions in the case of manifolds are replaced respectively by $T(\varpi, \vartheta^0, u, v)$ and τ respectively in the case of the set of germs of cross sections. We observe that, when M' is reduced to a point, $T(\varpi, \vartheta^0, u, v)$ and τ are respectively the coordinate neighbourhood and coordinate function in M. Because of this analogy we call τ the coordinate mapping with centre ϑ^0 induced by the coordinate system (x, y) in (M, M', ϖ).

Because we will only consider, henceforth, real analytic manifolds and real analytic mappings, we will drop the usage R in $H_n^R, (H_n^m)^R$ and $H^R(S)$ etc. Thus H_n is the vector space of convergent power series with real coefficients, and $H_n(u, v)$ is the subset of H_n, which was denoted by $H_n^R \cap H_n(u, v)$ so far. A germ of infinite analytic maps of $H(S)$ into $H(S')$ means a germ of real infinite analytic maps.

1.12

Definition. *Let σ be a set of germs of cross-sections of a fibred manifold (M, M', ϖ). Denote by $(n+m), n$ the dimensions of M, M' respectively. σ is said to depend on s functions in p variables around a fixed germs ϑ^0 of σ if there exist germs $\mathscr{F}$ and $\mathscr{G}$ of infinite analytic maps of $H(S)$ into H_n^m*
42 *and of H_n^m into $H(S)$, respectively, satisfying the following conditions:*

(i) *S has degree p and multiplicity s;*

(ii) *there exist strictly positive numbers* $\tilde{u}, \tilde{v}$ *and an integer* $\tilde{k}$ *such that for every a (with* $0 < a < \tilde{u}$*)* $\mathscr{F}$ *has a representative which maps* $H(S\,; a, a^{\tilde{k}}\tilde{v}$ *into* $\tau(T(\varpi, \vartheta^0, u', v') \cap \sigma)$ *where* τ *is the coordinate mapping induced by a suitable coordinate system in* (M, M', ϖ) *around* ϑ^0*;*

(iii) $\mathscr{G}\ o\ \mathscr{F}$ *is the germ of the identity mapping.*

Then we see easily that the germ $\mathscr{F}\ o\ \mathscr{G}$ has a representative defined on $H^m_n(a, a^\ell, v)$ and which is identity on $\tau(T(\varpi, \vartheta^0, a, a^\ell \tilde{v})) \cap \sigma)$ for sufficiently large ℓ and small a.

We remark that if the conditions (*ii*) and (*iii*) are satisfied by a coordinate system, then the same conditions hold for any arbitrary coordinate system of (M, M', ϖ) with centre ϑ^0 by suitably changing $\tilde{u}, \tilde{v}, \mathscr{F}$ and $\mathscr{G}$.

Proposition 11. *If* σ *is a set of germs of cross - sections of* (M, M', ϖ) *depending on s functions in p variables and also on* s' *functions in* p' *variables, then* $p = p'$ *and* $s = s'$.

Proof. By the above remark, we can, without loss of generality, assume that the coordinate systems in both the cases are the same. Then the germs $\mathscr{G}'\ o\ \mathscr{F}$ (resp. $\mathscr{G}\ o\ \mathscr{F}$) of $H(S)$ (resp. $H(S')$) onto $H(S')$ (resp. $H(S)$) are germs of infinite analytic maps and both $(\mathscr{G}'\ o`\mathscr{F})\ o\ (\mathscr{G}\ o\ \mathscr{F}$ and $(\mathscr{G}\ o\ \mathscr{F})\ o\ (\mathscr{G}'\ o\ \mathscr{F}')$ are germs of identity mappings. Hence, by Proposition 10, S, S' have the same degree and multiplicity. □

Chapter 2

Exterior differential systems

2.1

In this chapter, as we mentioned in the beginning of chapter I. We study 43
the construction and properties of the submanifolds of a given manifold which are integrals of a certain differential system. We make this more explicit in the following. We begin with certain notations which we employ throughout this and the following chapter.

All manifolds, submanifolds, differential forms which we consider in the following will be real analytic, so we omit the adjective real analytic. Let M be a manifold of dimension n. For any point $z \in M, (M)_z$ denotes the tangent vector space to M at z.

Let φ be a homogeneous differential form of degree h on M. φ associates to every point z an anti-symmetric h-tuple multilinear mapping φ_z on $(M)_z$. If $L^1, \ldots, L^h$ are tangent vectors in $(M)_z$ the value of the function φ_z on $(L^1, \ldots, L^h)$ is denoted by $\langle \varphi, L^1 \wedge \cdots \wedge L^h \rangle$. If φ has an expression $\varphi = \sum a^{i_1 \cdots i_h} du_{i_1} \wedge \cdots \wedge du_{i_h}$ in terms of a system of local coordinates then we have $\langle \varphi, L^1 \cdots L^h \rangle = \sum a^{i_1 \cdots i_h} \det \langle du_{i_\nu}, L_\mu \rangle$. For any subspace E of the tangent vector space $(M)_z$ of M at z, we denote by $\varphi | E$ the restriction of the function φ_z to E.

A q-dimensional subspace E of $(M)_z$ is called a q-dimensional *contact elements* at z. Let $\mathscr{G}^q_z(M)$ be the set of all contact elements of M at z. 44
Then $\mathscr{G}^q(M) = \cup \{ \mathscr{G}^q_z(M) : z \in M \}$ is the set of all q-dimensional contact

elements of M. $\mathscr{G}^q(M)$ can be provided with a structure of real analytic manifold such that each $\mathscr{G}^q_z(M)(z \in M)$, is a real analytic submanifold of $\mathscr{G}^q(M)$ and such that there is a real analytic projection mapping ρ of $\mathscr{G}^q(M)$ onto M, mapping every contact element onto its origin. Now we shall explicitly give the coordinate system in $\mathscr{G}^q(M)$.

Let $x_1, \dots, x_q$ be (real analytic) functions defined on an open set U of M such that $dx_1, \dots, dx_q$ are linearly independent at each point of U. Let $U(x_1, \dots, x_q)$ be the set of all q-dimensional contact elements E in $\mathscr{G}^q(M)$ such that $\rho(E)$, the origin of E, is in U and such that the restrictions $(dx_1|E, \dots, dx_q|E)$ are linearly independent. IF E is in $U(x_1, \dots, x_q)$ let $L^1(E), \dots, L^q(E)$ be a basis in E dual to $dx_1|E, \dots, dx_q|$ E. Clearly this choice of the dual basis depends on the choice of $(x_1, \dots,$ $x_q)$. Suppose that $(x_1, \dots, x_q; w_1, \dots, w_{n-q})$ be a coordinate system of M defined on U. Then, for every E in $U(x_1, \dots, x_q)$, $L^i(E)$ can be expressed by

$$\frac{\partial}{\partial x_i} + \sum_{\lambda=1}^{n-q} y^i_\lambda(E) \frac{\partial}{\partial w_\lambda}$$

where y^i_λ are functions defined on $U(x_1, \dots, x_q)$. The mapping which associates to every E in $U(x_1, \dots, x_q)$ the system (origin of $E, \dots, y^i_\lambda(E),$ $\dots)$ defines a coordinate system $(y_1 \, o \, \rho, \dots, y_n \, o \, \rho, \dots, y^i_\lambda(E), \dots)$ where
45 $(y_1, \dots, y_n)$ is a coordinate system on M defined in U. Thus $(y_1 \, o \, \rho, \dots,$ $y_n \, o \, \rho, \dots, y^i_\lambda(E), \dots)$ is a coordinate system in $\mathscr{G}^q(M)$. $\mathscr{G}^q(M)$ is the so called Grassman manifold.

2.2

Hereafter we consider only the domains in a Euclidean space. Let D be a domain in R^n; ler $\Lambda^k(D)$ denote the module of homogeneous exterior differential forms of degree k on D and the direct sum $\Lambda(D)$ of $\lambda^o(D), \dots, \Lambda^n(D)$, where n denote the dimension of D, is the algebra over $\Lambda^o(D)$ of homogeneous exterior differential forms on D. Here $\Lambda^0(D)$ denotes the ring of (real analytic) functions on D. The operators $\wedge$ and d always denote the exterior multiplication and the exterior derivation respectively in the algebra $\Lambda(D)$. If f is a real analytic mapping of a manifold M into another manifold M' then the inverse image on M of an exterior differential form ϕ on M' by f is denoted by $f^*\varphi$ and we have the equality $d(f^*\varphi) = f^*(d\varphi)$.

Let $(\sum)$ be a subset of $\wedge(D)$.

Definition. *A submanifold N of D is said to be in* integral submanifold *or an* integral *of* $(\sum)$ *if the restrictions of* φ *to N vanish for all* $\varphi \in (\sum)$.

Definition. *A q-dimensional contact element E of D is called an* integral element *of dimension q of the system* $(\sum)$ *if the restrictions of* φ *(in* $\sum$*) to E vanish. A 0-dimensional integral element is also sometimes called an* integral point.

The following proposition is an immediate consequence of this definition.

Proposition 1. *A submanifold N of D is an integral of the system* $(\sum)$ *of exterior differential forms if and only if for any point z in* $N, (N)_z$ *is an integral element of* $(\sum)$. 46

Now let N be a submanifold of D and let i denote the natural inclusion mapping of N into D. If $\varphi = \varphi^0 + \cdots + \varphi^n$, where $\varphi^j \in \Lambda^j(D)$, is an exterior differential form on $D, i^*\varphi$ is nothing but the restriction of φ to N. Therefore we remark that, if the restriction of φ to N also vanishes, because $i^*\varphi = 0$ implies that $i^*(d\varphi) = d(i^*\varphi) = 0$. If φ and ψ are two differential forms on D with $i^*\varphi = 0$, then $i^*(\psi \wedge \varphi) = (i^*\psi) \wedge (i^*\varphi) = 0$. If moreover $i^*\psi = 0$ then clearly $i^*(\alpha\varphi + \beta\psi) = 0$. Hence we conclude that the homogeneous ideal (closed under the operation d) generated by homogeneous parts of elements in $(\sum)$ in the exterior algebra $\Lambda(D)$ also possesses the same integrals as $(\sum)$.

Definition. *A homogeneous ideal* $(\sum)$ *in* $\Lambda(D)$ *is said to be closed if* $(d\sum) \subset (\sum)$.

Proposition 2. *If* $(\sum)$ *is a homogeneous ideal in* $\Lambda(D)$ *then the homogeneous ideal generated by* $(\sum)$ *and* $(d\sum)$ *is closed. This follows easily if we use the fact that* $d \circ d = 0$.

Definition. *A homogeneous ideal* $(\sum)$ *in the exterior algebra* $\Lambda(D)$ *is called an exterior differential system if (i)* $(\sum)$ *is closed and (ii)* $(\sum)$ *is finitely generated as an ideal.*

As remarked above, as far as the set of integrals are concerned, the
47 situation will not change when we replace a finite set $(\sum) = \{\varphi_1, \ldots, \varphi_h\}$ by the homogeneous ideal generated by $\{\varphi_1, \ldots, \varphi_h, d\varphi_1, \ldots, d\varphi_h\}$ in the algebra $\wedge(D)$. Then the ideal is closed by Proposition 2 and so hereafter we will consider only exterior differential systems instead of finite subsets in $\wedge(D)$.

Let $(\sum)$ be an exterior differential system. $(\sum)$ being a homogeneous ideal, it can be decomposed as $\sum = \sum^{(0)} + \sum^{(1)} + \cdots + \sum^{(n)}$ where $\sum^{(k)} = \sum \cap \Lambda^k(D)$. Let E be a fixed element in $\mathscr{G}^q_z(D)$. Any set of vectors $L^1, \ldots L^r$ in E and a differential form φ in $\sum^{(r+1)}$ define a linear functional α_φ on the tangent space $(D)_z$ at z, as follows:

$$\alpha_\varphi(L) = \langle \varphi, L^1 \wedge \cdots \wedge L^r \wedge L \rangle \quad (L \in (D)_z).$$

The subspace of the dual of $(D)_z$ generated by all the α_φ $\left(\varphi \in \sum^{(r+1)}; L^1, \ldots, L^r \in E; r = 0, 1, \ldots, q\right)$ is denoted by $J(E, \sum)$ (or simply by $J(E)$ when there is no possible confusion regarding $\sum$) and the dimension of $J(E, \sum)$ is denoted by $t(E, \sum)$ (or simply by $t(E)$). $J(E, \sum)$ is called the *space of polar forms* of $\sum$ at E. t can be regarded as an integral valued (not necessary real analytic) function on $(\mathscr{G}^q(D)$. The following are immediate consequences of this definition.

Proposition 3. *The subspace of the tangent space* $(D)_z$ *of* D *at* z*, spanned by an integral element* E *(subspace of* $(D)_z$*) and a tangent vector* L *is an integral element of* $(\sum)$ *if and only if* L *is a solution of the equation* $J(E) = 0$.

48 **Proposition 4.** *If* E *and* E' *are two integral elements of* $(\sum)$ *with* $E' \subset E$ *then* $J(E') \subseteq J(E)$ *and hence* $t(E') \leq t(E)$.

Definition. $J(E) = 0$ *is called the polar equation of* $\sum$ *at* E.

Definition. *Let* F *be a set of (real analytic) functions defined in a neighbourhood of a point* z *in* D; $F = 0$ *is said to be a regular local equation of a subset* N *of* $D (z \in N)$ *around* z *if*

(i) *there exists a neighbourhood U of z in D such that $U \cap N$ is a submanifold;*

(ii) $f = 0$ *on N for every f in F; and*

(iii) *there exists functions $f_1, \ldots, f_{n-h}$ in F (h being the dimension of the submanifold $U \cap N$) such that $df_1, \ldots, df_{n-h}$ are linearly independent at z.*

The set of all q-dimensional integral elements of $(\sum)$ will be denoted by $\ell^q \sum$. we shall now define the notions of ordinary and regular integral elements of $(\sum)$ by induction on the dimension q. $\ell^q \sum$ is provided with the topology induced by $\mathscr{F}^q D$.

Definition. *An integral point x of $\sum$ is said to be an* ordinary integral point *if* $\sum^{(0)} = 0$ *is a regular local equation of $\ell^o \sum$ around x. An integral point x is said to be a* regular integral point *if x is an ordinary integral point and the function t is a constant in a neighbourhood of x in $\ell^o \sum$. Suppose that the ordinary and regular integral elements of dimensions q', for $q' < q$, are defined.*

Definition. *A q-dimensional integral element is said to be an ordinary integral element of $(\sum)$ if it contains atleast one $(q-1)$-dimensional* **49**
regular integral element. A q-dimensional integral element is said to be a regular integral element of $(\sum)$ if it is an ordinary integral element and the function t is a constant in a neighbourhood of it in $\ell^q \sum$.

Example. *Let D be the plane R^2 represented by (x, y). Let $(\sum)$ be the differential system generated (as a closed ideal) in $\Lambda(D)$ by $\{x, dx, xdy, dx \wedge dy\}$. If $z \in D$ then clearly $J(z)$ is generated by $\{(dx)_z\}$ if $z \in \ell^o \sum$ since $x(z) = 0$ and by $\{(dx)_z, (dy)_z\}$ if $z \notin \ell^o \sum$. Hence any integral point z is a regular integral element of $(\sum)$.*

2.3

Let (M, M', ϖ) be a real analytic fibered manifold.

Definition. *A homogeneous differential form θ on M of degree r is said to be a fibred differential form if, for every $z \in M$ and every pair of sets of tangent vectors $(L', \dots, L^r)$ and $(L'^1, \dots, L'^r)$ in $(M)_z$ such that $\varpi(L^i) = \varpi(L'^i)$ $(i = 1, 2, \dots, r)$, we have*

$$\langle \theta, L^1 \wedge \cdots \wedge L^r \rangle = \langle \theta, L'^1 \wedge \cdots \wedge L'^r \rangle.$$

If z is any point in M, a fibred differential form on M defines an antisymmetric multilinear form $\theta[z]$, on the tangent space of M' at $\varpi(z)$ such that for any $L^1, \dots, L^r \in (M)_z$

$$\langle \theta, L^1 \wedge \cdots \wedge L^r \rangle = \langle \theta[z], \varpi(L^1) \wedge \cdots \wedge \widetilde{\omega}(L^r) \rangle.$$

50 If $(x'_1, \dots, x'_n)$ is a coordinate system in an open subset U of M' and if we set $x_i = x'_i o \varpi$ on $\varpi^{-1}(U)$ the fibred differential form θ has an expression of the form $\theta = \sum^{i_1 \cdots i_r} dx_{i_1} \wedge \cdots \wedge dx_{i_r}$. Then $\theta[z], z \in \varpi^{-1}(U)$ is given by $\theta\,[z] = \sum a^{i_1 \cdots i_r}(z)(dx'_{i_1})_{\varpi(z)} \wedge \cdots \wedge (dx'_{i_r})_{\varpi(z)}$. This expression shows clearly that $\theta[z]$ depends real analytically on z.

Let $(x_1, \dots, x_q)$ be a set of functions defined on a domain D of R^n such that $dx_1, \dots, dx_q$ are linearly independent at each point of D. Let $\mathscr{G}^q(D; x_1, \dots, x_q)$ denote the subset of $\mathscr{G}^q(D)$ consisting of all elements E in $\mathscr{G}^q(D)$ for which the restrictions of $dx_1, \dots, dx_q$ to E are linearly independent. Let ρ be the canonical projection of $\mathscr{G}^q(D; x_1, \dots, x_q)$ onto D which associates to every element E in $\mathscr{G}^q(D; x_1, \dots, x_q)$ its origin. When $\mathscr{G}^q(D; x_1, \dots, x_q)$ is provided with the manifold structure induced from $\mathscr{G}^q(D)$, it is easy to see that $(\mathscr{G}^q(D; x_1, \dots, x_q), D, \rho)$ is a fibred manifold. Given a homogeneous differential form $\varphi \in \sum^{(r+1)}(D)$ and a set of integers $(i_1, \dots, i_r)$ with $1 \leq i_1 < \cdots < i_r \leq q$, *a Pfaffian form*, denoted by $\varphi^{\{i_1 \cdots i_r\}}$ on $\mathscr{G}^q(D; x_1, \dots, x_q)$ is defined by the identity

$$\langle \varphi^{\{i_1 \cdots i_r\}}, L \rangle = \langle \varphi, L^{i_1}(E) \wedge \cdots \wedge L^{i_r}(E) \wedge d\rho L \rangle$$

where $L \in (\mathscr{G}^q(D))_E$ and $L^1(E), \dots, L^q(E)$ is a basis of E dual to the restrictions of $dx_1, \dots, dx_q$ to E. It is immediate to see that $\varphi^{\{i_1 \cdots i_r\}}$ is a
51 fibred differential form on $(\mathscr{G}^q(D; x_1, \dots, x_q))$ with respect to the fibred manifold $(\mathscr{G}^q(D; x_1, \dots, x_q); D, \rho)$.

Let $(\sum)$ be an exterior differential system on D and let $\varphi_1, \dots, \varphi_\ell$ be a set of homogeneous differential forms (of degree $d(1), \dots, d(\ell)$) in

$(\sum)$ which generate $(\sum)$ (as a closed ideal in $\Lambda(D)$). Let $\{\varphi_1, \dots, \varphi_h\}$ be the subset of $\varphi_1, \dots, \varphi_\ell$ for which $1 \le d(i) \le q$. As before choose a set $(x_1, \dots, x_q)$ of functions on D such that $dx_1, \dots, dx_q$ are linearly independent at each point of D and let $E \in \mathscr{G}^q(D; x_1, \dots, x_q)$.

Proposition 5. *If E is in $\ell^q \sum \cap \mathscr{G}^q(D; x_1, \dots, x_q)$ then $J(E)$ is generated by $\{\varphi^{i_1 \cdots i_{d(\sigma)-1}}$ [E] ; $\rho = 1, \dots, k, 1 \le i_j \le q\}$* $(1 \le i_1 < \cdots < i_{d(\rho)-1} \le q)$

Proof. By definition $J(E)$ is generated by all α_φ defined by $\alpha_\varphi(L) = \langle \varphi, L^1 \wedge \cdots \wedge L^r \wedge L\rangle$ where $\varphi \in \sum^{(r+1)}, L^1, \dots, L^r \in E$ and $L' \in (D)_z, z$ being the origin of E. If $L^1(E), \dots, L^q(E)$ is a basis of E dual to $dx_1\big|E, \dots, dx_q\big|E$ we can write $L^j = \sum b^j_i L^i(E)$. Therefore α_φ is in the space generated by $\varphi^{\{i_1 \cdots i_r\}}[E]$. ◻

On the otherhand, since $\deg \varphi \le q$ we see that $\alpha_\varphi = 0$ for $r \ge q$. Hence one can assume that $r < q$ and one can write $\varphi = \sum\limits_{\sigma=1}^{h} \psi_\sigma \wedge \varphi_\sigma + \sum_{f_j} \wedge \xi_j (f_j \in \sum^{(0)})$. Therefore, for $L \in (D)_z$

$$\begin{aligned}\langle \varphi^{\{i_1 \cdots i_r\}}[E], L\rangle &= \langle \varphi, L^{i_1}(E) \wedge \cdots \wedge L^{i_r}(E) \wedge L\rangle \\ &= \sum_{\sigma=1}^{h} \langle \psi_\sigma \wedge \varphi_\sigma, L^{i_1}(E) \wedge \cdots \wedge L^{i_r}(E) \wedge L\rangle \\ &\quad + \sum \langle f_j \wedge \xi_j, L^{i_1}(E) \wedge \cdots \wedge L^{i_r}(E) \wedge L\rangle.\end{aligned}$$

52

But since E is an integral element and $f_j \in \sum^{(0)}$ the second term in the right member vanishes. Hence

$$\begin{aligned}\left\langle \varphi^{\{i_1 \cdots i_r\}}, L\right\rangle &= \sum_{\sigma=1}^{h} \langle \psi_\sigma \wedge \varphi_\rho, L^{i_1}(E) \wedge \cdots \wedge L^{i_r}(E) \wedge L\rangle \\ &= \sum_{\sigma=1}^{h} \pm \langle \psi_\sigma, L^{h_1}(E) \wedge \cdots \wedge L^{h^\sigma_{r_1}}(E)\rangle \langle \varphi_\sigma L^{k_1}(E) \wedge \cdots \wedge L^{k^\sigma_{r_2}-1}(E) \\ &+ \sum_{\sigma=1}^{h} \pm \langle \psi_\sigma, L^{h_1}(E) \wedge \cdots \wedge L^{h^\sigma_{r_1}-1}(E) \wedge L\rangle \langle \varphi_\sigma, L^{k_1}(E) \wedge . \wedge L^{k^\sigma_{r_2}}(E)\end{aligned}$$

Here r_1^σ, r_2^σ are the degrees of ψ_σ and φ_σ respectively. Again E being an integral element, the latter term in the right member vanishes. Therefore we obtain

$$\langle \varphi^{\{i_1 \cdots i_r\}}, L \rangle = \sum_{\sigma=1}^{h} \pm * \langle \varphi_\sigma^{k_1 \cdots k_{r_2} \sigma_{-1}}, L \rangle.$$

This completes the proof of the proposition.

Let $(\sum)$ be a differential system on a domain D of R^n. If E is an integral element of $(\sum)$, let G_E^r (when there is no possible confusion, simply G) be the set of all r-dimensional contact elements of D contained in E. G can be provided with the structure of a real analytic manifold as follows. Let E' be an element of G and let $\varphi_1, \ldots, \varphi_h$ be a base of the dual E^* of E such that the restrictions of $\varphi_1, \ldots, \varphi_r$ to E' are linearly in-
53 dependent. Then a coordinate neighbourhood of E' in G can be defined to be $U(\varphi_1, \ldots, \varphi_r) = \{E'' \in G : \varphi_1 \big| E'', \ldots, \varphi_r \big| E''$ are linearly independent $\}$. The coordinate systems can be explicitly constructed without much difficulty. But we do not go into the details since we do not need the explicit coordinate systems. Clearly G_E^r is a submanifold of $\mathscr{G}^r(D)$.

Definition. *A subset A of manifold M is said to be a* (real analytic) subvariety (or a *real analytic subset) if for every a in M there exists a neighbourhood U of a in M and a finite number of real analytic functions $f_1, \ldots, f_h$ defined on U such that $A \cap U$ is the set of common zeros of $f_1, \ldots, f_h$.*

We say that a subvariety A is proper when $A \neq M$. Clearly this definition is local. Every real analytic subset is closed in M. If M is connected and A is proper then $M - A$ is everywhere dense.

Proposition 6. *Let E be an integral element of $\sum$. Then there exists a proper real analytic subset A of $G_E^r = G$ and an integer $k > 0$ such that $t(E') = k$ for every E' in $G - A$ and $t(E') \underset{\neq}{<} k$ for every E' in A. (We will denote this k by $t_r(E)$).*

Proof. Let $E' \in G$ and let $(x_1 \ldots, x_q, y_1, \ldots, y_m)$ be a coordinate system in D such that $dx_1 \big| E', \ldots, dx_r \big| E'$ are linearly independent.

$\mathscr{G}^r(D; x_1, \ldots, x_r)$ being an open subset of $\mathscr{G}^r(D)$ and the property of a real analytic subset being local, it is sufficient to prove that there exists a real analytic subset A' of $G \cap \mathscr{G}^r(D; x_1, \ldots, x_r)$ satisfying the assertion **54** of the proposition. □

We know that there exists a finite number of Pfaffian forms $\theta_1, \cdots,$ θ_h of the fibred manifold $\mathscr{G}^r(D; x_1, \ldots, x_r), D, \rho)$ such that $\theta_1[E'], \ldots,$ $\theta_h[E']$ generate $J(E')$ for any integral element E' in $\mathscr{G}^r(D; x_1, \ldots, x_r)$ in particular. for any E' in $G \cap \mathscr{G}^r(D; x_1, \ldots, x_r)$. We can write $\theta_\sigma = \sum\limits_{i=1}^{r} a^i_\sigma dx_i + \sum\limits_{\lambda=1}^{m} a^{r+\lambda}_\sigma dy_\sigma$. Hence for an element E' in $\mathscr{G}^r(D; x_1, \ldots, x_r) \cap G$, the dimension $t(E')$ of $J(E')$ is nothing but the rank of the matrix

$$\begin{pmatrix} \cdots a^i_1(E') \cdots a^{r+\lambda}_1(E') \cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots a^i_h(E') \cdots a^{r+\lambda}_h(E') \cdots \end{pmatrix}$$

Since $a^i_\sigma(E')$ and $a^{r+\lambda}_\sigma(E')$ are real analytic functions on a neighbourhood of E' in $G \cap \mathscr{G}^r(D; x_1, \ldots, x_r)$, it can be easily seen that t is a constant $k > 0$ except on a real analytic subset A' of $G \cap \mathscr{G}^r(D; x_1, \ldots, x_r)$ and this proves the required assertion.

It is easy to prove the following

Corollary to Proposition 6. Let E be an ordinary integral element. If E' is regular and is contained in E, then $t(E') = t_r(E)$ where $r = \dim(E')$.

We shall denote the set of all regular integral elements of dimension q of $(\sum)$ by $\mathscr{R}^q \sum$ and the set of all ordinary integral elements of dimension q of $(\sum)$ by $\mathfrak{G}^q \sum$.

2.4

Let $(x_1, \ldots, x_q, y_1, \ldots, y_m)$ be a coordinate system of D. Any differen- **55** tial form $\varphi \in \Lambda^r(D)$ and a finite set of integers $(i_1, \ldots, i_r)$ such that $1 \leq i_1 < \cdots < i_r \leq q$ define a function $\varphi[i_1, \ldots, i_r]$ $(r = 0, 1, \ldots, q)$ on $\mathscr{G}^q(D; x_1, \ldots, x_q)$ as follows: If $E \in \mathscr{G}^q(D; x_1, \ldots, x_q)$ and $\{L^1(E),$ $\ldots, L^q(E)\}$ is a basis of E dual to $dx_1\big|E, \ldots, dx_q\big|E$ then

$$\varphi[i_1, \ldots, i_r](E) = \langle \varphi, L^{i_1}(E) \cdots L^{i_r}(E) \rangle.$$

Let $F = F(\sum; x_1, \ldots, x_q)$ be the set of all $\varphi[i_1, \ldots, i_r]$ with $\varphi \in \sum^{(r)}$ $(1 \leq i_1 < \cdots < i_r \leq q; r = 0, 1 \ldots, q)$.

