Lectures on
Topics in Mean Periodic Functions
And The Two-Radius Theorem

By

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Introduction

Three different subjects are treated in these lectures. **1**

1. In the first part, an exposition of certain recent work of J.L. Lions on the transmutations of singular differential operators of the second order in the real case, is given. (J.L. Lions- Bulletin soc. Math. de France, 84(1956)pp. $9-95$)

2. The second part contains the first exposition of several new results on the theory of mean periodic functions F, of two real variables, that are solutions of two convolution equations: $T_1 * F = T_2 * F = 0$, in the case of countable and simple spectrum. These functions can be, at least formally, expanded in a series of mean-periodic exponentials, corresponding to different points of the spectrum. Having determined the coefficients of this development, we prove its uniqueness and convergence when T_1 and T_2 are sufficiently simple. The result is obtained by using an interpolation formula, in C^2, which is analogous to the Mittag-Leffler expansion, in C^1.

   The exposition and the proofs given here can probably later, be simplified, improved, and perhaps generalized. They should therefore be considered as a preliminary account only.

3. Finally, in the third part, I state and prove the two-radius theorem, which is the converse of Gauss's classical theorem on the spherical **2** mean for harmonic functions. The proof is the same as that recently published, (Comm. Math. Helvetici, 1959) in collaboration with J.L. Lions; it uses the theory of transmutations of singular differential

operators of the second order, and the fundamental theorem of mean-periodic functions in R^1.

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Part I

Transmutation of Differential Operators

Chapter 1

Riemann's Method

1 Riemann's Method for the Cauchy problem

Definition. A function is said to be (C, r) on a subset of A of R^n if all its 3
partial derivatives upto the order r exists and are continuous in A.

Let D be a region (open connected set) in R^2, the (x, y) plane. Let a, b, c be three functions which are $(C, 1)$ in D and u a function $(C, 2)$ in D and let L denote the differential operator

$$Lu = \frac{\partial^2 u}{\partial x \partial y} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + cu.$$

Let v be a function which is $(C, 2)$ in D and L^* be a differential operator of the same type as L:

$$L^* v = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial}{\partial x}(av) - \frac{\partial}{\partial y}(bv) + cv$$

then

$$vL(u) - uL^*(v) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \tag{1}$$

where M and N are $(C, 1)$ in D and are certain combinations of u, v and 4
their partial derivatives.

$$vL(u) = v\frac{\partial^2 v}{\partial x \partial y} + av\frac{\partial u}{\partial x} + bv\frac{\partial u}{\partial y} + cuv$$

$$\begin{aligned}
&= cuv + \frac{\partial}{\partial x}(auv) + \frac{\partial}{\partial y}(buv) + \frac{\partial}{\partial x}\left(v\frac{\partial u}{\partial y}\right) \\
&\qquad - u\frac{\partial}{\partial x}(av) - u\frac{\partial}{\partial y}(bv) - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} \\
&= u\left[\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial}{\partial x}(av) - \frac{\partial}{\partial y}(bv) + cv\right] \\
&\qquad + \frac{\partial}{\partial x}\left[auv + v\frac{\partial u}{\partial y}\right] - \frac{\partial}{\partial y}\left[u\frac{\partial v}{\partial x} - buv\right]
\end{aligned}$$

$$vL(u) - uL^*(v) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \qquad \text{where}$$

$$M = auv + v\frac{\partial u}{\partial y}, N = buv - u\frac{\partial v}{\partial x}.$$

The right hand member of (1) does not change if we replace M by $auv + \frac{1}{2}(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y})$ and N by $buv + \frac{1}{2}(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x})$. We prefer to have

$$M = \frac{1}{2}\frac{\partial}{\partial y}(uv) - uP(v) \tag{i}$$

$$N = \frac{1}{2}\frac{\partial}{\partial x}(uv) - uQ(v) \tag{ii}$$

where

$$P(v) = \frac{\partial v}{\partial y} - av \; , Q(v) = \frac{\partial v}{\partial x} - bv.$$

5 Let C be a closed curve lying entirely in the region D.

By Green's formula,

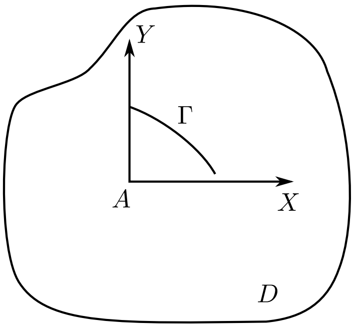
$$\int_C (\lambda M + \mu N) ds = \int\int_A \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy$$

where λ, μ denote the direction cosines of the interior normal to C and A denotes the region enclosed by C.

In view of equation (1),

$$\int_C (\lambda M + \mu N) ds = \int\int_A (vL(u) - uL^*(v)) dxdy \tag{2}$$

We shall consider this equation in the case when C consists of two straight lines AX, AY parallel to the axes of coordinates and a curve Γ, monotonic in the sense of AX and AY, joining X and Y. Suppose that $L(u) = 0$ and $L^*(v) = 0$ then $\int_C (\lambda M + \mu N) ds = 0$ i.e. $\int_A^X N dx - \int_Y^A M dy = \int_Y^X (\lambda M + \mu N) ds$ substituting for M and N from (i) and (ii) respectively,



$$\int_A^X N dx = \frac{1}{2}[(uv)_X - (uv)_A] - \int_A^X uQ(v) dx$$

and
$$-\int_Y^A M dy = \frac{1}{2}[(uv)_Y - (uv)_A] + \int_Y^A uP(v) dy.$$

If the functions u, v satisfy **6**

$$Lu = 0, L^*v = 0, P(v) = 0 \text{ on } AY \text{ and } Q(v) = 0 \text{ on } AX \tag{3}$$

then we obtain the Riemann's Resolution formula:

$$(uv)_A = \frac{1}{2}[(uv)_X + (uv)_Y] + \int_Y^X (\lambda M + \mu N) ds.$$

Let A be (x_0, y_0). Then $v = g(x, y; x_0, y_0)$ satisfying the conditions (3) is the Riemann's function for the equation $L(u) = 0$.

In the situation

$$L(u) = f(x, y), L^*v = 0, P(v) = 0 \text{ on } AY, Q(v) = 0 \text{ on } AX,$$

exactly similar computation gives the formula:

$$(uv)_A = \frac{1}{2}\left[(uv)_X + (uv)_Y\right] + \int_Y^X (\lambda M + \mu N)ds + \int\int_A v f(x,y)dxdy.$$

An important property of the Riemann's function.

Let now Γ consider of two straight lines XB, YB parallel to the axes of coordinates. Then

$$\int_Y^X (\lambda M + \mu N)ds = \int_B^X Mdy - \int_Y^B Ndx$$

We can write
$$M = -\frac{1}{2}\frac{\partial}{\partial y}(uv) + vP^*(u)$$
$$N = -\frac{1}{2}\frac{\partial}{\partial x}(uv) + vQ^*(u)$$

7 where $P^*(u) = \dfrac{\partial u}{\partial y} + au$, $Q^*(u) = \dfrac{\partial u}{\partial x} + bu$.

$$\int_B^X Mdy = -\frac{1}{2}\left[(uv)_X - (uv)_B\right] + \int_B^X vP^*(u)dx$$
$$-\int_Y^B Ndx = \frac{1}{2}\left[(uv)_B - (uv)_Y\right] - \int_Y^B vQ^*(u)dx$$

Thus in this case Riemann's resolution formula becomes

$$(uv)_A = \frac{1}{2}\left[(uv)_X + (uv)_Y\right] - \frac{1}{2}\left[(uv)_X - (uv)_B\right] + \int_B^X vP^*(u)dx + \frac{1}{2}\left[(uv)_B - (uv)_Y\right] - \int_Y^B vQ^*(u)dx$$

i.e., $$(uv)_A = (uv)_B - \int_Y^B vQ^*(u)dx + \int_B^X vP^*(u)dx$$

If $u = h(x,y;x_1,y_1), (x_1,y_1)$ being the point B, is such that

$$Lu = 0, P^*(u) = 0 \text{ on } BX, Q^*(u) = 0 \text{ on } BY,$$

we get $$(uv)_A = (uv)_B.$$

Choosing constant multipliers for $u = h(x, y; x_1, y_1)$ and $v = g$
8 $(x, y; x_0, y_0)$ in such a way that $u = 1$ at B and $v = 1$ at A, we have

$$h(x_0, y_0; x_1, y_1) = g(x_1, y_1; x_0, y_0) \tag{4}$$

This shows that the Riemann's function g, considered as a function of (x_0, y_0) satisfies the differential equation $Lu = 0$.

2 Proof for the Riemann's method

We have obtained (4) under the hypothesis that there exists functions $u = h(x, y; x_1, y_1)$ and $v = g(x, y; x_0, y_0)$ which are $(C, 2)$ in D and which satisfy

$$Lu = \frac{\partial^2 u}{\partial x \partial y} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + cu = 0$$
$$P^*(u) = \frac{\partial u}{\partial y} + au = 0 \text{ on } BX$$
$$Q^*(u) = \frac{\partial u}{\partial x} + bu = 0 \text{ on } BY \text{ and}$$
$$L^* v = \frac{\partial^2 v}{\partial x \partial y} + a^* \frac{\partial v}{\partial x} + b^* \frac{\partial v}{\partial y} + c^* v = 0 \text{ where}$$
$$a^* = -a, b^* = -b, c^* = -\frac{\partial a}{\partial x} - \frac{\partial b}{\partial y} + c$$

$$P(v) = \frac{\partial v}{\partial y} + a^*(v) = 0 \text{ on } AY$$
$$Q(v) = \frac{\partial v}{\partial x} + b^*(v) = 0 \text{ on } AX.$$

By change of notation and translation of the origin, the problem for 9
the existence of the Riemann's function $v = g(x, y)$ in D for the point

$(x_0, y_0) \in D$ for the differential equation $Lu = 0$ reduces to the solution of the problem 1:

$$\frac{\partial^2 u}{\partial x \partial y} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + cu = 0$$

with the conditions

$$\begin{aligned} u(0, y) &= \alpha(y) \\ u(x, 0) &= \beta(x)\ I \\ \alpha(0) &= \beta(0) = 1 \end{aligned}$$

where a, b, c are $(C, 1)$ in D and α, β are $(C, 1)$ functions of one real variable. The solution of the more general problem 2:

$$\frac{\partial^2 u}{\partial x \partial y} = \lambda\left[a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + cu\right]$$

with the conditions I will give for $\lambda = -1$ the solution of the problem (1). We shall now prove the existence of the unique solution for the problem 2 by using Piccard's method of successive approximations. Consider the series

$$u_0(x, y) + \lambda u_1(x, y) + \cdots + \lambda^n u_n(x, y) + \cdots \tag{5}$$

10 where $u_i(x, y)$ are defined by the following recurrence formula:

$$\frac{\partial^2 u_0}{\partial x \partial y} = 0\ u_0(0, y) = \alpha(y), u_0(x, 0) = \beta(x)$$

$$\frac{\partial^2 u_n}{\partial x \partial y} = a\frac{\partial u_{n-1}}{\partial x} + b\frac{\partial u_{n-1}}{\partial y} + cu_{n-1}, u_n(x, 0) = u_n(0, y) = 0 \text{ for } n \geq 1.$$

It suffices to take $u_0(x, y) = \alpha(y) + \beta(x) - 1$ and $u_n(x, y) = \int_0^x \int_0^y \phi_{n-1}(\xi, \eta) d\xi d\eta$ where

$$\phi_{n-1} = a\frac{\partial u_{n-1}}{\partial x} + b\frac{\partial u_{n-1}}{\partial y} + cu_{n-1}$$

We shall now prove the convergence of the series (5) by the process of majorisation which is classical. Suppose that a, b, c are $(C, 0)$ in D

and α, β are $(c, 1)$ of one real variable. Let K be a compact subset D containing the rectangle with sides parallel to the axes and $(0, 0)$ and (x, y) as opposite corners. Then there exists an M, A such that $|\alpha(\xi) + \beta(\eta) - 1| \leq M$,

$$|\frac{\partial}{\partial \xi}(\alpha(\xi) + \beta(\eta) - 1)| \leq M,$$

and
$$|\frac{\partial}{\partial \eta}(\alpha(\xi) + \beta(\eta) - 1)| \leq M,$$

for (ξ, η) in K, and $|a|, |b|, |c|, \leq A$ in K. Then $|\phi_0(x, y)| \leq 3$ A M. By the **11**
recurrence formula for $n = 1$,

$$|u_1(x, y)| \leq |\int_0^x \int_0^y \phi_0(\xi, \eta) d\xi d\eta| \leq 3AM|x||y|$$

$\left|\dfrac{\partial u_1}{\partial x}\right| \leq 3AM|y|$ and, $\left|\dfrac{\partial u_1}{\partial y}\right| \leq 3AM|x|$. Hence $|u_1(x, y)|, \left|\dfrac{\partial u_1}{\partial x}\right|, \left|\dfrac{\partial u_1}{\partial y}\right|$ are each $\leq 3AM(1 + |x|)(1 + |y|)$.

Computing $\phi_1(x, y)$, we have immediately,

$$|u_2(x, y)| \leq 9A^2 M \left(\frac{1 + |x|}{2!}\right)^2 \left(\frac{1 + |y|}{2!}\right)^2$$

$$\left|\frac{\partial u_2(x, y)}{\partial x}\right| \leq 9A^2 M(1 + |x|) \left(\frac{1 + |y|}{2}\right)^2$$

$$\left|\frac{\partial u_2(x, y)}{\partial y}\right| \leq 9A^2 M \left(\frac{1 + |x|}{2}\right)^2 (1 + |y|)$$

In general

$$|u_n(x, y)|, \left|\frac{\partial u_n}{\partial x}\right|, \left|\frac{\partial u_n}{\partial y}\right|, \leq \frac{M[3A(1 + |x|)(1 + |y|)]^n}{n!}$$

Comparing with the exponential series, this majorization proves that the series (3) as also the series obtained from (3) by differentiating each term once and twice are all convergent uniformly on each compact subset of D and absolutely in D, so that (3) converges to a function $(C, 2)$ in D. This function $u(x, y)$ is evidently the solution of problem 2 and the proof for the existence of Riemann's functions is complete.

Remark. From the recurrence formula it is clear that the function $u(x, y)$ 12
satisfies

$$u(x, y) = \alpha(y) + \beta(x) - 1 + \lambda \int_0^x \int_0^y \left[a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + cu \right]_{\substack{x=\xi \\ y=\eta}} d\xi \, d\eta$$

This integral equation is equivalent to the differential equation of problem 2 with conditions I. When a, b, c are $(C, 0)$ and α, β are $(C, 1)$, the solution $u(x, y)$ of the integral equation may not be $(C, 2)$. But then it is a solution of problem 2 in the sense of distributions.

Chapter 2

Transmutation of Differential Operators

Let L_1 and L_2 denote two differential operators on the real line R and 13
$\mathscr{E}^m(x \geq a)$ denote the space of functions m times continuously differentiable in $[a, \infty)$ furnished with the usual topology of uniform convergence of functions together with their derivatives upto the order m on each compact subset of $[a, \infty)$. Let E be a subspace of the topological vector space $\mathscr{E}^m(x \geq a)$.

Definition. A transmutation of the differential operator L_1 into the differential operator L_2 in E is a topological isomorphism X of the topological vector space E onto itself (i.e. a linear, continuous, one-to-one, onto map), such that

$$XL_1 = L_2X$$

X is said to transmute the operator L_1 into the operator L_2 in E and $L_2 = XL_1X^{-1}$ on E.

Transmutation in the regular case. Let $L_1 = D_x^2 - q(x), L_2 = D_x^2$ where q satisfies certain conditions of regularity. The construction of the transmutation operator in this case depends on the consideration of certain partial differential equation.

Problem 1. To determine $\Phi(x, y)$ in $x \geq a, y \geq a$ satisfying $\Phi_{xx} - \Phi_{yy} -$

$q(x)\Phi = 0$ with the boundary conditions

$$\Phi(x, a) = 0 = \Phi(a, y); \Phi_y(x, a) = f(x)$$

14 This mixed problem is equivalent to the Cauchy problem if we set

$$\begin{aligned} u(x, y) &= \Phi(x, y) \text{ for } x \geq a \\ &= -\Phi(2a - x, y) \text{ for } x \leq a \end{aligned}$$

We set without proof the following proposition.

Proposition 1. *(a) If $q \in \mathscr{E}^\circ(R)$, $f \in \mathscr{E}^\circ(x \geq a)$ with $f(a) = 0$, then problem 1 possesses a unique solution which is $(C, 1)$ in $x \geq a, y \geq a$ and satisfies the differential equation in the sense of distributions.*

(b) If $q \in \mathscr{E}^1(R)$ and if $f \in \mathscr{E}^2(x \geq a)$ with $f(a) = 0$, the solution of problem 1 is $(C, 2)$ in $x \geq a, y \geq a$. In the region $y \geq x$(or $x \geq y$), the solution is $(C, 3)$.

(c) If $q \in \mathscr{E}^2(R)$ with $q'(a) = 0$ and $f \in \mathscr{E}^3(x \geq a)$ with $f(a) = f''(a) = 0$, then the solution u of the problem is $(C, 4)$ in $y \geq a$. [Refer to E. Picard, 'Lecons sur quelques types simples d' equation aux derivces partielles', Paris, Gauthier-VIllars, 1927.]

With the help of this proposition we prove

Proposition 2. *If $q \in \mathscr{E}^2(R)$ with $q'(a) = 0$ and $f \in \mathscr{E}^4(x \geq a)$ with $f(a) = f''(a) = 0$ then $D^2 Af = ALf$ where $L = D^2 = q$ and A is defined by*

$$Af(y) = \frac{\partial}{\partial x}[\Phi(a, y)],$$

15 *$\Phi(x, y)$ being the solution of problem 1.*

Let $\psi(x, y) = L_x[\Phi(x, y)] = \frac{\partial^2 \Phi}{\partial x^2} - q(x)\Phi$ and $g(x) = L_x f(x)$. As Φ is $(C, 4)$ by Proposition 1 (c), and $f \in \mathscr{E}^4(x \geq a), \Psi(x, y)$ is $(C, 2)$ and $g \in \mathscr{E}^2(x \geq a)$ with $g(a) = 0$. Replacing f by g in proposition 1 (b), problem 1 possesses a unique solution. We verify below that this unique solution is $\Psi(x, y)$.

$$L_x \Psi - D_y^2 \Psi = L_x L_x \Phi - D_y^2 L_x \Phi = L_x[L_x \Phi - D_y^2 \Phi] = 0;$$

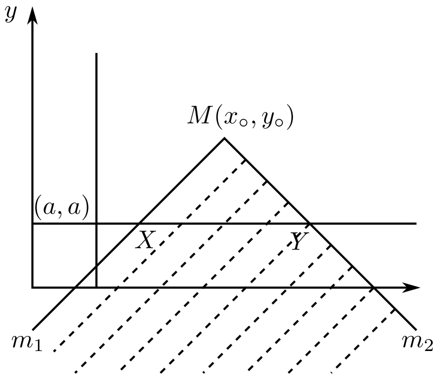
$$\Psi(x, a) = L_x[\Phi(x, a)] = L_x[0] = 0;$$
$$\frac{\partial}{\partial y}\Psi(x, y) = D_y L_x \Phi(x, y) = L_x D_y \Phi(x, y) \quad \text{so that}$$
$$\Psi_y(x, a) = L_x D_y \Phi(x, a) = L_x f(x) = g(x);$$
$$\Psi(a, y) = D_y^2[\Phi(a, y)] = 0.$$

Now by definition of A, $AL[f(y)] = A.g = \dfrac{\partial}{\partial x}\Psi(a, y)$

$$\Psi_x(x, y) = D_x L_x \Phi = D_y^2[\Phi_x(x, y)] \text{ gives}$$

$\Psi_x(a, y) = D_y^2[\Phi_x(x, y)] = D_y^2 A f(y)$. Hence we have proved that $AL = D_y^2 A$.

Computation of the solution $u(x, y), y \geq a$ of problem 1 by using Riemann's function. **16**



$X = (a - y_\circ + x_\circ, a)$
$Y = (a + x_\circ + y_\circ, a)$

Let $K(x, y; x_0, y_0)$ be the Riemann's function defined in the shaded part of the satisfying the conditions $\dfrac{\partial^2 K}{\partial x^2} - \dfrac{\partial^2 K}{\partial y^2} - q^*(x)K = 0$

$$\text{with } q^*(x) = q(x) \text{ if } x \geq a$$
$$= q(2a - x) \text{ if } x < a$$

and K on $Mm_1 = K$ on $Mm_2 = -\dfrac{1}{2}$.

In this case Riemann's method gives

$$u(x_0, y_0) = \int_{a-y_0+x_0}^{x_0+y_0-a} f^*(x) K(x, a; x_0, y_0) dx$$

where
$$\begin{aligned} f^*(x) &= f(x) \quad \text{if } x \geq a \\ &= -f(2a - x) \quad \text{if } x \leq a \end{aligned}$$

Hence

$$\begin{aligned} A[f(y_0)] &= \frac{\partial}{\partial x_0} u(a, y_0) \\ &= f(y_0) - 2 \int_a^{y_0} f(x) \frac{\partial}{\partial x_0} K(x, a; a, y_0) dx. \end{aligned}$$

17 **Problem 2.** To determine the function $\Phi(x, y)$ in $x \geq a, y \geq a$ satisfying the conditions $\Phi_{xx} - \Phi_{yy} - q(x)\Phi = 0; \Phi(a, y) = 0; \Phi_x(a, y) = g$ where $g \in \mathscr{E}^2(y \geq a)$ with $g(a) = 0$, and $\Phi(x, a) = 0$. Problem 2 , is the same as Problem 1 if in the boundary conditions the lines $x = a, y = a$ are interchanged. If Φ is the solution of Problem 2, we define a $g(x) = \dfrac{\partial}{\partial y}\Phi(x, a)$.

Proposition 3. *If $q \in \mathscr{E}^1(R)$ and $f \in \mathscr{E}^2(x \geq a)$ with $f(a) = 0$, then $aAf = Aaf = f$.*

Let Φ be the solution of Problem 1 and let $g(y) = Af(y) = \dfrac{\partial}{\partial x}\Phi(a, y)$. By Proposition 1 (b), g is $(c, 2)$ in $y \geq a$ and $g(a) = 0$. Hence Φ is the solution of Problem 2 and $a \cdot g(x) = \dfrac{\partial}{\partial y}\Phi(x, a) = f(x)$. This shows that $aAf = f$. Similarly $Aaf = f$.

Proposition 3 together with Proposition 2 shows that if $q \in \mathscr{E}^1(R)$ with $q'(a) = 0$ the map A, which is obviously linear is one-to-one of the space $E = \left\{f / f \in \mathscr{E}^4(x \geq a), f(a) = f''(a) = 0\right\}$ onto itself and verifies $ALf = D^2Af$. Further in view of the formula for Af on page 13 in items of Riemann's functions, A is continuous on E with the topology induced by $\mathscr{E}^4(x \geq a)$. As E is a closed subspace of the Frechet space

$\mathscr{E}^4(x \geq a), A$ is a topological isomorphism. Thus we have proved the existence of the transmutation operator A in E transmuting $D^2 - q(x)$ into D^2 when q is sufficiently regular.

We now consider the problem of transmuting more general differential operators $L_i = D^2 + r_i(x)D + s_i(x)$ $(i = 1, 2)$ into each other when r_i and s_i are regular (e. g. $r_i, s_1 \in \mathscr{E}(R)$). **18**

Proposition 4. *There exists an isomorphism $A_{L_1L_2}$ of E which satisfies*

$$A_{L_1L_2}L_1 = L_2A_{L_1L_2}.$$

The proposition will be proved if we prove the existence of transmutation X of the operator L_1 into the operator $D^2 - q_1$. For then the same method will give a transmutation of L_2 into $L_2^* = D^2 - q_2$ and each of the operators $D^2 - q_i (i = 1, 2)$ can be transmutated into the operator D^2 so that finally we obtain the required transmutations $A_{L_1L_2}$ by composing several transmutations.

Let $R_1(x) = \int_a^x r_1(\xi)d\xi$ then the verification of the following equations is straight forward:

$$L_1\left[e^{-\frac{1}{2}R_1(x)}f(x)\right] = e^{-\frac{1}{2}R_1(x)}L_1[f].$$

Hence we have $Xf(x) = e^{-\dfrac{1}{2}R_1(x)}f(x).L_1^* = D^2 - q_1$ where

$$q_1(x) = \frac{1}{4}r_1^2(x) + \frac{1}{2}r_1(x) - s_1(x) \in \mathscr{E}.$$

Application of transmutation to the Mixed Problems of differential equations.

If Λ is an elliptic operator inR^n (independent of the variable t which corresponds to time) we consider the problem of finding a function $u(x, t)$ $(x \in R^n, t$ time) which satisfies the differential equation **19**

$$\Lambda_x(x, t) + \left(\frac{\partial_2}{\partial t^2} + r(t)\frac{\partial}{\partial t} + s(t)\right)u(x, t) = 0$$

with the Cauchy data in a bounded domain $\Omega \subset R^n$ for $t = 0$ and also on the hemicylinder $\Omega^* \times [t \geq 0]$ where Ω^* denotes the frontier of Ω. In this problem the variable x which corresponds to space and the variable t which corresponds to time are strictly separated. Suppose that A_t is a transmutation in the variable t which transmutes $\frac{\partial^2}{\partial t^2} + r(t)\frac{\partial}{\partial t} + s(t)$ into $\frac{\partial^2}{\partial t^2}$ and let $V(x, t) = A_t u(x, t)$ when $u(x, t)$ is the solution of the differential equation. Then applying A_t to the left hand member of the equation we have

$$A_t \Lambda_x u(x, t) + A_t\left(\frac{\partial^2}{\partial t^2} + r(t)\frac{\partial}{\partial t} + s(t)\right)u(x, t) = 0$$

i.e.,
$$\Lambda_x v(x, t) + \frac{\partial^2 v(x, t)}{\partial t^2} = 0$$

and $v(x, t)$ satisfies Cauchy's data i.e., by means of the transmutation, consideration of the gives equation is reduced to the consideration of the wave equation.

Chapter 3

Transmutation in the Irregular Case

Introduction. Our aim in this chapter is to obtain a transmutation op- **20**
erator for a differential operator with regular coefficients. In order to reduce the mixed problem relative to the operator

$$\Lambda + \frac{\partial^2}{\partial t^2} + \frac{2p+1}{t}\frac{\partial}{\partial t}, \; p \text{ real or complex,}$$

where $\Lambda = -\Delta$(Δ being the Laplacian in the variables $x_1, \ldots, x_n$) to the mixed problem relative to the operator

$$\Lambda + \frac{\partial^2}{\partial t^2}$$

we shall construct an operator will transmute the operator

$$L_p = D^2 + \frac{2p+1}{x}D$$

into the operator D^2. The difficulty in this case arises due to the presence of the coefficient $\frac{1}{x}$ which has a singularity at $x = 0$. If we seek the solution of the problem in $x \geq a$, where $a > 0$, the method of the preceding chapter is perfectly valid without any change. But the important case is precisely the one in which $a = 0$.

We shall determine isomorphisms $B_p, \mathscr{B}_p$(of certain space which will be precisely specifies in the sequel) which satisfy 21

$$D^2 B_p = B_p L_p; \ \mathscr{B}_p D^2 = L_p \mathscr{B}_p.$$

The operator B_p for $-1 < \text{Re}p < -\frac{1}{2}$ and the operator $\mathscr{B}_p$ for $\text{Re}p > -\frac{1}{2}$ are classical. $\mathscr{B}_p$ is the Poisson's operator and B_p is the derivative of the Sonine operator.

1 The operator $\mathscr{B}_p$ for $\text{Re}p > -\frac{1}{2}$

It can be forseen that the operator $\mathscr{B}_p$ is defined in terms of the solution for $y > 0$ of the partial differential equation

$$\Phi_{xx} - \Phi_{yy} + \frac{2p+1}{x}\Phi_x = 0$$

with the conditions $\Phi(0, y) = g^*(y)$ being an even function $g^*(y) = g(y)$ for $y > 0$ and $g^*(y) = g(-y)$ for $y < 0$, and $\Phi_x(0, y) = 0$ Now we define $\mathscr{B}_p[g(x)] = \Phi(x, 0)$.

Changing the variables (x, y) to the variables (s, t) be means of the formulae $s\sqrt{2} = y + x$ and $t\sqrt{2} = y - x$, we see by simple computation that $u(s, t) = \Phi\left(\frac{s-t}{2}, \frac{s+t}{2}\right)$ is a solution of the partial differential equation

$$u_{st} - \frac{p+\frac{1}{2}}{s-t}u_s + \frac{p+\frac{1}{2}}{s-t}u_t = 0$$

22 with the conditions $u(s, s) = g^*(s\sqrt{2})$

$$(u_s - u_t)_{s=t} = 0$$

For $\alpha = p + \dfrac{1}{2}$, Poisson's solution has the form

$$u(s,t) = \frac{\Gamma(2\alpha)}{[\Gamma(\alpha)]^2} \int_0^1 g^* \left\{ [s + (t-s)\rho]\sqrt{2} \right\} (1-\rho)^{\alpha-1} \rho^{\alpha-1} d\rho$$

valid for $\mathrm{Re}\alpha > 0$ (or $\mathrm{Re}p > -\dfrac{1}{2}$). Then

$$\begin{aligned}\mathscr{B}[g(x)] = \Phi(x,0) = u(\frac{x}{\sqrt{2}}, \frac{-x}{\sqrt{2}}) \\ = \frac{\Gamma(2\alpha)}{[\Gamma(\alpha)]^2} \int_0^1 g^*[x(1-2\rho)](1-\rho)^{\alpha-1}\rho^{\alpha-1} d\rho\end{aligned}$$

setting $1 - 2\rho = r$,

$$\begin{aligned}\mathscr{B}_p[g(x)] &= \frac{\Gamma(2\alpha)}{[\Gamma(\alpha)]^2} \frac{1}{2^{2\alpha-2}} \int_0^1 (1-r^2)^{\alpha-1} g^*(rx) dr \\ &= \frac{2\Gamma(p+1)}{\sqrt{\pi}\Gamma(p+\frac{1}{2})} \int_0^1 (1-t^2)^{p-\frac{1}{2}} g(tx) dt.\end{aligned}$$

$$\left(\text{ Using } \Gamma(2\alpha) = \frac{\Gamma(\alpha)\Gamma(\alpha+\frac{1}{2})2^{2\alpha-1}}{\sqrt{\pi}}\right).$$

Note that $\mathscr{B}_p[g(0)] = 1$.

Proposition 1. *The mapping $f \to \mathscr{B}_p f$ is a linear continuous map of* 23
$F^2 = \left\{ f / f \in \mathscr{E}^2(x \geq 0), f'(0) = 0 \right\}$ into itself and satisfies

$$\mathscr{B}_p D^2 f = L_p \mathscr{B}_p f \text{ for any } f \in F^2.$$

As for $\mathrm{Re}p > -\frac{1}{2}$*, differentiation under sign of integration is permissible, $\mathscr{B}_p f \in \mathscr{E}^2(x \geq 0)$. Further $\left\{ \dfrac{d}{dx}\mathscr{B}_p[f] \right\}_{x=0} = f'(0)\,k$ (k being some constant),*

$$= 0.$$

Hence $\mathscr{B}_p f \in F^2$. Evidently $\mathscr{B}_p$ is linear. In order to prove continuity, since F^2 has a metrizable topology induced by that of $\mathscr{E}^2(x \geq 0)$, it is *sufficient to show that if a sequence* $\{f_n\}, n = 1, 2, \ldots, f_n \in F^2$*. converges to* 0 *in* F^2*, then* $\mathscr{B}_p f_n$ *converges to zero in* F^2*. But* $f_n \to 0$ *in* F^2 *implies* $f_n \to 0$ *uniformly on each compact set, in particular on the compact set* $[0, 1]$ *from which it follows that* $\mathscr{B}_p f_n \to 0$*. It remains to verify that* $\mathscr{B}_p$ *satisfies the given condition. Writing* $\beta_p = \frac{2\Gamma(p+1)}{\sqrt{\pi}\Gamma(p+1/2)}$

$$\begin{aligned}
\frac{1}{\beta_p}\left\{L_p \mathscr{B}_p f - \mathscr{B}_p D^2 f\right\} &= \frac{1}{\beta_p}\int_0^1 \Big\{t^2(1-t^2)^{p-\frac{1}{2}} f''(tx) \\
&\quad + \frac{(2p+1)t}{x}(1-t^2)^{p-\frac{1}{2}} f'(tx) - (1-t^2)^{p-\frac{1}{2}} f''(tx)\Big\} dt \\
&= \frac{2p+1}{x}\int_0^1 t(1-t^2)^{p-\frac{1}{2}} f'(tx)dt - \int_0^1 (1-t^2)^{p+\frac{1}{2}} f''(tx)dt. \\
&= \frac{2p+1}{x}\left\{-\frac{1}{2}\left[\frac{(1-t^2)^{p+\frac{1}{2}}}{p+\frac{1}{2}} f'(tx)\right]_0^1 + \frac{1}{2}\int_0^1 \frac{(1-t^2)^{p+\frac{1}{2}}}{x} f''(tx)dt\right\} \\
&\quad - \int_0^1 (1-t^2)^{p+\frac{1}{2}} f''(tx)dt \text{ (integrating the first integral by parts)}. \\
&= 0.
\end{aligned}$$

24 **Remark.** Let $J_p(x)$ denote the classical Bessel function and let

$$j_p(x) = 2^p\Gamma(p+1)x^{-p}J_p(x)$$

$j_p(x)$ is in F_2 and is the unique solution of the differential equation $\dfrac{d^2y}{dx^2} +$
25 $\dfrac{2p+1}{x}\dfrac{dy}{dx} + s^2y = 0$ with the conditions $(y)_{x=0} = 1$ and $\left(\dfrac{dy}{dx}\right)_{x=0} = 0$. Now $\cos sx \in F^2$. Let $\mathscr{B}_p(\cos sx) = g(x)$. Using $L_p\mathscr{B}_p = \mathscr{B}_pD^2$, we get

$$\begin{aligned}
L_p\mathscr{B}_p[\cos sx] = L_p[g(x)] &= \mathscr{B}_pD^2(\cos sx) \\
&= -s^2\mathscr{B}_p[\cos sx] = -s^2 g(x).
\end{aligned}$$

i.e., $$L_pg + s^2g = 0.$$

Further $g(0) = \cos 0 = 1$ and $g'(0) = 0$ since $g \in F^2$. This shows that

$$\mathscr{B}_p(\cos sx) = g(x)j_p(sx).$$

Hence we obtain the classical formula,

$$j_p(sx) = \frac{2\Gamma(p+1)}{\sqrt{\pi}\Gamma(p+\frac{1}{2})}\int_0^1 (1-t^2)^{p-\frac{1}{2}}\cos(stx)dt.$$

The operator $\mathscr{B}_p$ was considered by Poisson in this particular question of the transformation of the consine into the function j_p. *The operator B_p for* $-1 < \mathrm{Re}\, p < -\dfrac{1}{2}$

For $f \in \mathscr{E}^0(x \geq 0)$, the definition of B_p is

$$B_p[f(x)] = b_p x \int_0^x (x^2-y^2)^{\frac{-p-3}{2}} \cdot y^{2p+1} f(y)dy$$
$$= b_p \int_0^1 t^{2p+1}(1-t^2)^{-p-3/2} f(tx)dt$$

where $$1/b_p = \frac{1}{2\sqrt{\pi}}\Gamma(p+1)\Gamma\left(-p-\frac{1}{2}\right)$$

Proposition 2. *For* $-1 < \mathrm{Re}\, p < -\dfrac{1}{2}$ *and* $f \in F^2$, $f \to B_p f$ *is a linear continuous map of* F^2 *into itself satisfying* **26**

$$D^2 B_p f = B_p L_p f.$$

The proof of the fact that B_p is a linear continuous map of F^2 into itself is analogous to the one we have given in Proposition 1. We verify that B_p satisfies the given condition, again as in Proposition 1, by integration by parts

$$\frac{1}{b_p}\left\{B_p L_p - D^2 B_p\right\} f(x) = \int_0^1 t^{2p+1}(1-t^2)^{-p-3/2}$$
$$\left\{f''(tx) + \frac{2p+1}{tx} f'(tx) - t^2 f''(tx)\right\} dt$$
$$= \int_0^1 t^{2p+1}(1-t^2)^{-p-\frac{1}{2}}) f''(tx) + \frac{2p+1}{x}\int_0^1 t^{2p}(1-t^2)^{-p-3/2}) f'(tx)dt$$
$$= \left[t^{2p+1}(1-t^2)^{-p-\frac{1}{2}} f'(tx)\right]_0^1 - \int_0^1 \frac{f'(tx)}{x}\frac{d}{dt}$$

$$\left(\frac{t^{2p+1}}{(1-t^2)^{p+\frac{1}{2}}}\right)dt + \frac{2p+1}{x}\int_0^1 t^{2p}\frac{f'(tx)}{(1-t^2)^{p+3/2}}dt = 0.$$

27 The Sonine operator $\bar{B}_p$ for $-1 < \text{Re}p < \dfrac{1}{2}$.

$\bar{B}_p$ is defined for every $f \in \mathscr{E}^0(x \geq 0)$by

$$\begin{aligned}\bar{B}_p f &= x\bar{b}_p \int_0^1 \frac{t^{2p+1}}{(1-t^2)^{p+\frac{1}{2}}} f(tx)dt\\ &= \bar{b}_p \int_0^x y^{2p+1}(x^2-y^2)^{-p-\frac{1}{2}} f(y)dy\end{aligned}$$

where $$\bar{b}_p = \frac{\sqrt{\pi}}{\Gamma(p+1)\Gamma(-p+\frac{1}{2})}.$$

The integral converges if $-1 < \text{Re}p < \frac{1}{2}$ and we can differentiate under the sign fo integration

$$\frac{d}{dx}\bar{B}_p[(f(x))] = B_p[f(x)] \text{ for every } f \in \mathscr{E}^0(x \geq 0)$$

Relation between $\mathscr{B}_p$ and $\bar{B}_p$

When $-\dfrac{1}{2} < \text{Re}p < \dfrac{1}{2}$, both $\mathscr{B}$ and $\bar{B}_p$ are defined and it is easy to prove by direct computation Abel's functional equation

$$\bar{B}_p\mathscr{B}_p[f(x)] = \int_0^x f(y)dy.$$

In fact

$$\begin{aligned}\bar{B}_p\mathscr{B}_p[f(x)] &= \bar{b}_p \int_0^x y^{2p+1}(x^2-y^2)^{-p-\frac{1}{2}}\beta_p y^{-2p}\int_0^y f(z)(y^2-z^2)^{p-\frac{1}{2}}dz\\ &= \bar{b}_p\beta_p \int_0^y dy \int_0^y y(x^2-y^2)^{-p-\frac{1}{2}} f(z)dz\\ &= \bar{b}_p\beta_p \int_0^x [f(z)\int_z^x (x^2-y^2)^{-p-\frac{1}{2}}(y^2-z^2)^{p-\frac{1}{2}} ydy]dz\end{aligned}$$

28 Setting $x^2\sin^2\theta + z^2\cos^2\theta = y^2$, we have

$$x^2 - y^2 = (x^2 - z^2)\cos^2\theta, y^2 - z^2 = (x^2 - z^2)\sin^2\theta$$
$$ydy = 2(x^2 - z^2)\sin\theta\cos\theta d\theta$$

$$\begin{aligned}
&\int_z^x (x^2 - y^2)^{-p-\frac{1}{2}}(y^2 - z^2)^{p-\frac{1}{2}} ydy \\
&\qquad = \int_0^{\frac{\pi}{2}} \cos^{-2p}\theta\sin^{2p}\theta d\theta = \frac{1}{2}B\left(-p + \frac{1}{2}, p + \frac{1}{2}\right) \\
&\qquad = \frac{1}{2}\frac{\Gamma\left(-p + \frac{1}{2}\right)\Gamma\left(p + \frac{1}{2}\right)}{\Gamma(1)}
\end{aligned}$$

Hence $\bar{B}_p\mathscr{B}_p[f(x)] = \int_0^x f(z)dz$ so that $D\bar{B}_p\mathscr{B}_p[f(x)] = f(x)$.

