Lectures On
Fibre Bundles and Differential Geometry

By

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Notes by

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Contents

Chapter 1

Differential Calculus

1.1

Let k be a commutative ring with unit and A a commutative and associative algebra over k having 1 as its element. In Applications, k will usually be the real number field and A the algebra of differentiable functions on a manifold. **1**

Definition 1. A *derivation* X is a map $X : A \to A$ such that

i) $X \in \mathrm{Hom}_k(A, A)$, and

ii) $X(ab) = (Xa)b + a(Xb)$ for every $a, b \in A$.

If no non-zero element in k annihilates A, k can be identified with a subalgebra of A and with this identification we have $Xx = 0$ for every $x \in k$. In fact, we have only to take $a = b = 1$ in (*ii*) to get $X_1 = 0$ and consequently $Xx = xX(1) = 0$.

We shall denote the set of derivations by C. Then C is obviously an A-module with the following operations:

$$(X + Y)(a) = Xa + Ya$$
$$(aX)(b) = a(Xb) \quad \text{for} \quad a, b \in A \quad \text{and} \quad X, Y \in C.$$

We have actually something more: If $X, Y, \in C$, then $[X, Y] \in C$.

This bracket product has the following properties:

$$[X_1 + X_2, Y] = [X_1, Y] + [X_2, Y]$$
$$[X, Y] = -[Y, X]$$
$$\Big[X, [Y, Z]\Big] + \Big[Y, [Z, X]\Big] + \Big[Z, [X, Y]\Big] = 0,$$

2 for $X, Y, Z \in C$. The bracket is not bilinear over A, but only over k. We have

$$\begin{aligned}[X, aY](b) &= \{X(aY) - (aY)(X)\}(b) \\ &= (Xa)(Yb) + a[X, Y](b)\end{aligned}$$

so that $[X, aY] = (Xa)Y + a[X, Y]$ for $X, Y \in C, a \in A$. The skew commutativity of the bracket gives

$$[aX, Y] = -(Ya)X + a[X, Y].$$

When A is the algebra of differentiable functions on a manifold, C is the space of differentiable vector fields.

1.2 Derivation laws

Definition 2. A derivation law in a unitary A-module M is a map $D : C \to Hom_k(M, M)$ such that, if D_X denotes the image of $X \in C$ under this map, we have

i) $D_{X+Y} = D_X + D_Y$

$D_{aX} = aD_X$ for a $\in A, X, Y \in C$.

i.e., $D \in \mathrm{Hom}_A(C, \mathrm{Hom}_k(M, M))$.

3 ii) $D_X(au) = (Xa)u + aD_Xu$ for $a \in A, u \in M$.

In practice, M will be the module of differentiable sections of a vector bundle over a manifold V. A derivation law enables one thus to differentiate sections of the bundle in specified directions.

If we consider A as an A-module, then D defined by $D_X a = Xa$ is a derivation law in A. This will hereafter be referred to as the *canonical derivation* in A. Moreover, if V is any module over k, we may define on the A-module $A \bigotimes_k V$, a derivation law by setting $D_X(a \otimes v) = Xa \otimes v$ and extending by linearity. This shall also be termed the *canonical derivation* in $A \bigotimes_k V$.

There exist modules which do not admit any derivation law. For instance, let A be the algebra $k[t]$ of polynomial in one variable t over k; then C is easily seen to be the free A-module generated by $P = \partial/_{\partial t}$. Let M be the A-module $A/\mathfrak{N}$ where $\mathfrak{N}$ is the ideal of polynomials without constant term. If there were a derivation law is this module, denoting by e the identity coset of $A/_{\mathfrak{N}}$, we have

$$0 = D_p(te) = (P.t)e + t.D_p e = e,$$

which is a contradiction.

However, the situation becomes better if we confine overselves to free A-modules.

Theorem 1. *Let M be a free A-module, $(e_i)_{i \in I}$ being a basis. Given any* 4
system $(\omega_i)_{i \in I}$ of elements in $\mathrm{Hom}_A(C, M)$, there exists one and only one derivation law D in M such that $D_X e_i = \omega_i(X)$ for every $i \in I$.

Let u be an arbitrary element of M. Then u can be expressed in the term $u = \sum \lambda_i e_i$. If the conditions of the theorem have to be satisfied, we have to define $D_X u = \sum (X \wedge_i) e_i + \sum \lambda_i \lambda_i(X)$. It is easy to verify that this is a derivation law.

We shall now see that the knowledge of one derivation law is enough to compute all the possible derivation laws. In fact, if D, D' are two such laws, then $(D_X - D'_X)(au) = a(D_X - D'_X)(u)$. Since $D, D' \in \mathrm{Hom}_A(C, \mathrm{Hom}_k(M, M))$, it follows that $D - D' \in \mathrm{Hom}_A(C, \mathrm{Hom}_A (M, M))$. Conversely, if D is a derivation law and h any element of $\mathrm{Hom}_A(C, \mathrm{Hom}_A (M, M))$, then $D' = D + h$ is a derivation law as can be easily verified.

1.3 Derivation laws in associated modules

Given module M_i with derivation laws D^i, we proceed to assign in a canonical way derivation laws to module which are obtained from the M_i by the usual operations.

Firstly, if M is the direct sum of the modules M_i, then $D_X(m) = \sum D^i_X(m_i)$ where $m = \Sigma m_i$, gives a derivation law in M.

Since D^i_X are k-linear, we may define D in $M_1 \bigotimes\limits_k \cdots \bigotimes\limits_k M_p$ by setting $D_X(u_1, \otimes \cdots \otimes u_p) = \sum u_1 \otimes \cdots D^i_X u_i \otimes \cdots u_p$. Now, it is easy to see that this leaves invariant the ideal generated by elements of the form

$$u_1 \otimes \cdots a u_i \otimes \cdots \otimes u_p - u_1 \otimes \cdots a u_j \otimes \cdots \otimes u_p \text{ with } a \in A.$$

5 This therefore induces a k-linear map D_X of $M_1 \bigotimes\limits_A \cdots \bigotimes\limits_A M_p$ into itself, where

$$D_X(u_1 \otimes \cdots \otimes u_p) = \sum u_1 \otimes \cdots D^i_X u_i \otimes \cdots \otimes u_p$$

where $u_1 \otimes \cdots \otimes u_p \in M_1 \bigotimes\limits_A \cdots \bigotimes\limits_A M_p$.

It is easily seen that D is a derivation law.

We will be particularly interested in the case when $M_1 = M_2 = \cdots = M_p = M$. In this case, we denote $M_1 \otimes \cdots \otimes M_p$ by $T^p(M)$. Since we have such a law in each $T^p(M)$ (for $T^0(M) = A$, we take the canonical derivation law) we may define a derivation law in the tensor algebra $T^*(M)$ of M. If t, t' are two tensors, we still have

$$D_X(t \otimes t') = D_X t \otimes t' + t \otimes D_X t'.$$

Now let $\mathfrak{N}$ be the ideal generated in $T^*(M)$ by elements of the form $u \otimes v - v \otimes u$ with $u, v \in M$. It follows from the above equality that $D_X \mathfrak{N} \subset \mathfrak{N}$. Consequently D induces a derivation law in $T^*(M)/\mathfrak{N}$, which is the symmetric algebra over M. Again, if $\mathfrak{N}'$ is the ideal in $T^*(M)$ whose generators are of the form $u \otimes u, u \in M$, then it is immediate that $D_X \mathfrak{N}' \subset \mathfrak{N}'$. Thus we obtain a derivation law in exterior algebra $T^*(M)/_{\mathfrak{N}'}$ of M.

Let M, L be two A-modules with derivation laws D^M, D^L respectively. We define a derivation law D in $\mathrm{Hom}_A(L, M)$ by setting $(D_X h) = D_X^M h - hD_X^L$ for every $h \in \mathrm{Hom}_A(L, M)$ and $X \in C$.

In fact
$$\begin{aligned} D_X(ah) &= D_X^M(ah) - (ah)D_X^L. \\ D_X(ah)(l) &= D_X^M(ah)(l) - (ah)D_X^L(l) \\ &= D_X^M(a.h(l)) - a.h(D_X^L(l)) \\ &= (Xa)(h(l)) + aD_L^M(h(l)) - a.h(D_X^L(l)) \\ &= \{(Xa)h\}(l) + (aD_Xh) \text{ for every } \ I \in L. \end{aligned}$$

In particular, if $L = M$ with $D^L = D^M$, then we have 6

$$D_X h = \left[D_X^L, h\right] \ \text{ for every } h \in \mathrm{Hom}_A(L, L).$$

Moreover, this leads to a derivation law in the dual L^* of L by taking $M = A$ with the canonical derivation law. The corresponding law is

$$(D_X f)(u) = X(f(u)) - f(D_X^L(u)) \ \text{ for every } \ f \in L^* \ \text{ and } \ u \in L.$$

Now let L, M be modules with derivation laws D^L, D^M respectively. We may then define a derivation law in the A-module $\mathscr{F}^p(M, L)$ of multilinear forms on M of degree p with values in L in the following way:

$$(D_X\omega)(u_1, \ldots, u_p) = D_X^L\omega(u_1, \ldots, u_p) - \sum_{r=1}^{p} \omega(u_1, \ldots, D_X^M u_r, \ldots u_p).$$

If $\mathscr{U}^p(M, L)$ is the submodule of $\mathscr{F}^p(M, L)$ consisting of alternate forms , then it is easy to see that $D_\wedge \mathscr{U}^p(M, L) \subset \mathscr{U}^p(M, L)$. This leads to a derivation law in the A-modules $\mathscr{F}^p(M, A), \mathscr{U}^p(M, A)$, if we take 7
$L = A$ with the canonical derivation law.

Let M_1, M_2, M_3 be three modules with a bilinear product $M_1 \times M_2 \to M_3$, denoted $(u, v) \to uv$. If D^1, D^2, D^3 are the respective derivation laws, we say that the product is *compatible* with the derivation laws if

$$D_X^3(uv) = (D_X^1 u)v + u(D_X^2 v) \ \text{ for } \ X \in C, u \in M_1 \ \text{ and } \ v \in M_2.$$

This was the case the we took $M_1 = M_2 = M_3 = A$ with the canonical derivation law and the algebra-product. Again, we seen that the above condition is satisfied by $M_1 = M_2 = M_3 = T^*(M)$ with the associated derivation laws and the usual multiplication. Moreover, it will be noted that if M is an A-module with derivation law D, the condition

$$D_X(au) = (Xa)u + aD_Xu \text{ for every } a \in A, u \in M$$

expresses the fact that the map $A \times M \to M$ defining the module structure is compatible with the derivation laws in A and M. If we denote $\mathscr{U}^p(L, M_i)$ by $\mathscr{U}_i^p(i = 1, 2, 3)$, we may define for $\alpha \in \mathscr{U}_1^p \beta \in \mathscr{U}_2^q, \alpha \wedge \beta$ by setting

$$(\alpha \wedge \beta)(a_1, \ldots, a_{p+q}) = \sum \in_\sigma \alpha(a_{\sigma(1)}, \ldots, a_{\sigma(p)})\beta(a_{\sigma(p+1)}, \ldots, a_{\sigma(p+q)})$$

where the summation extends over all permutations σ of $(1, 2, \ldots, p+q)$ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(p)$ and $\sigma(p+1) < \sigma(p+2) < \cdots < \sigma(p+q)$ and $\in_\sigma$ is its signature. If the derivation laws are compatible with the product, we have

$$D_X^3(\alpha \wedge \beta) = (D_X^1\alpha) \wedge \beta + \alpha \wedge (D_X^2\beta) \text{ for every } X \in C.$$

1.4 The Lie derivative

8 Let V be a manifold, $\mathscr{U}$ the algebra of differentiable functions on V and $\mathscr{C}$ the module of derivations of $\mathscr{U}$ (viz. differentiable vector fields on V. Any one-parameter group of differentiable automorphisms on V generates a differentiable vector field on V. Conversely, every differentiable vector field X gives rise to a local one-parameter group $s(t)$ of local automorphisms of V. If ω is a p-co variant tensor, i.e., if $\omega \in \mathscr{F}^p(\mathscr{C}, \mathscr{U})$, then we may define differentiation of ω with respect to X as follows:

$$\theta_X\omega = \lim_{t \to o} \frac{s^*(t)\omega - \omega}{t},$$

where $s^*(t)$ stands for the k-transpose of the differential map lifted to $\mathscr{F}^p(\mathscr{C}, \mathscr{U})$. This is known as the *Lie derivative* of ω with respect to X and can be calculated to be

$$\theta_X\omega(u_1, \ldots, u_p) = X\omega(u_1, \ldots, u_p) - \sum_{i=1}^{p} \omega(u_1, \ldots [X, u_i], \ldots u_p).$$

It will be noted if $a \in \mathscr{U}$, then $\theta_{aX}\omega \neq a(\theta_X\omega)$ if $p > 0$. At a point $\xi \in V$, the Lie derivative $\theta_X\omega$, unlike the derivation law, does not depend only on the value of the vector field x at ξ.

Lemma 1. *Let D be a derivation law in the A-module M and α a multilinear form on C of degree p with values in M. Then the map $\beta : C^P \to M$ defined by*

$$\beta(Z_1,\ldots,Z_p) = D_X\alpha(Z_1,\ldots,Z_p) - \sum_{i=1}^{p} \alpha(Z_1,\ldots[X,Z_i],\ldots,Z_p)$$

is multilinear. **9**

In fact, $\beta(Z_1,\ldots Z_i+Z'_i,\ldots Z_p) = \beta(Z_1,\ldots Z_i,\ldots Z_p)+\beta(Z_1,\ldots Z'_i,\ldots Z_p)$ and

$$\begin{aligned}\beta(Z_1,\ldots aZ_i,\ldots Z_p) &= (Xa)\alpha(Z_1,\ldots Z_p) + a(D_X\alpha(Z_1,\ldots Z_p))\\ &\quad - \sum_{j=1}^{p} a\alpha(Z_1,\ldots[X,Z_j],\ldots Z_p) - \alpha(Z_1,\ldots(Xa)Z_i,\ldots Z_p)\\ &= a\beta(Z_1,\ldots Z_p) \text{ for every } Z_1,\ldots Z_p \in C, a \in A.\end{aligned}$$

Definition 3. The map $X \to \theta_X$ of C into $\mathrm{Hom}_k(\mathscr{F}^p(C,M)\mathscr{F}^p(C,M))$ defined by $(\theta_X\alpha)(Z_1,\ldots,Z_p) = D_X\alpha(Z_1,\ldots Z_p) - \sum_{i=1}^{p}\alpha(Z_1,\ldots[X,Z_i]\cdots Z_p)$ is called the *Lie derivation* in the A-module $\mathscr{F}^p(C,M)$ and $\theta_X\alpha$ is defined to be the *Lie derivative* of α with respect to X.

The Lie derivation satisfies the following

$$\begin{aligned}\theta_X(\alpha+\beta) &= \theta_X\alpha + \theta_X\beta\\ \theta_X(a\alpha) &= (Xa)\alpha + a(\theta_X\alpha)\\ \theta_{X+Y}(\alpha) &= \theta_X\alpha + \theta_Y\alpha\\ \theta_{\lambda X}(\alpha) &= \lambda(\theta_X(\alpha))\end{aligned}$$

for every $X, Y \in C$, $\alpha\beta \in \mathscr{F}^p(C,M)$ and $\lambda \in k$.

Thus θ looks very much like a derivation law, but $\theta_{aX} \neq a\theta_X$ in general. From the definition, it follows that if α is alternate (resp. symmetric), so is $\theta_X\alpha$. **10**

1.5 Lie derivation and exterior product

As in ch.1.3, let M_1, M_2, M_3 be A-module with derivations D_1, D_2, D_3 respectively. Let there be given a bilinear product $M_1 \times M_2 \to M_3$ with reference to which an exterior product $(\alpha, \beta) \to \alpha \wedge \beta$ of $\mathscr{U}^p(C, M_1) \times \mathscr{U}^q(C, M_2) \to \mathscr{U}^{p+q}(C, M_3)$ is defined. If the product is compatible with the derivation laws, we have, on direct verification,

$$\theta_X(\alpha \wedge \beta) = \theta_X \alpha \wedge \beta + \alpha \wedge \theta_X \beta$$
$$\text{for } \alpha \in \mathscr{U}^p(C, M_1), \beta \in \mathscr{U}^q(C, M_2) \text{ and } X \in C.$$

1.6 Exterior differentiation

We shall introduce an inner product in the A-module $\mathscr{F}^p(C, M)$. For every $X \in C$, the inner product is the homomorphism l_X of $\mathscr{F}^p(C, M)$ into $\mathscr{F}^{p-1}(C, M)$ defined by

$$(l_X \alpha)(Z_, \ldots Z_{p-1}) = \alpha(X, Z_1, \ldots, Z_{p-1})$$

for every $\alpha \in \mathscr{F}^p(C, M), Z_1, \ldots Z_{p-1} \in C$. If α is alternate it is obvious that $l_x \alpha$ is also alternate. When α is of degree $0, l_X \alpha = 0$. The inner product satisfies the following

1. $l_{aX} = a l_X$

   $l_{X+Y} = l_X + l_Y$, for $a \in A, X, Y \in C$.

2. If θ_X is the Lie derivation,

   11 $\theta_X l_Y - l_Y \theta_X = l_{[X,Y]}$ for $X, Y, \in C$.

3. If α is alternate, $l_X l_X \alpha = 0$.

4. Let M_1, M_2, M_3 be three A-modules with a bilinear product compatible with their laws D_1, D_2, D_3. Then we have, for $\alpha \in \mathscr{U}^p(C, M_1), \beta \in \mathscr{U}^q(C, M_2)$

$$l_X(\alpha \wedge \beta) = (l_X \alpha) \wedge \beta + (-1)^p \alpha \wedge l_X \beta.$$

in fact, let $Z_1, \ldots, Z_{p+q-1} \in C$. Then

$$\begin{aligned} l_X(\alpha \wedge \beta)(Z_1, \ldots, Z_{p+q-1}) &= (\alpha \wedge \beta)(X, Z_1, \ldots, Z_{p+q-1}) \\ = \sum_\sigma \in_\sigma \alpha(X, Z_{\sigma(1)}, \ldots, z_{\sigma(p-1)})&\beta(Z_{\sigma(p)}, \ldots Z_{\sigma(p+q-1)}) \\ + \sum_\tau \in_\tau \alpha(X_{\tau(1)}, \ldots X_{\tau(p)})&\beta(X, Z_{\tau(p+1)}, \ldots Z_{\tau(p+q-1)}) \end{aligned}$$

where σ runs through all permutations of $[1, p+q-1]$ such that $\sigma(1) < \cdots < \sigma(p-1)$ and $\sigma(p) < \cdots < \sigma(p+q-1)$, while τ runs through those which satisfy $\tau(1) < \cdots < \tau(p)$ and $\tau(p+1) < \cdots < \tau(p+q-1)$. The first sum is equal to $(l_X\alpha) \wedge \beta$ and the second to $(-1)^p \alpha \wedge (l_X\beta)$.

Theorem 2. *Let D be a derivation law in an A-module M. Then there exists one and only one family of k-linear maps $d : \mathscr{U}^p(C, M) \to \mathscr{U}^{p+1}(C, M)(p = 0, 1, 2, \ldots)$ such that $\theta(X) = dl_X + l_X d$ for every $X \in C$.*

We call this map the *exterior differentiation* in the module of alternate forms on C.

First of all, assuming that there exists such a map d, we shall prove that it is unique. If d' is another such map, we have **12**

$$(d - d')l_X + l_X(d - d') = 0$$

Hence $l_X(d - d')\alpha = (d' - d)l_X\alpha$ for every $\alpha \in \mathscr{U}^p(C, M)$. We shall prove that $d\alpha = d'\alpha$ for every $\alpha \in \mathscr{U}^p(C, M)$ by induction on p. When α is of degree 0, we have $(d' - d)l_x\alpha = 0 = l_X(d - d')\alpha$. This being true for every $X \in C, d\alpha = d'\alpha$. If the theorem were for $p = q - 1$, then $(d' - d)l_X\alpha = 0 = l_X(d - d')\alpha$. Again, since X is arbitrary, $d\alpha = d'\alpha$ which proves the uniqueness of the exterior differentiation.