Proposition 7. *$\theta^q \sum$ and $\mathscr{R}^q \sum$ are open subsets of $\ell^q \sum$. If $E^0 \in \theta^q \sum \cap \mathscr{G}^q(D; x_1, \ldots, x_q)$ then there exists a neighbourhood of E^0 in $\mathscr{G}^q$ (D) such that $\theta^q \sum \cap \mathscr{U}$ is a submanifold of $\mathscr{U}$ and $F = 0$ is a regular local equation of $\theta^q \sum$. Moreover, if E' is a regular $(p-1)$-dimensional integral element in E^0, then* $\dim(\theta^q \sum \cap \mathscr{U}) = \dim(\theta^{q-1} \sum \cap \mathscr{U}') - (q - 1) + (n - q - t(E'))$*, when $n = \dim D$ and $\mathscr{U}'$ is a sufficiently small neighbourhood of E'.*

Proof. The proposition is obviously true in the case $q = 0$. We proceed now by induction on q. Let us assume the proposition to have been proved for all dimensions $q' < q$. If E is an $\theta^q \sum$ there exists a subspace E' of E in $\mathscr{R}^{q-1} \sum$. Choose the coordinate system $(x_1, \ldots, x_q, y_1, \ldots, y_m)$ in D such that $dx_1 \big| E^0, \ldots, dx_q \big| E^0$ are linearly independent and such that

$$E' = \{L \in E : \langle dx_q, L \rangle = 0\}$$

56 We shall define a map π of $\mathscr{G}^q(D; x_1, \ldots, x_q)$ into $\mathscr{G}^{q-1}(D; x_1, \ldots x_{q-1})$ as follows : if $E \in \mathscr{G}^q(D; x_1, \ldots, x_q)$ then $\pi(E)$ is the space generated by $L^1(E), \ldots, L^{q-1}(E)$. For E'' in $\mathscr{G}^{q-1}(D; x_1, \ldots, x_{q-1})$ denote by $L^1(E''), \ldots, L^{q-1}(E'')$ the base of E'' dual to $dx_1 \big| E'', \ldots, dx_{q-1} \big| E''$. We can write

$$L^r(E'') = \frac{\partial}{\partial x_r} + w^r(E'') \frac{\partial}{\partial x_q} + y^r_\lambda(E'') \frac{\partial}{\partial y_\lambda} \; (r = 1, \ldots, q-1)$$

where $w^r(E''), y^r_\lambda(E'')$ form part of a coordinate system in $\mathscr{G}^{q-1}(D; x_1, \ldots, x_{q-1})$. Therefore the equations $w_1 = \cdots = w_{q-1} = 0$ define a submanifold W of $\mathscr{G}^{q-1}(D; x_1, \ldots, x_{q-1})$ and $\dim W + (q-1) = \dim_{E'}$ $(\ell^{q-1} \sum)$. Then it is clear that π maps $\mathscr{G}^q(D; x_1, \ldots, x_q)$ onto W and $\mathscr{G}^q(D; x_1, \ldots, x_q), W, \pi)$ is a fibred manifold. ◻

By the induction assumption, there exists a neighbourhood $\mathscr{U}'$ of E' in $\mathscr{G}^{q-1}(D)$ such that $\mathscr{U}' \cap \mathscr{R}^{q-1} \sum$ is a submanifold of $\mathscr{U}'$ and $F_{E'} = 0$ is a regular local equation. We now assert that, for a suitable $\mathscr{U}'$, $\mathscr{U}' \cap W \cap \mathscr{R}^{q-1} \sum$ is a submanifold of $\mathscr{R}^{q-1} \sum$. Let z_0 be the origin of

E^0. Given real numbers $a^1, \ldots, a^{q-1}$, and u, let $E_u \in \mathscr{G}^{q-1}(D)$, be generated by $L^j(E^0)+ua^jL^q(E^0)$ $(j = 1, \ldots, q-1)$. E_u being a subspace of E^0, it is an integral element. Therefore E_u is a real analytic curve in $\mathscr{R}^{q-1}\sum$ for sufficiently small u. Because of the choice of $E', (E_u)_{u=0} = E'$ and $w^j(E_u) = u\ a^j$. The last equality implies that $\langle dw^j, X^a\rangle = a^j$ where X^a is the tangent vector at $u = 0$ of the curve E_u in $\mathscr{R}^{q-1}\sum$. Because $a_1, \ldots, \ldots, a_{q-1}$ are arbitrary, it follows that $(dw^1)_E, \ldots, (dw^{q-1)}E'$, restricted to the tangent vector space to $\mathscr{R}^{q-1}\sum$ at E' are linearly indepen- **57**
dent. Hence $\mathscr{U}' \cap W \cap \mathscr{R}^{q-1}\sum$ is a submanifold N of $\mathscr{U}'$. Let $\mathscr{U}$ be a neighbourhood of E^0 in $\mathscr{G}^q(D)$ such that $\pi(\mathscr{U}) \subset \mathscr{U}'$. It is clear that any element E in $\mathscr{U}$ is in $\ell^q\sum$ if and only if its image $\pi(E)$ is in N and $L^q(E)$ is a zero of $J(\pi(E))$.Take real analytic functions $f_1, \ldots, f_a$ on $\mathscr{U}'$ where $a = \dim\mathscr{G}^{q-1}(D) - \dim(\ell^{q-1}\sum\cap\mathscr{U}')$ such that $df_1, \ldots, df_a$ are linearly independent at E' and $f_1 = \cdots = f_a = 0$ define $\ell^{q-1}\sum\cap\mathscr{U}'$.

By the induction assumption we can choose fibred differential forms $\theta_1, \ldots, \theta_{t)(E')}$ of $(\mathscr{G}^{q-1}(D; x_1, \ldots, x_{q-1}), D, \rho)$ such that $\theta_1[E''], \ldots, \theta_{t(E')}$ $[E'']$ are linearly independent and generate $J(E'')$ for integral elements E'' near E'. We can write

$$\theta_\sigma = \sum_{j=1}^{q} a^j_\sigma dx_j + \sum_{\lambda=1}^{m} a^{q+\lambda}_\sigma dy_\lambda.$$

We recall that the functions y^q_λ on $\mathscr{G}^q(D; x_1, \ldots, x_q)$ defined by $L^q(E) = \dfrac{\partial}{\partial x_q} + \sum\limits_\lambda y^q_\lambda(E)\dfrac{\partial}{\partial y_\lambda}$, form a part of the coordinate system in $\mathscr{G}^q(D; x_1, \ldots, x_q)$. Then the conditions $\pi(E) \in N$ and $\langle J(\pi(E)),\ L^q(E)\rangle = 0$ are analytically expressed by the conditions:

$$(I)\qquad \begin{cases} f_1 o\pi = \cdots = f_a o\pi = 0 \\ a^q_\sigma o\pi + (a^{q+\lambda}_\sigma o\pi)y^q_\lambda = 0 \quad (\sigma = 1, \ldots, t(E')) \end{cases}$$

Because $dx_1, \ldots, dx_q, \theta_1[E'], \ldots, \theta_t[E']$ are linearly independent, it is clear now that the differentials of the above functions at E are linearly independent at each E in $\mathscr{U}$, for $\mathscr{U}$ sufficiently small. Therefore **58**
$\ell^q\sum\cap\mathscr{U}$ form a submanifold of $\mathscr{G}^q(D)$. Since $\dim\pi^{-1}(E') = n - q$, we see easily that $\dim(\ell^q\sum\cap\mathscr{U})$ is equal to $\dim(\mathscr{U}' \cap \ell^{q-1}\sum) - (q - 1) +$

$(n - t(E') - q)$. By induction assumption we can choose $f_1, \ldots, f_a$ in $F_{E'}$. Then it is clear by the definition of F_E that $f_1 o\pi, \ldots, f_a o\pi$ are in F_E. If we choose θ_σ as constructed in in Proposition 5, we see easily that the function $\langle\theta_\sigma[\pi_0 E], L^q(E)\rangle$ are in F_E. Thus we can choose f_h and θ_σ in such a way that the equations (I) is a part of the equation $F_E = 0$. Therefore $F_E = 0$ is a regular local equation of $\theta^q \sum$ at E.

Because *pi* is continuous and $\mathscr{R}^{q-1} \sum$ is open in $\ell^{q-1} \sum$ it follows that $\theta^q \sum$ is open. Then, because of the definition of regular integral elements, $\mathscr{R}^q \sum$ is also open.

Proposition 8. *Let E be an ordinary integral element of dimension q. Let $E' \subset E$. Assume that there is a sequence of subspaces $E' = E^{(1)} \subset E^{(2)} \subset \cdots E^{(h)} = E$ of E such that* $\dim(E^{i+1)}) = \dim(E^{(i)}) + 1$ *and such that each $E^{(i)}$ $(i = 1, \ldots, h-1)$ is regular. Then for each neighbourhood $\mathscr{U}$ of E in $\ell^q \sum$ there is a neighbourhood $\mathscr{U}'$ of E' with the following property: for any $E'' \in \mathscr{U}' \cap \ell^r \sum$*$(r = \dim E')$ *there is an element $E_1 \in \mathscr{U} \cap \Theta^q \sum$ such that $E_1 \supset E''$.*

Proof. By an induction argument the problem can be reduced to the case when $r = q - 1$. Take a neighbourhood $\mathscr{U}_1'$ of E' and a system $\theta_1, \ldots, \theta_t$, of fibred differential forms on $(\mathscr{U}_1', D, \rho)$ such that for
59 any $E'' \in \mathscr{U}_1' \cap \ell^{q-1} \sum, \theta_1[E''], \ldots, \theta_t, [E'']$ are linearly independent and generate $J(E'')$. Then the proposition follows from the fact that a non-zero solution of $J(E'')$ together with E'' generate a q-dimensional integral element for E'' in $\ell^{q-1} \sum$. □

When $\dim E' = 0$, that is when E' is the origin of E, we have the following corollary.

Corollary to Proposition 8. Let E be an ordinary integral element of origin z. Then for neighbourhood $\mathscr{U}$ of E there is a neighbourhood U of z such that for any z' in $U \cap \ell^0 \sum$ there is an integral element $E'' \in \mathscr{U}$ with origin z'.

2.5

Let $(\sum)$ be a differential system on a domain D in R^n Then we pose the following definitions.

Definition. *Let E be a q-dimensional contact element of D. A flag on E is defined to be a finite sequence of subspaces of E:*

$$\{0\} = E_0 \subset E_1 \subset \cdots \subset E_q = E,$$

such that the dimension of E_r is $r(r = 0, 1, \ldots, q)$.

q is called the dimension of the flag and E_r is called the r^{th} component of the flag on E $(r = 0, 1, \ldots, q)$.

Definition. *Let E be an integral element of $(\sum)$. A flag in E with components E_r, is said to be normal if $t(E_r) = t_r(E)$ holds for $r = 0, 1, \ldots, q-1$ (cf. Proposition 6).*

Definition. *For an ordinary integral element E of $(\sum)$ a flag on E is called regular if each component E_r is a regular integral element $(r = 0, 1, \ldots, q-1)$.*

It is clear by definition and by Corollary to Proposition 6 that if E is **60**
an ordinary integral element of $(\sum)$, a regular flag on E is a normal flag.

Proposition 9. *Given a q-dimensional ordinary integral element of $(\sum)$ there exists atleast one regular flag on E.*

Proof. Take a regular E'_1 in E. There exists a neighbourhood U_1 of E'_1 in G^1_E such that any $E''_1 \in U_1$ is regular. Let $U'_2 = \{E''_2 \in G^2_E :$ there exists $E''_1 \in U_1$ such that $E''_1 \subset E''_2\}$. Clearly this is a non-empty open subset of G^2_E. There exists a regular $E''_2 \in U'_2$ such that $t(E''_2) = t_2(E)$. Thus we find $E''_1 \subset E''_2 \subset E$ such that E''_r is regular and r-dimensional for $r = 1, 2$. Now we proceed similarly by induction on r upto $q-1$ and thus the assertion is proved. □

Given an element E in $\mathscr{G}^q(D)$ the set $\widetilde{G(E)}$ of all flags on E, can be made into a real analytic manifold by considering it as a homogeneous space as follows: Let $G\ell(q)$ be the general linear group of E. Fix a flag $E_0 \subset E_1 \subset \cdots \subset E_q = E$ on E. We define a map f of $G\ell(q)$ into $\tilde{G}(E)$ as follows: For $A \in G\ell(q)$, $f(A)$ is the flag $AE_0 \subset AE_1 \subset \cdots \subset AE_q = E$. It is clear that f is surjective. Denoting by H the subgroup of all the elements which leaves each E_q invariant, we see easily that f induces a bijective mapping of $G\ell(q)/H$ onto $\tilde{G}(E)$. Hence we can identify $\tilde{G}(E)$ with $G\ell(q)/H$ and take its real analytic structure. It is easy to see the following:

61 **Proposition 10.** *The set of all normal flags on an integral element E is a non-empty open subset of $\tilde{G}(E)$.*

Proof is similar to that of Proposition 9.

Let E be an element of $\ell^q \sum$ of origin z. We assume that $\ell^0 \sum$ is a submanifold on a neighbourhood of z and $\sum^{(0)} = 0$ is a regular local equation of $\ell^0 \sum$ around z. We define a sequence of integers $s_r(E)$ ($r = -1, 0, \ldots, q$) by setting

$$\begin{aligned} s_{-1}(E) &= t_{-1}(E) = \text{ co-dimension of } \ell^0 \sum \text{around } z \\ s_r(E) &= t_r(E) - t_{r-1}(E) \text{ for } r = 0, 1, \ldots, q-1 \\ s_q(E) &= n - q - t_{q-1}(E). \end{aligned}$$

Hence $s_{-1}(E) + \cdots + s_q(E) = n - q$.

Proposition 11. *If $E \in \ell^q \sum$ satisfies the above conditions, then $s_r(E) \geq 0$ ($r = -1, \ldots, q$). In particular, if E is ordinary then $s_r(E) \geq 0$.*

Proof. Since $\sum^{(0)} = 0$ is a regular local equation $\ell^0 \sum$ is a submanifold around the origin z of E. By Proposition 9 there exists a normal flag on E, say $E_0 \subset E_1 \subset \cdots \subset E_q = E$ with $t_r(E) = t(E_r)$. By Proposition 4, $t(E_r) \geq t(E_{r-1})$. Hence $s_r(E) \geq 0$ for $r = 1, \ldots, q-1$. □

Case when $r = 0$ ($\sum$) being closed $f \in \sum^{(0)}$ implies $df \in \sum^{(1)}$. The set of all $(df)_z$, where $f \in \sum^{(0)}$, will generate a subspace A of the dual of E, whose dimension is $s_{-1}(E)$. Since ($\sum$) is closed $A \subset J(z)$ and hence $s_0(E) = t_0(E) - t_{-1}(E) \geq 0$.

62 **Case when $r = q$.** For every $L \in E$ and for any $\alpha \in J(E_{q-1}), \alpha(L) = 0$ implies $\dim J(E_{q-1}) \leq n - \dim E = n - q$ and hence $s_q = n - q - t_{q-1} \geq 0$.

From the second part of Proposition 7, we easily deduce the following

Corollary to Proposition 7. Let E be a q-dimensional ordinary integral element of $\sum$. Then the dimension of $\ell^q \sum$ is equal to $(n - s_{-1}(E)) + \sum_{r=1}^{q} r s_r(E)$, where $n = \dim D$.

2.6

Let $(\sum)$ be a differential system on D. Let D_1 be a submanifold of D. Denote by $(\sum_1)$ the differential system on D_1 generated by the restriction of $(\sum)$ to D_1. Then $\mathscr{G}^q(D_1)$ is a submanifold of $\mathscr{G}^q(D)$ and so $\ell^q \sum_1$ is a subset of $\mathscr{G}^q D$. We easily see the following:

$$\ell^q \sum_1 = \ell^q \sum \cap \mathscr{G}^q(D_1).$$

Let z be a point of D_1. Then the injection mapping $i : D_1 \to D$ induces a homomorphism i_z^* of $(D)_z^*$ onto $(D_1)_z^*$. Let E be an integral element of $(\sum_1)$ with z as its origin. Then by the remark above E is also an integral element of $(\sum)$. Therefore we have $J(E, \sum)$ and $J(E, \sum_1)$ which are subspaces of $(D)_z^*$ and $(D_1)_z^*$ respectively. Let A_z denote the kernel of i_z^*.

Proposition 12. *Notations being as above, i_z^* induces a surjective mapping of $J(E, \sum)$ onto $J(E, \sum_1)$. If, moreover, for any $\beta(\neq 0)$ in A_z there is an integral element E' of $(\sum)$ containing E and there is an L in E' with* 63
$\beta(L) \neq 0$, then i_z^* induces a bijective mapping of $J(E, \sum)$ onto $J(E, \sum_1)$.

Proof. The first assertion is an immediate consequence of the definitions of $J(E, \sum)$ and $J(E, \sum_1)$. As for the second, take β in $J(E, \sum) \cap A_z$. Then, because E' is an integral element, $\beta(L) = 0$ for any L in E'. Therefore $\beta = 0$ because of the assumption that there exists an L in E' with $\beta(L) \neq 0$ for non-zero β. □

Let f be a function on D such that $df\big|E_q \neq 0$ where E_q is a q-dimensional ordinary integral element of $(\sum)$ on D. Let us assume further that D_1 is the submanifold of D defined by the equation $f = 0$. Take a regular flag of $(\sum)$ on $E : E_0 \subset E_1 \subset \cdots \subset E_q$. Assume that $E_{q-1} = \{L \in E : \langle df, L\rangle = 0\}$. Then we have the following proposition:

Proposition 13. *$E_r (r \leq q-1)$, regarded as a contact element of D_1, is a regular integral element of $(\sum_1)$. Moreover, we have*

$$s_r\left(E_{q-1}, \sum_1\right) = s_r\left(E_q, \sum\right) \quad \textit{for } r = -1, \dots, q-2;$$
$$s_{q-1}\left(E_{q-1}, \sum_1\right) = s_{q-1}\left(E_q, \sum\right) + s_q\left(E_q, \sum\right).$$

Proof. First, we show that there is a neighbourhood U^r of E_r in $\mathscr{G}^r(D)$ such that for any E'' in $U^r \cap \ell^r \sum_1 t(E'', \sum_1) = t(E_r. \sum)$. Take a neighbourhood U of E_q in $\mathscr{G}^q D$ such that, for any E' in U, $df\big|E' \neq 0$. By Proposition 8 there is a neighbourhood U^r of E_r such that for any E'' in
64 $U^r \cap \ell^r \sum$, there is E' in $U \cap \ell^q \sum$ containing E'' as a subspace. Then if E'' is in $U^r \cap \ell^r \sum_1$, the conditions in the second part of Proposition 12 are satisfied for E'' and hence it follows that $t(E'', \sum_1) = t(E'', \sum)$ from Proposition 12. If we take U^r sufficiently small, then $t(E'', \sum) = t(E_r, \sum)$. Now the proof can be completed by induction on r. When $r = 0$, the condition $df\big|E_q \neq 0$ implies that $\ell^0 \sum_1 = \ell^0 \sum \cap D_1$ is a submanifold of D_1 on a neighbourhood of z having regular local equation $\sum_1^{(0)} = 0$. Thus E_0 is ordinary and $\dim \ell^0 \sum_1 + 1 = \dim \ell^0 \sum$. In particular, $s_{-1}(z, \sum_1) = s_{-1}(z, \sum)$. On the other hand we have already shown that $t(w, \sum_1) = t(z, \sum)$ for any integral point w of $(\sum_1)$ in a sufficiently small neighbourhood of E_0 in $\ell^0 \sum_1$. Hence E_0 is a regular integral point of $(\sum_1)$ and $s_0(z, \sum_1) = s_0(z, \sum)$. Assuming the case of all $r' < r$ to show that E_r is in $\mathscr{R}^r \sum_1$; it is only necessary to show that $t(E', \sum_1)$ remains constant when E' is an integral element of $\sum_1$ sufficiently near E_r. We have already shown this and moreover, $t(E', \sum_1) = t(E_r, \sum)$. So we have $s_r(E_{q-1}, \sum_1) = s_r(E_q, \sum)$ for $r \leq q-2$. By definition,

$$\begin{aligned} s_{q-1}\left(E_{q-1}, \sum_1\right) &= \dim D_1 - (q-1) - t\left(E_{q-2}, \sum_1\right) \\ &= \dim D - q - t\left(E_{q-1}, \sum\right) + \left(t\left(E_{q-1}, \sum\right) - t\left(E_{q-2}, \sum\right)\right) \\ &= s_q\left(E_q, \sum\right) + s_{q-1}\left(E_q, \sum\right). \end{aligned}$$

This completes the proof of the proposition. □

2.7 Differential systems with independent variables

Let (D, D', ϖ) be a fibred manifold where D and D' are domains in 65
Euclidean spaces R^{q+m} and R^q respectively. Let $(\sum)$ be a differential system defined on D. A pair consisting of a fibred manifold (D, D', ϖ) and a differential system $(\sum)$ on D is called a differential system with independent variables and is denoted by $[\sum; (D, D', \varpi)]$.

Definition . *A cross-section f of (D, D', ϖ) over an open set U of D' is said to be an integral of the differential system $[\sum, (D, D', \varpi]$ if the map f of U into D defines a submanifold of D which is an integral submanifold of $(\sum)$.*

Let $\mathscr{G}^r(D.D', \varpi)$ denote the set of all r-dimensional contact elements E in $\mathscr{G}^r(D)$ which are such that if z is the origin of $E, (d\varpi)_z$ is injective on E. We set

$$\ell^r \left[\sum, (D, D', \varpi\right] = \ell^r \sum \cap \mathscr{G}^r(D, D', \varpi);$$
$$\theta^r \left[\sum, (D, D', \varpi\right] = \theta^r \sum \cap \mathscr{G}^r(D, D', \varpi);$$
and
$$\mathscr{R}^r \left[\sum, (D, D', \varpi\right] = \mathscr{R}^r \sum \cap \mathscr{G}^r(D, D', \varpi).$$

Let E be a q-dimensional ordinary integral element of this system (an element of $\theta^q [\sum, (D, D', \varpi])$ with origin z. Let $(x_1, \ldots, x_q, y_1, \ldots, y_m)$ be a coordinate system of (D, D', ϖ) around z.

Definition . *A coordinate system $(x_1, \ldots, x_q, y_1, \ldots, y_m)$ of (D, D', ϖ)
around z is said to be regular with respect to an ordinary integral el-* 66
ement E of the system $[\sum, (D, D', \varpi)]$ if $x_i(z) = y_\lambda(z) = 0$ and if the following conditions are satisfied:

(i) $E_r = \{L \in E : \langle dx_{r+1}, L\rangle = \cdots = \langle dx_q, L\rangle = 0$ *is a regular integral element of* $(\sum)$ *for* $r = 0, 1, \ldots, q-1$*;*

(ii) $y_1, \ldots, y_{s_{-1}}$, *where* $s_{-1} = s_{-1}(E)$, *are in* $\sum^{(0)}$*;*

(iii) $(dx_1)_z, \ldots, (dx_q)_z, (dy_{t_r+1})_z, \ldots, (dy_m)_z$, *where* $t_r = t_r(E) = s_{-1}(E) + \cdots + s_r(E)$, *are linearly independent modulo* $J(E_r, \sum)$ *for* $r \leq q-1$*; and*

(iv) *E is equal to the tangent vector space at z to the submanifold $y_1 = \cdots = y_m = 0$.*

Put $S = (s_0, \ldots, s_q)$, where $s_r = s_r(E)$ (s_q may be zero; S is a system of characters (cf. 1.2). $A\xi$ in $H(S)$ has $s_0 + \cdots + s_q = m - s_{-1}$ components. Namely, if $s_{r+1} \neq 0$, $\xi_{s_0+s_1+\cdots+s_r+j}$ is in H_{r+1} for $j = 1, \ldots, s_{r+1}$. Let (x, y) be a fixed regular coordinate system and ξ in $H(S)$ be given.

Definition. *A cross-section f of (D, D', ϖ) over an open neighbourhood U of $\varpi(z)$ is said to have initial condition ξ with respect to the regular coordinate system (x, y) with center z if*

(i) *$f(\varpi(z))$ is in the domain of (x, y) and*

(ii) *when f is expressed by $y_\lambda = \eta_\lambda(x_1, \ldots, x_q)$ we have $\eta_1(x) = \cdots = \eta_{s_{-1}}(x) = 0$ and*

$$y_{s_{-1}+s_0+s_{r-1}+j}(x_1, \ldots, x_r, 0, \ldots, 0) = \xi_{s_0+\cdots+s_{r-1}+j}(x_1, \ldots, x_r).$$

67 We note that the last condition has a meaning since $s_{-1} + s_0 + \cdots + s_q = \dim D - q = m$.

Definition . *A mapping F of $H(S; u, v)$ into $H_q^m(u', v')$ is said to be a* solution mapping of the system $\left[\sum, (D, D', \tilde{\omega})\right]$ with respect to a regular coordinate system (x, y) if $y_\lambda = F_\lambda(\xi)$ is an integral of $(\sum)$ with the initial condition ξ.

2.8

Proposition 14. *For any ordinary integral element E of $[\sum, (D, D', \varpi)]$ of dimension $q = \dim D'$, there exists a regular coordinate system in (D, D', ϖ) with respect to E and $\sum$.*

Proof. Take a regular flag $E_0 \subset E_1 \subset \cdots \subset E_q = E$ on E. Take a coordinate system $(x'_1, \ldots, x'_q)$ around $\varpi(z)$ in D' (z being the origin of E) such that

$$E_r = \{L \in E : \langle dx_{r+1}, L\rangle = \cdots = \langle dx_q, L\rangle = 0\}$$

where $x_j = x'_j o \varpi$. Since z is an ordinary integral point there are functions $y_1, \ldots, y_{t_{-1}}$ in $\sum^{(0)}$ such that $(dy_1)_z, \ldots, (dy_{t_{-1}})_z$ are linearly independent. Then, since $dy_\sigma \big| E = 0, dy_1, \ldots, dy_{t_{-1}}, dx_1, \ldots, dx_q$ are linearly independent at z. Therefore $(x_1, \ldots, x_q, y_1, \ldots, y_{t_{-1}})$ can be completed into a coordinate system $(x_1, \ldots, x_q, y_1, \ldots, y_m)$ around z in (D, D', ϖ). This coordinate system satisfies (i) and (ii) of the definition of a regular coordinate system. Since $J(E_r, \sum)$ are ascending with $t_r = \dim(E_r, \sum)$ for $r = 1, 2, \ldots, q-1$ and since they contain $(dy_1)_z, \ldots, (dy_{t_{-1}})_z$ (because $(\sum)$ is closed), it is now clear that, by applying a linear transformation of 68
$x_1, \ldots, x_q\, y_1, \ldots, y_m$ which fixes each x_j (if necessary), we can construct a coordinate system satisfying (i), (ii), (iii) and (iv) of the definition of a regular coordinate system. ◻

Proposition 15. *Let $(x_1, \ldots, x_q, y_1, \ldots, y_m)$ be a regular coordinate system in (D, D', ϖ) with respect to an ordinary integral element E of $[\sum, (D, D', \varpi)]$. Denote by D_1 (resp. D'_1) the submanifold of D (resp. D') defined by $x_q = 0$ (resp. $x'_q = 0$, where $x_q = x'_q o \varpi$). Let $(\sum_1)$ be the restriction of $(\sum)$ to D_1. Let $E_{q-1} = \{L \in E : \langle dx_q, L\rangle = 0\}$. Then E_{q-1} is an ordinary integral element of $\left[\sum_1, (D_1, D'_1, \varpi)\right]$ and $(x_1, \ldots, x_{q-1}, 0, y_1, \ldots, y_m)$ is a regular coordinate system of (D_1, D'_1, ϖ) with respect to E_{q-1} and $(\sum_1)$*

Proof. E_{q-1} is a regular integral element of $(\sum_1)$ by Proposition 13 and the coordinate system $(x_1, \ldots, x_{q-1}, 0, y_1, \ldots, y_m)$ satisfies the condition (i) of a regular coordinate system with respect to E^{q-1} and $(\sum_1)$. The condition (ii) is clear, because $s_{-1}(E, \sum) = s_{-1}(E^{q-1}, \sum_1)$ by the same proposition. The restriction mapping i^*_z of $(D)^*_z$ onto $(D_1)^*_z$ induces an isomorphism of $J(E_r, \sum)$ onto $J(E_r, \sum_1)$ and the kernel of i^*_z is $(dx_q)_z$. Then the verification of the condition (iii) for $(x_1, \ldots, x_{q-1}, 0, y_1, \ldots, y_m)$ is immediate. ◻

2.9

We fix in this n^0 once for all a regular coordinate system (x, y) in $(D.D', \varpi)$ with respect to an ordinary integral element E^0 and $(\sum)$. So, 69

$\dim E^0 = \dim D$. We use the notations used in the definition of a regular coordinate system and Proposition 15. In particular $E_r = \{L \in E^0; \langle dx_{r+1}, L\rangle = \cdots = \langle dx_q, L\rangle = 0\}$. Set $S = (s_0, \ldots, s_q)$ where $s_r = s_r(E^0)$ and $S_1 = (s_0, \ldots, s_{q-2},\ s_{q-1} + s_q)$. We define an infinite analytic mapping P (everywhere defined) of $H(S)$ onto $H(S_1)$ by setting

$$p(\xi) = (\xi_1, \ldots, \xi_{t_{q-1}}, \xi_{t_{q-1}+1}(x_1, \ldots, x_{q-1}, 0), \ldots, \xi_m(x_1, \ldots, x_{q-1}, 0))$$

We remark that the elements of $H(S)$ (resp. $H(S_1)$) can be regarded as initial conditions for integrals of $[\sum, (D, D', \varpi)]$ (resp. $[\sum_1, (D_1, D'_1, \varpi)]$) as explained in n^0 2.7.

Take a cross-section f of (D, D', ϖ) over an open neighbourhood of $\varpi(z)$ (z being the origin of E), which is represented by $y_\lambda = \eta_\lambda(x_1, \ldots, x_q)$ ($\lambda = 1, \ldots, m$), with the initial condition ξ in $H(S)$. Now we state necessary conditions on η_λ so that f is an integral of $[\sum, (D, D', \varpi)]$ with initial condition ξ in $H(S)$.

I. $(x_1, \ldots, x_{q-1}, 0, y_1, (x_1, \ldots, x_{q-1}, 0), \ldots, y_m(x_1, \ldots, x_{q-1}, 0))$ should be an integral of $\left[\sum_1(D_1, D'_1, \varpi_1)\right]$ with the initial condition $P(\xi)$. This follows easily from the definitions involved.