2 Continuation of the operator B_p

For any $f \in \mathscr{E}(x \geq 0) = \mathscr{E}^\infty(x \geq 0)$, we define

$$T_1[f(t,x)] = D_t[tf(tx)],$$
$$T_2[f(t,x)] = D_t[t^3 T_1\{f(t,x)\}].$$

In general $$T_n[f(t,x)] = D_t\{t^3 T_{n-1}[f(t,x)]\}.$$

Lemma 1. *$T_n[f(t,x)] = t^{2n-2}g_n(t,x)$ where $g_n(t,x)$ is an indefinitely* 29
differentiable function in $[0,1]\times[0,\infty)$.

The proof of the lemma is trivial and is based on indication on n.

For $n = 1$, we have only to set $g_1(t,x) = D_t[tf(tx)]$. Assume that the lemma is true for $n-1$ so that

$$T_{n-1}[f(t,x)] = t^{2n-4}g_{n-1}(t,x).$$

Define

$$g_n(t,x) = 3g_{n-1}(t,x) + (2n-4)g_{(n-1)}(t,x) + tD_t\, g_{n-1}(t,x).$$

By definition

$$
\begin{aligned}
T_n[f(t,x)] &= 3t^2 T_{n-1}[f(t,x)] + t^3 D_t T_{n-1}[f(t,x)]\\
&= 3t^2 t^{2n-4} g_{n-1}(t,x) + t^3\left\{(2n-4)t^{2n-5} g_{n-1}(t,x) + t^{2n-4} D_t\, g_{n-1}(t,x)\right\}.\\
&= t^{2n-2} g_n(t,x).
\end{aligned}
$$

Corollary. *The integral* $\int_0^1 t^{2p-(2n-3)}(1-t^2)^{-p+\left(\frac{2n-3}{2}\right)} T_n f(x,t)dt$ *converges for* $-1 < \operatorname{Re} p < n - \dfrac{1}{2}$.

30 The corollary is immediate since the integral can be written as $\int_0^1 t^{2p+1}(1-t^2)^{-p+\frac{2n-3}{2}} g_n(t,x)$.

Proposition 1. *For* $-1 < \operatorname{Re} p < -\dfrac{1}{2}$, *and for* $f \in \mathscr{E}(x \geq 0)$,

$$
B_p[f(x)] = \frac{(-1)^n b_p}{(2p+1)(2p-1)\cdots(2p-\overline{2n-3})} \int_0^1 \frac{t^{2p-(2n-3)}}{(1-t^2)^{p-\frac{2n-3}{2}}} T_n f(t,x)dt
$$

The proof is based on induction on n and the following formula which is obvious:

$$
\frac{d}{dt}\left[\frac{t^{2p-\lambda}}{(1-t^2)^{p-\frac{\lambda}{2}}}\right] = \frac{(2p-\lambda)t^{2p-\lambda-1}}{(1-t^2)^{p-\lambda/2+1}} \tag{1}
$$

for any $\lambda B_p[f(x)] = b_p \int_0^1 t^{2p+1}(1-t^2)^{-p-3/2} f(tx)dt$

Let $n = 1$. If $p + \dfrac{3}{2} = p - \dfrac{\lambda}{2} + 1$, i.e., $\lambda = -1$

$$
\frac{t^{2p}}{(1-t^2)^{p+3/2}} = \frac{1}{(2p+1)}\frac{d}{dt}\left(\frac{t^{2p+1}}{(1-t^2)^{p+\frac{1}{2}}}\right)
$$

so that

$$
\begin{aligned}
B_p[f(x)] &= \frac{b_p}{2p+1}\int_0^1 \frac{d}{dt}\left(\frac{t^{2p+1}}{(1-t^2)^{p+\frac{1}{2}}}\right) t f(tx)dt \\
&= -\frac{b_p}{2p+1}\int_0^1 \frac{t^{2p+1}}{(1-t^2)^{p+\frac{1}{2}}} T_1[f(t,x)]dt
\end{aligned}
$$

(integrating by parts, the integrated part being zero since $\operatorname{Re} p + \frac{1}{2} < 0$ **31**
and $2\operatorname{Re} p + 2 > 0$). Thus the formula to be proved holds for $n = 1$. Assuming it for $n-1$, we establish it for n

$$
B_p[f(x)] = \frac{(-1)^{n-1} b_p}{(2p+1)(2p-1)\dots(2p-2n+5)} \int_0^1 \frac{t^{2p-(2n-5)}}{(1-t^2)^{p-\frac{2n-5}{2}}} T_{n-1}[f(t,x)]dt
$$

Using (1) with $p - \dfrac{2n-5}{2} = p - \dfrac{\lambda}{2} + 1$ i.e., $\lambda = 2n-3$, the integral on the right hand side equals

$$
\begin{aligned}
&\frac{1}{2p-2n+3}\int_0^1 \frac{d}{dt}\left[\frac{t^{2p-2n+3}}{(1-t^2)^{p-\frac{2n-3}{2}}}\right] t^3 T_{n-1}[f(t,x)]dt \\
&\quad = \frac{1}{2p-2n+3}\left\{\left[\frac{t^{2p+2}}{(1-t^2)^{p-\frac{2n-3}{2}}} g_{n-1}(t,x)\right]_0^1 - \int_0^1 \frac{t^{2p-2n+3}}{(1-t^2)^{p-\frac{2n-3}{2}}} T_n\left[f(t,x)\right]dt\right\} \\
&\quad = \frac{1}{2p-2n+3}\int_0^1 \frac{t^{2p-2n+3}}{(1-t^2)^{p-\frac{2n-3}{2}}} T_n[f(t,x)]dt,
\end{aligned}
$$

the integrated part being zero since $-1 < \operatorname{Re} p < -\dfrac{1}{2}$. We write **32**

$$
B_p^n[f(x)] = b_p^{(n)} \int_0^1 \frac{t^{2p-(2n-3)}}{(1-t^2)^{p-\frac{2n-3}{2}}} T_n[f(t,x)]dt \tag{2}
$$

where $$b_p^{(n)} = (-1)^n \frac{b_p}{(2p+1)(2p-1)(2p-3)\cdots(2p-2n+3)} \tag{3}$$

The integral is convergent for $-1 < \text{Re}p < n - \dfrac{1}{2}$ so that under this condition, we can differentiate under the sign of integration and $B_p^n f \in \mathscr{E}$. We obtain for each n a function which assigns to each p in $-1 < \text{Re}p < n - \dfrac{1}{2}$, a map B_p^n of $\mathscr{E}$ into itself which coincides with B_p if $-1 < \text{Re}p < -\dfrac{1}{2}$. It is easy to see that B_p^n is a linear map of $\mathscr{E}$ into itself. In order to show that it is continuous, as $\mathscr{E}$ is metrizable, it is enough to prove that if a sequence $\{f_j\}_{j=1,2,\ldots}$ tends to zero in $\mathscr{E}$, then $B_p^n f_j$ tends to 0 in $\mathscr{E}$. We have

$$\left|D_x^r B_p^n [f_j(x)]\right| \leq b_p^{(n)} M(r,x,j) \int_0^1 \frac{t^{2p-(2n-3)}}{(1-t^2)^{p-\frac{2n-3}{2}}} dt$$

where $$M(r,x,j) = \sup_{1 \leq t \leq 1} \left|D_x^r T_n \left[f_j(t,x)\right]\right|$$

33 Now $T_n[f_f(t,x)]$ is a polynomial in t,x with coefficients which are derivatives of order $\leq n$ of f_j and f_j together with all its derivatives converge to zero on each compact subset. Hence $M(r,x,j) \to 0$ as $j \to \infty$ uniformly for x on each compact subset *i.e.* $B_p^n f_j \to 0$ in $\mathscr{E}$. Thus we obtain a function $p \to B_p^n$ on $-1 < \text{Re}p < -\dfrac{1}{2}$ with values in $\mathscr{L}(\mathscr{E},\mathscr{E})$, the space of linear continuous maps of $\mathscr{E}$ into itself. We intend to prove that this function is analytic and can be continued in the whole complex plane into a function which ia analytic in the half plane $\text{Re}p > -1$ and meromorphic in $\text{Re}p < -1$ with a sequence of poles lying on the real axis. Before proving this continuation theorem we give first the definition of a vector valued analytic function and some of its properties which follow immediately from the definition.

Definition. Let 0 be an open subset of the complex plane and E a locally convex vector space. A function $f : 0 \to E$ is called analytic if for every e' in the topological dual E' of E (*i.e.* the space of linear continuous forms on E) the function $z \to< f(z), e' >$ is analytic in 0 where $<,>$ denotes the scalar product between E and E'.

Lemma 2. *If E is locally convex vector space in which closed convex envelope of a compact set is compact, a function $f : 0 \to E$ is analytic if f is continuous and for every e' in a total set M' of E', $z \to < f(z), e' >$ is analytic in* 0*.*

Let C be any simple closed curve lying entirely in 0, enclosing region (open connected set) contained in 0. For $e' \in M'$, the function $z \to < f(z), e' >$ is analytic in 0, so that $\int_c < f(z), e' > dz = 0$ i. e. $< \int_c f(z)dz, e' >= 0$. $\int_c f(z)dz$ is the integral of the continuous E-valued function f over the compact set C and is an element of E since E has the property that the closed convex envelope of any compact subset is compact. (For integration of a vector valued function, refer to N. Bourbaki, Elements de Mathematique, Integration, Chapter III). Hence $\int_c f(z)dz = 0$ since M' is total, so that $< \int_c f(z)dz,\ e' >= \int_c < f(z), e' > dz = 0$ for every $e' \in E'$. Also as f is continuous it follows that $z \to < f(z),\ e' >$ is continuous. This proves that $z \to < f(z), e' >$ is analytic for every $e' \in E'$ since the choice of C was arbitrary. 34

Proposition 2. *Suppose that* $-1 < \text{Re}p < n - \dfrac{1}{2}$. *Then*

a) $B^n_p f \in \mathscr{E}(x \geq 0)$ *for every* $f \in \mathscr{E}(x \geq 0)$

b) *The mapping* $f \to B^n_p f$ *is linear continuous of* $\mathscr{E}$ *into itself.*

c) *The function* $p \to B^n_p$ *on the strip* $-1 < \text{Re}p < n - \dfrac{1}{2}$ *with values in* $\mathscr{L}_s(\mathscr{E}, \mathscr{E})$ *is analytic where* $\mathscr{L}_s(\mathscr{E}, \mathscr{E})$ *is the space of linear continuous maps of* $\mathscr{E}$ *into* $\mathscr{E}$ *endowed with the topology of simple convergence.*

We have to prove only (c). We first observe that any linear continuous form on $\mathscr{L}_s(E, F)$ is given by finite linear combination of forms 35 of the type $u \to < ue, f' >$ with $e \in E$ and $f' \in F'$. The theorem will be proved if we show that the map $p \to < B^n_p f, T >$ is analytic in $-1 < \text{Re}p < n - \dfrac{1}{2}$ where f is any element of $\mathscr{E}$ and T any element of $\mathscr{E}'$; i.e. for fixed f we have to show that the map $p \to B^n_p f$ is analytic with values in $\mathscr{E}$. Now $\mathscr{E}$ ia a Hausdorff complete locally convex vector space and therefore closed convex envelope of each compact subset of $\mathscr{E}$ is compact (refer to N. Bourbaki, Espaces Vectoriels Topologiques, Ch.II, §4, Prop.2). Further it is easy to see that $p \to B_p^n f$ is continuous and that the set $\{\delta_x\}_{x \geq 0}$, where δ_x is the Dirac measure with support at x, is total in $\mathscr{E}'$. Applying the lemma, we see that in order to prove analyticity of the function $p \to B_p^n f$ we have only to prove that the function $p \to < B_p^n f, \delta_x >$ i.e. the function $p \to B_p^n f(x)$ (for f and x fixed) is analytic in $-1 < \text{Re} p < n - \dfrac{1}{2}$.

Observing that $b_p = \dfrac{2\sqrt{\pi}}{\Gamma(p+1)\Gamma(-p-\frac{1}{2})}$ is an entire function with zeros at $p = -1, -2, -3, \ldots$ and $p = -\dfrac{1}{2}, \dfrac{1}{2}, \dfrac{3}{2}, \ldots$ we see that b_p^n in (3) is an entire function. The integral in (2) on the other hand converges for $-1 < \text{Re} p < n - \dfrac{1}{2}$ and therefore is analytic in the same region, Hence $p \to B_p^n f(x)$ is analytic in the strip $-1 < \text{Re} p < n - \dfrac{1}{2}$.

36 **Corollary.** *The functions B_p^m and B_p^n where m and n are two distinct positive integers are identical in the intersection of their domains of definition.*

We have in fact two analytic functions B_p^m and B_p^n which coincide with B_p in $-1 < \text{Re} p < -\dfrac{1}{2}$ which is common to their domains of definitions and therefore the two functions coincide everywhere in the domain which is the intersection of their domains of definition due to analyticity. It follows from the corollary that we have a unique analytic function B_p defined for $\text{Re} p > -1$.

Remark. We have $B_{-1/2}$ = identity.

For $n = 1$,

$$B_p(f) = \frac{-b_p}{2p+1} \int_0^1 \frac{t^{2p+1}}{(1-t^2)^{p+\frac{1}{2}}} \frac{d}{dt}[t f(tx)] dt$$

and if $p = -\dfrac{1}{2}$, $\dfrac{-b_p}{2p+1} = 1$ and

$$B_{-\frac{1}{2}}[f(x)] = \int_0^1 \frac{d}{dt}[t\, f(tx)]dt = f(x).$$

Continuation of B_p for Rep $< -\dfrac{1}{2}$**.**

We define

$$U_1[f(t,x)] = D_t[(1-t^2)^{\frac{1}{2}} f(tx)].$$
$$U_2[f(t,x)] = D_t[(1-t^2)^{3/2}\ f(tx)].$$

In general $\quad U_n[f(t,x)] = D_t[(1-t^2)^{3/2} U_{n-1} f(tx)].$ 37

Lemma 3. *For every f in $\mathcal{E}(x \geq 0) = \mathscr{E}$ we have*

$$U_n[f(t,x)] = (1-t^2)^{3/2} D_t U_{n-1} f(t,x) - 3t(1-t^2)^{\frac{1}{2}} U_{n-1}[f(t,x)] \tag{4}$$

and $\quad U_n[f(t,x)] = (1-t^2)^{\frac{n-2}{2}} h_n(t,x) \quad (5)$

where $h_n(t,x)$ is indefinitely differentiable in $[0,1] \times [0,\infty)$.

Relation (4) is evident. (5) can be proved by induction on n. It is true for $n = 1$ if we set

$$h_1(t,x) = -tf(tx) + x(1-t^2)f'(t,x).$$

Assuming it for $n-1$, it is easy to verify that (5) holds for n if

$$h_n(t,x) = -nth_{n-1}(t,x) + (1-t^2)D_t h_{n-1}(t,x)$$

Corollary. *The integral* $\int_0^1 \frac{t^{2p+n+1}}{(1-t^2)^{p+\frac{n+1}{2}}}\ U_n[f(x,t)]dt$ *converges for* $-1 - \frac{n}{2} <$ Rep $< -\frac{1}{2}$.

Proposition 3. *If* $-1 <$ Rep $< -\dfrac{1}{2}$,

$$B_p[f(x)] = \frac{(-1)^n b_p}{(2p+2)(2p+3)\cdots(2p+n+1)} \int_0^1 \frac{t^{2p+n+1}}{(1-t^2)^{p+\frac{n+1}{2}}} U_n[f(t,x)]dt$$

The proof of this proposition is analogous to that of Proposition 1. 38

We prove it by induction on n and by using formula (1).

For $-1 < \text{Re}p < -\dfrac{1}{2}$,

$$B_p[f(x)] = b_p \int_0^1 t^{2p+1}(1-t^2)^{-p-\frac{3}{2}} f(tx)dt$$

39 Using (1) with $2p+1 = 2p-\lambda-1$ i. e. $\lambda = -2$, we have,

$$\frac{t^{2p+1}}{(1-t^2)^{p+2}} = \frac{1}{2p+2}\frac{d}{dt}\left(\frac{t^{2p+2}}{(1-t^2)^{p+1}}\right)$$

so that

$$\begin{aligned} B_p[f(x)] &= \frac{b_p}{2p+2}\int_0^1 \frac{d}{dt}\left(\frac{t^{2p+2}}{(1-t^2)^{p+1}}\right)(1-t^2)^{\frac{1}{2}} f(tx)dt \\ &= \frac{-b_p}{(2p+2)}\int_0^1 \frac{t^{2p+2}}{(1-t^2)^{p+1}} U_1[f(t,x)]dt, \end{aligned}$$

the integrated part being zero since $-1 < \text{Re}p < \dfrac{1}{2}$. The formula is proved for $n = 1$. We assume it for $n-1$

$$B_p[f(x)] = \frac{(-1)^{n-1} b_p}{(2p+2)(2p+3)\cdots(2p+n)} \int_0^1 \frac{t^{2p+n}}{(1-t^2)^{p+n/2}} U_{n-1}[f(t,x)]dt$$

Using (1) with $2p+n = 2p-\lambda-1$ i. e. $\lambda = -(n+1)$, we get

$$\frac{d}{dt}\left(\frac{t^{2p+n+1}}{(1-t^2)^{p+\frac{n+1}{2}}}\right) = (2p+n+1)\frac{t^{2p+n+2}}{(1-t^2)^{p+\frac{n+3}{2}}},$$

so that

$$B_p[f(x)] = \frac{-(-1)^{n-1} b_p}{(2p+2)\cdots(2p+n+1)} \int_0^1 \frac{t^{2p+n+1}}{(1-t^2)^{p+\frac{n+1}{2}}} D_t\left\{(1-t^2)^{\frac{3}{2}} U_{n-2} f(t,x)\right\} dt$$

we shall now set

$$n_{B_p}[f(x)] = (n)_{b_p} \frac{t^{2p+n+1}}{(1-t^2)^{p+\frac{n+1}{2}}} U_n[f(t,x)]dt \tag{6}$$

$$(n)_{b_p} = \frac{(-1)^n b_p}{(2p+2)(2p+3)\cdots(2p+n+1)} \tag{7}$$

$(n)_{b_p}$ is a meromorphic function of p, with poles at the points $p = \frac{-3}{2}, \frac{-5}{2}, \ldots.$

Proposition 4. *Suppose that p satisfies*

$$-1 - \frac{n}{2} < \text{Re} p < -\frac{1}{2} \tag{8}$$

and does not assume any of the values

$$-\frac{3}{2}, \frac{-5}{2}, \ldots \tag{9}$$

Then 40

a) ${}^n B_p f \in \mathscr{E}$ for each $f \in \mathscr{E}$

b) The mapping $f \to {}^n B_p f$ is linear continuous from $\mathscr{E}$ into $\mathscr{E}$.

c) The function $p \to {}^n B_p$ is meromorphic in the strip defined by (8) with poles situated at the points given by (9).

We omit the proof of *a*) and *b*) since it is exactly similar to that of Proposition 2. The proof of *c*) reduces as in Proposition 2, to showing that the function

$$p \longrightarrow {}^{(n)}b_p \int_0^1 \frac{t^{2p+n+1}}{(1-t^2)^{p+\frac{n+1}{2}}} U_n\,[f(t,x)]\,dt$$

is meromorphic in the strip (8) with poles at the points (9), which is obvious since the integral converges in (8) and is therefore analytic in (8) and n_{b_p} is meromorphic in (8) with poles given by (9).

Corollary. *If m, n are two distinct positive integers, the two functions m_{B_p}, n_{B_p} coincide in the common part of their domains of definition.*

This corollary is immediate like the corollary of Proposition 2. Consequently we have a unique function B_p defined for Re$p < -\dfrac{1}{2}$.

41 Propositions (2) and (4) give finally the continuation theorem for the operator B_p.

Theorem. *The function p–B_p defined initially in $-1 < -$ Re$p < -\dfrac{1}{2}$ with values in $\mathscr{L}(\mathscr{E}, \mathscr{E})$ endowed with the topology of simple convergence can be continued in the whole plane into a meromorphic function. The poles of this function are situated at the points $\dfrac{-3}{2}, \dfrac{-5}{2}, \dfrac{-7}{2}, \ldots$.*

Remark. The notion of an analytic function with values in a locally convex vector space E depends only on the system of bounded subsets of E. $\mathscr{E}$ being complete it is easy to see (in view of Theorem 1, page 21, Ch.III, Espaces vectorieles Topologiques by N. Bourbaki) that the space $\mathscr{L}(\mathscr{E}, \mathscr{E})$ when furnished with the topology of simple convergence has the same system of bounded sets as when furnished with the topology of uniform convergence on the system of bounded sets of $\mathscr{E}$. Hence in the theorem we can replace the topology of simple convergence by the topology of uniform convergence on bounded subsets of $\mathscr{E}$ or by any other locally convex topology which lies between these two topologies.

Let $\mathscr{E}_*$ and $\mathscr{D}_\circ$ be subspaces of $\mathscr{E}_{(x\geq 0)}$ defined by

$$\mathscr{E}_* = \left\{ f \mid f \in \mathscr{E}(x \geq 0),\ f^{2n+1}(0) = 0 \text{ for } n \geq 0 \right\}$$
$$\mathscr{D}_\circ = \{ f | f \in (x \geq 0),\ f^n(0) = 0 \text{ for } n \geq 0 \}$$

42 When $-1 <$ Re$p < -\dfrac{1}{2}$, we have

$$D^r B_p[f(x)] = b_p \int_0^1 t^{2p+1+r} (1-t^2)^{-p-\frac{3}{2}} f^r(tx)dt$$

so that

$$D^r B_p[f(0)] = b_{p,r} f^r(0) \tag{10}$$

where

$$b_{q,r} = \frac{\sqrt{\pi}\,\Gamma\left(p + \frac{r}{2} + 1\right)}{\Gamma\left(\frac{r+1}{2}\right)\Gamma(p+1)}$$

This shows that when r is even, $p \to b_{p,r}$ is an entire function of p and when r is odd it is a meromorphic function with poles at $\frac{-3}{2}, \frac{-5}{2}, \ldots$ (10) is therefore true for all p not equal to the exceptional values $\frac{-3}{2}, \frac{-5}{2}, \ldots$. Then $B_p[f]$ is in $\mathscr{E}_*$ or in $\mathscr{D}_\circ$ according as f is in $\mathscr{E}_*$ or $\mathscr{D}_\circ$. Therefore for $p \neq \frac{-3}{2}, \frac{-5}{2}, \ldots, B_p \in \mathscr{L}(\varepsilon_*, \varepsilon_*)$(or $\in \mathscr{L}(\mathscr{D}_o, \mathscr{D}_o)$) and $p \to B_p$ is meromorphic with poles at $p = \dfrac{-3}{2}, \ldots$. Actually the following stronger result holds.

Theorem. *The function $p \to B_p$ is an entire analytic function with values in $\mathscr{L}(\varepsilon_*, \varepsilon_*)$ (also in $(\mathscr{D}_o, \mathscr{D}_o)$).*

We have seen more than once, that in order to investigate the analyticity of the function $p \to B_p$, it is sufficient to do the same for the function $p \to B_p[f(x)]$, where $f \in \mathscr{E}$ and $x \geq 0$ are arbitrarily chosen and are fixed. In this case we study the behaviour of the function 43
$p - B_p[f(x)]$ where $f \in \mathscr{E}^*$ and $x \geq 0$, in the neighbourhood of the point $p_o = -\dfrac{2m+1}{2}$, (a supposed singularity of the function).

By Taylor's formula,

$$f(tx) = \sum_0^N \frac{x^n t^n}{n!} f^n(0) + \frac{x^{N+1}}{N} \int_o^t f(t\xi) f^{N+1}(\xi x) d\xi,$$

N being an arbitrary integer.

Suppose first that $-1 < \mathrm{Re} p < -\dfrac{1}{2}$. Then

$$B_p[f(x)] = \frac{\sqrt{\pi}}{\Gamma(p+1)} \sum_o^N \frac{\Gamma\left(p + \frac{n}{2} + 1\right)}{\Gamma\left(\frac{n}{2} + \frac{1}{2}\right) n!} x^n \; f^n(0)$$

$$+ b_p \frac{x^{N+1}}{N!} \int_o^1 t^{2p+1}(1-t^2)^{-p-\frac{3}{2}} dt \int_o^1 (t-\xi)^N f^{N+1}(\xi x) d\xi.$$

The right hand side of this formula is well defined when $-\dfrac{N+3}{2} <$ Rep $< -\dfrac{1}{2}$ and depends analytically on p and therefore coincides with the continuation of B already obtained. Choosing N such that $-\dfrac{N+3}{2} < p_o$, the integral on the right is analytic at the point p_o. Hence we have only to consider the finite sum at p_o. The function $\Gamma(p+\dfrac{n}{2}+1)$ has poles at the points p such that $p+\dfrac{n}{2}+1 = -\mu, \mu$ a positive integer. It has a
44 pole at p_0 if $\dfrac{n}{2} = -\mu-1-p_0 = m-\mu-\dfrac{1}{2}$ i.e. $n = 2(m-\mu)-1$. But when n is odd, $f^n(0) = 0$ since $f \in \mathscr{E}_*$. The finite sum is therefore analytic at p_0 and $B_p[f(x)]$ has false singularity at $-\dfrac{2m+1}{2}$ and the proof of the theorem is complete.

Theorem. *The formula*

$$D^2 B_p f = B_p L_p f \text{ holds for every } f \in \mathscr{E}^*$$

and for every complex number p. $L_p f$ *is in* $\mathscr{E}_*$ *if* $f \in \mathscr{E}_*$ *so that* $L_p \in \mathscr{L}(\mathscr{E}_*, \mathscr{E}_*)$ *and it is easy to verify that* $p \to L_p$ *is an entire function with value is* $\mathscr{L}(\mathscr{E}_*, \mathscr{E}_*)$. *The two entire functions* $p \to B_p L_p$ *and* $p \to D^2 B_p$ *coincide by Proposition 2, §1 in* $-1 <$ Rep $< -\dfrac{1}{2}$ *and are therefore identical in the whole plane.*

Remark. It is necessary to suppose that $f \in \mathscr{E}_*$. If f is only in $\mathscr{E}, L_p f$ is not in $\mathscr{E}$ (always).

3 Continuation of $\mathscr{B}_p$

We now consider the extension of the operator $\mathscr{B}_p$ initially defined for Rep $> -\dfrac{1}{2}$ by

$$\mathscr{B}_p[f(x)] = \beta_p \int_o^1 (1-t^2)^{p-\frac{1}{2}} f(tx) dt. \tag{I}$$

where

$$\beta_p = \frac{2\Gamma(p+1)}{\sqrt{\prod}\Gamma(p+\frac{1}{2})}$$

Changing the variable $t = \sin\Theta$, **45**

$$\mathscr{B}_p[f(x)] = \beta_p \int_o^{\frac{\Pi}{2}} f(x\sin\theta)\cos^{2p}\theta d\theta.$$

Let $M_p^1[f(x,\theta)] = f(x\sin\theta)$

$$+\frac{1}{(2p+1)(2p+2)}\frac{d}{d\theta}\left\{\sin\theta\frac{d}{d\theta}[\sin\theta f(x\sin\theta)]\right\}$$

and

$$N_p^1[f(x,\theta)] = \frac{1}{2p+1}\frac{d}{d\theta}\{\sin\theta f(x\sin\theta)\}$$

We determine by induction, the functions

$$M_p^n[f(x,\theta)] = M_p^{n-1}[f(x,\theta)] + \frac{1}{2p+2n}\frac{d}{d\theta}$$
$$\left\{\sin\theta\frac{d}{d\theta}\sin\theta N_p^{n-1}[f(x,\theta]\right\} + \frac{1}{(2p+2n-1)(2p+2n)}\frac{d}{d\theta}$$
$$\left\{\sin\theta\frac{d}{d\theta}[\sin\theta M_p^{n-1}(f(x,\theta))\right\}$$

and

$$N_p^n[f(x,\theta)] = N_p^{n-1}[f(x,\theta)] + \frac{1}{2p+2n-1}\frac{d}{d\theta}\left\{\sin\theta M_p^{n-1}[f(x,\theta)]\right\}$$

Proposition 1. *For* $\text{Re}p > -\dfrac{1}{2}$, $f \in \mathscr{E}(x \geq 0)$ *and for any n,*

$$\mathscr{B}_p[f(x)] = \beta_p \int_o^{\frac{\pi}{2}} \cos^{2p+2n}\theta M_p^n[f(x,\theta)]d\theta$$
$$+\beta_p\int_0^{\frac{\pi}{2}}\cos^{2p+2n+1}\theta N_p^n f(x,\theta)d\theta \quad (1)$$

The proof of this proposition is elementary and is based on induction on n and the process of integration by parts. 46

$$\mathscr{B}_p f(x) = \beta_p \int_0^{\frac{\pi}{2}} f(x\sin\theta)\cos^{2p+2}\theta d\theta + \beta_p \int_o^{\frac{\pi}{2}} f(x\sin\theta)\sin^2\theta\cos^{2p}\theta d\theta.$$

But as $\sin\theta\cos^{2p}\theta = \dfrac{d}{d\theta}(\dfrac{\cos^{2p+1}\theta}{-(2p+1)})$ integrating by parts the second integral, we get,

$$\mathscr{B}_p f(x) = \beta_p \int_o^{\frac{\pi}{2}} f(x\sin\theta)\cos^{2p+2}\theta d\theta + \beta_p \int_0^{\frac{\pi}{2}} \frac{1}{2p+1}\frac{d}{d\theta}(\sin\theta f(x\sin\theta))\cos^{2p+1}\theta d\theta$$

Now the second integral in this equation equals

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \frac{1}{2p+1}\frac{d}{d\theta}(\sin\theta f(x\sin\theta))\cos^{2p+3}\theta d\theta \\
&\quad + \int_0^{\frac{\pi}{2}} \frac{1}{2p+1}\frac{d}{d\theta}(\sin\theta f(x\sin\theta))\cos^{2p+1}\theta\sin^2\theta d\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{1}{2p+1}\frac{d}{d\theta}(\sin\theta f(x\sin\theta))\cos^{2p+3}\theta d\theta \\
&\quad + \int_0^{\frac{\pi}{2}} \frac{1}{(2p+1)(2p+2)}\cos^{2p+2}\theta\frac{d}{d\theta}\left\{\sin\theta\frac{d}{d\theta}(\sin\theta f(x\sin\theta))\right\}d\theta
\end{aligned}$$

47 so that

$$\begin{aligned}
\mathscr{B}_p f(x) = \beta_p \int_0^{\frac{\pi}{2}} \cos^{2p+2}\theta\Big\{ & f(x\sin\theta) \\
&+ \frac{1}{(2p+1)(2p+2)}\frac{d}{d\theta}(\sin\theta\frac{d}{d\theta}(\sin\theta f(x\sin\theta)))\Big\}d\theta \\
&+ \beta_p \int_0^{\frac{\pi}{2}} \cos^{2p+3}\theta\frac{1}{2p+1}\frac{d}{d\theta}(\sin\theta f(x\sin\theta))d\theta
\end{aligned}$$

Hence formula (1) holds for $n = 1$. Assuming it for $n - 1$, it can be verified for n by integration by parts.

We define for every integer $n > 0$,

$$\mathscr{B}_p^n f(x) = \beta_p \int_0^{\frac{\pi}{2}} \cos^{2p+2n} \theta M_p^n f(x,\theta) d\theta + \beta_p \int_o^{\frac{\pi}{2}} \cos^{2p+2n+1} \theta N_p^n f(x,\theta) d\theta$$

The two integrals converge for $\text{Re} p > -\frac{1}{2} - n$.

Proposition 2. *If p satisfies* 48

$$\text{Re} p > -\frac{1}{2} - n \tag{2}$$

and

$$p \neq -1, -2, -3, \ldots \tag{3}$$

Then

a) For $f \in \mathscr{E}(x \geq 0) = \mathscr{E}, \mathscr{B}_p^n f \in \mathscr{E}$.

b) The mapping $f \to \mathscr{B}_p^n f$ is linear continuous of $\mathscr{E}$ into itself.

c) The function $p \to \mathscr{B}_p^n$ is meromorphic in the half plane defined by (2) with values in $\mathscr{L}(\mathscr{E}, \mathscr{E})$ the poles being situated at the points (3). We shall give only the outline of the proof since it is analogous to that of Proposition (2), §2. The function $\beta_p = \dfrac{2\Gamma(p+1)}{\sqrt{\pi}\Gamma(p+\frac{1}{2})}$ is a meromorphic function of p which vanishes at $-\frac{1}{2}, -\frac{3}{2}, \ldots,$ and poles at $-1, -2, -3, \ldots$ and for x, θ fixed, $M_p^n f(x,\theta)$, and $N_p^n f(x,\theta)$ are meromorphic function with poles at $-\frac{1}{2}, -1, \frac{-3}{2}, -2, \ldots$. Hence for p fixed and $\neq -1, -2, \ldots,$ the function $(x,\theta) \to \beta_p M_p^n(x,\theta)$ and $(x,\theta) \to \beta_p N_p^n f(x,\theta)$ are indefinitely differentiable in $[0, \infty) \times (0, \frac{\pi}{2})$

so that (a) is true. The proof for (b) is similar to that of (b) of proposition 2, §2. For f, x and θ fixed, $M_p^n f(x, \theta)$ and $N_p^n f(x, \theta)$ are meromorphic functions with poles at $-1, -2, \ldots$ so that $p \to \mathscr{B}_p f(x)$ is meromorphic $\text{Re}\, p > -\frac{1}{2} - n$ with poles at (3). Further since $p \to \mathscr{B}_p$
49 is continuous in the region given by (2) and (3), (c) follows.

Corollary. *If m, n are two distinct positive integers, the functions $\mathscr{B}_p^m$ and $\mathscr{B}_p^m$ coincide in the intersection of their domains of definitions.*

In fact the two meromorphic functions coincide with $\mathscr{B}_p$ in the half plane $\text{Re}\, p > -\frac{1}{2}$ which is common to their domains of definitions and hence they coincide everywhere in the intersection of their domains of definition.

We have proved the following

Theorem. *The function $p \to \mathscr{B}_p$ defined in* $\text{Re}\, p > -\frac{1}{2}$ *by (I) with values in $\mathscr{L}(\mathscr{E}, \mathscr{E})$ can be continued analytically into a function meromorphic in the whole plane with poles at $-1, -2, -3, \ldots$.*

Remark 1. $\mathscr{B}_{-\frac{1}{2}=}$ identity.

Remark 2. For $Rep > -\frac{1}{2}$, we have

$$D^r \mathscr{B}_p f(x) = \beta_p \int_\circ^1 t^r (1 - t^2)^{p - \frac{1}{2}} f^r(tx)\, dt$$

$$\text{and} \qquad [D^r \mathscr{B}_p f(x)]_{x=0} = \beta_{p,r} f^r(0) \tag{4}$$

where $\beta_{p,r} = \dfrac{\Gamma(p+1)\Gamma\left(\dfrac{r+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\dfrac{r}{2} + p + 1\right)}$ is a meromorphic function of p with
50 poles at $-1, -2, \ldots$. By analytic continuation, the equation (I) holds for $p \neq -1, -2, \ldots$. Then if $f \in \mathscr{E}_*(resp.\ \mathscr{D}_o)$, $\mathscr{B}_p f \in \mathscr{E}_*(resp.\ \mathscr{D}_o)$ and $p \to \mathscr{B}_p$ is meromorphic with values in $\mathscr{L}(\mathscr{E}_*, \mathscr{E}_*)(resp.\ \mathscr{L}(\mathscr{D}_o, \mathscr{D}_o))$.

Theorem. *For any $p \neq -1, -2, \ldots$ we have*

$$\mathscr{B}_p D^2 f = D^2 \mathscr{B}_p f, \; f \in \mathscr{E}_*.$$

By Proposition 1, §1, the given equation holds for $\operatorname{Re} p > -\frac{1}{2}$ and hence for all $p \neq -1, -2, \ldots$ by analytic continuation.

Theorem. *For any complex $p \neq -1, -2, \ldots$ the operators B_p and $\mathscr{B}_p$ are isomorphisms of $\mathscr{E}_*$ (resp $\mathscr{D}_o$) onto itself which are inverses of each other.*

We have seen that $D\bar{B}_p \mathscr{B}_p = I$(identity) for $-\frac{1}{2} < \operatorname{Re} p < \frac{1}{2}$. Also for p in the same region $D\bar{B}_p = B_p$ so that $B_p \mathscr{B}_p = I$ for $-\frac{1}{2} < \operatorname{Re} p < \frac{1}{2}$ and the same holds for all $p \neq -1, -2, -3$, by analytic continuation. Similarly $\mathscr{B}_p B_p = I$ for $p \neq -1, -2, \ldots$.

Chapter 4

Transmutation in the Irregular Case

$\left[\textbf{Case of the general operator : } D^2 + \left(\dfrac{2p+3}{x}\right) + M(x)D + N(x)\right]$

1

Let M and N be two indefinitely differentiable functions, M odd (i.e., 51
$M^{(2n)}(0) = 0$ for every $n \geq 0$) and N even (i.e., $N^{2n+1}(0) = 0$ for every $n \geq 0$ i.e., $N \in \mathscr{E}_*$).

We consider the two following problems.

Problem 1. To find $u(x, y)$, indefinitely differentiable even in x and y which is a solution of

$$\frac{\partial^2 u}{\partial x^2} + \left(\frac{2p+1}{x} + M(x)\right)\frac{\partial u}{\partial x} + N(x)u - \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

with

$$u(0, y) = g(y), g \in \mathscr{E}_* \tag{2}$$

Problem 2. To find $v(x, y)$ indefinitely differentiable, even in x and y, solution of the same equation (1) with

$$v(x, 0) = f(x), f \in \mathscr{E}_* \tag{3}$$

We intend to prove the following

Theorem 1. *For $p \neq -1, -2, \ldots$, each of the two problems stated above admits a unique solution, which depends continuously on g (or f).*

52 Once we prove this theorem, we can define *Operators X_p and $\mathfrak{X}_p$*.

Definition 1. For $g \in \mathscr{E}_*$, and $p \neq -1, -2, \ldots$

$$\mathfrak{X}_p[g(x)] = u(x, 0),$$

where u is the solution of the Problem 1.