The existence is also proved by induction. Let $\alpha \in \mathscr{U}^o(C, M) = M$. Then we wish to define d_o such that $l_X d\alpha = (d\alpha)(X) = \theta_X\alpha$ (since $l_X\alpha = 0) = D_X\alpha$. Hence we can set $(d(\alpha)(X) = D_X\alpha$ for every $X \in C$. This is obviously A-linear since D_X is A-linear in X. Let us suppose that d has been defined on $\mathscr{U}^p(C, M)$ for $p = 0, 1, \ldots (q - 1)$ such that the formula is true. We shall define $d\alpha$ for $\alpha \in \mathscr{U}^p(C, M)$ by setting

$$(d\alpha(Z_1, \ldots, Z_{q+1}) = (\theta_{Z_1}\alpha)(Z_2, \ldots Z_{q+1}) - (dl_{Z_1}\alpha)(Z_2, \ldots Z_{q+1})$$

We have of course to show that the $d\alpha$ thus defined is an alternate form. That it is linear in $Z_2, \ldots Z_{q+1}$ follows from the induction assumption and the multilinearity of $\theta_{Z_1}\alpha$. We shall now prove that it is alternate. That means:

$$(d\alpha)(Z_1, \ldots, Z_{q+1}) = 0 \text{ whenever } Z_i = Z_j \text{ with } i \neq j.$$

13 Using the alternate nature of $\theta_{Z_1}\alpha$ and $dl_{Z_1}\alpha$, we see that it suffices to prove that

$$(\theta_{Z_1}\alpha)(Z_2, \ldots, Z_{q+1}) = (dl_{Z_1}\alpha)(Z_2, \ldots, Z_{q+1}) \text{ when } Z_1 = Z_2$$

Now
$$\begin{aligned}(\theta_{Z_2}\alpha)(Z_2, \ldots, Z_{q+1}) &= (l_{Z_2}\theta_{Z_2}\alpha)(Z_3, \ldots, Z_{q+1})\\ &= (\theta_{Z_2}l_{Z_2}\alpha)(Z_3, \ldots, Z_{q+1}) \text{ by } (2)\, of\ Ch.1.6\\ &= (dl_{Z_2}l_{Z_2}\alpha + l_{Z_2}dl_{Z_2}\alpha)\,(Z_3, \ldots, Z_{q+1})\\ &= (dl_{Z_2}\alpha)(Z_2, Z_3, \ldots, Z_{q+1}).\end{aligned}$$

Since $(d\alpha)$ is alternate, linearity in Z_1 follows from that in the other variables and additivity in Z_1. This completes the proof of the theorem.

Remark. If we take A to be the algebra of differentiable functions on a manifold V, and M to be A itself then the exterior differentiation defined above coincides with the usual exterior differentiation.

1.7 Explicit formula for exterior differentiation

Lemma 2. *The exterior differentiation defined above is given by*

$$\begin{aligned}(d\alpha)(Z_1, \ldots, Z_{p+1}) = &\sum_{i=1}^{p+1}(-1)^{i+1}D_{Z_i}\alpha(Z_1, \ldots Z_i, \ldots Z_{p+1})\\ &+ \sum_{i<j}(-1)^{i+j}\alpha([Z_i, Z_j], Z_1, \ldots \hat{Z}_i, \ldots \hat{Z}_j, \ldots Z_{p+1})\end{aligned}$$

14 *where the symbol $\wedge$ over a letter indicates that the corresponding elements is omitted.*

If d is defined as above and $\alpha \in \mathscr{U}^p(C, M)$, it is easy to see that

$$(l_{Z_1} d\alpha + dl_{Z_1}\alpha) = \theta_{Z_1}\alpha \text{ for every } Z_1 \in C.$$

By the uniqueness of exterior differentiation , we see that the above gives the formula for the exterior differentiation.

We shall use this explicit formula only when the degree of $\alpha \leq 2$. Then we have the formula:

1. $(d\alpha)(X) = D_X\alpha$ for $\alpha \in \mathscr{U}^0(C, M)$, $X \in C$.
2. $(d\alpha)(X, Y) = D_X\alpha(Y) - D_Y\alpha(X) - \alpha([X, Y])$ for $\alpha \in \mathscr{U}^1(C, M)$, $X, Y \in C$.
3. $(d\alpha)(X, Y, Z) = \sum\{D_X\alpha(Y, Z) - \alpha([X, Y], Z)\}$ for $\alpha \in \mathscr{U}^2(C, M)$, $X, Y, Z \in C$

where the summation extends over all cyclic permutations of (X, Y, Z).

1.8 Exterior differentiation and exterior product

We now investigate the behaviours of d with regard to the exterior product. Let M_1, M_2, M_3 be three modules with derivation laws D_1, D_2, D_3 and let $M_1 \times M_2 \to M_3$ be a linear product compatible with the derivation laws. Then we have, for $\alpha \in \mathscr{U}^p(C, M_1), \beta \in \mathscr{U}^q(C, M_2)$

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta.$$

Again, we prove this by induction, but this on $p+q$. When $p+q = 0$, the above formula just expresses the compatibility of the product with **15**
the derivation laws. Let us assume the theorem proved for $p+q = r-1$. We have

$$d(\alpha \wedge \beta)(Z_1, \ldots, Z_{p+q+1}) = (l_{Z_1} d(\alpha \wedge \beta))(Z_2, \ldots, Z_{p+q+1})$$

But,
$$\begin{aligned} l_{Z_1} d(\alpha \wedge \beta) &= \theta_{Z_1}(\alpha \wedge \beta) - d(l_{Z_1}(\alpha \wedge \beta)) \\ &= \theta_{Z_1}\alpha \wedge \beta + \alpha \wedge \theta_{Z_1}\beta - d(l_{Z_1}\alpha \wedge \beta + (-1)^p \alpha \wedge l_{Z_1}\beta) \end{aligned}$$

$$
\begin{aligned}
& \text{by(3) of Ch. 1.6} \\
&= \theta_{Z_1}\alpha \wedge \beta + \alpha \wedge \theta_{Z_1}\beta - dl_{Z_1}\alpha \wedge \beta - (-)^{p-1} l_{Z_1}\alpha \\
&\qquad \wedge d\beta - (-1)^{P} d\alpha \wedge l_{Z_1}\beta - \alpha \wedge dl_{Z_1}\beta \\
& \text{by induction assumption} \\
&= l_{Z_1} d\alpha \wedge \beta - (-1)^{p} d\alpha \wedge l_{Z_1}\beta + \alpha \wedge l_{Z_1} d\beta \\
&\qquad - (-1)^{p-1} l_{Z_1}\alpha \wedge d\beta \\
&= l_{Z_1}(d\alpha \wedge \beta) + (-1)^{p} l_{Z_1}(\alpha \wedge d\beta) \\
&= l_{Z_1}((d\alpha \wedge \beta) + (-1)^{p}(\alpha \wedge d\beta)),
\end{aligned}
$$

which proves our assertion.

Remark. If we take $M_2 = M_3 = M$ and $M_1 = A$ with the product $A \times M \to M$ defining the module structure, then, by the above formula, we obtain

$$
\begin{aligned}
d(a\alpha) &= (da) \wedge \alpha + a \wedge d\alpha \\
&= da \wedge \alpha + a.d\alpha
\end{aligned}
\left\{
\begin{array}{ll}
\text{for} & a \in \mathscr{U}^{\circ}(C, A) = A \\
\text{and} & \alpha \in \mathscr{U}^{p}(C, M).
\end{array}
\right.
$$

16 Thus the exterior differentiation is not A-linear.

1.9 The curvature form

It is well-know that the exterior differentiation in the algebra of differential forms on a manifold satisfies $dd = 0$. Let us compute in our general case value of $dd\alpha$ for $\alpha \in \mathscr{U}^{0}(C, M) = M$. Then one has

$$
\begin{aligned}
dd\alpha(X, Y) &= D_X(d\alpha)(Y) - D_Y(d\alpha)(X) - (d\alpha)([X, Y]) \\
&= D_X D_Y \alpha - D_Y D_X \alpha - D_{[X,Y]}\alpha
\end{aligned}
$$

Let $K(X, Y) = D_X D_Y - D_Y D_X - D_{[X,Y]}$. Then it is obvious that $K(X, Y)$ is a k-endomorphism of M. But, it actually follows on trivial verification that it is an A- endomorphism. On the other hand, we have

i) $K(X, Y) = -K(Y, X)$

ii) $K(X + X', Y) = K(X, Y) + K(X', Y)$, and

iii) $K(aX, Y)\alpha = aK(X, Y)\alpha$ for every $\alpha \in M$.

i) and *ii*) are trivial and we shall verify only *iii*).

$$\begin{aligned} K(aX, Y) &= D_{aX}D_Y - D_Y D_{aX} - D_{[aX,Y]} \\ &= aD_X D_Y - D_Y(aD_X) - D_{a[X,Y]} - (Ya)X \\ &= aD_X D_Y - (Ya)D_X - aD_Y D_X - aD_{[X,Y]} + (Ya)D_X \\ &= aK(X, Y). \end{aligned}$$

We have therefore proved that K is an alternate form of degree 2 over C with values in the module $Hom_A(M, M)$. **17**

Definition 4. The element K of $\mathscr{U}^2(C, \mathrm{Hom}_A(M, M))$ as defined above i.e.,

$$K(X, Y) = D_X D_Y - D_Y D_X -_{[X,Y]}$$

is called the *curvature form* of the derivation law D.

Examples. 1) Take the simplest case when $M = A$, with the canonical derivation law. Then

$$K(X, Y)u = XYu - YXu - [X, Y]u = 0$$

i.e., the curvature form is identically zero.

2) However, there are examples in which the curvature form is non-zero.

Let $A = k[x, y]$ with x, y transcendental over k. It is easy to see that C is the free module over A with $P\left(= \dfrac{\partial}{\partial x}\right), Q\left(= \dfrac{\partial}{\partial y}\right)$ as base.

We take M to be A itself but with a derivation law different from the canonical one. By Th.1, Ch. 1.2, if we choose $1 \in M$ and take $\omega \in \mathrm{Hom}_A(C, A)$, then there exists a derivation law D such that $D_X 1 = \omega(X)$. We shall define ω by requiring that $\omega(P) = y$ and $\omega(Q) = 1$. Then it follows that $D_P(1) = y, D_Q(1) = 1$.

$$K(P, Q)(1) = D_P D_Q(1) - D_Q D_P(1) - D_{[P,Q]}(1)$$

$$\begin{aligned} &= y - (Qy).1 - y(D_Q(1)) \\ &= -1. \end{aligned}$$

18 We now prove two lemmas which give the relation between Lie derivatives in two directions and the relation of Lie derivative to the exterior differentiation in terms of the curvature form.

Lemma 3. *$\theta_X\theta_Y\alpha - \theta_Y\theta_X\alpha = \theta_{[X,Y]}\alpha + K(X,Y)(\alpha)$ for every $\alpha \in \mathscr{U}^p$ (C, M).*

In fact, when α is of degree 0, the formula is just the definition of the curvature form. In the general case, this follows on straight forward verification.

Lemma 4. *$\theta_X d\alpha - d\theta_X\alpha = (l_X K) \wedge \alpha$ for every $\alpha \in \mathscr{U}^p(C, M)$.*

It will be noted that $K \in \mathscr{U}^2(C, \mathrm{Hom}_A(M, M))$ and hence $l_X K$ has values in $\mathrm{Hom}_A(M, M)$. Taking $M_1 = \mathrm{Hom}_A(M, M)$, $M_2 = M$, $M_3 = M$ in our standard notation, one has a bilinear product $M_1 \times M_2 \to M_3$ defined by $(h, u) \to$. The symbol $\wedge$ used in the enunciation of the lemma is with reference to this bilinear product.

Proof. As usual, we prove this by induction on p, the degree of α. When α is of degree 0, the formula reduces to

$$D_X(d\alpha(u)) - d\alpha([X, u]) - (dD_X\alpha)(u) = (l_X K \wedge \alpha)(u) \text{ for } u \in C.$$

i.e., $$D_X D_u\alpha - D_{[X,u]}\alpha - D_u D_X\alpha = (l_X K \wedge (\alpha)(u)$$

19 which is but the definition of K. Assuming the truth of the lemma for forms of degree $< p$, we have

$$\begin{aligned} l_Y\theta_X d\alpha &- l_Y d\theta_X\alpha - l_Y((l_X K) \wedge \alpha) \\ &= \theta_X l_Y d\alpha - l_{[X,Y]}d\alpha + dl_Y\theta_X\alpha - \theta_Y\theta_X\alpha \\ &\qquad - (l_Y l_X K) \wedge \alpha + (l_X K) \wedge (l_Y\alpha) \\ &= -\theta_X dl_Y\alpha + \theta_X\theta_Y\alpha - l_{[X,Y]}d\alpha + d\theta_X l_Y\alpha - dl_{[X,Y]}\alpha - \theta_Y\theta_X\alpha \\ &\qquad - (l_Y l_X K) \wedge \alpha + (l_X K) \wedge (l_Y\alpha) \\ &= -\theta_X dl_Y\alpha + d\theta_X l_Y\alpha + \theta_X\theta_Y\alpha - \theta_Y\theta_X\alpha \end{aligned}$$

$$
\begin{aligned}
&\quad - \theta_{[X,Y]}\alpha - (l_Y l_X K) \wedge \alpha + (l_X K) \wedge (l_Y \alpha) \\
&= -\theta_X d l_Y \alpha + d\theta_X l_Y \alpha + (l_X K) \wedge (l_Y \alpha) \text{ by lemma 3, Ch. 1.9} \\
&= 0 \text{ by induction assumption.}
\end{aligned}
$$

□

Remark. The maps θ_X, l_X, d are of degrees $0, -1, 1$ respectively considered as homomorphisms of the graded module $\mathscr{U}^*(C, M) = \sum_p \mathscr{U}^p(C, M)$. If φ is a map $\mathscr{U}^*(C, M) \to \mathscr{U}^*(C, M)$ of degree p, and ψ another of degree q, we define the commutator $[\varphi, \psi] = \varphi\psi - (-1)^{pq}\psi\varphi$. We have, in this notation, the following formulae :

i)	$[\theta_X, l_Y] = l_{[X,Y]}$	by (2) of Ch.1.6
ii)	$[d, l_X] = \theta_X$	by Th. 2, Ch. 1.6
iii)	$[\theta_X, \theta_Y] = \theta_{[X,Y]}$	for derivation laws of zero curvature
iv)	$[\theta_X, d] = 0$	by lemma 3 and 4, Ch.1.9
v)	$[l_X, l_Y] = 0$	by (3) of Ch.1.6

The operators θ_X and l_X have been generalised (see [15] to the 20
case $M = A$ by replacing X by an alternate form γ on C with values in C. This generalisation has applications in the study of variations of complex structures on a manifold.

A general formula for d^2 is given by

Lemma 5. *In our usual notation, $d^2\alpha = K \wedge \alpha$.*

The meaning of $\wedge$ has to be interpreted as in Lemma 4, Ch.1.9 . The proof is again by induction on the degree of α. If α is of degree $0, \alpha \in M$ and we have to show that $dd\alpha(X, Y) = K(X, Y)\alpha$.Assuming the lemma verified for forms of degree $< p$, we get

$$
\begin{aligned}
l_X dd\alpha &= \theta_X d\alpha - d l_X d\alpha \\
&= d\theta_X \alpha + (l_X K) \wedge \alpha - d\theta_X \alpha + dd l_X \alpha \text{ by Lemma 4, Ch. 1.9} \\
&= (l_X K) \wedge \alpha + K \wedge l_X \alpha \text{ by induction assumption} \\
&= l_X(K \wedge \alpha).
\end{aligned}
$$

Since X is arbitrary, $dd\alpha = K \wedge \alpha$ and the lemma is proved.

Lemma 6.

$$
\begin{aligned}
dK &= 0 \\
dK(X,Y,Z) &= \sum \{(D_X K)(Y,Z) - K([X,Y],z)\} \ \textit{by (3) of Ch. 1.7} \\
&= \sum \{D_X K(Y,Z) - K(Y,Z)D_X - K([X,Y],Z)\} \\
&= \sum \{D_X D_Y D_Z - D_X D_Z D_Y - D_X D_{[Y,Z]} - D_Y D_Z D_X + \\
&\quad D_Z D_Y D_X + D_{[Y,Z]} D_X - D_{[X,Y]} D_Z + D_Z D_{[X,Y]} + D_{[[X,Y]Z]}\} \\
&= 0
\end{aligned}
$$

21 *using Jacobi's identity where the summations extend over cyclic permutations of X, Y, Z.*

1.10 Relations between different derivation laws

We shall now investigate the relations between the exterior differentiations, curvature forms etc. corresponding to two derivation laws in the same module M. It has already been shown (Ch.1.2) that if D, D' are two such derivation laws, then there exists $h \in \mathrm{Hom}_A(M, M))$ such that $D'_X = D_X + h_X$. We denote the exterior differential operator, curvature form etc. Corresponding to D' by d', K' etc. Then d' is given by $d'\alpha = d\alpha + h \wedge \alpha$. (Here also, the *Lambda*-sign has to be interpreted as in lemma 4, Ch. 1.9). In fact, by definition, it follows that $\theta'_X \alpha = \theta_X \alpha + h_X \circ \alpha$. Hence

$$dl_X \alpha + h \wedge l_X \alpha + l_X d\alpha + l_X(h \wedge \alpha) = \theta'_X \alpha.$$

From Th. 2, Ch. 1.6 on the uniqueness of exterior differentiation, our
22 assertion follows.

With regard to the curvature form, we have the following formula:

$$K' = K + h \wedge h + dh.$$

For

$$K'(X,Y) = (D_X + h_X)(D_Y + h_Y) - (D_Y + h_Y)(D_X + h_Y) - D_{[X,Y]} - h_{[X,Y]}$$

$$= K(X, Y) + h_X h_Y - h_Y h_X + (D_X h_Y) - (D_Y h_Y) - h_{[X,Y]}$$
$$= K(X, Y) + h_X h_Y - h_Y h_X + dh(X, Y)$$
$$= K(X, Y) + (h \wedge h)(X, Y) + dh(X, Y).$$

The classical notation for $(h \wedge h)$ defined by $(h \wedge h)(X, Y) = h(X)h(Y) - h(Y)h(X)$ is $[\ h, h\ ]$. In that notation, we have

$$K' = K + [h, h] + dh.$$

1.11 Derivation law in C

When A is the algebra of differentiable functions on a manifold V, any derivation law in the A-module C of differentiable vector fields on V is called a *linear connection* on the manifold.

Let A be an algebra over k, D a derivation law in the A- module C of derivations of A. Let $\eta : C \to C$ be the identity mapping. Then exterior defferential of η is given by

$$\begin{aligned}(d\eta)(X, Y) &= D_X \eta(Y) - D_Y \eta(x) - \eta([X, Y])\\ &= D_X Y - D_Y X - [X, Y] \text{ for } X, Y \in C.\end{aligned}$$

The alternate linear form $d\eta = T$ is called the *torsion form* of the **23**
derivation law in C.

Regarding the action of d on T, we have the *Bianchi's identity*

$(dT)(X, Y, Z) = \sum K(X, Y)Z$, where the summation extends over all cyclic permutations of $(X, .Y, Z)$.

This is immediate from Lemma 5, Ch. 1.9

Let D be a derivation law in C. Then we can define a derivation law in the module of multilinear forms on C with values in an A- module M with derivation law D. For $\alpha \in \mathscr{F}^p(C, M)$, we define

$$(D_X \alpha)(Z_1, \ldots, Z_p) = D_X \alpha(Z_1, \ldots, Z_p) - \sum_{i=1}^{p} \alpha(Z_1, \ldots, D_X Z_i, \ldots, Z_p).$$

Moreover, we define for every $\alpha \in \mathscr{F}^p(C, M)$

$$(D_X \alpha)(Z_1, \ldots, Z_{p+1}) = D_{Z_1} \alpha(Z_1, \ldots, Z_p) - \sum_{i=2}^{p} \alpha(Z_2, \ldots, D_{Z_1} Z_i, \ldots, Z_{p+1})$$

Obviously $D\alpha \in \mathscr{F}^{p+1}(C, M)$. However, this operator D does not take alternate forms into alternate forms. We therefore set, for $\alpha \in \mathscr{U}^p(C, M)$

$$d'\alpha = \sum_{i=1}^{p+1} (-1)^{i+1} (D\alpha)(Z_i, Z_1, \ldots, \tilde{Z}_i, \ldots, Z_{p+1}).$$

It is easy to see that $d'\alpha \in \mathscr{U}^{p+1}(C, M)$.

Theorem 3. *If the torsion form is zero, then d' and the exterior deffer-*
24 *ential coincide.*

In fact, it is easy to verify that $l_X d' + d' l_X = \theta_X$ and the theorem then follows from Th.2, Ch.1.6

In particular, let K be the curvature form of the derivation law in C; since $dK = 0$, we have $d'K = 0$. It is easily seen that $dK = \sum (D_X K)(Y, Z)$. where the summation extends over cyclic permutations of X, Y, Z. Hence we have the *Second Bianchi Identity* :

If the torsion of the derivation law in C is 0, then $\sum (D_X K)(Y, Z) = 0$ where the summation is over all cyclic permutations of X, Y, Z.

1.12 Connections in pseudo-Riemannian manifolds

A differentiable manifold V together with a symmetric bilinear form is said to be *Riemannian* if the form is positive definite at all points. If the above form is only non-degenerate (not necessarily positive definite) at each point, the manifold is *pseudo-Riemannian.* Such a form defines a natural isomorphism of the module of vector fields on V onto its dual.

Accordingly, in our algebraic set-up, we define a *pseudo Riemannian form* on C to be a symmetric bilinear form on C with values in A such that the induced map $C \to C^*$ is bijective.

Theorem 4. *If g is a pseudo-Riemannian form on C then there exists one and only one derivation law D on C such that*

25 1) *the torsion form of D is zero,*

2) *$D_X g = 0$ for every X.*

Explicitly, 1) means that $Xg(Y,Z) - g(D_X Y, Z) - g(Y, D_X Z) = 0$ *for every* $Y, Z \in C$.