II. Consider the fibred manifold $(\mathscr{G}^{q-1}(D), D$, the canonical projection). Let $\alpha_1, \ldots, \alpha_{t_{q-1}}$ be fibred differential forms on $(\mathscr{G}^{q-1}(D), D$, canonical projection) defined in a neighbourhood of E_{q-1} such that
70 $\alpha_1[E'], \ldots, \alpha_{t_{q-1}}[E']$ generate $J(E', \sum)$ for any integral element E' near E_{q-1} in $\mathscr{G}^{q-1}(D)$ (cf. n^o2.5, and Proposition 5). Now suppose f, expressed by $y_\lambda = \eta_\lambda(x_1, \ldots, x_q)$ is an integral of $(\sum)$ over an open neighbourhood U of $\tilde{\omega}(z)$ in $D, M = f(U)$ is an integral submanifold of D. If $(x, \eta(x)) \in M$ denote by E_x the tangent vector space $(M)_{x,\eta(x)}$ to M at $(x, \eta(x))$. E_x is an integral element of (Σ). Let $L'(E_x), \ldots, L^q(E_x)$ be a basis of E_x such that $\tilde{\omega}(L'(E_x)), \ldots, \tilde{\omega}(L^q(E_x))$ are dual to $dx_1|\tilde{\omega}(E_x), \ldots, dx_q|\tilde{\omega}(E_x)$. Let E'_x be the subspace of E_x generated by $L'(E_x), \ldots, L^{q-1}(E_x)$. Clearly E'_x is an integral element of (Σ). $L^q(E_x)$ must be a solution of $J(E'_x, \Sigma) = 0$ and hence a solution of $\alpha_\sigma[E'_x](L^q(E_x)) = 0$

$(\sigma = 1, \ldots, t_{q-1})(*)$. Let us express this condition $(*)$ by using the coordinate system. We have, for $\sigma = 1, \ldots, t_{q-1}$,

$$\alpha_\sigma =' b^i_\sigma dx_i +' a^\lambda_\sigma dy_\lambda$$

where $'b^i_\sigma$ and $'a^\lambda_\sigma$ are real analytic functions on a neighbourhood of E_{q-1} in $\mathscr{G}^{q-1}(D)$. Since (x, y) is a regular coordinate system $(dx_1)_z, \ldots, (dx_q)_z, \alpha_1[E_{q-1}], \ldots, \alpha_{t_{q-1}}[E_{q-1}], (dy_{t_{q-1}}+1)_z \ldots, (dy_m)_z$ are linearly independent. Hence we can find real analytic functions $c^{\sigma'}_\sigma(\sigma, \sigma' = 1, \ldots, t_{q-1})$ on a neighbourhood of E_{q-1} such that

$$c^{\sigma'}_\sigma \alpha_{\sigma'} = dy_\sigma - d^i_\sigma dx_i - a^{t+\mu}_\sigma \mathrm{dy}_{t+\mu}$$
$$(\sigma, \sigma' = 1, \ldots, t_{q-1}, t = t_{q-1}, \mu = 1, \ldots, m - t)$$

Since $dx_1|E_{q-1}, \ldots, dx_{q-1}|E_{q-1}$ are linearly independent, we have 71
the coordinate system $(x, y, w^1, \ldots, w^{q-1}, \ldots, y^r_\lambda, \ldots)(\lambda = 1, \ldots, m;$ $r = 1, \ldots, q-1)$ on a neighbourhood of E_{q-1} in $\mathscr{G}^{q-1}(D)$ such that

$$L^r(E') = \frac{\partial}{\partial x^r} + w^r(E')\frac{\partial}{\partial x_q} + y^r_\lambda(E')\frac{\partial}{\partial y_\lambda}$$

where $L^1(E'), \ldots, L^{q-1}(E')$ is a basis of E' dual to $dx_1|E', \ldots,$ $dx_{q-1}|E'$. Then

$$L^r(E'_x) = L^r(E_x) = \frac{\partial}{\partial x_r} + \frac{\partial \eta_\lambda}{\partial x_r}\frac{\partial}{\partial y_\lambda}$$

and therefore E'_x has the coordinates $w^r(E'_x) = 0$, $y^r_\lambda(E'_x) = \dfrac{\partial \eta_\lambda}{\partial x_r}$, Let $A^{t+\mu}(x, y, w, \ldots, y^r_\lambda, \ldots)$, $B_\sigma(x, y, w, \ldots, y^r_\lambda, \ldots)$ be the expression of the functions $a^{t+\mu}_\sigma, b^q_\sigma$ in terms of the coordinate system. Then the condition $(*)$ can be expressed as

$$\frac{\partial \eta_\sigma}{\partial x_q} = B_\sigma\left(x, \eta, \ldots, \frac{\partial \eta_\lambda}{\partial x_r}, \ldots\right) + A^{t+\mu}_\sigma\left(x, \eta, \ldots, \frac{\partial \eta_\lambda}{\partial x_r}, \ldots\right)\frac{\partial \eta_{t+\mu}}{\partial x_q} \quad (r = 1, \ldots, q-1)$$

On the otherhand, by the initial condition of η, we have

$$\eta_{t+\mu}(x_1,\dots,x_q) = \xi_{s_o+\cdots+s_{q-1}+\mu}(x_1,\dots,x_q).$$

Hence, by setting $C^{\xi}_{\sigma}(x_1,\dots,x_q)\, y_1,\dots,y_t, y^r_k,\dots)$ $(k = 1,\dots,t; r = 1,\dots,q-1)$, the function obtained by substituting $y_{t+\mu} = \xi_{s_o+\cdots+s_{q-1}}$ $+ \mu(x_1,\dots,x_q), y^i_{t+\mu} = \dfrac{\partial\xi_{s_o+\cdots+s_{q-1}} + \mu}{\partial x_i}(i = 1,\dots,q)$ in $B_\sigma(x, y, \dots,$
72 $y^r_\lambda,\dots) + A^{t+\mu}(x, y,\dots, y^r_\lambda,\dots) y^q_{t+\mu}$ in the above differential equation, the condition $(*)$ can be expressed by

$$\frac{\partial\eta_\sigma}{\partial x_q} = C^{\xi}_{\sigma}\left(x_1,\dots,x_q,\eta_1,\dots\eta_t,\dots,\frac{\partial\eta_k}{\partial x^r},\dots\right)$$
$$(\sigma, k = 1,\dots,t; r = 1,\dots,q-1).$$

Now we claim that the above two conditions are also sufficient. More precisely we have the following:

Proposition 16. *Let f be a cross-section of $(D, D'\tilde{\omega})$ over an open neighbourhood U of $\tilde{\omega}(z)$, expressed by $y_\lambda = \eta_\lambda(x_1,\dots,x_q)$, with the initial condition ξ. Assume that the tangent space of $M = f(U)$ at the point over $\tilde{\omega}(z)$ is sufficiently near the ordinary integral element E $(\dim E = q)$. Then f is an integral of (Σ) if and only if the following two conditions are satisfied:*

(i) *$y_\lambda = \eta_\lambda(x_1,\dots,x_{q-1},0)$ represent an integral of (Σ_1) with the initial condition $P(\xi)$;*

(ii) *$(y_1 = \eta_1(x_1,\dots,x_q),\dots,y_t = \eta_t(x_1,\dots,x_q))$ is a solution of the system of equations*

$$(**)\qquad \frac{\partial y_\sigma}{\partial x_q} = C^{\xi}_{\sigma}\left(x_1,\dots,x_q,y_1,\dots,y_t,\dots,\frac{\partial y_k}{\partial x_r},\dots\right)$$

In order to prove the proposition we make the following preliminaries:

Let φ be a homogeneous differential form of degree h on D and let f be a cross-section of $(D, D', \tilde{\omega})$ over an open neighbourhood of $\tilde{\omega}(z)$ in D. Denote, as before, by M the image $f(U)$. Let $i_M : M \to D$ be the injection mapping. Then

$$i_M^* \varphi = \sum_{i_1 < \cdots < i_h} \varphi_M^{i_1 \cdots i_h} dx_{i_1} \wedge \cdots \wedge dx_{i_h},$$

where $x_i o i_M$ is also denoted by x_i. Then by definition (cf. $n^o 2.4$) of 73
$\varphi[i_1, \ldots, i_h]$ we have

$$\varphi_M^{i_1, \ldots i_h}(x) = (\varphi[i_1, \ldots, i_h])(E_x),$$

where again E_x is the tangent vector space to M at x. E_x is regarded as an element of $\mathscr{G}^q(D)$. Since $i_M^*(d\varphi) = d(i_M^* \varphi)$, the above equality implies

$$(\chi) \sum_{s=1}^{h+1} (-1)^s \frac{\partial \varphi^{i_1, \ldots \hat{i}_s \cdots i_{h+1}}}{\partial x_{i_s}} = ((d\varphi)[i_1, \ldots, i_{h+1}])(E_x)$$

For a fibred differential form α of the fibred manifold $\mathscr{G}^{q-1}D, D$, the canonical projection), we define a function $\tilde{\alpha}$ of $\mathscr{G}^q(d; x_1, \ldots, x_q)$ as follows: For E in $\mathscr{G}^q(D; x_1, , \ldots, x_q)$ let $\pi(E)$ be the subspace defined by $dx_q = 0$, or equivalently generated by $L^1(E), \ldots, L^{q-1}(E)$. Set $\tilde{\alpha}(E) = (\alpha[\pi(E)])(L^q(E))$. In this notation the condition $(*)$ can be expressed as

$$(*\ 1) \qquad \tilde{\alpha}_\sigma(E_x) = 0 \ (\sigma = 1, \ldots, t_{q-1} = t),$$

where we adopt same notations as before. We choose real analytic functions $f_1, \ldots, f_k$ (defined on a neighbourhood of $E_{q-1} = \pi(E)$) from $F(\sum, x_1, \ldots, x_q)$ (cf. Proposition 7), such that $f_1 = \cdots = f_k = 0$ is a regular local equation of $\ell^{q-1} \sum$ on a neighbourhood of E_{q-1} in $\mathscr{G}^{q-1}D$. We recall that $f_1 o\pi = \cdots = f_k o \pi = \tilde{\alpha}_1 = \cdots = \tilde{\alpha}_t = 0$ is a regular local equation of $\ell^q \sum$ on a neighbourhood $\mathcal{U}$ of E (cf. Proof of Proposition 7). Take $\mathcal{U}$ so small that, besides the above property, we have real ana- 74
lytic functions $C_\sigma^{\sigma'}$, as in (#), defined on $\mathcal{U}$ and such that every element E of $\mathcal{U} \cap \ell^q \sum$ is ordinary and $\pi(E_1)$ is regular.

proof of the proposition 16. We have only to show that the conditions are sufficient. We assume that the tangent vector space of M at the point over $\tilde{\omega}(z)$ is in the above neighbourhood $\mathcal{U}$ of E. To prove that M is an integral submanifold is the same as proving that E_x are in $\ell^q \sum$. By the condition (2) which is equivalent to (∗) (and hence to (∗′)), $\tilde{\alpha}_\sigma(E_x) = 0$. Hence it remains to show that $f_i o\pi(E_x) = \cdots = f_k o\pi(E_x) = 0$. Set $g_\theta(x_1, \ldots, x_q) = f_\theta o\pi(E_x)$ for $\theta = 1, \ldots, k$. Since f_θ are in $F(\sum, x_1, \ldots, x_{q-1})$ each f_θ is expressed as

$$f_\theta = \varphi[i_1, \ldots, i_h], \varphi \in \sum^{(h)},$$

where $$1 \leq i_1 \cdots < i_h \leq q-1.$$

Therefore, by (χ), we obtain

$$(\chi\chi)(-1)^h \frac{\partial g_\theta}{\partial x_q} = \sum_{s=1}^{h} (-1)^s \frac{\partial \varphi_M^{i_1 \cdots \hat{i}_s \cdots i_h q}}{\partial x_{i_s}} - ((d\varphi)[i_1, \ldots, i_h, q])(E_x)$$

Since φ and $d\varphi$ are in $(\sum)$ and since

$$f_1 o\pi = \cdots = f_k o\pi = \tilde{\alpha}_1 = \cdots = \tilde{\alpha}_t = 0$$

is a regular local equation of $\ell^q \sum$ around E, we have

$$\varphi[i_1, \ldots, \hat{i}_s, \ldots, i_h, q] = X_s^\theta.(f_\theta o\pi) + Y_s^\lambda.\tilde{\alpha}_\lambda,$$
$$d\varphi[i_1, \ldots, i_h, q] =' X^\theta.(f_\theta o\pi) +' Y^\lambda.\tilde{\alpha}_\lambda,$$

75 where $X_s^\theta, ' X^\theta, Y_s^\lambda, ' Y^\lambda$ are real analytic functions on the neighbourhood of E. (This is here, where we use the closeness of $(\sum)$ most essentially). Therefore, by the definition of $\varphi_M^{i_1, \ldots i_h}$, the equation $(\chi\chi)$ can be written in the form

$$\frac{\partial g_\theta}{\partial x_q} = W_{\theta g_{\theta'}}^{\theta'} + Z_\theta^{\theta', r} \frac{\partial g_{\theta'}}{\partial x_r}.$$

In the above $\tilde{\alpha}(E'_x)$ and their derivatives do not appear because they are already known to be zero. This equation is of Cauchy Kowalewski type. Hence the solution $g_\theta(x_1, \ldots, x_q)$ is uniquely determined by functions $g_\theta(x_1, \ldots, x_{q-1}, 0)$. But $g_\theta(x_1, \ldots, x_{q-1}, 0)$ are zero, because of the condition (1). Then $g_\theta = 0$ is clearly the solution, so we proved that $g_\theta(x_1, \ldots, x_q) = 0$. Therefore M is an integral submanifold and f is an integral of $(\sum)$.

2.10

Now we are in a position to formulate and prove the main theorems of this Chapter. Let $[\sum, (D, D', \tilde{\omega})]$ be a defferential system with independent variables. Set $q = \dim D'$. Let E^o be a q-dimensional *ordinary* integral element of the system. Take a regular coordinate system $(x_1, \ldots, x_q, y_1, \ldots, y_m) = (x, y)$ in $(D, D', \tilde{\omega})$ with respect to E^o and $\sum$, which is known to exist by Proposition 14. To E^o we have already associated a system if characters S, and $H(S)$ was considered to be initial conditions for cross sections over open neighbourhoods of $\tilde{\omega}(z)$ where z is the origin of E^o (cf, 2.7). We remark that $x_i(z) = y_\lambda(z) = 0$.

Under the above notations and assumptions, we have the following theorem:

Theorem 1. (i) *For any given ξ in $H(S)$, the germ of integrals of $[\sum, (D, D', \tilde{\omega})]$ over $\tilde{\omega}(z)$ with the initial condition ξ (with respect to (x, y)) is unique if it exists;* 76

(ii) *there exists a solution mapping F with respect to (x, y) such that the mapping $\xi \to F(\xi) - F(0)$ is finite analytic;*

(iii) *if $y_\lambda = 0$ is an integral, then there exists a regular solution mapping and any two such regular solution mappings are equivalent and determine a germ of infinite analytic mapping.*

Let p be the largest integral such that $s_p \neq 0$, $s_{p+1} = \cdots = s_q = 0$.

Theorem 2. *Let M_0 be a germ of integral submanifolds of $[\sum, (D, D', \tilde{\omega})]$ at z. Assume that the tangent vector space E^0 to M_0 at z is an ordinary integral element of $(\sum)$. Then the set of germs over $\tilde{\omega}(z)$ of integrals of $[\sum, (D, D', \tilde{\omega})]$ depend on s_p functions in p variables around M_0.*

The proofs of theorems 1 and 2 are given in the following section.

2.11

We reduce the problem to the case of $(q-1)$-variables. For this purpose we introduce the mapping P of $H(S)$ into $H(S')$, where $S=(s_0,\ldots,s_q)$ and $S'(s_0,\ldots,s_{q-2},s_{q-1}+s_q)$, defined by

$$P_\sigma(\xi)(x_1,\ldots,x_{q-1})=\xi_\sigma(x_1,\ldots,x_{q-1}), \sigma\le s_o+\cdots+s_{q-1}$$
$$P_{s_o+\cdots+s_{q-1}+\sigma}(\xi)(x_1,\ldots,x_{q-1})$$
$$=\xi_{s_o+\cdots+s_{q-1}+\sigma}(x_1,\ldots,x_{q-1},0) \text{ for } 1\le\sigma\le s_q$$

77 Let $(D_1,D'_1,\tilde{\omega})$ be the subfibred manifold of $(D,D',\tilde{\omega})$ where D_1 (resp. D'_1) is defined by the coordinate system $(x_1,\ldots,x_{q-1},0,y_1,\ldots,y_m)$ (resp. $(x_1,\ldots,x_{q-1},0)$) and $\tilde{\omega}_1$ is the restriction of $\tilde{\omega}$ to D_1. Let $(\sum_1)$ be the differential system on D_1 generated by the restriction of $(\sum)$ to D_1 and let E be an ordinary integral element of $[\sum,(D,D',\tilde{\omega})]$.

Now according to Proposition 16 a section f of $(D,D',\tilde{\omega})$ represented by $y_\lambda=\eta_\lambda(x_1,\ldots,x_q)$ with the initial condition ξ at $\tilde{\omega}(z)$ is an integral of $[\sum,(D,D',\tilde{\omega})]$ if and only if the following two conditions are satisfied:

(1) $y_\lambda=\eta_\lambda(x_1,\ldots,x_{q-1},0)$ represents an integral of $[\sum_1,(D_1,D'_1,\tilde{\omega}_1)]$ with the initial condition $P(\xi)$ at $\tilde{\omega}(z)$

(2) $y_k=\eta_k(x_1,\ldots,x_q)$ for $k=1,\ldots t=t_{q-1}$, is the system of solutions of the Cauchy -Kowalewski system of equations

$$(\mathscr{G}^\xi)\frac{\partial y_\sigma}{\partial x_q}=A^\xi_\sigma(x,y_1,\ldots,y_t,\frac{\partial y'_\sigma}{\partial x_r}(\sigma,\sigma'=1,\ldots,t;r=1\cdots q-1)$$

Here $A^\xi_\sigma(x_1,\ldots,x_q,y_1,\ldots,y_t,\ldots,y^r_{\sigma'},\ldots)$ is equal to

$$A_\sigma\Big(x_1,\ldots,x_q,y_1,\ldots,y_t,\ldots,y^r_\sigma,\ldots,\xi_{t+1}(x_1,\ldots,x_q),$$
$$\ldots,\xi_m(x_1,\ldots,x_q),\ldots,\frac{\partial\xi_{t+\mu}}{\partial x_j},\ldots\Big)$$

with $A_\sigma(x_1,\ldots,x_q,y_1,\ldots,y_t,\ldots,y^r_{\sigma'},\ldots,y_{t+1},\ldots y_m,\ldots,y^j_{t+\mu},\ldots)$ are analytic functions of all their arguments for $t=s_{-1}+\cdots+s_{q-1};r=$

$1, \ldots, q-1; j = 1, \ldots, q'\sigma, \sigma' = 1, \ldots, t$. $(\mathscr{G}^{\xi})$ is a system of equations of Cauchy-Kowalewski type with parameter in which the derivatives of the parameters also applear . There exists a unique solutions of the system $(\mathscr{G}^{\xi})$ by Theorem 5 of Chapter I. Hence, by Theorems 5 and 6 of Chapter *I*, there exists a solution mapping F'' of $(\mathscr{G}^{\xi})$ and $\exists''$ represents a germ of infinite analytic maps of $H^t_{q-1} + H^{m-t}_q$ into H^t_q. **78**

We complete the proof of Theorem 1 by induction on q.

(i) Let f be a germ of integrals of $[\sum, (D, D', \tilde{\omega})]$ over $\tilde{\omega}(z)$ with the initial condition ξ. Then the restriction f_1 of the cross-section f to D_1 is a germ of integrals of $[\sum_1, (D, D'_1, \tilde{\omega}_1)]$ with the initial condition $P(\xi)$ at $\tilde{\omega}(z)$. So f_1 is uniquely determined by the induction assumption. Then f is unique since $\eta_1(x), \ldots, \eta_t(x)$ is the solution of $(\mathscr{G}^{\xi})$ with the prescribed initial functions $\eta_\sigma(x_1, \ldots, x_{q-1}, 0)$ and $\eta_{t+\tau}(x) = \xi_\tau(x)$.

(ii) Let E' be the subspace of E defined by $dx_q = 0$. By Proposition 16, E' is an ordinary integral element of $[\sum_1(D_1, D'_1, \tilde{\omega}_1)]$ and $(x_1, \ldots, x_{q-1}, 0, y)$ is a regular coordinate system of $(D_1, D'_1, \tilde{\omega}_1)$ with respect to E' and $(\sum_1)$. Let F' be a solution mapping of $[\sum_1(D_1, D'_1, \tilde{\omega}_1)]$ with respect to $(x_1, \ldots, x_{q-1}, 0, y)$. F' is a mapping of $H(S'; u_1, v_1)$ into $H^m_{q-1}(u'_1, v'_1)$ with suitable u_1, v_1, u'_1, v'_1, clearly we can choose u'_1 arbitrarily small. Since the mapping $\zeta \to F'(\zeta) - F'(0)$ is finite analytic, choosing v_1 small, we can make sup $\left\{\left|F'(\zeta) - F'(0)\right|; \zeta \in H(S'; u_1, v_1)\right\}$ arbitrarily small. On the otherhand $[F(0)](0) = 0$ because the cross-section $y_\lambda = F_\lambda(0)$ must pass through z. Therefore we can choose v'_1 arbitrarily small (cf. Proposition 1, Chapter I). Hence we can assume without loss of generality that $\mathscr{F}''$ has a representative F'' which maps $H^t_{q-1}(u'_1, v'_1) +$ **79**
$H^{m-1}_q u'_1, v'_1)$ into $H^t_q(u', v')$. Set $u = \max(u_1, u'_1), v = \min(v_1, v'_1)$. Let F be the mapping of $H(S; u, v)$ into $H^m_q(u', v')$ defined by

$$F_\sigma(\xi) = F''((F'_1(P(\xi)), \ldots, F'_t(P(\xi))), (\xi_{t+1}, \ldots, \xi_m))(\sigma = 1 \cdots t),$$

$F_{t+\lambda}(\xi) = \xi_{t+\lambda}(\lambda = 1 \cdots m-t)$. Then by Proposition 16 it is easy to verify that F satisfies the conditions of the solution mapping except for the fact that the mapping $\xi \to F(\xi) - F(0)$ is infinite analytic. But this follows from the remark in section 1.9 (see page 25),

(iii) is clear from (ii).

Proof of Theorem 2. Choose (x, y) such that M_0 is represented by $y_\lambda = 0$. We define a germ $\mathscr{Y}$ of finite analytic mappings of H^m_q into $H(S)$ as follows: Take v sufficiently small and for each $\eta \in H^m_q$ let f_η be the germ of cross-sections of $(D, D', \tilde{\omega})$ represented by $y_\lambda = \eta_\lambda(x_1, \ldots, x_q)$ around $\tilde{\omega}(z)$. Denote by $G(\eta)$ the initial condition of f_η with respect to (x, y). Now it is clear that $\mathscr{Y}_0\mathscr{F}$ is identity since $\mathscr{Y}$ associates the initial condition. The definition of $\mathscr{F}$ implies the second condition of the definition of the parametrization. On the otherhand the choice of S and p implies that $H(S) \cong H^{s_p}_p$. This proves the Theorem 2.

Proposition 17. *Under the notation of Theorem 1, let M be the integral of* $[\sum, (D, D', \tilde{\omega})]$ *with initial condition* 0. *Then the tangent vector space of M at z is equal to E^o.*

80 *Proof.* (by induction on q) Denote by E the tangent vector space. By induction assumption applied to $[\sum_1, (D_1, D'_1, \tilde{\omega}_1)]$,, we can assume that $L_r(E) = L_r(E^o)$ for $r = 1, \ldots, q-1$. Since M has initial condition 0, the condition (iv) of the regular coordinate system implies that $\langle dy_{t+\rho}, L_q(E)\rangle = \langle dy_{t+\rho}, L_q(E^o)\rangle$ for $\rho = 1, \ldots, m-t$. Since $L_q(E)$ and $L_q(E^o)$ are solutions of $J(E_{q-1}, \sum) = 0$, the condition (iii) of the regular coordinate system implies that $L_q(E) = L_q(E^o)$. Hence $E = E^0$. □

corollary or proposition 17. Let E be an ordinary integral element (with origin z) of a differential system $(\sum)$. Then there is an integral submanifold M of $(\sum)$ such that the tangent vector space to M at z is equal to E.

Chapter 3

Prolongation of Exterior Differential Systems

3.1

In the chapter, we shall introduce the notion of jets of mappings of one 81
manifold into another and give the construction of prolongation of a given differential system. For this purpose we make use of the notion of ℓ- jets of mappings.

Let M' and M be two infinitely differentiable (C^∞) manifolds and let x' and x denote points of M' and M respectively. Let f be a C^∞ mapping of an open neighbourhood of x' in M' into M. We shall introduce an equivalence relation in the set of all such C^∞ maps f. The open neighbourhood of x' may depend on the function f. Let $(w_1, \ldots, w_{n'})$ and $(x_1, \ldots, x_n)$ be coordinate systems at x' in M' and x in M respectively. Let ℓ be an integer ≥ 0.

Definition. *Two C^∞ mappings f and g, of open neighbourhood of x' in M' into M, are said to be ℓ-equivalent, and is denoted by $f\tilde{\ell}g$, if , for every $h \leq \ell$,*

$$\frac{\partial^h f_i}{\partial w_{i_1} \cdots \partial w_{i_h}}(x') = \frac{\partial^h g_i}{\partial w_{i_1} \cdots \partial w_{i_h}}(x')$$

for all $(i_1, \ldots, i_h)$ where f_i and g_i are the components of f and g respec-

tively.

Clearly $\tilde{\ell}$ is an equivalence relation and this definition of equivalence
82 is independent of the choice of the coordinate systems as can be easily verified.

Definition. *An equivalence class of C^∞ mappings such as above under $\tilde{\ell}$ is called an ℓ*jet of mappings *at x'.*

An ℓ-jet of mappings at x' containing a mapping f is denoted by $j^\ell_{x'}(f)$.

Example. *Let M' be the real line R and x' the origin. Then any C^∞ mapping f of an open neighbourhood of* 0 *into M is a C^∞ curve through the point $x = f(0)$ in M. A jet is thus a generalisation of the notion of high order of contact of two curves.*

Let $J^\ell(M', M) = \cup\left\{j^\ell_{x'}(f) : x' \in M'\right\}$ be the set of all ℓ-jets of C^∞ mappings of M' into M. Let α (resp. β) be the mapping of $J^\ell(M', M)$ onto M'(resp. M) which associates to every jet $j^\ell_{x'}(f)$ in $J^\ell(M', M)$ the point x' of M' (resp. x of M). the point x' is called the source and the point $x = f(x')$ the target of the jet $j^\ell_{x'}(f)$.

We can provide $J^\ell(M', M)$ with the structure of a C^∞ manifolds as follows: Let $X \in J^\ell(M', M)$ with $\alpha(X) = x'\beta(X) = x$, and suppose V' and V be coordinate neighbourhoods of x' and x in M' and M respectively. Let $(w_1, \ldots, w_{n'})$ and $(x_1, \ldots, x_n)$ be the coordinate systems at x' and x in V' and V respectively. Denote by $\mathscr{H}$ the set

$$\mathscr{H} = \left\{X' \in J^\ell(M', M) : \alpha(X') \in V' \text{ and } \beta(X') \in V\right\}$$

$\mathscr{H}$ can be taken as a coordinate neighbourhood of X in $J^\ell(M', M)$ defin-
83 ing the manifold structure. The explicit coordinate system at X in $\mathscr{H}$ can be given as follows:

Suppose $X' = j^\ell_y(f) \in \mathscr{H}$ with $\alpha(X') = y$. The mapping

$$X' \to \left(\alpha(X'), \beta(X'), \ldots, \frac{\partial^h f_j}{\partial w_{i_1} \cdots \partial w_{i_h}}(y), \ldots\right).$$

of $\mathscr{H}$ into $V' \times V \times \left\{(\ldots, P^{i_1\cdots i_h}_j, \ldots)\right\}$, where $h \leq \ell, 1 \leq i_1, \ldots, i_h \leq n, j = 1, \ldots, m)$, is objective. Here the functions $P^{i_1,\ldots i_h}_j$ are assumed to be symmetric with respect to $i_1 \cdots i_h$. Clearly this mapping is well defined independent of the choice of the respective f of $j^\ell_y(f)$. Thus a coordinate system at X' is

$$\alpha(X'), \beta(X'), \ldots, \frac{\partial^h f_j}{\partial w_{i_1} \cdots \partial w_{i_h}}(y), \ldots$$

The change of coordinate can again be verified to be C^∞. Therefore, this defines a C^∞ manifold structure on $J^\ell(M', M)$.

When M' and M are real analytic manifolds, $J^\ell(M', M)$ can also be made a real analytic manifold in the same way. This is the case in which we will be interested in. It is, now, easy to see that $(J^\ell(M', M), M' \times M, \alpha \times \beta)$ is a fibre bundle over $M' \times M$ with projection mapping $\alpha \times \beta$ and the structure group a linear group.

Let $(M, M', \tilde{\omega})$ be a fibred manifold. Let us denote by $J^\ell(M, M', \tilde{\omega})$ the set of all jets $X = j^\ell_{x'}(f)$ in $J^\ell(M', M)$ of cross-sections f of $(M, M', \tilde{\omega})$ over open neighbourhoods of x' in M'. $J^\ell(M, M', \tilde{\omega})$ is a real analytic submanifold of $J^\ell(M', M)$ as is clear from the following: **84**

Let (x, y) be a coordinate system in $(M, M', \tilde{\omega})$, and let $\mathcal{V}$ be the coordinate neighbourhood in $J^\ell(M', M)$ associated with (x, y). Then $\mathcal{V} \cap J^\ell(M, M', \tilde{\omega})$ is a submanifold. In fact, let $X \in \mathcal{V}$ with $\alpha(X) = x^o$ be represented by a mapping $(X) \to (f(x), g(x))$. X has the coordinate $x^0, f(x^0), g(x^0), \ldots, \dfrac{\partial^h f_j}{\partial x_{i_1} \cdots \partial x_{i_h}}(x^o), \ldots, \dfrac{\partial^h g_\lambda}{\partial x_{i_1} \cdots \partial x_{i_h}}(x^o), \ldots)$ in $\mathscr{V}$. Then X is in $J^\ell(M, M', \tilde{\omega})$ if and only if $f(x^o) = x^o, \dfrac{\partial f_j}{\partial x_i}(x^o) = \delta^j_i, \dfrac{\partial^{h_{f_j}}}{\partial x_{i_1} \cdots \partial x_{i_h}}(x^o) = 0$ for $h \geq 2$. Thus $\mathscr{V} \cap J^\ell(M, M', \tilde{\omega})$ is a submanifold. Moreover, it has a coordinate system $(x, y, y^{i_1 \cdots i_h})$. More explicitly $X = j^\ell_x(g)$, where $g(x') = (x', g_\lambda(x')$ is a cross-section, has the coordinates: $y_\lambda = g_\lambda(x), y^{i_1 \cdots i_h} = \partial^{h_{g_\lambda}} / \partial x_{i_1} \cdots \partial x_{i_h}$. Thus coordinate system in $J^\ell(M, M', \tilde{\omega})$ is called the coordinate system corresponding to the coordinate system (x, y) of $(M, M', \tilde{\omega})$.

Let U' be an open set of a manifold M' and M be another manifold. For a mapping f of U' into M we denote by $j^\ell(f)$ the submanifold of

$J^\ell(M', M)$ defined by the mapping

$$U' \to J^\ell(M', M)$$
$$x' \to j^\ell_{x'}(f)$$

This mapping is injective because $\alpha o j^\ell_{x'}(f) = x'$. If f is a cross-
85 section of a fibred manifold $(M, m', \tilde{\omega})$ over an open set U' of M' then
$j^\ell(f)$ is a cross-section of the fibred manifold $(J^\ell(M, M'\tilde{\omega},), M', \alpha)$.