Definition 2. For $f \in \mathscr{E}_*$, and $p \neq -1, -2, \ldots$

$$X_p[f(y)] = v(0, y)$$

where v is the solution of the second problem. Assuming Theorem 1, definitions (1) and (2) give immediately

Theorem 2. *For $p \neq -1, -2, \ldots, X_p$ and $\mathfrak{X}_p$ are in $L(\mathscr{E}_*, \mathscr{E}_*)$.*

We now prove

Theorem 3. *For $p \neq -1, -2, \ldots$, we have*

$$D^2 X_p = X_p \Lambda_p \tag{4}$$

$$\mathfrak{X}_p D^2 = \Lambda_p \mathfrak{X}_p \tag{5}$$

where
$$\Lambda_p = D^2 + \frac{2p+1}{x} D + M(x) D + N(x) \tag{6}$$

Applying the operator $\dfrac{\partial^2}{\partial y^2}$ to the two sides of

$$\frac{\partial^2 v}{\partial x^2} + \left(\frac{2p+1}{x} + M(x)\right) \frac{\partial v}{\partial x} + N(x) v - \frac{\partial^2 v}{\partial y^2} = 0$$

53 *and setting* $\dfrac{\partial^2 v}{\partial y^2} = V$, *we have*

$$(\Lambda_p)_x V - \frac{\partial^2 V}{\partial y^2} = 0,$$

and $V(x,0) = \dfrac{\partial^2 v}{\partial y^2}(x,0) = (\Lambda_p)_x(v(x,0)) = \Lambda_p[f(x)]$; *and by the definition of* X_p, $V(0,y) = X_p\Lambda_p[f(x)]$.

But we have also $\dfrac{\partial^2 V}{\partial y^2}(0,y) = D_y^2[X_p f(y)]$. Thus (4) is proved. Similarly (5) can be proved. But (5) follows from (4) and the

Theorem 4. *For* $p \neq -1,-2,\ldots,$ *the operators* X_p, $\mathfrak{X}_p$ *are isomorphisms of* $\mathscr{E}_*$ *onto itself and* $X_p\mathfrak{X}_p = \mathfrak{X}_p X_p =$ *the identity.*

Let $g \in \mathscr{E}_*$ and u be the solution of Problem 1. Then $u(x,0) = \mathfrak{X}_p[g(x)]$. In order to find $X_p\mathfrak{X}_p[g(x)]$ it is sufficient to determine the solution $v(x,y)$ of

$$\frac{\partial^2 v}{\partial y^2} + (\frac{2p+1}{x} + M(x))\frac{\partial v}{\partial x} + N(x)v - \frac{\partial^2 v}{\partial y^2} = 0$$

with $v(x,0) = \mathfrak{X}_p[g(x)] = u(x,0)$.

Then, in view of Theorem 1, $u(x,y) = v(x,y)$. Hence $X_p\mathfrak{X}_p[g(y)] = u(0,y) = g(y)$ similarly $\mathfrak{X}_p X_p =$ the identity. 54

The Theorem 1. We now proceed to solve Problem 1. Let $(B_p)_x[u(x,y)] = w(x,y)$ where $(B_p)_x$ denotes the operator B_p relative to the variable x. Applying $(B_p)_x$ to the two sides of equation (1), i. e., of

$$(L_p)_x[u(x,y)] + M(x)\frac{\partial u}{\partial x} + N(x)u - \frac{\partial^2 u}{\partial y^2} = 0$$

and using $D^2 B_p = B_p L_p$, we obtain

$$\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} + (B_p)_x[M(x)\frac{\partial u}{\partial x} + N(x)u] = 0$$

and we have also $w(0,y) = g(y) = u(0,y)$ since $B_p f(0) = f(0)$.

We intend to put this equation which is equivalent to equation (1) in a proper form, We first prove

Proposition 1. *$B_p A \mathscr{B}_p f(x) = A(x) f(x) + x \int_o^x T_p[A](x,y) f(y) dy$ where $A(x)$ and $f(x) \in \mathscr{E}_*$ and T_p is a map defined in $\mathscr{E}_*$ with values in the space of even indefinitely differentiable functions of the variables x and y.*

For $-1 < \text{Rep} < \dfrac{1}{2}$,

$$\overline{B}_p f(x) = \overline{b}_p \int_o^x (x^2 - y^2)^{-p-\frac{1}{2}} y^{2p+1} f(y) dy$$

where $$\overline{b}_p = \frac{\sqrt{\pi}}{\Gamma(p+1)\Gamma(-p+\frac{1}{2})}, \overline{B}_p f(0) = 0, D\overline{B}_p = B_p$$

and for $$\text{Rep} > -\frac{1}{2},$$

$$\mathscr{B}_p g(x) = \beta_p x^{-2p} \int_o^x (x^2 - y^2)^{p-\frac{1}{2}} f(y) dy$$

55 where $$\beta_p = \frac{2\Gamma(p+1)}{\sqrt{\pi}\Gamma(p+\frac{1}{2})}.$$

Hence for $-\dfrac{1}{2} < \text{Rep} < \dfrac{1}{2}$, we compute

$$\begin{aligned}\overline{B}_p A \mathscr{B}_p f(x) &= \overline{b}_p \beta_p \int_o^x (x^2 - z^2)^{-p-\frac{1}{2}} z^{2p+1} A(z) \\ &\qquad \left[z^{-2p} \int_o^z (z^2 - y^2)^{p-\frac{1}{2}} f(y) dy \right] dz \\ &= \overline{b}_p \beta_p \int_o^x f(y) \left[\int_y^x (x^2 - z^2)^{-p-\frac{1}{2}} (z^2 - y^2)^{p-\frac{1}{2}} A(z) z dz \right] dy.\end{aligned}$$

Setting $z^2 = x^2 \sin^2\theta + y^2 \cos^2\theta$, we get

$$\overline{B}_p A \mathscr{B}_p f(x) = \overline{b}_p \beta_p \int_o \phi_p(x,y) f(y) dy$$

where, $$\phi_p(x,y) = \int_o^{\frac{\pi}{2}} \sin^{2p}\theta \cos^{-2p}\theta A\left[\sqrt{x^2 \sin^2\theta + y^2 \cos^2\theta} \right] d\theta$$

which converges for $-\dfrac{1}{2} < \text{Re}p < \frac{1}{2}$. It follows that

$$D\overline{B}_p A\mathscr{B}_p f(x) = \overline{b}_p\beta_p\phi_p(x,x)f(x) + \overline{b}_p\beta_p\int_0^x \frac{\partial}{\partial x}\phi_p(x,y)f(y)dy$$

But we have **56**

$$\phi_p(x,x) = A(x)\frac{1}{2}\Gamma(p+\frac{1}{2})\Gamma\left(-p+\frac{1}{2}\right)$$

so that $$\overline{b}_p\beta_p\phi_p(x,x) = A(x).$$

Hence $B_p A\mathscr{B}_p f(x) = A(x)f(x) + \gamma_p\int_0^x \dfrac{\partial}{\partial x}\phi_p(x,y)f(y)dy$ where

$$\gamma_p = \frac{2}{\Gamma(p+\frac{1}{2})\Gamma(-p+\frac{1}{2})}.$$

Now $\dfrac{\partial}{\partial x}\phi_p(x,y) = \int_0^{\frac{\pi}{2}} \sin^{2p+2}\theta\cos^{-2p}\theta\dfrac{A'(z)}{z}xd\theta$ so that if we give

Definition 3. For $f \in \mathscr{E}_*$

$$T_p[f](x,y) = \gamma_p\int_0^{\frac{\pi}{2}} \sin^{2p+2}\theta\cos^{-2p}\theta f_1(z)d$$

where $f_1(z) = \dfrac{f'(z)}{z}, z = \left[x^2\sin^2\theta + y^2\cos^2\theta\right]^{\frac{1}{2}}, \gamma_p = \overline{b}_p\beta_p$, then we can write

$$B_p A\mathscr{B} f(x) = A(x)f(x) + x\int_o^x T_p[A](x,y)f(y)dy \tag{7}$$

(7) is valid for $|\text{Re}p| < \dfrac{1}{2}$ and $f \in \mathscr{E}_*$.

Remark. We have $\gamma_p = 0$ for $p = \pm\dfrac{1}{2}, \pm\dfrac{3}{2}, \ldots$ so that $T_{-\frac{1}{2}} = 0$. Again **57**
for $p = -\frac{1}{2}, B_p = \mathscr{B}_p =$ the identity. Hence (7) is valid for $p = -\frac{1}{2}$. Now $w(x,y) = (B_p)_x u(x,y)$ so that $u(x,y) = (\mathscr{B}_p)_x w(x,y)$ and

$(B_p)_x M(x)\dfrac{\partial u}{\partial x} = (B_p)_x M(x) D_x(\mathscr{B})_x w(x,y)$. We shall denote by $\phi(x)$, the function $w(x,y)$ considered as a function of x; let $(\mathscr{B}_p)_x w(x,y) = \mathscr{B}_p \phi = \psi$. Then $B_p\psi = \phi, B_p MD\mathscr{B}_p\phi = B_p MD\psi = B_p(D(M\psi))B_p(M'\psi) = B_p[D(xM^*\psi)] - B_p(M'\psi)$ where $M^*(x) = \dfrac{M(x)}{x} \in \mathscr{E}_*$ and $B_p[MD\psi] = B_p[xDM^*\psi] + B_p[(M^* - M')\psi]$. But for $-1 < \mathrm{Re}\, p < -\dfrac{1}{2}$,

$$B_p[xDF] = b_p x \int_o^1 t^{2p+2}(1-t^2)^{-p-3/2} F'(tx)dt = xD(B_p[F])$$

Hence $B_p(MD\psi) = xDB_pM^*\psi + B_p[(M^* - M)\psi]$

Now by (7),

$$\begin{aligned} B_p[MD\psi] &= xD\left\{M^*\phi + x\int_o^x T_p[M^*](x,y)\phi(y)dy\right\} \\ &\quad + (M^* - M')\phi + x\int_o^x T_p[M^* - M](x,y)\phi(y)dy. \\ &= x^2 T_p[M^*](x,x)\phi(x) + x\int_o^x S_p[M](x,y)\phi(y)dy + M(x)\phi'(x) \end{aligned}$$

58 where $S_p[M](x,y) = T_p[2M^* - M](x,y) + x\dfrac{\partial}{\partial x} T_p[M^*](x,y)$

As

$$\begin{aligned} T_p[M^*](x,x) &= \gamma_p \frac{M^{*'}(x)}{x} \int_o^{\frac{\pi}{2}} \sin^{2p+2}\theta \cos^{-2p}\theta d\theta \\ &= \left(p + \frac{1}{2}\right)\frac{M^{*'}(x)}{x}, \\ B_p MD\mathscr{B}_p[\phi] &= M(x)\phi'(x) + \left(p + \frac{1}{2}\right)[M'(x) - M^*(x)]\phi(x) \\ &\quad + x\int_o^x S_p[M](x,y)\phi(y)dy \end{aligned}$$

Also $B_p[N(x)\psi] = B_p N\mathscr{B}_p\phi = N(x)\phi(x) + x\int_o^x T_p[N](x,y)\phi(y)dy$

Hence the differential equation in w has the form

$$\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} + M(x)\frac{\partial w}{\partial x} + \left(P + \frac{1}{2}\right)[M'(x) - M^*(x)]w$$
$$+ N(x)w + x\int_o^x \left\{S_p[M](x,\xi) + T_p[N](x,\xi)\right\} w(\xi,y)d\xi = 0$$

Let $Q_p(x) = N(x) + \left(p + \frac{1}{2}\right)\left(M'(x) - \frac{M(x)}{x}\right) \in \xi_*$ and $R_p(x,\xi) = S_p[M](x,\xi) + T_p[N](x,\xi)$. We see that Problem 1 is equivalent to the determination of the indefinitely differentiable solution even in x and y the integro differentiable equation

$$\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} M(x)\frac{\partial w}{\partial x} + Q_p(x)w + x\int_o^x R_p(x,\xi,)w(\xi,y)d\xi = 0$$

with the condition $w(0,y) = g(y)$.

It is easy and classical to transform this problem to a purely integral equation of Valterra type and the solution is obtained by the method of successive approximation. It can be verified that all the conditions of w are verified . A process completely analogous gives the solution of Problem 2. **59**

3 Continuation of T_p

For $f \in \mathscr{E}_*$ fixed we define by induction the functions $g_n(x,y,\theta)$ as follows

$$g_o(x,y,\theta) = f_1(z) = \frac{f'(z)}{z} \text{ where } z = \sqrt{x^2\sin^2\theta + y^2\cos^2\theta}$$
$$\sin^{2p}\theta\cos\theta g_1(x,y,\theta) = \frac{\partial}{\partial\theta}\left[\sin^{2p+1}\theta g_o(x,y,\theta)\right]$$

In general

$$\sin^{2p-2n+2}\theta\cos\theta\, g_n(x,y,\theta) = \frac{\partial}{\partial\theta}\Big[\sin^{2p-2n+3}\theta g_{n-1}(x,y,\theta)$$

i.e., $$g_1(x,y,\theta) = (2p+1)f_1(z) + (x^2-y^2)\sin^2\theta f_2(z)$$

where $$f_2(z) = \frac{f_1'(z)}{z} \in \mathscr{E}_*.$$

In fact we can prove immediately by induction on n the following

Lemma 1. *$g_n(x,y,\theta)$ is a linear combination of $f_1, f_2, \ldots, f_{n+1}$ with*
60 *coefficients which are polynomials in x^2, y^2 and $\sin^2\theta$ where $f_n(z) = \dfrac{1}{z} f_{n-1}(z) \in \mathscr{E}_*$.*

Corollary. *The functions $g_n(x,y,\theta)$ are indefinitely differentiable in x, y, θ and even in x and y for $(x,y) \in R^2$ and $\theta \in [0, \frac{\pi}{2}]$.*

Definition.

$$T_p^{(n)}[f](x,y) = \frac{(-1)^n \gamma_p}{(2p-1)(2p-3)\cdots(2p-2n+1)} \int_0^{\frac{\pi}{2}} \cos^{-2p+2n}\theta \sin^{2p-2n+2}\theta g_n(x,y,\theta) d\theta$$

The integral converges for $\frac{2n-3}{2} < \operatorname{Re} p < \frac{2n+1}{2}$ and $T_p^{(n)} \in \mathscr{L}(\mathscr{E}_*, E)$ where E is the space of indefinitely differentiable functions of two variables x, y which are also even in x and y with the usual topology of uniform convergence on every compact subset of R^2 of functions together with their derivatives. It can be verified that $p \to T_p^{(n)}$ is an analytic function for p in the strip in view of the fact that $p \to \dfrac{\gamma_p}{(2p-1)\cdots(2p-2n+1)}$ is an entire function.

Lemma 2. *In each strip $\frac{2n-3}{2} < \operatorname{Re} p < \frac{2n-1}{2}$, we have $T_p^{(n)} = T_p^{(n-1)}$*

$$T_p^{(n-1)}[f](x,y) = \frac{(-1)^{n-1}\gamma_p}{(2p-1)(2p-3)\cdots(2p-2n+3)} \int_0^{\frac{\pi}{2}} \cos^{-2p+2n-2}\theta \sin^{2p-2n+4}\theta g_{n-1}(x,y,\theta) d\theta$$

61 The integral can be written as

$$\int_0^{\frac{\pi}{2}} \cos^{-2p+2n-2}\theta \sin\theta \sin^{2p-2n+3}\theta g_{n-1}(x,y,\theta)d\theta$$
$$= \frac{-1}{(2p-2n+1)}\int_0^{\frac{\pi}{2}} \cos^{-2p+2n-1}\theta \frac{\partial}{\partial\theta}\left\{\sin^{+2p-2n+3}\theta g_{n-1}(x,y,\theta)\right\} d\theta$$

(integrating by parts, the integrated part being zero for $\frac{2n-3}{2} < \text{Re} p < \frac{2n-1}{2}$). The lemma is now evident by the recurrence formula for g_n. Thus we have

Proposition. *The function $p \to T_p$ defined for $-\frac{3}{2} < \text{Re} p < \frac{1}{2}$ admits an analytic continuation in the half plane $Rep > \frac{-3}{2}$, with values in $\mathscr{L}(\varepsilon_*, E)$. The explicit definition of T_p is given by the formula for $T_P^{(n)}$ for suitable n.*

Analytic continuation of T_p in the half plane $\text{Re} p < \frac{-3}{2}$ is obtained by exactly similar process. We introduce functions

$$h_o(x,y,\theta) = f_1(z), z = (x^2\sin^2\theta + y^2\cos^2\theta)^{\frac{1}{2}},$$
$$\cos^{-2p-2}\theta\sin\theta h_1(x,y,\theta) = \frac{\partial}{\partial\theta}\Big[\cos^{-2p-1}\theta h_o(x,y,\theta)\Big],$$

and by induction,

$$\cos^{-2p-2n}\theta\sin\theta h_n(x,y,\theta) = \frac{\partial}{\partial\theta}\Big[\cos^{-2p-2n+1}\theta h_{n-1}(x,y,\theta)\Big],$$

It is easy to see by induction on n that $h_n(x,y,\theta)$ is a linear com- 62
bination of $f_1, f_2, \ldots, f_{n+1}$, with coefficients which are polynomials in $x^2, y^2, \cos^2\theta$ so that the functions $h_n(x,y,\theta)$ are indefinitely differentiable in x, y, θ and are even in x and y. If we set

$$(n)T_p f(x,y) = \frac{(-1)^n\gamma_p}{(2p+3)(2p+5)\cdots(2p+2n+1)}$$

$$\int_0^{\frac{\pi}{2}} \sin^{2p+2n+2}\theta \cos^{-2p-2n}\theta h_n(x,y,\theta)d\theta$$

for $n \geq 1$, then ${}^{(n)}T_p \in \mathscr{L}(\varepsilon_*, E)$ for $-n - \dfrac{3}{2} < \text{Rep} < -n + \dfrac{1}{2}$ and $p \rightarrow^{(n)} T_p$ is analytic in this strip with values in $\mathscr{L}(\varepsilon_*, E)$ and that for $-n - \dfrac{1}{2} < Rep < -n + \dfrac{1}{2}, {}^{(n-1)}T_p =^{(n)} T_p$. Finally we have the

Theorem. *The function $p \rightarrow T_p$ initially defined for $-\dfrac{3}{2} < Rep < \dfrac{1}{2}$, admits an analytic continuation into an entire function with values in $\mathscr{L}(\varepsilon_*, E)$ and we have the explicit formula for this continuation.*

In particular, we have, by analytic continuation, $T_p[f](x,x) = (p + \dfrac{1}{2})f_1(x)$ for any p.

Corollary. *For $p \neq -1, -2, \ldots,$ the formula*

$$B_p A \mathscr{B}_p f(x) = A(x)f(x) + x\int_o^x T_p[A](x,y)f(y)dy$$

is valid.

63 The first member $B_p A \mathscr{B}_p f(x)$ is defined and is analytic except for $p = -1, -2, \ldots$. The second member is an entire function of p, and the two members are equal for $|Rep| < \dfrac{1}{2}$ so that the corollary follows.

Remark. If $A(x)$ is a constant, we have $T_p[A](x,y) = 0$ and the formula of the corollary is equivalent to $B_p \mathscr{B}_p$ = the identity.

Part II

Topics In Mean-Periodic Functions

Chapter 1

Expansion of a Mean-periodic Function in Series

Introduction.

There is no connection between Part I and Part II of this lecture course. But both these parts are essential for the two-radius theorem which will be proved in the last part. 64

The theory of Mean periodic functions was founded in 1935 (refer to "Functions Moyenne-periodiques" by J. Delsarte in Journal de Mathematique pures et appliques, Vol. 14, 1935). A mean periodic function was then defined as the solution f of the integral equation

$$\int_a^b K(\xi)f(x+\xi)d\xi = 0 \tag{1}$$

where K is " density " given by a continuous function $K(x)$, given on a bounded interval $[a, b]$. It was obvious that the study of an ordinary differential equation with constant coefficients or of periodic functions with period $b - a$ was a problem exactly of the same type as the study of equation (1). It was proved in the paper mentioned above that if K satisfied certain conditions and if f were a solution of (1) i.e. f were mean periodic with respect to K then f is developable in a series of exponential functions which converges towards f uniformly on each interval on 65 which f is continuous. Equation (1) in modern notation is essentially a convolution equation

$$K * f = 0$$

if we replace f by $\overset{\vee}{f}(\overset{\vee}{f}(x) = f(-x))$.

L.Schwartz in 1947 applied the theory of distributions to the theory of Mean periodic functions. He calls a mean periodic function a continuous function f which is a solution of

$$\mu * f = 0$$

where μ is measure with compact support. More generally a continuous function f is mean periodic with respect to the distribution T with compact support if $T * f = 0$. He has also given a new definition of mean periodic function which is important from the topological point of view and the theory developed in his paper (Annals of Mathematics, 1947) is complete in the case of one variable. J.P. Kahane has given a special and delicate development of the theory which gives connection between Mean periodicity and almost periodicity. The extension to the case of several variables is certainly difficult and the first known result in R^n is due to B. Malgrange.

For $x \in R^1, \lambda \in C$ we have,

$$T * (\overset{\vee}{e^{\lambda x}}) = e^{-\lambda x} M(\lambda)$$

66 where $M(\lambda) = \langle T, e^{\lambda\xi} \rangle$ is the Fourier - Laplace transform of the distribution T. When T is a density K or a measure, μ,

$$M(\lambda) = \int_a^b K(\xi) e^{\lambda\xi} d\xi \quad \text{or} \quad M(\lambda) = \int_a^b e^{\lambda\xi} d\mu(\xi)$$

respectively and for periodic function of period a, we have $M(\lambda) = e^{a\lambda} - 1$. In the case of the differential operator with constant coefficients, $M(\lambda)$ is the characteristic polynomial of the operator. In all these cases $M(\lambda)$ is an entire function of λ, of exponential type and behaves like a polynomial if Re λ is bounded and the (simple) zeros of this function give the exponential functions (or the exponential polynomials in the case of multiple zeros) which are mean periodic with respect to the distribution T. It is clear that any linear combination of mean periodic exponentials is also mean periodic and the first problem is to determine, for any mean - periodic function f, the coefficients of the mean periodic exponentials in the expansion of the function.

1 Determination of the coefficients in the formal series

Let T be a distribution with compact support and $M(\lambda)$ its Fourier-Laplace transform. The set of zeros of $M(\lambda)$ will be called the *spectrum* of $M(\lambda)$ and will be denoted by (σ).

For the sake of simplicity we suppose that these zeros are simple. **67**
Let F be a function sufficiently regular for the validity of the scalar product

$$\langle T, \int_o^x e^{\lambda(x-\xi)} F(\xi) d\xi$$

If T is a measure with compact support, F need be only an integrable function and if T is a distribution with compact support and order m, F can be any $(m-1)$ times continuously differentiable function.

Let $F(x) = e^{\alpha x}$ where $\alpha \in C$ is fixed

$$\int_o^x e^{\lambda(x-\xi)} e^{\alpha\xi} d\xi = \frac{e^{\lambda x} - e^{\alpha x}}{\lambda - \alpha}$$

so that $$\langle T, \int_o^x e^{\lambda(x-\xi)} e^{\alpha\xi} d\xi\rangle = \frac{M(\lambda) - M(\alpha)}{\lambda - \alpha} = \tau_\alpha(\lambda) \cdot \tau_\alpha(\lambda)$$

is an integral function of exponential type and for $\alpha, \beta \in (\sigma)$,

$$\begin{aligned} \tau_\alpha(\beta) &= 0 \quad \text{if} \quad \beta \neq \alpha \\ &= M'(\alpha) \quad \text{if} \quad \alpha = \beta. \end{aligned}$$

Let $t_\alpha(\lambda) = \dfrac{1}{M'(\alpha)} \tau_\alpha(\lambda)$ so that $t_\alpha(\beta) = \delta_\alpha^\beta \quad \begin{aligned} &= 1 \text{if } \alpha = \beta \\ &= 0 \text{ if } \alpha \neq \beta \end{aligned}$

Then $t_\alpha(\lambda)$ is an entire function of exponential type and therefore by the theorem of Paley-Wiener, there exists a distribution T_α with compact **68** support whose Fourier-Laplace transform is

$t_\alpha(\lambda)$ i.e. $\langle T_\alpha, e^{\lambda x}\rangle = t_\alpha(\lambda)$.

Thus we see that the system of distributions $\{T_\alpha\}_{\alpha\in(\sigma)}$ and the functions $\{e^{\alpha x}\}_{\alpha\in(\sigma)}$ form a biorthogonal system relative to the distribution T.

If

$$F(x) = \sum_{\alpha\in(\sigma)} c_\alpha e^{\alpha x} \text{ then } c_\alpha = \langle T_\alpha, F\rangle$$
$$= \frac{1}{M'(\alpha)}\langle T, \int_o^x e^{\alpha(x-\xi)}F(\xi)d\xi\rangle.$$

For any F which satisfies suitable regularity conditions (stated at the beginning) we consider the formal expansion

$$F(x) \sim \sum_{\alpha\in(\sigma)} c_\alpha e^{\alpha x}$$

where $$c_\alpha = \frac{1}{M'(\alpha)}\langle T, \int_o^x e^{\alpha(x-\xi)}F(\xi)d\xi\rangle$$

and we have immediately two problems.

Problem 1. If F is mean periodic relatively to a distribution T i.e. if $T * F = 0$ and if we compute the coefficients (c_α) and construct the series $\sum\limits_{\alpha\in(\sigma)} c_\alpha e^{\alpha x}$.

Then what is the significance of this series? If the series converges (in some sense) what is the connection between the sum of the series and the given function F? In particular does there exist a one-to-one
69 correspondence between the mean periodic function F and the system of coefficients $(c_\alpha)_{\alpha\in\sigma}$?

Problem 2. If F is given one the smallest closed interval which contains the support of T it is possible to compute the c_α by the formula. In this case is it possible to extend F into a mean periodic function on R'?

2 Examples

Example 1. We consider a periodic function $F(x)$ with $[0, 1]$ as the interval of periodicity. In this case $T = \delta_1 - \delta_0 (\delta_a$ being the Direct measure at a) and $T * F = F(x + 1) - F(x) = 0$ and $M(\lambda) = e^{\lambda} - 1$ so that the spectrum (σ) is given by $= 2k\pi i, k$ an integer; $M'(\lambda) = e^{\lambda}, M'(\alpha) = 1$ for every $\alpha \in (\sigma)$. We have

$$\begin{aligned} c_\alpha = \langle T_\alpha, F\rangle = \langle T, \int_o^x e^{\alpha(x-\xi)} F(\xi) d\xi \\ = \int_o^1 e^{\alpha(1-\xi)} F(\xi) d\xi = \int_o^1 e^{-2ik\pi\xi} F(\xi) d\xi. \end{aligned}$$

This is the classical formula for the coefficients of the Fourier series for $F(x)$ on the interval $[0, 1]$ and the answers to Problem 1 and Problem 2 are classical.

Example 2. Let T be a distribution in R^1 with compact support which has the property

$$T * F = F(x + 1) - kF(x) - \int_0^1 K(\xi) F(x + \xi) d\xi \tag{2}$$

where k is a constant $\neq 0$ and $K(x)$ is continuously differentiable and 70
$T * F$ is a function defined by

$$T * F(x) = \langle T_\xi, F(x + \xi)\rangle = T_\xi . F(x + \xi).$$

We shall study in this case the spectrum and the formal development in series of exponentials for a function F mean periodic with respect to T.

a) Let K be continuous in $[0, 1]$.

The definition $T * e^{\lambda x} = e^{+\lambda x} M(\lambda)$ gives

$$M(\lambda) = e^{\lambda} - k - \int_o^1 K(\xi) e^{\lambda\xi} d\xi \tag{3}$$

or $$e^{-\lambda} M(\lambda) - 1 = -ke^{-\lambda} - \int_o^1 K(\xi) e^{\lambda(\xi-1)} d\xi$$

and if $\lambda = \lambda_o + i\lambda_1$,

$$\left|e^{-\lambda}M(\lambda) - 1\right| \leq \left|k\right|e^{-\lambda o} + \int_o^1 \left|K(\xi)\right|e^{\lambda_o(\xi-1)}d\xi$$

The second member of this inequality $\to 0$ when $\lambda_0 \to +\infty$ so that the real parts of the zeros of $M(\lambda)$ are bounded above.

71 Similarly from $\left|M(\lambda) + k\right| \leq e^{\lambda o} + \int_o^1 \left|K(\xi)\right|e^{\lambda o\xi}d\xi$, we see that the real parts of the zeros are bounded below. Thus the spectrum lies in a vertical band of complex numbers with real parts bounded.

b) Let K be once continuously differentiable in $[0, 1]$. Integrating by parts the integral in (3),

$$M(\lambda) = e^{\lambda} - k - \frac{K(1)e^{\lambda} - K(0)}{\lambda} + \frac{1}{\lambda}\int_o^1 K'(\xi)e^{\lambda}d\xi = e^{\lambda} - k - \frac{M_1(\lambda)}{\lambda}$$

where $M_1(\lambda)$ is an entire function of exponential type which remains bounded when the real part λ_0 of λ remains bounded. The zeros of $M(\lambda)$ are therefore asymptotic with the zeros of $e^{\lambda}-k$ i.e. with $\alpha+2\pi ih$, where α is a determination of $\log\alpha$ and $h = 0, \pm1, \pm2, \ldots$. Let α_h be the zero of $M(\lambda)$ near $\alpha+2\pi ih$. Then the convergence of the series $\sum\limits_{h=-\infty}^{+\infty} \dfrac{1}{|\alpha + 2\pi ih|^2}$ implies the convergence of $\sum\limits_{h=-\infty}^{+\infty} \dfrac{1}{|\alpha_h|^2}$.

c) Let F be $(C, 2)$ in $[0, 1]$ and K be $(C, 1)$. We consider the expansion of $F(x)$ in terms of the exponentials $\{e^{\alpha_h x}\}_{h=0,\pm1,\pm2,\ldots}(\alpha_h \in (\sigma))$, mean periodic with respect to T.

$$F(x) \sim \sum_{h--\infty}^{+\infty} A_h e^{\alpha n^x} \tag{4}$$

where $$A_h = \frac{1}{M'(\alpha_h)}\langle T, \int_o^x e^{\alpha_h(x-\xi)}F(\xi)d\xi \tag{5}$$

72 We have $M'(\lambda) = e^{\lambda} - \int_o^1 K(\xi)e^{\lambda\xi}d\xi$

since $M(\alpha_h) = 0$,

$$e^{\alpha_h} = k + \int_o^1 K(\xi)e^{\alpha_h\xi}d\xi \tag{6}$$

and $$M'(\alpha_n) = k + \int_o^1 (1-\xi)K(\xi)e^{\alpha_h \xi} d\xi$$

Integrating by parts integral and observing that $K'(\xi)$ is bounded in $[0,1]$ since it is continuous, it is clear that $M'(\alpha_h) = k + \dfrac{M_2(\alpha_h)}{\alpha_h}$ where $M_2(\alpha_h)$ is bounded when $Re\alpha_h$ is bounded. Also from (a), we know that if $|h| \to \infty, |\alpha_h| \to \infty$ with $|Re\alpha_h|$ bounded. Hence

$$A_h = \frac{B_h}{M'(\alpha_h)} \sim \frac{1}{K} B_h \quad \text{as} \quad |h| \to \infty \cdots \tag{7}$$

where $$B_h = \langle T, G_h(x)\rangle, G_h(x) = \int_o^x e^{\alpha_h(x-\xi)} F(\xi) d\xi$$

$$\langle T, G_h(x)\rangle = T * G_h(0) = G_h(1) - kG_h(0) - \int_o^1 K(\xi) G_h(\xi) d\xi$$

$$B_h = \int_o^1 e^{\alpha_h(1-\xi)} F(\xi) d\xi - \int_o^1 \int_o^\xi K(\xi) e^{\alpha_h(\xi-\eta)} F(\eta) d\eta d\xi$$

Now **73**

$$\begin{aligned}\int_o^1 e^{\alpha_h(1-\xi)} F(\xi) d\xi &= -\frac{F(1)}{\alpha_h} + \frac{e^{\alpha_h} F(o)}{\alpha_h} + \frac{1}{\alpha_h} \int_o^1 e^{\alpha_h(1-\xi)} F'(\xi) d\xi \\ &= -\frac{1}{\alpha_h}[F(1) - e^{\alpha_h} F(0)] - \frac{1}{\alpha_h^2}[F'(1) - e^{\alpha_h} F'(0)] \\ &\quad + \frac{1}{\alpha_h^2} \int_o^1 e^{\alpha_h(1-\xi)} F''(\xi) d\xi\end{aligned}$$

and $$\begin{aligned}&- \int_o^1 \int_o^\xi K(\xi) e^{\alpha_h(\xi-\eta)} F(\eta) d\eta d\xi \\ &= \frac{1}{\alpha_h} \int_o^1 K(\xi) F(\xi) d\xi - \frac{F(0)}{\alpha_h} \int_o^1 e^{\alpha_h \xi} K(\xi) d\xi \\ &\quad - \frac{1}{\alpha_h} \int_o^1 \int_o^\xi e^{\alpha_h(\xi-\eta)} K(\xi) F'(\eta) d\eta d\xi \\ &= \frac{1}{\alpha_h} \int_o^1 K(\xi) F(\xi) d\xi - \frac{F(0)}{\alpha_h} \int_o^1 e^{\alpha_h \xi} K(\xi) d\xi\end{aligned}$$

$$+\frac{1}{\alpha_h^2}\int_o^1 K(\xi)F'(\xi)d\xi - \frac{F'(0)}{\alpha_h^2}\int_o^1 e^{\alpha_h\xi}K(\xi)d\xi$$
$$-\frac{1}{\alpha_h^2}\int_o^1\int_o^\xi e^{\alpha_h(\xi-\eta)}K(\xi)F''(\eta)d\eta d\xi$$

74 whence

$$B_h = \frac{1}{\alpha_h}\left\{-F(1)+e^{\alpha_h}F(0)+\int_o^1 K(\xi)F(\xi)d\xi - F(0)\int\limits_o^1 e^{\alpha_h\xi}K(\xi)d\xi\right.$$
$$\frac{1}{\alpha_h 2}\left\{-F'(1)+e^{\alpha_h}F'(0)+\int\limits_o^1 e^{\alpha_h(1-\xi)}F''(\xi)d\xi\right.$$
$$+\int_o^1 K(\xi)F'(\xi)d\xi - F'(0)\int_o^1 e^{\alpha_h\xi}K(\xi)d\xi$$
$$\left.-\int_o^1\int_o^\xi e^{\alpha_h(\xi-\eta)}K(\xi)F''(\eta)d\eta d\xi\right\}$$

substituting for e^{α_h} from (6) the first bracket above becomes
$-F(1)+kF(0)+F(0)\int_o^1 K(\xi)e^{\alpha_h\xi}d\xi+\int_o^1 K(\xi)F(\xi)d\xi$
$-F(0)\int_o^1 e^{\alpha_h\xi}K(\xi)d\xi = -(T*F)(0) = 0$ since F is assumed to be mean periodic with respect to T. Hence $B_h = o\left(\frac{1}{\alpha_h^2}\right)$ provided F' and F'' are bounded in $[0,1]$. As $|Re\alpha_h|$ remains bounded when $|h|\to\infty$, from (7) it is immediate that the series (4) is comparable with $\sum\limits_{h=-\infty}^{+\infty}\frac{1}{\alpha_h^2}$ and therefore converges uniformly and absolutely on each compact subset of R^1 to a continuous function $F_1(x)H(x) = F(x)-F_1(x)$ is mean periodic with respect to T and the coefficients $c_{\alpha_h}(\alpha_h\in\sigma)$ in the expansion for $H(x)$ in mean periodic exponentials are all zero. Hence $H = 0$ by the uniqueness of development. (Problem 1).

Chapter 2

Mean Periodic Function in R^2

We shall study function of two variables mean periodic with respect 75
to two distributions, and discuss problems 1 and 2 stated in Chapter 1 (page 56). The general case being difficult we shall give solutions of the problems in some special cases when the distributions are of particular type with the spectrum satisfying certain conditions.