In fact, by straight forward computation, it can be found that if there exists one such derivation law D, it must satisfy the equation

$$\begin{aligned} 2g(D_X Y, Z) &= Xg(Y,Z) - Zg(Y,X) + Yg(Z,X) \\ &\qquad - g(Y,[X,Z]) + g([Z,Y],X) - g(Z,[Y,X]). \end{aligned}$$

Since the map $C \to C^*$ induced from g is bijective, this equation determines D uniquely and conversely if we define D by this equation, it can be easily verified that D is a derivative law in C and that it satisfies the conditions of the theorem

1.13 Formulae in local coordinates

Finally we translate some of our formulae in the case of a differentiable manifold in terms of local coordinates. As far as local coordinates are concerned, we may restrict ourselves to an open subset V of R^n. Let $\mathscr{U}$ denote the algebra of differentiable functions on V and $\mathscr{C}$ the $\mathscr{U}$-module of vector fields on V. Let $(x^1, x^2, \ldots {}^n)$ be a system of coordinates. The partial derivatives $P_i = \dfrac{\partial}{\partial x^i}$ have the following properties:

1) $(P_i)_{i=1,2,\ldots n}$ is a base of $\mathscr{C}$ over $\mathscr{U}$.

2) $P_i x^j = \delta_{ij}$

3) $[P_i, P_j] = 0$ for any i, j. **26**

Also $(dx^1, \ldots dx^n)$ form a base for $\mathscr{C}^*$ over $\mathscr{U}$ dual to (P_i) i.e. $(dx^i)(P_j) = \delta_{ij}$. Since the $\mathscr{U}$ -module $\mathscr{C}$ is free over $\mathscr{U}$, all the associated modules such as $T^p(\mathscr{C}), \mathscr{U}^p(\mathscr{C}, \mathscr{U})$ are all free over $\mathscr{U}$. Thus $\mathscr{U}^p(\mathscr{C}, \mathscr{U})$ has a basis consisting of elements $dx^{\lambda(1)} \wedge \cdots \wedge dx^{\lambda(p)}, \lambda \in S$ where S where S is the set of all maps $\lambda : [1,p] \to [1,n]$ such that $\lambda(1) < \lambda(2) < \cdots < \lambda(p)$. If M is an $\mathscr{U}$-module, any alternate form $\in \mathscr{U}^p(\mathscr{C}, M))$ can be written in the form $\sum_{\lambda\in S} \omega^\lambda \wedge dx\,(1)_{\wedge dx}(2)_{\wedge\cdots\wedge dx}\lambda(p)$

with $\omega^\lambda \in M$. (The exterior product is with respect to the bilinear product $M \times A \to M$ defining the structure of $\mathscr{U}$-module).

Let M be a free $\mathscr{U}$ -module of finite rank with a derivation law D. Let $e_\alpha(\alpha = 1, 2, \ldots m)$ be a base of M. We set

$$D_{P_i} e_\alpha = \sum \Gamma^\beta_{i\alpha} e_\beta, \Gamma^\beta_{i\alpha} \in \mathscr{U}$$

The functions $\Gamma^\beta_{i\alpha}$ completely determine the derivation law. Conversely, given $\Gamma^\beta_{i\alpha} \in \mathscr{U}$, we may define, for

$$u = \sum_{\alpha=1}^{m} \rho^\alpha l_\alpha, \rho^\alpha \in \mathscr{U}.$$

$$D_{P_i} u = \sum_\alpha (p_i \rho^\alpha) e_\alpha + \sum_{\alpha\beta} \rho^\alpha \Gamma^\beta_{i\alpha} e_\beta$$

and extend D to the whole of ξ by linearity. It is easy to see that is a

27 derivation law. The above becomes in the classical notation

$$D_{P_i} u = \sum_\alpha \frac{\partial \rho^\alpha}{\partial x^i} e_\alpha + \sum_{\alpha,\beta} \rho^\alpha \Gamma^\beta_{i,\alpha} e_\beta$$

From this we get if $u \in M$, then

$$du = \sum_i dx_i \wedge (D_{P_i} u)$$

Let $\omega \in \mathscr{U}^p(C, M)$. We have seen that

$$\omega = \sum_{\lambda \in S} \omega^\lambda \wedge dx^{(1)} \wedge \cdots \wedge dx^{(p)} \text{ with } \omega^\lambda \in M.$$

Using the fact that $\theta_{P_i}(\alpha \wedge \beta) = (\theta_{P_i}\alpha) \wedge \beta + \alpha \wedge (\theta_{P_i}\beta)$, in order to compute $\theta_X \omega$, it is enough to compute $\theta_{P_i}\omega^\lambda$ and $\theta_{P_i} dx^j$. But $\theta_{P_i}\omega^\lambda = D_{P_i}\omega^\wedge$ which we have fond out earlier. On the other hand,

$$\begin{aligned}(\theta_{P_i} dx^j(P_k) &= P_i((dx^j)(P_k)) - dx^j([P_i, P_k]) \\ &= 0.\end{aligned}$$

Chapter 2

Differentiable Bundles

2.1

We give in this chapter, mostly without proofs, certain definitions and results on fibre bundles which we require in the sequel. **28**

Definition 1. *A differentiable principal fibre bundle is a manifold P on which a Lie group G acts differentiable to the right, together with a differentiable map p of P onto a differentiable manifold X such that*

P.B. *for every $x_0 \in X$, there exist an open neighbourhood U of x_0 in X and a diffeomorphism γ of $U \times G \to p^{-1}(U)$ (which is an open submanifold of P) satisfying $p\gamma(x, s) = x, \gamma(x, st) = \gamma(x, s)t$ for $x \in U$ and $s, t \in G$.*

X shall be called the *base* , p the *projection* and P the bundle. For any $x \in X$, $p^{-1}(x)$ shall be called the *fibre over x* and for $\xi \in P$, the fibre over $p\xi$ is the *fibre through ξ*.

The following properties follow immediately from the definition:

a) Each fibre is stable under the action of G, and G acts with out fixed points on P, i.e. if $\xi s = \xi$ for some $\xi \in P$ and $s \in G$, then $s = e$;

b) G acts transitively on each fibre, i.e. if ξ, η are such that $p\xi = p\eta$, then there exists $s \in G$ such that $\xi = \eta$; **29**

c) For every $x_0 \in X$, there exist an open neighbourhood V of x_0 and a differentiable map $\sigma : V \to P$ such that $p\sigma(x) = x$ for every $x \in V$. We have only to choose for V the neighbourhood U of condition (P.B), and define for $x \in V, \sigma(x) = \gamma(x, e)$. A continuous (resp. differentiable) map $\sigma : V \to P$ such that $p\sigma(x) = x$ for every $x \in V$ is called a *cross-section resp. differentiable cross-section*) over V.

d) For every $x_0 \in X$, there exist an open neighbourhood V of x_0 and a differentiable map ρ of $p^{-1}(V)$ into G such that $\rho(\xi s) = \rho(\xi)s$ for every $\xi \in p^{-1}(V)$ and $s \in G$. Choose for V as before the neighbourhood U of (P.B.). If π is the canonical projection $U \times G \to G$, the map $\rho = \pi \circ \gamma^{-1}$ of $p^{-1}(V)$ into G satisfies the required condition. It is also obvious that ρ is bijective when restricted to any fibre.

Conversely we have the following

Proposition 1. *Let G be a Lie group acting differentiably to the right on a differentiable manifold P. Let X be another differentiable manifold and p a differentiable map $P \to X$. If conditions* $(b), (c), (d)$ *are fulfilled, then P with $p : P \to X$ is a principal bundle over X.*

For every $x_0 \in X$, we can find an open neighbourhood V of x_0, a differential map $\sigma : V \to P$ and a homomorphism $\rho : p^{-1}(V) \to G$
30 such that $p\sigma(x) = x, \rho(\xi s) = \rho(\xi)s$ and moreover $\rho\sigma(x) = e$ for every $x \in V, \xi \in p^{-1}(V)$. We define $\gamma : U \times G \to P$ by setting $\gamma(x, s) = \sigma(x)s$ for $x \in V, s \in G$. If θ is the map $p^{-1}(V) \to V \times G$ defined by $\theta : \xi \to (p\xi, \rho\xi)$ it is easy to verify that $\theta\gamma = \gamma\theta =$ Identity, using the fact that $\rho\sigma(x) = e$ for every $x \in U$. Both θ and γ being differentiable, our assertion is proved.

2.2 Homomorphisms of bundles

Definition 2. A *homomorphism h of a differentiable principal bundle P into another bundle P' (with the same group G) is a differentiable map $h : P \to P'$ such that $h(\xi s) = h(\xi)s$ for every $\xi \in P$, $s \in G$.*

It is obvious that points on the same fibre are taken by h into points of P' on the same fibre. Thus the homomorphism h induces a map $\underline{h} : X \to X'$ such that the diagram

$$\begin{tikzcd} P \arrow[r, "h"] \arrow[d, "p"'] & P' \arrow[d, "p'"] \\ X \arrow[r, "h"] & X' \end{tikzcd}$$

is commutative. The map $\underline{h} : X \to X'$ is easily seen to be differentiable. This is called the *projection* of h.

Definition 3. A homomorphism $h : P \to P'$ is said to be an *isomorphism* if there exists a homomorphism $h' : P' \to P$ such that hoh'l $h'oh$ are identities on P', P respectively.

Proposition 2. *If P and P' are differentiable principal bundles with the* 31
same base $\underline{X}$ and group G, then every homomorphism $h : P \to P'$ whose projection $\overline{h}$ is a diffeomorphism of X onto X, is an isomorphism.

In the case when $\underline{P}$ and P' have the same base X, an isomorphism $h : P \to P'$ for which $\overline{h}$ is identity will be called an isomorphism *over* X.

2.3 Trivial bundles

If G is a Lie group and X a differentiable manifold, G acts on the manifold $X \times G$ by the rule $(x, s)t = (x, st)$. $X \times G$ together with the natural projection $X \times G \to X$ is a principal bundle. Any bundle isomorphic to the above is called *trivial principal bundle*.

Proposition 3. *Let P be a principal bundle over X with group G. Then the following statements are equivalent:*

1) *P is a trivial bundle.*

2) *There exists a differentiable section of P over X.*

3) *There exists a differentiable map $\rho : P \to G$ such that $\rho(\xi s) = \rho(\xi)s$ for every $\xi \in P$, $s \in G$.*

2.4 Induced bundles

Let P be a principal bundle over X with group G. Let q be a differentiable map of a differentiable manifold Y into X. The subset P_q of $Y \times P$ consisting of points (y, ξ) such that $q(y) = p(\xi)$ is a closed submanifold. There is also a canonical map $p_q : P_q \to Y$ defined by $P_q(y, \xi) = y$. The
32 group G acts on P_q with the law $(y, \xi)s = (y, \xi s)$. It is easy to see that P_q together with p_q is a principal bundle with base Y and group G. This is called the bundle induced from P by the map q. There then exists clearly a canonical homomorphism $h : P_q \to P$ such that

$$\begin{tikzcd} P_q \arrow[r, "h"] \arrow[d, "p_q"'] & P \arrow[d, "p"] \\ Y \arrow[r, "h=q"] & X \end{tikzcd}$$

is commutative. h is defined by $h(y, \xi) = \xi$. By proposition (3). Ch.2.3, the principal bundle P_q is trivial if and only if there exists a differentiable cross section for P_q over Y. This is equivalent to saying that there exists a differentiable map $\lambda : Y \to P$ such that $p = p \circ \lambda$, i.e., the diagram is commutative.

$$\begin{tikzcd} & P \arrow[d, "p"] \\ Y \arrow[ur, "\lambda"] \arrow[r, "q"'] & X \end{tikzcd}$$

We now assume that q is surjective and everywhere of *rank* $= \dim X$. If P_q is trivial we shall say that P is *trivialised* by the map q.

Let q be a differentiable map $Y \to X$ which trivialises P. Consider the subset Y_q of $Y \times Y$ consisting of points (y, y') such that $q(y) = q(y')$. This is the graph of an equivalence relation in Y. Since q is of *rank* $= \dim X$, Y_q is a closed submanifold of $Y \times Y$.

33 Let λ be a map $Y \to P$ such that $q = p \circ \lambda$. If $(y, y') \in Y_q$, then $\lambda(y), \lambda(y')$ are in the same fibre and hence there exists $m(y, y') \in G$ such that $\lambda(y') = \lambda(y)m(y, y')$. Thus we have a map $m : Y_q \to G$ such that for $(y, y') \in Y_q$, we have $\lambda(y') = \lambda(y)m(y, y')$. This map is

easily seen to be differentiable. Obviously we have, for $(y, y'), (y', y'') \in Y_q, m(y, y')m(y', y'') = m(y, y'')$.

Definition 4. Let Y be differentiable manifold and q a differentiable map of Y onto X everywhere of *rank* $= \dim X$. The manifold Y_q is defined as above. Any differentiable map $m : Y_q \to G$ is said to be a *multiplicator* with value in G if it satisfies

$$m(y, y')\, m(y', y'') = m(y, y'') \text{ for } (y, y'), (y'.y'') \in Y_q.$$

We have seen that to every trivialisation of P by q corresponds a multiplicator with values in G. However, this depends upon the particular lifting λ of q. If μ is another such lifting with multiplicator n, there exists a differentiable map $\rho : Y \to G$ such that $m(y', y)\rho(y) = \rho(y')n(y', y)$ for every $(y, y')eY_q$. Accordingly, we define an equivalence relation in the set of multiplicators in the following way:

The multiplicators m, n are *equivalent* if there exists a differentiable map $\rho : Y \to G$ such that $m(y, y')\rho(y') = \rho(y)n(y, y')$ for every $(y, y') \in Y_q$. Hence to every principal bundle P trivialised by q corresponds a class of multiplicators $m(P)$. It can be proved that if $m(P) = m(P')$, then P and P' are isomorphic over X. Finally, given a multiplicator m, there exists bundle a P trivialised by q for which $m(P) = m$. In
fact, in the space $Y \times G$, introduce an equivalence relation R by defining **34**
$(y, s) \sim (y', s')$ if $(y, y') \in Y_q$ and $s' = m(y', y)s$. Then the quotient $(Y \times G)/R$ can be provided with the structure of a differentiable principal bundle over X trivialised by q. The multiplicator corresponding to the map $\lambda : Y \to (Y \times G)/R$ defined by $y \to (y, e)$ is obviously m.

2.5 Examples

1) Given a principal bundle P over X, we may take $Y = P$ and $q = p$. Then λ =Identity is a lifting of q to P. The corresponding multiplicator m is such that $y' = y\, m(y, y')$ where $p(y) = p(y')$

2) Let $(U_i)_{i \in I}$ be an open cover of X such that there exists a cross section σ_i of P over each U_i. Take for Y the open submanifold of $X \times I$ (with

I discrete) consisting of elements (x, i) such that $x \in u_i$. Define $q(x, i) = x$. This is obviously surjective and everywhere of rank $= \dim X$. Then P is trivialised by q, since the map $\lambda(x, i) = \sigma_i(x)$ of $Y \to P$ is a lifting of q. The manifold Y_q may be identified with the submanifold of $X \times I \times I$ consisting of elements x, i, j such that $x \in \cup_i \cap \cup_j$. If m is the corresponding multiplicator, we have $\lambda(x, i) = \lambda(x, j) m(x, j, i)$. This can be written as $\sigma_i(x) = \sigma_j(x) m_{ji}(x)$ where the multiplicator m is looked upon as a family of maps $m_{ji} : U_j \cap U_i \to G$ such that $m_{ji} m_{ik}(x) = m_{jk}(x)$ for every $x \in U_i \cap U_j \cap U_k$. Such a family of maps is called a set of *transition functions*. Two
35 sets of transition functions $\{m_{ij}\}, \{n_{ij}\}$ are equivalent if and only if there exists a family of differentiable maps $\rho_i : U_i \to G$ such that $m_{ji}(x)\rho_i(x) = \rho_j(x) n_{ji}(x)$ for every $x \in U_i \cap U_j$. Conversely, given a set of transition functions $\{m_{ij}\}$ with respect to a covering U_i of V, we can construct a bundle P over V such that P is trivial over each U_i and there exists cross - sections σ_i over U_i satisfying $\sigma_i(x) = \sigma_j(x) m_{ij}(x)$ for every $x \in U_i \cap U_j$

Let G be the sheaf of germs of differentiable functions on X with values in G. The compatibility relations among transition functions

$$\text{viz }. m_{ji}(x) m_{ik}(x) = m_{jk}(x) \textit{ for every } x \in U_i \cap U_j \cap U_k$$

only state that a set of transition functions is a $1-$cocycle of the covering $(U_i)_{i \in I}$ with values in the sheaf $\underline{G}$. Two such cocycles are equivalent (in the sense of multiplicators) if and only if they differ by a coboundary. In other words, the set of equivalent classes of transition functions for the covering $(U_i)_{i \in I}$ is in one-one correspondence with $H^1((U_i)_{i \in I}, G)$. It will be noted that there is no group structure in $H^1((U_i)_{i \in I}, G)$ in general. It can be proved by passing to the direct limit that there is a one-on correspondence between classes of isomorphic bundles over X and elements of $H^1(X, \underline{G})$.

2.6 Associated bundles

36 Let P be a differentiable principal fibre bundle over X with group G. Let

F be a differentiable manifold on which G acts differentiably to the right. Then G also acts on the manifold $P \times F$ by the rule $(\xi, u)s = (\xi s, us)$ for every $s \in G$.

Definition 5. A *differentiable bundle* with fibre type F *associated to* P is a differentiable manifold E together with a differentiable map $q : P \times F \to E$ such that $(P \times F, q)$ is a principal fibre bundle over E with group G.

Let G act on a differentiable manifold F to the right. Then we can construct a differentiable bundle associated to P with fibre F. We have only to take $E = \dfrac{(P \times F)}{G}$ under the action of G defined as above and q to be the canonical projection $P \times F \to E$. The differentiable structure in E is determined by the condition: $(P \times F, q)$ is a differentiable principal bundle over E.

Now let E be a differentiable bundle associated to P with fibre type F and group G. Then there exists a canonical map $p_E : E \to X$ such that $p_E q(\xi, u) = p\xi$ for $(\xi, u) \in P \times F$ where p, q are respectively the projections $P \to X$ and $P \times F \to E$. X is therefore called the base manifold of E and p_E the projection of E. For every $x \in X, p_E^{-1}(x)$ is called the fibre over x. Let U be an open subset of X. A continuous (resp. differentiable) map $\sigma : U \to E$ such that $p_E \sigma(x) = x$ for every $x \in U$ is called a *section (resp differentiable section)* of E over U.

Let σ be a differentiable section of P over an open subset of U of
X. This gives rise to a diffeomorphism γ of $U \times F$ onto $p_E^{-1}(U)$ defined **37**
by $\gamma(x, v) = q(\sigma(x), v)$ for $x \in U, v \in F$. On the other hand, we also have $p_E \gamma(x, v) = x$. In particular , if P is trivial, there exists a global cross-section σ (Prop.3, Ch. 2.3) and hence γ is a diffeomorphism of $U \times F$ onto E.

We finally prove that all fibres in E are diffeomorphic with F. In fact for every $z \in P$. the map $F \to E$ (which again we denote by z) defined by $z(v) = q(z, v)$ is a diffeomorphism of F Onto $p_E^{-1}(p(z))$. Such a map $z : F \to E$ is called a *frame* at the point $x = p(z)$. Corresponding to two different frames z, z' at the same point x, we have two different diffeomorphisms z, z' of F with $p_E^{-1}(x)$. If $s \in G$ such that $z's = z$, then the we have $z(v) = z'(vs^{-1})$.

Examples. (1) Let V be a connected differentiable manifold and P the universal covering manifold of V. Let $p : P \to V$ be a covering map. Then the fundamental group π_1 of V acts on P and makes of P a principal bundle over V with group π_1. Moreover, any covering manifold is a bundle over V associated to the universal covering manifold of V with discrete fibre. On the other hand, any Galois covering of V may be regarded as a principal bundle over V with a quotient of π_1 as group.

(2) Let B be a closed subgroup of a Lie group G. Then B is itself a Lie group and it acts to the right on G according to the following rule: $G \times B \to G$ defined by $(s, t) \to st$. Consider the quotient space $V = G/B$ under the above action. There exists one and only one structure of a differentiable manifold on V such that G is a differen-
38 tiable bundle over V with group B ([29]). It is moreover easy to see that left translations of G by elements of G are bundle homomorphisms of G into itself. The projections of these automorphisms to the base space are precisely the translations of the left coset space G/B by elements of G.

(3) Let G be a Lie group and B a closed subgroup. Let H be a closed subgroup of B. As in (2), B/H has the structure of a differentiable manifold and B acts on B/H to the right according to the rule:

$$q(b)b' = b^{-1}q(b) = q(b'^{-1}b),$$

where $b, b' \in B$ and q is the canonical projection $B \to B/H$. We define a map $r : G \times B/H \to G/H$ by setting $r(s, bH) = sbH$. It is easy to see that this makes $G \times B/H$ a principal bundle over $\dfrac{G}{H}$. In other words, G/H is a bundle associated to G with base G/B and fibre B/H.

2.7 Vector fields on manifolds

Let V be a differentiable manifold and $\mathscr{U}(V)$ the algebra of differentiable functions on V. At any point $\xi \in V$, a tangent vector U is a map $U : \mathscr{U}(V) \to R$ satisfying $U(f + g) = Uf + Ug; Uf = 0$ when f is constant; and $U(fg) = (Uf)g(\xi) + f(\xi)(Ug)$ for every $f, g \in \mathscr{U}(V)$. The tangent vectors U at a point ξ form a vector space T_ξ. A vector field X is an assignment to each ξ in V of a tangent vector X_ξ at ξ. A vector field X may also be regarded as a map of $\mathscr{U}(V)$ into the algebra of real valued functions on V by setting $(Xf)(\xi) = X_\xi f$. A vector field X is a said to be differentiable if $X\mathscr{U}(V) \subset \mathscr{U}(V)$. Hence the set of differentiable vector 39
fields on V is the module $\mathscr{C}(V)$ of derivations of $\mathscr{U}(V)$.