3.2

Consider now a fibred manifold $(D, D', \tilde{\omega})$ where D and D' are domains in Euclidean spaces $\underline{R}^p \times \underline{R}^m$ and $\underline{R}^p$ respectively, and where $\tilde{\omega}$ is the projection. We shall denote by $\prod^{[1]}(\ell)$ the set of all differential forms ω of degree 1 on $J^\ell(D, D', \tilde{\omega})$ such that for any cross- section f of $(D, D', \tilde{\omega})$ over an open set of D', the restriction $\omega | j^{\ell(f)}$ is zero. This can equivalently be expressed by saying $j^\ell(f)^*(\omega) = 0$ when $j^\ell(f)$ is regarded as cross- section of $(J^\ell(D, D', \tilde{\omega}), D', d)$.

Proposition 1. $\prod^{[1]}(\ell)$ *is finitely generated over the ring of real analytic functions* $\wedge^0(J^\ell(D, D', \tilde{\omega}))$.

More precisely, if (x, y) is a coordinate system in $(D, D', \tilde{\omega})$ and if $(x', x, y, \ldots, y^{i_1 \ldots i_h}, \ldots)$ is the corresponding coordinate system of $J^\ell(D, D', \tilde{\omega})$ then $\prod^{[1]}(\ell)$ is generated over $\wedge^0(J^\ell(D, D', \tilde{\omega}))$ by

$$\omega_\lambda = dy_\lambda - y^i_\lambda dx_i$$
$$\omega^{i_1 \cdots i_h}_\lambda = dy^{i_1 \cdots i_h}_\lambda - y^{i_1 \cdots i_h i}_\lambda dx_i \ (h \leq \ell - 1)$$

Proof. First of all we shall show that $\omega^{i_1 \cdots i_h}_\lambda$ are in $\prod^{[1]}(\ell)$. Let f be a cross-section of $(D, D', \tilde{\omega})$ over an open set U' of D' represented by
86 $y_\lambda = f_\lambda(x_1, \ldots, x_q)$ then $j^\ell(f)$ is represented by

$$y_\lambda = f_\lambda(x);$$
$$y^{i_1 \cdots i_h}_\lambda = \frac{\partial^h f_\lambda}{\partial x_{i_1} \cdots \partial x_{i_h}}$$

□

Hence

$$j^{\ell}(f)^{*}\omega_{\lambda} = df_{\lambda} - \left(\frac{\partial f_{\lambda}}{\partial x_{i}}\right) dx_{i} = 0;$$

$$j^{\ell}(f)^{*}\omega_{\lambda}^{i_{1}\cdots i_{h}} = d\left(\frac{\partial^{h} f}{\partial x_{i}\cdots\partial x_{i_{h}}}\right) - \left[y_{\lambda}^{i_{i}\cdots i_{h}.i}\, j^{\ell}(f)\right] dx_{i}$$

$$= d\left(\frac{\partial^{h} f}{\partial x_{i_{1}}\cdots\partial x_{i_{h}}}\right) - \frac{\partial^{h+1} f_{\lambda}}{\partial x_{i_{1}}\cdots\partial x_{i_{1}}\partial x_{i}} dx_{i} = 0.$$

Conversely, let $\omega \in \prod^{[1]}(\ell)$. We know that $\left\{dx_{i}, dy_{\lambda}^{i_{1}\cdots i_{h}}\right\}$ form a basis of Pfaffian forms on D. Hence any $\omega \in \prod^{[1]}(\ell)$ can be expressed as

$$\omega = a^{i} dx_{i} + b^{\lambda}_{i_{1}\cdots i_{\ell}} dy_{\lambda}^{i_{1}\cdots i_{\ell}} + c^{\lambda}_{i_{1}\cdots i_{h}}\omega_{\lambda}^{i_{1}\cdots i_{h}} (h \leq \ell - 1)$$

Let f be any cross - section of $(D, D', \tilde{\omega})$ represented by $y_{\lambda} = f_{\lambda}(x)$. Then we obtain

$$j^{\ell}(f)^{*}(\omega) = [a^{i} o j^{\ell}(f) dx_{i}] + \left[b^{\lambda}_{i_{1}\cdots i_{\ell}} o j^{\ell}(f)\right] \frac{\partial^{\ell+1} f}{\partial x_{i_{1}}\cdots\partial x_{i_{\ell}}\partial x_{i}} dx_{i}$$

These $j^{\ell}(f)^{*}(\omega)$ are differential forms on D'. A necessary and sufficient condition for $j^{\ell}(f)^{*}(\omega) = 0$ is that

$$\left[a^{i} o j^{\ell}(f)\right] dx_{i} + \left[b^{\lambda}_{i_{1}\cdots i_{\ell}} o j^{\ell}(f)\right] \frac{\partial^{\ell+1} f}{\partial x_{i_{1}}\cdots\partial x_{i_{\ell}}\partial x_{i}} dx_{i} = 0.$$

This equality holds for any cross-section f if and only if $a^{i} = 0$, **87**
$b^{\lambda}_{i_{1}\cdots i} = 0$. Therefore we have $\omega = c^{\lambda}_{i_{1}\cdots i_{h}}\omega_{\lambda}^{i_{1}\cdots i_{h}}$ so much so that $\omega_{\lambda}^{i_{1}\cdots i_{h}}$ generate $\prod^{[1]}(\ell)$.

Let us denote by $\prod(\ell)$ or by $\prod(\ell; (D, D'\tilde{\omega}))$ if there is any possibility of confusion, the differential system generated by $\prod^{[1]}(\ell)$ over the ring $\wedge^{0}(J^{\ell}(D, D', \tilde{\omega}))$.

Proposition 2. *Let F be a cross-section of the fibred manifold $(J^{\ell}(D, D', \tilde{\omega}), D', \alpha)$ over an open set U of D'. Then there exists a cross-section f of $(D, D'\tilde{\omega})$ over U such that $F = j^{\ell}(f)$ if and only if F is an integral of $\prod(\ell)$.*

Proof. It is immediate that F is an integral of $\prod(\ell)$ if $F = j^{\ell}(f)$ because of the definition of $\prod(\ell)$. □

Conversely, F being a cross-section it can be expressed by $y_\lambda = F_\lambda(x), y_\lambda^{i_1\cdots i_h} = F_\lambda^{i_1\cdots i_h}(x)$ so that we can write

$$F^*\omega_\lambda^{i_1\cdots i_h} = dF_\lambda^{i_1\cdots i_h} - F_\lambda^{i_1\cdots i_h i}(x)dx_i.$$

F being an integral of $\prod(\ell), F^*\omega_\lambda^{i_1\cdots i_h} = 0$ and therefore we obtain that $F_\lambda^{i_1\cdots i_h i}(x) = \dfrac{\partial F_\lambda^{i_1\cdots i_h}}{\partial x_i}(x)$. Therefore $F = j^{\ell}(f)$ where f is represented by $y_\lambda = f_\lambda(x)$ over the open set U of D' and this proves the existence of a section f of $(D, D', \tilde{\omega})$ such that $F = j^{\ell}(f)$.

Suppose $(D, D', \tilde{\omega})$ is a fibred manifold with $\dim D' = p$. Then we can identify $J^1(D, D', \tilde{\omega})$ with of $\mathscr{G}^p(D, D'.\tilde{\omega})$ canonically by means of
88 the following map: Let $X \in J^1(D, D', \tilde{\omega})$. If $X = J_z^1(f)$ we associate to X the element $(df)_{z'}((D')_{z'})$ of $\mathscr{G}^p(D, D', \tilde{\omega})$. Here we observe the fact that $(df)_{z'}$, is injective. It is clear that this canonical identification is independent of the choice of the representative section f of the jet X.

3.3

We shall define the notion of ℓ jets of differential forms on a C^∞ manifold M. Let $z \in M$ and $(x_1, \ldots, x_n)$ be a coordinate system at z in M. Let φ and θ be two differential forms of the same degree (say a) having the following expressions with respect to the coordinate system $(x_1, \ldots, x_n)$:

$$\varphi = \sum_{i_1<\cdots<i_a} \varphi^{i_1\cdots i_a} dx_{i_1} \wedge \cdots \wedge dx_{i_a}$$

and

$$\theta = \sum_{i_1<\cdots<i_a} \theta^{i_1\cdots i_a} dx_{i_1} \wedge \cdots \wedge dx_{i_a}$$

respectively.

Definition. *φ is said to be $\ell-$ equivalent to θ, and is denoted by $\varphi\tilde{\ell}$, if*

$$j_z^{\ell}\left(\varphi^{i_1\cdots i_a}\right) = j_z^{\ell}\left(\theta^{i_1\cdots i_a}\right)$$

It is easy to verify that $\tilde{\ell}$ is an equivalence relation and is independent of the choice of the coordinate system.

Definition. *An equivalence class of differential forms is called an $\ell-$* jet of differential forms *on M at z and is denoted by $j^{\ell}_{z}(\varphi)$.*

The following are almost immediate consequences of this definition.

(1) $j^{\ell+1}_{z}(\varphi) = j^{\ell+1}_{z}(\theta)$ implies $j^{\ell}_{z}(d\varphi) = j^{\ell}_{z}(d\theta)$. **89**

(2) If M' and M are two C^{∞} manifolds, f and g are two C^{∞} maps of M' into M, and φ and θ are two differential forms on M such that $j^{\ell}_{z'}(f) = j^{\ell}_{z'}(g)$ and $j^{\ell}_{z}(\varphi) = j^{\ell}_{z}(\theta)$ where $z = f(z') = g(z')$ then $j^{\ell-1}_{z'}(f^{*}\varphi) = j^{\ell-1}_{z'}(g^{*}\theta)$.

Now consider an exterior differential system $[\sum, (D, D', \tilde{\omega})]$ with independent variables. We pose the following definition.

Definition. *An $\ell-$ jet $X \in J^{\ell}(D, D', \tilde{\omega})$ is an integral $\ell-$ jet of the system $[\sum, (D, D', \tilde{\omega}]$ if the jet $X = j^{\ell}_{z}(f)$ satisfies $j^{\ell-1}_{z'}(f^{*}\varphi) = o_{z'}$ the zero $\ell-$ jet of differential forms at z', for every $\varphi \in (\sum)$.*

This is again independent of the choice of f. By the canonical identification of $J^{1}(D, D'\tilde{\omega})$ and $\mathscr{Y}^{p}(D, D', \tilde{\omega})$, the notion of integral 1-jets is equivalent to the notion of p-dimensional integral elements.

Fix a coordinate system (x, y) of $(D, D', \tilde{\omega})$. For any φ in $\wedge^{a}(D)$ and for set of integers $i_1 < \cdots < i_a$ define a mapping $F^{i_1\cdots i_a}_{\varphi}$ of $J^{\ell}(D, D', \tilde{\omega})$ into $J^{\ell}(D', R)$ by setting $F^{i_1\cdots i_a}_{\varphi}(X) = j^{\ell-1}_{\alpha(X)}(\varphi^{i_1\cdots i_a}_{f})$ where $X = j^{\ell}_{\alpha(X)}(f) \in J^{\ell}(D, D\tilde{\omega})$ and $\varphi^{i_1\cdots i_a}_{f}$ are the coefficients in $f^{*}\varphi = \sum \varphi^{i_1\cdots i_a}_{f} dx_{i_1 \wedge\cdots\wedge dx_{i_a}}$. Let $(x, \ldots, w^{i_1\cdots i_r}, \ldots)r \leq \ell - 1$, be a coordinate system in $J^{\ell-1}(D'R)$ where $w^{i_1\cdots i_r}$ is given by $w^{i_1\cdots i_r}(L) = \dfrac{\partial r_g}{\partial x_i \cdots \partial x_{i_r}}$ for any $j^{\ell-1}_{z}(g) = L \in j^{\ell-1}(D', R)$. Denote $w^{i_1\cdots i_r}(F^{k_1\cdots k_a}_{\varphi}(X))$ by $F^{k_1\cdots k_a; i_1\cdots i_r}_{\varphi}(X)$.

In particular $F^{k_1\cdots k_a}_{\varphi} = \varphi^{k_1\cdots k_a}_{f}$ (the case $r = 0$). It can be verified that **90**
$F^{k_1\cdots k_a; i_1\cdots i_\varphi}$ is a real analytic function on $J^{\ell}(D, D'\tilde{\omega})$, when the form φ is real analytic.

Remarks. (1) *Given a differential system* $[\sum, (D, D'\tilde{\omega})]$ *with independent variables, an* $\ell-$ *jet* X *in* $J^{\ell}(D, D', \tilde{\omega})$ *is an integral* $\ell-$ *jet of the system if and only if each* $F_{\varphi}^{k_1\cdots k_a;i_1\cdots i_r}(X) = 0$ *for any* $\varphi \in \sum^{(a)}(0 \leq k_1, \ldots, k_a \leq p; 1 \leq i_1, \ldots i_r \leq p; a = 0, 1, \ldots; r \leq l-1)$*. This is an immediate consequence of the definition of an integral* $\ell-$ *jet of such a differential system.*

(2) $F_{\varphi}^{k_1\cdots k_a;i_1\cdots i_r}$ *is symmetric with respect to* $i_1, \ldots i_r$ *and anti-symmetric with respect to* $k_1 \cdots k_a$.

We shall denote by $\mathscr{F}^{\ell}(\sum)$ *the set of all* $F_{\varphi}^{k_1\cdots k_a;i_1\cdots i_r}$ *where* $\varphi \in \sum^{(a)}, (a = 0, 1, \ldots$ *and* $r \leq \ell-1)$*. Therefore an* $\ell-$ *jet* X *is an integral of* $[\sum, (D, D', \tilde{\omega})]$ *if and only if* $F(X) = 0$ *for every* $F \in \mathscr{F}^{\ell}(\sum)$.

(3) *If* φ, ψ *are two differential forms of degree* a*, then* $F_{\varphi+\psi}^{k_1\cdots k_a;i_1\cdots i_r} = F_{\varphi}^{k_1\cdots k_a;i_1\cdots i_r} + F_{\psi}^{k_1\cdots k_a;i_1\cdots i_r}$.

(4) *If* φ *is a differential form of degree* a_1 *and* ψ *is a differential form of degree* a_2*, then, setting*

$$a = a_1 + a_2 F_{\varphi+\psi}^{k_1\cdots k_a;i_1\cdots i_r} = 0 \mod F_{\psi}^{h_1\cdots h_{a_2};j_1\cdots j_s}, s \leq r).$$

Denote by $\rho_{\ell}^{\ell'}(\ell' \geq \ell)$ the natural projection of $J^{\ell'}(D, D', \tilde{\omega})$ onto $J^{\ell}(D, D', \tilde{\omega})$. If we write ${}^{(\ell)}F_{\varphi}^{k_1\cdots k_a;i_1\cdots i_r}$ the function $F_{\varphi}^{k_1\cdots k_a;i_1\cdots i_r}$ on $J^{\ell}(D, D', \tilde{\omega})$ for the sake of precision, then

$$(\ell')F_{\varphi}^{k_1\cdots k_a;i_1\cdots i_r} = {}^{(\ell)}F_{\varphi}^{k_1\cdots k_a;i_1\cdots i_r} o \rho_{\ell}^{\ell'}$$

91 Because of this relation, there will be no confusion when we omit the index (ℓ) in ${}^{(\ell)}F_{\varphi}^{k_1\cdots k_a;i_1\cdots i_r}$. Thus $F_{\varphi}^{k_1\cdots k_a;i_1\cdots i_r}$ is a function on J^{ℓ} $(D, D'\tilde{\omega})$ for $\ell \geq r+1$.

Proposition 3. *If* φ *is a differential form of degree* a*, then for* $r \leq \ell - 2$ *we have the identity*

$$dF_{\varphi}^{k_1\cdots k_a;i_1\cdots i_r} \equiv F_{\varphi}^{k_1\cdots k_a;i_1\cdots i_r i} dx_i \quad (\text{mod } \prod(\ell)).$$

Proof. Suppose $j^\ell_z(f) = x \in J(D, D', \tilde{\omega})$; then we have

$$f^*\varphi = \sum_{k_1<\cdots<k_a} \varphi_f^{k_1\cdots k_a} dx_{k_1} \wedge \cdots \wedge dx_{k_a}$$

□

Therefore, $F_\varphi^{k_1\cdots k_a;i_1\cdots i_r}(X) = \left(\dfrac{\partial\varphi_f^{k_1\cdots k_a}}{\partial x_{i_1}\cdots\partial x_{i_r}}\right)_{z=\alpha(X)}$ On the otherhand, because $dx_i, \omega_\lambda, \ldots \omega_\lambda^{i_1\cdots i_a}(a \leq \ell - 1), dy_\lambda^{i_1\cdots i_\ell}$ form a base for Pfaffian forms on $J^\ell(D, D', \tilde{\omega})$ and $\omega_\lambda^{i_1\cdots i_a}$ generate $\prod(\ell)$, we can write

$$dF_\varphi^{k_1\cdots k_a;i_1\cdots i_r} \equiv A_j dx_j + A^\lambda_{j_i\cdots j} dy_\lambda^{j_1\cdots j_1}$$

modulo $\prod(\ell)$. But for $r = \ell - 2, F_\varphi^{k_1\cdots k_a;i_1\cdots i_r}$ are functions only of the arguments $x, y, \ldots, y_\lambda^h, \ldots, y_\lambda^{h_1\cdots h_{r+1}}$. Hence the terms $A^\ell_{j_1\cdots j_\ell} dy_\lambda^{j_1\cdots j_\ell}$ do not appear in the expression of $dF_\varphi^{k_1\cdots k_a;i_1\cdots i_r}$, i.e., $dF_\varphi^{k_1\cdots k_a;i_1\cdots i_r} \equiv A_j dx_j$ modulo $\prod(\ell)$.

Now $j^\ell(f)$ being a cross-section of $J^\ell(D, D', \tilde{\omega})$ over an open neighbourhood of $z = \alpha(X)$ we obtain 92

$$j^\ell(f)^*\left(dF_\varphi^{k_1\cdots k_a;i_1\cdots i_r}\right) \equiv \left[A^j o j^\ell(f)\right] dx_j \text{ modulo } [j^\ell(f)^*\left(\prod(\ell)\right)]$$

Since $j^\ell(f)^*(\prod(\ell)) = 0$ the above congruence becomes an inequality,

$$(j^\ell(f))^*\left(dF_\varphi^{k_1\cdots k_a;i_1\cdots i_r}\right) = [A^j o j^\ell(f)]dx_j.$$

On the other hand

$$\begin{aligned} j^\ell(f)^*(dF_\varphi^{k_1\cdots k_a;i_1\cdots i_r}) &= d(F_\varphi^{k_1\cdots k_a;i_1\cdots i_r} o j^\ell(f)) \\ &= d\left(\frac{\partial^r\varphi_\ell^{k_1\cdots k_a}}{\partial x_{i_\ell}\cdots\partial x_{i_r}}\right) \\ &= \frac{\partial^{r+1}\varphi_f^{k_1\cdots k_a}}{\partial x_{i_1}\cdots\partial x_{i_r}\partial x_j} dx_j \end{aligned}$$

so much so that we obtain

$$A^j o j^\ell(f) = \frac{\partial^{r+1}\varphi_f^{k_1\cdots k_a}}{\partial x_{i_1}\cdots\partial x_{i_r}} = F_\varphi^{k_1\cdots k_a;i_1\cdots i_r,j} o j^\ell(f)$$

Therefore $j^\ell(f)^*(dF_\varphi^{k_1\cdots k_a;i_1\cdots i_r} - F_\varphi^{k_1\cdots k_a;i_1\cdots i_r j}dx_j) = 0$ for any cross-section f. Hence, by the definition of $\prod(\ell)$

$$dF_\varphi^{k_1\cdots k_a;i_1\cdots i_r} \equiv F_\varphi^{k_1\cdots k_a;i_1\cdots i_r}dx_j \quad (\text{mod } \prod(\ell)).$$

93 **Proposition 4.** *For any $\varphi \in \wedge'(D)$ we have the relation*

$$F_\varphi^{i;k_1\cdots k_\nu j} - F_\varphi^{j;k_1\cdots k_\nu i} = F_\varphi^{ji;k_1\cdots k_\nu} (\nu \leq \ell - 2)$$

Proof. Let f be a cross-section of the fibred manifold $(D, D', \tilde{\omega})$; we can then write

$$f^*\varphi = \varphi_f^i dx_i \text{ and } f^*(d\varphi) = \frac{1}{2}\Big(\frac{\partial\varphi_f^i}{\partial x_j} - \frac{\partial\varphi_f^j}{\partial x_i}\Big)dx_j \wedge dx_i$$

□

But, on the otherhand we have $f^*(d\varphi) = \dfrac{1}{2}(d\varphi)_f^{ji}dx_j \wedge dx_i$ and therefore it follows that

$$(d\varphi)_f^{ji} = \frac{\partial\varphi_f^i}{\partial x_j} - \frac{\partial\varphi_f^j}{\partial x_i}.$$

But

$$\begin{aligned} F_{d\varphi}^{ji;k_1\cdots k}\left(j_{\alpha(X)}^\ell(f)\right) &= \frac{\partial^\nu}{\partial x_{k_1}\cdots x_{k_\nu}}\left[(d\varphi)_f^{ji}\right] \\ &= \frac{\partial^{\nu+1}}{\partial x_{k_1}\cdots\partial x_{k_\nu}\partial x_j}\left[\varphi_f^i\right] - \frac{\partial^{\nu+1}}{\partial x_{k_1}\cdots\partial x_{k_\nu}\partial x_i}\left[\varphi_f^j\right] \\ &= F_\varphi^{i;k_1\cdots k_\nu j} - F_\varphi^{j;k_1\cdots k_\nu i}. \end{aligned}$$

Hence the required identity.

By the same method as in the proof of Proposition 3, we prove the following proposition:

Proposition 5. *If φ is a form of degree a, then on $J^1(D, D', \tilde{\omega})$ we have*

$$\varphi \equiv F_{\varphi}^{k_1 \cdots k_a} dx_{k_1} \wedge \cdots \wedge dx_{k_a} \Big(\mod \prod(1) \Big).$$

Consider an exterior differential system $[\sum, (D, D'\tilde{\omega})]$ with independent variables. Since $(\sum)$ is finitely generated as an ideal in $\wedge(D)$ (closed for the operator d of exterior derivation), we see that $\mathscr{F}^{\ell}(\sum)$ is also finitely generated. Let $\prod(\ell)$ be the exterior differential system constructed on $(J^{\ell}(D, D', \tilde{\omega}), D', \alpha)$. These considerations lead to the following definition. **94**

Definition . *The differential system generated by* $\left\{\prod(\ell), \mathscr{F}^{\ell}, \beta^* \sum\right\}$ *on* $J^{\ell}(D, D', \tilde{\omega})$ *is called the* standard prolongation *of* $(\sum)$ *to the space of* $\ell-$ *jets and is denoted by* $P_S^{\ell}[\sum, (D, D', \tilde{\omega})]$.

The following is a consequence of this definition and the Proposition 2. (Remark that $\alpha = \tilde{\omega} \circ \beta$).

Proposition 6. *For an integral f of the system* $[\sum, (D, D', \tilde{\omega})]$ *the cross-section* $j^{\ell}(r)$ *of* $(J^{\ell}(D, D', \tilde{\omega}), D', \alpha)$ *is an integral of* $P_S^{\ell}[\sum; (D, D', \tilde{\omega})]$. *Conversely for any integral F of* $P_S^{\ell}[\sum; (D, D', \tilde{\omega})]$ *there exists a unique cross- section f of* $(D, D', \tilde{\omega})$ *such that* $F = j^{\ell}(f)$*; when this is so f is an integral of* $(\sum)$.

3.4 Admissible Restriction

Let D, D_1 be two domains in Euclidean spaces such that $D_1 \subseteq D$ and $(\sum)$ be a differential system on D.

Definition. *A differential system* $(\sum_1)$ *on* D_1 *is said to be an admissible restriction of* $(\sum)$ *to* D_1 *when*

(i) $(\sum_1)$ is generated by $i^*(\sum), i$ being the injection map of D_1 into D

(ii) there exist functions $f_1, \ldots, f_a$ in $\sum^{(0)}$ such that $df_1, \ldots, df_a$ are linearly independent at each point of D and D_1 is the set of common zeros of $f_1, \ldots, f_a$. **95**

Proposition 7. *If $(D_1, \sum_1)$ is an admissible restriction of $(\sum)$ on D to D_1 and $(D_2, \sum_2)$ is an admissible restriction of $(\sum_1)$ on D_1 to D_2, then $(D_2, \sum_2)$ is an admissible restriction of $(\sum)$ to D_2.*

This follows immediately from the above definition.

Remarks. (1) *The condition (ii) of the above definition implies that $\ell^0 \sum$ is subset of D_1.*

Suppose we denote by di_q the injective mapping of $\mathscr{G}^q(D_1)$ into $\mathscr{G}^q(D)$ induced by the injection i of D_1 into D, di_q defines an isomorphism of $\ell^q \sum_1$ onto $\ell^q \sum_1$ onto $\ell^q \sum$.

(2) *Any integral of $(\sum)$ is contained in D_1. A submanifold of D_1 is an integral of $(\sum_1)$ if only if it is an integral of $(\sum)$.*

Proposition 8. *An integral element E of $(\sum_1)$ is an ordinary (resp. regular) with respect to $(\sum_1)$ if and only if $di_q(E)$ is ordinary (resp. regular) with respect to $(\sum)$.*

Proof. The proof is by induction on the dimension q of E. The proposition is trivial in the case $q = 0$ because of the definition of an ordinary (resp. regular) integral point. Let us suppose that the proposition holds for all $q' < q$. If $E \in \theta^q \sum_1$ then E contains a $(q-1)$ dimensional regular integral element E' of $(\sum_1)$. By induction assumption
96 $E' \in \mathscr{R}^{q-1} \sum_1$ if and only if $di_q(E') \in \mathscr{R}^{q-1} \sum$. Hence $E \in \theta^q \sum_1$ if and only if $di_q(E) \in \theta^q \sum$. Since $\ell^q \sum \subseteq \mathscr{G}^q D_1 \subseteq \mathscr{G}^q D, i^*(J(E; \sum)) = J(i_q E; \sum)$, and $J(E; \sum) \ni (df_1)_z, \ldots, (df_a)_z$, where z is the origin of E and $f_1, \ldots, f_a \in \sum^o$ such that D_1 is defined by $f_1 = \cdots = f_a = 0$, it follows by the definition of regular elements that E is regular if and only if $i_q E$ is regular. □

Proposition 8′ . *Let $(D, D', \tilde{\omega})$ be a fibered manifold. Let D_1 be a submanifold of D such that $(D, D', \tilde{\omega})$ is a fibered manifold. Assume that $\sum$ is a differential system on D such that its restriction to D_1 is an admissible restriction. Denote by i' the canonical injection of $J^\ell(D_1, D', \tilde{\omega})$ into $J^\ell(D, D', \tilde{\omega})$, where $p = \dim D'$. Then i' induces an isomorphism of $P^\ell_S[\sum_1, (D, D', \tilde{\omega})]$ onto an admissible restriction of $P^\ell_S[\sum, (D_1, D', \tilde{\omega})]$.*

Proof. We can take a coordinate system $(x, y_1, \ldots, y_m)$ of $(D, D', \tilde{\omega})$ such that $y_1, \ldots, y_s \in \sum(0)$ and D_1 is defined by the equation: $y_1 = \cdots = y_s = 0$. Then our assertion follows easily by direct verification. □

Let $(D, D', \tilde{\omega})$ be a fibred manifold then we can define a mapping of $J^{\ell+m}(D, D', \tilde{\omega})$ into $J^m(J^\ell(D, D', \tilde{\omega}), D', \alpha)$ by associating to each jet $X = j^{\ell+m}(f)$ in $J^{\ell+m}(D, D', \tilde{\omega})$ the jet $j^m_{z'}(j^\ell(f))$. Since $j^{\ell+m}(f)$ is a cross-section of $(J^\ell(D, D', \tilde{\omega}), D', \alpha)$ the following diagram is commutative.

$$J^{\ell+m}(D, D', \tilde{\omega}) \to J^m(J^\ell(D, D', \tilde{\omega}), D', \alpha)$$

$$\begin{array}{ccc} J^{\ell+m}(d, D', \tilde{w}) & \longrightarrow & J^m(J^\ell(D, D', \tilde{w}), D', \alpha) \\ & \searrow \quad \swarrow & \\ & D' & \end{array}$$

$$\begin{array}{ccc} j^{\ell+m}_{z'}(f) & \longrightarrow & j^m_{z'}(j^\ell(f)) \\ & \searrow \quad \swarrow & \\ & z' & \end{array}$$

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Then we claim that the standard prolongation $\prod(\ell + m; (D, D', \tilde{\omega}))$ is an admissible restriction $p^m_S[\prod(\ell; (D, D', \tilde{\omega})); J^\ell(D, D', \tilde{\omega})]$. For simplicity, we set $\prod(\ell) = \prod(\ell; (D, D', \tilde{\omega}))$, $\prod(\ell+m) = \prod(\ell+m; (D, D', \tilde{\omega}))$, $J^r = J^r(D, D', \tilde{\omega})$.

Let (x, y) be a fixed coordinate system of $(D, D', \tilde{\omega})$. Then $(x, y, \ldots, y^{i_1\cdots i_a}_\lambda, \cdots)$ $a \leq \ell$, is a coordinate system in $J^\ell(D, D', \tilde{\omega})$ and $(x, y, \ldots, y^{i_1\cdots i_b})$ $b \leq \ell + m$, is a coordinate system in $J^{\ell+m}(D, D', \tilde{\omega})$. Let w_σ denote $y^{i_1\cdots i_a}_\lambda (a \leq \ell)$; then a coordinate system in $J^m(J^\ell(D, D', \tilde{\omega}), D', \alpha)$ will be $(x, y, \ldots, w_\sigma, \ldots w^{j_1\cdots j_c}_\sigma, \ldots)(c \leq m)$. We shall write $y^{i_1\cdots i_a; j_1\cdots j_c}_\lambda$ instead of $w^{j_1\cdots j_c}_\sigma$. Then the canonical injection mapping i is defined by: $y^{i_1\cdots i_a; j_1\cdots j_c}_\lambda oi = y^{i_1\cdots i_a; j_1\cdots j_c}_\lambda$ Since $P^m_S[\prod(\ell), (J^\ell, D', \alpha)]$ is generated by $\prod(m; (J^\ell, D', \alpha))$, which we shall denote by $\widetilde{\prod(m)}, \beta^* \prod(\ell)$ and $\mathscr{F}^m$ $(\prod(\ell))$, we shall compute each of these.