Definition. A continuous function F on R^2 is said to be mean periodic with respect to $T_1, T_2 \in \varepsilon'(R^2)$ if it verifies simultaneously the convolution equations

$$T_1 * F = 0, T_2 * F = 0.$$

The Fourier- Laplace transforms

$$M_i(\lambda) = \langle T_i, e^{<\lambda, x>} \rangle$$

where $x = (x_1, x_2) R^2, \lambda = (\lambda_1, \lambda_2) \in C^2, < \lambda, x >= \lambda_1 x_1 + \lambda_2 x_2$ of $T_i (i = 1, 2)$ are entire functions of λ_1, λ_2 of exponential type. The *spectrum* (σ) is defined by $M_i(\lambda) = 0, i = 1, 2,$. In all that follows we shall restrict ourselves to the case when i) (σ) is countable and (ii) (σ) is simple i.e.
$\alpha \in (\sigma)$ is a simple zero of $M_1(\lambda)$ and $M_2(\lambda)$ and the Jacobian 76

$D(\lambda) = \begin{vmatrix} \dfrac{\partial M_1}{\partial \lambda_1} & \dfrac{\partial M_1}{\partial \lambda_2} \\ \dfrac{\partial M_2}{\partial \lambda_1} & \dfrac{\partial M_2}{\partial \lambda_2} \end{vmatrix}$ does not vanish at any $\alpha \in (\sigma)$. We define

$$t_\alpha(\lambda) = \frac{1}{D(\alpha)(\lambda_1 - \alpha_1)(\lambda_2 - \alpha_2)} \quad \frac{J(M, M)}{J(\lambda, \alpha)}$$

where

$$\frac{J(M, M)}{J(\lambda, \alpha)} = \begin{vmatrix} M_1(\lambda_1, \lambda_2) - M_1(\alpha_1, \lambda_2) & M_1(\alpha_1, \lambda_2) - M_1(\alpha_1, \alpha_2) \\ M_2(\lambda_1, \lambda_2) - M_2(\alpha_1, \lambda_2) & M_2(\alpha_1, \lambda_2) - M_2(\alpha_1, \alpha_2) \end{vmatrix}$$

is the determinant of Jacobi. It is clear that

$$\begin{aligned} t_\alpha(\beta) &= 0 \quad \text{for} \quad \alpha, \beta \in (\sigma), \alpha \neq \beta \\ &= 1 \quad \text{for} \quad \alpha, \beta \in (\sigma), \alpha = \beta \end{aligned}$$

For $\alpha_\in(\sigma)$, $t_\alpha(\lambda)$ is an entire function of λ of exponential type and by Paley-Wiener theorem, $t_\alpha(\lambda)$ is the Fourier Laplace transform of a distribution $T_\alpha \in \mathscr{E}^1(R^2)$. $\{T_\alpha\}_{\alpha \in (\sigma)}$ together with the exponentials $\left\{e^{\langle \alpha, x \rangle}\right\}_{\alpha \in (\sigma)}$ mean periodic with respect to T_1, T_2 form a biorthogonal system, i. e.

$$\begin{aligned} \langle T_\alpha, e^{<\beta, x>} \rangle &= 0 \text{ for } \alpha, \beta \in (\sigma), \alpha \neq \beta \\ \langle T_\alpha, e^{<\alpha, x>} \rangle &= 1 \quad \text{for} \quad \alpha \in (\sigma) \end{aligned} \tag{1}$$

77 If F is mean periodic with respect to T_1, T_2 and if we suppose the existence of an expansion of F in a series of mean periodic exponentials

$$F(x) \sim \sum_{\alpha \in (\sigma)} c_\alpha \quad e^{<\alpha, x>} \tag{2}$$

we have formally

$$c_\alpha = \langle T_\alpha, F \rangle \tag{3}$$

Let $s_\alpha(\lambda) = \dfrac{1}{D(\alpha} \dfrac{J(M, M)}{J(\lambda, \alpha)}$ so that

$$s_\alpha(\lambda) = (\lambda_1 - \alpha_1)(\lambda_2 - \alpha_2) t_\alpha(\lambda)$$

For α fixed in (σ), $s_\alpha(\lambda)$ is an entire function of λ of exponential type. Let S_α be the distribution in $\mathscr{E}'(R^2)$ whose Fourier-Laplace image is $s_\alpha(\lambda)$. Denoting $\dfrac{\partial}{\partial x_i}$ by $D_i (i = i, 2)$, we have

$$\begin{aligned}\langle (D_1 + \alpha_1)(D_2 + \alpha_2) T_\alpha, e^{\lambda x}\rangle &= (D_1 D_2 + \alpha_1 D_2 + \alpha_2 D_1 + \alpha_1\alpha_2) T_\alpha, e^{\lambda x}\rangle \\ &= \langle T_\alpha, (D_1 D_2 - \alpha_1 D_2 - \alpha_2 D_1 + \alpha_1\alpha_2) e^{\lambda x}\rangle\end{aligned}$$

(by the definition of the derivative of a distribution $\langle D^p T, \phi\rangle = (-1)^p$ 78
$\langle T, D^p \phi\rangle$, refer to 'theory des distributions' by L. Schwartz)

$$= \langle T_\alpha, (\lambda_1 - \alpha)(\lambda_2 - \alpha_2) e^{\lambda x}\rangle = (\lambda_1 - \alpha_1)(\lambda_2 - \alpha_2) t_\alpha(\lambda)$$

Hence

$$S_\alpha = (D_1 + \alpha_1)(D_2 + \alpha_2) T_\alpha \tag{4}$$

(4) is equivalent to

$$e^{-\langle\alpha, x\rangle} D_1 D_2 \left[e^{<\alpha, x>} T_\alpha\right] = S_\alpha \tag{5}$$

For

$$\langle e^{-\langle\alpha, x\rangle}\left[D_1 D_2\, e^{\langle\alpha, x\rangle} T_\alpha\right], \phi\rangle = \langle T_\alpha, e^{<\alpha, x>} D_1 D_2 e^{-\langle\alpha, x\rangle}\phi\rangle$$

(by the definition of the multiplicative product of a distribution by a function and the derivative of a distribution; refer to 'Theory des distributions')

$$\begin{aligned}&= \langle T_\alpha, (\alpha_1\alpha_2 - \alpha_2 D_1 - \alpha_1 D_2 + D_1 D_2)\phi\rangle \\ &= \langle (D_1 + \alpha_1)(D_2 + \alpha_2) T_\alpha, \phi >=< S_\alpha, \phi\rangle\end{aligned}$$

Let $G(x)$ be any solution of the partial differential equation

$$e^{\langle\alpha, x\rangle} D_1 D_2 \left[e^{-\langle\alpha, x\rangle} G(x)\right] = F(x) \tag{6}$$

$$\text{or} \qquad DG = F \tag{6'}$$

where $D = \dfrac{\partial^2}{\partial x_1 \partial x_2} - \alpha_1 \dfrac{\partial}{\partial x_2} - \alpha_2 \dfrac{\partial}{\partial x_1} + \alpha_1\alpha_2$ is a differential operator 79 with constant coefficients. For $F \in \mathscr{E}$, we can choose $G \in \mathscr{E}$. We can actually take G to be

$$G(x_1, x_2) = \int_{a_1}^{x_1} \int_{a_2}^{x_2} e^{\alpha_1(x_1-\xi_1)+\alpha_2(x_2-\xi_2)} F(\xi_1, \xi_2) d\xi_1 d\xi_2 \tag{7}$$

where a_1, a_2 are arbitrary.

Thus we obtain from

$$\begin{aligned}\langle T_\alpha, F\rangle &= \langle T_\alpha, e^{\langle\alpha,x\rangle} D_1 D_2 \left[e^{-\langle\alpha,x\rangle} G(x)\right]\rangle \\ &= e^{-\langle\alpha,x\rangle} D_1 D_2 \left[e^{\langle\alpha,x\rangle} T_\alpha\right], G\rangle = \langle S_\alpha, G\rangle,\end{aligned}$$

the formula

$$c_\alpha =< S_\alpha, \int_{a_1}^{x_1} \int_{a_2}^{x_2} e^{\alpha_1(x_1-\xi_1)+\alpha_2(x_2-\xi_2)} F(\xi_1, \xi_2) d\xi_1 d\xi_2 > \tag{8}$$

which is a natural generalization of the formula know in the case of R^1.

80 **Remark.** The distributions S are completely explicit. We have

$$S_\alpha(\lambda) = \frac{1}{D(\alpha)} M_1(\lambda_1, \lambda_2) M_2(\alpha_1, \lambda_2) - M_2(\lambda_1, \lambda_2) M_1(\alpha_1, \lambda_2)$$

and $S_\alpha = \dfrac{1}{D(\alpha)} \left[T_1 * \sum_2 - T_2 * \sum_1\right]$ where T_1 and T_2 are the given distributions and $\sum_2$ for instance is a distribution in the variable x_2 determined by its Fourier-Laplace image $M_2(\alpha_1, \lambda_2)$.

Theorem. *The mean periodic exponentials $\{e^{<\alpha,x>}\}_{\alpha\in(\sigma)}$ form a free system of functions in $\mathscr{E}$.*

In fact they form a biorthogonal system with the distributions $\{T_\alpha\}_{\alpha\in(\sigma)}$ in $\mathscr{E}'$ and it is clear from (1) that no $e^{\langle\alpha,x\rangle}$ is in the closed subspace generated by $\left\{e^{<\beta,x>}\right\}_{\substack{\beta\in(\sigma)\\ \beta\neq\alpha}}$.

Remark. It is possible to choose the solution $I_\alpha(f) = G$ of (6) such that for $F \in \mathscr{E}$, the function $\lambda \to I_\alpha(F)$ from C^2 to $\mathscr{E}$ is an entire function. In fact it suffices to take G as in (7). In this case

$$c_\alpha \quad e^{\langle\alpha,x\rangle} = \frac{e^{\langle\alpha,x\rangle}}{D(\alpha)} \langle S_\alpha, I_\alpha(F)\rangle$$

81 $\lambda \to S_\lambda$ is an entire function of λ with values in $\mathscr{E}'$ and $\lambda \to I_\lambda(F)$ is an entire function of λ with values in $\mathscr{E}$. Hence

$$\mathscr{F}(x,\lambda) = \frac{e^{\langle\lambda,x\rangle}\langle S_\lambda, I_\lambda(F)\rangle}{M_1(\lambda)M_2(\lambda)}$$

is a meromorphic function of λ in which the 'local residues' in the sense of Poincare are precisely $c_\alpha \quad e^{<g\alpha,x>}$.

The Problem is now as follows.

If F is given in $\mathscr{E}$ mean periodic with respect to T_1 and T_2 then is F then sum of the series $\sum\limits_{\alpha\in(\sigma)} e_\alpha \quad e^{<\alpha,x>}$?

If for x fixed, the integrals of $\mathscr{F}(x,\lambda)$ over a system of varieties in C^2 tending to infinity have for limit $F(x)$ and if the global Cauchy theorem were true in C^2, then the answer to this question would be in affirmative. In this manner, the problem is equivalent to the global Cauchy theorem in C^2 and the answer is completely unknown.

Chapter 3
The Heuristic Method

1

We consider the problem of continuation of a function given on a subset of R^1 into a function defined on the whole of the real line, mean periodic with respect to a distribution with compact support (Problem 2, page 56) $T \in \mathscr{E}'(R^1)$ has its support contained in a finite closed interval say [0, 1]. Then the distributions $\{T_\alpha\}$, for α of T have their supports contained in [0, 1]. Let F be a function given in $[0,1]$ so that the computation of the coefficients $c_\alpha = \langle T_\alpha, F\rangle$ is possible. The heuristic point of view is the following. We suppose that problem 2 is completely solved for the distribution T and for the function F i.e. any function F given on the minimal set $[0,1]$ admits of an extension $\tilde{F}$ in the whole of R^1 as a function mean periodic with respect to T. F is not necessarily infinitely differentiable, even if F may be in $[0,1]$ but it is probably sufficiently regular piecewise. We have $T * F = 0$ and in a certain sense $\tilde{F} = \sum\limits_{\alpha\in(\sigma)} c_\alpha e^{\alpha x}$. In particular, for $x \in (0,1)$ we have 82

$$F(x) = \tilde{F}(x) = \sum_{\alpha\in(\sigma)} c_\alpha \ e^{\alpha x} \tag{1}$$

Now we suppose the possibility of the change of the distribution T, 83
with its compact support keeping in a fixed interval $[0,1]$ and we shall obtain several representations of F in series of mean periodic exponentials (relative to several distributions). For instance we suppose that T

depends on a parameter ε which tends to zero, but that the support of T is contained in $[0, 1]$. In such a situation, the spectrum (σ) and therefore each term in the right hand member of (1) depends on ε whereas the left hand member is fixed whatever ε. Thus we have

$$d\left[\sum_{\alpha\in(\sigma)} c_\alpha\, e^{\alpha x}\right] = 0 \quad x \in (0, 1)$$

where the differential d is related to the variation of the distribution T. Formal computation gives

$$\sum_{\alpha\in(\sigma)} d[c_\alpha]e^{\alpha x} + \sum_{\alpha\in(\sigma)} x\, e^{\alpha x}\, c_\alpha d\alpha = 0$$

But applying our heuristic concept to the distribution T and the function $xe^{\alpha x}$ we obtain

$$xe^{\alpha x} = \sum_{\beta\in(\sigma)} k_{\alpha\beta}\, e^{\beta x}, \text{ for } x \in (0, 1),$$

which gives

$$\sum_{\alpha\in(\sigma)} \left\{dc_\alpha + \sum_{\beta\in(\sigma)} c_\beta k_{\beta\alpha} d\beta\right\}^{c^{\alpha x}} = 0$$

84 so that

$$dc_\alpha + \sum_{\beta\in(\sigma)} c_\beta\ k_{\beta\alpha}\, d\beta = 0 \tag{2}$$

But

$$c_\alpha = \langle T_\alpha, \tilde{F}\rangle \quad \text{where}$$
$$\langle T_\alpha, e^{\lambda x}\rangle = t_\alpha(\lambda) = \frac{M(\lambda)}{(\lambda-\alpha)M'(\alpha)}$$

As support $T_\alpha \subset [0, 1], c_\alpha = \langle T_\alpha, R\rangle$

so that

$$dc_\alpha = \langle dT_\alpha, F\rangle$$

since F is independent of ε and (2) becomes

$$\langle dT_\alpha + \sum_{\beta\in(\sigma)} k_{\beta\alpha} \; T_\beta \; d\beta, F\rangle = 0 \tag{3}$$

(3) holds for any arbitrarily chosen F on $[0,1]$ and for the distribution $dT_\alpha + \sum_{\beta\in(\sigma)} k_{\beta\alpha} \; T_\beta \; d\beta$ with support contained in $[0,1]$ whenever the scalar product is meaningful $\Big\{$ whenever F is sufficiently regular in $[0,1]$ in order that the scalar product be defined $\Big\}$. Hence 85

$dT_\alpha + \sum_{\beta\in(\sigma)} k_{\beta\alpha} \; T_\beta \; d_\beta = 0$ (in the sense of distributions)

Taking Fourier-Laplace image,

$$dT_\alpha(\lambda) + \sum_{\beta\in(\sigma)} k_{\beta\alpha} \; T_\beta(\lambda) \; d\beta = 0$$

Now

$$k_{\beta\alpha} = \frac{1}{M'(\alpha)}\langle T, \int_0^x \; e^{\alpha(x-\xi)}\xi \; e^{\beta\xi} \; d\xi\rangle$$

But

$$e^{\alpha x}\int_0^x \; e^{(\beta-\alpha)\xi} d\xi = \frac{xe^{\beta_x}}{(\beta-\alpha)} - e^{-\alpha x}\int\limits_o^x \frac{e^{(\beta-\alpha)\xi}}{(\beta-\alpha)} d\xi$$

$$= \frac{xe^{\beta x}}{\beta-\alpha} - \frac{e^{\beta x}-e^{\alpha x}}{(\beta-\alpha)^2}$$

and
$$k_{\beta\alpha} = \frac{1}{M'(\alpha)}\left\{\frac{1}{(\beta-\alpha)}\langle T, xe^{\beta x}\rangle - \frac{M(\beta)-M(\alpha)}{(\beta-\alpha)^2}\right\}$$

$$= \frac{1}{M'(\alpha)}\left\{\frac{M'(\beta)}{\beta-\alpha} - \frac{M(\beta)-M(\alpha)}{(\beta-\alpha)^2}\right\}$$

Hence 86

$$k_{\beta\alpha} = \frac{1}{\beta-\alpha}\;\frac{M'(\beta)}{M'(\alpha)} \;\text{ for }\; \beta\neq\alpha$$

and
$$k_{\alpha\alpha} = \frac{1}{2}\;\frac{M''(\alpha)}{M'(\alpha)}$$

We have also

$$dT_\alpha(\lambda) = d\frac{M(\lambda)}{(\lambda-\alpha)M'(\alpha)}$$
$$= \frac{d[M(\lambda)]}{(\lambda-\alpha)M'(\alpha)} + \frac{M(\lambda)}{(\lambda-\alpha)^2M'(\alpha)}d\alpha - \frac{M(\lambda)}{(\lambda-\alpha)[M'(\alpha)]^2}d[M'(\alpha)]$$

and

$$d[M'(\alpha)] = d[M'(\lambda)]_{\lambda=\alpha} + M''(\alpha)d\alpha$$

This gives

$$d[T_\alpha(\lambda)] + \frac{1}{2}\frac{M''(\alpha)}{M'(\alpha)}\,\frac{M(\lambda)}{(\lambda-\alpha)M'(\alpha)}d\alpha + \sum_{\substack{\beta\in(\sigma)\\ \beta\neq\alpha}} \frac{M(\lambda)}{(\lambda-\beta)(\beta-\alpha)M'(\alpha)}d\beta = 0$$

or

$$\frac{d[M(\lambda)]}{(\lambda-\alpha)M'(\alpha)} - \frac{M(\lambda)(d[M'(\lambda)])}{(\lambda-\alpha)[M'(\alpha)]^2}\lambda$$
$$= \alpha + \left\{\frac{M(\lambda)}{(\lambda-\alpha)^2M'(\alpha)} - \frac{M(\lambda)M''(\alpha)}{(\lambda-\alpha)[M'(\alpha)]^2}\right\}$$
$$d\alpha + \frac{1}{2}\,\frac{M''(\alpha)}{M'(\alpha)}\,\frac{M(\lambda)}{(\lambda-\alpha)M'(\alpha)}d\alpha + \sum_{\substack{\beta\in(\sigma)\\ \beta\neq\alpha}} \frac{M(\lambda)}{(\lambda-\beta)(\beta-\alpha)M'(\alpha)}d\beta = 0$$

87 After simplification we finally obtain the formula

$$\frac{d[M(\lambda)]}{M(\lambda)} - \frac{(d[M'(\lambda)])_{\lambda=\alpha}}{M'(\alpha)} + \left\{\frac{1}{\lambda-\alpha} - \frac{1}{2}\,\frac{M''(\alpha)}{M'(\alpha)}\right\}$$
$$d\alpha + \sum_{\substack{\beta\in(\sigma)\\ \beta\neq\alpha}} \frac{\lambda-\alpha}{(\lambda-\beta)(\beta-\alpha)}d\beta = 0\ldots\ldots \qquad (\mathscr{F})$$

2

The formula ($\mathscr{F}$) is obtained by a purely heuristic process and the following example will serve as a partial verification of the computation.

Suppose that $M(\lambda)$ is a polynomial. (In this case T is a differential operator with constant coefficients)

$M(\lambda) = \prod\limits_{i=1}^{n}(\lambda - \beta_i)$. Let $\beta_i \neq \beta_j$ for $i \neq j$

$$\frac{dM(\lambda)}{M(\lambda)} = -\sum_{j=1}^{n} \frac{1}{\lambda - \beta_j} d\beta_j$$

$$M'(\lambda) = \sum_j (\lambda - \beta_1)(\lambda - \beta_2) \ldots (\widehat{\lambda - \beta_n}) \ldots (\lambda - \beta_n)$$

and

$$(d[M'(\lambda)])_{\lambda=\alpha=\beta_i} = -\Bigg(\sum_{j,k} (\lambda - \beta_1) \ldots (\widehat{\lambda - \beta_j}) \ldots (\widehat{\lambda - \beta_k}) \ldots$$

$$(\lambda - \beta_n) d\beta_k\Bigg)_{\lambda=\beta_i} - \sum_j (\beta_i - \beta_1) \ldots (\widehat{\beta_i - \beta_j}) \ldots (\beta_i - \beta_k) d\beta_i$$

But **88**

$$M'(\beta_i) = M'(\alpha) = (\beta_i - \beta_1) \ldots (\widehat{\beta_i - \beta_i}) \ldots (\beta_i - \beta_n)$$

$$\frac{(d[M'(\lambda)])_{\lambda=\alpha}}{M'(\alpha)} = -\sum_{k \neq i} \frac{1}{\beta_i - \beta_k} d\beta_k - \Big[\sum_{j \neq i} \frac{1}{\beta_i - \beta_j}\Big] d\beta_i$$

$$M''(\lambda) = (\lambda - \beta_1) \ldots (\widehat{\lambda - \beta_j}) \ldots (\widehat{\lambda - \beta_k}) \ldots (\lambda - \beta_n)_{k \neq j}$$

so that $$M''(\lambda) = 2\sum_{k \neq i} (\beta_i - \beta_j) \ldots (\widehat{\beta_i - \beta_j}) \ldots (\widehat{\beta_i - \beta_k}) \ldots (\beta_i - \beta_n)$$

and $$\frac{M''(\alpha)}{M'(\alpha)} = 2\sum_{k \neq i} \frac{1}{\beta_i - \beta_k}.$$

Thus we can write

$$\frac{d[M(\lambda)]}{M(\lambda)} = -\sum_{\beta \in (\sigma)} \frac{1}{\lambda - \beta} d\beta$$

$$\frac{(d[M'(\lambda)])_{\lambda=\alpha}}{M'(\alpha)} = -\sum_{\substack{\beta \\ \beta \neq \alpha}} \frac{1}{\alpha - \beta} d\beta - \frac{1}{2} \frac{M''(\alpha)}{M'(\alpha)} d\alpha$$

and 89

$$\frac{d[M(\lambda)]}{M(\lambda)} - \frac{(d[M'(\lambda)])_{\lambda=\alpha}}{M'(\alpha)} - \frac{1}{2}\frac{M''(\alpha)}{M'(\alpha)}d\alpha$$
$$= -\sum_{\beta\neq\alpha}\frac{1}{\lambda-\beta}d\beta + \sum_{\beta\neq\alpha}\frac{1}{\alpha-\beta}d\beta - \frac{1}{\lambda-\alpha}d\alpha$$
$$= -\frac{1}{\lambda-\alpha}d\alpha + \sum_{\substack{\beta\\ \beta\neq\alpha}}\frac{\lambda-\alpha}{(\lambda-\beta)(\alpha-\beta)}d\beta$$
$$= -\frac{1}{\lambda-\alpha}d\alpha - \sum_{\substack{\beta\\ \beta\neq\alpha}}\frac{\lambda-\alpha}{(\lambda-\beta)(\beta-\alpha)}d\beta$$

This is exactly the formula ($\mathscr{F}$).

Remark. It is important to note that in this case, the support of T is only $\{0\}$.

3 The general formula in R^2 by the heuristic process

Let $T_1, T_2 \in \mathscr{E}'(R^2)$ with their spectrum (σ) simple. Suppose that F is a function given on a subset E of R^2, which depends on the supports of T_1 and T_2 so that the computation of $c_\alpha =< T_\alpha, F >, \alpha \in (\sigma)$ is possible (page 64). We can write formally

$$F \approx \sum_{\alpha\in(\sigma)} c_\alpha\, e^{<\alpha,x>}$$

90 suppose that the distributions vary in certain family which depends on a parameter their supports fixed. Then $dF = 0$ gives

$$\sum_{\alpha\in(\sigma)} dc_\alpha e^{<\alpha,x>} + \sum_{\beta\in(\sigma)}\sum_{i=1}^{2} c_\beta\, x_i\, d\beta_i e^{<\beta,x} = 0\, \beta = (\beta_1, \beta_2)$$

But

$$x_i e^{\langle\beta,x\rangle} = \sum_{\beta\in(\sigma)} k_{\beta\alpha_i} e^{<\alpha,x>} \text{ on } E,$$

and so

$$dc_\alpha + \sum_{\beta\in(\sigma)} \sum_{i=1}^{2} k_{\beta\alpha_i} d\beta_i \; c_\beta = 0.$$

The coefficients c_α are given by $c_\alpha = \langle F, T_\alpha \rangle$ and

$$dT_\alpha + \sum_{\beta\in(\sigma)} \sum_{i=1}^{2} k_{\beta\alpha_i} \; T_\beta \; d\beta_i = 0$$

Taking Fourier-Laplace transform,

$$d(t_\alpha(\lambda)) + \sum_{\beta\in(\sigma)} \sum_{i=1}^{2} k_{\beta\alpha_i} \; t_\beta(\lambda) d\beta_i = 0$$

Now $k_{\beta\alpha_i} = \langle x_i e^{<\beta,x\rangle}, T_\alpha \rangle = \dfrac{\partial}{\partial\lambda_i}\left[< T_\alpha, e^{<\lambda,x>}\right]_{\lambda=\beta} = \left[\dfrac{\partial}{\partial\lambda_i} t_\alpha(\lambda)\right]_{\lambda=\beta}$ and the general formula is

$$d[t_\alpha(\lambda)] + \sum_{\beta\in(\sigma)} \sum_{i=1}^{2} \; t_\beta(\lambda) \frac{\partial}{\partial\lambda_i} t_\alpha(\lambda)_{\lambda-\beta} d\beta_i = 0 \qquad (\mathscr{G})$$

where $\left\{t_\alpha(\lambda)\right\}_{\alpha\in(\sigma)}$ is a biorthogonalising system of functions on the 91
spectrum: $t_\alpha(\beta) = \begin{cases} 0, & \beta \neq \alpha \\ 1, & \beta = \alpha \end{cases}$

4

We now again consider the formula ($\mathscr{F}$) and give another interpretation of the formula by making precise the variation of T. Let $U \varepsilon \mathscr{E}'(R^1)$ be a distribution with support contained in $[0, 1]$ and ε a parameter infinitely small. The distribution $T - \varepsilon U$ has Fourier-Laplace transform $M(\lambda) - \varepsilon A(\lambda)$ where $A(\lambda)$ is the Fourier-Laplace transform of U and support $(T - \varepsilon U) \subset [0, 1]$. If $\{\alpha\}$ is the spectrum of $T, \{\alpha + d\alpha\}$ is the spectrum of $T - \varepsilon U$. $M(\alpha + d\alpha) - \varepsilon A(\alpha + d\alpha) = 0$ give

$$M(\alpha) + d\alpha M'(\alpha) + \cdots - \in \{A(\alpha) + d\alpha A'(\alpha) + \cdots\} = 0$$

using $M(\alpha) = 0$ and neglecting terms of higher order, we have

$$M'(\alpha)d\alpha - \in A(\alpha) = 0 \text{ so that } d\alpha = \frac{\in A(\alpha)}{M'(\alpha)}$$

substituting in $(\mathscr{F})$ for $d\alpha$,

$$\frac{A(\lambda)}{M(\lambda)} = \frac{A(\lambda)}{(\lambda-\alpha)M'(\alpha)} + \sum_{\substack{\beta\varepsilon(\sigma)\\ \beta\neq\alpha}} \left(\frac{1}{\lambda-\beta} + \frac{1}{\beta-\alpha}\right)\frac{A(\beta)}{M'(\beta)} - \frac{1}{2}\,\frac{M''(\alpha)A(\alpha)}{[M'(\alpha)]^2} - \lim_{\varepsilon\to 0}\frac{1}{2}\left\{\frac{d[M'(\lambda)]_{\lambda=\alpha}}{M'(\alpha)}\right\}$$

92 But $M'(\lambda) + dM'(\lambda) = M' - \varepsilon A'(\lambda)$ and $d[M'(\lambda)] = -\varepsilon A'(\lambda)$ so that finally

$$\frac{A(\lambda)}{M(\lambda)} = \frac{A(\alpha)}{(\lambda-\alpha)M'(\alpha)} + \sum_{\substack{\beta\in(\sigma)\\ \beta\neq\alpha}} \left(\frac{1}{\lambda-\beta} + \frac{1}{\beta-\alpha}\right)\frac{A(\beta)}{M'(\beta)} - \frac{1}{2}\,\frac{M''(\alpha)A(\alpha)}{[M'(\alpha)]^2} + \frac{A'(\alpha)}{M'(\alpha)}\cdots\cdots \quad (\mathscr{F}_1)$$

Remark. The series $\sum\limits_{\substack{\beta\varepsilon(\sigma)\\ \beta\neq\alpha}} \left(\frac{1}{\lambda-\beta} + \frac{1}{\beta-\alpha}\right)\frac{A(\beta)}{M'(\beta)} = 0$ for $\lambda = \alpha$, and it is easy to prove that

$$\lim_{\lambda\to\alpha}\left[\frac{A(\lambda)}{M(\lambda)} - \frac{A(\alpha)}{(\lambda-\alpha)M'(\alpha)}\right] = \frac{A'(\alpha)}{M'(\alpha)} - \frac{1}{2}\,\frac{M''(\alpha)A(\alpha)}{[M'(\alpha)]^2}$$

We have in fact, in the neighbourhood of α

$$A(\lambda) = A(\alpha) + (\lambda-\alpha)A'(\alpha) + \cdots\cdots$$
$$M(\lambda) = (\lambda-\alpha)M'(\alpha) + \frac{1}{2}(\lambda-\alpha)^2 M''(\alpha) + \cdots\cdots$$
$$\frac{A(\lambda)}{M(\lambda)} = \frac{1}{(\lambda-\alpha)}\;\frac{A(\alpha) + (\lambda-\alpha)A'(\alpha) + \cdots\cdots}{M'(\alpha) + \frac{1}{2}(\lambda-\alpha)M''(\alpha) + \cdots\cdots}$$

$$\frac{A(\lambda)}{M(\lambda)} - \frac{A(\alpha)}{(\lambda-\alpha)M'(\alpha)} = \frac{1}{\lambda-\alpha}\left\{\frac{1}{M'(\alpha)}[A(\alpha)+(\lambda-\alpha)A'(\alpha)+\cdots]\right.$$
$$\left.[1-\frac{1}{2}(\lambda-\alpha)\frac{M''(\alpha)}{M'(\alpha)}+\cdots] - \frac{A(\alpha)}{M'(\alpha)}\right\}$$
$$= \frac{1}{\lambda-\alpha}\left\{\frac{1}{M'(\alpha)}[A(\alpha)+(\lambda-\alpha)A'(\alpha)][1-\frac{1}{2}(\lambda-\alpha)\right.$$
$$\left.\frac{M''(\alpha)}{M'(\alpha)}+\cdots] - \frac{A(\alpha)}{M'(\alpha)}\right\}$$
$$= \frac{1}{\lambda-\alpha}\left\{(\lambda-\alpha)\left[\frac{A'(\alpha)}{M'(\alpha)} - \frac{1}{2}\,\frac{A(\alpha)M''(\alpha)}{[M'(\alpha)]^2}+\cdots\right]\right\}$$
$$= \frac{A'(\alpha)}{M'(\alpha)} - \frac{1}{2}\,\frac{A(\alpha)M''(\alpha)}{[M'(\alpha)]^2} + (\lambda-\alpha)\left\{\cdots\right\}$$

which gives the required result. **93**

Remark 2. The formula ($\mathscr{F}$) is exactly a formula of Mittag-Leffler; the principal part of $\dfrac{A(\lambda)}{M(\lambda)}$ in the neighbourhood of the simple pole β is $\dfrac{A(\beta)}{(\lambda-\beta)M'(\beta)}$; the term $\frac{A(\beta)}{(\beta-\alpha)M'(\beta)}$ is a corrective term which gives the convergence of the series $\sum\limits_{\beta}$, by reason of the convergence of $\sum \dfrac{1}{|\beta|^2}$ (which itself is a consequence of the fact that $M(\lambda)$ is of exponential type) only if $A(\beta)$ is bounded on (σ). We can also write

$$A(\lambda) = \frac{M(\lambda)A(\alpha)}{(\lambda-\alpha)M'(\alpha)} + \sum_{\substack{\beta\varepsilon(\sigma)\\ \beta\neq\alpha}}\left(\frac{1}{\lambda-\beta}+\frac{1}{\beta-\alpha}\right)\frac{M(\lambda)}{M'(\beta)}A(\beta)$$

and we consider this formula as an interpolation formula for $A(\lambda)$, with **94**
the interpolation function

$$t_\beta(\lambda) = \frac{M(\lambda)}{(\lambda-\beta)M'(\beta)}$$

Now we see that the computation of §1 is in a certain sense the converse of the theorem of the Mean periodic functions, viz a mean periodic function admits of an expansion in mean periodic exponentials (the spectrum being simple otherwise mean periodic exponential-monomials) (Problem 1). Mittag-Leffler's theorem is used in the proof of this theorem. Conversely problem (2) if solved gives the Mittag-Leffler's theorem.

5

We shall now consider the formula ($\mathscr{G}$), in the case of R^2 and C^2. For convenience we shall first fix the following notation.

Let S denote the function from C^2 into C^2 given by $(\lambda_1, \lambda_2) - - - -(S_1(\lambda_1, \lambda_2), S_2(\lambda_1, \lambda_2))$. S is an entire analytic function on C^2 into itself since S_1 and S_2 are so. For $\lambda, \alpha, \beta \varepsilon C^2$ let $[\lambda - \alpha] = (\lambda_1 - \alpha_1)(\lambda_2 - \alpha_2)$ and

$$[\alpha - \beta] = (\alpha_1 - \beta_1)(\alpha_2 - \beta_2).$$

Let

$$D(\alpha) = \frac{\partial(S_1, S_2)}{\partial(\alpha_1, \alpha_2)} = \begin{vmatrix} \frac{\partial S_1}{\partial \alpha_1} & \frac{\partial S_1}{\partial \alpha_2} \\ \frac{\partial S_2}{\partial \alpha_1} & \frac{\partial S_2}{\partial \alpha_n} \end{vmatrix}$$

and $$\frac{J(S, A)}{J(\lambda, \alpha)} = \begin{vmatrix} S_1(\lambda_1, \lambda_2) & A_1(\alpha_1, \lambda_2) \\ S_2(\lambda_1, \lambda_2) & A_2(\lambda_1, \lambda_2) \end{vmatrix}$$

95 where S and A are two entire functions of C^2 into itself and λ, α are two points of C^2. If $S = A$ then

$$\frac{J(S, A)}{J(\lambda, \alpha)} = \begin{vmatrix} S_1(\lambda_1, \lambda_2) & S_1(\alpha_1, \lambda_2) \\ S_2(\lambda_1, \lambda_2) & S_2(\lambda_1, \lambda_2) \end{vmatrix}$$

The point α is a zero of the function S if and only if $S_1(\alpha_1, \alpha_2) = 0 = S_2(\alpha_1, \alpha_2)$ i.e., if $\alpha \varepsilon (\sigma)$ and

$$D(\alpha) = \lim_{\lambda \to \alpha} \frac{1}{(\lambda - \alpha)} \frac{J(S, S)}{J(\lambda, \alpha)}$$

96 The set (α) is countable and the zeros $\alpha \varepsilon (\sigma)$ are simple (i.e. $D(\alpha) \neq 0$ for $\alpha \varepsilon (\sigma)$).

We shall now compute

a) $d\beta_i (i = 1, 2)$,

b) $\dfrac{\partial}{\partial\lambda_i}(t_\alpha(\lambda))_{\lambda=\beta}$,

c) $d[t_\alpha(\lambda)]$ which occur in $(\mathscr{G})$.

a) Let ε be a small parameter which has 0 for limit. We substitute for the function S, a neighbouring function $S - \varepsilon A$ where $A = (A_1, A_2)$ is another entire analytic function of C^2 into itself and we compute the variations of the spectrum (σ).

$$S_1(\alpha_1 + d\alpha_1, \alpha_2 + d\alpha_2) - \in A_1(\alpha_1 + d\alpha_1, \alpha_2 + d\alpha_2) = 0$$
$$S_2(\alpha_1 + d\alpha_1, \alpha_2 + d\alpha_2) - \in A_2(\alpha_1 + d\alpha_1, \alpha_2 + d\alpha_2) = 0$$

But (α) is a simple zero of the functions S. Hence neglecting the terms of the second order in ε, we have,

$$\frac{\partial S_1}{\partial\alpha_1} d\alpha_1 + \frac{\partial S_2}{\partial\alpha_2} = \in A_1(\alpha_1, \alpha_2)$$
$$\frac{\partial S_2}{\partial\alpha_1} d\alpha_1 + \frac{\partial S_2}{\partial\alpha_2} d\alpha_2 = \in A_2(\alpha_1, \alpha_2)$$

and by the Cramer's rule,

$$d\alpha_1 = \frac{\in}{D(\alpha)} \frac{J(A, \frac{\partial S}{\partial\lambda_2})}{J(\alpha, \alpha)}, d\alpha_2 = \frac{\in}{D(\alpha)} \frac{J(\frac{\partial S}{\partial\lambda_1}, A)}{J(\alpha, \alpha)}$$

b) As β is a zero of the the function S, it is obvious that, for $\beta \neq \alpha$, the terms of $\left(\dfrac{\partial}{\partial\lambda_i} t_\alpha(\lambda)\right)_{\lambda=\beta}$ which are different from zero arise from the differentiation of S_1 and S_2 and we have immediately, 97

$$\left(\frac{\partial}{\partial\lambda_i} t_\alpha(\lambda)\right)_{\lambda=\beta} = \frac{1}{D(\alpha)[\beta - \alpha]} \frac{J\left(\frac{\partial S}{\partial\lambda_i}, S\right)}{(\beta, \alpha)}, i = 1, 2$$

substituting for $d\beta$, we obtain

$$\sum_{i=1}^{2}\left(\frac{\partial}{\partial\lambda_i}t_\alpha(\lambda)\right)_{\lambda=\beta} d\beta_i = \frac{\in}{D(\alpha)D(\beta)[\beta-\alpha]}$$
$$\frac{J\left(\frac{\partial S}{\partial\lambda_1}, S\right)}{J(\beta,\alpha)}\frac{J\left(A, \frac{\partial S}{\partial\lambda_2}\right)}{J(\beta,\beta)} + \frac{J\left(\frac{\partial S}{\partial\lambda_2}, S\right)}{J(\beta,\alpha)}\frac{J\left(\frac{\partial S}{\partial\lambda_1}, A\right)}{J(\beta,\beta)}$$

The curly bracket equals

$$\left[\frac{\partial}{\partial\beta_1}S_1(\beta_1,\beta_2)S_2(\alpha_1,\beta_2) - \frac{\partial}{\partial\beta_1}S_2(\beta_1,\beta_2)S_1(\alpha_1,\beta_2)\right]\times$$
$$\left[A_1(\beta_1,\beta_2)\frac{\partial}{\partial\beta_2}S_2(\beta_1,\beta_2) - A_2(\beta_1,\beta_2)\frac{\partial}{\partial\beta_2}S_1(\beta_1,\beta_2)\right]$$
$$+\left[\frac{\partial}{\partial\beta_2}S_1(\beta_1,\beta_2)S_2(\alpha_1,\beta_2) - \frac{\partial}{\partial\beta_2}S_2(\beta_1,\beta_2)S_1(\alpha_1,\beta_2)\right]\times$$
$$\frac{\partial}{\beta_1}S_1(\beta_1,\beta_2)A_2(\beta_1,\beta_2) - \frac{\partial}{\partial\beta_1}S_2(\beta_1,\beta_2)A_1(\beta_1,\beta_2)$$
$$= \frac{\partial}{\partial\beta_1}S_1(\beta_1,\beta_2)\frac{\partial}{\partial\beta_2}S_2(\beta_1,\beta_2)A_1(\beta_1,\beta_2)S_2(\alpha_1,\beta_2)$$
$$- \frac{\partial}{\partial\beta_1}S_1(\beta_1,\beta_2)\frac{\partial}{\partial\beta_2}S_2(\beta_1,\beta_2)A_2(\beta_1,\beta_2)S_1(\alpha_1,\beta_2)$$
$$+ \frac{\partial}{\partial\beta_2}S_1(\beta_1,\beta_2)\frac{\partial}{\partial\beta_2}S_2(\beta_1,\beta_2)A_2(\beta_1,\beta_2)S_1(\alpha_1,\beta_2)$$
$$- \frac{\partial}{\partial\beta_2}S_1(\beta_1,\beta_2)\frac{\partial}{\partial\beta_2}S_1(\beta_1,\beta_2)A_1(\beta_1,\beta_2)S_2(\alpha_1,\beta_2)$$
$$= D(\beta)\frac{J(A,S)}{J(\beta,\alpha)}.$$

98 Finally,

$$t_\beta(\lambda)\sum_{i=1}^{2}\frac{\partial}{\partial\lambda_i}t_\alpha(\lambda)d\beta_i$$
$$= \frac{1}{D(\beta)[\lambda-\beta]}\frac{J(S,S)}{J(\lambda,\beta)}\frac{\in}{D(\alpha)D(\beta)[\beta-\alpha]}D(\beta)\frac{J(A,S)}{J(\beta,\alpha)}$$
$$= \frac{\in}{D(\alpha)D(\beta)[\beta-\alpha]}\frac{1}{\lambda-\beta}\frac{J(S,S)}{J(\lambda,\beta)}\frac{J(A,S)}{J(\beta,\alpha)}$$

99
c)

$$t_\alpha(\lambda) = \frac{1}{D(\alpha)(\lambda-\alpha)}\frac{J(S,S)}{J(\lambda,\alpha)}$$
$$d\{t_\alpha(\lambda)\} = \frac{1}{D(\alpha)(\lambda-\alpha)}d\left\{\frac{J(S,S)}{J(\lambda,\alpha)}\right\} + d\left\{\frac{1}{D(\alpha)(\lambda-\alpha)}\right\}\frac{J(S,S)}{J(\lambda,\alpha)}$$

The differential of

$$\frac{J(S,S)}{J(\lambda,\alpha)} = \begin{vmatrix} S_1(\lambda_1,\lambda_2) & S_1(\alpha_1,\lambda_2) \\ \\ S_2(\lambda_1,\lambda_2) & S_2(\alpha_1,\lambda_2)\end{vmatrix}$$

is the sum of two determinants, one of which is obtained from the differentiation of the first column and the second by the differentiation of the second column. The first determinant gives in $d\{t_\alpha(\lambda)\}$ the term

$$\frac{-\in}{D(\alpha)(\lambda-\alpha)}\frac{J(A,S)}{J(\lambda,\alpha)}$$

and this term is, in a certain sense, a principal term, because it is composed of variations A_1 and A_2 of S_1 and S_2 respectively, in the generic sense, that is which depend on the two independent variables : λ_1, λ_2. In the second determinant, the second column has elements

$$d[S_1(\alpha_1,\alpha_2)] \text{ and } d[S_2(\alpha_1,\lambda_2)]$$

which are equal to 100

$$-\in A_1(\alpha_1,\lambda_2) + \frac{\partial}{\partial\alpha_1}S_1(\alpha,\lambda_2)d\alpha_1$$

and $-\varepsilon A_2(\alpha_1,\lambda_2) + \frac{\partial}{\partial\alpha_1}S_1(\alpha,\lambda_2)d\alpha_1$ respectively, in which $\frac{\varepsilon}{D(\alpha)}\frac{J(A,\frac{\partial S}{\partial\lambda_2})}{J(\alpha,\alpha)}$ has to be substituted for $d\alpha_1$ and $\frac{\varepsilon}{D(\alpha)}\frac{J(A,\frac{\partial S}{\partial\lambda_1},A)}{J(\alpha,\alpha)}$ has to be substituted for $d\alpha_2$. Other terms is $d[t_\alpha(\lambda)]$ artist from the denominator $D(\alpha)[\lambda-\alpha]$ in which again we have to substitute for $d\alpha_1$ and $d\alpha_2$. For us, it is sufficient to write the principal term and to write $\Re$ for the others which are, in fact, of irregular formation, depend on only, finite in number and contain A_1 and A_2 (variations of S_1 and S_2) in a non-generic manner. Then the formula is reduced to

$$\frac{J(A,S)}{J(\lambda,\alpha)} = \sum_{\substack{\beta\neq\alpha\\ \beta\varepsilon(\sigma)}} \frac{[\lambda-\alpha]}{D(\beta)[\lambda-\beta][\beta-\alpha]} \frac{J(S,S)}{J(\lambda,\beta)} \frac{J(A,S)}{J(\beta,\alpha)} + (\mathscr{R}) \qquad (\mathscr{G}_1)$$

in which the terms in $\Re$ are i) finite in number, ii) non-generic in A, iii)
101 depend only on λ and α. It is good to note that in dimension 1, formula $\mathscr{G}_1$ becomes

$$A(\lambda) = \sum_{\substack{\beta\neq\alpha\\ \beta\in(\sigma)}} \frac{\lambda-\alpha}{D(\beta)(\lambda-\beta)(\beta-\alpha)} S(\lambda)A(\beta) + \cdots\cdots\cdots$$

where the terms not written are exactly

$$\frac{S(\lambda)A(\alpha)}{D(\alpha)(\lambda-\alpha)} + \frac{S(\lambda)A'(\alpha)}{D(\alpha)} - \frac{1}{2}\frac{S(\lambda)A(\alpha)D'(\alpha)}{D^2(\alpha)}$$

That is the formula ($\mathscr{F}_1$) of §4 which is analogous to $\mathscr{G}_1$.