If p is a differentiable map from a differentiable manifold V into another manifold V', we define a map $p^* : \mathscr{U}(V') \to \mathscr{U}(V)$ by setting $p^* f = fop$. Furthermore, if $\xi \in V$, then a linear map of T_ξ into $T_{p\xi}$ (which is again denoted by p) is defined by $(pU)g = U(p^* g)$.

Now let G be a Lie group acting differentiably to the right on a manifold V. As usual, the action is denoted $(\xi, s) \to \xi s$. For every $(\xi, s) \in V \times G$, there are two inclusion maps

$$\begin{aligned} &V \to V \times G \text{ defined by } \eta \to (\eta, s); \text{ and} \\ &G \to V \times G \text{ defined by } t \to (\xi, t) \end{aligned}$$

These induce injective maps $T_\xi \to T_{(\xi,s)}, T_s \to T_{(\xi,s)}$ respectively. The image of $d\xi \in T_\xi$ in $T_{(\xi,s)}$ is denoted by $(d\xi, s)$. The image of $ds \in T_s$ in $T_{(\xi,s)}$ is denoted by (ξ, ds). We set $(d\xi, ds) = (d\xi, s) + (\xi, ds)$. The image of $d\xi \in T_\xi$ by the map $\eta \to \eta s$ of V into Vwill be denoted by $d\xi s$. Similarly, the image of vector $ds \in T_s$ by the map $t \to \xi t$ will be denoted ξds. Therefore the image of the vector $(d\xi, ds) \in T_{(\xi,s)}$ by the map $(\xi, s) \to \xi s$ is $d\xi s + \xi ds$. In particular, the group G acts on itself and such expressions as dst and tds will be used in the above sense. The following formulae are easy to verify:

1) $(d\xi s)t = (d\xi)(st)$ 40

2) $(\xi ds)t = \xi(dst)$

3) $(\xi t)ds = \xi(tds)$ for $\xi \in V$ and $s, t \in G$.

Let G be a Lie group with unit element e. The space T_e of vectors at e will be denoted by $\mathscr{Y}$. A vector field X on G is said to be left invariant if $sX_t = X_{st}$ for every $s, t \in G$. Every left invariant vector field is differentiable and is completely determined by its value at e. Given $a \in \mathscr{Y}$, we define a left invariant vector field I_a by setting $(I_a)_s = sa$. This gives a natural isomorphism of $\mathscr{Y}$ onto the vector space of left invariant vector fields on V. There is also a similar isomorphisms of $\mathscr{Y}$ onto the space of right invariant vector fields. In the same way, when G acts on a manifold V to the right, for every $a \in \mathscr{Y}$, we define a vector field Z_a on V, by setting $(Z_a)_\xi = \xi$ a for $\xi \in V$. Thus $\mathscr{Y}$ defines a vector space of vector fields on V.

2.8 Vector fields on differentiable principal bundles

Let P be a differentiable principal over a manifold V with group G. Then the projection $p : P \to V$ gives rise to a homomorphism $p^* : \mathscr{U}(V) \to \mathscr{U}(P)$. Since p is onto , p^* is injective. This defines on every $\mathscr{U}(P)$- module a structure of an $\mathscr{U}(V)$module. It is clear that any element $h \in p^*\mathscr{U}(V)$ is invariant with respect to the action of G on P. Conversely, if $h \in \mathscr{U}(P)$, whenever h is invariant with respect to G then $h \in p^*\mathscr{U}(V)$. In fact if $f \in \mathscr{U}(V)$ is defined by setting $f(x) = h(z)$ where
41 z is any element in $p^{-1}(x)$, then f coincides locally with the composite of a differentiable cross- section and h, and is consequently differentiable. For every $\xi \in P$, there exists a natural linear map of $\mathscr{Y}$ into T_ξ taking $a \in \mathscr{Y}$ onto $\xi a \in T_\xi$. It is easy to verify that the sequence of linear maps

$$(0) \to \mathscr{Y} \to T_\xi \to T_{p(\xi)} \to (0)$$

is exact. The image of $\mathscr{Y}$ in T_ξ, i.e. the space of vectors ξ a with $a \in \mathscr{Y}$ is the space of vectors *tangential* to the fibre ξG and will be denoted by $\mathfrak{N}_\xi$.

Definition 6. A vector field X on P is said to be *tangential to the fibres* if for every $x \in P$, $p(X_\xi) = 0$.

A vector field X is tangential to the fibres if and only if $X_\xi \in \mathfrak{N}_\xi$ for every $\xi \in P$. It is immediate than an equivalent condition for a vector field X to be tangential to the fibres is that $X(p^* \mathscr{U}(V)) = (0)$. We denote the set of vector fields tangential to the fibres by $\mathfrak{N}$. Then $\mathfrak{N}$ is an $\mathscr{U}(P)$-submodule of $\mathscr{C}(P)$. We have the following

Proposition 4. *If $a_1, \ldots, a_r$ is a base for $\mathscr{Y}$, then $Z_{a_l}, \ldots Z_{a_r}$ is a basis for $\mathfrak{N}$ over $\mathscr{U}(P)$.*

In fact, for every $a \in \mathscr{Y}$, $(Z_a)_\xi = \xi a \in \mathfrak{N}_\xi$, i.e., $Z_a \in \mathfrak{N}$. Moreover, if $f_1, \ldots, f_r \in \mathscr{U}(P)$ are such that $\sum\limits_{i=1}^{r} f_i Z_{a_i} = 0$, then $\sum\limits_{i=1}^{r} f_i(\xi)(\xi a_i) = 0$ for every $\xi \in P$. Since $\{\xi a_i\}_{i=1}^{r}$ form a bias for $\mathfrak{N}_\xi$, $f_i(\xi) = 0$ for $i = 1, 2, \ldots, r$ and $Z_{a_i}, \ldots Z_{a_r}$ are linearly independent. On the other hand, 42
if $X \in \mathfrak{N}$, at each point $\xi \in P$, we can write $X_\xi = \sum\limits_{i=1}^{r} f_i(\xi)(\xi)\xi a_i$, and hence $X = \sum\limits_{i=1}^{r} f_i Z_{a_i}$ where the f_i are scalar functions on P. Let $\xi \in P$ and $g_1, \ldots, g_r \in \mathscr{U}(P)$ such that $(Z_{a_i})_\xi g_j = \delta_{ij}$ for every $i.j$; then the functions f_i are solutions of the system of linear equations:

$$Xg_j = \sum_{i=1}^{r} f_1(Z_{a_i} g_j).$$

Since the coefficient are differentiable and the determinant $|Z_{a_i} g_j| \neq 0$ in a neighbourhood of ξ, f_i are differentiable at ξ.

2.9 Projections vector fields

Let P be a principal bundle over V and X a vector field on P. It is in general not possible to define the image vector field pX on V. This is however possible if we assume that for all ξ in the same fibre , the image pX_ξ in the same. This yields the following.

Definition 7. A vector field X on P is said to be *projectable* if $p(X_\xi) = p(X_{\xi_s})$ for every $\xi \in P, s \in G$. 43

Proposition 5. *A vector field X on P is projectable if and only if $Xp^*\mathscr{U}(V) \subset p^*\mathscr{U}(V)$.*

This follows from the fact that

$$X_\xi(p^* f) = p(X_\xi) f = p(X_{\xi s}) f = X_{\xi s}(p^* f)$$

for every $f \in \mathscr{U}(V)$.

Proposition 6. *A vector field X is projectable if and only if $Xs - X$ is tangential to the fibre, ie. $Xs - X \in \mathfrak{N}$ for every $s \in G$.*

In fact, if X is projectable , we have

$$\begin{aligned} p(Xs - X)_\xi &= p(X_{\xi s^{-1}} s - X_\xi) \\ &= p(X_{\xi s^{-1}}) - p(X_\xi) \\ &= 0. \end{aligned}$$

Hence $(Xs - X)_\xi \in \mathfrak{N}(\xi)$ for every $\xi \in P$.
The converse is also obvious from the above.

Definition 8. The *projection pX* of a projectable vector field X is defined by $(pX)_{p\xi} = pX_\xi$ for every $\xi \in P$.

Since $p^*\{(pX)f\} = X(p^* f)$ for every $f \in \mathscr{U}(V)$ we see that pX is a differentiable vector field on V. We shall denote the space of projectable vector field by $\wp$. It is easy to see that if $X, Y \in \wp$, then $X + Y \in \wp$ and $p(X + Y) = pX + pY$. Moreover $\wp$ is a submodule of $\mathscr{C}(P)$ regarded as an $\mathscr{U}(V)$-module (but *not* an $\mathscr{U}(P)$-submodule). For $f \in \mathscr{U}(V)$ and $X \in \wp$, we have $p((p^* f)X) = f(pX)$. Thus $p : \wp \to \mathscr{C}(V)$ is an $\mathscr{U}(V)$-homomorphism and the kernel is just the module $\mathfrak{N}$ of vector fields on P tangential to the fibre. Furthermore, for every $X, Y \in \wp$, we have $[X, Y] \in \wp$ and $p[X, Y] = [pX, pY]$.

Proposition 7. *If V is paracompact, every vector field on V is the image*
44 *of a projectable vector field on P. i.e., $\wp \to \mathscr{C}(V)$ is surjective.*

Let $x \in V$ and U a neighbourhood of x over which P is trivial. It is clear that any vector field $\underline{X}$ in U is the projection of a vector field X in

$p^{-1}(U)$. *Using the fact that V is paracompact we obtain that there exists a locally finite cover* $(U_i)_{i\in I}$ *of V and a family of projectable vector fields* $X_i \in \wp$ *such that* $pX_i = \underline{X}_i$ *on* U_i *where* X_i *coincides on* U_i *with a given vector field X on V. Let* $(\varphi_i)_{i\in I}$ *be a differentiable partition of unity for V with respect to the above cover. Then* $X = \sum_{i\in I}(p^*\varphi_i)X_i$ *is well-defined and is in* $\wp$ *with projection* $pX = \sum_{i\in I}\varphi_i pX_i = \underline{X}$.

Example. Let G be a Lie group acting differentiably to the right on a differentiable manifold V. The action of G on V is given by the map $p : V \times G \to V$. Consider the manifold $V \times G$ with the above projection onto V. G acts to the right on $V \times G$ by the rule

$$(x, s)t = (xt, t^{-1}s) \text{ for every } x \in V, s, t \in G.$$

The map $\gamma : V \times G \to V \times G$ defined by $\gamma(x, s) = (xs, s^{-1})$ for $x \in V, s \in G$ is a diffeormorphism. We also have

$$\gamma(x, st) = \gamma(x, s)t$$

This show that $V \times G$ is a trivial principal bundle over V with group G and projection p. A global cross -section is given by $\sigma : x \to (x, e)$.

Let $\mathscr{Y} = T_e$ be the space of vectors at e and I_a the left invariant vector field on G whose value at e is a. Let $(0, I_a)$ be the vector field on $V \times G$ whose value at (x, s) is (x, sa). Then for every $a \in \mathscr{Y}, (0, I_a)$ is projectable; as a matter of fact , it is even right invariant. For, 45

$$\begin{aligned}(O, I_a)_{(x,s)}t &= (x, sa)t\\ &= (x, t^{-1}sa)\\ &= (0, I_a)_{(x,t)t^{-1}s}\\ &= (0, I_a)_{(x,s)t}\end{aligned}$$

We moreover see that $p(0, I_a) = Z_a$ since we have $p(x, sa) = xsa = (Z_a)_{xs}$.

We define a bracket operation $[a, b]$ in $\mathscr{Y}$ by setting $[I_a, I_b] = I_{[a,b]}$. Then we have in the above situation

$$[(0, I_a), (0, I_b)] = (0, I_{[a,b]}).$$

Then it follows that

$$\begin{aligned}[Z_a, Z_b] &= [p(0, I_a), p(0, I_b)] \\ &= p[(0, I_a), (o, I_b)] \\ &= Z_{[a,b]}.\end{aligned}$$

Chapter 3

Connections on Principal Bundles

3.1

A connection in a principal bundle P is, geometrically speaking, an assignment to each point ξ of P of a tangent subspace at ξ which is supplementary to the space $\mathfrak{N}_\xi$. This distribution should be differentiable and invariant under the action of G. More precisely, 46

Definition 1. A *connection* Γ on the principal bundle P is a differentiable tensor field of type $(1,1)$ such that

1) $\Gamma(X) \subset \mathfrak{N}$ for every $X \in \mathscr{C}(P)$

2) $\Gamma(X) = X$ for every $X \in \mathfrak{N}$

3) $\Gamma(X) = \Gamma(X)s$ for every $s \in G$.

Thus, at each point ξ, Γ is a projection of T_ξ onto $\mathfrak{N}_\xi$. Condition 3) is equivalent to $\Gamma(d\xi)s = \Gamma(d\xi s)$ for every $\xi \in P, d\xi \in T_\xi$ and $s \in G$.

Examples. 1) If G is a discrete group, the submodule $\mathfrak{N}$ of $\mathscr{U}(P)$ is (0). Thus any projection of $\mathscr{C}(P)$ onto $\mathfrak{N}$ has to be 0. Hence the only connection on a Galois covering manifold is the (0) tensor.

2) Let V be a differentiable manifold and G a Lie group. Consider the trivial principal bundle $V \times G$ over V. The tensor Γ defined by

$$\Gamma(dx, ds) = (x, ds)$$

is easily seen to be a connection.

47 3) Let P be a Lie group and G a closed subgroup. Consider the principal bundle P over P/G. Let $\wp$ be the tangent space at e to P, and $\mathscr{Y}$ that at e to G. Then $\mathscr{Y}$ can be identified with a subspace of $\wp$. For $a \in \mathscr{Y}$, the vector field Z_a is the left invariant vector field I_a on P. We shall now assume that in $\wp$, there exists a subspace $\mathfrak{M}$ such that

1) $\wp = \mathscr{Y} \oplus \mathfrak{M}$
2) for every $s \in G$, $s^{-1}\mathfrak{M}s \subset \mathfrak{M}$.

($\mathfrak{M}$ is only a subspace supplementary to $\mathscr{Y}$, which is invariant under the adjoint representation of G in $\wp$. Such a space always exists if we assume that G is compact or semisimple).

Under the above conditions, there exists a connection Γ and only one on P such that $\Gamma(\mathfrak{M}) = (0)$ and $\Gamma(tb) = t(\Gamma b)$ for every $b \in \wp$ and $t \in G$. The kernel of Γ is the submodule of $\mathscr{C}(P)$ generated by the left invariant vector field I_a on P with $a \in \mathfrak{M}$. Moreover, Γ is left invariant under the action of G. Conversely, every left invariant connection corresponds to such an invariant subspace $\mathfrak{M}$ supplementary to $\mathscr{Y}$ in $\wp$.

Theorem 1. *If V is paracompact, for every differentiable principal bundle P over V, there exists a connection on P.*

We have seen that there exists a connection on a trivial bundle. Using the paracompactness of V, we can find a locally finite open cover $(U_i)_{i\in I}$ of V such that P is trivial over each U_i and such that there exist tensor fields Γ_i of type $(1, 1)$ on P whose restrictions to $P^{-1}(U_i)$ are
48 connections. Set $\theta_{ij} = \Gamma_i - \Gamma_j$ for every $i, j \in I$. Then the θ_{ij} satisfy the following equations:

$$\theta_{ij}\ d\ \xi \in \mathfrak{N}_\xi, \theta_{ij}(\xi\ a) = 0, \text{ and } \theta_{ij}(d\xi s) = (\theta_{ij} d\xi)s$$

for every $\xi \in p^{-1}(U_i \cap U_j), d\xi \in T_\xi, a \in \mathscr{Y}, s \in G$.

Now, let $(\varphi_i)_{i \in I}$ be a partition of unity with respect to the above covering, i.e., support of $\varphi_i \subset U_i$ and $\sum \varphi_i = 1$. Denote $p^* \varphi_i$ by $\tilde{\varphi}_i$. Then $\tilde{\varphi}_i$ is a partition of unity with respect to the covering $p^{-1}(U_i) . \tilde{\varphi}_k \theta_{ik}$ is a tensor on P for every k and hence

$$\zeta_i = \sum_k \tilde{\varphi}_k \theta_{ik}$$

is a tensor having the following properties:

$$\zeta_i d\xi \in \mathfrak{N}_\xi, \zeta_i \xi\, a = 0 \text{ and } \zeta_i(d\xi s) = (\zeta_i d\xi)s.$$

The tensor $\Gamma_i - \zeta_i$ is easily seen to be a connection on $p^{-1}(U_i)$. Since $\tilde{\varphi}_k$ is a partition of unity, it follows that $\Gamma_i - \zeta_i, \Gamma_j - \zeta_j$ coincide on $p^{-1}(U_i \cap U_j)$ for every $i, j \in I$. Therefore the tensor field Γ on P defined by $\Gamma = \Gamma_i - \zeta_i$ on $p^{-1}(U_i)$ is a connection on P.

3.2 Horizontal vector fields

Let Γ be a connection on P. By definition, Γ is a map $\mathscr{C}(P) \to \mathfrak{N}$. The kernel $\mathfrak{I}$ of this map is called the module of *horizontal vector fields*. It is clear that Γ maps $\wp$ into itself. Hence we have $\wp = \mathfrak{N} \oplus (\wp \cap \mathfrak{I})$. It is easily seen that the vector fields belonging to the $\mathscr{U}(V)$-module $\wp \cap \zeta$
are invariant under the action of G. We have defined the projection p of 49
$\wp$ onto $\mathscr{C}(V)$. The kernel of this projection is $\mathfrak{N}$ since the restriction of p to $\wp \cap \mathfrak{I}$ is bijective.

We shall now see that the module $\mathscr{C}(F)$ of all vector fields on P is generated by projectable vector fields.

Theorem 2. *Let V be a paracompact manifold and P a principal bundle over V. Then the map $\eta : \mathscr{U}(P) \underset{\mathscr{U}(V)}{\otimes} \wp \cap \mathfrak{I} \to \mathfrak{I}$ defined by $\eta(\sum f_s \otimes X_s) = \sum f_s X_s$ is bijective.*

The proof rests on the following

Lemma 1. *$\mathscr{C}(V)$ is a module of finite type over $\mathscr{U}(V)$.*

In fact, by Whitney's imbedding theorem ([28]) there exists a regular, proper imbedding of each connected component of V in R^{2n+1} where $n = \dim V$. This gives us a map $f : V \to R^{2n+1}$ defined by $y \to (f,(y), \ldots, f_{2n+1}(y))$ which is of maximal rank. Let $(U_j)_{j\in J}$ be an open covering of V such that on every $U_j, (df_i)_{i=1,2,\ldots 2n+1}$ contains a base for the module of differential forms of degree 1 on U_j. Let $\mathfrak{S}$ be the set of maps $\alpha : [1, n] \to [1, 2n+1]$ such that $\alpha(i+1) < \alpha(i)$ for $i = 1, 2, \ldots (n-1)$. We shall denote by U_α the union of the open sets of the above covering in which $df_{\alpha_1} \ldots df_{\alpha_n}$ are linearly independent. Thus we arrive at a finite cover $(U_\alpha)_{\alpha\in\mathscr{C}}$ having the above property. Using a partition of unity for this cover, any form ω on V can be written as a linear combination of the df_i. Using the map f, we can introduce a
50 Riemannian metric on V which gives an isomorphism of $\mathscr{C}(V)$ onto the module of differential forms of degree 1. This completes the proof of the lemma.

Proof of the theorem. *Let $X_1, \ldots X_{2n+1}$ be a set of generators for the module $\mathscr{C}(V)$ and $(U_\alpha)_{\alpha\in\mathfrak{S}}$ a finite covering such that for every $\alpha, X_{\alpha(1)}, \ldots X_{\alpha(n)}$ form a base for the module of vector fields on U_α. We shall now prove that the map η is injective.*

Let $(\varphi_\alpha)_{\alpha\in\mathfrak{S}}$ be a partition of unity for this covering and X_s be vector fields $\in \wp \cap \zeta$ such that $pX_s = X_s$. Then $\varphi_\alpha X_s = \sum\limits_{i=1}^{n} g^i_{\alpha,s} X_{\alpha(i)}$, with $g^i_{\alpha,s} \in \mathscr{U}(V)$ and hence $\varphi_\alpha X_s = \sum\limits_{i=1}^{n} g^i_{\alpha,s} X_{\alpha(i)}$ using the structure of $\mathscr{U}(V)$-module on $\mathscr{C}(P)$. If $u = \sum\limits_{s=1}^{2n+1} f_s \otimes X_s$ with $f_s \in \mathscr{U}(P)$ is such that $\eta(u) = 0$, then $\eta(\varphi_\alpha\, u) = 0$ for every α. Therefore $\sum\limits_{s} f_\alpha\, g^i_{\alpha,s} = 0$ on U_α and consequently on V. But $\varphi_\alpha\, u = \sum\limits_{s,i} f_s\, g^i_{\alpha,\beta} \otimes X_{\alpha(i)}$; it follows that $u = \sum \varphi_\alpha\, u = 0$. The proof that η is surjective is similar.