$\prod(\ell)$ is generated by $\{dy_\lambda - y^i_\lambda dx_i, dy^{i_1\cdots i_a}_\lambda - y^{i_1\cdots i_a i}_\lambda dx_i$ $(a \le \ell - 1)$ **98**
and $\widetilde{\prod_i(m)}$ is generated by $dy^{i_1\cdots i_h}_\lambda - y^{i_1\cdots i_h;i} dx_i; (h \le \ell - 1), dy^{i_1\cdots i_h;j_1\cdots j_c}_\lambda - y^{i_1\cdots i_h;j_1\cdots j_c,j}_\lambda dx_j$ $(h \le \ell, c \le m - 1)$.

$\beta^* \prod(\ell)$ is generated by

$$\omega^{i_1\cdots i_a}_\lambda = dy^{i_1\cdots i_a}_\lambda - y^{i_1\cdots i_a i}_\lambda dx_i \ (0 \le a \le \ell - 1)$$

We shall now calculate $\mathscr{F}^m(\prod(\ell))$. Take a jet $X \in J^m(J^\ell, D'\alpha)$, say $X = j^m_x(g)$ where g is represented by $(x, \ldots, g_\lambda(x), \ldots, g^{i_1\cdots i_a}_\lambda(x), \ldots)(a \le \ell)$. We have

$$\begin{aligned} g^*(d\omega^{i_1\cdots i_a}_\lambda) &= -dg^{i_1\cdots i_a k}_\lambda dx_k \\ &= -\frac{1}{2}\Big(\frac{\partial g^{i_1\cdots i_a k}_\lambda}{\partial x_j} - \frac{\partial g^{i_1\cdots i_a j}}{\partial x_k}\Big) dx_j \wedge dx_k \end{aligned}$$

Therefore we obtain $F^{i;j_1\cdots j_c}_{\omega^{i_1\cdots i_a}_\lambda} = y^{i_1\cdots i_a;ij_1\cdots j_c}_\lambda - y^{i_1\cdots i_a i;j_1\cdots j_c}_\lambda$,

$$\begin{aligned} 2F^{jk}_{d\omega} i_1 \cdots i_a(X) &= -y^{i_1\cdots i_a k;j}_\lambda + y^{i_1\cdots i_a j;k}_\lambda, \\ 2F^{jk;j_1\cdots j_c}_{d\omega\lambda} i_1 \cdots i_a(X) &= -y^{i_1\cdots i_a k;jj_1\cdots j_c}_\lambda + y^{i_1\cdots i_a j;kj_1\cdots j_c}_\lambda \end{aligned}$$

Now it can be verified that $\prod(\ell + m)$ is an admissible restriction of $p^m_S(\prod(\ell); (J^\ell, D', \alpha))$ to the submanifold defined by the equations

$$\begin{aligned} y^{i_1\cdots i_a j;j,j_1\cdots j_b}_\lambda - y^{i_1\cdots i_a j;i,j_1\cdots j_b}_\lambda &= 0 \\ y^{i_1\cdots i_a;ij_1\cdots j_c}_\lambda - y^{i_1\cdots i_a i;j_1\cdots j_c}_\lambda &= 0 \end{aligned}$$

99 Similarly it can be proved that, given a differential system $[\sum, (D, D', \tilde{\omega})], P^{\ell+m}_S[\sum, (D, D', \tilde{\omega})]$ is an admissible restriction of $P^m_S(P^\ell_S(\sum))$.

3.5

We consider certain special types of exterior differential systems and their prolongation.

Definition. *An exterior differential system* $[\sum, (D, D', \tilde{\omega})]$ *with independent variables is called a* normal differential system *if it satisfies the following conditions.*

(1) $\sum^{(0)} = 0$;

(2) there exist Pfaffian forms $\theta_{1,\ldots,}\theta_a$ on D which form a system of generators of $\sum^{(1)}$ and which are such that $\theta_{1,\ldots,}\theta_a, dx_{1,\ldots,}dx_p$ are linearly independent at each point of D, where $(x_1, \ldots, x_p)$ is a coordinate system in D';

(3) $\sum^{(2)} \equiv 0 \mod (\sum^{(1)}, dx_1, \ldots, dx_p)$;

(4) $\sum$ is generated as an ideal, without the operator d, by $\sum^{(1)}$ and $\sum^{(2)}$ over $\wedge^0(D)$.

Let $(x, y) = (x_1, \ldots, x_p, y_1, \ldots, y_m)$ be a coordinate system of $(D, D', \tilde{\omega})$. By a linear change of coordinates y and by restricting to a neighbourhood of a given point if necessary, we can assume that $dx_1, \ldots, dx_p, dy_1, \ldots, dy_m, \theta_1, \ldots, \theta_a (m' = m - a)$ are linearly independent at each point of D. Therefore we can write

$$dy_{m'+b} = c_b^{b'}\theta_{b'} + A_b^\lambda y_\lambda + B_b^i dx_i (1 \le i \le p, 1 \le \lambda \le m', 1 \le b, b' \le a).$$

Then the determinant of the matrix $(C_b^{b'})$ is non zero. Hence $C_b^{b'}\theta_{b'}$ **100**
$\in \sum^{(1)}$ and generate $\sum^{(1)}$. Therefore we can assume without loss of generality that

$$\theta_b = dy_{m'+b} - A_b^\lambda dy_\lambda - B_b^i dx_i.$$

Now we calculate $F_{\theta_b}^{k;i_1,\ldots i_r} = F_b^{k;i,\ldots i_r}$. Because

$$\theta_b \equiv (y_{m'+b}^i - A_b^\lambda y_\lambda^i - B_b^i) dx_i \mod (dy_\sigma - y_\sigma^j dx_j), (\sigma = 1, \ldots, m)$$

Proposition 5 implies that

$$F_b^k = y_{m'+b}^k - A_b^\lambda y_\lambda^k - B_b^k.$$

Hence there are functions $E_b^{k;i}$ on $J'(D, D', \tilde{\omega})$ such that

$$dF_b^k \equiv \left(y_{m'+b}^{ki} - A_b^\lambda y_\lambda^{ki} - B_b^{k;i}\right) dx_i \quad (\text{mod } \prod(2))$$

Therefore Proposition 3 implies that

$$F_b^{k;i} = y^{ki}_{m'+b} - A^\lambda_b y^{ki}_\lambda - B^{k;i}_b$$

By the repetition of the same argument , we find that, for $r \leq \ell - 1$

$$F_b^{k;i_1\cdots i_r} = y^{ki_1\cdots i_r}_{m'+b} - A^\lambda_b y^{ki_1\cdots i_r}_\lambda - B^{k;i_1\cdots i_r}_b$$

where $B^{k;i_1\cdots i_r}_b$ are functions on J^r. Since $F^{k;i_1\cdots i_r}_b$ is symmetric with respect to $i_1 \cdots i_r$, so is $B^{k;i_1\cdots i_r}_b$. Take φ in $\sum^{(2)}$. By the condition (3) of the normal systems,

$$\varphi \equiv \varphi' = A^{i\lambda}_\varphi dx_i \wedge dy_\lambda + \frac{1}{2} B^{ij}_\varphi dx_i \wedge dx_j \quad (\text{mod } \Sigma^{(1)})$$

101 where $\lambda = 1, \ldots, m'$ and $B^{ij}_\varphi + B^{ji}_\varphi = 0$. Then $\varphi' \in \sum^{(2)}$ and

$$F^{kj;i_1\cdots i_r}_\varphi \equiv F^{kj;i_1\cdots i_r}_\varphi \quad \text{mod } (F^{h;j_1\cdots j_\nu}_b, \nu \leq r)$$

We find, by the same argument as we used to calculate $F^{k;i_1\cdots i_r}_b$, that

$$F^{kj;i_1\cdots i_r}_{\varphi'} = A^{k\lambda}_\varphi y^{ji_1\cdots i_r}_\lambda - A^{j\lambda}_\varphi y^{ki_1\cdots i_r}_\lambda + B^{kj;i_1\cdots i_r}_\varphi$$

where $B^{kj;i_1\cdots i_r}_\varphi$ are functions on J^r and symmetric with respect to $i_1 \cdots i_r$. We set

$${}'F^{kj;i_1\cdots i_r}_\varphi = F^{kj;i_1\cdots i_r}_{\varphi'}.$$

Now, $P^\ell_S(\sum)$ is generated by $\{F^{k_1\cdots k_a;j_1\cdots j_r}_\psi(\psi \in \sum^{(a)}, r \leq \ell - 1), \prod(\ell), (\sum)\}$ and their exterior derivatives. By the remark (4) on the functions $F^{k_1\cdots k_a;i_1\cdots i_r}_\psi$ we can restrict ψ to a system of generators of the ideal $(\sum)$. By Proposition 5, we can omit $(\sum)$ for $\ell \geq 1$. Therefore, the condition (4) of the definition of normal system shows that $p^\ell_S(\sum)$ is generated by

$$\begin{cases} F^{k;i_1\cdots i_r}_b & (r \leq \ell - 1, b = 1, \ldots, a), \\ F'^{kj;i_1\cdots i_r}_\varphi & (r \leq \ell - 1; \varphi \in \sum^{(2)}), \\ \prod(\ell) & \end{cases}$$

and their exterior derivatives. By Proposition 4 we have

$$F_b^{k;ji_1\cdots i_{r-1}} - F_b^{j;ki_1\cdots i_{r-1}} = F_{d\theta_b}^{jk;i_1\cdots i_{r-1}}$$

102 Since $F_b^{k;i_1\cdots i_r}$ is symmetric with respect to $i_1 \ldots i_r$ and since $d\theta_b \in \sum^{(2)}$, we have proved the following

Proposition 9. *When $[\sum, (D, D', \tilde{\omega})]$ is a normal exterior differential system, $P_S^\ell \sum$ is generated by*

$$\begin{cases} F_b^{k;i_1\cdots i_r} & (1 \le k \le i_1 \le \cdots \le i_r \le p, 0 \le r \le \ell - 1, 1 \le b \le a), \\ {}'F_\varphi^{k;i_1\cdots i_r} & (1 \le k, i_1 \ldots, i_r \le p, 0 \le r \le \ell - 1, \varphi \in \sum^{(2)}), \\ \prod(\ell) & \end{cases}$$

and their exterior derivatives.

3.6

Let $[\sum, (D, D', \tilde{\omega})]$ be an exterior differential system with independent variables. Let it be normal. If X in $J(D, D', \tilde{\omega})$ is an integral point of the standard prolongation $P_S^\ell(\sum)$, let $J(X)$ be the space of polar forms of X with respect to $P_S^\ell(\sum)$. By definition $J(X)$ is the linear subspace of the dual of $(J^\ell)_X$, the tangent vector space of $J^\ell(D, D', \tilde{\omega})$ at X, generated by $\{(\psi)_X : \psi \in (P_S^\ell \sum_k)^{(1)}\}$. This is equivalent to say that $J(X)$ is generated by $\{(dF_B^{k;i_1\cdots i_r})_X, 0 \le b \le a, (dF_\varphi^{kj;i_1\cdots i_r})_X, (\prod^{(1)}(\ell))_X (r \le \ell - 1)\}$. But by Proposition 3 we have for $r \le \ell - 2$

$$dF_\psi^{k_1\cdots k_a;i_1\cdots i_r} \equiv F_\psi^{k_1\cdots k_a;i_1\cdots i_r i} dx_i \pmod{\prod(\ell)}.$$

X being an integral point, $(F_b^{k;i_1\cdots i_r})_X = 0$ and so $(dF_b^{k;i_1\cdots i_r})_X \equiv \pmod{\prod(\ell)}$ for $r \le \ell - 2$. Also, for any 103

$$\begin{aligned} \varphi \in \textstyle\sum^{(2)}, (dF_\varphi^{kj;i_1\cdots i_r})_X &\equiv (F_\varphi^{kj;i_1\cdots i_r i})_X dx_i \\ &\equiv 0 \pmod{\prod(\ell)}. \end{aligned}$$

From this we may conclude that $J(X)$ has for generators the set $\left\{(dF_b^{k;i_1\cdots i_{\ell-1}})_X (k \le i_1 \le \cdots \le i_{\ell-1} \le p) \text{ and } \left(d' F^{kj;i_1\cdots i_{\ell-1}}\right)_X (1 \le k, j, i_1 \cdots i_{\ell-1} \le p), (\prod^{(1)}(\ell))_X\right\}$.

Proposition 10. *For any $\theta_b (1 \le b \le a)$. We have*

$$dF_b^{k;i_1\cdots i_{\ell-1}} \equiv \left\{(dy)_{m'+b}^{ki_1\cdots i_{\ell-1}} - A_b^\lambda y_\lambda^{ki_1\cdots i_{\ell-1}} + B_b^{k;i_1\cdots i_{\ell-1}i}\right\} dx_i \quad (\mathrm{mod} \prod(\ell))$$

and for any $\varphi \in \sum^{(2)}$

$$d'F_\varphi^{kj;i_1\cdots i_{\ell-1}} \equiv \left(A_{\varphi y_\lambda}^{k\lambda j i_1\cdots i_{\ell-1}} - A_{\varphi y_\lambda}^{j\lambda k i_i \cdots i_{\ell-1}} + B_\varphi^{kj;i_1\cdots i_{\ell-1}}\right) dx_i \quad (\mathrm{mod} \prod(\ell)).$$

Proof. $J^{\ell+1}$ can be considered as a fibre space over J^ℓ with the natural projection $\rho_\ell^{\ell+1}$. Hence, by lifting, $dF_b^{k;i_1\cdots i_{\ell-1}}$ can be considered as a pfaffian form on $J^{\ell+1}(D, D', \tilde{\omega})$. Then we can write

$$\begin{aligned} dF_b^{k;i_1\cdots i_{\ell-1}} &\equiv F_b^{k;i_1\cdots i_{\ell-1}i} dx_i \quad (\mathrm{mod} \prod(\ell+1)) \\ &= \left\{y_{m'+b}^{ki_1\cdots i_{\ell-1}i} - A_{by\lambda}^{\lambda k i_1\cdots i_{\ell-1}i} + B_b^{k;i_1\cdots i_{\ell-1}i}\right\} dx_i \quad (\mathrm{mod} \prod(\ell+1)) \\ &= dy_{m'+b}^{ki_1\cdots i_{\ell-1}} - A_b^\lambda dy_\lambda^{ki_1\cdots i_{\ell-1}} + B_b^{k;i_1\cdots i_{\ell-1}i} dx_i \quad (\mathrm{mod} \prod(\ell+1)) \end{aligned}$$

104 because $dy_\sigma^{ki_1\cdots i_{\ell-1}} - y_\sigma^{ki_1\cdots i_{\ell-1}i} dx_i \in \prod(\ell+1)$. Thus if Ω denotes

$$dy_{m'+b}^{ki_1\cdots i_{\ell-1}} - A_b^\lambda dy_\lambda^{ki_1\cdots i_{\ell-1}} + B_b^{k;i_1\cdots i_{\ell-1}i} dx_i,$$

then $dF_b^{k;i_1\cdots i_{\ell-1}} - \Omega$ is a form on J^ℓ and is in $\prod(\ell+1)$. Therefore $J^\ell(f)^*(dF_b^{k;i_1\cdots i_{\ell-1}} - \Omega) = 0$ for any cross-section f of $(D, D', \tilde{\omega})$. As proved before, it follows that $dF_b^{k;i_1\cdots i_{\ell-1}} - \Omega \in \prod(\ell)$ and this completes the proof of the first assertion. The second assertion can also be proved on the same lines. □

Let us denote by $G^{(\ell)}$ the subspace of Pfaffian forms on J^ℓ generated by

$$\eta_b^{k;i_1\cdots i_{\ell-1}} = dy_{m'+b}^{ki_1\cdots i_{\ell-1}} - A_b^\lambda dy_\lambda^{ki_1\cdots i_{\ell-1}} + B_b^{k;i_1\cdots i_{\ell-1}i} dx_i$$
$$(1 \le k \le i_1 \le \cdots \le i_{\ell-1} \le p, b = 1, \ldots, a)$$

and by $A_o^{(\ell)}$ the subspace generated by

$$\xi_\varphi^{kj;i_1\cdots i_{\ell-1}} = A_\varphi^{k\lambda} dy_\lambda^{ji_1\cdots i_{\ell-1}} - A_\varphi^{j\lambda} y_\lambda^{ki_1\cdots i_{\ell-1}} + B_\varphi^{kj;i_1\cdots i_{\ell-1}i} dx_i$$

$$\left(1 \le k, j, i_1, \ldots, i_{\ell-1} \le p, \varphi \in \textstyle\sum^{(2)}\right)$$

By the remark preceding Proposition 10 by Proposition 10 it follows that $J(X) = (G^{(\ell)})_X + (\prod(\ell))_X + (A_o^{(\ell)})_X$. On the otherhand $dy_\sigma^{i_1\cdots i_r} - y_\sigma^{i_1\cdots i_r i} dx_i (1 \le i_1 \le \cdots \le i_r \le p.r \le \ell - 1, \sigma = 1, \ldots m)$, which form a sys- 105
tem of generators of $\prod(\ell)$, together with the generators of $G^{(\ell)}$ are linearly independent modulo $dx_1, \ldots, dx_p, \ldots, dy_\lambda^{i_1\cdots i_\ell}, \ldots (\lambda = 1, \ldots, m')$. Moreover $\xi_\varphi^{kj;i_1\cdots i_{\ell-1}}$ are linear combinations of $dx_1, \ldots, dx_p, dy_\lambda^{i_1\cdots i_\ell}$. Therefore $J(X) = (G^{(\ell)})_X + (\prod(\ell))_X + (A_o^{(\ell)})_X$ is a direct sum decomposition of the vector space $J(X)$. Moreover it is clear that $\dim(G^{(\ell)})_X = aC_\ell^{p+\ell-1}$ and $\dim(\prod(\ell))_X = m \sum\limits_{r=0}^{\ell} C_r^{p+r-1}$, where C_r^{p+r-1} are the Binomial coefficients.

Now, we shall show a similar decomposition of the space of polar forms $J(E)$ of a q-dimensional integral element E of the standard prolongation $P_S^\ell \sum$. Let X be the origin of E. Let us first recall the definition of $J(E)$. Take any system of generators $\psi_1, \ldots, \psi_n$ of the ideal $P_S^\ell \sum$, where ψ_τ is homogeneous of degree a_τ. Let $L^1, \ldots, L^q$ be a base of the vector space E. Then $J(E)$ is generated by $f_\tau^{h_1\cdots h_a}\tau^{-1}(1 \le h_1, \ldots, h_{a_\tau - 1} \le q, \tau = 1, \ldots, N)$ defined by

$$f_\tau^{h_1\cdots h_a}\tau^{-1}(L) = \langle \psi_\tau, L^{h_1} \wedge \cdots \wedge L^{h_a}\tau^{-1} \wedge L \rangle.$$

Since $P_S^\ell \sum$ is generated by $(P_S^\ell \sum)^{(1)}$ and $\prod^2(\ell)$, $J(E)$ is generated by $J(X)$ together with all the f defined by

$$f(L) = \langle dy_\sigma^{i_1\cdots i_r i} \wedge dx_i, L' \wedge L \rangle, (\sigma = 1, \ldots, m; 1 \le i_1, \ldots, i_r \le p, L' \in E).$$

If $r \le \ell - 2$, then $dy_\sigma^{i_1\cdots i_r i} \wedge dx_i \equiv y_\sigma^{i_1\cdots i_r ij} dx_j \wedge dx_i = 0 \pmod{\prod(\ell)}$, because $y_\sigma^{i_1\cdots i_r ij} = y_\sigma^{i_1\cdots i_r ji}$. Thus $J(E)$ is generated by $J(X)$ together with 106

$$\zeta_\sigma^{i_1\cdots i_{\ell-1}, L} = \langle dx_i, L \rangle dy^{i_1\cdots i_{\ell-1} i} - \langle dy^{i_1\cdots i_{-1} i}, L \rangle dx_i$$

where $L \in E$.

Let $\mathscr{G}^q J^\ell(dx_1, \ldots dx_q)$ denote the subspace of $\mathscr{G}^q J^\ell(D, D', \tilde{\omega})$ consisting of all the elements E such that the restrictions $dx_1|E, \ldots, dx_q|E$ are linearly independent. $\mathscr{G}^q J^\ell(dx_1, \ldots, dx_q)$ is an open submanifold of

$\mathscr{G}^q J^\ell(D, D', \tilde{\omega})$. Let $L^1(E), \dots, L^q(E)$ be a dual base in E of $dx|E, \dots,$ $dx_q|E$. We introduce the following functions on $\mathscr{G}^q J^\ell(dx_1, \dots, dx_q)$: for any $E \in \mathscr{G}^q J^\ell(dx_1, \dots, dx_q)$ let $w^{q'}_{i.q}(E) = \langle dx_i, L^{q'}(E)\rangle$,

$$w^{i_1\cdots i_\ell;q'}_{\sigma\ q}(E) = \langle dy^{i_1\cdots i_\ell}_{\sigma}, L^{q'}(E)\rangle.$$

Now if the integral element E is in $\mathscr{G}^q J^\ell(dx_1, \dots, dx_q)$ the above argument proves that $J(E)$ is generated by $J(X)$ together with

$$\zeta^{i_1\cdots i_{-1};q'}_{\sigma,q} = \zeta^{i_1\cdots i_{-1};L^{q'}(E)}_{\sigma,q} = w^{q'}_{i,q}(E) dy^{i_1\cdots i_{\ell-1} i;q'}_{\sigma} dx_i$$
$$(\sigma = 1, \dots, m; q' = 1, \dots, q)$$

We shall introduce the following notation to facilitate the writing of the above identities. We shall denote by I_r (or by I when there is no possible confusion) any set of indices $(i_1, \dots, i_r)$ for $r = 0, 1, \dots, \ell$. We
107 can now write all the identities above in the compact form as follows:

$$F^{k;I}_b = y^{kI}_{m'+b} - A^\lambda_b y^{kI}_\lambda + B^{k;I}_b;$$
$$dF^{k;I}_b \equiv dy^{kI}_{m'+b} - A^\lambda_b dy^{kI}_\lambda + B^{k;Ii}_b dx_i \quad (\operatorname{mod} \textstyle\prod(\ell))$$

where $B^{k;I}_b$ are functions on $J^{r+1}(D, D', \tilde{\omega})$.

$$'F^{k;I}_\varphi = A^{k\lambda}_\varphi y^{jI}_\lambda A^{j}_\lambda y^{kI}_\lambda + B^{kj;I}_\varphi;$$
$$d'F^{k;I}_\varphi = A^{k\lambda}_\varphi dy^{jI}_\lambda - A^{j\lambda}_\varphi dy^{kI} + B^{kj;Ii}_\varphi dx_i \quad (\operatorname{mod} \textstyle\prod(\ell))$$

where $B^{kj;I}_\varphi$ are functions on $J^r(D, D', \tilde{\omega})$. The generators of $G^{(\ell)}$ are

$$\eta^{k;I}_b = dy^{kI}_{m'+b} - A^\lambda_b dy^{kI}_\lambda + B^{k;Ii}_b dx_i,$$
$$(1 \leq b \leq a; I = (i_1, \dots, i_{\ell-1}); 1 \leq k \leq i_1 \leq \cdots \leq i_{\ell-1} \leq p) \text{ and}$$
$$\xi^{kj,I}_\varphi = A^{k\lambda}_\varphi dy^{jI}_\lambda - A^{j\lambda}_\varphi dy^{kI}_\lambda + B^{kj;Ii}_\varphi dx_i$$

$(\varphi \in \sum^{(2)}, I = (i_1, \cdots, i_{\ell-1}), 1 \leq k, j, i_1 \cdots \leq p)$ are generators of the subspace $A^{(\ell)}_0$ of Pfaffian forms.

If X is an integral point, it is clear that the dimensions of the spaces $(G^{(\ell)})_X$ and $(\prod(\ell))_X$ do not change when X is moved in a sufficiently small neighbourhood of $\ell^o p_S^\ell \sum$. The direct sum decomposition $J(X) =$
108 $(G^{(\ell)})_X + (\prod(\ell))_X + (A_o^{(\ell)})_X$ shows that any change in the dimension of $J(X)$ is due only to the change in the dimension of $(A_o^{(\ell)})_X$.

If E is any q-dimensional integral element of $p_S^\ell \sum$ and if X is the origin of E, it has already been proved that $J(E)$ is generated by $J(X)$ together with

$$\zeta_{\sigma,q}^{I;q'} = w_{i,q}^{q'} dy_\sigma^{Ii} - w_{\sigma,q}^{Iiq'} dx_i$$

on $\mathscr{G}^q J^\ell(dx_1, \ldots, dx_q), I = (i_1 \cdots i_{\ell-1})$.

Remark. *For any q_1 with $q' \leq q_1 < q$ let E^{q_1} be the subspace of E^q spanned by $L^1(E), \ldots, L^{q_1}(E)$ and let η be the natural projection of E^q onto E^{q_1}. Clearly we have $w_{i,q}^{q'} = w_{i,q_1}^{q'} o\eta, w_{\sigma,q}^{i_1\cdots i_{\ell;q'}} = w_{\sigma,q_1}^{i_1\cdots i_{\ell;q'}} o\eta$ and hence we can simply write $w_i^{q'} w_\sigma^{i_1\cdots i_{\ell;q1}}$ in place of $w_{i,q}^{q'}$ and $w_{\sigma,q}^{i_1\cdots i_{\ell;q'}}$ without any ambiguity . Also, we can write $\zeta_\sigma^{I;q'}$ instated of $\zeta_{\sigma,q}^{I;q'}$. For any $\ell \geq 2$ we have*

$$dy_{m'+b}^{i_1\cdots i_{\ell-1}i} \wedge dx_i \equiv A_b^\lambda dy_\lambda^{i_1\cdots i_{\ell-1}i} \wedge dx_i - B_b^{i_1;i_2\cdots i_{\ell-1}ij} dx_i \wedge dx_j$$
$$\left(\text{mod } \left(p_S^\ell \sum\right)^{(1)}, \left(p_S^\ell \sum\right)^{(0)}\right)$$

But $B_b^{i_1\cdots i_{\ell-1}ij} dx_i \wedge dx_j = 0$ since $B_b^{i_1\cdots i_{\ell-1}ij}$ are symmetric in i, j. Therefore $J(E)$ is generated by $J(X)$ and $\zeta_\lambda^{I;q'}(\lambda = 1, \ldots, m')$.

Let $A_q^{(\ell)}$ denote the subspace of 1-forms on $\mathscr{G}^q J^\ell(dx_1, \ldots dx_q)$ generated by $A_o^{(\ell)}, \zeta_\lambda^{I;1}, \ldots \zeta_\lambda^{I;q}(I = (i_1, \ldots, i_{\ell-1}); \lambda = 1, \ldots, m')$.

Then we prove the following: 109

Proposition 11. *Let E be a q-dimensional integral element of the standard prolongation to the space of ℓ- jets of a normal system $[\sum, (D, D', \tilde{\omega})]$ $(0 \leq q \leq p = \dim D')$. Denote by X the origin of E. Then we have the direct sum decomposition*

$$J(E) = \left(G^{(\ell)}\right)_X + \left(\prod(\ell)\right)_X + \left(A_q^{(\ell)}\right)_E$$

$L^1(E), \dots L^q(E)$ being a basis of E dual to $dx_1|E, \dots dx_q|E$ we see that $w_i^{q'} = \langle dx_i, L^{q'}(E)\rangle = \delta_i^{q'} (i \leq q)$ so that the Pfaffian forms $\zeta_\sigma^{I;q'} (I = (i_1, \dots, i_{\ell-1}))$ have the reduced form $\zeta_\sigma^{I;q'} = dy_\sigma^{Iq'} + w_{q+u}^{q'} dy_\sigma^{Iq+u} - w_\sigma^{Ii;q'} dx_i$. Denote by ${}'A_q^{(\ell)}$ the subspace of Pfaffian forms generated by the following set:

$$
\begin{aligned}
{}'\xi_\varphi^{kj;I} &= A_\varphi^{k\lambda} dy_\lambda^{jI} - A_\varphi^{j\lambda} dy_\lambda^{kI},\\
{}'\zeta_\sigma^{I;q'} &= dy_\sigma^{Iq'} + w_{q+u}^{q'} dy_\sigma^{Iq+u}
\end{aligned}
$$

where $\varphi \in \sum^{(2)}; \lambda, \sigma = 1, \dots, m'; q' = 1, \dots, q; I = (i_1, \dots, i_{\ell-1})$ with $1 \leq i_1, \dots, i_{\ell-1}\ j, k \leq p$. Let $t'(E)$ denote the dimension of $({}'A_q^{(\ell)})_E$. If we denote by Ω the space generated by $(dx_1)_X, \dots (dx_p)_X$, X being the origin of E, then the definition shows that $t'(E) = \dim((A_q^{(\ell)})_E + \Omega/\Omega)$. Thus $t'(E)$ is defined independent of the choice of the coordinate system (because of Proposition 11).

3.8

In this section we establish an inequality regarding $t'(E)$. If E is a q-dimensional integral element of $p_S^\ell \sum$ and if $L^1(E), \dots, L^q(E)$ is a
110 basic of E dual to $dx_1|E, \dots, dx_q|E$, denote by E' the subspace of E spanned by $L^1(E), \dots L^{q-1}(E)$. Suppose ρ is the natural projection of $J^\ell(D, D', \tilde{\omega})$ onto $J^{\ell-1}(D, D', \tilde{\omega})$. It is clear that $E'' = d\rho.E'$ is an integral of $P_S^{\ell-1} \sum$ on $J^{\ell-1}(D, D', \tilde{\omega})$. Then we have the

Proposition 12. *If E is a q-dimensional integral element of $P_S^\ell \sum$ then the following inequality holds:*

$$
\begin{aligned}
t'(E) \leq{} & \dim(A_q^{(\ell)})_E\\
& \dim(A_{q-1}^{(\ell)})_{E'} + n_{\ell-1} - t'(E''),
\end{aligned}
$$

where $n_{\ell-1} = mc_{\ell-1}^{p+\ell-2}$ where c_s^r denotes the Binomial coefficient.