It is natural to say that $\mathscr{G}_1$ is a generalisation, in C^2, and for the function S of the formula of Mittag-Leffler in C^1.

6

We now proceed to give an example in which the preceding results, which are purely formal at present, are correct.

Let T_1, T_2 in $\mathscr{E}'(R^2)$ be finite linear combinations of Dirac measures, situated at *rational points*.

Their Fourier-Laplace transforms will be of the form

$$S_1(\lambda_1,\lambda_2) = \sum a_{pq} \exp\left(\frac{p\lambda_1 + q\lambda_2}{N}\right) \qquad (1)$$

$$S_2(\lambda_1,\lambda_2) = \sum b_{pq} \exp\left(\frac{p\lambda_1 + q\lambda_2}{N}\right) \qquad (2)$$

102 where N is a fixed integer, and where p, q are also integers and the to

summations are finite.

When (λ_1, λ_2) is in spectrum, the point (X_1, X_2) with $X_1 = \exp \dfrac{\lambda_1}{N}$, $X_2 = \exp \dfrac{\lambda_2}{N}$, have to satisfy the two algebraic equations

$$\sum a_{pq} X_1^p X_2^q = 0 \tag{3}$$

and
$$\sum b_{pq} X_1^p X_2^q = 0 \tag{4}$$

The two algebraic curves represented by (3) and (4) intersect in a finite number, say M, of points. We suppose that all these points (ξ_1, ξ_2) are simple and *finite*.

Let $\xi_1 = \exp \dfrac{\alpha_1}{N}, \xi_2 = \exp \frac{\alpha_2}{N}$. The spectrum (σ) is defined by

$$\beta_1 = \alpha_1 + 2hn\pi i\ ,\ \beta_2 = \alpha_2 + 2kn\pi i$$

where h and k are integers and the general solutions of

$$T_1 * F = 0 = T_2 * F$$

can be formally expressed as

$$F(x_1, x_2) = \sum_\alpha \exp(\alpha_1 x_1 + \alpha_2 x_2) f_{\alpha_1 \alpha_2}(x_1, x_2)$$

where the M functions $f_{\alpha_1\alpha_2}(x_1, x_2)$ are periodic in x_1 and x_2 with pe- **103**
riod $\dfrac{1}{N}$. The developments of the $f_{\alpha_1\alpha_2}(x_1, x_2)$ in Fourier series give immediately the development of F in mean periodic exponentials of the spectrum (σ) and the computation of the coefficients by the use of the distributions T_α is perfectly correct in this case.

Let the convex envelope of the supports of T_1 and T_2 be the rectangle

$$0 \leq x_1 \leq a\ ,\ 0 \leq x_2 \leq b$$

and let T_1, T_2 have Fourier-Laplace transforms

$$S_1(\lambda_1, \lambda_2) = \sum a_{pq} \exp\left(\frac{pa\lambda_1}{m} + \frac{qb\lambda_2}{n}\right) \tag*{(1)$'$}$$

$$S_2(\lambda_1, \lambda_2) = \sum b_{pq} \exp\left(\frac{pa\lambda_1}{m} + \frac{qb\lambda_2}{n}\right) \qquad (2)'$$

respectively, where m, n are positive integers and p, q are integers satisfying

$$0 \leq p \leq m\ ,\ 0 \leq q \leq n$$

setting $\exp \dfrac{a\lambda_1}{m} = X_1$, $\exp \dfrac{b\lambda_2}{n} = X_2$, we obtain (3) and (4).

If the coefficients a, b are generic, the equations (3), (4) have degree
104 n in X_2, the coefficients relatively to X_2 being the polynomial in X_1 of degree m. We eliminate X_2 by Sylvester's resultant

$$\begin{vmatrix}
A_0 & A_1 & \cdots & A_n & 0 & 0 & \cdots 0 \\
0 & A_0 & \cdots & A_{n-1} & A_n & 0 & \cdots 0 \\
\cdots & & \cdots & & \cdots & & \cdots \\
\cdots & & \cdots & & \cdots & & \cdots \\
0 & 0 & \cdots & A_1 & A_2 & \cdots & A_n \\
B_0 & B_1 & \cdots & B_1 & 0 & \cdots & 0 \\
0 & B_0 & \cdots & B_{n-1} & B_n & \cdots & 0 \\
\cdots & & \cdots & & \cdots & & \cdots \\
0 & 0 & \cdots & & B_1 & \cdots & B_n
\end{vmatrix} = 0$$

The determinant is given by the elimination of $X_2^0, X_2^1, \ldots, X_2^{2n-1}$ between the $2n$ equations

$$\begin{gathered}
A_0 X_2^{2n-1} + A_1 X_2^{2n-2} \cdots + A_n X_2^{n-1} = 0 \\
A_0 X_2^{2n-2} + \cdots + A_n X_2^{n-2} = 0 \\
\cdots\ \cdots\ \cdots\ \cdots\ \cdots \\
A_0 X_2^n + A_1 X_2^{n-1} + \cdots + A_n = 0 \\
B_0 X_2^{2n-1} + \cdots\ \ \cdots + B_n X_2^{n-1} = 0 \\
\cdots \qquad \cdots \qquad \cdots \\
B_0 X_2^n + \cdots + B_n = 0
\end{gathered}$$

105 in which the A_i and the B_i are polynomials in X_1 of degree m so that the determinant of Sylvestor is a polynomial in X_1 of degree $2mn$ and the

number M of common points of (3) and (4) is $2mn$. Any point (ξ_1, ξ_2) is given by $\xi_1 = \exp \dfrac{a\alpha_1}{m}, \xi_2 = \exp \dfrac{b\alpha_2}{n}$ and the spectrum (σ) consists of (λ_1, λ_2) where $\lambda_1 = \alpha_1 + \dfrac{2hm\pi i}{a}, \lambda_2 = \alpha_2 + \dfrac{2kn\pi i}{b}$ and $F(x, y) = \sum\limits_{\alpha_1, \alpha_2} \exp(\alpha_1 x_1 + \alpha_2 x_2) f_{\alpha_1 \alpha_2}(x_1, x_2)$ with $f_{\alpha_1, \alpha_2}(x_1, x_2)$ periodic in x_1 of period $\dfrac{a}{m}$ and in x_2 of period $\dfrac{b}{n}$.

We know that for $\alpha \varepsilon (\sigma)$,

$$c_\alpha = \langle S_\alpha, \int_P^{x_1} \int_Q^{x_2} e^{\alpha_1(x_1 - \xi_1) + \alpha_2(x_2 - \xi_2)} F(\xi_1, \xi_2) d\xi_1 d\xi_2 \rangle$$

where $S_\alpha = \frac{1}{D(\alpha)} \{T_1 * \sum_2 - T_2 * \sum_1\}$ and if

$$\langle T_1, F \rangle = \sum_i \rho_i F(a_i, b_i)$$
$$\langle T_2, F \rangle = \sum_j \sigma_j F(c_j, d_j) \qquad \text{then,}$$

$\sum_1, \sum_2$ are defined by **106**

$$\langle \Sigma_1, F \rangle = \Sigma_i \rho_i e^{\alpha_2 b_i} F(a_i, 0)$$
$$\langle \Sigma_2, F \rangle = \Sigma_j \sigma_j e^{\alpha_2 d_j} F(c_j, 0)$$

so that

$$\langle S_\alpha, F \rangle = \sum_{i,j} \rho_i \sigma_j e^{\alpha_2 d_j} F(a_i + c_j, b_i) - e^{\alpha_2 b_i} F(a_i + c_j, d_j)$$

As the points (a_i, b_i) and (c_j, d_j) are in the rectangle $0 \leq x_1 \leq a$, $0 \leq x_2 \leq b$, it is clear that in the computation of c_{α_1, α_2}, the values of the function F in the rectangle

$$0 \leq x_1 \leq 2a \, , \; 0 \leq x_2 \leq b \tag{5}$$

are used. This rectangle can be divided in $M = 2mn$ small rectangles

$$\frac{ap}{m} \leq x_1 \leq \frac{a(p+1)}{m}, \; \frac{bq}{n} \leq x_2 \leq \frac{b(q+1)}{n}$$

for $\qquad p = 0, 1, 2, \ldots, (2m - 1) \quad q = 0, 1, 2, \ldots, (n - 1)$

setting 107

$$\phi_{pq}(x_1, x_2) = F\left(\frac{ap}{m} + x_1, \frac{bq}{n} + x_2\right)$$
$$p = 0, 1, \ldots, 2m - 1$$
$$q = 0, 1, \ldots, n - 1,$$

where $0 \leq x_1 \leq \dfrac{a}{m}, 0 \leq x_2 \leq \dfrac{b}{n}$ we obtain

$$\phi_{pq}(x_1, x_2) = \sum_{(\alpha_1, \alpha_2)} \exp\left(\frac{a\alpha_1 p}{m} + \frac{b\alpha_2 q}{n}\right)$$
$$\exp(\alpha_1 x_1 + \alpha_2 x_2)\, f_{\alpha_1 \alpha_2}(x_1, x_2) \qquad (6)$$
$$p = 0, 1, \ldots, 2m - 1\ ;\ q = 0, 1, \ldots, (n - 1)$$

(since $f_{\alpha_1, \alpha_2}\left(\frac{ap}{m} + x_1, \frac{bq}{n} + x_2\right) = f_{\alpha_1, \alpha_2}(x_1, x_2)$).

The numbers of equations is (6) is equal to the number of unknowns

$$\exp(\alpha_1 x_1 + \alpha_2 x_2) f_{\alpha_1, \alpha_2}(x_1, x_2)$$

and the solution of the linear system gives periodic functions in the rectangle $0 \leq x_1 \leq \dfrac{a}{m}, 0 \leq x_2 \leq \dfrac{b}{n}$. Thus we see that

108 i) The continuation of F in all the plane is completely known.

ii) The coefficients of the mean periodic exponentials in the development of F are determined by the process of the Jacobi determinant.

iii) The expansion converges as the Fourier series of the function $f_{\alpha_1, \alpha_2}(x_1, x_2)$.

For instance if $F(x_1, x_2)$ is given and continuous in the rectangle (5) together with their derivatives of order 1, 2, 3, 4, we have uniform convergence in any compact set contained in the interior of the rectangle (5).

The heuristic computation has given us the following important result:

The formula of the type of Mittag-Leffler in C^2, is valid if S_1 and S_2 are two linear combinations of exponentials

$$\begin{aligned} S_1(\lambda_1,\lambda_2) &= \sum a_{pq}\exp\Big(\frac{pa\lambda_1}{m}+\frac{qb\lambda_2}{n}\Big)\\ S_2(\lambda_1,\lambda_2) &= \sum b_{pq}\exp\Big(\frac{pa\lambda_1}{m}+\frac{qb\lambda_2}{n}\Big)\\ p &= 0,1,\ldots,m-1\\ q &= 0,1,\ldots,n-1 \end{aligned}$$

where the coefficients a_{pq}, b_{pq} are generic **109**

7 The formula ($\mathscr{F}$) for a polynomial

We shall now give the formula $\mathscr{F}$ in the case of a polynomial. It is possible to establish the formula for more than one variable but the proof and computation will be very long and therefore for the sake of simplicity we consider only the case of one variable.

Let $M(\lambda)$ be a polynomial in e^λ. $M(\lambda) = P(X) = a_0+a_1X+\cdots+a_nX^n$ with $X = e^\lambda$. Let $A_1, A_2, \ldots, A_n$ be n distinct roots of $P(X) = 0$. The spectrum (σ) is in this case composed of n arithmetical progressions $\beta_j(h) = \alpha_j + 2h\pi i$ where $e^{\alpha_j} = A_j$, $j = 1, 2, \ldots, n$, h an integer. In the formula $\mathscr{F}$:

$$\frac{dM(\lambda)}{M(\lambda)} - \frac{d[M'(\lambda)]_{\lambda=\alpha}}{M'(\lambda)} + \left[\frac{1}{1-\lambda} - \frac{1}{2}\frac{M''(\alpha)}{M'(\alpha)}\right]d\alpha \\ + \sum_{\beta\neq\alpha}\frac{\beta-\alpha}{(\lambda-\beta)(\beta-\alpha)}d\beta = 0,$$

the summation $\sum\limits_{\beta\neq\alpha}$ can be divided into n summations with respect to h, corresponding to n arithmetic progressions which constitute (σ). Let $\alpha = \alpha_j$, be fixed. We first consider the summation for $\beta(h) = \alpha_k +$

$2h\pi i,\ k \neq j$, i.e.,

$$\sum_{h=-\infty}^{+\infty} \frac{1}{(\lambda - \alpha_k - 2h\pi i)(\alpha_k - \alpha_j - 2h\pi i)} \tag{1}$$

110 The well known classical formula (Mittag-Laffler) give

$$\frac{1}{e^u - 1} = -\frac{1}{2} + \frac{1}{u} + \sum_{h=-\infty}^{h=+\infty}{}^{*} \left(\frac{1}{u - 2h\pi i} + \frac{1}{2h\pi i} \right);$$

setting $u = \lambda - \alpha_k$ and $u = \alpha_j - \alpha_k$ in this, we have

$$\frac{1}{e^{\lambda - \alpha_k} - 1} = -\frac{1}{2} + \frac{1}{\lambda - \alpha_k} + \sum_h^{*} \left(\frac{1}{\lambda - \alpha_k - 2h\pi i} + \frac{1}{2h\pi i} \right)$$

and $\dfrac{1}{e^{\alpha_j - \alpha_k} - 1} = -\frac{1}{2} + \frac{1}{\alpha_j - \alpha_k} + \sum_h^{*} \left(\frac{1}{\alpha_j - \alpha_k - 2h\pi i} + \frac{1}{2h\pi i} \right)$

By subtraction,

$$\begin{aligned} \frac{1}{e^{\lambda - \alpha_k} - 1} - \frac{1}{e^{\alpha_j - \alpha_k} - 1} &= \frac{\alpha_j - \lambda}{(\lambda - \alpha_k)(\alpha_j - \alpha_k)} \\ &\quad + \sum^{*} \left\{ \frac{1}{\lambda - \alpha_k - 2\pi i} - \frac{1}{\alpha_j - \alpha_k - 2h\pi i} \right\} \\ &= (\alpha_j - \lambda) \sum_{h=-\infty}^{+\infty} \frac{1}{(\lambda - \alpha_k - 2\pi i)(\alpha_j - \alpha_k - 2h\pi i)} \end{aligned}$$

Hence

$$\begin{aligned} &\sum_{h=-\infty}^{\infty} {+\infty} \frac{1}{(\lambda - \alpha_k - 2\pi i)(\alpha_h - \alpha_j - 2h\pi i)} \\ &= \frac{1}{(\lambda - \alpha_j)} \left[\frac{1}{e^{\lambda - \alpha_k} - 1}\ \frac{1}{e^{\alpha_j - \alpha_k}} - 1 \right] = \frac{1}{\lambda - \alpha_j} \left[\frac{A_k}{X - A_k} - \frac{A_k}{A_j - A_k} \right] \\ &= \frac{A_k [X - A_j]}{(\lambda - \alpha_j)(X - A_k)(A_k - A_j)}. \end{aligned}$$

111 But $d\beta_k(h) = d\alpha_k$ is independent of h and the part of the summation

$(\lambda-\alpha_j)\sum\limits_{\beta\neq\alpha}\dfrac{d\beta}{(\lambda-\alpha)(\beta-\alpha)}$ which is under consideration is

$$\frac{(X-A_j)A_k\, d\alpha_k}{(X-A_k)(A_k-A_j)}$$

For the part of the summation corresponding to the arithmetical progression $\beta_j(h)=\alpha_j+2hg\pi i, h\neq 0$, it is necessary to compute $\sum\limits_h^* \dfrac{1}{(\lambda-\alpha_j-2h\pi i)2h\pi i}$ setting $z=\lambda-\alpha_j$, we have,

$$\begin{aligned}\sum_h^* \frac{1}{(z-2h\pi i)(2h\pi i)} &= \frac{1}{z}\sum_h^*\left[\frac{1}{z-2h\pi i}+\frac{1}{2h\pi i}\right]\\ &= \frac{1}{2z}+\frac{1}{z}\left(\frac{1}{e^z-1}-\frac{1}{z}\right)\\ &= \frac{1}{(\lambda-\alpha_j)}\left[\frac{A_j}{X-A_j}-\frac{1}{\lambda-\alpha_j}\right]+\frac{1}{2(\lambda-\alpha_j)}\end{aligned}$$

Hence the second part of the summation in gives

$$\frac{A_j\, d\alpha_j}{X-A_j}-\frac{d\,\alpha_j}{\lambda-\alpha_j}+\frac{d\,\alpha_j}{2}.$$

For any $h, dA_k=e^{\alpha_k}d\alpha_k=A_k d\alpha_k$. ($\mathscr{F}$) now becomes 112

$$\begin{aligned}\frac{d\,M(\lambda)}{M(\lambda)}-\frac{(d\,M'(\lambda))_{\lambda=\alpha_j}}{M'(\alpha_j)}-\frac{1}{2}\frac{M''(\alpha_j)}{M'(\alpha_j)}\,d\,\alpha_j\\ +\frac{d\alpha_j}{2}+\frac{dA_j}{X-A_j}+\sum_{k\neq j}\frac{(X-A_j)dA_k}{(X-A_k)(A_k-A_j)}=0 \quad (1)\end{aligned}$$

Now $M(\lambda)=P(X)$

$$M'(\lambda)=XP'(X); M''(\lambda)=X^2P''(X)+XP'(X).$$

We have

$$d[A_jP'(A_j)]=(d[XP'(X)])_{X=A_j}+[A_jP''(A_j)+P'(A_j)]dA_j$$

and also, $d[A_jP'(A_j)] = dA_j\ P'(A_j) + A_j\ d[P'(A_j)]$
$= dA_j\ P'(A_j) + A_j(d\ P'(X))_{X=A_j} + A_j\ P''(A_j)dA_j$

Hence

$$(d[XP'(X)])_{X=A_j} = A_j(d[P'(X)])_{X=A_j};$$

also $$\frac{(dM'(\lambda))_{\lambda=\alpha_j}}{M'(\alpha_j)} = \frac{(d[XP'(X)])_{X=A_j}}{A_j\ P'(A_j)} = \frac{(d\ P'(X))_{X=A_j}}{P'(A_j)}$$

and $$-\frac{1}{2}\frac{M''(\alpha_j)}{M'(\alpha)} = -\frac{1}{2}\frac{P''(A_j)}{P'(A_j)}\ A_j\ d\alpha_j - \frac{1}{2}\frac{dA_j}{A_j}$$
$$= -\frac{1}{2}\frac{P''(A_j)}{P'(A_j)}\ dA_j - \frac{1}{2}\ d\alpha.$$

113 From (1) we obtain,

$$\frac{d\ P(X)}{P(X)} - \frac{d\ P'(X)_{X=A_j}}{P'(A_j)} + \left[\frac{1}{X - A_j} - \frac{1}{2}\frac{A''(A_j)}{P'(A_j)}\right] dA_j + \sum_{k\neq j} \frac{(X - A_j)dA_k}{(X - A_k)(A_k - A_j)} = 0$$

which is the formula ($\mathscr{F}$) for the polynomial $P(X)$.

8

We have seen that the formula ($\mathscr{G}_1$) holds for the Fourier - Laplace transforms of distributions which are finite linear combinations of Dirac measures placed at rational points in a rectangle in R^2 such that the convex envelops of the supports of these distributions (i.e. the set of rational points) is precisely the rectangle. We shall now prove that ($\mathscr{G}_1$) holds for Fourier - Laplace transforms of two distributions T_1 and T_2 in $\varepsilon'(R^2)$ which differ slightly (in fact by a measure defined by a density) from a finite linear combination of Dirac measures. Let T_1 and T_2 be defined by

$$T_1 * F = a_1F(x, y) + b_1F(x + 1, y) + c_1F(x, y + 1) + d_1F(x + 1, y + 1)$$

$$+\int_o^1\int_o^1 k_1(\xi,\eta)F(x+\xi,y+\eta)d\,\xi\,d\,\eta$$

$$T_2 * F = a_2F(x,y) + b_2F(x+1,y) + c_2F(x,y+1) + d_2F(x+1,y+1)$$
$$+\int_o^1\int_o^1 k_2(\xi,\eta)F(x+\xi,y+\eta)d\,\xi\,d\eta$$

where **114**

i) a_i, b_i, c_i, d_i $(i = 1, 2)$ are all not zero

ii) $k_i (i = 1, 2)$ are continuous.

Let $\mathscr{F}\mathscr{L}T_i = M_i(\lambda,\mu)$ be Fourier - Laplace transforms of $T_i (i = 1, 2)$

$$M_1(\lambda,\mu) = a_1 + b_1\,e^{\lambda} + c_1\,e^{\mu} + d_1\,e^{\lambda+\mu} + \iint k_1(\xi,\eta)e^{\lambda\xi+\mu\eta}d\,\xi\,d\,\eta$$
$$M_2(\lambda,\mu) = a_2 + b_2\,e^{\lambda} + c_2\,e^{\mu} + d_2\,e^{\lambda+\mu} + \iint k_2(\xi,\eta)e^{\lambda\xi+\mu\eta}d\,\xi\,d\,\eta$$

Let (σ) denote the spectrum.

Properties of the spectrum (σ) of T_1 and T_2.

We have

$$M_1(\lambda,\mu)\,e^{-\lambda-\mu} - d_1 = a_1e^{-\lambda-\mu} + b_1\,e^{-\mu} + c_1\,e^{-\lambda} + \iint k_1(\xi,\eta)e^{\lambda(\xi-1)+\mu(\eta-1)}d\xi d\eta$$

Let **115**

$$(\lambda,\mu) \in (\sigma), \lambda = \lambda_o + i\,\lambda_i, \mu = \mu_o + i\,\mu_1 |M_1(\lambda,\mu)e^{-\lambda-\mu} - d_1| \leq |a_1|e^{-\lambda_o-\mu_o}$$
$$+ |b_1|e^{-\mu_o} + |c_1|\,e^{-\lambda_o}\int_o^1\int_o^1 |k_1(\xi,\eta)|e^{\lambda_o(\xi-1)+\mu_o(\eta-1)}d\,\xi\,d\,\eta$$

For $(\lambda,\mu) \in (\sigma)$, the left hand member is $-d_1$ and as λ_o, μ_o both tend to $+\infty$, the right hand member tends to 0. This cannot happen as

$d_1 \neq 0$. Hence for $(\lambda, \mu) \in (\sigma)$, λ_o, μ_o both cannot tend to $+\infty$. Similarly the pairs $(\lambda_o, -\mu_o), (-\lambda_o, \mu_o)$ and $(-\lambda_o, -\mu_o)$ cannot each tend to positive infinity. We have only to consider for $i = 1, 2$,

$$M_i(\lambda, \mu)e^{-\lambda} - b_i = a_i e^{-\lambda} + d_i\, e^{\mu} + \int_o^1 \int_o^1 k_i(\xi, \eta) e^{\lambda(\xi-1)+\mu\eta} d\,\xi\, d\,\eta$$

$$\begin{aligned} M_i(\lambda, \mu)e^{-\mu} - c_i &= a_i e^{-\mu} + b_i\, e^{\lambda-\mu} + d_i\, e^{\lambda} \\ &\quad + \int_o^1 \int_o^1 k_i(\xi, \eta) e^{\lambda\xi+\mu(\eta-1)} d\,\xi\, d\,\eta \end{aligned}$$

$$M_i(\lambda, \mu)\, e^{-\mu} - d_i = b_i e^{\lambda} + c_i e^{\mu} + d_i e^{\lambda+\mu} + \int_o^1 \int_o^1 k_i(\xi, \eta) e^{\lambda\xi+\mu\eta}\, d\,\xi\, d\,\eta.$$

Proposition 1. *If each of $b_2 d_1 - b_1 d_2, c_1 d_2 - c_2 d_1, c_1 a_2 - a_1 c_1, b_1 a_2 - b_2 a_1$ is distinct from zero, then the spectrum (σ) of T_1, T_2 is contained in a*
116 *vertical band in C^2 (i.e. the projection (λ_o, μ_o) of $(\lambda, \mu) \in (\sigma)$ in the real plane remains bounded).*

In view of the observations made in the preceding paragraph we have only to show that $(\lambda, \mu) \in (\sigma), \lambda = \lambda_o + i\lambda_1, \mu = \mu_o + i\,\mu_1$, one of λ_o, μ_o cannot tend to infinity while the other remains bounded. Eliminating $e^{-\mu}$ from the equation

$$\begin{aligned} M_1(\lambda, \mu)e^{-\lambda-\mu} - d_1 &= a_1\, e^{-\lambda-\mu} + b_1\, e^{-\mu} + c_1\, e^{-\lambda} \\ &\quad + \int_o^1 \int_o^1 k_1(\xi, \eta) e^{\lambda(\xi-1)+\mu(\eta-1)}\, d\,\xi\, d\,\eta \\ M_2(\lambda, \mu)e^{-\lambda-\mu} - d_2 &= a_2\, e^{-\lambda-\mu} + b_2\, e^{-\mu} + c_2\, e^{-\lambda} \\ &\quad + \int_o^1 \int_o^1 k_2(\xi, \eta) e^{\lambda(\xi-1)+\mu(\eta-1)}\, d\,\xi\, d\,\eta \end{aligned}$$

we have

$$
\begin{aligned}
b_2M_1(\lambda,\mu)e^{-\lambda-\mu} - b_1M_2(\lambda,\mu)e^{-\lambda-\mu} - b_2d_1 + b_1d_2 \\
&= (b_2a_1 - b_1a_2)e^{-\lambda-\mu} + (b_2c_1 - b_1c_2)e^{-\lambda} \\
&\quad + \int\limits_o^1 \int\limits_o^1 [b_2k_1(\xi,\eta) - b_1k_2(\xi,\eta)]e^{\lambda(\xi-1)+\mu(\eta-1)} d\xi d\eta
\end{aligned}
$$

Hence

$$
\begin{aligned}
|b_2M_1(\lambda,\mu)e^{-\lambda-\mu} - b_1M_2(\lambda,\mu)e^{-\lambda-\mu} - b_2d_1 + b_1d_2| \\
&\leq |b_2a_1 - b_1a_2|e^{-\lambda_o-\mu_o} + |b_2c_1 - b_1c_2|e^{-\lambda_o} \\
&\quad + \int\limits_o^1 \int\limits_o^1 |b_2k_1(\xi,\eta) - b_1k_2(\xi,\eta)|e^{\lambda_o(\xi-1)+\mu_o(\eta-1)} d\,\xi\, d\,\eta
\end{aligned}
$$

For $(\lambda,\mu) \in (\sigma), M_i(\lambda,\mu) = 0$ so that passing to the limit as $\lambda_o \to$ **117**
$+\infty$ on (σ) while μ_o remains bounded, we obtain $b_2d_1 - b_1d_2 = 0$ which is supposed to be not true.

Similarly we can show that

$\mu_o \to +\infty$ with λ_o remaining bounded implies that $c_1d_2 - c_2d_1 = 0$;

$\lambda_o \to -\infty$ with μ_o remaining bounded implies that $c_1a_2 - c_2a_1 = 0$;

and $\mu_o \to -\infty$ with λ_o remaining bounded implies that $a_1b_2 - a_2b_1 = 0$.

Remark. For the hyperbolas

$$a_1 + b_1X + c_1Y + d_1XY = 0 \qquad (H_1)$$

and

$$a_2 + b_2X + c_2Y + d_2XY = 0 \qquad (H_2)$$

The condition $a)$ $b_2d_1 - b_1d_2 = 0$ expresses that (H_1) and (H_2) have the same horizontal asymptote $b)$ $c_2d_1 - c_1d_2 = 0$ implies that (H_1) and (H_2) have the same vertical asymptote; $c)$ $c_2a_1 - c_1a_2 = 0$ implies that (H_1) and (H_2) have a common point on oY; $d)$ $b_2a_1 - b_1a_2 = 0$ implies that (H_1) and (H_2) have a common point on oX.

We assume that the conditions $a), b), c), d)$ are not valid. Besides we suppose that the hyperbolas (H_1) and (H_2) are not tangent to each other.

Proposition 2. *If $k_i(i = 1, 2)$ are twice continuously differentiable in* 118
$0 \leq x \leq 1, 0 \leq y \leq 1$, then the spectrum (σ) is asymptotic to (λ, μ) where

$$\left.\begin{aligned} \lambda &= \alpha_1 + 2h\pi i \\ \mu &= \beta_1 + 2k\pi i \end{aligned}\right\}$$

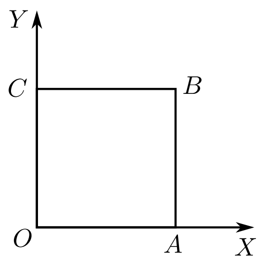
and

$$\begin{aligned} \lambda &= \alpha_2 + 2h'\pi i \\ \mu &= \beta_2 + 2k'\pi i \end{aligned}$$

where h, k, h', k' are integers and (ξ_i, η_i) with $e^{\alpha_i} = \xi_i, e^{\beta_i} = \eta_i$ $(i, 1, 2)$ are the distinct common points of (H_1) and (H_2)

$$M_1(\lambda, \mu) = \alpha_1 + \beta_1 e^{\lambda} + \gamma_1 e^{\mu} + \delta_1 e^{\lambda+\mu} + \frac{1}{\lambda\mu} \int\limits_o^1 \int\limits_o^1 k_1(\xi, \eta) \frac{\partial^2}{\partial\xi\partial\eta} \left[e^{\lambda\xi+\mu\eta}\right] d\,\xi\, d\,\eta$$

setting $H(x, y) = e^{\lambda x+\mu y}$ and applying Green's formula,



$$\int\limits_o^1 \int\limits_o^1 \{k_1(\xi, \eta) \frac{\partial^2}{\partial\xi\partial\eta} H(\xi, \eta) - H(\xi, \eta) \frac{\partial^2}{\partial\xi\partial\eta} k_1(\xi, \eta)\}\, d\,\xi\, d\,\eta$$
$$= \int_Q \left[H(\xi, \eta) \frac{\partial k_1}{\partial\xi}\, d\xi + k_1(\xi, \eta) \frac{\partial H}{\partial\eta}\, d\,\eta\right]$$

where $Q_\uparrow$ denotes the perimeter of the square $0 \leq x \leq 1, 0 \leq y \leq 1$ in the sense OABC. The integral over the vertical sides AB, CD will be denoted by $\int_\uparrow$ and it equals

$$\int_\uparrow k_1(\xi,\eta)\frac{\partial H}{\partial \eta}\,d\eta = \int_\uparrow k_1\,dH_1 = \int_\uparrow d(k_1H) - \int_\uparrow H dk_1$$
$$= -\int_\uparrow H\,\frac{\partial k_1}{\partial \eta}\,d\eta + (k_1H)_{(1,1)} - (k_1H)_{(1,0)}$$
$$- (k_1H)_{(0,1)} + (k_1H)_{(0,0)}$$

Finally **119**

$$M_1(\lambda,\mu) = a_1 + b_1e^\lambda + c_1e^\mu + d_1e^{\lambda+\mu}$$
$$+ \frac{1}{\lambda\mu}\Big[k_1(1,1)e^{\lambda+\mu} - k_1(1,0)e^\lambda - k_1(0,1)e^\mu + k_1(0,0)$$
$$\int_{Q_\uparrow} e^{\lambda\xi+\mu\eta}\left[\frac{\partial k_1}{\partial \xi}d\xi - \frac{\partial k_1}{\partial \eta}d\eta\right] + \int_o^1\int_o^1 e^{\lambda\xi+\mu\eta}\frac{\partial_1^k}{\partial\xi\partial\eta}d\,\xi\,d\,\eta$$

and we have a similar expression for $M_2(\lambda,\mu)$ i.e.

$$M_1(\lambda,\mu) = a_1 + b_1e^\lambda + c_1e^\mu + d_1e^{\lambda+\mu} + \frac{\bar{M}_1(\lambda,\mu)}{\lambda\mu}$$
$$M_2(\lambda,\mu) = a_2 + b_2e^\lambda + c_2e^\mu + d_2e^{\lambda+\mu} + \frac{\bar{M}_2(\lambda\mu)}{\lambda\mu}$$

where the functions $\bar{M}_1(\lambda,\mu)$ and $\bar{M}_2(\lambda,\mu)$ are entire functions of exponential type which remains bounded when (λ,μ) lie in a vertical band of C^2 so that $|\frac{M_1(\lambda,\mu)}{\lambda\mu}| \to 0.$ as $\lambda,\mu \to \infty$ in a vertical band and in particular when $(\lambda,\mu) \in (\sigma)$ by proposition 1. Thus the spectrum (σ) is asymptotic with the solutions of **120**

$$\phi_1(\lambda,\mu) = a_1 + b_1e^\lambda + c_1e^\mu + d_1e^{\lambda+\mu} = 0$$
$$\phi_2(\lambda,\mu) = a_2 + b_2e^\lambda + c_2e^\mu + d_2e^{\lambda,\mu} = 0$$

Setting $e^\lambda = X$ and $e^\mu = Y$ we obtain (H_1) and (H_2) and the required result follows.

Corollary. *$|D(\lambda,\mu)|, D(\lambda,\mu)$ being the Jacobian of $M_1(\lambda,\mu), M_2(\lambda,\mu)$ possesses a positive lower bound on (σ). $D(\lambda,\mu)$ it asymptotic with the Jacobian of $\phi_1(\lambda,\mu)$ and $\phi_2(\lambda,\mu)$ which is*

$$\frac{\partial^2(1_1,\varphi_2)}{(\lambda,\mu)} = \begin{vmatrix} b_1e^{\lambda} + d_1e^{\lambda+\mu} & c_1e^{\mu} + d_1e^{\lambda+\mu} \\ b_2e^{\lambda} + d_2e^{\lambda+\mu} & c_2e^{\mu} + d_2e^{\lambda+\mu} \end{vmatrix}$$

When $(\lambda,\mu) \in (\sigma), (e^{\lambda}, e^{\mu}) = (\xi_i, \eta_i)\, i = 1, 2, (\xi_i, \eta_i)$ being the common points of (H_1) and (H_2). Then

$$\Big[\frac{\partial(\varphi_1,\phi_2)}{\partial(\lambda,\mu)}\Big]_{(\lambda,\mu)\in(\sigma)} = \xi_i\,\eta_i\Big[(b_1c_2-b_2c_1)+(b_1d_2-b_2d_1)\xi_i+(d_1c_2-d_2c_1)\eta_i\Big]$$

for $i = 1, 2$.

121 But the expression in the square bracket is precisely the Jacobian of (H_1) and (H_2) at the common point (x_i, η_i) which is distinct from zero since the two hyperbolas do not touch each other. In view of the conditions of proposition 1 and the remarks following the proposition $\xi_i\eta_i \neq 0$. Thus $|D(\lambda,\mu)|$ for $(\lambda,\mu) \in (\sigma)$ is asymptotic with two nonzero values. Further by the usual hypothesis that the spectrum (σ) is 'simple' i.e. $D(\lambda,\mu) \neq 0$ for any $(\lambda,\mu) \in (\sigma)$ it follows that $|D(\lambda,\mu)|$ possesses a positive (strictly) lower bound.