Corollary. *If V is paracompact, the module of horizontal vector fields on P is generated over $\mathscr{U}(P)$ by the projectable and horizontal vector fields.*

3.3 Connection form

Le P be a principal bundle over V and Γ a connection on P. If $d\xi \in T_\xi$, then $\Gamma\, d\xi \in \mathfrak{N}_\xi$ and $a \to \xi a$ is an isomorphism of $\mathscr{Y}$ onto $\mathfrak{N}_\xi$. We define a differential form γ on P with values in $\mathscr{Y}$ by setting $\gamma(d\xi) = a$ where **51**
$\Gamma(d\xi) = \xi\, a$. In order to prove that γ is differentiable, it is enough to prove that γ takes differentiable vector fields into differentiable functions with values in $\mathscr{Y}$. We have $\gamma(Z_a) = a$ for every $a \in \mathscr{Y}$ and $\gamma(X) = 0$ for every $X \in \mathfrak{I}$. Since the module $\mathscr{C}(P)$ is generated by $\mathfrak{I}$ and the vector fields $Z_a, \gamma(X)$ is differentiable for every differentiable vector field X on P. Thus corresponding to every connection Γ on P, there exists one and only one form with values in $\mathscr{Y}$ such that $\Gamma(d\xi) = \xi\gamma(d\xi)$ for every $d\,\xi \in T_\xi$. It is easily seen that γ satisfies

1) $\gamma(\xi a) = a$ for every $\xi \in P$ and $a \in \mathscr{Y}$

2) $\gamma(d\xi s) = s^{-1}\gamma(d\xi)s$ for $d\xi \in T_\xi$ and $s \in G$.

A $\mathscr{Y}$- valued form on P satisfying 1) and 2) is called a *connection form*. Given a connection form γ on P, it is easy to see that there exists one and only one connection Γ for which γ is the associated form, i.e., $\Gamma(d\xi) = \xi\gamma(d\xi)$ for every $d\xi \in T_\xi$.

3.4 Connection on Induced bundles

Let P, P' be two principal bundles over V, V' respectively. Let h be a homomorphism of P' into P. If γ is a connection form on P, the form $h^*\gamma$ on P' obviously satisfies conditions (1) and (2) and is therefore a connection form on P'.

In particular, if P' is the bundle induced by a map $q : V' \to V$, then γ induces a connection on P'.

Let P be a differentiable principal bundle over V, and q a map **52**
$Y \to V$ which trivialises P, ρ being a lifting of q to P with m as the multiplicator. As in Chapter 2.4, we denote by Y_q the subset of $Y \times Y$ consisting of points (y, y') such that $q(y) = q(y')$. Now $\omega = \rho^*\gamma$ is a differential form on Y. We have $\rho(y') = \rho(y)\, m(y, y')$ for every $(y, y') \in Y_q$.

Differentiating we obtain for every vector (dy, dy') at $(y, y') \in Y_q$

$$\rho(dy') = \rho(dy)m(y, y') + \rho(y)m(dy, dy')$$

Since $\omega(dy') = \gamma(\rho dy')$ we have

$$\begin{aligned}\gamma(\rho dy') &= m(y, y')^{-1}\omega(dy)m(y, y') + \gamma(\rho(y)m(y, y')m(y, y')^{-1}m(dy, dy'))\\ &= m(y, y')^{-1}\omega(dy)m(y, y') + m(y, y')^{-1}m(dy, dy')\end{aligned}$$

Hence $m(dy, dy') = m(y, y')\omega(dy') - \omega(dy)m(y, y')$.

Conversely if ω is a differential form on Y satisfying

$$m(dy, dy') = m(y, y')\omega(dy') - \omega(dy)m(y, y')$$

for every $(y, y') \in Y_q$, then there exists one and only one connection form γ on P such that $\omega = \rho^*\gamma$.

In particular, when the trivialisation of P is in terms of a covering $(U_i)_{i\in I}$ of V with differentiable sections $\sigma_i(Ch.2.5)$, the connection form γ on P gives rise to a family of differential forms $\omega_i = \sigma_i^*\gamma$ on U_i. From
53 the equations $\sigma_i(x) = \sigma_j(x)m_{ji}(x)$ defining the transition functions m_{ji}, we obtain on differentiation,

$$\omega_i(dx) = m_{ji}(x)^{-1}\omega_j(dx)m_{ji}(x) + m_{ji}(x)^{-1}m_{ji}(dx) \text{ for } x \in U_i,$$

i.e.,
$$m_{ji}(dx) = m_{ji}(x)\omega_i(dx) - \omega_j(dx)m_{ji}(x).$$

Conversely given a family of differentiable forms ω_i on the open sets of a covering $(U_i)_{i\in I}$ of V satisfying the above, there exists one and only one connection form γ on P such that $\omega_i = \sigma_i^*\gamma$ for every i.

3.5 Maurer-Cartan equations

To every differentiable map f of a differentiable manifold V into a Lie group G, we can make correspond a differential form of degree 1 on V with values in the Lie algebra $\mathscr{Y}$ of G defined by $\alpha(\xi)(d\xi) = f(\xi)^{-1}f(d\xi)$ for $\xi \in V$ and $d\xi \in T_\xi$. We shall denote this form by $f^{-1}df$. This is easily seen to be differentiable.

If we take $f : G \to G$ to be the identity map, then we obtain a canonical differential form ω on G with values in $\mathscr{Y}$. Thus we have $\omega(ds) = s^{-1}ds \in \mathscr{Y}$. Also $\omega(tds) = \omega(ds)$ for every vector ds of G. That is, ω is a left invariant differential form. Moreover, it is easy to see that any scalar left invariant differential form on G is obtained by composing ω with elements of the algebraic dual of $\mathscr{Y}$. In what follows, we shall provide $A \otimes \mathscr{Y}$ with the canonical derivation law (Ch.1.2).

Proposition 1 (Maurer-Cartan)**.** *The canonical form ω on G satisfies $d\omega + [\omega, \omega] = 0$.* **54**

We recall that the form $[\omega, \omega]$ has been defined by $[\omega, \omega](X, Y) = [\omega(X), \omega(Y)]$ for every vector fields $X, Y \in \mathscr{C}(G)$. Since the module of vector fields over G is generated by left invariant vector fields, it suffices to prove the formula for left invariant vector fields $X = I_a, Y = I_b$ $a, b \in \mathscr{Y}$. We have

$$d(\omega)(I_a, I_b) = I_a\omega(I_b) - I_b\omega(I_a) - \omega([I_a, I_b])$$

But $[I_a, I_b] = I_{[a,b]}$ and $\omega[I_a, I_b] = \omega(I_{[a,b]}) = [a, b] = [\omega(I_a), \omega(I_b)]$ since $\omega(I_a) = a, \omega(I_b) = b$.

Corollary. *If f is a differentiable map $V \to G$, then the form $\alpha = f^{-1}df$ satisfies $d\alpha + [\alpha, \alpha] = 0$.*

In fact, $\alpha(d\xi) = f^{-1}(\xi)f(d\xi) = \omega(f(d\xi)) = (f^*\omega)(d\xi)$ and hence $\alpha = f^*\omega$. The above property of α is then an immediate consequence of that of ω.

Conversely, we have the following

Theorem 3. *If α is a differential form of degree 1 on a manifold V with values in the Lie algebra $\mathscr{Y}$ of a Lie group G satisfying $d\alpha + [\alpha, \alpha] = 0$, then for every $\xi_o \in V$ and a differentiable map $f : U \to G$ such that $f^{-1}(\xi)f(d\xi) = \alpha(d\xi)$ for every vector $d\xi$ of U.*

Consider the form $\beta = p_1^*\alpha - p_2^*\omega$ on $V \times G$ where $p_1 : V \times G \to V$ **55**
and $p_2 : V \times G \to G$ arc the two projections and ω the canonical left invariant form on G. If β is expressed in terms of a basis $\{a_1, \ldots a_r\}$ of

$\mathscr{Y}$, the component scalar differential forms β_i are everywhere linearly independent. This follows from the fact that on each $p_1^{-1}(\xi), \beta$ reduces to $p_2^* \omega$. We shall now define a differentiable and involutive distribution on the manifold $V \times G$. Consider the module $\mathfrak{M}$ of vector fields X on $V \times G$ such that $\beta_i(X) = 0$ for every β_i. We have now to show that $X, Y \in \mathfrak{M} \Rightarrow [X, Y] \in \mathfrak{M}$. But this is an immediate consequence of the relations $d\,\alpha = -[\alpha, \alpha]$ and $d\omega = [\omega, \omega]$.

By Frobenius' theorem (see [11]) there exists an integral submanifold W (of dim = dim V) of $V \times G$ in a neighbourhood of (ξ_o, e). Since for every vector $(\xi_o, a) \neq 0$ tangent to $p_1^{-1} \xi_o, \beta(\xi_o, a) = a \neq 0$ there exist a neighbourhood U of ξ_o and a differentiable section σ into $V \times G$ over U such that $\sigma(U) \subset W$. Define $f(\xi) = p_2 \sigma(\xi)$. By definition of W, it follows that $\beta\sigma(d\xi) = 0$. This means that

$$\alpha p_1 \sigma(d\xi) - \omega p_2 \sigma(d\xi) = 0$$

i.e., $$\alpha(d\xi) = \omega(p_2 \sigma(d\xi)) = (f^* \omega) d\xi = f^{-1}(\xi) f(d\xi).$$

Remark. When we take G = additive group of real numbers, the above theorem reduces to the Poincare's theorem for 1-forms. Concerning the uniqueness of such maps f, we have the

Proposition 2. *If f_1, f_2 are two differentiable maps of a connected man-*
56 *ifold V into a Lie group G such that $f_1^{-1} d f_1 = f_2^{-1} d f_2$, then there exists*
an element $s \in G$ such that $f_1 = f_2 s$.

Define a differentiable function $s : V \to G$ by setting

$$s(\xi) = f_1(\xi) f_2^{-1}(\xi) \text{ for every } \xi \in V.$$

Differentiating this, we get

$$s(d\xi) f_2(\xi) + s(\xi) f_2(d\xi) = f_1(d\xi).$$

Hence $s(d\xi) = 0$. Therefore s is locally a constant and since V is connected s is everywhere a constant.

Regarding the existence of a map f in the large, we have the following

Theorem 4. *If α is a differential form of degree 1 on a simply connected manifold V with values in the Lie algebra $\mathscr{G}$ of a Lie group G, such that $d\alpha + [\alpha, \alpha] = 0$, then there exists a differentiable map f of V into G such that $f^{-1}df = \alpha$.*

The proof rests on the following

Lemma 2. *Let m_{ij} be a set of transition functions with group G of a simply connected manifold V with respect to a covering (U_i). If each m_{ij} is locally a constant, then there exists locally constant maps $\lambda_i : U_i \to G$ such that $m_{ij}(x) = \lambda_i(x)^{-1}\lambda_j(x)$ for every $x \in U_i \cap U_j$.*

In fact, there exists a principal bundle P over V with group G considered as a discrete group and with transition functions m_{ij} (Ch.2.5). P is then a covering manifold over V and hence trivial. Therefore the 57
maps $\lambda_i : U_i \to G$ exist satisfying the conditions of the lemma (Ch.2.5).

Proof of the theorem. *Let $(U_i)_{i\in I}$ be a covering of V such that on each U_i, there exists a differentiable function f_i satisfying $f_i(x)^{-1} f_i(dx) = \alpha(dx)$ for every $x \in U_i$. We set $m_{ij}(x) = f_i(x) f_j(x)^{-1}$. It is obvious that the m_{ij} form a set of locally constant transition functions. Let λ_i be the maps $U_i \to G$ of the lemma. Then $\lambda_i(x) f_i(x) = \lambda_j(x) f_j(x)$ for every $x \in U_i \cap U_j$. The map $f : V \to G$ which coincides with $\lambda_i f_i$ on each U_i is such that $f^{-1}df = \alpha$.*

3.6 Curvature forms

Definition 2. Let γ be a connection form on a principal bundle P over V. Then the alternate form of degree 2 with values in the Lie algebra $\mathscr{G}$ of the structure group G defined by $K = d\gamma + [\gamma, \gamma]$ is said to be the *curvature form* of γ. A connection on P is said to be *flat* if the form $K \equiv 0$.

Theorem 5. *The following statements are equivalent:*

a) The connection γ is flat i.e., $d\gamma + [\gamma, \gamma] = 0$.

b) If X, Y are two horizontal vector fields on p, so is $[X, Y]$.

c) *For every $x_o \in V$, there exists an open neighbourhood U of x_o and a differentiable section σ on U such that $\sigma^*\gamma = 0$.*

Proof. a) $\Rightarrow b$) is obvious from the definition.

b) $\Rightarrow c$)

□

58 Let $a_1, a_2, \ldots a_n$ be a basis of $\mathscr{Y}$ and let $\gamma = \sum \gamma_i a_i$ where γ_i are scalar differential forms. Since $\gamma_i(Za_j) = \delta_{ij}$, it follows that $\{\gamma_1, \ldots \gamma_r\}$ form a set of differential forms of maximal rank r. From b), it follows that the module $\mathfrak{J}$ of horizontal vector fields is stable under the bracket operation. Hence $\mathfrak{J}$ forms a distribution on P which is differentiable and involutive. Let $\xi_o \in P$ such that $p(\xi_o) = X_o$ and let W be an integral manifold W of dim $=$ dim V in a neighbourhood of ξ_o. As in Th.3, Ch.3.5, W is locally the image of a differentiable section σ over an open neighbourhood of x_o. Since all the tangent vectors of W are horizontal, we have $\gamma\sigma(dx) = 0$ for every vector dx on U.
c) $\Rightarrow a$)

Let σ be a differentiable section over an open subset U of V such that $\sigma^*\gamma = 0$. For every $\xi \in p^{-1}(U)$, define a differentiable function $\rho : p^{-1}(U) \to G$ by the condition $\xi = \sigma(p(\xi))\rho(\xi)$. Differentiating this, we obtain

$$d\xi = \sigma p(d\xi)\rho(\xi) + \sigma p(\xi)\rho(d\xi)$$

Hence $\gamma(d\xi) = \rho^{-1}(\xi)\gamma(\sigma p(d\xi))\rho(\xi) + \rho(\xi)^{-1}\rho(d\xi)$ by conditions (1) and (2) for connection forms. i.e. $\gamma(d\xi) = \rho(\xi)^{-1}\rho(d\xi)$ and a) follows from cor. to prop.1, Ch.3.5.

59 **Theorem 6.** *If there exists a flat connection on a principal bundle P over a simply connected manifold V, then P is a trivial bundle. Moreover, a differentiable cross-section σ can be found over V such that $\sigma^*\gamma = 0$.*

By $Th.4, Ch.3.5$, there exists an open covering $(U_i)_{i\in I}$ and cross sections $\sigma_i : U_i \to P$ such that $\sigma_i^*\gamma = 0$. Let $\sigma_i(x)m_{ij}(x) = \sigma_j(x)$ where the corresponding transition functions are m_{ij}. Then we have

$$\gamma(\sigma_i(dx)m_{ij}(x) + \sigma_i(x)m_{ij}(dx)) = \gamma\sigma_j(dx).$$

Therefore, $m_{ij}(dx) = 0$, i.e. the m_{ij} are locally constant. Lemma 2, Ch.3.5 then gives a family λ_i of locally constant maps $U_i \to G$ such that

$$m_{ij} = \lambda_i^{-1}\lambda_j$$

It is easily seen that $\sigma_i\lambda_i^{-1} = \sigma_j\lambda_j^{-1}$ on $U_i \cap U_j$. Define a cross section σ on V by setting $\sigma = \sigma_i\lambda_i^{-1}$ on every U_i. Then we have

$$\begin{aligned}\gamma(\sigma(dx)) &= \gamma(\sigma_i(dx)\lambda_i^{-1}(x))\\ &= \lambda_i(x)(\gamma\sigma_i(dx))\lambda_i^{-1}(x)\\ &= 0 \text{ for } x \in U_i.\end{aligned}$$

Hence the bundle is trivial and $\sigma^*\gamma = 0$.

Proposition 3. *If X is a vector field on P tangential to the fibre, then* 60
$K(X, Y) = 0$ for every vector field Y on P.

In fact, since $\wp = \wp \cap \mathfrak{I} \oplus \mathfrak{R}$ and $\wp$ generates $\mathscr{C}(P)$, it is enough to prove the assertion for $Y \in \mathfrak{R}$ and $Y \in \wp \cap \zeta$. In the first case $X = Z_a$ and $Y = Z_b$, we have

$$\begin{aligned}K(Z_a, Z_b) &= (d\gamma + [\gamma, \gamma])(Z_a, Z_b)\\ &= Z_a\gamma(Z_b) - Z_b\gamma(Z_a) - \gamma[Z_a, Z_b] + [\gamma(Z_a), \gamma(Z_b)]\\ &= -\gamma(Z_{[a,b]}) + [a, b]\\ &= 0\end{aligned}$$

In the second case, Y is invariant under G and it is easy to see that $[Z_a, Y] = 0$. Since $\gamma(Y) = 0$, we have

$$\begin{aligned}K(Z_a, Y) &= (d\gamma + [\gamma, \gamma])(Z_a, Y)\\ &= Z_a\gamma(Y) - Y\gamma(Z_a) - \gamma[Z_a, Y] + [\gamma(Z_a), \gamma(Y)]\\ &= 0.\end{aligned}$$

Proposition 4.

$$K(d_1\xi s, d_2\xi s) = s^{-1}K(d_1\xi, d_2\xi)s \ \ \textit{for} \ \ \xi \in P, d_1\xi, d_\alpha\xi \in T_\xi \ \ \textit{and} \ \ s \in G.$$

By Prop.3, Ch.3.6, it is enough to consider the case when $d_1\xi, d_2\xi \in \mathfrak{I}_\xi$. Extend $d_1\xi, d_2\xi$ to horizontal vector fields X_1, X_2 respectively. Then we have

$$\begin{aligned} K(d_1\xi s, d_2\xi s) &= K(X_1 s, X_2 s)(\xi) \\ &= -\gamma[X_1 s, X_2 s](\xi) \\ &= -\gamma([X_1, X_2]s)(\xi) \\ &= -\gamma([X_1, X_2]_\xi s) \\ &= -s^{-1}\gamma([X_1, X_2]_\xi)s \\ &= s^{-1}K(X_1, X_2)(\xi)s \\ &= s^{-1}(d_1\xi, d_2\xi)s \text{ for every } s \in G. \end{aligned}$$

61

3.7 Examples

1. Let $\dim V = 1$. Then $\mathscr{C}(V)$ is a face module generated by a single vector field. But $\mathscr{C}(V)$ is $\mathscr{U}(V)$-isomorphic to $\mathfrak{I} \cap \wp$ and hence $\mathfrak{I} = \mathscr{U}(P) \bigotimes_{\mathscr{U}(V)} \mathfrak{I} \cap \wp$ is also $\mathscr{U}(P)$ - free, and of rank 1. Since K is alternate, $K \equiv 0$. In other words, if the base manifold of P is a curve, any connection is flat.

2. Let G be a closed subgroup of a Lie group H and let $\mathscr{Y}, \mathscr{F}$ be their respective LIe algebras. We have seen that (Example 3, Ch.3.1) if $\mathfrak{M}$ is a subspace of $\mathscr{F}$ such that $\mathscr{F} = \mathfrak{M} \oplus \mathscr{Y}$ and $s^{-1}\mathfrak{M}s \subset \mathfrak{M}$ for every $s \in G$, then the projection $\Gamma : \mathscr{F} \to \mathfrak{M}$ gives rise to a connection which is left invariant by element of G. Denoting by I_a the left invariant vector fields on H whose values at e is $a \in \mathscr{F}$, we obtain

$$\begin{aligned} K(I_a, I_b) &= -\gamma([I_a, I_b]) + [\gamma(I_a), \gamma(I_b)] \\ &= -\gamma I_{[a,b]} + [\Gamma_e a, \Gamma_e b] \\ &= -\Gamma_e([a, b]) + [\Gamma_e(a), \Gamma_e(b)]. \end{aligned}$$

62 Thus K is also left invariant for elements of G. Moreover Γ is flat if and only if Γ_e is a homomorphism of $\mathscr{J}$ onto $\mathscr{U}$. This again is true if and only if $\mathfrak{M}$ is an ideal in $\mathscr{J}$.

Chapter 4

Holonomy Groups

4.1 Integral paths

We shall consider in this chapter only connected base manifolds. **63**

Definition 1. A *path* ψ in a manifold W is a continuous map ψ of the unit interval $I = [0, 1]$ into W. A path ψ is said to be *differentiable* if ψ can be extended to a differentiable map of an open neighbourhood of I into W. $\psi(0)$ is called the origin and $\psi(1)$ the extremity, of the path. The path $t \to \psi(1 - t)$ is denoted by ψ^{-1}. If γ is a connection on a principal bundle P over X, then a differentiable path ψ in P is said to be *integral* if $\psi^* \gamma = 0$. If ψ is integral, so is ψ^{-1}.

Let ψ be a path in X. A path $\hat{\psi}$ in P such that $po\ \hat{\psi} = \psi$ is called a lift of ψ. It is easy to see that if $\hat{\psi}$ is an integral lift of ψ, so is $\hat{\psi}s$, where $\hat{\psi}s$ is defined by $\hat{\psi}s(t) = \hat{\psi}(t)s$. If $\hat{\psi}$ is an integral lift of ψ, then $\hat{\psi}^{-1}$ is an integral lift of ψ^{-1}.