Proof. In general we denote by I or I_χ indices $(i_1, \dots, i_{\ell-1})$. Let X be the origin of E. Let $\left(dy_{\lambda_\chi}^{I_\chi}\right)_{\rho(X)}$ be the maximum number of linearly independent elements dy_λ^I modulo $(A_{q-1}^{(\ell-1)})_{E''}$. We denote by J indices

$(j_1, \dots, j_{\ell-2})$. We can write

$$(dy^I_\lambda)_{\rho(X)} = a^\chi dy^{I\chi}_{\lambda_\chi} + b^\varphi_{kj;J}\xi^{kj;J}_\varphi + c^{\mu'}_{J,q'}\zeta^{J;q'}_\mu,$$

$(q' = 1, \dots, q-1; \mu = 1, \dots, m')$. We claim that, for any $s(1 \leq s \leq p)$

$$(\#)\ (dy^{Is}_\lambda)_X = a^\chi(dy^{I\chi s}_{\lambda\chi})_X + b_{kj;J}({}'\xi^{kj;Js}_\varphi)_{E'} + c^\mu_{J';q'}({}'\zeta^{Js,q'}_\mu)_{E'}$$

To see this, we consider a linear mapping of the vector space generated by all $(dy^{I'}_\mu)_{\rho(X)}$ onto the space generated by all $(dy^{I's}_\mu)_X$, sending $(dy^{I'}_\mu)_{\rho(X)}$ to $\left(dy^{I's}_\mu\right)_X$. It is clear that this map maps ${}'\xi^{kj;J}_\varphi$ upon ${}'\xi^{kj;Js}_\varphi$. Since $w^{q'}_{q+u}(E) = w^{q'}_{q+u}(d\rho, E)$ it follows that ${}'\zeta^{J;q'}_\mu$ is mapped upon ${}'\zeta^{Js;q'}_\mu$. **111**
Thus we have proved the equality (#). E being an integral element of $P^\ell_S \sum$, taking the value of $(dy^{Is}_\lambda)_X$ at $L^q(E)$ we obtain by (#) that

$$\begin{aligned}\langle(dy^{Is}_\lambda)_X, L^q(E)\rangle &= w^{Is;q}_\lambda(E) = a^\chi w^{I\chi s;q}_{\lambda\chi}(E)\\ &- b^\varphi_{kj;J}(B^{kj;Jsi}_\varphi(X)\langle dx_i, L^q(E)\rangle) + c^\mu_{J;q'}(w^{Jsi;q'}_\mu\langle dx_i, L^q(E)\rangle),\end{aligned}$$

since $\xi^{kj;J}_\varphi = {}'\xi^{kj;J}_\varphi + B^{kj;J}_\varphi$ vanishes at $L^q(E)$. Using the relations $\langle dx_i, L^q(E)\rangle = \delta^q_i$ we can write

$$\begin{aligned}w^{Is;q}_\lambda(E) &= a^\chi w^{I\chi s;q}_{\lambda\chi}(E) - b^\varphi_{kj;J}(B^{kj;Jsq}_\varphi(X) + B^{kj;Js'(q+u)}_\varphi w^q_{q+u}(E))\\ &+ c^\mu_{J;q'}\left(w^{Jsq;q'}_\mu + w^{Js'(q+u);q'}_\mu w^q_{q+u}(E)\right).\end{aligned}$$

But

$$\begin{aligned}\zeta^{I;q}_\lambda &= dy^{Iq}_\lambda + w^q_{q+u}(E)dy^{Iq+u}_\lambda - w^{Ii;q}_\lambda(E)dx_i\\ &= a^\chi\zeta^{I\chi;q}_{\lambda\chi} + b^\varphi_{kj;J}(\xi^{kj;Jq}_\varphi + w^q_{q+u}(E)\xi^{kj;J,(q+u)}_\varphi)\\ &+ c^\mu_{J;q'}(\zeta^{Jq;q'}_\mu + w^q_{q+u}(E)\zeta^{J'(q+u);q'}_\mu)\end{aligned}$$

where $q' = 1, \dots, q-1$. Thus $\zeta^{I;q}_\lambda$ is in the space generated by $\zeta^{I\chi;q}_{\lambda\chi}$ and $(A^{(\ell)}_{q-1})_E$. Hence we obtain

$$\begin{aligned}\dim(A^{(\ell)}_q)_E &\leq \dim(A^{(\ell)}_{q-1})_{E'} + \text{the number of indices } \chi\\ &= \dim(A^{(\ell)}_{q-1})_{E'} + m.c^{p+\ell-2}_{\ell-1} - t'(E'').\end{aligned}$$

This completes the proof of the proposition. □ **112**

3.9

Let $[\sum, (D, D', \tilde{\omega})]$ be a differential system with independent variables which is a normal system. Then we pose the following definition.

Definition. *For any point $z \in D$, a pair (z, E^{χ}) of z and a q-dimensional contact element E^{χ} to D' at $\tilde{\omega}(z)$ is called a q-dimensional reduced contact element of D and z is called the origin of it.*

Let $\chi_{\mathscr{G}}q = \chi_{\mathscr{G}}q(D, D', \tilde{\omega})$ be the set of all reduced contact elements $(z, E^{\chi})(z \in D)$. $\chi_{\mathscr{G}}$ is a submanifold of $D \times \mathscr{G}^q D'$. For, let ρ denote the mapping of $\chi_{\mathscr{G}}$ into D' which assigns to every element (z, E^{χ}) the origin of E^{χ}. We observe that $\chi_{\mathscr{G}}q = \{(z, E^{\chi}) \in D \times \mathscr{G}^q D' : \tilde{\omega}(z) = \rho(E^{\chi})\}$. This condition defines the structure of a real analytic submanifold of $D \times \mathscr{G}^q D'$.

To every reduced contact element (z, E^{χ}) we associate a certain homogeneous ideal $A(z, E^{\chi})$ in a certain symmetric algebra $\underline{\mathrm{R}}(V)$ on a module V over $\underline{\mathrm{R}}$. We shall first construct the symmetric algebra $\underline{\mathrm{R}}(V)$. Consider the tangent vector space $(D')_{\tilde{\omega}(z)}$ which is a p-dimensional vector space over $\underline{\mathrm{R}}$. Denote this by V^p. Let $V^{m'}$ denote the quotient module $[\wedge^1_z(D)/\tilde{\omega}^*(\wedge^1 D')_{\tilde{\omega}(z)} + (\sum^{(1)})_z]$ over the ring $\wedge^o_z(D) = \underline{\mathrm{R}}$, where $\wedge^1_z(D)$ is the conjugate space of $(D)_z$. Let V denote the direct sum $V^p \oplus V^{m'}$. Now we denote by $\underline{\mathrm{R}}(V)$ the symmetric algebra over V. If
113 we choose a coordinate system $(x_1, \ldots, x_p, y_1, \ldots, y_m)$ in $(D, D', \tilde{\omega})$ such that dy_λ are linearly independent modulo $(dx_1, \ldots, dx_p; \sum^{(1)})$, then V^p is spanned by $\dfrac{\partial}{\partial x_1}, \ldots, \dfrac{\partial}{\partial x_p}$ and $V^{m'}$ by $[dy_1], \ldots, [dy_{m'}]$ where $[dy_\mu] = dy_\mu \mod \{\tilde{\omega}^*(\wedge^1(D'))_{\tilde{\omega}(z)} + (\sum^{(1)}_{\tilde{\omega}(z)})\}$ $(\mu = 1, \ldots, m')$. If we set $\dfrac{\partial}{\partial x_i} = X^i$ and $[dy_\mu] = Y_\mu$ the elements of the symmetric algebra $\underline{\mathrm{R}}(V)$ can be expressed as polynomials in $X^1, \ldots, X^p, Y_1, \ldots, Y_{m'}$. In view of the fact that $\underline{\mathrm{R}}(V)$ depends on $z \in D$, we may, when there is any possible ambiguity write $\underline{\mathrm{R}}(z)$.

$(\sum)$ being a normal differential system any $\varphi \in \sum^{(2)}$ can be written as

$$\varphi \equiv A^{i\lambda}_{\varphi} dx_i \wedge dy_\lambda + \frac{1}{2} B^{ij}_{\varphi} dx_i \wedge dx_j \pmod{\textstyle\sum^{(1)}}$$

where $A^{i\lambda}_\varphi, B^{ij}_\varphi$ are real analytic functions on D. Then we have that

$$({}'\widetilde{\xi}^{kj}_\varphi)_z = A^{k\lambda}_\varphi(z)dy'^{j}_\lambda - A^{j\lambda}_\varphi(z)dy'^{k}_\lambda.$$

We associate to every $\varphi \in \sum^{(2)}$ an element

$$\tilde{\xi}^{kj}_\varphi = \tilde{\xi}^{kj}_\varphi(z) = A^{kj}_\varphi(z)Y_\lambda X^j - A^{j\lambda}_\varphi(z)Y_\lambda X^k$$

of $\underline{\mathrm{R}}(z)$. By a change of coordinate systems $(x, y) \to (x', y')$ such that dy'_λ are linearly independent modulo $(dx'_1, \ldots, dx'_p, \sum^{(1)})$ we have $(dx'_i)_z = a^j_i(dx_j)_z(dy'_\lambda)_z \equiv b^\mu_\lambda(dy_\mu)_z \mod ((\sum^{(1)})_z(dx_i)_z)$.

Hence obtain

$$X^i = a^i_j{}'X^j, Y'_\lambda = b^\mu_\lambda Y_\mu,$$

where $${}'X^j = (\frac{\partial}{\partial x'_j})_z, Y'_\lambda = (dy'_\lambda) \mod ((\sum^{(1)})_z, (dx_i)_z).$$

If we express $\varphi \in \sum^{(2)}$ in the new coordinate system as **114**

$$\varphi \equiv {}'A^{i\lambda}_\varphi dx'_i \wedge dy' + \frac{1}{2}{}'B^{ij}_\varphi dx'_i \wedge dx'_j \quad (\text{mod } \sum^{(1)})$$

then we obtain $A^{i\lambda}_\varphi(z) = {}'A^{j\lambda}_\varphi a^i_j b^\lambda_\mu$. In the new coordinate system the element of the symmetric algebra associated to $\varphi \in \sum^{(2)}$ is

$${}'\tilde{\xi}^{kj}_\varphi = {}'A^{k\lambda}_\varphi Y'_{\lambda'}{}'X^j - {}'A^{j\lambda}_\varphi Y'_\lambda{}'X^k.$$

We shall obtain the relation between the $\tilde{\xi}^{kj}_\varphi$ and the ${}'\tilde{\xi}^{kj}_\varphi$.

$$\begin{aligned}\tilde{\xi}^{kj}_\varphi &= A^{k\lambda}_\varphi Y_\lambda X^j - A^{j\lambda}_\varphi Y_\lambda X^k\\ &= {}'A^{k'\lambda'}_\varphi a^k_{k'} b^\lambda_{\lambda'} Y_\lambda X^j - {}'A^{j'\lambda'}_{\varphi'} a^j_{j'} b^\lambda_{\lambda'} Y_\lambda X^k\\ &= {}'A^{k'\lambda'}_\varphi a^k_{k'} a^j_{j'} Y'_{\lambda'}{}'X^{j'} - {}'A^{j'\lambda'}_\varphi a^j_{j'} a^k_{k'} Y_{\lambda'}{}'X^{k'}.\end{aligned}$$

Therefore $\tilde{\xi}^{kj}_\varphi = a^j_{j'} a^k_{k'}{}'\tilde{\xi}^{k'j'}_\varphi$. Therefore the ideal $A(z)$ in $\underline{\mathrm{R}}(v)$ generated $\tilde{\xi}^{kj}_\varphi(\varphi \in \sum^{(2)})$ does not depend on the choice of the coordinate system.

Consider a q-dimensional reduced contact element (z, E^{χ}).

We remark that any L^{χ} in E^{χ} is in $(D')_{\tilde{\omega}(z)} = V^p \subseteq \underline{R}(z)$. Therefore, for any $Y \in V^{m'} \subseteq \underline{R}(z)$, the multiplication $L^{\chi}Y \in \underline{R}(z)$ is defined. Now let $A(z, E^{\chi})$ be the ideal in $\underline{R}(z)$ generated by $A(z)$ and by all $L^{\chi}.Y(L^{\chi} \in E^{\chi}, y \in V^{m'})$. Let $(x_1, \ldots, x_p)$ be a coordinate system in D'. Set $X^i = (\dfrac{\partial}{\partial x_i})_z$. Then L^{χ} is expressed as

$$L^{\chi} = \sum_{j=1}^{p} \langle dx_j, L^{\chi} \rangle X^j.$$

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Remark. *The symmetric algebra $\underline{R}(z)$ being a polynomial algebra in the indeterminates $X^1, \ldots, X^p, Y_1, \ldots, Y_m$ we can decompose it into direct sum of submodules of bidegree $(\ell, h)(\ell$ = degree in X^k and h = degree in $Y_\lambda)$ and written $\underline{R}(z) = \sum \underline{R}^{(\ell,h)}(z)$. An ideal I in $\underline{R}(z)$ is said to be a homogeneous ideal if $I = \sum I^{(\ell,h)}$ where $I^{(\ell,h)} = I \cap \underline{R}^{(\ell,h)}$ where $\underline{R}^{(\ell,h)}(z)$ is the submodule of all homogeneous polynomials of bidegree (ℓ, h). $A(z, E^{\chi})$ is a homogeneous ideal, so we have $A(z, .E^{\chi}) = \sum A^{(\ell,h)}(z, E^{\chi})$.*

Proposition 13. *Let E be a q-dimensional integral element of the system $P^{\ell}_{S}[\sum, (D, D', \tilde{\omega})]$ with independent variables. If E^{χ} denotes $d\alpha(E)$ then $t'(E)$ = dimension of $A^{(\ell,1)}(\beta(X), E^{\chi})$ where X is the origin of E.*

Proof. Since $\alpha = \tilde{\omega} o \beta, (\beta(X), E^{\chi})$ is a reduced contact element. The proposition asserts that $t'(E) = \dim A^{(\ell,1)}(z, E^{\chi}), z = \beta(X)$. Let $X_1, \ldots, X_p$ be a coordinate system at $\tilde{\omega}(z)$. Then the generators of $A^{(\ell,1)}(z, E^{\chi})$ are

$$\begin{cases} \tilde{\xi}^{kj;i_1\cdots i_{\ell-1}}_{\varphi} = A^{k\lambda}_{\varphi}(z) Y_\lambda X^j X^{i_1} \cdots X^{i\ell-1} - A^{j\lambda}_{\varphi}(z) Y_\lambda X^k X^{i_1} \cdots i_{\ell-1} X; \\ L^{\chi} Y_\lambda X^{i_{\ell-1}} \cdots X^{i\ell-1} = \sum_{i=1}^{p} \langle dx_i, L^{\chi} \rangle Y_\lambda X^{i_1} \cdots X^{i_{\ell-1}} (L^{\chi} \in E^{\chi}) \end{cases}$$
$$(\varphi \in \sum\nolimits^{(2)}, 1 \leq k, j, i_1, \ldots, i_{\ell-1} \leq p, \lambda = 1, \ldots, m).$$

We can assume if necessary, by a change of coordinate system that
116 $dx_i|E^{\chi}, \ldots, dx_q|E^{\chi}$ are linearly independent. Let $L^1(E^{\chi}), \ldots, L^q(E^{\chi})$ be a basis of E^{χ} dual to $dx_1|E^{\chi}, \ldots, dx_q|E^{\chi}$. Then

$$\begin{aligned}
L^{q'}(E^{\mathcal{X}})Y_\lambda X^{i_1}\cdots X^{i_{\ell-1}} &= Y_\lambda X^{i_1}\cdots X^{i_{\ell-1}}X^{q'} \\
&\quad + \sum_{h=1}^{\ell-q}\langle dx_{q+h}, L^{q'}(E^{\mathcal{X}})\rangle Y_\lambda X^{i_1}\cdots X^{i\ell-1}X^{q+h} \\
&= Y_\lambda X^{i_1}\cdots X^{i_{\ell-1}}X^{q'} + \sum_{h=1}^{\ell-q}\omega^{q'}_{q+h}(E^{\mathcal{X}})Y_\lambda X^{i_1}\cdots X^{i_{\ell-1}}X^{q+h}
\end{aligned}$$

On the other hand the generators of $'A^{(\ell)}_q$ are

$$\begin{aligned}
{}'\xi^{kj;i_1\cdots i_{\ell-1}}_{\varphi'} &= A^{k\lambda}_{\varphi}dy^{jii_1\cdots i_{\ell-1}}_{\lambda} - A^{j\lambda}_{\varphi}dy^{ki_1\cdots i_{\ell-1}}_{\lambda} \\
{}'\zeta^{i_1\cdots i_{\ell-1}q'}_{\mu'} &= dy^{i_1\cdots i_{\ell-1}q'}_{\mu} + w^{q'}_{q+h}dy^{i_1\cdots i_{\ell-1}q+h}_{\mu} \\
&(\varphi\in\textstyle\sum^{(2)}, 1\le k, j, i_1\cdots i_{\ell-1}\le p, \lambda = 1,\ldots,m)
\end{aligned}$$

and $t'(E) = \dim('A^{(\ell)}_q)_E$ by definition. □

Setting $(dy^{i_1\cdots i_\ell}_\lambda)_X = X^{i_1\cdots j_\ell}_\lambda$, where X is the origin of E, we see that $x^{i_1\cdots i_\ell}_\lambda = X^{j_1\cdots j_\ell}_\lambda$ for any permutation $j_1,\ldots,j_\ell$ of $i_1,\cdots,i_\ell$. The generators of $('A^{(\ell)}_q)_E$ are, therefore,

$$\begin{aligned}
{}'\xi^{kj;i_1\cdots i_{\ell-1}}_{\varphi}(E) &= A^{k\lambda}_{\varphi}(z)X^{ji_1\cdots i_{\ell-1}}_{\lambda} - A^{j\lambda}_{\lambda}(z)X^{ki_1\cdots i_{\ell-1}}_{\lambda} \\
{}'\zeta^{i_1\cdots i_{\ell-1}q'}_{\mu}(E) &= X^{i_1\cdots i_{\ell-1}q'}_{\lambda} + w^{q'}_{q+h}X^{i_1\cdots i_{\ell-1}q+h}_{\lambda}.
\end{aligned}$$

Let f denote a homomorphism of the submodule $('A^{(\ell)}_q)_E$ of the dual of $(J^\ell)_z$ generated by $\{'\xi^{kj;i_1\cdots i_{\ell-1}}_{\varphi}(E), '\zeta^{i_1\cdots i_{\ell-1}q'}_{\mu}(E)\}$ into the module $\underline{\mathrm{R}}^{(\ell,1)}(z)$ defined by

$$f(X^{i_1\cdots i_\ell}_\lambda) = Y_\lambda X^{i_1}\cdots X^{i_\ell}.$$

It is easy to verify that f is an isomorphism of $('A^{(\ell)}_q)_E$ onto $A^{(\ell,1)}$ **117**
$(z, E^{\mathcal{X}})$ and hence $t'(E) = \dim A^{(\ell,1)}(z, E^{\mathcal{X}})$.

3.10 Some results from the theory of ideals in polynomial rings

Let $X_1\cdots X_p$ be p indeterminates. Order the set of all monomials in $X_1\cdots X_p$ lexicographically. Denote by M the set of first k elements in the set of all monomials of degree ℓ. Let $Q_\ell(k, p)$ denote the number of elements in the set $MX_1 U \cdots UMX_p$.

We shall state two theorems, without proof, regarding the function $Q_\ell(k, p)$. For proof one can refer to "Lectures on commutative algebra" by S.Bochner (1938).

Theorem 1. *Given p and k, there exists an integer $Q_\ell(k, p)$ such that $Q_\ell(k, p) = Q(k, p)$ for sufficiently large ℓ.*

Definition. *$Q(k, p)$ is called Macauly function.*

If I is any homogeneous ideal in a polynomial ring $K[X_1 \cdots X_p]$, let $I^{(\ell)} = I \cap K^{(\ell)}[X_1 \cdots X_p]$ where $K^{(\ell)}[X_1 \cdots X_p]$ is the submodule of homogeneous polynomials of total degree ℓ. Set $\phi^\ell(I)$ = the dimension of $I^{(\ell)}$.

Theorem 2 (Hilbert)**.** *For any homogeneous ideal I of $K[X_1 \cdots X_p]$ there exists an integer $\ell_0(I)$ satisfying the following conditions:*

(i) $\phi(I^{(\ell+1)}) > Q(\phi^{(\ell)}, p)$ *for* $\ell < \ell_0(I)$,

(ii) $\phi(I^{(\ell+1)}) = Q(\phi^{(\ell)}, p)$ *for* $\ell \geq \ell_0(I)$.

118 Let us write the symmetric algebra $\underline{R}(z)$ as a direct sum of the submodules $\underline{R}^{(\ell,h)}(z)$ of homogeneous elements of bidegree (ℓ, h) (degree ℓ in $X^1 \cdots X^p$ and degree h in $Y_1, \ldots, Y_{m'}$). $\underline{R}(z) = \sum \underline{R}^{(\ell,h)}(z)$. Set $\underline{R}^{(\ell)}(z) = \sum\limits_{\ell'+h=\ell} \underline{R}^{(\ell',h)}(z)$ for $\ell \geq 0$.

Definition. *To every reduced contact element (z, E^ψ) associate an ideal $B(z, E^\chi)$ defined to be $\underline{R}^{(0,1)}(z)A(z, E^\chi)$, the product being in the sense of multiplication in the symmetric algebra.*

Definition. *For any integer $\ell \geq 0$ define*

$$\phi^\ell(z, E^\chi) = \dim(A(z, E^\chi) \cap \underline{R}^{(\ell+1)}(z))$$
$$\psi^\ell(z, E^\chi) = \dim(B(z, E^\chi) \cap \underline{R}^{(\ell+1)}(z))$$

By definition it follows that $\phi^\ell(z, E^\chi) - \psi^\ell(z, E^\chi)$ is the dimension of $A^{(\ell,1)}(z, E^\chi)$.

Proposition 14. *There exists a proper subvariety S_ℓ of $\chi_{\mathscr{G}}q$ and a constant $\phi^\ell_q(\sum, (D, D', \tilde{\omega})) = \phi^\ell_q(\sum)$ depending only on $(\sum)$ and $(D, D', \tilde{\omega})$ such that*

$$\phi^\ell (z, E^\chi) < \phi^\ell_q \left(\sum\right) \text{ for any } (z, E^\chi) \in S_\ell$$
$$\phi^\ell(z, E^\chi) = \phi^\ell_q \left(\sum\right) \text{ for any } (z, E^\chi) \notin S_\ell.$$

Proof. Let S_ℓ be the set of all reduced contact element (z, E^χ) such that the function $\phi^\ell(z', E'^\chi)$ in a nighbourhood of (z, E^χ) in $\chi_{\mathscr{G}}q$ is not a constant. Let $(x_1, \ldots, x_p, y_1, \ldots, y_m)$ be a coordinate system in a neighbourhood of z in $(D, D', \tilde{\omega})$. Then we know that

$$\bigcup_{1 \le i_1 < \cdots < i_q \le p} \chi_{\mathscr{G}} q_{D'}(dx_{i_1}, \ldots, dx_{i_q}) = \chi_{\mathscr{G}} q$$

and that $\bigcap \chi_{\mathscr{G}} q D'(dx_{i_1}, \ldots, dx_{i_q}) \neq \phi$. Therefore it is sufficient to show 119
that $S''_\ell = S_\ell \bigcap \chi \mathscr{G}^q D'(dx_1, \ldots dx_q)$ is a proper subvariety, outside of which the function ϕ^ℓ is a constant. For $(z, E^\chi) \in \chi \mathscr{G}^q D'(dx_1, \ldots, dx_q)$ the set of generators of the ideal $A^\ell(z, E^\chi)$ are

$$\begin{cases} \xi^{kj;i_1\cdots i_{\ell-1}}_\varphi = A^{k\lambda}_\varphi(z) Y_\lambda X^j X^{i_1} \cdots X^{i_{\ell-1}} - A^{j\lambda}_\varphi Y_\lambda X^k X^{i_1} \cdots X^{i_{\ell-1}}, \\ L^\chi Y_\lambda = \sum^p_{i=1} \langle dx_i, L^\chi \rangle X^i Y_\lambda (L^\chi \in E^\chi). \end{cases}$$

□

$\phi^\ell(z, E^\chi)$ is the number of linearly independent such generators. Let N be the maximum of the function $\phi^\ell(z, E^\chi)$ on ${}^\chi\mathscr{G}^q D'(dx_1, \ldots, dx_q)$. Take a subset $g_1, \ldots, g_N$ of the above system of generators and let f be the determinant of the submatrix of degree N in the matrix of coefficients in $g_1, \ldots, g_N$. Let $f_1, \ldots, f_r$ be the set of all f obtained by this process. Then it is clear that S''_ℓ is the set of common zeros of $f_1, \ldots, f_r$ and that S''_ℓ has the required properties. Therefore S_ℓ is a proper subvariety. Then our assertion follows easily.

Proposition 15. *There exists a proper subvariety S'_ℓ of ${}^\chi\mathscr{G}^q D'$ and a constant $\psi^\ell_q(\sum, (D, D', \tilde{\omega})) = \psi^\ell_q(\sum)$ depending only on $\sum$ and $(D, D', \tilde{\omega})$ such that*

$$\psi^\ell_q(z, E^\chi) < \psi^\ell_q \left(\sum\right) \text{ for any } (z, E^\chi) \in S',$$

$$\psi^\ell_q(z, E^\chi) = \psi^\ell_q\left(\sum\right) \textit{ for any } (z, E^\chi) \notin S'_\ell,$$

120 *The proof is on the same lines as for Proposition 14.*

$$\text{Set} \qquad t^\ell_q(\Sigma) = \phi^\ell_q(\Sigma) - \psi^\ell_q(\Sigma).$$

Proposition 16. *If $\phi^\ell(z, E^\chi) = \phi^\ell_q(\Sigma)$, then $t^\ell(z, E^\chi) = t^\ell_q(\Sigma)$.*

Proof. By definition, $B(z, E^\ell) = Y_1 A(z, E^\chi) + \cdots + Y_m A(z, E^\chi)$ and $A(z, E^\chi)$ is generated by elements of type (1, 1). It follows, then, that $B^{(\ell,j)}(z, E^\chi) = A^{(\ell,j)}(z, E^\chi)$ for $j \geq 2$. By the same argument as in the proof of Proposition 14, it follows that $\dim A^{(\ell,j)}(z, E^\chi)$ is upper-semi continuous. Hence the fact that dim $\phi^\ell(z', E^\chi) = \sum\limits_{k+j=\ell+1} \dim A^{(k,j)}$ $(z', 'E^\chi)$, is constant on an open set implies that $\dim A^{(A+1-j,j)}(z', 'E^\chi)$ is also a constant. Therefore the function $\psi^\ell(z', 'E^\chi) = \sum\limits_{1 \leq j \leq \ell-1}$ dim $(A^{(\ell+1-j,j)})(z', 'E^\chi)$ is a constant on a neighbourhood of (z, E^χ). Hence

$$\psi^\ell(z, E^\chi) = \psi^\ell_q(\Sigma) \quad \text{and so} \quad t^\ell(z, E^\chi) = t^\ell_q(\Sigma)$$

□

Proposition 17. *$\phi^{\ell+1}_q(\Sigma) \geq Q(\phi^\ell_q(\Sigma), P + m')$ for any ℓ. There is an integer $\ell_0(\Sigma)$ such that*

$$\phi^{\ell+1}_q(\Sigma) = Q(\phi^\ell_q(\Sigma), p + m') \quad \textit{for} \quad \ell \geq \ell_0(\Sigma)$$

Proof. It is sufficient, by Theorem 2, to show the following: There is an ideal I in a ring of polynomials in $p + m$ variables over a field K such that $\phi^\ell_q(\Sigma)$ is equal to the dimension of the vector space $I^{(\ell)}$ over K. To construct such a field K and I, take a coordinate system $(x_1, \ldots, x_p, y_1, \ldots, y_m)$ in $(D, D', \tilde{\omega})$ and let x^i, Y_λ be as before. For a
121 connected open set $\mathscr{D}$ of ${}^\chi\mathscr{G}^q(dx_1, \ldots, dx_q)$, denote by $k(\mathscr{D})$ the field of quotient of the ring of real analytic functions of $\mathscr{D}$. We remark that

$$\xi^{kj}_\varphi = A^{k\lambda}_\varphi Y_\lambda X^j - A^{j\lambda}_\varphi Y_\lambda X^k, \eta^{q^1}_\lambda = Y_\lambda X^{q^1} + w^{q^1}_{q+h} Y_\lambda X^{q+h}$$

can be considered, by restriction, as elements in the polynomial ring $K(\mathscr{D})[X^1, \ldots, X^p, Y_1, \ldots, Y_m,]$. Let $I(\mathscr{D})$ be the ideal generated by ξ^{kj}_{φ}, $\eta^{q^1}_{\lambda}$ in $K(\mathscr{D})[X, Y]$. For $\mathscr{D} \subset \mathscr{D}^1$, we have the canonical injective mapping $k(\mathscr{D}^1) \to K(\mathscr{D})$. It is clear by definition that the image of $I(\mathscr{D}^1)$ generate $I(\mathscr{D})$ over $K(\mathscr{D})$. Then it follows easily that $\dim_{K(\mathscr{D}')}(I(\mathscr{D}')^{(\ell)}) = \dim_{K(\mathscr{D})}(I(\mathscr{D})^{(\ell)})$. Now, set $K = K(\chi\mathscr{G}^q(dx_1, \ldots, dx_q)), I = I({}^{\chi}\mathscr{G}^q (dx_1, \ldots, dx_q))$. For any fixed ℓ, take (z, E^{χ}) such that $\phi^{\ell}(z, E^{\chi}) = \phi^{\ell}_q(\Sigma)$. Take a sufficiently small open connected neighbourhood $\mathscr{D}$ of (z, E^{χ}). Then it is easy to verify that $\dim_{K(\mathscr{D})} I(\mathscr{D})^{(\ell)} = \phi^{\ell}_q(\Sigma)$. By the above remark this implies that $\dim_k(I^{(\ell)}) = \phi^{\ell}_q(\Sigma)$. This finishes the proof of Proposition 17. ◻

3.11

In this section we introduce the notions of ℓ- stable and ℓ- regular reduced constant elements and P-regular points, and prove some of their properties.