Let A_1 and A_2 be Fourier- Laplace transforms of two densities U_1 and U_2 with supports in the square $0 \leq x \leq 1, 0 \leq y \leq 1$

$$U_1 * F = \iint U_1(\xi,\eta)F(x+\xi, y+\eta)d\,\xi\, d\eta$$
$$U_2 * F = \iint U_2(\xi,\eta)F(x+\xi, y+\eta)d\xi\, d\eta$$

We wish to prove the following

Theorem. *The formula*

$$\frac{J(A,M)}{J(\lambda,\alpha)} = \sum_{\substack{\beta\neq\alpha \\ \beta\in(\sigma)}} \frac{[\lambda-\alpha]}{D(\beta)[\lambda-\beta][\beta-\alpha]}\frac{J(M,M)}{J(\lambda,\beta)}\frac{J(A,M)}{J(\beta,\alpha)} + \mathscr{R}$$

holds if M_1 and M_2 are Fourier- Laplace transforms of distributions T_1 and T_2 given by

$$T_i * F = a_iF(x,y) + b_iF(x+1,y) + c_iF(x,y+1) + d_iF(x+1,y+1) + \iint k_i(\xi,\eta)F(x+\xi,y+\eta)d\xi\eta \quad i = 1,2$$

(provided the functions $k_i(\xi,\eta)$ and the coefficients $a_i, b_i, c_i, d_i, i = 1,2,$ satisfy the required conditions in order that the spectrum (σ) should have the desirable properties; cf. propositions proved above). 122

If T_1, T_2, U_1, U_2 be replaced by $T_1(\eta), T_2^{(n)}, U_1^{(n)}, U_2^{(n)}$ respectively where

$$T_i^n * F = a_iF(x,y) + b_iF(x+1,y) + c_iF(x,y+1) + d_iF(x+1,y+1) + \sum_{p,q=0}^{n-1} \frac{1}{n^2} k_i\left(\frac{p}{n},\frac{q}{n}\right) F\left(x+\frac{p}{n}, y+\frac{q}{n}\right), i = 1,2; \quad \text{and}$$

$$U_i^n * F = \sum_{p,q=0}^{n-1} \frac{1}{n^2} U_1\left(\frac{p}{n},\frac{q}{n}\right) F\left(x+\frac{p}{n}, y+\frac{q}{n}\right),$$

then we know that the theorem holds for $M_1^n, M_2^n, A_1^n, A_2^n$ (with the obvious notation: $\mathscr{F}\mathscr{L}T_1^n = M_1^n \cdots$ etc). Hence the first step in the proof is to study the spectrum (σ^n) of T_1^n and T_2^n and its relation to (σ) as $n \to \infty$.

Proposition 3. *(σ^n) is contained in a vertical band which is independent of n.*

The proof of the proposition is analogous to that of Proposition 1 and we shall give only a partial verification. For instance we prove that if $(\lambda,\mu) \in (\sigma^n)$ and $\lambda = \lambda_o + i\lambda, \mu = \mu_o + i\,\mu_1$, then i) λ_o, μ_o cannot both tend to $+\infty$; ii) λ_o cannot tend to $+\infty$ when $\mu_o \to -\infty$, iii) when $|\lambda_o| < m < \infty$, then μ_o cannot tend to $+\infty$. 123

i) For $(\lambda,\mu) \in (\sigma^n)$, we have $M_i^n(\lambda,\mu) = 0, i = 1,2$. If $\lambda_o \to +\infty, \mu_o \to \infty$, we write

$$-d_1 = a_1e^{-\lambda-\mu} + b_1e^{-\mu} + c_1e^{-\lambda} + e^{-\lambda-\mu}\sum \frac{1}{n^2}k_1\left(\frac{p}{n},\frac{q}{n}\right)\exp\frac{\lambda p + \mu q}{n}$$

$$|d_1| \le |a_1|e^{-\lambda_o-\mu_o} + |b_1|e^{-\lambda_o} + |c_1|e^{-\lambda_o} + e^{-\lambda_o-\mu_o} L_1 \int_o^1 \int_o^1 e^{\lambda_o\xi+\mu_o\eta}\, d\xi d\eta$$

where $$L_1 = \sup_{\substack{0\le\xi\le 1\\ 0\le\eta\le 1}} k_1(\xi,\eta)$$

since $(x,y)\to e^{\lambda_o x+\mu_o y}$ is an increasing function of (x,y) when $\lambda_o,\mu_o>0$ which may be assumed to be the case since λ_o,μ_o both tend to $+\infty$.

Or

$$|d_1| \le |a_1|e^{-\lambda_o-\mu_o} + |b_1|e^{-\mu_o}|c_1|e^{-\lambda_o} + e^{-\lambda_o-\mu}L_1\frac{(e^{\lambda_o-1})(e^{\mu_o-1})}{\lambda_o\mu_o}$$
$$\to 0 \text{ as } \lambda_o,\mu_o\to+\infty.$$

124 This contradicts the hypothesis that $d_1\neq 0$.

ii) If $\lambda_o\to+\infty,\mu_o\to-\infty$, we write

$$-b_1 = a_1e^{-\lambda} + c_1e^{\mu-\lambda} + d_1e^{\mu} + e^{-\lambda}\sum_p\sum_q \frac{1}{n^2}k_1(\frac{p}{n},\frac{q}{n})\exp\frac{\lambda p+\mu q}{n}$$

$$|b_1| \le |a_1|e^{-\lambda_o} + |c_1|e^{\mu_o-\lambda_o} + |d_1|e^{\mu_o} + e^{-\lambda_o}\sum_{p=0}^{n-1}\sum_{q=0}^{n-1}\frac{1}{n^2}|k_1(\frac{p}{n},\frac{q}{n})|\exp\frac{\lambda_o p}{n}$$

since $\exp\frac{\mu_o q}{n}\le 1$

i.e., $$|b_1| \le |a_1|e^{-\lambda_o} + |c_1|e^{\mu_o-\lambda_o} + |d_1|e^{\mu_o} + e^{-\lambda_o}L_1\int_o^1\int_o^1 e^{\lambda_o\xi}\,d\eta d\xi$$
$$= |a_1|e^{-\lambda_o} + |c_1|e^{\mu_o-\lambda_o} + |d_1|e^{\mu_o} + e^{-\lambda_o}L_1\frac{e^{\lambda_o}-1}{\lambda_o}$$

$\to 0$ as $\lambda_o\to+\infty$ and $\mu_o\to-\infty$ which is not possible since $b_1\neq 0$.

iii) Suppose that $|\lambda_o|<m_1$ and $\mu_o\to+\infty$. We write

$$0 = a_1e^{-\mu} + b_1e^{\lambda-\mu} + c_1 + d_1e^{-\mu} + e^{-\mu}\sum_{p=0}^{n-1}\sum_{q=0}^{n-1}k_1\left(\frac{p}{n},\frac{q}{n}\right)\exp\frac{\lambda p+\mu q}{n}$$
$$0 = a_2e^{-\mu} + b_2e^{\lambda-\mu} + c_2 + d_2e^{-\mu} + e^{-\mu}\sum_{p=0}^{n-1}\sum_{q=0}^{n-1}k_2\left(\frac{p}{n},\frac{q}{n}\right)\exp\frac{\lambda p+\mu q}{n}$$

125 Eliminating e^{λ} from these we get

$$|c_1d_2 - c_2d_1| \leq |a_1d_2 - a_2d_1|e^{-\mu_o} + |b_1d_2 - b_2d_1|$$
$$e^{\lambda_o-\mu_o} + e^{-\mu_o}L_2\int\limits_o^1\int\limits_o^1 e^{\mu_o\eta}d\xi d\eta$$

where L_2 depends upon the supremum of $|d_1k_2(x,y) - d_2k_1(x,y)|e^{\lambda_o x}$ for $0 \leq x \leq 1, 0 \leq y \leq 1$. i.e. the last term is majorised by $e^{-\mu_o}\ L_2\dfrac{(e^{\mu_o}-1)}{\mu_o}$. Hence right hand side of the above inequality $\to 0$ when $|\lambda_o| < m$ and $\mu_o \to +\infty$. But this contradicts the assumption that $c_1d_2 - c_2d_1 \neq 0$.

Summation of the series $S_n(\lambda,\alpha_n)$.

$$S_n(\lambda,\alpha_n) = \sum_{\beta^n\in(\sigma^n)}\frac{[\lambda-\alpha^n]}{D(\beta^n)[\lambda-\beta^n][\beta^n-\alpha^n]}\frac{J(M^n,M^n)}{J(\lambda,\beta^n)}\frac{J(A^n,M^n)}{J(\beta^n,\alpha^n)}+\mathscr{R}^n$$

(where $\mathscr{R}^n$ can be got from $\mathscr{R}$ by replacing M, A by M^n, A^n respectively), where $\lambda = (\lambda_1,\lambda_2)$ denotes a point in C^2. The functions $M_1^n(\lambda), M_2^n(\lambda), A_1^n(\lambda), A_2^n(\lambda)$ are periodic in λ_1,λ_2 with periods $2\pi ni$. The same statement holds for the Jacobian $D_n(\lambda)$ of the functions $M^n(\lambda)$ as also the two determinants of Jacobi:

$$\frac{J(M^n,M^n)}{J(\lambda,\alpha)} \text{ and } \frac{J(A^n,M^n)}{J(\beta,\alpha)}$$

considered as functions of couples of points of $C^2, (\lambda,\alpha)$ and (β,α), 126
$\lambda,\alpha,\beta \in C^2$. These are therefore periodic functions of the four complex variables with the same period $2n\pi i$. Moreover M_1^n, and M_2^n are polynomials in $\exp\dfrac{\lambda_1}{n}, \exp\dfrac{\lambda_2}{n}$ so that the zeros β_n in the spectrum (σ^n) can be arranged in $2n^2$ classes, each of these classes being situated in a plane parallel to the purely imaginary plane of C^2 and forming in this plane a network of squares of sides $2\pi n$. By virtue of this remark, the series $S_n(\lambda,\alpha^n)$, which is absolutely convergent can be broken up in $2n^2$

partial sums corresponding to $2n^2$ classes in which the spectrum (σ^n) is divided. In each of these partial sums the factors

$$\frac{1}{D_n(\beta^n)}\frac{J(M^n,M^n)}{J(\lambda,\beta^n)}\frac{J(A^n,M^n)}{J(\beta^n,\alpha^n)}$$

has the same value for all the terms of the partial sum (λ, α^n are fixed) due to the periodicity of the functions $D_n, M_1^n, M_2^n, A_1^n, M_2^n$. We shall then choose a representative β^n in each class which will be fixed (for the class λ, σ^n, we choose α^n itself as its representative). It is necessary to calculate, for each class

$$\sum_{h=-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty}\frac{\lambda_1-\alpha_1^n}{(\lambda_1-\beta_1^n-2nh\pi i)(\beta_1^n+2nh\pi i-\alpha_1^n)}\frac{\lambda_2-\alpha_2^n}{(\lambda_2-\beta_2^n-2nk\pi i)(\beta_2^n+2nk\pi i-\alpha_2^n)}$$

127 if α^n does not belong to the class considered and

$$\sum_{h=-\infty}^{+\infty\prime}\sum_{k=-\infty}^{+\infty\prime}\frac{\lambda_1-\alpha_1^n}{(\lambda_1-\alpha_1^n-2nh'\pi i)(2nh\pi i)}\frac{\lambda_2-\alpha_2^n}{(\lambda_1-\alpha_1^n-2nk\pi i)(2nk\pi i)}$$

if α^n belongs to the class considered. ($\sum'\sum'$ denotes that that value $h=0, k=0$ is excluded in the summation). The calculation of these two sums is classical and gives

$$\frac{1}{n^2}\left[\frac{1}{\exp\left(\frac{\lambda_1-\beta_1^n}{n}\right)-1}-\frac{1}{\exp\frac{\alpha_1^n-\beta_1^n}{n}-1}\right]\left[\frac{1}{\exp\left(\frac{\lambda_2-\beta_2^n}{n}\right)-1}-\frac{1}{\exp\frac{\alpha_2^n-\beta_2^n}{n}-1}\right]$$

in the first case and

$$\frac{1}{n^2}\left[\frac{1}{\exp\left(\frac{\lambda_1-\beta_1^n}{n}\right)-1}-\frac{n}{\lambda_1-\alpha_1^n}+\frac{1}{2}\right]\left[\frac{1}{\exp\frac{\lambda_2-\alpha_2^n}{n}-1}-\frac{n}{\lambda_2-\alpha_2^n}+\frac{1}{2}\right]$$

is the second.

As $\dfrac{J(A^n, M^n)}{J(\lambda^n, \alpha^n)} = 0$ since $\alpha^n \in e(\sigma^n)$, we see that the series $S_n(\lambda, \alpha^n)$ can now be put in the form of a finite sum (with $2n^2 - 1$ terms) each of these terms corresponding to classes in which the spectrum (σ^n) is divided. Denoting the classes by capital letters,

$$a_n = \text{ class of } \alpha^n, \mathscr{B}_n = \text{ class of } \beta^n,$$

the representatives α^n, β^n being fixed in their class, we can write **128**

$$S_n(\lambda, \alpha^n) = \frac{1}{n_2} \sum_{\mathscr{B}^n \neq a^n} \left[\frac{1}{\exp\left(\frac{\lambda_1 - \beta_1^n}{n}\right) - 1} - \frac{1}{\exp\left(\frac{\alpha_1^n - \beta_1^n}{n}\right) - 1} \right]$$
$$\left[\frac{1}{\exp\left(\frac{\lambda_2 - \beta_2^n}{n}\right) - 1} - \frac{1}{\exp\left(\frac{\alpha_2^n - \beta_2^n}{n}\right) - 1} \right]$$
$$\frac{1}{D_n(\beta^n)} \frac{J(M^n, M^n)}{J(\lambda, \beta^n)} \frac{J(A^n, M^n)}{J(\beta^n, \alpha^n)} \tag{1}$$

Behaviour of $S_n(\lambda, \alpha^n)$ for n large.

Applying Taylor's formula for the function $\dfrac{z}{e^z - 1}$ which is holomorphic in the neighbourhood of the origin, we have

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \frac{x^2}{2\pi i} \int_0 \frac{dz}{z(z - x)(e^z - 1)}$$

where C is the circumference of a circle with centre origin and radius $< 2\pi$ (2π is the radius of convergence of the Taylor's series of the function about the origin). Let $x = \dfrac{\lambda}{n}$ with $|\lambda| < \pi n$ so that $|x| < \pi$. Dividing by λ, we have

$$\frac{1}{n(e^{1/n} - 1)} - \frac{1}{\lambda} + \frac{1}{2\pi} = \frac{\lambda}{2\pi\, i\, n^2} \int\limits_C \frac{dz}{z(z - \frac{\lambda}{n})(e^z - 1)}, \lambda \neq 0$$

similarly for $|\mu| < \pi\, n$, **129**

$$\frac{1}{n(e^{\mu/n}-1)}-\frac{1}{\mu}+\frac{1}{2n}=\frac{\mu}{2\pi\, i\, n^2}\int\limits_C \frac{dz}{z(z-\frac{\mu}{n})(e^z-1)}, \mu \neq 0$$

subtracting,

$$\frac{1}{n(e^{\lambda/n}-1)}-\frac{1}{n(e^{\mu/n}-1)}-\left(\frac{1}{\lambda}-\frac{1}{\mu}\right)=\frac{1}{2\pi in}\int\limits_C\left[\frac{\frac{\lambda}{n}}{z-\frac{\lambda}{n}}-\frac{\frac{\mu}{n}}{z-\frac{\lambda}{n}}\right]$$

$$\frac{dz}{z(e^z-1)}=\frac{\lambda-\mu}{2\pi in^2}\int\limits_C\frac{dz}{\left(z-\frac{\lambda}{n}\right)\left(z-\frac{\mu}{n}\right)(e^z-1)}$$

The length of C is $< 4\pi^2, |\frac{\lambda}{n}| < \pi$ and $|\frac{\mu}{n}| < \pi$. Hence $|z-\frac{\lambda}{n}| \geq \pi, |z-\frac{\mu}{n}| \geq \pi$; let M denote $\max\limits_{|z|=R} |\frac{1}{e^z-1}|, (R < 2\pi)$. Then

$$\left|\frac{1}{n(e^{\lambda/n}-1)}-\frac{1}{n(e^{\mu/n}-1)}-\left(\frac{1}{\lambda}-\frac{1}{\mu}\right)\right| \leq C_o\ \frac{|\lambda-\mu|}{n^2}$$

with $C_o = \dfrac{2M}{\pi}$ and $|\lambda| < \pi n, |\mu| \leq \pi n$. Changing λ into $\lambda_1-\beta_1^n, \mu$ into $\alpha_1^n-\beta_1^n$ or λ into $\lambda_2-\beta_2^n, \mu$ into $\alpha_2^n-\beta_2^n$, we have the majorisation

$$\left|\frac{1}{n}\left[\frac{1}{\exp\left(\frac{\lambda_1-\beta_1^n}{n}\right)-1}-\frac{1}{\exp\left(\frac{\alpha_1^n-\beta_1^n}{n}\right)-1}\right]-\frac{\lambda_1-\alpha_1^n}{(\lambda_1-\beta_1^n)(\beta_1^n-\alpha_1^n)}\right|$$
$$\leq \frac{c_0}{n^2}|\lambda_1-\alpha_1^n|$$

$$\left|\frac{1}{n}\left[\frac{1}{\exp\left(\frac{\lambda_2-\beta_2^n}{n}\right)-1}-\frac{1}{\exp\left(\frac{\alpha_2^n-\beta_2^n}{n}\right)-1}\right]-\frac{\lambda_2-\alpha_2^n}{(\lambda_2-\beta_2^n)(\beta_2^n-\alpha_2^n)}\right|$$
$$\leq \frac{c_0}{n^2}|\lambda_2-\alpha_2^n|$$

130 provided that

$$\left.\begin{array}{l}|\lambda_1-\beta_1^n| \leq \pi_n, |\lambda_2-\beta_2^n| \leq \pi n\\ |\beta_1-\alpha_1^n| \leq \pi_n, |\beta_2^n-\alpha_2^n| \leq \pi n\end{array}\right\} \tag{2}$$

[Note that $|\lambda_1 - \alpha_1|$ and $|\lambda_2 - \alpha_2|$ are independent of ρ_1, β_2 and therefore fixed in the summation (1)].

Using the majorisation (2), we shall study (1) and compare it with a finite sum

$$\tau(\lambda, \alpha) = \frac{1}{n^2} \sum_{\beta \neq \alpha} \left[\frac{1}{\exp\left(\frac{\lambda_1 - \beta_1}{n}\right) - 1} - \frac{1}{\exp\left(\frac{\alpha_1 - \beta_1}{n}\right) - 1} \right] \left[\frac{1}{\exp\left(\frac{\lambda_2 - \beta_2}{n}\right) - 1} - \frac{1}{\exp\left(\frac{\lambda_2 - \rho_2}{n}\right) - 1} \right] \frac{1}{D(\beta)} \frac{J(M, M)}{J(\lambda, \beta)} \frac{J(A, M)}{J(\beta, \alpha)}.$$

The summation in β is made for $\beta \in (\sigma)$ which are "near" to those β^n **131**
which are the chosen representatives of the $2n^2 - 1$ classes $\mathscr{B}_n \neq a_n$. It is necessary for this to compare the functions M and M^n, A and $A^n, (\sigma)$ and (σ^n).

Comparison of M with M^n and of A with A^n.

$$\left.\begin{aligned} M_1(\lambda) &= a_1 + b_1 e^{\lambda_1} + c_1 e^{\lambda_2} + d_1 e^{\lambda_1 + \lambda_2} + N_1(\lambda) \\ M_2(\lambda) &= a_2 + b_2 e^{\lambda_1} + c_2 e^{\lambda_2} + d_2 e^{\lambda_1 + \lambda_2} + N_2(\lambda) \end{aligned}\right\} \tag{3}$$

$$\left.\begin{aligned} M_1^n(\lambda) &= a_1 + b_1 e^{\lambda_1} + c_1 e^{\lambda_2} + d_1 e^{\lambda_1 + \lambda_2} + N_1^n(\lambda) \\ M_2^n(\lambda) &= a_2 + b_2 e^{\lambda_1} + c_2 e^{\lambda_2} + d_2 e^{\lambda_1 + \lambda_2} + N_2^n(\lambda) \end{aligned}\right\} \tag{4}$$

Hence it is sufficient to compare N and N^n. The calculation will be similar for A and A^n.

We shall now suppose (for simplifying the proof) that the functions k_1, k_2, a_1, a_2 in R^2 are indefinitely differentiable with compact support **132**
contained in the square $0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1$. Then the functions $A(\lambda), N(\lambda)$ decrease rapidly when λ recedes to infinity keeping itself in a vertical plane. Hence the functions such as

$$\frac{1}{n^2} k_1\left(\frac{y_1}{n}, \frac{y_2}{n}\right) \exp\left(\frac{\lambda_1 y_1 + \lambda_2 y_2}{n}\right) = K_1^n(y_1, y_2)$$

are indefinitely differentiable with compact support contained in the square $0 \leq y_1 \leq n, 0 \leq y_2 \leq n$ and we can write for example

$$N_1^n(\lambda) = \sum_{p,q} K_1^n(p, q)$$

where the summation is made over all the couples of integers (p, q). In order to evaluate this sum it suffices to apply Poisson's formula. The Fourier transform of $K_1^n(y_1, y_2)$ is (μ_1, μ_2 being real):

$$\begin{aligned}\mathscr{F}[K_1^n] &= \iint K_1^n(y_1, y_2)\exp[-2\pi i(\mu_1 + \mu_2 y_2)]dy_1\, dy_2\\ &= \frac{1}{n^2}\iint K_1(y_1, y_2)\exp[\left(\frac{\lambda_1}{n} - 2\pi\mu_1\right)y_1 + \left(\frac{\lambda_2}{n} - 2\pi i\mu_2\right)y_2]dy_1 dy_2\\ &= \iint k_1(u)\exp\langle\lambda - 2\pi i\mu n, u\rangle du = N_1(\lambda - 2\pi i\mu n)\end{aligned}$$

Hence by Poisson's formula,

$$\left.\begin{aligned}N^n(\lambda) &= \sum_{h,k} N(\lambda_1 - 2\pi ihn, \lambda_2 - 2\pi ikn)\\ A^n(\lambda) &= \sum_{h,k} A(\lambda_1 - 2\pi ihn, \lambda_2 - 2\pi ikn)\end{aligned}\right\} \tag{5}$$

$$\left.\begin{aligned}N^n(\lambda) - N(\lambda) &= \sum_{h,k}{}' N(\lambda_1 - 2\pi ihn, \lambda_2 - 2\pi ikn) = M^n(\lambda) - M(\lambda)\\ A^n(\lambda) - A(\lambda) &= \sum_{h,k}{}' A(\lambda_1 - 2\pi ihn, \lambda_2 - 2\pi ikn)\end{aligned}\right\} \tag{6}$$

133 where the accent indicates that in the summation the couple $h = 0,\ k = 0$ is excluded.

Majorisation of the difference $N^n - N$ and $A^n - A$.

As N decreases rapidly in the vertical bound, we have

$$|N(\lambda_1, \lambda_2)| \le \frac{c_1(r)}{|\lambda_1\lambda_2|^r}$$

in a vertical band where r is a positive integer, arbitrarily large and where $c_1(r)$ is a constant which depends on r and the function $k_1(x_1, x_2)$. Hence

$$|M_1^n(\lambda) - M_1(\lambda)| \le c_1(r)\sum_{h,k}{}' \frac{1}{|\lambda_1 - 2\pi inh|^r|\lambda_2 - 2\pi ink|^r}$$

(for λ_1 and λ_2 different from the multiples of $2\pi in$; this restriction is artificial).

Majorising the second member for $r \geq 3$, it is easy to verify that

$$|M_1^n(\lambda) - M_1(\lambda)| \leq \frac{c_2(r)}{(2\pi n)^r} \tag{7}$$

if

$$|Im\lambda_1| \leq \pi n, |Im\lambda_2| \leq \pi n \tag{8}$$

and we have analogous inequalities for the functions M_2, A_1, A_2 under the same conditions (8). We can always denote by $c_2(r)$ the positive constant figuring in the numerator of the second member of (1) by taking the same constant for the functions. 134

The volume V_n; zeros of $M(\lambda)$ in the interior of V_n.

We know that (σ) and (σ^n) are in a fixed vertical band $\mathscr{B}$ independent of n. We shall intersect the vertical band by a horizontal band in C^2 defined by (8). Its section by a vertical plane is a square of side $2\pi n$. Such a square contains one and only one point of each of the $2n^2$ classes in which (σ^n) is decomposed, since each of these $2n^2$ classes form, in its plane, which is vertical, a network of squares of side $2\pi n$. It follows that the volume V_n in C^2 contains exactly $2n^2$ points of the spectrum (σ^n). We wish to find the points of (σ) which are also in V_n. We first recall a classical result due to Kronecker.

Let f, g, h be three functions continuously differentiable in a region in R^3. Let V be a volume contained in the region with boundary S. Then

$$m = -\frac{1}{4\pi}\iint\limits_S (A\cos\lambda + B\cos\mu + C\cos\nu)dS$$

where $$A = \left[f\frac{D(g,h)}{D(y,z)} + g\frac{D(h,f)}{D(y,z)} + h\frac{D(f,g)}{D(y,z)}\right]\frac{1}{[f^2+g^2+h^2]^{3/2}}$$

B, C being analogously defined and $\cos\lambda, \cos\mu, \cos, \gamma$ are the direction cosines of the interior normal to S and m equals the difference between the number of solutions lying in V of the systems $f = g = h = 0$ for 135 which $\dfrac{D(f,g,h)}{D(x,y,z)} > 0$ and the number of solutions for which $\dfrac{D(f,g,h)}{D(x,y,z)} < 0$. We shall use the analogue of this proposition in $R^4 = C^2$ i.e.,

If f_1, f_2, f_3, f_4 are four functions which are $(C, 1)$ in a region of R^4, then

$$m = -\frac{1}{2\pi^2}\iiint(A_1\cos\lambda_1 + A_2\cos\lambda_2 + A_3 + A_4\cos\lambda_4)dS$$

with
$$A_1 = \left[\sum_{f_1}\frac{D(f_2, f_3, f_4)}{D(x_2, x_3, x_4)}\right]\frac{1}{\left[\sum\limits_{1=1}^{4} f_i^2\right]^2}$$

and A_2, A_3, A_4 similarly defined, where m is defined as above for the system of equations $f_1 = f_2 = f_3 = f_4 = 0$ and the Jacobian $\frac{D(f_1,f_2,f_3,f_4)}{D(x_1,x_2,x_3,x_4)}$. The integral on the right hand side will be called the Kronecker integral.

We consider the following analytic transformation of C^2 into itself,

$$(\lambda_1, \lambda_2) \to (M_1(\lambda_1, \lambda_2), M_2(\lambda_1, \lambda_2))$$

136 This can be considered as a transformation of R^4 into itself. Instead of the four variables which are the real and imaginary parts of both λ_1, λ_2, we take $\lambda_1, \bar{\lambda}_1, \lambda_2, \bar{\lambda}_2$. Then

$$dM_1 = \frac{\partial M_1}{\partial\lambda_1}d\lambda_1 + \frac{\partial M_1}{\partial\lambda_2}d\lambda_2 \text{ and}$$
$$dM_2 = \frac{\partial M_2}{\partial\lambda_1}d\lambda_1 + \frac{\partial M_2}{\partial\lambda_2}d\lambda_2,$$

as M_1, M_2 are analytic. The volume elements which correspond to each other by this transformation are proportional to

$$d\lambda_1 \wedge d\bar{\lambda}_1 \wedge d\lambda_2 \wedge d\bar{\lambda}_2 \text{ and } dM_1 \wedge d\bar{M}_1 \wedge dM_2 \wedge d\bar{M}_2$$

Now

$$d\bar{M}_1 = \frac{\partial\bar{M}_1}{\partial\lambda_1}d\bar{\lambda}_1 + \frac{\partial\bar{M}_1}{\partial\lambda_2}d\bar{\lambda}_2 \text{ and}$$
$$d\bar{M}_2 = \frac{\partial\bar{M}_2}{\partial\lambda_1}d\bar{\lambda}_1 + \frac{\partial\bar{M}_2}{\partial\lambda_2}d\bar{\lambda}_2$$

Hence

$$
\begin{aligned}
dM_1 \wedge d\bar{M}_1 \wedge dM_2 \wedge d\bar{M}_2 \\
&= \left[\frac{\partial M_1}{\partial \lambda_1} \overline{\frac{\partial M_1}{\partial \lambda_1}} \frac{\partial M_2}{\partial \lambda_2} \overline{\frac{\partial M_2}{\partial \lambda_2}} - \frac{\partial M_1}{\partial \lambda_1} \overline{\frac{\partial M_1}{\partial \lambda_2}} \frac{\partial M_2}{\partial \lambda_2} \overline{\frac{\partial M_2}{\partial \lambda_1}} - \frac{\partial M_1}{\partial \lambda_2} \overline{\frac{\partial M_1}{\partial \lambda_1}} \frac{\partial M_2}{\partial \lambda_1} \right. \\
&\qquad \left. \overline{\frac{\partial M_2}{\partial \lambda_2}} + \frac{\partial M_1}{\partial \lambda_2} \overline{\frac{\partial M_1}{\partial \lambda_2}} \frac{\partial M_2}{\partial \lambda_1} \overline{\frac{\partial M_2}{\partial \lambda_1}} \right] d\lambda_1 \wedge d\bar{\lambda_1} \wedge \lambda_2 \wedge d\bar{\lambda_2} \\
&= \frac{D(M_1, M_2)}{D(\lambda_1, \lambda_2)} \overline{\frac{D(M_1, M_2)}{D(\lambda_1, \lambda_2)}} d\lambda_1 \wedge d\bar{\lambda_1} \wedge d\lambda_2 \wedge d\bar{\lambda_2}
\end{aligned}
$$

Thus the Jacobian of the transformation under consideration is always real and ≥ 0. Taking real and imaginary parts of M_1 and M_2 the system $M_1 = 0, M_2 = 0$ is equivalent to the four equations $f_i = 0$ $i = 1, 2, 3, 4$. The Kronecker integral which can be briefly denoted by $-\dfrac{1}{2\pi^2}$ $\iiint K(M_1, M_2) dS$ in this case gives m exactly equal to the number of solutions of the system $f_i = 0$ in the volume V enclosed by S (since the Jacobian does not change sign) i.e., the number of elements of $(\sigma) inV$. 137

Proposition 4. *The volume V_n contains $2n^2$ points of σ for n sufficiently large.*

We know that V_n contains $2n^2$ points of (σ^n). Using Kronecker's result it follows that

$$2n^2 = \int_{\partial V_n} K(M_1^n, M_2^n) dS$$

where ∂V_n denotes the boundary of V_n whose measure is of the form $c_3 n^2$, where c_3 is a fixed constant (viz. the product of $4\pi^2$ by the length of the parameter of the right section of the vertical band $\mathscr{B}$) and the proposition will be proved if we know that

$$2n^2 = \int_{\partial V_n} K(M_1, M_2) dS \tag{i}$$

Consider the difference $K(M_1, M_2) - K(M_1^n, M_2^n)$ in ∂V_n. Let $\lambda_1 = x_1 + ix_2, \lambda_2 = x_3 + ix_4, M_1 = f_1 + if_2, M_2 = f_3 + if_4, M_1^n = f_1^n + if_2^n, M_2^n =$ 138

$f_3^n + i f_4^n$ where the f's are real valued functions of the four real variables x_1, x_2, x_3, x_4. For $(\lambda_1, \lambda_2) \in V_n, |M_i - M_i^n| \leq \dfrac{c_2(r)}{(2\pi n)^r} i = 1, 2$. Hence

$$|f_i - f_i^n| \leq |\frac{c_2(r)}{(2\pi n)^r}\ i = 1, 2, 3, 4.$$

Similarly $|\dfrac{\partial f_i}{\partial x_j} - \dfrac{\partial f_i^n}{\partial x_j}| \leq \dfrac{c_2'(r)}{(2\pi n)^r}$ for $(\lambda_1, \lambda_2) \in V_n$, since the partial derivatives of the f_i with respect to x_j are exactly of the same form as the f_i.

We consider the difference $K(M_1, M_2) - K(M_1^n, M_1^n)$ on ∂v_n. In $K(M_1, M_2)$ the f_i and $\dfrac{\partial f_i}{\partial x_j}$ may be regarded as a finite set of variables $u_1, u_2, \ldots$ and $K(M_1^n, M_2^n)$ is the same function with the variables f_i and $\dfrac{\partial f_i}{\partial x_j}$ replaced by f_i^n and $\dfrac{\partial f_i^n}{\partial x_j}$ respectively or the variables $u_1, u_2, \ldots$ replaced by $u_1^n, u_2^n, \ldots$ respectively. Hence applying mean value theorem of differential calculus,

$$|K(M_1, M_2) - K(M_1^n, M_2^n)| \leq \frac{c_3(r)}{(2\pi n)^r} L \text{ on } \partial V_n \tag{ii}$$

where L depends on the maximum modulus of the derivatives of K
139 with respect to $u_1, u_2, \ldots$ over a region which contains $(u_1, u_2, \ldots)$ as also $(u_1^n, u_2^n, \ldots)$ while $(\lambda_1, \lambda_2) = (x_1, x_2, x_3, x_4)$ varies in ∂V_n. Since $|u_i - u_i^n| = 0(\dfrac{1}{n^r})$ in V_n and therefore on ∂V_n, for $n \geq N_1$, and since $\dfrac{\partial K}{\partial u_i}$ are continuous functions of $(u_1, u_2, \ldots)$ in estimating L it is enough to consider the maximum modulus of $\dfrac{\partial K}{\partial u_i}$ when $(\lambda_1, \lambda_2) \in \partial V_n$. Now the numerator of $K(u_1, u_2, \ldots)$ is a homogeneous polynomial in all the u's of total degree 4 with coefficients which are functions of λ_1, λ_2 with maximum moduls ≤ 1 and the denominator is $2\pi^2(u_1^2 + u_2^2 + u_3^2 + u_4^2)$ or $2\pi^2[|M_1(\lambda)|^2 + |M_2(\lambda)|^2]$. Both the numerator and denominator as also their partial derivatives with respect to u_i are uniformly bounded on ∂V_n since they are uniformly bounded on $\mathscr{B}$ and V_n is a closed subset of $\mathscr{B}$. Hence L can be found to be a fixed positive number which

does not depend on n if we prove that the denominator of K i. e. $2\pi^2[|M_1(\lambda)|^2 + |M_2(\lambda)|^2]$ is bounded below uniformly for $(\lambda_1, \lambda_2) \in \partial V_n$ by a fixed number > 0 which does not depend on n.

First we consider two vertical parts of ∂V_n, denoted by $(\partial V_n)_1$ i. e. parts contained in $\partial\mathscr{B}$. On $\partial\mathscr{B}$ real parts of λ_1, λ_2 are constant and the imaginary parts vary from $-\infty$ to $+\infty$. We can suppose that the principal parts

$$\phi_1(\lambda_1, \lambda_2) = a_1 + b_1 e^{\lambda_1} + c_1 e^{\lambda_2} + d_1 e^{\lambda_1+\lambda_2} \text{ and}$$
$$\phi_2(\lambda_1, \lambda_2) = a_2 + b_2 e^{\lambda_1} + c_2 e^{\lambda_2} + d_2 e^{\lambda_1+\lambda_2}$$

of $M_1(\lambda)$ and $M_2(\lambda)$ respectively do not vanish on $\partial\mathscr{B}$ (we have only to 140
choose $\mathscr{B}$ suitably) and have their moduli bounded below by $m_1 > 0$ if $(\lambda_1, \lambda_2) \in \partial\mathscr{B}$ and $|Im\lambda_1| \leq \pi$ and $|Im\lambda_2| \leq \pi$. But ϕ_1 and ϕ_2 are periodic in λ_1 and λ_2 with periods $2\pi i$ so that $|\phi_1| > m_1$ and $|\phi_2| > m_1$ on $\partial\mathscr{B}$. Now given $\dfrac{m_1}{2} > 0$, we can find a compact set K_1 such that $\left|M_i(\lambda_1, \lambda_2) - \phi_i(\lambda_1, \lambda_2)\right| < \dfrac{m_1}{2}, i = 1, 2$ for $(\lambda_1, \lambda_2) \notin K_1$ by Proposition 2. Hence for $(\lambda_1, \lambda_2) \in (\partial V_n)_1$ and $(\lambda_1, \lambda_2) \notin K_1, |M_i(\lambda_1, \lambda_2)| > \dfrac{m_1}{2}, i = 1, 2$. As M_1, M_2 do not vanish on $\partial\mathscr{B}$ and therefore on $(\partial V_n)_1 \cap K_1, |M_1|$ and $|M_2| > m_2 > 0$ on $(\partial V_n)_1 \cap K_1$ so that if $m = Min(\dfrac{m_1}{2}, m_2) > 0, 2\pi^2(|M_1(\lambda)|^2 + |M_2(\lambda)|^2) > 2\pi^2 m^2 > 0$ on $(\partial V_n)_1$. On the horizontal part $(\partial V_n)_2$ of ∂V_n i. e. where $|Im\lambda_1| = \pi_n$ and $|Im\lambda_2| = \pi_n$, since the $a's, b's \cdots etc.$ are generic, we may suppose that ϕ_1 and ϕ_2 do not vanish on $(\partial V_n)_2$ so that ϕ_1 and ϕ_2 have a lower bound $m' > 0$ on $(\partial V_n)_2$ (since $(\partial V_n)_n$ is compact) and m' is independent of n because of the periodicity 141
of ϕ_1 and ϕ_2. Now given $\dfrac{m'}{2}$, there exists a compact set outside which $|M_i - \phi_i| < \dfrac{m'}{2}, i = 1, 2$. Also for $n > N_2, (\partial V_n)_2$ lies out side this compact set so that $|M_1|$ and $|M_2| \geq \dfrac{m'}{2} > 0$ on $(\partial V_n)_2$ for $n \geq N_2$. Thus $(|M_1(\lambda)|^2 + |M_2(\lambda)|^2 \geq Min(m^2, \dfrac{m^{12}}{2} > 0)$ on ∂V_n for $n \geq N_2$.

Thus the moduli of the denominator and numerator of $K(u_1, u_2, \ldots)$ as also their partial derivatives are bounded above uniformly on ∂V_n and the modulus of the denominator is bounded below by a strictly positive number on ∂V_n for $n \geq N_2$, so that $|\dfrac{\partial K}{\partial u_i}|$ is bounded above by a fixed number on ∂V_n independent of $n \geq N_2$. Hence for all $n \geq N_3 = Max(N_1, N_2)$ there exists a fixed positive L satisfying (ii) independent of $n \geq N_3$. Hence

$$\Big|\int\limits_{\partial V_n} K(M_1, M_2)dS - \int\limits_{\partial V_n} K(M_1^n, M_2^n)dS\Big| \leq \frac{c_3(r)}{(2\pi n)^r} Lc_3 n^2$$
$$= \frac{c_4(r)}{n^{r-2}} \text{ for } n \geq N_3, \text{ and } \frac{c_4(r)}{n^{r-2}} < 1 \text{ for } n \geq N_4.$$

But each of the integrals equals an integer and their difference has to be zero for $n \geq N = Max\{N_3, N_4\}$, and $\int_{\partial V_n} K(M_1, M_2)dS = \int\limits_{\partial V_n} K(M_1^n, M_2^n)dS = 2n^2$.

142 **Proposition 5.** *For n sufficiently large ($n \geq n_o$ fixed), there exists a one-to-one correspondence between the $2n^2$ points of (σ) and of (σ^n) contained in the volume V_n such that the distance between the corresponding points of (σ) and of (σ^n) is uniformly majorised in V_n by $\dfrac{g_5(r)}{n^{r/2}}$ where $c_5(r) > 0$ depends only on the maximum modulus of the real and imaginary parts of M_1 and M_2 as also their partial derivatives with respect to real coordinates.*

As M^n converges to M uniformly on each compact subset of C^2, it is easy to see using Kronecker's integral that in any arbitrary neighbourhood of a point of (σ), there exists a point of (σ^n) for n sufficiently large. Also by Proposition 4, for n sufficiently large, both (σ) and (σ^n) have the same number $(= 2n^2)$ of points in V_n. Now we show that if we describe a sphere of radius $\dfrac{c_5(r)}{n^{r/2}}$ about each of the $2n^2$ points of (σ) in V_n, then there exists in the interior of each of these spheres a point of (σ^n) lying in V_n.