Theorem 1. *If ψ is a differentiable path in V with origin at $x \in V$, then for every $\xi \in P$ such that $p(\xi) = x$, there exists one and only one integral lift of ψ with origin ξ.*

Let I' be an open interval containing I to which ψ can be extended. **64**
The map $\psi : I' \to V$ defines an induced bundle P_ψ with base I' and a

canonical homomorphism h of P_ψ into P such that the diagram

$$\begin{tikzcd} P_\psi \arrow[r, "h"] \arrow[d, "q"'] & P \arrow[d, "p"] \\ I' \arrow[r, "\psi"] & V \end{tikzcd}$$

is commutative. The connection γ on P also induces a connection $h^*\gamma$ on P_ψ. Since I' is a curve, by Ex.1 of Ch.3.7 and Th.5 of Ch.3.6, there exists a section σ such that the inverse image by σ of $h^*\gamma$ is zero. i.e. $\sigma^* h^* \gamma = 0$. Define $\hat{\psi} : I' \to P$ by setting $\hat{\psi} = h\sigma$ on I'. Then it is obviously an integral lift of ψ. If $s \in G$ is such that $\xi = \hat{\psi}(0)s$, then $\hat{\psi}s$ is an integral lift of ψ with origin ξ.

If $\hat{\varphi}, \hat{\psi}$ are two such lifts, then we may define a map $s : I \to G$ such that

$$\hat{\psi}(t) = \hat{\varphi}(t)s(t) \text{ for every } t \in I.$$

Since $\hat{\varphi}, \hat{\psi}$ are differentiable, it can be proved that s is also differentiable using the local triviality of the bundle. Differentiation of the above yields $\hat{\psi}(dt) = \hat{\varphi}(dt)s(t) + \hat{\varphi}(t)s(dt)$. Hence

$$\gamma(\hat{\varphi}(dt)s(t)) + \gamma(\hat{\varphi}(t)s(dt)) = 0.$$

Using conditions (1) and (2) of connection form, we get $s(dt) = 0$.
65 Hence s is a constant. Since $\hat{\varphi}(0) = \hat{\psi}(0)$, the theorem is completely proved.

4.2 Displacement along paths

Let P_x be the fibre at x and ψ a path with origin x and extremity y. For every $\xi \in P_x$, there exists one and only one integral lift $\hat{\psi}$ of ψ whose origin is ξ. The extremity of $\hat{\psi}$ is an element of the fibre at P_y. We shall denote this by $\tau_\psi \xi$. Then τ_ψ is said to be a displacement along ψ. Any displacement along the path ψ commutes with the operations of G in the sense that $\tau_\psi(\xi s) = (\tau_\psi \xi)s$ for every $\xi \in P_x$; therefore τ_ψ is differentiable and is easily seen to be a bijective map $P_x \to P_y$. Trivially, $(\tau_\psi)^{-1} = \tau_{\psi^{-1}}$. It is also obvious that the displacement is independent of the parameter, i.e., if θ is a differentiable map of I into I such that

1) $\theta(0) = 0, \theta(1) = 1$;

2) $\psi'\theta = \psi$

then $\tau_{\psi'} = \tau_\psi$

Given a differentiable path ψ, we may define a new path ψ' by suitably changing the parameter so that ψ' has all derivatives zero at origin and extremity. By the above remark, we have $\tau_{\psi^{-1}} = \tau_\psi$.

4.3 Holonomy group

Definition 2. A chain of paths in V is a finite sequence of paths $\{\psi_p, \ldots, \psi_1\}$ where ψ_i is a path such that $\psi_{i+1}(0) = \psi(1)$ for $i \leq p - 1$.

We define the origin and extremity of the chain ψ to be respectively **66**
$\psi_1(0)$ and $\psi_p(1)$. Given two chains

$\psi = \{\psi_p, \psi_{p-1}, \ldots \psi_1\}, \varphi = \{\varphi_q, \varphi_{q-1}, \ldots \varphi_1\}$ such that origin of ψ = extremity of φ, we define $\psi\varphi = \{\psi_p, \ldots \psi_1, \varphi_q, \ldots \varphi_1\}$. φ^{-1} is defined to be $\{\varphi_1^{-1}, \ldots \varphi_q^{-1}\}$ and it is easily seen that $(\psi\varphi)^{-1} = \varphi^{-1}\psi_1^{-1}$.

A displacement τ_ψ along a chain $\psi = \{\psi_p, \ldots \psi_1\}$ in the base manifold V of a principal bundle P is defined by $\tau_\psi = \tau_{\psi p} \circ, \ldots \circ \tau_{\psi 1}$.

Let $x, y \in V$. Define $\Phi(x, y)$ to be the displacements τ_ψ where ψ is a chain with $\psi(0) = x, \psi(1) = y$. It can be proved, by a suitable change of parameters of the paths ψ_i (ch. 4.2), that there exists a differentiable path φ such that $\tau_\varphi = \tau_\psi$. The following properties are immediate consequences of the definition:

1) $\alpha \in \Phi(x, y) \Longrightarrow \alpha^{-1} \in \Phi(y, x)$.

2) $\alpha \in \Phi(x, y), \beta \in \Phi(y, z) \Longrightarrow \beta\alpha \in \Phi(x, y)$.

3) $\Phi(x, y)$ is non empty since V is connected.

We shall denote $\Phi(x,x)$ by $\Phi(x)$. $\Phi(x)$ is then a subgroup of the group of automorphisms of P_x commuting with operations of G. This is called the *holonomy group* at x with respect to the given connection on the principal bundle P. If we restrict ourselves to all chains homotopic to zero at x, then we get a subgroup $\Phi_r(x)$ of $\Phi(x)$, viz., the subgroup of displacements τ_ψ where ψ is a closed chain at x homotopic to zero. This is called the *restricted holonomy group* at X. This is actually a normal
67 subgroup of $\Phi(x)$, for if ψ is homotopic to 0 and φ any continuous path $\varphi\psi\varphi^{-1}$ is again homotopic to zero.

Proposition 1. *For every $x \in V$, there exists a natural homomorphism of the fundamental group $\prod_1(V,x)$ onto $\Phi(x)/\Phi_r(x)$.*

Let ψ be a closed path at x. Then there exists a differentiable path ψ' homotopic to ψ. We map ψ on the coset of $\Phi_r(x)$ containing $\tau_{\psi 1}$. If φ and ψ are homotopic, so are φ' and ψ' i.e. $\psi'\varphi'^{-1}$ is homotopic to zero. Then $\tau_{\psi'\varphi'-1} = \tau_{\psi'}\tau_{\varphi}^{-1} \in \Phi_r$. Thus we obtain a canonical map $\theta : \prod_1(V,x) \to \Phi(x)/\Phi_r(x)$ which is easily seen to be a homomorphism of groups. That this is surjective is immediate.

In general, θ will not be injective. As the fundamental group of manifolds which are countable at ∞ is itself countable, so is$\Phi(x)/\Phi_r(x)$. Hence from the point of view of structure theory, a study of $\Phi_r(x)$ is in most cases sufficient.

4.4 Holonomy groups at points of the bundle

If x, y are two points of the connected manifold V and φ a path joining x and y, then there exists an isomorphism $\Phi(y) \to \phi(x)$ defined by $\tau_\psi \to \tau_{\varphi-1}\tau_\psi\tau_\varphi$ for every path closed at y. The image of $\Phi_r(y)$ under this isomorphism is contained in $\Phi_r(x)$. However, this isomorphism is not canonical depending as it does on the path φ. The situation can be improved by association to each point of the bundle P a holonomy group which is a subgroup of G.

Let A_x be the group of automorphisms of P_x commuting with oper-
68 ations of G. Let $\xi \in P_x$, Then we define an isomorphism $\lambda_\xi : A_x \to G$ by requiring $\alpha\xi = \xi\lambda_\xi(\alpha)$ for every $\alpha \in A_x$. That λ_ξ is a homomorphism

and is bijective is trivial. At every $\xi \in P$, we define the *holonomy group* $\overline{\varphi}(\xi)$ to be $\lambda_\xi \Phi(p\xi)$. Similarly the *restricted holonomy group* $\Phi_r(\xi)$ at ξ is $\lambda_\xi \Phi_r(p\xi)$.

Theorem 2.

For any two points $\xi, \eta \in P, \Phi(\xi), \Phi(\eta)$ are conjugate subgroups of G. Moreover, if ξ, η lie on an integral path, then $\Phi(\xi) = \Phi(\eta)$.

If ξ, η are on the same integral path, then $\eta = \tau\varphi\xi$ for some path φ with origin $p\xi = x$ and extremity $p\eta = y$. It is sufficient to prove that $\Phi(\xi) \subset \Phi(\eta)$. Let ψ be a closed chain at x. Then

$$\begin{aligned}
\xi(\lambda_\xi \tau_\psi) &= \tau_\psi \xi \\
&= \tau_\psi \tau_\varphi^{-1} \eta \\
&= \tau_\varphi^{-1} \tau'_\psi \eta \text{ where } \psi' = \varphi\psi\varphi^{-1} \\
&= \tau_\varphi^{-1}(\eta \lambda_\eta(\tau_{\psi'})) \\
&= \xi \lambda_\eta(\tau'_\psi)
\end{aligned}$$

Hence $\lambda_\eta(\tau_{\psi'}) = \lambda_\xi(\tau_\psi) \in \Phi(\eta)$ and our assertion is proved.

Finally, if ξ, η are arbitrary, we may join $p\xi, p\eta$ by a path φ. Let $\hat{\varphi}$ be an integral lift of φ through ξ and η' its extremity. Then there exists $s \in G$ such that $\eta' = \eta s$. We have already proved that $\Phi(\xi) = \Phi(\eta s)$. **69**
Now

$$\begin{aligned}
(\eta s)(\lambda_{\eta s}(\alpha)) &= \alpha(\eta s) \\
&= (\eta \lambda_\eta(\alpha))s
\end{aligned}$$

Hence $\Phi(\xi) = s^{-1}\Phi(\eta)s$, which completes the proof of theorem 2.

4.5 Holonomy groups for induced connections

Let h be a homomorphism of a principal bundle P' over V' into a principal bundle P over V. Let h be the projection of h. Let γ be a connection form on P and $h * \gamma$ the induced connection on P'. If $\xi' \in P'$ and $h(\xi') = \xi$, then $\Phi(\xi') \subset \Phi(\xi)$. Moreover, $\Phi(\xi')$ is the set of $\lambda_\xi(\tau_{h_{\psi'}})$

where ψ' is a closed chain at $p\xi' = x'$. For, let $s \in \Phi(\xi')$ corresponding to the closed chain ψ' at x'. Then $\xi' s = \tau_{\psi'}\xi'$.

$$\begin{aligned}\xi s &= h(\tau_{\psi'}\xi')\\ &= \tau_{\psi}\xi \quad \text{where} \quad \psi = \underline{h}\psi'\\ &= \xi\lambda_{\xi}(\tau_{\psi}).\end{aligned}$$

Therefore $s = \lambda_{\xi}(\tau_{\psi})$ and it is obvious that any such $\lambda_{\xi}(\tau_{\psi})$ belongs to $\Phi(\xi')$.

In particular, let V' be the universal covering manifold of $V, \underline{h}$ the covering map, P' a principal bundle over V' and h a homomorphism
70 $P' \to P$ whose projection is $\underline{h}$. Then for $\xi \in P'$, $\Phi(\xi')$ is the set of
$\lambda_{\xi}\tau_{\psi}$ where ψ is the image under $\underline{h}$ of a closed chain at $p\xi'$. But V' being simply connected, ψ is any closed chain at ξ homotopic to zero and hence $\Phi(\xi') = \Phi_r(\xi)$.

4.6 Structure of holonomy groups

Theorem 3.

For every $\xi \in P, \Phi_r(\xi)$ is an arcwise connected subgroup of G.

If ψ is a closed path at $x \in V$ which is homotopic to zero, then it can be shown that there exists a *differentiable* map $\varphi : I' \times I' \to V(I'$ being a neighbourhood of $I)$ such that $\varphi(t, 0) = \psi(t)$ and $\varphi(t, 1) = x$ for every $t \in I$. This can be lifted into a map $\hat{\varphi} : I' \times I' \to P$ such that

1) $p\hat{\varphi}(t, \theta) = \varphi(t, \theta)$ for every $t, \theta \in I$;

2) For every $\theta \in I, t \to \hat{\varphi}(t, \theta)$ is an integral path;

3) $\hat{\varphi}(0, \theta) = \xi$ with $p(\xi) = x$.

In fact, $\varphi : I' \times I' \to V$ induces a bundle P_{φ} over $I' \times I'$. Let γ_{φ} be the induced connection on P_{φ}. For every $\theta \in I'$, the path $t \to (t, \theta)$ in $I' \times I'$ can be lifted to an integral path $\hat{\sigma}_{\theta}$ in P_{ϕ} with origin at $((0, \theta), \xi) \in P_{\varphi}$. The origin depends differentiably on θ and therefore the map σ of $I' \times I'$

defined by $\sigma(t,\theta) = \sigma_\theta(t)$ is differentiable. Hence $\rho \circ \sigma = \hat{\varphi}$ satisfies the conditions 1), 2) and 3).

Now we define a function $s : I \to G$ by requiring that $\xi s(\theta) = \hat{\varphi}(1,\theta)$. Obviously $s(1) = e$ and $s(0)$ is the element of the restricted holonomy group at ξ corresponding to the path ψ. It is also easy to verify that $s(\theta)$ is a differentiable function of θ using the local triviality of P. Therefore any point of $\Phi_r(\xi)$ can be connected to e by a differentiable arc. **71**

Corollary. *$\Phi_r(\xi)$ is a Lie subgroup of G.*

This follows from the fact that any arcwise connected subgroup of a Lie group is itself a Lie group [31].

4.7 Reduction of the structure group

Theorem 4.

The structure group of P can be reduced to $\Phi(\xi)$ for any $\xi \in P$.

For this we need the following

Lemma 1. *Let $M(\xi)$ be the set of points ξ' of P which are extremities of integral paths emanating from ξ. There exists an open covering $(U_i)_{i\in I}$ of V and cross-sections σ_i over U_i such that $\sigma_i(x) \in M(\xi)$ for every $x \in U_i$.*

Let $x = p\xi$ and $x_0 \in V$. Consider the path ψ connecting x and x_0. Then the integral lift $\hat{\psi}$ of ψ with origin ξ has extremity ξ_0 with $p\xi_0 = x_0$. Let U be a neighbourhood of x_0 in which are defined n linearly independent vector fields $\underline{X}_1, \ldots \underline{X}_n$ where $n = \dim V$. Let $X_1, X_2, \ldots X_n$ be horizontal vector fields on $p^{-1}(U)$ such that $pX_i = \underline{X}_i$ for every i. By the theorem on the existence of solution for differential equations, there exists a neighbourhood of $(0,0,\ldots)$ in $R \times R^n$ in which is defined a differentiable map ξ having the properties

1) $\xi(0,a) = \xi_0$ **72**

2) $\xi'(t,a) = \sum ta_i(X_i)_{\xi(t,a)}$.

From the uniqueness of such solutions, we get

$$\xi(\lambda t, a) = \xi(t, \lambda a)$$

for sufficiently small values of λ, t and a. Thus there exists a differentiable map ξ having the properties 1) and 2) above on $[0, 2] \times W$ where W is a neighbourhood of $(0, \ldots)$ in R^n. Since the X_i are horizontal, $\xi(t, a) \in M(\xi)$. Now consider the differentiable map $g : W \to P$ defined by $g(a) = \xi(1, a)$. The map $p.g$ is of maximal rank. Moreover since $\dim V = n$, this is locally invertible. In other words, in a neighbourhood U' of X_0 we have a differentiable map $f : U' \to W$ such that $p \circ g \circ f =$ Identity, i.e., gof is a cross section over U, and

$$(\text{ gof })(y) = \xi(1, f(y)) \in M(\xi) \quad \text{for every} \quad y \in U'.$$

Proof of theorem 4. We have only to take for transition functions the functions m_{ij} such that

$$\sigma_i(x) = \sigma_j(x) m_{ji}(x) \quad \text{for every} \quad x \in U_i \cap U_j$$

where the σ_i have the properties mentioned in the lemma. Then it is obvious that $m_{ij}(x) \in \Phi(\sigma_i(x_0)) = \Phi(\xi)$ by theorem 2, Ch. 4.3.

Now let V be simply connected. Then $\Phi(\xi) = \Phi_r(\xi)$. Since $\Phi(\xi)$
73 is a Lie subgroup of G, it follows [11] that the m_{ji} are differentiable functions as maps: $U_i \cap U_j \to \Phi(\xi)$ also. If ρ_i are the diffeomorphisms $U_i \times G \to p^{-1}(U_i)$ defined by $\rho_i(\xi, s) = \sigma_i(\xi)$s for $\xi \in U_i, s \in G$. Consider the set W_i consisting of $\rho_i(\xi, s)$ with $\xi \in U_i, s \in \Phi(\xi)$. It is clear that $W_i \cap p^{-1}(U_j) = W_j \cap p^{-1}(U_i)$. Moreover if we provide each W_i with the structure of a differentiable manifold by requiring that the ρ_i be a diffeomorphism, then the differentiable structures on $W_i \cap W_j$ agree. In other words, $M(\xi) = \bigcup_{i \in I} W_i$ has a differentiable structure (with which it is a submanifold of P). $\Phi(\xi)$ acts on $M(\xi)$ differentiably to the right and makes of it a principal bundle over V with projection p. We denote the inclusion map $M(\xi) \to P$ by f.

Proposition 2. *Let $d\eta$ be a vector at the point η of the manifold $M(\xi)$ and $\mathscr{Y}(\xi)$ the Lie subalgebra of $\mathscr{Y}$ corresponding to $\Phi(\xi)$, then $\gamma(f d\eta) \in \mathscr{Y}(\xi)$.*

Since the connection form is 0 on horizontal vectors, it is enough to consider the case when $f d\eta$ is tangential to the fibre, i.e. is of the form sa with $a \in \mathscr{Y}(\xi)$, $s \in \Phi(\xi)$. In this case $\gamma(sa) = a \in \mathscr{Y}(\xi)$ which proves the assertion.

If we define $\gamma^*(d\eta) = \gamma(f d\eta)$ for every vector $d\eta$ of $M(\xi)$ then γ^* is again a connection form on the principal bundle $M(\xi)$. For every point $\eta \in M(\xi)$, the holonomy group corresponding to γ^* is $\Phi(\xi)$.

4.8 Curvature and the holonomy group

Theorem 5.

For $\xi \in P, \Phi_r(\xi) = (e)$ if and only if the curvature form is identically zero. 74

Consider the universal covering manifold $\tilde{V}$ of V with covering map $\underline{h}$. Let $\tilde{P} = P_{\underline{h}}$ be the induced bundle with h as the homomorphism $\tilde{P} \to P$. Let $\tilde{\xi} \in \tilde{P}$ such that $h\tilde{\xi} = \xi$. Then by Ch.4.5., the holonomy group $\Phi(\tilde{\xi})$ with respect to the induced connection $\tilde{\gamma} = h^*\gamma$ on $\tilde{P}$ is $\Phi_r(\xi)$. Therefore, if $\Phi_r(\xi) = (e), M(\tilde{\xi})$ is the image of a cross-section $\tilde{\sigma}$ of $\tilde{P}$ over V. By prop. 4, Ch.3.6, $\tilde{\gamma}\tilde{\sigma}(d\,\tilde{x}) = 0$ for every $\tilde{x} \in \tilde{V}$. Hence by th.5, Ch.3.6 the curvature is 0. Since $\tilde{V}$ is locally diffeomorphic with V the curvature form of γ is 0. Conversely, if $K = 0, \tilde{K} = 0$ and by th.6, ch.3.6, there exists a section $\tilde{\sigma}$ of $\tilde{P}$ over $\tilde{V}$ such that $\tilde{\gamma}\tilde{\sigma} = 0$ and such that $\tilde{\sigma}(\tilde{x}) = \tilde{\xi}$. Hence the holonomy group at $\tilde{\xi}$ reduces to $\{e\}$, i.e., $\Phi_r(\xi) = \{e\}$.

Theorem 6 (Ambrose-Singer; [1])**.**

The Lie algebra of the restricted holonomy group $\Phi_r(\xi)$ at $\xi \in P$ is the subspace of the Lie algebra $\mathscr{G}$ of G generated by the values $K(d_1\eta, d_2\eta)$ of the curvature form with $d_1\eta, d_2\eta \in T_\eta, \eta \in f M(\xi)$. We may assume that V is simply connected in which case $\Phi(\xi) = \Phi_r(\xi)$, the general case being an easy consequence. Moreover, since it is enough to take $d_1\eta, d_2\eta$ to be horizontal, we may consider the values of K on the manifold $M(\xi)$. We may therefore restrict ourselves to the case when $\Phi_r(\xi) = G$, and $M(\xi) = P$. Let $\mathscr{I}$ be the *subalgebra* of $\mathscr{G}$ generated by

the values of K. Let $\mathscr{L}$ be the set of vector fields X on P such that $\gamma(X)$ is a function with values in $\mathscr{I}$. This is an $\mathscr{U}(P)$- submodule. It is easily
75 seen that $\mathscr{L}$ has everywhere rank $= \dim V + \dim \mathscr{I}$, using the fact that $\gamma(Z_a) = a \in \mathscr{I}$ for $a \in \mathscr{I}$. Moreover,

$$\begin{aligned}\gamma([X,Y])\xi &= -K(X,Y)\xi - X\gamma(Y)\xi + Y\gamma(X)\xi + [\gamma(X),\gamma(Y)]\xi \\ &\in \mathscr{I} \quad \text{for every} \quad X, Y \in \mathscr{L} \quad \text{and} \quad \xi \in P.\end{aligned}$$

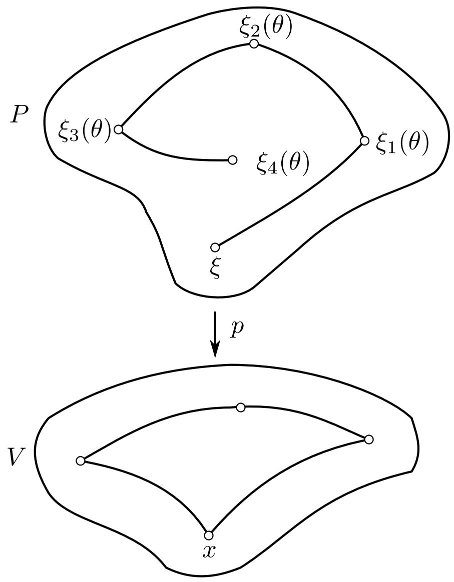
By Frobenius' theorem, there exists an integral manifold for the distribution given by $\mathscr{L}$. Let L be the maximal integral manifold for $\mathscr{L}$ containing ξ. Since the integral paths for γ with origin at ξ are integral with respect to $\mathscr{L}$ we must have $L = P$. Hence $\dim \mathscr{I} = \dim \mathscr{G}$, or $\mathscr{I} = \mathscr{G}$.