Definition . *A reduced contact element (z, E^{χ}) is said to be $\ell-$ stable (with respect to $[\Sigma, (D, D', \tilde{\omega})]$) if the function ϕ^{ℓ} remains a constant in a neighbourhood of (z, E^{χ}) in ${}^{\chi}\mathscr{G}^q$.*

Proposition 18. *A reduced contact element z, E^{χ} is $\ell-$ stable if and only if $\phi^{\ell}(z, E^{\chi}) = \phi^{\ell}_q(\Sigma)$, where $q = \dim E^{\chi}$. The set of non-ℓ-stable* 122 *$q-$dimensional reduced contact elements is a proper subvariety S^q_{ℓ} of ${}^{\chi}\mathscr{G}^q$. If (z, E^{χ}) is ℓ-stable then $t^{\ell}(z, E^{\chi}) = t^{\ell}_q(\Sigma)$.*

Proof. The first two assertions are immediate corollaries of Proposition 14. The last assertion follows from the first and Proposition 16. ◻

Proposition 19. *There exists an integer $\ell_0(\Sigma)$ depending only on (Σ) with the following property: If (z, E^{χ}) is ℓ- stable and if $\ell \geq \ell_0(\Sigma)$ then (z, E^{χ}) is $(\ell + 1)$ stable.*

Proof. Take $\ell_0(\Sigma)$ as in Proposition 17. Then

$$\phi^{\ell+1}_q(\Sigma) = Q(\phi^{\ell}_q(\Sigma), p + m^1)$$

$$= Q(\phi_q^\ell(z, E^\chi), p + m^1) \text{ by } \ell - \text{ stability of } (z, E)$$
$$\leq \phi_q^{\ell+1}(z, E) \text{ by the theorem of Hilbert .}$$

This inequality, together with the inequality $\phi_q^{\ell+1}(\Sigma) \geq \phi_q^{\ell+1}(z, E^\chi)$ (Proposition 14). shows that (z, E^χ) is $(\ell + 1)$- stable. □

The Proposition 19 simply states that for sufficiently large integer ℓ, ℓ- stability implies $(\ell + 1)$- stability. In other words for sufficiently large $\ell, S^q_{\ell+1} \subseteq S^q_\ell$.

Let us denote by $S(\ell) = S(\ell; [\Sigma, D, D', \tilde{\omega}])$ the set of all z in D such that any reduced contact element with origin at z is not an ℓ-stable element.

Proposition 20. *$S(\ell)$ is a proper subvariety of D.*

123 This follows from the following much more general lemma.

Lemma. *Let (M, M', ρ) be a fibred manifold, M'' be a proper subvariety of M. Assume that, for every $x \in M', \rho^{-1}(x)$ be connected. Then the set $M'' = \left\{x \in M' : \rho^{-1}(x) \subset M''\right\}$ is a proper subvariety of M'.*

Proof. Let U be an open subset of M and set $V = \rho(U)$. We claim first that $x \in M'' \cap V$ if and only if $\rho^{-1}(x) \cap V \subset M''$. Namely, if $\rho^{-1}(x) \cap V \subset M'', \rho^{-1}(x) \cap M''$ contains interior points of $\rho^{-1}(x)$. Then, $\rho^{-1}(x) \cap M''$ being a subvariety and $\rho^{-1}(x)$ being connected, it follows that $\rho^{-1}(x) \cap M'' = \rho^{-1}(x)$, i.e.. $x \in M''' \cap V$. converse is trivial. □

Take z_0 in M'. Let $x_0 \in M$ be such that $\rho(x_0) = z_0$. Take an open neighbourhood U of x_0 with the following conditions;

(i) A coordinate system (x, y) of (M, M', ρ) is defined on U.

(ii) there exist real analytic functions $f_1, \ldots, f_k$ on U such that $M'' \cap U$ is equal to the common zeros of $f_1, \ldots, f_k$. For each fixed y and $\lambda = 1, \ldots, k$, we define a function f^y_λ on $\rho(U) = V$ by $f^y_\lambda(x) = f_\lambda(x, y)$. Then by the remark made at the beginning, $M''' \cap V$ is equal to the common zeros of f^y_λ. Therefore M''' is a subvariety. Since $\rho(M - M'') \subseteq M' - M'''$ and since $M - M''$ is not empty, M''' is a proper subset.

Proof of Proposition 20. Denote by $S_q(\ell)$ the set of point z such that any $q-$dimensional reduced contact element at z is not $\ell-$stable. Set $M = D, M' = D', M'' = S_q(\ell)$. Applying the lemma, we find that $S_q(\ell)$ **124**
is a subvariety. Therefore $S(\ell) = \bigcup_{q=0}^{P} S_q(\ell)$ is a subvariety

Definition. *Given an integer $\ell > 0$, a reduced contact element (z, E^{χ}) is said to be ℓ- regular if (z, E^{χ}) is $\ell'-$ stable for $\ell' \geq \ell$.*

Let $\mathscr{V}(\ell, q)$ be the set of all non ℓ- regular reduced contact elements of dimension q. Clearly $\mathscr{V}(\ell, q) = \bigcup_{\ell' \geq \ell} S^q_{\ell'} = \bigcup_{0 \leq h \leq \max(0, \ell_0(\Sigma) - \ell)} S^q_{\ell+h}$(cf. Proposition 19). This shows that $\mathscr{V}(\ell, q)$ is again a proper subvariety of ${}^{\chi}\mathscr{G}^q D'$.

Now let us introduce the notion of reduced flags.

Definition. *A reduced flag ${}^qF^{\chi}$ on a q-dimensional reduced contact element (z, E^{χ}) is a finite sequence of reduced contact elements $\left\{z, (z, E_1^{\chi}), \ldots, (z, E_q^{\chi})\right\}$ such that $E_0^{\chi} \subset E_1^{\chi} \subset \cdots \subset E_q^{\chi} = E^{\chi}$ is a flag on E^{χ}.*

Let ${}^q\mathscr{F}^{\chi}$ denote the set of all reduced flags on q-dimensional contact elements of $(D, D', \tilde{\omega})$. $q\mathscr{F}^{\chi}$ is contained in the product space $D\times$(space of all flags on q-dimensional contact elements of D') $= D \times M$.

Let ρ be the map which associates to each flag its origin. Then ${}^q\mathscr{F}^{\chi} = \{qF^{\chi} : \tilde{\omega}(z) = \rho({}^qF^{\chi})\}$. This defines the structure of a real analytic submanifold of $D \times M$ on ${}^q\mathscr{F}^{\chi}$.

Definition. *A reduced flag ${}^qF^{\chi} = \left\{(z, E_k^{\chi})(k = 0, 1, \ldots, q)\right\}$ is said to be ℓ- regular if each component (z, E_k^{χ}) is ℓ- regular.*

Let $S\,{}^q\mathscr{F}^{\chi}(\ell)$ be the set of all non ℓ- regular reduced flags ${}^qF^{\chi}$ on $(D, D', \tilde{\omega})$. $S\,{}^q\mathscr{F}^{\chi}(\chi)$ is a proper real analytic subvariety of $q\mathscr{F}^{\chi}$. For if **125**
ρ_k denotes the projection $q\mathscr{F}^{\chi} \to \chi^k_{\mathscr{G}}$ associating to each flag in ${}^q\mathscr{F}^{\chi}$ its *kth* component then $S\,{}^q\mathscr{F}^{\chi}(\ell) = \bigcup_{k=0}^{q} \rho_k^{-1}(\gamma(\ell, k))$.

Definition . *A point $z \in D$ is said to be P- regular of weight ℓ with respect to (Σ) if there exists a reduced flag F^{χ} on $(z, (D')\tilde{\omega}(z))$ which is $\ell-$ regular (with respect to (Σ)).*

Let $S[\ell]$ denote the set of all points z in D which are not P-regular of weight ℓ. Denote by p the dimension of D'. Let us denote the map $\rho : p_{\mathscr{F}\chi} \to D$ which associates to every reduced flag its origin. Then since $S(\ell) = \left\{z \in D : \rho^{-1}(z) \subset S^{\chi}\mathscr{F}^{p}(\ell)\right\}$, it follows by the lemma that $S[\ell]$ is a proper subvariety. We remark that $S(\ell) \supseteq S(\ell+1) \supseteq \cdots$

Definition. *A point z in D is said to be P-regular of if there exists an integer $\ell \geq 0$ such that z is P-regular of weight ℓ with respect to (Σ) if there exists a reduce flag F^{χ} on $(,(D')\tilde{\omega}(z))$ which is ℓ which is $\ell-$ regular (with respect to (Σ)).*

Let $S[\ell]$ denote the set of all points z in D which are not P- regular of weight ℓ. Denote by p the dimension of D'. Let us denote the map $\rho :^{q} \mathscr{F}^{\chi} \to D$ which associates to every reduced flag its origin. The since $S(\ell) = \left\{z \in D : \rho^{-1}(z) \subset S^{\chi}\mathscr{F}^{p}(\ell)\right\}$, it follows by the lemma that $S[\ell]$ is a proper subvariety. We remark that $S(\ell) \supseteq S(\ell+1) \supseteq \ldots$

Definition. *A point z in D is said to be P- regular if there exists an integer $\ell \geq 0$ such that is z is $P-$ regular of weight ℓ.*

Let $S = S(\Sigma, (D, D', \tilde{\omega}))$ be the set of all points z in D which are not $P-$ regular. Clearly $S = \bigcap_{\ell} S(\ell)$. Hence the set of all non P-regular points z in D is a proper subvariety of D. For any $z \notin S$ we can construct a reduced flag $F^{\chi} = \left\{(z, E_q^{\chi})\right\}_{q=0,1,\ldots,p}$ on $(z, (D')_{\tilde{\omega}(z)})$ such that $\phi^{\ell}(z, E_q^{\chi}) = \phi_q^{\ell}(\Sigma)$ for sufficiently large ℓ.

3.12

Let (z, E^{χ}) be a fixed reduced contact element and let L^{χ} be in $(D')_{\tilde{\omega}(z)} = \underline{R}^{(1,0)}(z)$. Then the multiplication $L^{\chi} f \in \underline{R}(z)$ is defined for any $f \in \underline{R}(z)$. We now pose the following definition:

Definition. *L^{χ} is called ℓ_1-prime to (z, E^{χ}) if the conditions $f \in \underline{R}^{(\ell,1)}(z)$,*
126 *$l \geq \ell_1$ and $L^{\chi} f \in A(z, E^{\chi})$ imply $f \in A(z, E^{\chi})$.*

Proposition 21. *Let $(z, e^{\chi}) \in \chi_{\mathscr{G}} q$ be a reduced contact element. Then there exists an integer $\ell_1 = \ell_1(z, E^{\chi})$ satisfying the following condition:*

the set of vectors $L^{\chi} \in \underline{R}^{1,0}(z)$ which are ℓ_1- prime to (z, E^{χ}) is everywhere dense in $\underline{R}^{1,0}(z)$

Proof. since $\underline{R}(z)$ is a Noetherian ring we can write $A(z, E^{\chi}) = A = \mathscr{G}_1 \cap \cdots \cap \mathscr{G}_a$ where $\mathscr{G}_1, \ldots, \mathscr{G}_a$ are primary ideals in $\underline{R}(z)$. We shall denote by R the algebra $\underline{R}(z)$, for simplicity. It is known from the theory of ideals in polynomial rings that the set $\mathscr{G}_i$ of all elements u in R for which there exists an integer ℓ such that $u^{\ell} \in {}_i$ is a prime ideal in R. We can also assume that there exists an integer n_i such that $\mathscr{U}_i^{n_i} \subseteq \mathscr{G}_i$ for $i = 1, \ldots, a$. We can assume further that $\mathscr{G}_1, \ldots, \mathscr{G}_b$ are the primary ideals such that for any integer $\ell, \mathscr{G}_i \not\supseteq R^{(\ell,1)}(1 \leq i \leq b)$; but for each $\mathscr{G}_i$ among $\mathscr{G}_{b+1}, \ldots, \mathscr{G}_a$ there exists an integer ℓ_i such that $\mathscr{G}_i \supseteq R^{(\ell_i,1)}$ and hence $\mathscr{G}_i \supseteq R^{(\ ,1)}$ for any $\ell \geq \ell_i$.

b may be zero and in this case take $\tilde{\ell} = \max(\ell_1, \ldots, \ell_a)$. Then $A \supseteq R^{(\ell,1)}$ for any $\ell \geq \tilde{\ell}$ and the proposition follows in this case. Hence we may assume $b > 0$, define $\tilde{\ell} = \max(\ell_{b+1,\ldots,\ell_a})$ if $b < a$ and $\tilde{\ell} = 1$ if $b = a$. For any $\ell \geq \tilde{\ell}$ we have $\mathscr{G}_1 \cap \cdots \cap \mathscr{G}_b \cap R^{(\ell,1)} = A \cap R^{(\ell,1)}$. We claim that $\mathscr{G}_i \cap R^{(1,0)}$ is a proper subspace of $R^{(1,0)} = (D')_{\tilde{\omega}(z)}$ for $i = 1, \ldots, b$. For let, if possible, $\mathscr{G}_i \supseteq R^{(1,0)}$ so that $\mathscr{G}_i \supseteq \mathscr{G}_i^{n_i} \supseteq R^{(n_i,0)}$ which is a contradiction to the choice of $\mathscr{G}_1, \ldots, \mathscr{G}_b$. $\square$

Now take a vector L^{χ} in $R^{(1,0)}$ such that $L^{\chi} \notin \cup \mathscr{G}_i(1 \leq i \leq b)$. Fix 127
$\ell \geq \tilde{\ell}$. Then the condition $L^{\chi} f \notin A^{(\ell+1,1)}$ implies $L^{\chi}.f \in \ell_i$ $(i \leq i \leq b)$ because we can write $A^{(\ell+1,1)} = \mathscr{G}_1 \cap \cdots \cap \mathscr{G}_b R^{(\ell+1,1)}$. Then since $L^{\chi} \in \mathscr{G}_i$ it is known from the theory of ideals in polynomial rings that $f \in \mathscr{G}_i$ $(1 \leq i \leq b)$ which means that $f \in \mathscr{G}_1 \cap \cdots \cap \mathscr{G}_b \cap R^{(\ell,1)} = A^{(\ell,1)}$. Therefore the complementary set of $(\mathscr{G}_1 \cup \cdots \cup \mathscr{G}_b) \cap (D')_{\tilde{\omega}(z)}$, which is everywhere dense in $(D')_{\tilde{\omega}(z)}$ consists of all vectors $\tilde{\ell}$- prime to (z, E^{χ}).

Let $(D, D', \tilde{\omega})$ be a fibred manifold and (Σ) a normal differential system on it. We pose the following definition.

Definition. *If z is a point D, a coordinate system $(x_1, \ldots, x_p)$ of D' at $\tilde{\omega}(z)$ is said to be ℓ_1- regular when the following conditions are satisfied (at z with respect to Σ):*

(i) *z is a P-regular point of weight ℓ_1;*

(ii) *the reduced flag* $F^{\chi} = \left\{(z, E^{\chi}_{q}); (q = 0, 1, \ldots, p-1)\right\}$ *where each* E^{χ}_{q} *is the subspace of* $(D')_{\tilde{\omega})(z)}$ *consisting of vectors* L *such that* $\langle dx_{q+1}, L\rangle = \cdots = \langle dx_p, L\rangle = 0$*, is an* ℓ_1*-regular reduced flag;*

(iii) *If* $(L^1, \ldots, L^p)$ *denote a base of* $(D')_{\tilde{\omega}(z)} = M$ *dual to* $dx_1|M, \ldots$, $dx_p|M$*, each* L^q *is* ℓ_1*- prime to* $(z, E^{\chi}_{q-1})(q = 1, \ldots, p)$*.*

Theorem 3. *If* z *is a* P*-regular point of weight* ℓ_0*(with respect to* $[\Sigma, (D, D', \tilde{\omega})]$*), there exists an* ℓ_1*- regular coordinate system of* D' *at*
128 $\tilde{\omega(z)}$ *(with respect to* $[\Sigma, (D, D', \tilde{\omega})]$*) for sufficiently large* ℓ_1*.*

Proof. Since z is a P-regular point of weight ℓ_0 it is also a P-regular point of weight ℓ for $\ell \geq \ell_0$ so much so that we can assume without loss of generality that $\ell_o \geq \ell_o(\Sigma)$ .There exists an ℓ_0 - regular flag $F^{\chi} = \{(z, E^{\chi}_{q}) : (q = 1, \ldots, p)\}$ on $(z, (D')_{\tilde{\omega}(z)})$ by definition. The set of non ℓ_0 -regular reduced flags being a proper real analytic subvariety, the set of ℓ_0-regular reduced flags is open in the manifold of all reduced flags. Hence there exists a neighbourhood U_q of (z, E^{χ}_{q}) in $\chi\mathscr{G}q$ such that any flag $'F^{\chi}$ with its qth component in U_q for each q is also ℓ_0-regular. Consider the vectors $L^1, \ldots, L^p$ such that $L^q \in E^{\chi}_{q}$ and E^{χ}_{q} is generated by E^{χ}_{q-1} and L^q. we can choose a neighbourhood U_q of (z, E^{χ}_{q}) and a neighbourhood V_q of L^q in such a way that the space $''E^{\chi}_{q}$ spanned by $'E^{\chi}_{q-1} \in U_{q-1}$ and $'L^q \in V_q$ is in U_q. Let $'E^{\chi}_{q} = \tilde{\omega}(z)$. Then by Proposition 21 there exists a vector $'L^1 \in V_1 \cap (D')_{\tilde{\omega}(z)}$ which is ℓ^0 -prime to $(z, ''E^{\chi}_{0})$ and which is such that $''E^{\chi}_{1}$ spanned by $''E^{\chi}_{0}$ and $'L^1$ is in U_1, where $\ell^0 = \ell_1(z, ''E^{\chi}_{0})$. Proceeding in this manner inductively we can choose $'L^q \in V_q \cap (D')_{\tilde{\omega}(z)}$ such that □

(i) $\{(z, ''E^{\chi}_{q})\}$ is a reduced flag on $(z, (D')_{\tilde{\omega}(z)})$, which is ℓ_0 -regular ,

(ii) $'L^q$ is ℓ^{q-1}-regular to $(z, ''E^{\chi}_{q-1}$ where $\ell^{q-1} = \ell_1(z, ''E^{\chi}_{q-1})$, and

(iii) $''E^{\chi}_{q}$ is generated by $''E^{\chi}_{q-1}$ and $'L^q$.

129 Set $\ell_1 = \max(\ell^0, \ldots, \ell^{p-1})$. Then we have a reduced flag $\{(z, ''E^{\chi}_{q})\}$ such that

(i) it is ℓ_1 regular

(ii) $''E^{\chi}_{q}$ is spanned by $''E^{\chi}_{q-1}$ and $'L^{q}$; $'L^{q}$ is ℓ_1- prime to $(z, ''E^{\chi}_{q-1})$.

Take a coordinate system $(x_1, \ldots, x_p)$ such that $dx_{q+1}|''E^{\chi}_{q} = \cdots = dx_p|''E^{\chi}_{q} = 0$. Let $L', \ldots, L^p$ be a basis of $(D')_{\tilde{\omega}(z)}$ dual to $dx_1, \ldots, dx_p$. Now we have $L^q = a^{q'} L^q + v^q (v^q \in ''E^{\chi}_{q-1}), a^q \neq 0$. Because $'L^q$ is ℓ_1-prime to (z, E^{χ}_{q-1}), it is easy to see that L^q is ℓ_1-prime to $''E^{\chi}_{q-1}$ and this completes the proof.

Theorem 4. *There exists an integer $\tilde{\ell}\,(\sum)$ such that for $\ell \geq \tilde{\ell}\,(\sum)$*

$$t^{\ell}_{q}\left(\sum\right) = t^{\ell}_{q-1}\left(\sum\right) + n_{\ell-1} - t^{\ell-1}_{q-1}\left(\sum\right)$$

where $$n_{\ell-1} = \dim \underline{R}^{(\ell-1,1)}(z), 1 \leq q \leq p.$$

Proof. There exists point z which is P-regular of weight ℓ_1 for a sufficiently large ℓ_1. By Theorem 3 there exists a coordinate system $(x_1, \ldots, x_p)$ of D' at $\tilde{\omega}(z)$ which is P-regular. Hence there exists an $\tilde{\ell}$-regular reduced flag F^{χ} for an integer $\tilde{\ell}$ satisfying the following conditions. □

(i) if (z, E^{χ}_{q}) is the qth component of F^{χ}, E^{χ}_{q} is the space of all vectors L such that $\langle dx_{q+1}, L\rangle = \ldots = \langle dx_p, L\rangle = 0$;

(ii) the base $X^1, \ldots, X^p$ of $(D')_{\tilde{\omega}(z)}$ dual to $dx_1, \ldots, dx_p$ is such that X^p is $\tilde{\ell}$- prime to E^{χ}_{q-1}. Then $\phi^{\ell}(z, E^{\chi}_{q}) = \phi^{\ell}_{q}(\sum)$ and by Proposition 16 $\psi^{\ell}(z, E^{\chi}_{q}) = \psi^{\ell}_{q}(\Sigma)$ for $\ell \geq \tilde{\ell}$. Hence $t^{\ell}_{q}(\Sigma) = \phi^{\ell}_{q}(\Sigma) - \psi^{\ell}_{q}(\Sigma) = \dim A^{(\ell,1)}(z, E^{\chi}_{q})$ for $\ell \geq \tilde{\ell}. A^{(\ell,1)}(z, E^{\chi}_{q})$is generated as a vector space by 130

$$\begin{cases} \tilde{\xi}^{kj}_{\varphi}(z) X^{i_1} \ldots X^{i_{\ell-1}}, \\ L X^{i_1} \ldots X^{i_{\ell-1}} Y_{\lambda}, L \in E^{\chi}_{q} \subseteq \underline{\mathrm{R}}^{(1,0)}(z) \end{cases}$$

Where $\xi^{kj}_{\varphi}(z) = A^{k\lambda}_{\varphi}(z) Y_{\lambda} X^j - A^{j\lambda}_{\varphi}(z) Y_{\lambda} X^k$ and $X^i = \left(\dfrac{\partial}{\partial x_i}\right)_z \in (D')_z = \underline{\mathrm{R}}^{(1,0)}(z)$. Hence

$$\xi^{kj}_{\varphi} X^{i_i} \ldots X^{i_{\ell-1}}, Y_{\lambda} X^{q'} X^{i_1} \ldots X^{i_{\ell}-1} (q' = 1, \ldots, q)$$

generate $A^{(\ell,1)}(z, E^{\chi}_{q})$ and hence $A^{(\ell,1)}(z, E^{\chi}_{q}) = A^{(\ell,1)}(z, E^{\chi}_{q-1}) + X^q \underline{\mathrm{R}}^{(\ell-1,1)}(z)$. But this need not be a direct sum. Let $v_1, \ldots, v_{\gamma}$ be a set of maximum number of linearly independent elements in $\underline{R}^{(\ell-1,1)}(z)$ modulo $A^{(\ell-1,1)}(z, E^{\chi}_{q-1})$ .Then

$$A^{(\ell,1)}(z, E^{\chi}_{q}) = A^{(\ell,1)}(z, E^{\chi}_{q-1}) + X^{q}\underline{R}V_1 + \ldots + X^{q}\underline{R}V_{\nu}$$

where $\underline{R}$ is the field of real numbers. We claim that this is a direct sum decomposition. For, let a non-trivial relation

$$X^{q}b_1V_1 + \ldots + X^{q}b_{\nu}V_{\nu} + a = 0, a \in A^{(\ell,1)}(z, E^{\chi}_{q-1})$$

hold. That is, $X^{q}(b_1V_1+\ldots+b_{\nu}V_{\nu}) \in A^{(\ell,1)}(z, E^{\chi}_{q-1})$ Since X^{q} is $\tilde{\ell}-$ prime to (z, E^{χ}_{q-1}), it follows then that $b_1V_1 + \ldots + b_{\nu}v_n \in A^{(\ell-1,1)}, (z, E^{\chi}_{q-1})$. Then by the choice of $V_i, b_i = 0$, and so $a = 0$. Therefore $t^{\ell}_{q-1}(\Sigma) + \nu =$
131 $t^{\ell}_{q}(\Sigma)$. But by definition $\nu = n_{\ell-1} - t^{\ell-1}_{q-1}(\Sigma)$. Therefore we obtain the required equality

$$t^{\ell}_{q}(\Sigma) = t^{\ell}_{q-1}(\Sigma) + n_{q-1} - t^{\ell-1}_{q-1}(\Sigma) \quad \text{for} \quad \ell \geq \tilde{\ell} \qquad \text{q.e.d.}$$

By the same argument employed in the last part of the above proof, we have the following :

Proposition 22. *Let (z, E^{χ}_{q-1}) be a $(q-1)-$dimensional reduced contact element. Assume that $L^{\chi} \in (D')_{\tilde{\omega}(z)}$ is ℓ_1-prime to (z, E^{χ}_{q-1}). Denote by E^{χ}_{q} the subspace generated by $E^{\chi}_{q-1}$and L^{χ}. Assume that E^{χ}_{q} is q-dimensional. Then for $\ell \geq \ell_1$*

$$t^{\ell}(z, E^{\chi}_{q}) = t^{\ell}(z, E^{\chi}_{q-1}) + n_{\ell-1} - t^{\ell-1}(z, E^{\chi}_{q-1}).$$

3.13

Definition. *We say that a reduced flag $\{(z, E^{\chi}_{q}); q = 0, 1, \ldots p\}$ is weakly ℓ- stable when $t^{\ell}_{q}(z, E^{\chi}_{q}) = t^{\ell}_{q}(\Sigma)$ for $q = 0, 1, \ldots, p-1$.*

Clearly, the set of weakly ℓ -stable reduced flags is open, and contains the set of ℓ-stable reduced flags.

Proposition 23. *Let $F = \{E_q\}$ be an integral flag of $[p^\ell_S\Sigma, (j^\ell, D', \alpha)]$ satisfying the following conditions:*

(i) $\ell \geq \ell_1(\Sigma)$;

(ii) $(p^\ell_S\Sigma)^{(1)}_X \cap \Omega_X = \{0\}$, Where X is the origin of F and

$$\Omega_X = \alpha^*(\wedge^1_{\alpha(X)}(D'));$$

(iii) the reduced flag $F^\chi = \{(z, E^\chi_q) : q = 0, 1, \ldots, p\}$ is weakly ℓ-stable and weakly $(\ell - 1)-$ stable, where $z = \beta(X)$ and $z, E^\chi_q = d\alpha E_q$. Then $\dim(A^\ell)_{Eq} = f^\ell_q(z, E^\chi) = t^\ell_q(\Sigma)$ for $q = 0, 1, \ldots, p - 1$ (cf. Proposition 11).

Proof. We write $A^\ell(E_q)$ instead of $(A^\ell_q)E_q$. The proof is by induction on the dimension q. When X is an integral point, we have $A^\ell(X) \subseteq J(X) = (P^\ell_S\Sigma)^{(1)}_X$. Hence by (ii) $\dim A^\ell(X) = \dim(A^\ell(X) + \Omega_X/\Omega_X = t'(X) = t^\ell_o(x)$ (cf. Proposition 13). The proposition is therefore proved in the case $q = 0$. Let us assume that the proposition holds in the case $(q - 1)$. Now since $(z, E^\chi_q)$is weakly-ℓ-stable, 132

$$\begin{aligned} t^\ell_q(\Sigma) = t^\ell_q(z, E^\chi_q) = t'(E_q) \quad &\text{(by Proposition 13)} \\ \leq \dim A^\ell(E_q) & \\ \dim A^\ell(E_{q-1}) + n_{\ell-1} - t'(d\rho^\ell_{\ell-1}E_{q-1}) \quad &\text{(by Proposition 12)} \end{aligned}$$

where $n_{\ell-1} = \dim \underline{\mathrm{R}}^{(\ell-1,1)}_(z)$ and $\rho^\ell_{\ell-1}$ is the projection of J^ℓ onto $J^{\ell-1}$. By induction assumption and Proposition 13 the latter member of the above inequality is equal to

$$t^\ell_{q-1}(\Sigma) + n_{\ell-1} - t^\ell_{q-1}(z, E^\chi_{q-1})$$

Hence, since the flag is weakly -$(\ell - 1)$-stable we obtain $t^\ell_q(\Sigma) \leq \dim A^\ell$ $(E_q) \leq t^\ell_{q-1}(\Sigma) + n_{l-1} - t^{\ell-1}_{q-1}(\Sigma)$. But for $\ell \geq \ell_1(\Sigma), t^\ell_q(\Sigma) = t^\ell_{q-1}(\Sigma) + n_{\ell-1} - t^{\ell-1}_{q-1}(\Sigma)$ by Theorem 4. Therefore $t^\ell_q(\Sigma) = \dim A^\ell(E_q)$ and this proves the proposition. ◻

Let $[\Sigma, (D, D', \tilde{\omega})]$ be a differential system with independent variables. Suppose that $\dim D' = p$. We pose the following definition:

Definition. *The system $[\Sigma, (D, D', \tilde{\omega})]$ is said to be in involution at an integral point $z \in D$ if the following conditions are satisfied:* **133**

(i) any integral element, of dimension p of $[\Sigma, (D, D', \tilde{\omega})]$ with origin at z, is an ordinary integral element,

(ii) there exists a p-dimensional integral element with origin at z of $[\Sigma, (D, D', \tilde{\omega})]$.

Definition. *An integral point z of $[\Sigma, (D, D', \tilde{\omega})]$ is a normal integral point if the following conditions are satisfied:*

(i) z is an ordinary integral point;

(ii) there exists a neighbourhood U of z in D such that for any $z' \in U \cap v^{\circ}\Sigma$, we have

$$(\Sigma^{(1)})_{z'} \cap \tilde{\omega}^*(\wedge^{(1)}_{\tilde{\omega}(z)}(D')) = \{0\}.$$

Proposition 24. *If $[\Sigma, (D, D', \tilde{\omega})]$ is in involution at an integral point Z in D then Z is a normal integral point.*

Proof. By definition z is an ordinary integral point and there exists an integral element E of dimension p at z. Let $(x_1, \ldots, x_p)$ be a coordinate system of D' at $\tilde{\omega}(z)$. Then E being an integral element of $[\Sigma, (D, D', \tilde{\omega})]$ $dx_1|E, \ldots, dx_p|E$ are linearly independent. Because of corollary to proposition 8 (Chapter II) there exists a neighbourhood U of Z in D such that for any $Z' \in \cup \cap \vartheta^0\Sigma$ there is an integral element E' of Z' such that $dx_1|E', \ldots, dx_p|E'$ are linearly independent,
134 Take an element $a_1(dx_1)_{z'} + \ldots + a_p(dx_p)_{z'}$ in $(\Sigma^{(1)})_{z'}, \cap \tilde{\omega}^*(\wedge_{\tilde{\omega}(z)}D')|E'$. But since E' is an integral element of $(\sum), \Sigma^{(1)}|E' = 0$ so much so that $a_1(dx_1|E') + \ldots + a_p(dx_p|E') = 0$, that is $a_1 = \ldots = a_p = 0$. ◻

Definition. *Z in D is called P-weakly -regular of weight ℓ_0 with respect to $[\Sigma, (D, D', \tilde{\omega})]$) when there is a reduced flag over $(z, (D')_{\tilde{\omega}(z)})$ which is ℓ - weakly -stable for $\ell \geq \ell_0$. Therefore z is a normal point.*

Clearly , a P-regular point of weight ℓ_0 is a ℓ-weakly - regular point of weight ℓ_0 .