Let $S(\alpha, \in)$ denote the sphere of centre α and radius ε. The points of (σ) are asymptotic with (σ') which consist of points

$$(\lambda_1\lambda_2), \lambda_1 = \alpha_1' + 2h\pi i \qquad \qquad \lambda_1 = \beta_1' + 2h'\pi i$$
$$\lambda_2 = \alpha_2' + 2k\pi i \quad \text{and} \quad \lambda_2 = \beta_2' + 2k\pi i$$

(by Proposition 2). It follows therefore that for ε sufficiently small, the sphere $S(\alpha,\varepsilon)$ does not contain any other point of (σ) so that $\int\limits_{\partial S(\alpha,\varepsilon)}$ **143**
$K(M_1, M_2)dS = 1 \forall \alpha \varepsilon (\sigma)$. We shall establish the proposition by comparing this integral with $\int\limits_{\partial S(\alpha,\varepsilon)} K(M_1^n, M_2^n)dS$.

The denominator of $K(M_1, M_2)$ which is $2\pi^2[|M_1(\lambda)|^2 + |M_2(\lambda)|^2]^2$ is infinitely small of fourth order in ε on $\partial S(\alpha,\varepsilon)$. We have $\lambda = (\lambda_1, \lambda_2) = (x_1, x_2, x_3, x_4) = x$ and

$$M_1(\lambda) = f_1(x) + i f_2(x), M_2(\lambda) = f_3(x) + i f_4(x).$$

As $x = \alpha$ is a zero of each of f_j,

$$f_j(x) = \varepsilon \sum \frac{x_i - x_i^\alpha}{\varepsilon} \frac{\partial f_j}{\partial x_i^\alpha} + \varepsilon^2 \sum \frac{(x_i - x_i^\alpha)}{\varepsilon} \frac{(x_k - x_k^\alpha)}{\varepsilon} \left[\frac{\partial^2 f_j}{\partial x_i \partial x_k}\right]_{x=x'}$$

where x' is some point of $S(\alpha,\varepsilon)$

$$\begin{aligned} |M_1(\lambda)|^2 + |M_2(\lambda)|^2 &= f_1^2 + f_2^2 + f_3^2 + f_4^2 \\ &= \varepsilon^2 \sum_{j=1}^{4} \left(\sum_i \frac{x_i - x_i^\alpha}{\varepsilon} \frac{\partial f_j}{\partial x_i^\alpha}\right)^2 + \varepsilon^3 \psi(x,\alpha) \cdots (i) \end{aligned}$$

For $\alpha\varepsilon(\sigma)$ and $x\varepsilon S(\alpha,\varepsilon)$, $|\psi(x,\alpha)|$ is bounded above by M say, uniformly for $\alpha\varepsilon(\sigma)$ since the f_j as also their partial derivatives are uniformly bounded in the vertical band $\mathscr{B}$. As α is a simple zero $M_1(\alpha)$ and $M_2(\alpha)$, $\sum_j \left(\sum\limits_i \frac{x_i - x_i^\alpha}{\varepsilon} \frac{\alpha f_j}{\partial x_i^\alpha}\right)^2$ is positive definite for $x\varepsilon\partial S(\alpha,\varepsilon)$ and has a **144**
strictly positive minimum $A(\alpha)$ depending α. We know that at a great distance in $\mathscr{B}$, M_1 and M_2 behave as their principal parts ϕ_1 and ϕ_2 respectively and the same is true for their corresponding real and imaginary parts as also their first partial derivatives with respect to real coordinates. We observe that for $x\varepsilon\partial S'(\alpha',\varepsilon)$, $\sum_j \left[\sum\limits_i \frac{x_i - x_i^\alpha}{\varepsilon} \frac{\alpha g_j}{\partial x_i^\alpha}\right]^2$ (where g_j are the real and imaginary parts of ϕ_1 and ϕ_2 and other obvious notation) is a positive definite quadratic form with strictly positive minimum

$B(\alpha')$. But g_j's are periodic in x_3 and x_4 as also their partial derivatives and therefore $B(\alpha')$ has a lower bound $m' > 0$ independent of $\alpha'\varepsilon(\sigma')$. Let α, α' denote points of (σ) and (σ') respectively which are very near to each other (at a great distance in $\mathscr{B}$). We have

$$A(\alpha) = \min_{x\varepsilon\partial S(\alpha,\varepsilon)} \sum_j \left[\sum_i \frac{x_i - x_i^\alpha}{\varepsilon} \frac{\partial f_j}{\partial x_i^\alpha}\right]^2$$

$$b(\alpha') = \min_{x\varepsilon\partial S'(\alpha',\varepsilon')} \sum_j \left[\sum_i \frac{x_i - x_i^\alpha}{\varepsilon} \frac{\partial g_j}{\partial x_i^{\alpha'}}\right]^2$$

145 Let

$$B(\alpha) = \min_{x\varepsilon\partial S(\alpha,\varepsilon)} \left\{\sum_j \left[\sum_i \frac{x_i - x_i^\alpha}{\varepsilon} \frac{\partial g_i}{\partial x_i^\alpha}\right]^2\right\}.$$

The $g_j's$ are periodic functions of x_3 and x_4 and hence are uniformly continuous in $\mathscr{B}$; so are $\dfrac{\partial g_j}{\partial x_i}$. Hence $B(\alpha')$ is a uniformly continuous function of $\alpha'\varepsilon\mathscr{B}$; i.e. given $\dfrac{m'}{4} > 0$, there exists a $\delta > 0$ such that $|\alpha - \alpha'| < \delta$ implies that $|B(\alpha) - B(\alpha')| < \dfrac{m'}{4}$. Now given δ, there exists a compact set K_1 such that $|\alpha - \alpha'| < \delta$ for $\alpha \notin K_1$ and given $\dfrac{m'}{4}$, there exists a compact set K_2 such that $|A(\alpha) - B(\beta)| < \dfrac{m'}{4}$ for $\alpha \notin K_2$. Hence for $\alpha \notin K_1 \cup K_2$,

$$\begin{aligned} A(\alpha) &> B(\alpha') - |B(\alpha) - B(\alpha')| - |A(\alpha) - B(\alpha)| \\ &> m' - \frac{m'}{4} - \frac{m'}{4} = \frac{m'}{2} > 0. \end{aligned}$$

Also $K_1 \cup K_2$ contain only a finite number of $\alpha\varepsilon(\sigma)$ lying in V_n. Hence $A(\alpha) > M'' > 0$ for $\alpha\varepsilon K_1 \cup K_2, \alpha\varepsilon V_n$. From (i),.

$$|M_1(\lambda)|^2 + || |M_2(\lambda)|^2 > m_1\varepsilon^2 - \varepsilon^3 M \qquad \text{where}$$

$$m_1 = \min\left\{\frac{m'}{2}, m''\right\}$$

146 Choosing ε small enough, $\varepsilon^3 M < \varepsilon^2 \dfrac{m_1}{2}$, so that $|M_1(\lambda)|^2 + |M_2(\lambda)|^2$ has a strictly positive lower bound $m\varepsilon^2$ for $\lambda\varepsilon\partial S(\alpha, \varepsilon)$ which does not depend on $\alpha\varepsilon(\sigma)$.

In order to compare the integrals $\int\limits_{\partial S(\alpha,\varepsilon)} K(M_1, M_2) dS$ and $\int\limits_{\partial S(\alpha,\varepsilon)} K(M_1^n, M_2^n) dS$ for $\alpha\varepsilon V_n$, we adopt the procedure in proposition 4 in which integrand K is treated as a function of $u_1, u_2, \ldots$ which are f_i and $\dfrac{\partial f_i}{\partial x_j}$ and apply the mean value theorem for differential calculus to the difference $K(u_1, u_2, \ldots) - K(u_1^n, u_2^n, \ldots) = K(M_1, M_2) - K(M_1^n, M_2^n)$. We suppose that $S(\alpha, \varepsilon) \subset V_n$ for $\alpha\varepsilon V_n$ which is possible if ε is sufficiently small.

$$K(M_1, M_2) - K(M_1^n, M_2^n) = \sum (u_i - u_i^n) \frac{\partial K}{\partial u_i},$$

since $|u_i - u_i^n| = 0\left(\dfrac{1}{n^r}\right)$,

$$|K(M_1, M_2) - K(M_1^n, M_2^n)| < \frac{c'(r)}{(2\pi n)^r} L$$

where L depends on the maximum modules of $\dfrac{\partial K}{\partial u_i}$ where $(\lambda_1, \lambda_2) = x\varepsilon\partial S(\alpha, \varepsilon)$. The derivatives of the numerator of K are uniformly bounded in $\mathscr{B}$ and therefore on $\partial S(\alpha, \varepsilon)$. In the derivatives of the denominator appears the term $\left[|M_1(\lambda)|^2 + |M_2(\lambda)|^2\right]^{-3}$, partially compensated in the numerator by terms which involve derivatives of $|M_1(\lambda)|^2 + |M_2(\lambda)|^2$ or $u_1^2 + u_2^2 + u_3^2 + u_4^3$. These letter term are uniformly majorised in V_n 147
and on $\partial S(\alpha, \varepsilon)$ by quantities which are of the first order in ε. Thus the term in the denominator are uniformly bounded in V_n and therefore on $\partial S(\alpha, \varepsilon)$ by a quantity of order ε^{-5} since $[|M_1(\lambda)|^2 + |M_2(\lambda)|^2]$ is uniformly bounded below on all $\partial S(\alpha, \varepsilon), \alpha\varepsilon V_n$, by $m\varepsilon^2$ with $m > 0$. The measure of $\partial S(\alpha, \varepsilon)$ being proportional to ε^3, we have for all the spheres $S(\alpha, \varepsilon)$ situated in V_n,

$$\left| \int\limits_{\partial S(\alpha,\varepsilon)} K(M_1, M_2) ds - \int\limits_{\sigma S(\alpha,\varepsilon)} K(M_1^n, M_2^n) ds \right| < \frac{c''(r)}{\varepsilon^2 n^r}$$

If $\varepsilon = \left[\dfrac{c''(r)}{n^r}\right]^{\frac{1}{2}} = \dfrac{c_5(r)}{n^{r/2}}$ and $n > n_0$ in order that be sufficiently small we shall have the right hand side of the above inequality < 1 and hence equal to zero as it is an integer and

$$\int\limits_{\partial S(\alpha,\varepsilon)} K(M_1^n, M_2^n)dS = \int\limits_{\partial S(\alpha,\varepsilon)} K(M_1, M_2)ds = 1.$$

The passage to the limit.

We now compare the series

$$S(\lambda,\alpha) = \sum_{\beta\neq\alpha} \frac{[\lambda-\alpha]}{D(\beta)[\lambda-\beta][\beta-\alpha]} \frac{J(M,M)}{J(\lambda,\beta)} \frac{J(A,M)}{J(\beta,\alpha)}$$

148 with

$$S_n(\lambda,\alpha^n) = \sum_{\substack{\beta^n\neq\alpha^n \\ \beta^n\varepsilon(\sigma^n)}} \frac{[\lambda-\alpha^n]}{D_n(\beta^n)[\lambda-\beta^n][\beta^n-\alpha^n]} \frac{J(A^n,M^n)J(A^n,M^n)}{J(\lambda,\alpha^n)J(\beta^n,\alpha^n)}$$

using the finite form of $S_n(\lambda,\alpha^n)$ given by (1). We first remark that the preceding properties permit on e to establish a sequence of one-to-one correspondences

$$a_n \leftrightarrow \alpha^n \leftrightarrow \alpha$$

among the set of $2n^2$ classes of zeros of M^n, the set of M^n which are in V_n and the set of zeros of M which are in the same volume, the distance between the two zeros of M and M^n being estimated in Proposition 5.

Let W_n be the volume which consists of points of $\mathscr{B}$ satisfying

$$|Im\lambda_1| \leq \pi n^{\frac{1}{4}}, |Im\lambda_2| \leq \pi n^{\frac{1}{4}}$$

λ being fixed, we can suppose that W_n contains the point λ for n sufficiently large (since one can always suppose that for λ fixed, $\mathscr{B}$ contains λ). We choose a_n, α, α^n such that $\alpha, \alpha^n \varepsilon W_n$. Then

$$|\lambda_1-\alpha_1|, |\lambda_2-\alpha_2|, |\lambda_1-\alpha_1^n|, |\lambda_2-\alpha_2^n|$$

149 are majorised by $c_6 n^{\frac{1}{4}}$.$|D(\beta)|$ is bounded away from zero for $\beta\varepsilon(\sigma)$ and therefore $|D(\beta^n)|$ by proposition 5 for $\beta^n \varepsilon V_n$ and therefore $|D_n(\beta^n)|$ for n sufficiently large and $\beta^n \varepsilon V_n$. Also A^n, M^n, A, M as also their derivatives are uniformly bounded so that by (7) and (8),

$$\frac{1}{D_n(\beta^n)}\frac{J(M^n,M^n)}{J(\lambda,\beta^n)}\frac{J(A^n,M^n)}{J(\beta^n,\alpha^n)} - \frac{1}{D(\beta^n)}\frac{J(M,M)}{J(\lambda,\beta^n)}\frac{J(A,M)}{J(\beta^n,\alpha^n)} = 0\left(\frac{1}{n^r}\right)$$

for $\beta, \beta^n \varepsilon V_n$.

Similarly by Proposition 4 and 5, it is clear that

$$\frac{1}{D_n(\beta^n)}\frac{J(M^n,M^n)}{J(\lambda,\alpha^n)}\frac{J(A^n,M^n)}{J(\alpha^n,\beta^n)} = \frac{1}{J(\beta^n)}\frac{J(M,M)}{J(\lambda,\beta)}\frac{J(A,M)}{J(\beta,\alpha)} + A_n \tag{9}$$

with

$$|A| < \frac{c_7(r)}{n^{r/2}}. \tag{10}$$

The constant $c_7(r)$ being the same for all the terms of (1).

The second factor

$$\frac{1}{n^2}\left[\frac{1}{\exp\frac{\lambda_1-\beta_1^n}{n}-1} - \frac{1}{\exp\frac{\alpha_1^n-\beta_1^n}{n}-1}\right]\left[\frac{1}{\exp\frac{\lambda_2-\beta_2^n}{n}-1} - \frac{1}{\exp\frac{\alpha_2^n-\beta_2^n}{n}-1}\right] \tag{11}$$

of the general term in (1) can be written as (because of (2))

$$\left[\frac{\lambda_1-\alpha_1^n}{(\lambda_1-\beta_1^n)(\beta_1^n-\alpha_1^n)} + B_n\right]\left[\frac{\lambda_2-\alpha_2^n}{(\lambda_2-\beta_2^n)(\beta_2^n-\alpha_2^n)} + C_n\right] \tag{12}$$

with 150

$$|B_n| \leq \frac{c_0}{n^2}|\lambda_1-\alpha_1^n|, |C_n| \leq \frac{c_0}{n^2}|\lambda_2-\alpha_2^n| \tag{13}$$

In (12), the term

$$\frac{\lambda_1-\alpha_1^n}{(\lambda_1-\beta_1^n)(\beta_1^n-\alpha_1^n)} \text{ may be replaced by } \frac{\lambda_1-\alpha_1}{(\lambda_1-\beta_1)(\beta_1-\alpha_1)}$$

$$= \frac{1}{\lambda_1-\beta_1} + \frac{1}{\beta_1-\alpha_1}\cdots \tag{14}$$

But when β describes (α),

$$\lambda_1 - \beta_1, \lambda_2 - \beta_2, \beta_1 - \alpha_1, \beta_2 - \alpha_2$$

have a strictly positive minimum. Hence (11) can be written as

$$\left[\frac{\lambda_1 - \alpha_1}{(\lambda_1 - \beta_1)(\beta_1 - \alpha_1)} + B_n + B'_n\right]\left[\frac{\lambda_2 - \alpha_2}{(\lambda_2 - \beta_2)(\beta_2 - \alpha_2)} + C_n + C'_n\right] \quad (15)$$

with the conditions (13) and

$$|B'_n| < \frac{c_g(r)}{n^{r/2}}, |C'_n| < \frac{c_g(r)}{n^{r/2}}$$

151 We shall now put the second member of (9) as also (15) in place of (11) in the general term of (1). Then the principle term is evidently

$$\sum_{\substack{\beta\neq\alpha\\ \beta\varepsilon V_n}} \frac{1}{J(\beta)} \frac{J(M,M)}{J(\lambda,\beta)} \frac{J(A,M)}{J(\beta,\alpha)} \frac{[\lambda-\alpha]}{[\lambda-\beta][\beta-\alpha]}$$

where the summation is extended to $2n^2 - 1$ points of β contained in V_n (and distinct from α). The corrective terms are of different kinds. There are two terms of type

$$\sum_{\substack{\beta\varepsilon V_n\\ \beta\neq\alpha}} \left\{\frac{\lambda_1 - \alpha_1}{(\lambda_1 - \beta_1)(\beta_1 - \alpha_1)}\right\}\left\{\frac{1}{D(\beta)} \frac{J(M,M)}{J(\lambda_1,\beta_1)} \frac{J(A,M)}{J(\beta,\alpha)}\right\} C_n \quad (16)$$

The second bracket is uniformly bounded in $\mathscr{B}$ and

$$|C_n(\lambda_1 - \alpha_1)| \leq \frac{c_0}{n^2}|\lambda_1 - \alpha_1||\lambda_2 - \alpha_2| \leq \frac{c_9}{n^{3/2}}$$

Now we prove that

$$\sum_{\substack{\beta\varepsilon V_n\\ \beta\neq\alpha}} \frac{1}{(\lambda_1 - \beta_1)(\beta_1 - \alpha_1)} = 0(n) \quad (16)'$$

such that (16) will have the majorisation $\dfrac{c_{10}(\lambda,\alpha)}{\sqrt{n}}$ where $c_{10}(\lambda,\alpha)$ depends upon the shortest distance of λ_1 from the set of β_1 and the shortest
152 distance of α_1 from the set of $\beta_1 \neq \alpha_1$. Let $n = m^4$. The number of terms of $(16)'$ in $W_n = V_m$ is $2m^2 = 2\sqrt{n}$. Let $U_n = V_{m^2}$. Let d denote the minimum if the shortest distances of λ from α and of $\beta \neq \alpha$ from α. Then

$$\left|\sum_{\substack{\beta\varepsilon U_n\\ \beta\neq\alpha}} \frac{1}{(\lambda_1-\beta_1)(\beta_1-\alpha_1)}\right| \leq \frac{1}{d^2}2n.$$

There are $2n^2 - 2n$ points of (σ) in $V_n - U_n$ and $|\beta| > \pi\sqrt{n}$ for $\beta\varepsilon V_n - U_n$. Also $|\lambda_1| < c_6 n^{1/4}, |\alpha_1| <_6 n^{1/4}$ gives $|\lambda_1 - \beta_1| > \pi\sqrt{n} - c_6 n^{1/4}$ and $|\beta_1 - \alpha_1| > \pi\sqrt{n} - c_6 n^{1/4}$ so that

$$\left|\frac{1}{(\lambda_1-\beta_1)(\beta_1-\alpha_1)}\right| \leq \frac{1}{(\pi\sqrt{n} - c_6 n^{1/4})(\pi\sqrt{n} - c_6 n^{1/4})}$$

or
$$\frac{1}{(\lambda_1-\beta_1)(\beta_1-\alpha_1)} = 0\left(\frac{1}{n}\right) \text{ for } \beta\varepsilon V_n - U_n$$

Hence

$$\sum_{\beta\varepsilon V_n - U_n} \frac{1}{(\lambda_1-\beta_1)(\beta_1-\alpha_1)} = (2n^2 - 2n)\, 0\left(\frac{1}{n}\right) = 0(n)$$

and

$$\begin{aligned}\sum_{\substack{\beta\varepsilon V_n\\ \beta\neq\alpha}} \frac{1}{(\lambda_1-\beta_1)(\beta_1-\alpha_1)} &= \sum_{\substack{\beta\varepsilon U_n\\ \beta\neq\alpha}} \frac{1}{(\lambda_1-\beta_1)(\beta_1-\alpha_1)}\\ &\qquad + \sum_{\beta\varepsilon V_n - U_n} \frac{1}{(\lambda_1-\beta_1)(\beta_1-\alpha_1)}\\ &= 0(n).\end{aligned}$$

The terms of the type 153

$$\frac{\lambda_1-\alpha_1}{(\lambda_1-\beta_1)(\beta_1-\alpha_1)}\frac{1}{D(\beta)}\frac{J(M,M)}{J(\lambda,\beta)}\frac{J(A,M)}{J(\beta,\alpha)}C'_n \tag{17}$$

have majorisation of the form $\dfrac{c_{11}(\lambda,\alpha)}{n^{\left(\frac{2r-5}{4}\right)}}$ and the terms such as

$$\frac{1}{D(\beta)}\frac{J(M,M)}{J(\lambda,\beta)}\frac{J(A,M)}{J(\beta,\alpha)}B'_n C'_n \tag{18}$$

$$\frac{1}{D(\beta)}\frac{J(M,M)}{J(\lambda,\beta)}\frac{J(A,M)}{J(\beta,\alpha)}B_n C'_n \tag{19}$$

$$\frac{1}{D(\beta)}\frac{J(M,M)}{J(\lambda,\beta)}\frac{J(A,M)}{J(\beta,\alpha)}B'_n C'_n \tag{20}$$

have respectively the evident majorisation

$$\frac{c_{12}}{n^{7/2}}, \frac{c_{13}(r)}{n^{\frac{r}{2}+\frac{7}{4}}}, \frac{c_{14}(r)}{n^r}$$

Then term $\dfrac{[\lambda-\alpha]}{[\lambda-\beta][\beta-\alpha]}$ $\quad A_n$ is majorised by $\dfrac{c_{15}(r,\lambda,\alpha)}{n^{\frac{r-1}{2}}}$ where the constant c_{15} depends on λ,α and contains in the denominator the shortest distance of λ_1,λ_2 from the set of β_1,β_2 respectively and the shortest
154 distance of α_1,α_2 from the set of $\beta_1\neq\alpha_1,\beta_2\neq\alpha_2$ respectively.

Similarly the terms

$$\frac{\lambda_1-\alpha_1}{(\lambda_1-\beta_1)(\beta_1-\alpha_1)}A_n C_n, \frac{\lambda_1-\alpha_1}{(\lambda_1-\beta_1)(\beta_1-\alpha_1)}A_n C'_n$$
$$A_n B_n C_n, A_n B_n C'_n, A_n B'_n C'_n$$

give by summation, the majorisation of the form

$$\frac{c_{15}(r,\lambda,\alpha)}{n^{\frac{r+1}{2}}}, \frac{c_{16}(r,\lambda,\alpha)}{n^{r-\frac{5}{2}}}, \frac{c_{17}(r)}{n^{\frac{r+7}{2}}}, \frac{c_{18}(r)}{n^{\frac{3r}{2}}}.$$

Conclusion. For r sufficiently large, and for λ,α fixed, we have

$$n \overset{\text{Lt}}{\to} \infty \left\{ S_n(\lambda,\alpha^n) - \sum_{\substack{\beta\neq\alpha \\ \beta\varepsilon V_n}} \frac{[\lambda-\alpha]}{[\lambda-\beta][\beta-\alpha]}\frac{J(M,M)}{J(\lambda,\beta)}\frac{J(A,M)}{J(\beta,\alpha)} \right\} = 0$$

Now the same summation, extended to β exterior to the volume V_n is majorised by $\dfrac{c_{19}(\lambda,\alpha)}{n}$ (this is obvious if we consider the asymptotic behaviour of σ described by Proposition 2).

Hence for λ, α fixed and r sufficiently large

$$n \overset{\text{Lt}}{\rightarrow} \infty S_n(\lambda, \alpha) = \sum_{\substack{\beta \neq \alpha \\ \beta \varepsilon V_n}} \frac{1}{D(\beta)} \frac{[\lambda - \alpha]}{[\lambda - \beta][\beta - \alpha]} \frac{J(M, M)}{J(\lambda, \beta)} \frac{J(A, M)}{J(\beta, \alpha)}.$$

But $S_n(\lambda, \alpha^n) = \dfrac{J(A^n, M^n)}{J(\lambda, \alpha^n)} - \mathscr{R}^n$ which tends to $\dfrac{J(A, M)}{J(\lambda, \alpha)} - \mathscr{R}$ as 155
$n \to \infty$ for λ, α fixed. Hence

$$\frac{J(A, M)}{J(\lambda, \alpha)} = \sum_{\substack{\beta \neq \alpha \\ \beta \varepsilon (\sigma)}} \frac{[\lambda - \alpha]}{D(\beta)[\lambda - \beta][\beta - \alpha]} \frac{J(M, M)}{J(\beta, \alpha)} \frac{J(A, M)}{J(\lambda, \alpha)} + \mathscr{R}$$

and we have proved the formula $(\mathscr{G}_1)$ for the distributions T_1 and T_2.

9 The fundamental theorem of Mean periodic functions in the case of two variables

The fundamental theorem for Mean periodic functions, Viz. expansion of a mean periodic function in term of mean periodic exponentials in the case of one variable is well-known. But its analogue in R^n is not know. We shall prove it over for a function mean periodic relative to two special kinds of distributions in R^2. Even as in the case of R^1, the theorem is proved by making use of Mittag-Leffler theorem in C^1, the proof given here depends upon the formula $(\mathscr{G}_1)$ which may be considered as an analogue of Mittage-Leffler theorem in C^2.

Let $T_1, T_2 \varepsilon \mathscr{E}'(R^2)$ be defined as in the preceding article by

$$T_i * F = a_i F(x, y) + b_1 F(x + 1, y) + c_1 F(x, y + 1) + d_1 F(x + 1, y + 1)$$

$+ \int\limits_o^1 \int\limits_0^1 k_i(\xi, \eta) F(x + \xi, y + \eta) d\xi d\eta, i = 1, 2$, with supports in the rectangle $\mathscr{R}_1 : 0 \leq x \leq 1, 0 \leq y \leq 1$ where the densities $k_i(x, y)$ and the 156
coefficients $a_i, b_i, \ldots (i = 1, 2)$ satisfy all the necessary conditions in order that all the results of §8 should hold. Let $u_1, u_2 \varepsilon \mathscr{D}(\mathscr{R}_1)$ (the space of indefinitely differentiable functions with compact support) so that their

Fourier-Laplace transforms $A_1(\lambda_1,\lambda_2), A_2(\lambda_1,\lambda_2)$ are functions of exponential type which decrease more rapidly than any power of $\dfrac{1}{|\lambda_1|+|\lambda_2|}$, as $|\lambda_1|, |\lambda_2| \to \infty$ in any vertical band; then we have

$$\frac{J(A,S)}{J(A,\alpha)} = \sum_{\substack{\beta\varepsilon\sigma\\ \beta\neq\alpha}} \frac{[\lambda-\alpha]}{D(\beta)[\lambda-\beta][\beta-\alpha]}\frac{J(S,S)}{J(\lambda,\beta)}\frac{J(A,S)}{J(\lambda,\beta)} + \mathscr{R} \tag{1}$$

where $S_i(\lambda) = \mathscr{F}\mathscr{L}T_i, i = 1,2$, and (σ) is the spectrum. The series (1) converges uniformly on each compact set. Let $\varphi(x)\varepsilon\mathscr{D}\Phi(\lambda)\mathscr{F}\mathscr{L}\phi$ is a function rapidly decreasing in any vertical band.

Setting $\bar{A}_i(\lambda) = \Phi(\lambda)A(\lambda), \bar{S}_i(\lambda) = \Phi(\lambda)S(\lambda), i = 1,2$, we obtain, (after multiplying both sides of (1) by $\Phi(\lambda)$),

$$\frac{J(\bar{A},S)}{J(\lambda,\alpha)} = \sum \frac{[\lambda-\alpha]}{D(\beta)[\lambda-\beta][\beta-\alpha]}\frac{J(\bar{S},S)}{J(\lambda,\beta)}\frac{J(A,S)}{J(\beta,\alpha)} + \bar{\mathscr{R}} \tag{2}$$

The series (2), like (1), converges uniformly on each compact set. Moreover we can prove the following

157 **Proposition 1.** *The series (2) converges in the sense of $L^1(R^2)$ in every plane of C^2 for which the real parts of λ_1,λ_2 are fixed (i.e. for (λ_1,λ_2) in a vertical plane).*

As $A_1(\lambda), A_2(\lambda)$ decrease rapidly and $S_1(\lambda), S_2(\lambda)$ are bounded in a vertical band and (σ) is contained in one such band,

$$\left|\frac{J(A,S)}{J(\alpha,\beta)}\right| < \chi_\alpha(\beta)$$

where $\chi_\alpha(\beta)$ decreases rapidly in $\beta\varepsilon(\sigma)$. Similarly $\left|\dfrac{J(\bar{S},S)}{J(\lambda,\beta)}\right| < c_1|\Phi(\lambda)|$ where $c_1 > 0$ is a constant which depends only on two vertical bands, one containing (σ) and the other containing λ. Further by the corollary of proposition 2, §8, $|D(\beta)| \geq k > 0$ on (σ). Hence

$$\left|\frac{1}{D(\beta)}\frac{J(\bar{S},S)}{J(\lambda,\beta)}\frac{J(A,S)}{J(\beta,\alpha)}\right| < \frac{1}{k}\chi_\alpha(\beta)|\Phi(\lambda)|$$

We now consider the factor

$$\frac{[\lambda-\alpha]}{[\lambda-\beta][\beta-\alpha]} = \frac{1}{(\lambda_1-\beta_1)(\lambda_1-\beta_2)} + \frac{1}{(\lambda_1-\beta_1)(\beta_2-\alpha_2)} + \frac{1}{(\lambda_2-\beta_2)(\beta_1-\alpha_1)} + \frac{1}{(\lambda_1-\beta_1)(\beta_2-\alpha_2)} \quad (3)$$

For (λ_1, λ_2) fixed, we investigate the term of (2) in the four cases **158**

i) $|\lambda_1-\beta_1| \geq 1, |\lambda_2-\beta_2| \geq 1$

ii) $|\lambda_1-\beta_1| < 1, |\lambda_2-\beta_2| < 1$

iii) $|\lambda_1-\beta_1| < 1, |\lambda_2-\beta_2| \geq 1$

iv) $|\lambda_1-\beta_1| \geq 1, |\lambda_2-\beta_2| < 1$

For $\beta\varepsilon(\sigma)$ verifying (i),

$$|\frac{[\lambda-\alpha]}{[\lambda-\beta][\beta-\alpha]}| \leq 1 + \frac{1}{|\beta_1-\alpha_1|} + \frac{1}{|\beta_2-\alpha_2|} + \frac{1}{|\beta_1-\alpha_1||\beta_2-\alpha_2|}$$

The general term of the series (2) corresponding to such a β is majorised by

$$c_1'(1+\frac{1}{|\beta_1-\alpha_1|})(1+\frac{1}{|\beta_2-\alpha_2|})\chi_\alpha(\beta)|\Phi(\lambda)|$$

Summing for β since $\chi_\alpha(\beta)$ decrease rapidly on (σ), this part of the series is majorised in modulus by $c_3(\alpha)|\Phi(\lambda)|$. (ii):- For λ fixed, there are only a finite number of terms verifying this condition.

Now **159**

$$\frac{J(\overline{S}_1, S)}{J(\lambda,\beta)} = \begin{vmatrix} \overline{S}_1(\lambda_1,\lambda_2) - \overline{S}_1(\beta_1,\lambda_2), S_1(\beta_1,\lambda_2) - S_1(\beta_1,\beta_2) \\ \overline{S}_2(\lambda_1,\lambda_2) - \overline{S}_2(\beta_1,\lambda_2), S_2(\beta_1,\lambda_2) - S_2(\beta_1,\beta_2) \end{vmatrix}$$

and

$$\overline{S}_1(\lambda_1,\lambda_2) - \overline{S}_1(\beta_1,\lambda_2) = \int\limits_{\beta_1}^{\lambda_1} \frac{\partial}{\partial\rho_1}[\overline{S}_1(\rho_1,\lambda_2)]d\rho_1$$

Then

$$\frac{J(\bar{S}_1, S)}{J(\lambda,\beta)} = \int\limits_{\beta_1}^{\lambda_1}\int\limits_{\beta_2}^{\lambda_2} \frac{\partial}{\partial\rho_1}\bar{S}_1(\rho_1,\lambda_2)\frac{\partial}{\partial\rho_2}S_2(\beta_1,\rho_2)$$
$$- \frac{\partial}{\partial\rho_1}\bar{S}_2(\rho_1,\lambda_2)\frac{\partial}{\partial\rho_2}S_1(\beta_1,\rho_2)\Big\} d\rho_1 d\rho_2$$

Let $\psi(\lambda)$ denote the maximum modulus of the functions $\dfrac{\partial}{\partial\rho_1}\bar{S}_1(\rho_1,\lambda_2)$, $\dfrac{\partial}{\partial\rho_1}\bar{S}_2(\rho_1,\lambda_2)$ when (ρ_1,ρ_2) is such that $|\lambda_1-\rho_1| < 1, |\lambda_2-\rho_2| < 1$. Then $\psi(\lambda)$ is a rapidly decreasing function in a vertical band, and

$$\left|\frac{J(\bar{S}_1, S)}{J(\lambda,\beta)}\right| \leq c_2|\psi(\lambda)||\lambda_1-\beta_1||\lambda_2-\beta_2|$$

and the terms of (2) for which (ii) holds are majorised by $c_3(\alpha)\psi_1(\lambda)$
160 where $\psi_1(\lambda)$ is a rapidly decreasing function in a vertical band.

(iii) : - Writing $\dfrac{J(\bar{S}, S)}{J(\lambda,\beta)}$ is the form

$$\int\limits_{\beta_1}^{\lambda_1}\left\{\frac{\partial}{\partial\rho_1}\bar{S}_1(\rho_1,\lambda_2)S_2(\beta_1,\lambda_2) - \frac{\partial}{\partial\rho_1}\bar{S}_2(\rho_1,\lambda_2)S_1(\beta_1,\lambda_2)\right\} d\rho_1$$

we have in this case the terms of (2) majorised by $c\chi_\alpha(\beta)\psi_2(\lambda)\left|\dfrac{[\lambda-\alpha]}{[\beta-\alpha]}\right|$ where $\psi_2(\lambda)$ decreases rapidly in a vertical band and the sum of the series of this part is majorised by $c(\alpha)\psi_3(\lambda)$ where $c(\alpha)$ is a constant which depends on α and $\psi_3(\lambda)$ decreases rapidly in a vertical band and we have similar majorisation in case (*iv*). Thus the sum of (2) being majorised by a rapidly decreasing function of λ in a vertical plane, the series converges in the sense of $L^1(R^2)$ in a vertical plane.

In view of Proposition 1, we can apply the inverse of Fourier Laplace transformation to both sides of (2) and we obtain

$$\bar{u} = \sum_{\substack{\beta\in\sigma \\ \beta\neq\alpha}} \frac{J(A,S)}{J(\beta,\alpha)}\bar{s}_{\alpha\beta}(x,y) + \bar{v} \tag{4}$$

where $\qquad \mathscr{F}\mathscr{L}\bar{u} = \dfrac{J(\bar{A}, S)}{J(\lambda, \alpha)},$

$$\mathscr{F}\mathscr{L}\bar{s}_{\alpha\beta} = \frac{[\lambda - \alpha]}{D(\beta)[\lambda - \beta][\beta - \alpha]} \frac{J(\bar{S}, S)}{J(\lambda, \beta)} = \bar{\mathscr{S}}_{\alpha\beta}(\lambda),$$

$\mathscr{F}\mathscr{L}\bar{R} = \bar{v}$ and the series converges uniformly on each compact sub- **161**
set of R^2. Then for any continuous function F on R

$$\bar{u} * F = \sum_{\substack{\beta \neq \alpha \\ \beta \in (\sigma)}} \frac{J(A, S)}{J(\beta, \alpha)} \bar{s}_{\alpha\beta} * F + \bar{v} * F \tag{5}$$

Using (3), we write $\dfrac{[\lambda - \alpha]}{[\lambda - \beta][\beta - \alpha]} \frac{J(\bar{S}, S)}{J(\lambda, \beta)} = \sum_{j=1}^{4} \bar{\mathscr{S}}^{j}_{\alpha\beta}(\lambda)$ where

$$\bar{\mathscr{S}}^{1}_{\alpha\beta}(\lambda) = \frac{1}{D(\beta)(\lambda_1 - \beta_1)(\lambda_2 - \beta_2)} \quad \frac{J(\bar{S}, S)}{J(\lambda, \beta)}$$

$$\bar{\mathscr{S}}^{2}_{\alpha\beta}(\lambda) = \frac{1}{D(\beta)(\beta_1 - \alpha_1)(\lambda_2 - \beta_2)} \quad \frac{J(\bar{S}, S)}{J(\lambda, \beta)}$$

$$\bar{\mathscr{S}}^{3}_{\alpha\beta}(\lambda) = \frac{1}{D(\beta)(\beta_2 - \alpha_2)(\lambda_1 - \beta_1)} \quad \frac{J(\bar{S}, S)}{J(\lambda, \beta)}$$

$$\bar{\mathscr{S}}^{4}_{\alpha\beta}(\lambda) = \frac{1}{D(\beta)(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)} \frac{J(\bar{S}, S)}{J(\lambda, \beta)}$$

Let $\mathscr{F}\mathscr{L}s^{-j}_{\alpha\beta} = \bar{\mathscr{S}}^{j}_{\alpha\beta}(\lambda) j = 1, 2, 3, 4.$ Then **162**

$$\begin{aligned}
\bar{s}_{\alpha\beta} * F &= \sum_{j=1}^{4} s^{-j}_{\alpha\beta} * F \\
&= \bar{\mathscr{S}}^{1}_{\alpha\beta}(\lambda) = \frac{1}{D(\beta)(\lambda_1 - \beta_1)(\lambda_2 - \beta_2)} \Phi(\lambda)\{S_1(\lambda_1, \lambda_2) S_2(\beta_1, \beta_2) \\
&\quad - S_2(\lambda_1, \lambda_2) S_1(\beta_1, \lambda_2)\} \\
&= \Phi(\lambda) t\beta(\lambda).
\end{aligned}$$

Let $\mathscr{F}\mathscr{L}T_\beta = t_\beta(\lambda)$. Then $s^{-1}{}_{\alpha\beta} * F = \varphi * T_\beta * F$

$$(\lambda_1 - \beta_1) t_\beta(\lambda) = \frac{1}{D(\beta)} \Big[S_1(\lambda_1, \lambda_2) \frac{S_2(\beta_1, \lambda_2) - S_2(\beta_1, \beta_2)}{\lambda_2 - \beta_2}$$

$$S_2(\lambda_1,\lambda_2)\frac{S_1(\beta_1,\lambda_2)-S_1(\beta_1,\beta_2)}{\lambda_2-\beta_2}\Bigg]$$

$$\frac{S_2(\beta_1,\lambda_2)-S_2(\beta_1,\beta_2)}{\lambda_2-\beta_2},\ \frac{S_1(\beta_1,\lambda_2)-S_1(\beta_1,\beta_2)}{\lambda_2-\beta_2}$$

are entire functions of λ_2 of exponential type and are Fourier Laplace transforms of distributions s_1 and s_2 in the variable x_2 with support compact in which β_1 appears as a parameter. Then

$$(\lambda_1-\beta_1)t_\beta(\lambda) = \mathscr{F}\mathcal{L}\left\{\frac{1}{D(\beta)}T_1 * (\delta_{x_1}\otimes s_2) - T_2 * (\delta_{x_1}\otimes s_1)\right\}$$

where δ_{x_1} is the Dirac measure in the space of x_1. Setting $T_\beta * F = G$, we have

$$\frac{\partial G}{\partial x_1} - \beta_1 G = \frac{1}{D(\beta)}\Big[T_1 * (\delta_{x_1}\otimes s_2) * F - T_2 * (\delta_{x_1}\otimes s_1) * F\Big] = 0$$

Similarly

$$(\lambda_2-\beta_2)t_\beta(\lambda) = \frac{1}{D(\beta)}\Bigg[S_2(\lambda_1,\lambda_2)\frac{S_1(\lambda_1,\lambda_2)-S_1(\beta_1,\lambda_2)}{\lambda_1-\beta_1}$$
$$S_1(\lambda_1,\lambda_2)\frac{S_2(\lambda_1,\lambda_2)-S_2(\beta_1,\lambda_2)}{\lambda_2-\beta_2}\Bigg] \tag{3.1}$$

163 gives

$$\frac{\partial G}{\partial x_2} - \beta_2 G = 0$$

Hence

$$G = k\exp <\beta, x>$$

For $x = 0, G(0) = T_\beta * F(0) = \langle T_\beta, F\rangle$ so that

$$\begin{aligned} T_\beta * F &= \langle T_\beta, F\rangle \exp\langle\beta, x\rangle \quad \text{and}\\ s^{-1}_{\alpha\beta} * F &= \phi * T_\beta * F\langle T_\beta, F\rangle\phi * \exp\langle\beta, x\rangle\\ &= \Phi(\beta)\langle T_\beta, F\rangle \exp\langle\beta, x\rangle\\ \mathscr{S}^2_{\alpha\beta}(\lambda) &= \frac{1}{D(\beta)(\lambda_2-\beta_2)(\beta_1-\alpha_1)} \end{aligned}$$

$$\left\{\bar{S}_1(\lambda_1,\lambda_2)S_2(\beta_1,\lambda_2)-\bar{S}_2(\lambda_1,\lambda_2)S_1(\beta_1,\lambda_2)\right\}$$

$\dfrac{S_1(\beta_1,\lambda_2)}{\lambda_2-\beta_2}$ and $\dfrac{S_2(\beta_1,\lambda_2)}{\lambda_2-\beta_2}$ are entire functions of exponential type and the distributions S_1 and S_2 with compact support in the variable x_2 have these functions as Fourier Laplace transforms

$$s^{-2}_{\alpha\beta}=\frac{1}{D(\beta)(\beta_1-\alpha_1)}\left[\phi*T_1*(\delta_{x_1}\otimes s_2)-\phi*T_2*(\delta_{x_1}\otimes S_1)\right]$$

and $\vec{s}^{2}_{\alpha\beta}*F=0$.