Finally it remains to prove that the subspace $\mathscr{M}$ of $\mathscr{U}$ generated by the values of K is itself a Lie subalgebra.

Since $K(d_1\eta s, d_2\eta s) = s^{-1}K(d_1\eta, d_2\eta)s \in \mathscr{I}$, it follows that $s^{-1}\mathfrak{R}s \subset \mathfrak{R}$ for every $s \in G$. Therefore for every $X \in \mathscr{I}$, $[X, \mathfrak{R}] \subset \mathfrak{R}$. In particular, $[\mathscr{M}, \mathscr{M}] \subset \mathscr{M}$.

We now give a geometric interpretation of $K(d_1\xi, d_2\xi)$ for $\xi \in P$. Firstly, we may assume $d_1\xi, d_2\xi$ to be both horizontal vectors (Prop. 3, Ch.3.6). Let $x = p\xi, d_1x = pd_1\xi$ and $d_2x = pd_2\xi$. Extend d_1x, d_2x to vector fields $\underline{X}_1$ and $\underline{X}_2$ on V such that $[\underline{X}_1, \underline{X}_2] = 0$. Let X_1, X_2 be the corresponding projectable, horizontal vector fields on P. Let $\eta \to F_i(\eta, t)$ be the automorphism of parameter t defined on a neighbourhood of ξ by the vector field $X_i (i = 1, 2)$. We set $\xi_1(\theta) = F_1(\theta, \xi), \xi_2(\theta) = F_2(\theta, \xi_1(\theta)), \xi_3(\theta) = F_2(-\theta, \xi_2(\theta))$ and $\xi_4(\theta) = F_1(-\theta, \xi_3(\theta))$. The chain
76 of paths $F_1(t/\theta, \xi)$, $F_2(t/\theta, \xi_1(\theta)), F_1(-t/\theta, \xi_2(\theta))$, $F_2(-t/\theta, \xi_3(\theta))$ is not closed in general though its projection on V is closed (since $[X_1, X_2] = 0$). Let $s(\theta^2)$ be the element of G. such that $\xi_4(\theta) = \xi s(\theta^2)$. Then $[X_1, X_2]_\xi = \xi s'(0)$ and we have

$$\begin{aligned}[X_1, X_2]_\xi &= \gamma([X_1, X_2]_\xi) \\ &= -K(X_{1\xi}, X_{2\xi}) \\ &= -K(d_{1\xi}, d_{2\xi}).\end{aligned}$$


$\xi_2(\theta)$
P
$\xi_3(\theta)$
$\xi_4(\theta)$
$\xi_1(\theta)$
ξ
p
V
x

Chapter 5

Vector Bundles and Derivation Laws

5.1

Let P be a differentiable principal bundle over a manifold V with group **77**
G. Consider a vector space L of finite dimension over the field R of real numbers. Let $s \to s_L$ be a linear representation of G in $L.G$ acts differentiably on L (regarded as a manifold) to the right by the rule $vs = s_L^{-1}v$.

Let E be a fibre bundle associated to P with fibre L and q the map $P \times L \to E$. For every $\xi \in P$, the map $v \to q(\xi, v)$ is a bijection of L onto the fibre of E at $p\xi$, or a frame of L. On each fibre E_x at $x \in V$ of the associated bundle E, we may introduce the structure of a vector space by requiring that the frame defined by any point $\xi \in p^{-1}(x)$ be linear. It is clear that such a structure does exist and is unique. Let U be an open subset of V over which P is trivial and ρ a diffeomorphism $U \times L \to p^{-1}(U)$ defined by $\rho(x, v) = q(\sigma(x), v)$ where σ is a differentiable cross section of P over U. Then we have

1) $p\,\rho(x, y) = x$

2) $\rho(x, v + v') = \rho(x, v) + \rho(x, v')$

3) $\rho(x, \lambda v) = \lambda\rho(x, v)$

for every $x \in U, v, v' \in L$ and $\lambda \in R$.

Conversely, let E be a differentiable manifold and $p : E \to V$a
78 differentiable map. Assume each $p^{-1}(x) = E_x$ to be a vector space over R. The manifold E (together with p) is called a *differentiable vector bundle* (or simply a vector bundle) over V if the following condition is satisfied. For every $x_0 \in V$, there exist an open neighbourhood V of x_0 and a differentiable isomorphism ρ of $U \times R^n$ onto $p^{-1}(U)$ such that

1) $p\rho(x, v) = x$

2) $\rho(x, v + v') = \rho(x, v) + \rho(x, v')$

3) $\rho(x, \lambda v) = \lambda\rho(x, v)$

for every $x \in V, v, v' \in L$ and $\lambda \in R$.

Proposition 1. *Every vector bundle of dimension n over V is associated to a principal bundle over V with group $GL(n, R)$.*

In fact, let E be a vector bundle of dimension n over V and $q : E \to V$ the projection of E. For every $x \in V$, define P_x to be the set of all linear isomorphisms of R^n onto E_x and $P = \bigcup_{x \in V} P_x$. The group $G = GL(n, R)$ acts on P by the rule $(\xi s)v = \xi(sv)$ for every $v \in R^n$. Define $p : P \to V$ by $p(P_x) = x$. By definition of vector bundles, there exist an open covering $U_\alpha \subset V$ and a family of diffeomorphisms $\rho_\alpha : U_\alpha \times R^n \to q^{-1}(U_\alpha)$ satisfying conditions 1), 2) and 3). For every α, the map $\gamma_\alpha : U_\alpha \times G \to p^{-1}(U_\alpha)$ defined by $\gamma_\alpha(x, s)v = \rho_\alpha(x, sv)$ for $x \in U_\alpha$, $s \in G$ is bijective. We put on P a differentiable structure by requiring that all γ_α be diffeomorphisms. It is easy to see that on the overlaps the differentiable structures agree. Moreover,

$$\begin{aligned} \gamma_\alpha(x, st)v &= \rho_\alpha(x, (st)v) \\ &= \rho_\alpha(x, s(tv)) \\ &= \gamma_\alpha(x, s)(tv) \end{aligned}$$

79 for $x \in U_\alpha, s, t \in G$ and $v \in R^n$. Thus P is a principal bundle over V with group G. Let q' be the map of $P \times R^n$ onto E defined by $q'(\xi, v) = \xi v$ for $\xi \in P, v \in R^n$. It is easy to see that $P \times R^n$ is a principal bundle over E with projection q'. Hence E is an associated bundle of P.

5.2 Homomorphisms of vector bundles

Definition 1. Let V and V' be two differentiable manifolds, E a vector bundle over V and E' a vector bundle over V'. A *homomorphism* h of E into E' is a differentiable map $h : E \to E'$ such that such that, for every $x \in V$, $p'h(E_x)$ reduces to a point $x' \in V'$ and the restriction h_x of h to E_x is a linear map of E_x into E'_x.

The map $\underline{h} : V \to V'$ defined by the condition $p'h = \underline{h}p$ is differentiable and is called the *projection* of h. If $V = V'$, by a homomorphism of E into E', we shall mean hereafter a homomorphism having the identity map $V \to V$ as projection. In that case, $h : E \to E'$ is *injective* or *surjective* according as all the maps $h_x : E_x \to E'_x$ are injective or surjective.

A vector bundle of dimension n over V is said to be *trivial* if it is isomorphic to the bundle $V{\times}R^n$ (with the natural projection $V{\times}R^n \to V$). Let P be a principal bundle over V to which the vector bundle E is **80**
associated. If P is trivial, so is E and every section of P defines an isomorphism of $V \times L$ onto E.

Let E, E', E'' be vector bundles over V. Then a sequence of homomorphisms

$$E' \xrightarrow{h} E \xrightarrow{k} E''$$

is said to be *exact* if, for every $x \in V$, the sequence

$$E'_x \xrightarrow{h_x} E_x \xrightarrow{k_x} E''_x$$

is exact

5.3 Induced vector bundles

Let E be a vector bundle over a manifold V and q a differentiable map of a manifold Y into V. Let E_q be the subset of the product $Y{\times}E$ consisting of elements (y, η) such that $q(y) = p(\eta)$. Define $p : E_q \to Y$ by setting $p'(y, \eta) = y$. If $p'(y, \eta) = p'(y', \eta')$, then $y = y'$ and $p'^{-1}(y)(y \in Y)$ by setting

$$(y, \eta) + (y, \eta') = (y, \eta + \eta')$$
$$\lambda(y, \eta) = (y, \lambda\eta)$$

for $\eta, \eta' \in E$ and $\lambda \in R$. It is clear that this makes of E_q a differentiable vector bundle over Y. This is called the *bundle* over Y *induced* by q. Let P be a principal bundle over X and E a vector bundle associated to
81 P. If P_q is the principal bundle over Y induced by q (Ch. 2.4) then E_q is associated to P_q.

5.4 Locally free sheaves and vector bundles

Let M be a differentiable manifold and ε a sheaf over M. If $V \subset U$ are two open sets in M, φ_{VU} denotes the restriction map $\varepsilon(U) \to \varepsilon(V)$ and for every $x \in U, \varphi_{xU}$ denotes the canonical map of $\varepsilon(U)$ into the stalk ε_x at x.

Let $\mathscr{U}$ be the sheaf of differentiable real valued functions on M. For every open set $U \subset M, \varepsilon(U)$ is the algebra of real valued differentiable functions on U. A sheaf ε over M is called a *sheaf* of $\mathscr{U}$*- modules* if, for every open set $U \subset M, \mathscr{U}(U)$ is an $\mathscr{U}(U)$-module and if the restriction maps satisfy the condition:

$$\varphi_{Vu}(f\sigma) = (\varphi_{VU} f)(\varphi_{VU}\sigma)$$

whenever $V \subset U$ are open sets in M, $f \in \mathscr{U}(U)$ and $\sigma \in \varepsilon(U)$.,

Let ε and ε' be two sheaves of $\mathscr{U}$-modules over $M.A$ homomorphism h of ε into ε' is a family of maps $h_U : Hom_{\mathscr{U}(U)}(\varepsilon(U), \varepsilon'(U))(U$ open subset of M) such that if $V \subset U$ are two open sets in $M, \varphi_{VU} h_U = h_V \varphi_{VU}$.

Let E be a differentiable vector bundle over M. The sheaf of differentiable sections of E is a sheaf ε of $\mathscr{U}$-modules over M; for every open set $U \subset M, \varepsilon(U)$ is the $\mathscr{U}(U)$- module of differentiable sections of E over U.

Definition 2. A sheaf ε of $\mathscr{U}$-modules over M is said to be *free* of rank
82 n if ε is isomorphic to the sheaf $\mathscr{U}^n$ of differentiable maps $M \to R^n$.

A sheaf ε of $\mathscr{U}$-modules over M is said to be *locally free* of rank n if every point $x \in M$ has a neighbourhood U such that ε restricted to U is a free sheaf of rank n.

For every vector bundle E over M, the sheaf ε of differentiable sections of E is locally free of rank= dimension of E.

Proposition 2. *A sheaf ε of $\mathscr{U}$-modules over M is locally free of rank n if and only if for every $x \in M$, there exists a neighbourhood U of x and elements $\sigma_1, \sigma_2, \ldots \sigma_n \in \varepsilon(U)$ such that for every open set $V \subset U$, $(\varphi_{VU}\sigma_1, \varphi_{VU}\sigma_2, \ldots \varphi_{VU}\sigma_n)$ is a base of $\varepsilon(V)$ over $\mathscr{U}(V)$.*

That a locally free sheaf has such a property is an immediate consequence of the definition. Conversely, if such elements $\sigma_1, \sigma_2, \ldots \sigma_n$ over U exist, then, for every open set $V \subset U$,the map h_V of $\mathscr{U}(V)^n$ into $\varepsilon(V)$ defined by $h_V(f_1, f_2, \ldots f_n) = f_1\sigma_1 + f_2\sigma_2 + \ldots f_n\sigma_n$ is bijective and the maps h_V define an isomorphism of the sheaf $\mathscr{U}^n$ of differentiable maps $U \to R^n$ onto the sheaf ε restricted to U.

Let E and E' be two vector bundles over M. To every homomorphism $h : E \to E'$ there corresponds in an obvious way a homomorphism $\tau h : \varepsilon \to \varepsilon'$ and the assignment $E \to \varepsilon$ of locally free sheaves to vector bundles is a functor T from the category of vector bundles over M into the category of locally free sheaves over M.

Moreover, if

$$0 \to E' \xrightarrow{h} E \xrightarrow{k} E'' \to 0$$

is an exact sequence, then the sequence **83**

$$0 \to TE' \xrightarrow{\tau h} Te \xrightarrow{\tau k} TE'' \to 0$$

is also exact (in the first sequence, 0 denotes the vector bundle of dimension 0 over M).

We shall now define a functor from the category of locally free sheaves into the category of vector bundles over M. Let ε be a locally free sheaf over M. Let $\mathscr{U}_x$ be the stalk at $x \in M$ for the sheaf $\mathscr{U}$ and m_x the ideal of germs of $f \in \mathscr{U}_x$ such that $f(x) = 0$. Then we have the exact sequence

$$0 \to m_x \to \varepsilon_x \to R \to 0.$$

If ε_x is the stalk at x for the sheaf ε, then we have correspondingly the exact sequence

$$0 \to m_x\varepsilon_x \to \varepsilon_x \to \varepsilon_x/m_x\varepsilon_x \to 0.$$

Let $E_x = \varepsilon_x/m_x\varepsilon_x, E = \bigcup_{x\in M} E_x$ and define $p : E \to M$ by the condition $p(E_x) = (x)$. Let U be an open subset of M and $\sigma_1, \sigma_2, \ldots \sigma_n \in \varepsilon(U)$ satisfying the condition of prop.2. Then, for every $x \in U, \varphi_{xU}\sigma_1$, $\varphi_{xU}\sigma_2, \ldots \varphi_{xU}\sigma_n$ is a basis of ε_x over $\mathscr{U}_x$. Let $\sigma_i^*(x)$ be the image of $\varphi_{xU}\sigma_i$ in the quotient space E_x. Then $\sigma_1^*(x), \sigma_2^*(x), \ldots \sigma_n^*(x)$ also is a base of E_X over R. The map $\rho_U : U \times R^n \to p^{-1}(U) \subset E$ defined by $\rho(x, a_1, a_2, \ldots a_n) = \sum a_i\sigma_i^*(x)$ is obviously a bijection. There exists on
84 E one and only one differentiable structure such that every map ρ_U is a diffeomorphism. With this structure, E is a differentiable vector bundle $T^*\varepsilon$over M. Let $\varepsilon, \varepsilon'$ be two locally free sheaves of $\mathscr{U}-$ modules over M. To every homomorphism $h : \varepsilon \to \varepsilon'$ corresponds in an obvious way a homomorphism $T^*h : T^*\varepsilon \to T^*\varepsilon'$ and the assignment $\varepsilon \to T^*\varepsilon$ is a functor T^* from the category of locally free sheaves to the category of vector bundles over M. Moreover, if

$$0 \to \varepsilon' \to \varepsilon \to \varepsilon'' \to 0$$

is an exact sequence of locally free sheaves, then

$$0 \to T^*\varepsilon' \to T^*\varepsilon \to T^*\varepsilon'' \to 0$$

is an exact sequence of vector bundles.

Proposition 3. *For every vector bundle E over M, there exists a canonical isomorphism of E onto T^*TE. For every locally free sheaf of $\mathscr{U}-$ modules ε over M, there exists a canonical isomorphism of ε onto $TT^*\varepsilon$. (We shall later identify E with T^*TE and E with TT^*E means of these isomorphisms.)*

For instance, if $\varepsilon = TE$ and $E' = T^*\varepsilon$, the isomorphism $E \to E'$ maps $u \in E_x$ into $\sigma^*(x)$ where σ is a differentiable section over an open neighbourhood of x such that $\sigma(x) = u$. In particular T defines a bijection of the set of classes of isomorphic vector bundles over M onto the set of classes of isomorphic locally free sheaves of $\mathscr{U}$ −modules over M.

Many vector bundles are defined in differential geometry from a locally free sheaf, using the functor T^*. For instance the *tangent bundle* for a manifold M corresponds the sheaf $\mathscr{C}$ over M such that $\mathscr{C}(U)$ is the module of differentiable vector fields over the open set $U \subset M$, 85

5.5 Sheaf of invariant vector fields

Let P be a differentiable principal bundle over a manifold V with group G. Let $\mathcal{J}$ be the sheaf over V such that for every open set $U \subset V, \mathcal{J}(U)$ is the spec of invariant differentiable vector fields on $p^{-1}(U) \subset P$. Clearly each $\mathcal{J}(U)$ is a module over the algebra $\mathscr{U}(U)$ of differentiable functions on U and $\mathcal{J}$ is sheaf of $\mathscr{U}$-modules.

Let U be an open subset of V such that P is trivial over U and such that the module $\mathscr{C}(U)$ of vector fields on U is a free module over $\mathscr{U}(U)$. Let $\underline{X}_1.\underline{X}_2, \ldots \underline{X}_m$ be a base of $\mathscr{C}(U)$ over $\mathscr{U}(U)$. Let $I_1, I_2, \ldots I_n$ be a base of right invariant vector fields on G. Then $((\underline{X}_i, 0), (0, I_j))$ for $i = 1, 2, \ldots m, j = 1, 2, \ldots n$ is a base of the $\mathscr{U}(U)$-module of vector fields on $U \times G$ invariant under G acting to the right. This base satisfies the condition of Prop.2. Therefore the sheaf of invariant vector fields on $U \times G$ is a free sheaf of rank $= m + n$ over the sheaf $\mathscr{U}$, restricted to U. Hence,*$\mathcal{J}$ is a locally free sheaf of $\mathscr{U}$ modules of rank equal to* $\dim V + \dim G$.

By Chap.5.4, there exists a vector bundle $j = T^*\mathcal{J}$ of dimension $= \dim V + \dim G$ canonically associated to $\mathcal{J}$. Any point in the fibre J_x of J at $x \in V$ may be interpreted as a family of vectors L along the fibre P_x satisfying the condition 86

$$L_{\xi s} = L(\xi)s \text{ for } \xi \in P_x, s \in G.$$

Let $\mathscr{K}$ be the sheaf of $\mathscr{U}$-modulo over V such that, for every open subset $U \subset V, \mathscr{K}(U)$ is the sub-module of $\mathcal{J}(U)$ consisting of invariant vector fields *tangential* to the fibres. Then $\mathscr{K}$ is locally free of rank $= \dim G$ and there exists a canonical injection $\mathscr{K} \to \mathcal{J}$. The vector bundle $K = T^*\mathscr{K}$ associated to $\mathscr{K}$ is a vector bundle of dimension $= \dim G$ over V. Any point in the fibre K_x of K at $x \in V$ may be regarded as a vector field of the manifold P_x invariant under the action of G.

For every open set $U \subset V$, any $X \in \mathcal{J}(U)$ is projectable and the projection defines a homomorphism $P_U \in \mathrm{Hom}_{\mathscr{U}(U)}(\mathcal{J}(U), \mathscr{C}(U))$. The family of homomorphisms p_U gives rise to a homomorphism $p : \mathcal{J} \to \mathscr{C}$. Since p_U is surjective when U is paracompact, $p : \mathcal{J} \to \mathscr{C}$ is surjective. On the other hand, since the kernel of p_U is $\mathscr{K}(U)$, the kernel of p is the image of $\mathscr{K} \to \mathcal{J}$. Therefore we have the exact sequence of locally free sheaves over V:

$$0 \to \mathscr{K} \to \mathcal{J} \to \mathscr{C} \to 0.$$

This gives rise to an exact sequence of vector bundles over V:

$$0 \to K \to J \to C \to 0$$

87 where C denotes the tangent bundle of V.