3.14

Theorem 5. *Let* $[\Sigma, (D, D', \tilde{\omega})]$ *be a normal differential system with independent variables and* $P^{\ell}_{S}\Sigma$ *be its standard prolongation to* $J^{\ell}(D, D', \tilde{\omega})$*. Let* $z \in D$ *be a P-weakly -regular point of weight* ℓ_0 *and X be an integral point of* $[P^{\ell}_{S}\Sigma, (J^{\ell}, D', \alpha)]$ *such that* $\beta(X) = z$*. Then,if* $\ell \geq \max[\ell_0 + 1, \ell_1(\Sigma)]$ *the system* $[P^{\ell}_{S}\Sigma, (J^{\ell}, D', \alpha)]$ *is in involution at X if and only if X is normal with respect to* $P^{\ell}_{S}\Sigma$*.*

Proof. In view of the Proposition 24, it is sufficient to prove that if X is a normal integral point then $P^{\ell}_{S}\Sigma$, is in involution at X.

Take a p-dimensional integral element E of $[P^{\ell}_{S}\Sigma, (J, D', \tilde{\omega})]$ at X. Since z is a P-regular point of weight ℓ_0 there exists an ℓ_0 -regular reduced flag $F^{\chi} = \{(z, E^{\chi}_{q})\}$; that is each component is ℓ'-stable for any $\ell' \geq \ell_0$. Since $d\alpha$ is an isomorphism of E onto $(D')_{\tilde{\omega}(z)}$ there exists a subspace $E_q \subset E$ such that $D\alpha$ is an isomorphism of E_q onto E^{χ}_{q}. Now $F = \{E^{q}\}$ is a flag on E at X. We claim that each component E_q of 135
F is regular; that is there a neighbourhood $\mathcal{U}_q$ of E_q in $\mathscr{G}^{q} J^{\ell}(D, D', \tilde{\omega})$ such that for any element $E'_q \in \mathscr{U} \cap \vartheta^{q}(p^{\ell}_{S}\Sigma)$ we have $\dim J(E'_q)$. Let $\Omega(X) = \alpha^{*}(\wedge^{(1)}_{\alpha(X)}(D'))$. □

By condition (ii) of normality of X there exists a neighbourhood U of X in J^{ℓ} such that for any X' in $\mathscr{U} \cap \vartheta^{0}(P^{\ell}_{S}\Sigma)$ we have $((P^{\ell}_{S}\Sigma)^{(1)} \cap \Omega(X') = \{0\}$. Choose $\mathcal{U}_q$ so small that

i) for any $E'_q \in \mathcal{U}_q$, the origin X' of E'_q is in U;

ii) $(\beta(X''), d\alpha(E'_q))$ is ℓ_0 -weakly -regular .

Hence all the conditions of Proposition 23 are satisfied and so

$$\dim J(E'_q) = \dim(G^{\ell})_X + \dim(\pi^{(\ell)})_X + t^{\ell}_{q}(\Sigma)$$

The right member of this equation is independent of the choice of E'_q in $\mathcal{U}_q$: This proves that E is an ordinary integral element.

Now it only remains to show that there exists a P-dimensional integral element of $P^{\ell}_{S}\Sigma$ and or origin at X. Since X is normal $J(X) \cap \Omega(X) = \{0\}$ and therefore $\dim J(X) = dim[(J(X) + \Omega(X))/\Omega(X)]$.

Let $(x_1, \ldots, x_p)$ be a coordinate system of D' at $\alpha(X)$ such that E^X_q is generated by the first q elements in the base of E^X_q to $dx_1, \ldots, dx_p$. Let L_1 be a solution of the system of equations

$$\langle J(X), L\rangle = 0,$$
$$\langle dx_1, L\rangle = 1, \langle dx_2, L\rangle = \ldots = \langle dx_p, L\rangle = 0.$$

136 Let E_1 denote the one dimensional contact element spanned by L_1. E_1 is an one dimensional integral element of $P^\ell_S\Sigma$. Again

$$J(E_1) = (G^\ell)_X + (\pi^{(\ell)})_X + A^\ell(E_1)$$

$\dim A^\ell(E_1) = t^\ell_1(\Sigma)$ and so $J(E_1) \cap \Omega(X) + \{0\}$. Let L^2 be the solution of the equations

$$\langle J(E), L\rangle = 0,$$
$$\langle dx_1, L\rangle = 0, \langle dx_2, L\rangle = 1, \langle dx_3, L\rangle = \ldots = \langle dx_p, L\rangle = 0.$$

Repeating this process we construct a p-dimensional integral element of $P^\ell_S\Sigma$. This completes the proof of the theorem.

Remark. *In Theorem 5, the assumption that z is P-regular can be replaced by the following assumption: There is a reduced flag on*

$$(z, (D')_{\tilde{\omega}(z)})$$

such that each of its component ℓ-weakly -stable and $(\ell - 1)$ -weakly stable. The reason is that the former assumption is used in the proof of Theorem 5 only to the existence of a reduced flag having the property in the latter assumption.

3.15

Let $[\Sigma, (D, D', \tilde{\omega})]$ be a differential system with independent variables. We set $J^\ell = J^\ell(D, D', \tilde{\omega})$. The standard prolongation $P^\ell_S\Sigma$ on J^ℓ is gen-
137 erated by $(P^\ell_S\Sigma)^{(0)}, \pi^{(\ell)}$ as an ideal closed under d. Here $\prod(\ell)$ and the operator d do not depend on the system given. Thus $P^\ell_S\Sigma$ is completely

determined by $(P^{\ell}_{S}\Sigma)^{(0)}$. More generally, let us consider a submodule F of $\wedge^{0} J^{\ell}$ .We shall construct a submodule F' of $\wedge^{0} j^{\ell+1}$ out of F having the following property : When this construction is applied to $(P^{\ell}_{S}\Sigma)^{(0)}$, we obtain $(P^{\ell+1}_{S}\Sigma)^{(0)}$. To make the matter more general, we will consider subsheaves of the sheaf of germs of (real analytic) functions, instead of submodules.

Let $(X_1, \ldots, X_p, Y_1 l, \ldots, Y_m) = (X, Y)$ be a coordinate system in $(D, D', \tilde{\omega}).J^{\ell}$ has the coordinates system $(X, Y, \ldots, Y^{i,\ldots i_{\nu}}_{\lambda}, \ldots)$ $(\nu \leq \ell)$ associated with (X, Y). Let f be a function defined on an open set U in the domain of the coordinate system. Consider f as a function on $(\rho^{\ell+1}_{\ell})^{-1}(U) = U'$. Then we have $df \equiv f^{j} dx_j$ (mod $\prod(\ell + 1)$) where f^{j} are functions on U' This follows from the facts that df is a linear combination of $dx_j, dy_\lambda, dy^{i_1 \ldots i_\nu}_{\lambda} (\nu \leq \ell)$ and that $dy^{i,\ldots i_\nu}_{\lambda} \equiv y^{i_1 \ldots i_\nu j}_{\lambda} dx_j$ (mod $\prod(\ell + 1)$). We set $D^{j} f = f^{j}$. Clearly

$$D^{j}(\alpha f + \beta g) = \alpha D^{j}(f) + \beta D^{j}(g), D^{j}(fg) = (D^{j} f)g + f(D^{j} g)$$

Where $\alpha, \beta \in \underline{R}$. When (X, Y) is changed to (X', Y'), have $dx_j = a^{k}_{j} dx'_k$. Since $\prod(\ell + 1)$ does not depend upon the choice of the coordinate system, df$\equiv (D^{j} f)dx_j = (D^{j} f)a^{k}_{j} dx'_k (mod \prod(\ell + 1))$. Therefore

$$D'^{j}(f) = a^{j}_{k}(D^{k}(f)).$$

Let F be an ideal of $\wedge^{0}(U)$. Denote by $P(F)$ the ideal in $\wedge^{0}(U')$ generated by $F o \rho^{\ell+1}_{\ell}$ and $D^{j}(F)$, where $j = 1, \ldots, P$ and f runs through F. The above rule for change of D^{j} under coordinate transformation shows that $P(F)$ is independent of the choice if coordinate system employed to construct $P(F)$. **138**

Let $(M, M', \tilde{\omega})$ be a fibered manifold. Denote by $\mathscr{O} J^{\ell}$ the sheaf of germs of real analytic functions on J^{ℓ}. $\mathscr{O} J^{\ell}$ is a sheaf of rings. Let $\mathscr{U}$ be an open set of J^{ℓ}. Let Φ be a subsheaf of ideals of $\mathscr{O} J^{\ell} | \mathscr{U}$, the restriction of $\mathscr{O} J^{\ell}$ to $\mathscr{U}$. For each open set $U \subset \mathscr{U}$ of J^{ℓ} , denote by $\Gamma(U, \Phi)$ the ring of cross -sections of Φ over.U. For each open set V of $J^{\ell+1}$ such that $\rho^{\ell+1}_{\ell}(V) = W \subset \mathscr{U}$, denote by $\Psi(V)$ the ideal in $\wedge^{0}(V) = \Gamma(V, \mathscr{O} J^{\ell+1})$ generated by the restriction of $P(\Gamma((\rho^{\ell+1}_{\ell})^{-1}(W), \Phi))$. If $V' \subset V$, the restriction mapping sends $\Psi(V)$ into $\Psi(V')$. Hence the system $\Psi(V)$ defines a subsheaf of ideals of $\mathscr{O} J^{\ell+1} | (\rho^{\ell+1}_{\ell})^{-1}(\mathscr{U},)$ which will be denoted by $P(\Phi)$. $P(\Phi)$ is called the standard prolongation of Φ . Let us assume now that $M = D, M' = D'$. Let $\left[(P^{\ell}_{S}\Sigma)^{(0)}\right]$ the subsheaf of ideals in $\mathscr{O}J^{\ell}$ generated by $[(P^{\ell}_{S}\Sigma)^{(0)}]$ the subsheaf of ideals in $\mathscr{O}J^{\ell}$ generated by $(P^{\ell}_{S}\Sigma)^{(0)}$. Then

Proposition 25. $P([(P^{\ell}_{S}\Sigma)^{(0)}]) = [(P^{\ell+1}_{S}\Sigma)^{(0)}]$.

Proof. By definition, $(P^{\ell}_{S}\Sigma)^{(0)}$ is generated by $F^{k_1,\ldots k_a;i_1\ldots i_r}_{\varphi}(\varphi \in \Sigma^{(a)}; 1 \leq k_1, \ldots, k_a; i_1, \ldots, i_r \leq p; r \leq \ell - 1)$.

By Proposition 3, $D^{j}(F^{k_1\ldots k_a;j_1\ldots i_r}_{\varphi}) = F^{k_1\ldots k_a;i_1\ldots i_r j}_{\varphi}$. Therefore then
139 ideal $(P^{\ell+1}_{S}\Sigma)^{(0)}$ is generated by $D^{j}f$ and f, where f runs through $(P^{\ell}_{S}\sum)^{(0)}$. Hence our equality follows from the definition of P. □

Now we pose the following

Definition. *By a partial differential equation of order k on* $(M, M', \tilde{\omega})$, *we mean an open set* $\mathcal{U}$ *in* $J^{k}(M, M', \tilde{\omega})$ *and a subsheaf of ideals* Φ *in* $\mathscr{O}J^{k}|\mathcal{U}$ *such that* Φ *is locally finitely generated. By the* ℓ *-th standard prolongation of the partial differential equation* Φ, *we mean the open set* $(\rho^{\ell+1}_{\ell})^{-1}(\mathcal{U})$ *and the subsheaf of ideals* $P(\cdots(P(\Phi))\cdots) = P^{\ell}(\Phi)$, *where we operate* P ℓ*-times.*

It is clear by the definition that the standard prolongation of a partial differential equation is again a partial differential equation. It will be easy to see that our definition of partial differential equations is equivalent to the usual one when $(M, M', \tilde{\omega}) = (D, D', \tilde{\omega})$. Also it will be clear by Proposition 25 that the notion of partial differential equations and their prolongations includes the notion of exterior differential systems and their prolongations.

3.16

Let X be a point of $\mathscr{U}$. If $f(X) = 0$ for any f in $\Gamma(U, \Phi)$ and for any open neighbourhood U of X in $\mathscr{U}$, then we say that X is an integral jet of the equation Φ. Denote by $\vartheta^{o}\Phi$ the set of integral jets of Φ. It is clear that $\vartheta^{o}\Phi$ is a subvariety of $\mathscr{U}$. Take an open set D in $\mathscr{U}$ such that a coordinate system in $(M, M', \tilde{\omega})$ is defined on D. Set $D' = \tilde{\omega}(D)$.

Then $(D, D', \tilde{\omega})$ is a fibered manifold. Assume that $\Gamma(D, \Phi)$ is finitely generated and that $\Phi|D$ is generated by $\Gamma(D, \Phi)$. For any X in J^k we 140
can choose such a D containing X, because Φ locally finitely generated. Denote $\sum(D, \Phi)$ the exterior differential system generated by $\Gamma(D, \Phi)$ and $\prod(k). \sum(D, \Phi)$ is called a defferential system associated with Φ. Thus $P^k \sum$ is associated with $\left[(P^k \sum)^{(0)}\right]$. Let g be a cross-section of $(M, M', \tilde{\omega})$ over an open set in M'. We say that g is an integral of Φ when $j^k(g) \subset \vartheta^0\Phi$. It is equivalent to say that, for any $\sum(D, \Phi)$ such that D intersects with the image of g, the restriction of g is an integral of $\sum(D, \Phi)$. This follows from Proposition 2. By the proposition we also have the following: Let G be an integral of an associated system $\sum(D, \Phi)$. Then there is an integral g of Φ such that $G = j^k(g)$. Thus the problem of finding integrals of associated differential systems.

Proposition 26. *Let Φ be a partial differential equation of order k. Let $[\sum, (D, D', \tilde{\omega})] \sum(D, \Phi)$ be an associated differential system. Then $\sum((\rho_k^{k+\ell})^{-1}(D), P^\ell(\Phi))$ is isomorphic to an admissible restriction of $P_S^\ell\, [\sum, (D, D', \tilde{\omega})]$.*

Proof. We can assume without loss of generality that $M' = D'$. Also it is easy to reduce the proof to the case $\ell = 1$, by Proposition 8′. So we assume that $\ell = 1$. As mentioned in *p*.96, there is a canonical injection ℓ of $J^{k+1}(M, D', \tilde{\omega})$ into $J'(J^k(M, D', \tilde{\omega}), D', \alpha)$. ℓ induces an isomorphism of $\prod(k+1; (M, D'\tilde{\omega}))$ to an admissible restriction of $P'_S\, [\prod(k; (M, D', \tilde{\omega})), (D, D', \alpha)]$. $P'_S\, [\sum, (D, D', \alpha)]$ is generated (with 141
d) by $\Gamma(D, \Phi) o \rho_\ell^{\ell+1}, F^k_{d\varphi} (\varphi \in \Gamma(D, \Phi))$, and by $P^1_S [\prod(k; (M, D', \tilde{\omega})), (D, D', \alpha)]$. By Proposition 5

$$d\varphi \equiv F^k_{d\varphi} dx_k \mod \prod(1; (J^k(M, D', \tilde{\omega}), D', \alpha)).$$

□

Then since $\vartheta * (\prod(1; (J^k(M, D', \tilde{\omega}), D', \alpha)) = \prod(k+1; (M, D', \tilde{\omega}))$ as proved in p.97, it follows that $F^k_{d\varphi} o \ell = D^k \varphi$. Therefore ϑ induces an isomorphism of $\sum((\rho_k^{k+1})^{-1}(D), P(\Phi))$ to an admissible restriction $P^1_S\, [\sum, (D, D', \alpha)]$.

We say that an integral jet X of Φ is ordinary when X is ordinary with respect to an associated differential system $\sum(D, \Phi)$ (such that $D \ni X$).

When this is so, X is an ordinary integral point of any associated system $\sum(D_1, \Phi)$ such that $D_1 \ni X$. The definition implies immediately the following: When X is an ordinary integral jet of Φ and U is a suitable open neighbourhood of X, $\ell^0\Phi \cap U$ is submanifolds of U and $\Gamma(U, \Phi) = 0$ is its regular local equation. We say that Φ is in involution at an integral jet X, when an associated differential system is in involution at X.

Let Φ, Ψ be partial differential equations of order k on $(M, M', \tilde{\omega})$. Denote by $\mathscr{U}$, ω the sets in J^k on which Φ, Ψ are given, respectively. We say $\Phi \subset \Psi$, when $\mathscr{U} \subset \omega$ and $\Phi_x \subset \Psi_x$ for any x in ω. We say that Φ is a restriction of Ψ, when $\mathscr{U} \subset \omega$ and $\Phi_x = \Psi_x$ for any x in $\mathscr{U}$.

Our main purpose is to prove the following

142 **Theorem 6.** *Let $(M, M', \tilde{\omega})$ be a fibered manifold. Assume that a partial differential equation Φ^ℓ of order ℓ on $(M, M', \tilde{\omega})$ is given for any $\ell \geq \ell_0$. Let g^0 be a cross-section of $(M, M', \tilde{\omega})$ over an open neighbourhood of a point x^0 in M'. Assume the following: for any $\ell \geq \ell_0$*

(i) *g^0 is an integral of Φ^ℓ,*

(ii) *$\Phi^{\ell+1} \supseteq p(\Phi^\ell)$ on a neighbourhood of $X^\ell = j^\ell_{x^0}(f^0)$,*

(iii) *X^ℓ is an ordinary integral jet of Φ^ℓ,*

(iv) *for a suitable open neighbourhood U of X^{ℓ_0}, $(\ell^o\Phi^{-o} \cap U, \alpha(U), \alpha)$ is a fibered manifold,*

(v) *$(\ell^0\Phi^{\ell+1} \cap V, \ell^0\Phi^\ell \cap V', \rho^{\ell+1}_\ell)$ is a fibered manifold for suitable open neighbourhoods V, V' of $X^{\ell+1}, X^\ell$, respectively.*

Then there is an integer ℓ_1 such that $\Phi^{\ell+1}$ and $P(\Phi^\ell)$ are equal in a neighbourhood of $X^{\ell+1}$ and such that Φ^ℓ is in involution at X^ℓ for any $\ell \geq \ell_1$

3.17

In this article, we keep the notations in the above theorem and assume that the assumption is satisfied. Choose a coordinates system (x, y) in $(M, M', \tilde{\omega})$ defined on a neighbourhood of $g^0(x^0)$ such that $x^0 = (0)$

and the cross-section g^0 is represented by $y_\lambda = 0$. Let f be a function defined in a neighbourhood of X^ℓ. Expanding f in the power series in $x, y, \ldots, y_\lambda^{i_1\cdots i_\nu}, \ldots (1 \leq \nu \leq \ell)$, we have

$$f = a^\lambda_{i_1\cdots i_\ell}(x,y) y_\lambda^{i_1\cdots i_\ell} + f' o \rho^\ell_{\ell-1} + h$$

where f' is a function on $J^{\ell-1}$ and h is a function on J^ℓ, each of whose terms is of atleast degree two in $y_\lambda^{i_1\cdots i_\nu}$. The function $a^\lambda_{i_1\cdots i_\ell}(x,y)y_\lambda^{i_1\cdots i_\ell}$ is called the principal part of f and is denoted by $R(f)$, or $R^\ell(f)$. 143

Lemma. *Under the above notations,*

$$R^{\ell+1}(D^j f) = a^\lambda_{i_1\cdots i_\ell}(x,y) y^{i_1\cdots i_\ell j}$$

Proof. Because of the above expansion of f, we have

$$\begin{aligned} df &= a^\lambda_{i_1\cdots i_\ell}(x,y) dy_\lambda^{i_1\cdots i\ell} + y_\lambda^{i_1\cdots i\ell} + da^\lambda_{i_1\cdots i\ell} + d(f' o \rho^\ell_{\ell-1}) + dh \\ &\equiv (a^\lambda_{i_\ell\cdots i\ell} y_\lambda^{i_1\cdots i_\ell j} + y_\lambda^{i_1\cdots i_\ell} D^j(a^\lambda_{i_1\cdots i_\ell}) + (D^j f') o \rho^{\ell+1}_\ell \\ &\qquad + (D^j h) dx_j \pmod{\textstyle\prod(\ell+1)}). \end{aligned}$$

□

Therefore

$$D^j f = a^\lambda_{i_1\cdots i_\ell}(x,y) y_\lambda^{i_1\cdots i_\ell j} + y_\lambda^{i_1\cdots i_\ell} D^j(a^\lambda_{i_1\cdots i_\ell}) + (D^j f') o \rho^{\ell+1}_\ell + D^j h.$$

Then our conclusion follows immediately

Introduce indeterminates $Z^1, \ldots, Z^p, Y_1, \ldots, Y_m$, where $p = \dim M'$, $m = \dim M - p$. For each f as above, we set

$$F^\ell_f = a^\lambda_{i_1\cdots i_\ell}(0,0) Y_\lambda Z^{i_\ell} \cdots Z^{i\ell} \in \underline{R}[Z,Y]$$

where $\underline{R}[Z,Y]$ is the ring of polynomials in $Z^1, \ldots, Z^p, Y_1, \ldots, Y_m$. Denote by A^ℓ the ideal in $\underline{R}[Z,Y]$ generated by all $F^{\ell'}_f$ where $\ell \geq \ell'$ and f is function defined on a neighbourhood of X^ℓ which is a cross-section of Φ^ℓ. Clearly $A^0 \subseteq \cdots \subseteq A^\ell \subseteq A^{\ell+1} \subseteq \cdots \underline{R}[Z,Y]$ being a Noetherian ring there exists on integer ℓ_2 such that $A^{\ell+1} = A^\ell$ for $\ell \geq \ell_2$. This together 144 with the above lemma means the following: For any cross-section f of $\Phi^{\ell+1}$ defined on a neighbourhood of $X^{\ell+1}$ there exists h_j which is a cross section of Φ^ℓ such that $F_f^{\ell+1} = \sum Z^j F_{h_j}^\ell$. Therefore by the above lemma, it follows that the principal part of $f - \sum D^j h_j$ vanishes at $X^{\ell+1}$. Since $X^{\ell+1}$ has the coordinates $x = y = \cdots = y_\lambda^{i_1 \cdots i_\nu} = \cdots = 0$, this means that $d(f - \sum D^j h_j)_X \ell + 1$ is in $\rho_\ell^{\ell+1*}(\wedge_x \ell(J^\ell))$. Then the condition (iii) and (v) imply that $\Phi^{\ell+1}$ is generated by Φ^ℓ and $D^j(\Phi^\ell)$ locally at $X^{\ell+1}$. Therefore $\Phi^{\ell+1}$ is equal to $P(\Phi^\ell)$ on a neighbourhood of $X^{\ell+1}$ for $\ell \geq \ell_2$.

3.18

Take a sufficiently small open neighbourhood D^ℓ of X^ℓ (which we will change if necessary), and set $W^\ell = \ell^0 \Phi^\ell \cap D^\ell$. $(D^\ell, \alpha(D^\ell), \alpha)$ and $(W^\ell, \alpha(D^\ell), \alpha)$ are fibered manifolds by (iv) and (v). Denote by $\sum_\ell$ the restriction of $\sum(D^\ell, \Phi^\ell)$ to W^ℓ. By (iii) there are $f_1, \ldots, f_a$ in $\Gamma(D^\ell, \Phi^\ell)$ such that $d f_1, \ldots, d f_a$ are linearly independent mod. $d(\Gamma(D^{\ell-1}, \Phi^{\ell-1})o$ $\rho_{\ell-1}^\ell = 0$ is a regular local equation of W^ℓ on D^ℓ. If h is in $\Gamma(D^{\ell-1}, \Phi^{\ell-1})$, then the definition of $D^j h$ together with (ii) imply that $d(h \circ \rho_{\ell-1}^\ell)_X \in \prod(\ell)_X$ for any X in W^ℓ. Therefore $(\sum(D^\ell; \Phi^{(\ell)}))^{(1)}$ is generated by $(d f_1)_{X,\ldots,}(d f_a)_X, (\prod(\ell))_x$. Denote by ℓ the injection of W^ℓ into D^ℓ. Then the conclusion just reached shows that

$$\left(\Sigma_\ell^{(1)}\right)_X = \ell^* \left(\prod(\ell)\right)_X$$

145 Denote by Ω_X the subspace of $\wedge_X^1(D)$ generated by $(dx_1)_{X,\ldots,}(dx_p)_X$. We claim that $\ell^*(\prod(\ell))_X \cap \ell^* \Omega_X = 0$. Namely, if $\ell^*(\prod(\ell))_X \cap \ell^* \Omega_X \neq 0$, then there is no integral element E of $\sum_\ell$ with origin X such that $\dim(d\alpha(E)) = p$. This means that $\ell^0 P_S^1(\sum; (W^\ell, \alpha(D^\ell), \alpha))$ has no points with origin X. By Proposition 26, it follows that $X \notin \rho_\ell^{\ell+1}(\ell^0 p(\Phi^\ell))$. By (ii) this contradicts to (v). Thus $\ell^*(\prod(\ell)) \cap \ell^* \Omega = \sum_\ell^{(1)} \cap \ell^* \Omega = 0$. This shows in particular that X^ℓ is a normal integral point of $\sum_\ell$.

We will show that $\left[\sum_\ell, (W^\ell, \alpha(D^\ell), \alpha)\right]$ is a normal differential system. Since $\prod(\ell)$ is normal. the conditions (1), (3), and (4) in the definition of normal system (p.99) is trivial. As for the condition (2), we already showed that $\sum_\ell^{(1)} \cap \ell^* \Omega = 0$. Since $\prod(\ell)$ is generated by

$dy_\lambda^{i_1\cdots i_\nu} - y_\lambda^{i_1\cdots i_\ell i} dx_i (0 \leq \nu \leq \ell - 1)$, it is clear that $(\prod(\ell))_X + \Omega_X = (\rho^\ell_{\ell-1})^* \wedge'_X (D^{\ell-1})$. Therefore $(\sum^{(1)}_\ell)_x + \ell^*(\Omega_X) = (\rho^\ell_{\ell-1})^*(\wedge'_X(W^{\ell-1}))$. Hence $\dim(\sum^{(1)}_\ell)_X = \dim(w^{\ell-1}) - p$, which is independent of X in W^ℓ. This proves that the condition (2) is satisfied. Thus $\left[\sum, (W^\ell, \alpha(D^\ell), \alpha)\right]$ is a normal system.

We will show that X^{ℓ_2} is P-weakly-regular with respect to $\left[\sum_2, (W^{\ell_2}, \alpha(D^{\ell_2}, \alpha)\right]$,where ℓ_2 is the integer chosen in §3.17. Take z in W^{ℓ_2} sufficiently near X^{ℓ_2}. Take an integral point Y of $P^\ell_s \sum \ell_2$ over 146
z. Y is in $J^\ell(W^{\ell_2}, \alpha(D^{\ell_2}), \alpha) \subseteq J^\ell(J^{\ell_2}(M, M', \tilde{\omega}), M', \alpha)$. Denote by ℓ the canonical injection of $J^{\ell_2+\ell}(M, M', \tilde{\omega})$ into $J^\ell(J^{\ell_2}(M, M', \tilde{\omega}, M'\alpha)$. Then by Proposition 26 we can choose Y in such a way that there is X in $W^{\ell_2+\ell}$ near $X^{\ell_2+\ell}$ such that $Y = \ell(X)$. By Proposition 11 and 13, applied to the case $\dim E = 0$, it follows that $t^\ell_0(z; \sum_{\ell_2}) = \dim((P^\ell_S(\sum_{\ell_2})^{(1)})_Y + \Omega_Y/\Omega_Y) - c$,where c is a constant independent of z. Therefore , by Proposition 26 and 8′, $t^\ell_0(z; \sum_{\ell_2}) = \dim((\sum(P^\ell \Phi^{\ell_2})))^{(1)}_X + \Omega_X/\Omega_X) - c'$. By the choice of and by Proposition 8′, it follows that $t^\ell_0(z; \sum_{\ell_2} = \dim(\sum^{(1)}_{\ell_2+\ell})_{\overline{X}} c''$, because $(\sum^{(1)}_{\ell_2+\ell})_X \cap \Omega_X = 0$. As is already shown $\dim(\sum^{(1)}_{\ell_2} + \ell)_X$ is a constant independent of X in $W^{\ell_2+\ell}$. Hence $t^\ell_0(X^{\ell_2}; \sum \ell_2) = t^\ell_0(\sum \ell_2)$. Let us assume as an induction assumption that there is a sequence $X^{\ell_2} = E^\chi_0 \subset E^\chi_1 \subset \cdots \subset E^\chi_q$ such that $t^\ell_r((X^{\ell_2}, E^\chi_r); \sum \ell_2) = t^\ell_r(\sum \ell_2)$ for $r = 0, 1, \ldots, q$ and for sufficiently large ℓ. By Proposition 22, there is $E^\chi_{q+1} \supseteq E_q$ such that $t^\ell_{q+1}((X^{\ell_2}; E^\chi_q); \sum_{\ell_2}) = t^\ell_q((X^{\ell_2}, E^\chi_q); \sum_{\ell_s}) + n_{\ell-1} - t^{\ell-1}((X^{\ell_2}, E^\chi_q); \sum_{\ell_2})$ for large ℓ. Hence by theorem 4 for such E^χ_{q+1} we have the equality $t^\ell_{q+1}((X^{\ell_2}, E^\chi_{q+1}); \sum \ell_2) = t^\ell_{q+1}(\sum \ell_2)$ for large ℓ. Thus X^{ℓ_2} is P-weakly regular of weight ℓ_1, for sufficiently large $\ell_1 (\geq \ell_2)$ with respect to $\sum_{\ell_2}$. By Proposition 8, a differential system is in involution if and only if its admissible restriction is in involution. Therefore by Theorem 5, Φ^ℓ is in involution at X^ℓ for $\ell \geq \ell_1$. Thus 147
Theorem 6 is completely proved.

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