Similarly $\vec{s}^{3}_{\alpha\beta}*F=0$, $\vec{s}^{4}_{\alpha\beta}*F=0$ and we obtain finally,

$$\begin{aligned}\bar{u}*F&=\sum_{\beta\in(\sigma)}\Phi(\beta)\frac{J(A,S)}{J(\beta,\alpha)}\langle T_\beta,F\rangle\exp\langle\beta,x\rangle+\bar{v}*F\\&=\sum_{\beta\in(\sigma)}\langle T_\beta,F\rangle\exp\langle\beta,x\rangle+\bar{v}*F.\end{aligned}$$

We shall now verify that $\bar{v}*F=0$. $\mathscr{F}\mathcal{L}\bar{v}=\bar{\mathcal{R}}=\Phi(\lambda)\mathcal{R}$ where $\mathcal{R}$ **164**
contains only a finite number of terms of 'irregular type' in the formula $\mathscr{G}_1$, (refer to page 80). The terms in $\mathcal{R}$ come from

a) $D(\alpha)[\lambda-\alpha]d\left\{\dfrac{1}{D(\alpha)[\lambda-\alpha]}\right\}\dfrac{J(S,S)}{J(\lambda,\alpha)}$ and

b) the determinant $\begin{vmatrix}S_1(\lambda_1,\lambda_2),-\varepsilon A_1(\alpha_1,\lambda_2)+\frac{\partial}{\partial\alpha_1}S_1(\alpha_1,\lambda_2)d\alpha_1\\ S_2(\lambda_1,\lambda_2),-\varepsilon A_2(\lambda_1,\lambda_2)+\frac{\partial}{\partial\alpha_1}S_1(\alpha_1,\lambda_2)d\alpha_2\end{vmatrix}$

multiplied by $D(\alpha)[\lambda-\alpha]$ in which for $d\alpha_1$ and $d\alpha_2$ we have to substitute $\dfrac{\varepsilon}{D(\alpha)}\quad\dfrac{J\left(A,\dfrac{\partial S}{\partial\alpha_2}\right)}{J(\alpha,\alpha)}$ and $\dfrac{\varepsilon}{D(\alpha)}\quad\dfrac{J\left(\dfrac{\partial S}{\alpha_1},A\right)}{J(\alpha,\alpha)}$.$(b)$ gives in $\mathscr{R}$ a term of the form $K_\alpha[\lambda-\alpha]\dfrac{J(S,S)}{J(\lambda,\alpha)}$ where k_α depends only in α and this function in the ideal generated by $S_1(\lambda_1,\lambda_2)$ and $S_2(\lambda_1,\lambda_2)$ in the ring of entire functions. Similarly in (a) the term $\dfrac{d[D(\alpha)]}{[D(\alpha)]^2[\lambda-\alpha]}$ multiplied by

$D(\alpha)[\lambda-\alpha]\dfrac{J(S,S)}{J(\lambda,\alpha)}$ gives a function in the same ideal. The other term 165
in (a) viz.

$D(\alpha)[\lambda-\alpha]\dfrac{J(S,S)}{J(\lambda,\alpha)} \quad \dfrac{d[\lambda-\alpha]}{D(\alpha)[\lambda-\alpha]^2}$ equals
$\dfrac{1}{[\lambda-\alpha]} \quad \dfrac{J(S,S)}{J(\lambda,\alpha)}[(\lambda_1-\alpha_1)d\alpha_2+(\lambda_2-\alpha_2)d\alpha]$.Now
$\dfrac{J(S,S)}{J(\lambda,\alpha)} \quad \dfrac{1}{\lambda_1-\alpha_1}$ and $\dfrac{J(S,S)}{J(\lambda,\underline{\alpha})} \quad \dfrac{1}{\lambda_2-\alpha_2}$ belong to the ideal. This can be seen as in the study of $\mathscr{S}^2_{\alpha\beta}(\lambda)$ and $\mathscr{S}^3_{\alpha\beta}(\lambda)$. Thus $\bar{\mathcal{R}}$ lies in the ideal so that

$$\bar{\gamma}*F=0$$

We have

$$\bar{u}*F=\sum_{\beta\in(\sigma)}\frac{J(\bar{A},S)}{J(\beta,\alpha)}\langle T_\beta,F\rangle\exp\langle\beta,x\rangle \tag{6}$$

in which the series converges uniformly on compact of R^2 and where $\mathscr{F}\mathcal{L}\bar{u}=\dfrac{J(\bar{A},S)}{J(\lambda,\alpha)}$.

We shall now prove that the continuous function F mean periodic with respect to T_1 and T_2 a uniquely determined by the system of coefficients $c_\beta=\langle T_\beta,F\rangle$ corresponding to the mean periodic exponentials $(e^{<\beta,x>})_{\beta\varepsilon\sigma}$ by establishing the following

Representation theorem. If F is mean periodic with respect to T_1 and T_2 and $\phi\varepsilon\mathscr{D}(\mathscr{R}^2)$ and $u\in\mathscr{D}(\mathscr{R}_1)$ then

$$\phi*u*F=\sum_{\beta\in(\sigma)}\psi(\beta)<T_\beta,F>e^{<\beta,x>} \tag{7}$$

166 where $\psi(\lambda)=\mathscr{F}\mathcal{L}\phi*u$ and the series converges uniformly in every compact subset of R^2.

For $\phi\varepsilon\mathscr{D}(R^2)$ and $u_1,u_2\varepsilon\mathscr{D}(\mathscr{R}_1)$ we had $\mathscr{F}\mathcal{L}\phi*u_i=\bar{A}_i(\lambda_1,\lambda_2)i=1,2$ and

$$\frac{J(\bar{A},S)}{J(\lambda,\alpha)}=\bar{A}_1(\lambda_1,\lambda_2)S_2(\alpha_1,\lambda_2)-\bar{A}_2(\lambda_1,\lambda_2)S_1(\alpha_1,\lambda_2)$$

Let $\phi * u_1 * F = G_1, \phi * u_2 * F = G_2$. G_i are in $\mathscr{E}(R^2)$ and are mean periodic with respect to T_1, T_2. Let $\mathscr{F}\mathcal{L}\mu_i = M_i(\alpha_1, \lambda_2)$. Then

$$\mu_2 * G_1 - \mu_1 * G_2 = \sum_{\beta\in\sigma} \frac{J(\bar{A}, S)}{J(\beta, \alpha)} \langle T_\beta, F\rangle e^{\langle\beta,x\rangle}$$

Setting in this equation first $u_1 = u\varepsilon\mathscr{D}(\mathcal{R}_1), u_2 = 0$ and then $u_1 = 0$, $u_2 = -u$, and writing $\phi * u * F = G$ we obtain

$$\mu_1 * G = \sum_{\beta\in\sigma} \Psi(\beta) S_1(\alpha_1, \beta_2)\langle T_\beta, F\rangle e^{\langle\beta,x\rangle} \tag{8}$$

$$\mu_2 * G = \sum_{\beta\in\sigma} \Psi(\beta) S_2(\alpha_1, \beta_2)\langle T_\beta, F\rangle e^{\langle\beta,x\rangle} \tag{9}$$

Let H denote the sum of the series on the right hand side of (7). This series converges uniform;y on every compact set since $\psi(\lambda)$ is a rapidly decreasing function in any vertical band and (σ) is contained in such **167**
a band. Hence convolution with H is obtained by convoling with each term of the series then summing. But $\mu_1 * H$ and $\mu_2 * H$ so obtained are nothing but the series (8) and (9) respectively. Therefore $\mu_1 * G = \mu_1$ and $\mu_2 * G = \mu_2 * H$. Hence $G = H$ if we show that the two homogeneous equations $\mu_1 * L = 0 = \mu_2 * L = 0$ have $L = 0$ for the unique solution in $\mathscr{E}(R^2)$.

Now $\mu_1 = \delta_{x_1} \otimes s^{(1)}_{x_2} \mu_2 = \delta_{x_1} \otimes s^{(2)}_{x_2}$ where s^1 and s^2 are distributions in the variable x_2 having for Fourier Laplace transforms $S_1(\alpha_1, \lambda_2)$ and $S_2(\alpha_1, \lambda_2)$ respectively in which α_1 is a parameter. Hence $L(X_1, x_2)$ considered as a function of x_2 for x_1 fixed is means periodic with respect to $s^1_{x_2}$ and $s^2_{x_2}$. The spectrum (σ_{α_1}) in C for the variable λ_2 defined by the equations $S_1(\alpha_1, \lambda_2) = S_2(\alpha_1, \lambda_2) = 0$

$$L(x_1, x_2) = \sum_{\alpha_2\in\sigma_{\alpha 1}} c_{\alpha_2}(x_1) e^{\alpha_2 x_2}$$

where σ_{α_1} is the set of α_2 such that $(\alpha_1, \alpha_2)\varepsilon(\sigma)$ by the fundamental theorem of mean periodic functions in R^1.

Taking $\beta = (\beta_1, \beta_2)\varepsilon(\sigma)$ sum that $\alpha_1 \neq \beta_1$ we obtain again

$$L(x_1, x_2) = \sum_{\beta_2\in\sigma_{\beta_1}} d_{\beta_2}(x_1) e^{\beta_2 x_2}.$$

The spectrum (σ) is therefore decomposed into a countable union of subset each subset consisting of all $\alpha\varepsilon(\sigma)$ with the same first co-ordinate α_1 and corresponding to each such subset we have the expansion of $L(x_1, x_2)$ in mean periodic exponentials in x_2 with coefficients in which the fixed co-ordinate x_1 occurs as a parameter. But all these expansions have to be the same and therefore there will be at leat one α_2 common to all $\sigma_{\alpha 1}$ i.e σ is decomposed in countable subset of elements with the same first co-ordinate and we can choose one element from each of these such that the second co-ordinates of all these chosen elements are the same. But this is impossible for (σ). For (σ) is indefinitely near to (σ°) which is the set of common zeros of the principal parts of $S_1(\lambda_1, \lambda_2)$ and $S_2(\lambda_1, \lambda_2)$ { refer to §8 } and the above type of decomposition of (σ^0) is not possible for the following reason. 168

Setting $X_i = e^{\lambda_i}$ in these principal parts of S_1 and S_2, (σ^0) is given in terms of the points of intersection of two rectangular hyperbolas

$$a_1 + b_1X_1 + c_1X_2 + d_1X_1X_2 = 0$$
$$a_2 + b_2X_1 + c_2X_2 + d_2X_1X_2 = 0$$

169 These have two distinct points of intersection (X'_1, X'_2) and (Y'_1, Y'_2) let $X'_1 = e^{\alpha'_1}, X'_2 = e^{\alpha'_2}; Y'_1 = e^{\beta'_1} Y'_2 = e^{\beta'_2}$. Then (σ^0) is defined by

$$(\alpha'_1 + 2h\pi i,_2 + 2k\pi i) \text{ and } (\beta'_1 + 2h'\pi i, \beta_2 + 2k'\pi i)$$

where h, k, h', k' very in the set of rational integers Z. If the situation described for (σ) exists in the case of (σ°) then for all distinct first co-ordinates

$$\alpha'_1 + 2h\pi i,\ \beta'_1 + 2h'\pi i \quad h, h'\varepsilon Z,$$

there will exist a single 2*nd* co-ordinate such that the points formed will lie in (σ_0)*i.e* there exist integers k, k' such that

$$\alpha_2 + 2k\pi i = \beta + 2k'\pi i$$

This gives $Y'_2 = Y'_2$. But it is clear that the two points of intersection (of the two coines) which are assumed to be distinct (see §8) cannot have the same ordinates.

Thus we have proved that the only solution for the con-volution equation (8) $\mu_1 * L = \mu_2 * L = 0$ is $L \equiv 0$ and the theorem is proved.

The Uniqueness Theorem.

If all the coefficients $c_\beta = \langle T_\beta, F\rangle$ are zero then $F = 0$. 170

$c_\beta = 0$ imply that $\phi * u * F = 0\,\forall\phi\varepsilon\mathscr{D}(R^2), \forall u\varepsilon\mathscr{D}(\mathcal{R}_\infty)$. Letting $\phi \to \delta$ the Dirac measure $u * F = 0\ \forall u\varepsilon\mathscr{D}(\mathscr{R}_1)$. Letting u tend towards the Dirac at a point x^0 in the interior of the rectangle $\mathscr{R}_1$, we have

$$F(x + x^0) = 0 \text{ i.e } F = 0.$$

Part III

The Two-Radius Theorem

Chapter 1

1

The subject of this part is the two-radius theorem, which is the converse of the classical theorem of Gauss on the spherical mean of the harmonic functions in R^n 171

Let k_n denote the area of the sphere $\sum_{n-1}$ of radius 1 in $R^n(\sum_{n-1} = \{x/x = (x_1, x_2, \ldots x_n)\varepsilon R^n, \sum x_i^2 = 1)$. Let $f(x)$ be a function which is $(C, 2)$ in R^n. The spherical mean of $f(x)$ on the surface of the sphere with center x and radius r is by definition

$$M(x, r) = \frac{1}{k_n r^{n-1}} \int_{S_r^{n-1}(x)} f(\xi) d\sigma \tag{1}$$

where $S_r^{n-1}(x)$ is the sphere of center x and radius r in R^n, ξ is the generic point of the sphere and $d\sigma$ the element of area of the sphere. We have also

$$M(x, r) = \frac{1}{k_n} \int_{\sum_{n-1}} f(x + r\overrightarrow{u}) d\omega \tag{2}$$

where $d\omega$ is the element of the sphere $\sum_{n-1}$ and $\overrightarrow{u}$ is the unit vector at the origin, whose other extremity describes $\sum_{n-1}$. 172

Proposition 1 (Poisson)**.** *The function $M(r, x)$ is a solution of the partial differential equation*

$$\triangle_x [M(x, r)] = \frac{\partial^2 M}{\partial r^2} + \frac{n-1}{r} \quad \frac{\partial M}{\partial r} \tag{3}$$

As $\triangle$ is a convolution operator in R^n, we have

$$\triangle_x[M(x,r)] = \frac{1}{k_n r^{n-1}} \int_{S_r^{n-1}(x)} \triangle_\xi [f(\xi)] d\sigma$$

From 2,

$$\frac{\partial M}{\partial r} = \frac{1}{k_n} \int_{\sum_{n-1}} \frac{\partial}{\partial r} [f(x + r\vec{u})] d\omega$$
$$\frac{1}{k_n r^{n-1}} \int_{S_r^{n-1}(x)} \frac{d}{d\nu} [f(\xi)] d\sigma$$

where ν is the exterior normal to $S_r^{n-1}(x)$ at the point ξ. By Green's formula, we also have,

$$\frac{\partial M}{\partial r} = \frac{1}{k_n^r{}^{n-1}} \int\int \triangle_\xi [f(\xi)] dV = \frac{1}{r^{n-1}} J$$

173 where $J = \dfrac{1}{k_n} \int\int \triangle_\xi [f(\xi)] dV$ where the integral is taken over the volume of the solid sphere in R^n with $S_r^{n-1}(x)$ as boundary. Then

$$\frac{\partial^2 M}{\partial r^2} = -\frac{(n-1)}{r^n} J + \frac{1}{r^{n-1}} \frac{\partial J}{\partial r}$$

But
$$\frac{\partial J}{\partial r} = \frac{1}{k_n} \int_{S_r^{n-1}(x)} \triangle_\xi [f(\xi)] d\sigma$$

since $dV = d\sigma dr$.

Thus

$$\frac{\partial^2 M}{\partial r^2} + \frac{n-1}{r} \frac{\partial M}{\partial r} = \frac{1}{k_n r^{n-1}} \int\limits_{S_r^{n-1}} \triangle_\xi [f(\xi)] d\sigma$$
$$= \triangle_x [M(x,r)].$$

Remark. For $r = 0$, obviously

$$M(x, 0) = f(x).$$

And $$\left[\frac{\partial M}{\partial r}\right]_{r=0} = \lim_{r=0} \left\{ \frac{1}{k_n r^{n-1}} \int\int_{S_r^{n-1}} \triangle_\xi [f(\xi)] dV \right.$$

The integral in this expression is of order r^n

$$\left[\frac{\partial M}{\partial r}\right]_{r=0} = 0$$

Hence the solution $M(x,r)$ of (3), verifies the Cauchy conditions **174**

$$M(x,0) = f(x), [\frac{\partial}{\partial r} M(x,r)]_{r=0} = 0$$

2 Study of certain Cauchy-Problems

Let $\mathscr{E}_*$ denote the vector space of *even* functions indefinitely differentiable in R with the usual topology (that induced by $\mathscr{E}$). Let

$$L = D^2 + \frac{q(x)}{x} D$$

be a differential operator where $D = \dfrac{d}{dx}, q\varepsilon\mathscr{E}_*$ (with $q(0) \neq -\dfrac{n+1}{2}, n$ integer ≥ 1). Then (by the results obtained in part I) there exists an isomorphism B of $\mathscr{E}_*$ onto itself the property

$$D^2 B = BL \text{ and } B_0[f(\xi)] = f(0)$$

Problem 1. To find a function $F(x,y)$ which is twice differentiable in R^2 and which is a solution of the Cauchy-problem

$$\frac{\partial^2 F}{\partial x^2} + \frac{q(x)}{x} \frac{\partial F}{\partial x} = \frac{\partial^2 F}{\partial y^2} + \frac{q(y)}{y} \frac{\partial F}{\partial y}$$

with $$F(x,0) = f(x), \qquad f \in \mathscr{E}_*$$

and $$\left[\frac{\partial F}{\partial y}\right]_{y=0} = 0$$

Suppose that there exists a solution $F(x,y)$ which is an even function of x for y fixed and also an even function of y for x fixed. **175**

Let

$$G(x,y) = B_x B_y [F(\xi,\eta)]$$

where B_x operates on ξ and B_y on η. $G(x,y)$ is an element in $\mathscr{E}_*$ as a function of y for x fixed and of x for y fixed. We have

$$\begin{aligned}\frac{\partial^r G}{\partial x^2} &= D_x^2 B_\xi B_y[F(\xi_1,\eta)] = B_y D_x^2 B_\xi[F(\xi_1,\eta)]\\ &= B_y B_x L_\xi[F(\xi_1,\eta)] = B_y B_x L_\eta[F(\xi,\eta_1)]\\ &= B_x B_y L_\eta[F(\xi,\eta_1)] = B_x D_y^2 B_\eta[F(\xi,\eta_1)]\\ &= D_y^2 B_x B_\eta[F(\xi,\eta_1)] = \frac{\partial 2_G}{\partial y^2}.\end{aligned}$$

Also

$$G(x,0) = B_x B_o[F(\xi,\eta)] = B_x[F(\xi,\eta)] = B_x[f(\xi)] = g(x).$$
$$\left[\frac{\partial G(x,y)}{\partial y}\right]_{y=0} = 0.$$

176 Hence Problem 1 is reduced to the problem of finding a solution of the following Cauchy problem

$$\frac{\partial^2 G}{\partial x^2} = \frac{\partial^2 G}{\partial y^2}$$
$$G(x,0) = g(x) \in \mathscr{E}_*$$
$$\left[\frac{\partial G}{\partial y}\right]_{y=0} = 0.$$

It is wellknown that this problem has the unique solution

$$G(x,y) = \frac{1}{2}\left[g(x+y) + g(x-y)\right] \tag{4}$$

Then there exists a unique solution of Problem (1), defined by

$$F(x,y) = \frac{1}{2}\mathscr{B}_x\mathscr{B}_y[g(\xi+\eta) + g(\xi-\eta)] \tag{5}$$

where $\mathscr{B} = B^{-1}$ is an isomorphism of $\mathscr{E}_*$.

Remark. As g is even, $G(x,y)$ is symmetric in x and y. Then $F(x,y)$ is also symmetric in x and y.

Definition 1. For any $f \in \mathscr{E}_*$ let, $M_{x,y}[f(\xi)]$ denotes the solution of problem (1):

$$M_{xy}[f(\xi)] = \frac{1}{2}\mathscr{B}_x\mathscr{B}_y\left\{B_{\xi+\eta}[f] + B_{\xi-\eta}[f]\right\} \tag{6}$$

Problem 2. To determine $F(x, y), (r \in R^1, r \geq 0)$ which satisfies the differential equation 177

$$A_x[F(\xi, r)] = L_r[F(x, \rho)],$$

where A is an elliptic differential operator in R^n with indefinitely differentiable coefficients, and the Cauchy conditions

$$F(x, 0) = f(x); \left[\frac{\partial F}{\partial r}\right]_{r=0} = 0.$$

The existence and uniqueness of the solution of Problem 2 defines the operator $\mathfrak{M}_r$ by

$$\mathfrak{M}_r[f(\xi)] = F(x, r).$$

Remark. By Poisson's theorem, when $A = \Delta$ and $q(r) = n - 1$, then the solution of Problem 2 is given by the spherical mean.

Proposition 2. *The operator $\mathfrak{M}$ commutes with A.*

Let

$$G(x, r) = A_x[F(\xi, r)] = A_x\mathfrak{M}_r[f(\xi)].$$

We have

$$\begin{aligned} A_x[G(\xi, r)] &= A_xA_\xi[F(\xi_1, r)] \\ &= A_xL_r[F(\xi, \rho)] = L_rA_x[F(\xi, \rho)] = L_r\left[G(x, \rho)\right] \end{aligned}$$

Moreover,

$$G(x, 0) = A_x[F(\xi, 0] = A_x[f(\xi)]$$

and $\left[\dfrac{\partial G}{\partial r}\right]_{r=0} = 0$ since $G(x, r) \in \mathscr{E}_*$, as a function of r for x fixed. 178

Hence

$$G(x, r) = \mathfrak{M}_rA_\xi[f(\xi_1)]$$

and the proposition is proved.

Iteration of the operator

For $f \in \mathscr{E}(R^n)$, let $F(x, r) = \mathfrak{M}_r[f(\xi)]$ be the solution of Problem 2. By iteration we consider the Cauchy problem,

$$A_x[\mathscr{F}(\xi, s)] = L_s[\mathscr{F}(x, \sigma)]$$

with

$$\mathscr{F}(x, 0) = F(x, r)$$
$$\left[\frac{\partial \mathscr{F}}{\partial s}\right]_{s=0} = 0$$

in with r is a positive parameter. The solution $\mathscr{F}$ is a function of x, r, s,

$$\mathscr{F}(x|r, s) = \mathfrak{M}_s\left[F(\xi, s)\right] = \mathfrak{M}_s\mathfrak{M}_r[f(\xi)] \tag{7}$$

Proposition 3. *For x fixed in R^n, $\mathscr{F}(x|r, s)$ is a solution of Problem* 1, *i. e.*

$$L_r\left[\mathscr{F}(x|\rho, s)\right] = L_s\left[\mathscr{F}(x|r, \sigma)\right]$$

and

$$\mathscr{F}(x|r, 0) = F(x, r)$$
$$\left[\frac{\partial \mathscr{F}}{\partial s}\right]_{s=0} = 0.$$

179 We compute

$$\begin{aligned} L_r\left[\mathscr{F}(x|\rho, s)\right] &= \mathfrak{M}_s\Big\{L_r\left[F(\xi, \rho)\right]\\ &= \mathfrak{M}_s\left\{A_\xi\left[F(\xi_1, r)\right]\right\} = A_x\mathfrak{M}_s[F(\xi, r)]\\ &\qquad\text{(by Proposition 2)}\\ &= A_x\left[\mathscr{F}(\xi|r, s)\right] = L_s\left[\mathscr{F}(x|r, \sigma)\right]. \end{aligned}$$

By Definition 1, we have

$$\mathscr{F}(x|r, s) = M_{r,s}[F(x, \theta)].$$

Hence we obtain the formula

$$\mathfrak{M}_s\mathfrak{M}_r = \mathfrak{M}_r\mathfrak{M}_s = M_{r,s}[\mathfrak{M}_0] \tag{8}$$

We now consider the solution

$$F(x, r) = \mathfrak{M}_r[f(\xi)]$$

of Problem 2 and suppose that the it satisfies for $a > 0$ fixed the condition **180**

$$F(x, a) = f(x) \tag{9}$$

for any x.

Condition (9) expresses the fact for $r = a$ fixed, the value $F(x, a)$ reproduces the initial value $f(x)$ of the function for $r = 0$ in the space R^n.

Remark. If $A = \Delta$, and $L = D^2 + \dfrac{n-1}{r}D$ then $F(x, r) = M(x, r)$ and the condition (9) is 'Gauss's condition' for the fixed radius a.

In view of (9) and (7) we have

$$\mathscr{F}(x, a, s) = F(x, s)$$

where x and s are arbitrary.

By definition (1),

$$M_{a,s}[F(x, \theta)] = F(x, s) \tag{10}$$

where the left-hand side the operator M operates on the variable θ (Equation (10) thus gives a 'transposition' from R^n to R^1). Using (6), (10) becomes,

$$2F(x, s) = \mathscr{B}_a\mathscr{B}_s\,\{B_{\alpha+\sigma}F(x, \theta) + B_{\alpha-\sigma}[F(x, \theta)]\}$$

which gives

$$2B_s[F(x, \sigma)] = \mathscr{B}_a\,\{B_{\alpha+s}F(x, \theta) + B_{\alpha-s}F(x, \theta)\}$$

since B is the inverse of $\mathscr{B}$. **181**

Setting

$$K(x,s) = B_s[F(x,\sigma)].$$

we have

$$2K(x,s) = \mathscr{B}_a\{K(x,\alpha+s) + K(x,\alpha-s)\} \tag{11}$$

in which a is fixed.

Since $K(x,s)$ is even in s, (11) can be written as

$$2K(x,s) = \mathscr{B}_a\{K(x,\alpha+s) + K(x,s-\alpha)\} \tag{11$'$}$$

so that the function $K(x,s)$ is mean periodic in s as the equation (11)′ is clearly an equation of convolution in s.

Remark. In order to obtain the spectrum, we have to substitute $e^{\lambda s}$ in place of $K(x,s)$ in (11)′ which leads to

$$1 = \mathscr{B}_a[\cos h\alpha]$$

3 The generalized two-radius theorem

Let $a, b \in R^1$; $a, b, > 0$; $a \neq b$

Definition 2. A function $f(x)$ which is $(C,2)$ in R^n possesses the two
182 radius property with respect to the elliptic operator A and the singular operator L, if

$$F(x,a) = F(x,b) = f(x) \tag{12}$$

for $x \in R^n$, where

$$F(x,r) = \mathfrak{M}_r[f(\xi)]$$

is the solution of Problem 2.

Remark. If $A = \Delta$ and $L = D^2 + \dfrac{n-1}{r}D$. Condition (12) is the 'Gauss's condition' for two fixed radii a and b.

Now in place of (11)′ we have two equations

$$2K(x,s) = \mathscr{B}_a\{K(x,s+\alpha) + K(x,\mathscr{S}-\alpha)\}$$

$$2K(x,s) = \mathscr{B}_b\{K(x,s+\alpha) + K(x,s-\alpha)\}$$

so that $K(x,s)$ is mean periodic in s with respect to two distribution, and by the classical result of the theory of mean periodicity in R^1, the elements λ in the spectrum $\sigma(a,b)$ have to satisfy two equations

$$\left.\begin{aligned} 1 &= \mathscr{B}_a[\cos h\lambda\alpha] \\ 1 &= \mathscr{B}_b[\cos h\lambda\alpha] \end{aligned}\right\} \tag{13}$$

Now we show that $\lambda = 0$ is a double solution of (13).

By definition $D^2B = BL$. **183**

If $B_s[1] = \varphi(s)$, $D^2[\varphi(s)] = 0$ since $L_s(1) = 0$. As $\varphi(s)$ is even, $\varphi(s)$ has to be a constant equal to $\varphi(0)$. But $\varphi(0) = B_0[1] = 1$ so that $\varphi(s) \equiv 1$. Since $\mathscr{B}$ is the inverse of B, and $B_s[1] = 1$, we have $\underline{\mathscr{B}_s[1] \equiv 1}$.

As $\mathscr{B}_a[\cos h\lambda\alpha]$ is an even function of λ, (13) possesses the double solution $\lambda = 0$.

We shall hereafter restrict ourselves to the following hypothesis

Hypothesis (H)- The equations

$$1 = \mathscr{B}_a[\cos h\lambda\alpha] = \mathscr{B}_b[\cos h\lambda\alpha]$$

have the only double solution $\lambda = 0$. In this case $K(x,s)$ is necessarily of the form

$$K(x,s) = k_1(x) + sk_2(x) \tag{14}$$

by the fundamental theorem of the mean periodic functions in R^1. But $K(x,s)$ is even in s so that $k_2(x) \equiv 0$ in R^n and we have

$$K(x,s) = k_1(x) \tag{15}$$

By inversion, $K(x,s) = B_s[F(x,\sigma)]$ gives

$$F(x,s) = k_1(x)\mathscr{B}_s[1] = k_1(x)$$

and for $s = 0, F(x,s) = f(x)$. Hence **184**

$$F(x,s) = f(x) \tag{16}$$

for $x \in R^n$ and $s > 0$.

But $F(x, s)$ is a solution of

$$A_x\,[F(\xi, s)] = L_s\,[F(x, \sigma)]$$

and $F(x, s) = f(x)$ gives necessarily

$$A_x\,[f(\xi)] = 0 \tag{17}$$

Thus we have in conclusion the

Theorem. *The hypothesis* (H) *and the condition*

$$F(x, a) = F(x, b) = F(x),$$

gives

$$F(x, s) = f(x) \text{ for any } s \geq 0$$

and

$$A_x\,[f(\xi)] = 0.$$

Corollary. *A is the Laplacian* Δ, *then* f *is a harmonic function.*

4 Discussion of the hypothesis H

In general, the hypothesis (H) is satisfied because the equations

$$1 = \mathscr{B}_a\,[\cos h\lambda\alpha] = \mathscr{B}_b\,[\cos h\lambda\alpha]$$

185 are a system of two equations with only one unknown λ. But it is necessary to investigate certain exceptional values of $a, b(a \neq b, a > 0, b > 0)$ for which the two-radius Theorem is false. The question of existence of such exceptional couples (a, b) is difficult in the general case but in the case of $A = \Delta$ of R^3 and $L = D^2 + \dfrac{2}{r}D, (n = 3)$, we can assert that there do not exists any such exceptional couples and the two-radius is always true in R^3.

Remark. This discussion is completely independent of the function f and consequently the couples of exceptional values (a, b) are also independent of $f(x)$.

Results of the discussion in the case $A = \Delta$.

In this case, $\mathscr{B} = \mathscr{B}_p$ in the notation of part I with $p = \dfrac{n-2}{2}$.

We know that $\mathscr{B}[\cosh \lambda x] = j_p(\lambda ix)$ where $j_p(z) = 2^p \Gamma(p+1)$ $z^{-p} J_p(z)$. The function which assumes the value 1 for $z = 0$. In this case the equations under consideration are

$$j_P(\lambda ia) = j_p(\lambda ib) = 1 \tag{18}$$

Thus it is sufficient to consider in C^1, the equation

$$j_p(z) = 1 \tag{19}$$

and to examine whether there exists two roots of (19) with the same argument. **186**

It is easy to see that the set of points ζ in C^1 which are roots of (19) have for axes of symmetry the two axes 0ξ and 0η if $z = \xi + i\eta$. This set is countable and contains the origin ($\xi = \eta = 0$). By an intricate discussion based on the asymptotic expansion of the Bessel functions, it is even possible to prove that for a given p (i.e. for a given dimension n of the space) the number of couples of roots of (19) which have the same argument is necessarily *finite*. Hence for any dimension n the number of exceptional ratios $\dfrac{a}{b}$ is necessarily finite.

In the case $n = 3$ i.e. $p = \dfrac{1}{2}$,

$$j_p(z) = \frac{\sin z}{z}$$

(for any odd n, $j_p(z)$ has an expression which depends algebraically on z, $\sin z$ and $\cos z$).

Thus $\sin z = z$ gives

$$\sin \xi \cosh \eta = \xi$$

$$\cos\xi \sinh\eta = \eta.$$

Eliminating the hyperbolic functions and the circular functions, we obtain respectively

$$\frac{\xi^2}{\sin^2\xi} - \frac{\eta^2}{\cos^2\xi} = 1$$

and

$$\frac{\xi^2}{\cosh^2\eta} + \frac{\eta^2}{\sinh^2\eta} = 1$$

so that

$$\eta = \pm\xi\cot\xi\left[1 - \frac{\sin^2\xi}{\xi^2}\right]^{\frac{1}{2}} \tag{A}$$

and

$$\xi = \pm\eta\coth\eta\left[\frac{\sinh^2\eta}{\eta^2} - 1\right]^{\frac{1}{2}} \tag{B}$$

187 The equations define two real curves in the plane (ξ, η), and the roots $z = \xi + i\eta$ of (19) are a subset of the set points of intersection of two curves.

As $0\xi, 0\eta$ are the axes symmetry of the two curves it is sufficient to examine the situation for $\xi \geq 0, \eta \geq 0$ (A) can be written as

$$\xi = f_1(\eta) f_2(\eta) \tag{B$'$}$$

where $f_1(\eta) = \eta\coth\eta,\ f_2(\eta) = \left[\dfrac{\sinh^2\eta}{\eta^2} - 1\right]^{\dfrac{1}{2}}.$

We have,

$$\begin{aligned} f_1'(\eta) = \coth\eta - \frac{\eta}{\sinh^2\eta} &= \frac{1}{\sinh^2\eta}[\sinh\eta\cosh\eta - \eta] \\ &= \frac{1}{2\sinh^2\eta}[\sinh 2\eta - 2\eta] > 0 \end{aligned}$$

and

$$f_2'(\eta) = \frac{1}{f_2(\eta)}\left[\frac{\sinh\eta\cosh\eta}{\eta^2} - \frac{\sinh^2\eta}{\eta^3}\right]\frac{\sinh\eta\cosh\eta}{\eta^3 f_2(\eta)}[\eta - \tanh\eta] > 0.$$

188 Thus $f_1(\eta)$ and $f_2(\eta)$ are increasing functions of $\eta > 0$ and so is their product $f_1(\eta) f_2(\eta)$.

$$\frac{d\xi}{d\eta} = f_1(\eta)\frac{\sinh\eta\cosh\eta}{\eta^3 f_2(\eta)}(\eta - \tanh\eta) + f_2(\eta)\frac{1}{2\sinh^2\eta}(\sinh 2\eta - 2\eta)$$

and $\xi/\eta = \coth\eta f_2(\eta)$.

Hence

$$\frac{d\xi}{d\eta}/\frac{\xi}{\eta} = \frac{1}{[\eta f_2(\eta)]^2}\sinh\eta\cosh\eta(\eta - \tanh\eta) + \frac{\sinh 2\eta - 2\eta}{\sinh 2\eta}$$

But $[\eta f_2(\eta)]^2 = \sinh^2\eta - \eta^2$ and finally

$$\begin{aligned}\frac{d\xi}{d\eta}/\frac{\xi}{\eta} - 1 &= \sinh\eta\cosh\eta\frac{\eta - \tanh}{\sinh^2\eta - \eta^2} - \frac{\eta}{\sinh\eta\cosh\eta}\\ &= \frac{\eta^3 + \eta\sinh^4\eta - \sinh^3\eta\cosh\eta}{\sinh\eta\cosh\eta(\sinh^2\eta - \eta^2)}.\end{aligned}$$

As $\frac{d}{d\eta}(\eta\sinh\eta - \cosh\eta) = \eta\cosh\eta > 0, \eta\sinh\eta - \cosh\eta$ and therefore $F(\eta) = \eta^3 + \sinh^3\eta(\eta\sinh\eta - \cosh\eta)$ is an increasing function of η. 189
But $F(0) = 0$ so that $F(\eta) > 0$ for $\eta > 0$. Hence

$$\frac{d\xi}{d\eta} > \frac{\xi}{\eta} \text{ on B for } \xi > 0, \eta > 0$$

from which it is clear that there does not exist any point (ξ, η) on B, $\xi > 0, \eta > 0$, the tangent at which passes through the origin. Then any chord through the origin can cut the curve only in one point which is not the origin. But the roots of sinz = z lie on the curve and it is impossible to find out distinct roots other than zero which have the same argument. Finally, for $n = 3$ there are no exceptional ratios $\frac{a}{b}$ and the two-radius theorem is completely proved in R^3.

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