Theorem 1. *The vector bundle K is a vector bundle associated to P with typical fibre $\mathscr{Y}$, the action of G on $\mathscr{Y}$ being given by the adjoint representation.*

Let $E = (P \times \mathscr{Y})/G$ the vector bundle over V canonically associated to P and q the map $P \times \mathscr{Y} \to E$. We shall define a canonical isomorphism of E onto K. Let U be an open subset of V and $\sigma \in \varepsilon(U)$ a differentiable section of E on U. We define a differentiable map $\tilde{\sigma}$ of $p^{-1}(U)$ into $\mathscr{Y}$ by the condition $q(\xi, \tilde{\sigma}\xi) = \sigma(p\xi)$ for every $\xi \in p^{-1}(U)$. Then $\tilde{\sigma}(\xi s) = s^{-1}\tilde{\sigma}(\xi)s$ for every $s \in G$ and $\xi \in p^{-1}(U)$. Therefore the vector field X^σ on $p^{-1}(U)$ defined by $X^\sigma_\xi = \xi(\tilde{\sigma\xi})$ belongs to $\mathscr{K}(U)$. It is easy to verify that the map $\sigma \to X^\sigma$ is an isomorphism λ_U of the $\mathscr{U}(U)$-module $\varepsilon(U)$ onto the $\mathscr{U}(U)$-module $\mathscr{K}(U)$. The family λ_U (U open in V) is an isomorphism of the sheaf ε of differentiable sections of E onto the sheaf $\mathscr{K}$. Hence (λ_U) defines an isomorphism λ of E onto K. Let $(\xi, a) \in P \times \mathscr{Y}$. Then $\lambda q(\xi, a) = \varphi_{xU} X$ where U is an open neighbourhood of x and $X \in \mathscr{K}(U)$ a vector field satisfying the condition $X_\xi = \xi a$.

The isomorphism λ will be used later to identify the bundle $E = (P \times \mathscr{Y})/G$ with K. The vector bundle K is called the *adjoint bundle of* P.

Definition 3. Let 88
$$0 \to E' \xrightarrow{h} E \xrightarrow{k} E'' \to 0$$
be an exact sequence of vector bundles over the manifold V. A *splitting* of this exact sequence is an exact sequence $0 \to E'' \xrightarrow{\mu} E \xrightarrow{\lambda} E' \to 0$, such that λh is the identity on E' and $k\mu$ the identity on E''. Any homomorphism λ (resp μ) of E into E' (resp of E'' into E) such that λh (resp $k\mu$) is the identity determines a splitting. We shall now interpret the splitting of the exact sequence

$$\text{(S)} \qquad 0 \to K \to J \to C \to 0.$$

For every open set $U \subset V$, the module $\mathcal{J}(U)$ (resp. $\mathscr{K}(U)$) will be identified with the module of differentiable sections of J (resp. . K) over U.

Theorem 2. *There exists one and only one bijection ρ of the set of connections on P onto the set of splittings $J \to K$ of (S) such that, if Γ is a connection on P,*
$$\Gamma(X) = (\rho\Gamma) \circ X$$

If Γ is a connection on P, then, for every open set $U \subset V$, the restriction of the tensor Γ to $U' = p^{-1}(U)$ is a projection of the module $\mathscr{C}(U')$ of vector fields on U' onto the module $\mathfrak{N}(U')$ of vector fields on U' tan- 89
gential to the fibres which is invariant under the action of G and induces a projection $\lambda_U : \mathcal{J}(U) \to \mathscr{K}(U)$. The family (λ_U) is a homomorphism of the sheaf $\mathcal{J}$ onto the sheaf $\mathscr{K}$ and defines a homomorphism $\rho\Gamma : J \to K$ which is a splitting of (S). For every invariant vector field X on P, we have $(\rho\Gamma) \circ X = \lambda_V(X) = \Gamma(X)$. Since two homomorphisms $\lambda, \lambda' : J \to K$ such that $\lambda \circ X = \lambda' \circ X$ for every differentiable section X of J coincide, the map ρ is completely determined by the condition $\Gamma(X) = (\rho\Gamma) \circ X$. Moreover, since two connections Γ and Γ' such that $\Gamma(X) = \Gamma'(X)$ for every invariant vector field X coincide, ρ is injective. It

remains to prove that ρ is surjective. Let λ be a splitting of (S). For every open set $U \subset V$, Let λ_U be the projection of $\mathscr{J}(U)$ onto $\mathscr{K}(U)$ defined by setting $\lambda_U X = \lambda \circ X$ for every $X \in \mathscr{J}(U)$. Assume P to be trivial over U and let $U' = p^{-1}(U)$. The injection $\mathscr{J})(U) \to \mathscr{C}(U')$ defines an isomorphism of the $\mathscr{U}(U')$-module $\mathscr{U}(U') \bigotimes_{\mathscr{U}(U)} \mathscr{J}(U)$ onto $\mathscr{C}(U')$. The injection $\mathscr{K}(U) \to \mathfrak{N}(U')$ defines an isomorphism of the $\mathscr{U}(U')$-module $\mathscr{U}(U') \bigotimes_{\mathscr{U}(U)} \mathscr{K}(U)$ onto $\mathfrak{N}(U')$. Therefore λ_U is the restriction of a projection λ of $\mathscr{C}(U')$ onto $\mathfrak{N}(U')$ which, regarded as tensor over U' is invariant under the action of G. Hence λ_V is the restriction to $\mathscr{J}(V)$ of a projection $\Gamma : \mathscr{C}(P) \to \mathfrak{N}(P)$ which is a connection on P. For every invariant vector field X on $P, \Gamma(X) = \lambda_V x = \lambda \circ X$. Therefore $\lambda = \rho\Gamma$.

Remark. Let $0 \leftarrow K \overset{\lambda}{\leftarrow} J \overset{\mu}{\leftarrow} C \leftarrow 0$ be the splitting of (S) corre-
90 sponding to a connection Γ on P. For every vector field $\underline{X}$ on the base manifold, regarded as a section of C, the invariant vector field $\mu \circ \underline{X}$ is the horizontal vector field on P (for the connection Γ), having $\underline{X}$ as projection (cf. Atiyah [2]).

5.6 Connections and derivation laws

Let V and V' be two differentiable manifolds, and E, E' two differentiable vector bundles of dimension n over V and V' respectively. Let h be a homomorphism of E' into E with projection $\underline{h} : V' \to V$. Assume the map $h : E'_{x'} \to E_{\underline{h}(x')}$ to be bijective for every $x' \in V'$. If σ is a section of E over V, then exists one and only one section σ' of E' over V' such that $h\sigma' = \sigma\underline{h}$. If σ is differentiable, so is σ'. Since $\underline{h}$ is a differentiable map of V' into V, any module over the algebra $\mathscr{U}(V')$ of differentiable functions over V' can be regarded as a module over the algebra $\mathscr{U}(V)$ of differentiable functions over V. Then the map $\sigma \to \sigma'$ is a homomorphism of $\varepsilon(V)$ into $\varepsilon(V')$ regarded as $\mathscr{U}(V)$-modules.

In particular, let P be a principal differentiable bundle with group G over V and let $p : P \to V$ be the projection. Let $s \to s_L$ be a linear representation of G in a vector space L. Assume E to be a vector bundle associated to P with typical fibre L and q to be the map $P \times L \to E$.

Taking $V' = P, E' = P \times L, h = q, \underline{\mathrm{h}} = p$, for every section σ of E the section σ' is defined by $q\sigma'(\xi) = \sigma p(\xi)$ for $\xi \in P$. Let $\mathscr{L}(P)$ be **91**
the space of differentiable maps of P into L. Since to any section σ' of $E' = P \times L$ corresponds a map $\tilde{\sigma} \in \mathscr{L}(P)$ such that $\sigma'(\xi) = (\xi, \tilde{\sigma}(\xi))$, we obtain a *homomorphism* $\lambda : \sigma \to \tilde{\sigma}$ *of the* $\mathscr{U}(V)$*-module* $\varepsilon(V)$ *into* $\mathscr{L}(P)$ *such that*

$$q(\xi, \tilde{\sigma}\xi) = \sigma_p(\xi)$$

for every $\xi \in P$.

Definition 4. A differentiable function on P with values in the vector space L is said to be a *G-function* if $f(\xi s) = s_L^{-1} f(\xi)$ for every $\xi \in P, s \in G$.

We shall denote the space of G-functions $P \to L$ by $\mathscr{L}_G(P)$.

Proposition 4. *The homomorphism* $\lambda : \varepsilon(V) \to \mathscr{L}(P)$ *is injective and* $\lambda\, \varepsilon(V) = \mathscr{L}_G(P)$.

If $\sigma \in \varepsilon(V)$ and $\lambda\sigma = 0$, then $\sigma(\xi) = 0$ for every $\xi \in P$. Therefore $\sigma = 0$ since p is surjective. On other hand, if $\sigma \in \varepsilon(V)$ and $\tilde{\sigma} = \lambda\sigma$, we have $q(\xi s, \tilde{\sigma}(\xi s)) = \sigma p(\xi s) = \sigma p(\xi) = q(\xi, \tilde{\sigma}\xi) = q(\xi s, s_L^{-1}\tilde{\sigma}(\xi))$ for every $\xi \in P$ and $s \in G$. Hence $\tilde{\sigma} \in \mathscr{L}_G(P)$. Conversely, let $f \in \mathscr{L}_G(P)$. The map $\xi \to q(\xi, f\xi)$ maps each fibre into a point of E_X and therefore can be written σ_p, with $\sigma \in \varepsilon(V)$. We have $f = \lambda\sigma$.

We shall now show how a connection on P gives rise to a derivation law in the module of sections of the vector bundle E over V. By means of the map $\varepsilon(V) \to \mathscr{L}(P)$, the derivation law in the $\mathscr{U}(V)$-module $\varepsilon(V)$ will be deduced from a derivation law in the $\mathscr{U}(P)$-module $\mathscr{L}(P)$. **92**

To the representation $s \to s_L$ of G in L corresponds a representation in L of the Lie algebra $\mathscr{Y}$ of left invariant vector fields on G. For every $a \in \mathscr{Y}$, define a_L by setting $a_L v = av - v$ for every $v \in L$, the vectors at 0 of L being identified with elements of L. Then it is easy to see that $[a, b]_L = a_L b_L - b_L a_L$. Hence $a \to a_L$ is a linear representation of $\mathscr{Y}$ in L.

Let $\mathscr{Y}(P)$ be the space of differentiable functions on P with values in $\mathscr{Y}$. The linear map $a \to a_L$ of $\mathscr{Y}$ into $\hom_R(L, L)$ defines an

$\mathscr{U}(P)$-linear map $\alpha \to \alpha_L$ of $\mathscr{Y}(P)$ into $\hom_{\mathscr{U}(P)}(\mathscr{L}(P), \mathscr{L}(P))$ where $(\alpha_L f)(\xi) = \alpha(\xi)_L f(\xi)$ for $\xi \in P, \alpha \in \mathscr{Y}(P)$ and $f \in \mathscr{L}(P)$.

We have already seen (Ch. 1) that there are canonical derivation laws in the modules $\mathscr{Y}(P)$ and $\mathscr{L}(P)$ and hence in $\hom_{\mathscr{U}(P)}(\mathscr{L}(P), \mathscr{L}(P))$ also. It is easy then to see that

$$(X\alpha)_L f = X(\alpha_L f) - \alpha_L(Xf)$$

for every $\alpha \in \mathscr{Y}(P)$ and $f \in \mathscr{L}(P)$. We now define in $\mathscr{L}(P)$ a new derivation law in terms of a given connection form γ on the bundle by setting

$$D_X f = Xf + \gamma(X)_L f$$

for every $X \in \mathscr{C}(P)$ and $f \in \mathscr{L}(P)$. This differs from the canonical derivation law in $\mathscr{L}(P)$ by the map $\gamma : X \to \gamma(X)_L$ of $\mathscr{C}(P)$ into
93 $\hom_{\mathscr{U}(P)}(\mathscr{L}(P), \mathscr{L}(P))$. The curvatures form the canonical derivation law has been shown to be zero in Ch.1.9. Hence if K is the curvature form of the derivation law and $\mathcal{K}$ the curvature form of the connection from γ, we have

$$\begin{aligned} K(X,Y) &= D_X D_Y - D_Y D_X - D_{[X.Y]} \\ &= X\gamma(Y)_L - Y\gamma(X)_L - \gamma([X,Y])_L + [\gamma(X)_L, \gamma(Y)_L] \\ &= X\gamma(Y)_L - Y\gamma(X)_L - \gamma([X,Y])_L + [\gamma(X), \gamma(Y)]_L \\ &= (d\gamma(X,Y))_L + [\gamma(X), \gamma(Y)]_L \\ &= K(X,Y)_L \end{aligned}$$

Theorem 3. *For a given connection form γ on P, there exists one and only one derivation law D in the module of sections on the vector bundle E on V such that for every section σ and every projectable vector field X on P, we have*

$$D_{\tilde{p}X}\sigma = X\tilde{\sigma} + \gamma(X)_L\tilde{\sigma}$$

The proof is an immediate consequence of the two succeeding lemmas

Lemma 1. *If X is a vector field on P tangential to the fibres and $f \in \mathscr{L}(P)$, then $D_X f = 0$.*

In fact, if $X = Z_a$ where $a \in \mathscr{Y}$, we have

$$D_{Z_a} f = Z_a f + (\gamma(Z_a))_L f = Z_a f + a_L f$$

Hence 94

$$D_{Z_a} f(\xi) = (\xi a) f + a_L f(\xi).$$

If $g(s) = f(\xi s) = s^{-1} f(\xi)$, then $(\xi a) f = ag = -a_L f(\xi)$. Therefore $D_{Z_a} f = 0$ and the lemma is proved since the module of vector fields tangential to the fibres is generated by the vector fields $Z_a (a \in \mathscr{Y})$.

Lemma 2. *If X is projectable and $f \in \mathscr{L}_G(P)$, then $D_X f \in \mathscr{L}_G(P)$.*

By lemma 1, it is enough to consider the case when X is a horizontal projectable vector field. Then $\gamma(X) = 0$ and hence we have

$$(D_X f)(\xi s) = (X f)(\xi s) = X_{\xi s} f = (X_\xi s) f = X_\xi h,$$

where $h(\xi) = f(\xi s) = s^{-1} f$. Therefore

$$(D_X f)(\xi s) = s^{-1}(X_\xi f) = s^{-1}(D_X f)(\xi)$$

for $s \in G$, i.e., $D_X f \in \mathscr{L}_G(P)$.

The converse cannot be expected to be true in general. For, if E is a trivial vector bundle over V with group reduced to $\{e\}$, it is clear that we can have only the trivial connections in P, whereas there are nonzero derivation laws in V. However, we have the

Theorem 4. *Let G be the group of automorphisms of a vector space L, P a principal differentiable bundle with group G over a manifold V* 95
and E a differentiable vector bundle over V associated to P with typical fibre L. Let p be the projection of P onto V. For every derivation law D in the $\mathscr{U}(V)$-module $\varepsilon(V)$ of differentiable sections of E, there exists one and one connections form γ on P such that

$$\text{(C)} \qquad \widetilde{D_{pX}\sigma} = X\tilde{\sigma} + \gamma(X)_L \tilde{\sigma}$$

for every $\sigma \in \varepsilon(V)$ and every projectable vector field X on P.

We denote as before by q the map of $P \times L$ onto E. For every $\sigma \in \varepsilon(V)$ and every vector dx at a point $x \in V$, let D_{dx} be the value $(D_{\underline{X}}\sigma)(x)$, where $\underline{X}$ is any vector field on V such that $\underline{X}_x = dx$. Let $d\xi$ be a vector with origin at $\xi \in P_x$. We define a map $r(d\xi)$ of $\varepsilon(V)$ into the fibre E_x by setting

$$r(d\xi)\sigma = D_{pd\xi}\sigma - q(\xi, d\xi\tilde{\sigma})$$

for every $\sigma \in \varepsilon(V)$. Since $r(d\xi)(f\sigma) = f(x)(r(d\xi)\sigma)$ for every $f \in \mathscr{U}(V)$ and every $\sigma \in \varepsilon(V)$, $r(d\xi)\sigma$ depends only upon $\sigma(x)$ or upon $\tilde{\sigma}(\xi)$. Therefore, there exists an endomorphism α of L such that $r(d\xi)\sigma = q(\xi, \alpha\tilde{\sigma}(\xi))$ for every $\sigma \in \varepsilon(V)$. Since G is the group of automorphisms of L, there exists an element $\gamma(d\xi)$ in the Lie algebra $\mathscr{Y}$ of G such that $r(d\xi)\sigma = q(\xi, \gamma(d\xi)_L\tilde{\sigma}(\xi))$ for every $\sigma \in \varepsilon(V)$, and γ is clearly a form
96 degree 1 on P with values in y satisfying the condition (C). From (C) we deduce that $\gamma(X)$ is differentiable whenever X is a differentiable projectable vector field on P. Therefore γ is differentiable. It is the easy to verify that γ is a connection form on P. Any connection form on P satisfying the condition (C) coincides with γ since the representation of $\mathscr{Y}$ in L is faithful and since a form on P is determined by its values on projectable vector fields.

5.7 Parallelism in vector bundles

Let E be a vector bundle over V associated to the principal bundle P with typical fibre L, γ a connection form on P, D the corresponding derivation law in the module of sections $\varepsilon(V)$ of E, q the usual map $P \times L \to E, \varphi$ a differentiable path in V with origin at $x \in V$, and $\hat{\varphi}$ an integral lift of φ in P with respect to γ. For every $y \in E_x$, there exists $v \in L$ such that $q(\hat{\varphi}(0), v) = y$.

Then it is easy to see that the path $q(\hat{\varphi}(t), v)$ in E depends only on y and φ. The path $q(\hat{\varphi}(t), v)$ is called the integral lift of φ with origin y. The vectors $q(\xi(t), v)$ are sometimes said to be parallel vectors along φ.

Let σ be a section of the bundle E and dx a vector at $x \in V$. We would like to define the derivation of σ with respect to dx in terms of parallelism. Let φ be a differentiable path with origin x and $\varphi'(0) = dx$

and $\hat{\varphi}$ an integral lift of φ in P. Let $v(t) = \tilde{\sigma}(\hat{\varphi}(t))$ so that $q(\hat{\varphi}(t), v(t)) = \sigma(\varphi(t))$. Define $y(t) = q(\hat{\varphi}(0), v(t))$. Then we have

$$D_{dx}\sigma = \lim_{t\to 0}\frac{1}{t}\{y(t) - y(0)\}.$$

In fact, if **97**

$$\xi = \hat{\varphi}(0) \quad \text{and} \quad d\xi = \hat{\varphi}'(0), \text{we have}$$

$$\begin{aligned}
\lim_{t\to 0}\left\{\frac{1}{t}(y(t) - y(0))\right\} &= q(\hat{\varphi}(0), \lim_{t\to 0}\frac{1}{t}\{v(t) - v(0)\}) \\
&= q(\hat{\varphi}(0), \lim_{t\to 0}\frac{1}{t}\{\tilde{\sigma}(\hat{\varphi}(t) - \tilde{\sigma}(\hat{\varphi}(0))\}) \\
&= q(\hat{\varphi}(0), d\xi\tilde{\sigma}) \\
&= q(\hat{\varphi}(0), d\xi\tilde{\sigma} + \gamma(d\xi)_L\tilde{\sigma}) \\
&= D_{dx}\sigma.
\end{aligned}$$

For every $Y \in E_x$, let $(\xi, v) \in P \times L$ such that $q(\xi, v) = Y$. Then $q(d\xi, v) \in T_Y$. This gives in particular, a map of the space ζ_ξ of horizontal vectors with origin at ξ into T_Y, which is clearly injective. If we compose this with the projection, we obtain the bijection $p : \zeta_\xi \to T_x$. We define a vector at Y in E to be *horizontal* if it is of the form $q(d\xi, v)$ with $q(\xi, v) = Y$ and $d\xi \in S_\xi$. It is easy to see that this definition does not depend on the particular pair (ξ, v). With this definition, for every integral $\hat{\varphi}$ in P and every $u \in L$, the path $Y(t) = q(\hat{\varphi}(t), v)$ has horizontal agents at all points. The space of horizontal vectors at Y is a subspace of T_Y supplementary to the subspace of vectors $\in T_Y$ tangents to the fibres through Y.

Theorem 5. *Let σ be a section of E over V. Then $\sigma(dx)$ is horizontal if and only if $D_{dx}\sigma = 0$.* **98**

Let $d\xi$ be a horizontal tangent vector at $\xi \in p^{-1}(x)$ whose projection is dx. Then we have

$$\begin{aligned}
D_{dx}\sigma &= q(\xi, d\xi\tilde{\sigma} + \gamma(d\xi)\tilde{\sigma}) \\
&= q(\xi, d\xi\tilde{\sigma}).
\end{aligned}$$

But $\tilde{\sigma}$ has been defined by

$$q(\xi, \tilde{\sigma}(\xi)) = \sigma(p\xi).$$

Hence $q(\xi, \tilde{\sigma}(d\xi)) + q(d\xi, \tilde{\sigma}(\xi)) = \sigma(pd\xi) = \sigma dx$. If $\sigma(dx)$ is horizontal, so is $q(\xi, \tilde{\sigma}(d\xi))$. But $p_E.q(\xi, \tilde{\sigma}(d\xi)) = x$, i.e., $\tilde{\sigma}(d\xi) = \tilde{\sigma}(\xi)$. Since $d\xi\tilde{\sigma} = \tilde{\sigma}(d\xi) - \tilde{\sigma}(\xi)$, it follows that $d\xi\tilde{\sigma} = 0$. Therefore $q(\xi, d\xi\tilde{\sigma}) = 0$. The converse is also immediate.

This theorem enables us to define integral paths in E with respect to a given derivation law in $\varepsilon(V)$.

5.8 Differential forms with values in vector bundles

Definition 5. Let E be a differentiable vector-bundle over the manifold V. *A differential form* α of degree n on V *with values in the vector bundle* $\underline{\mathrm{E}}$ is an n-form on the module $\mathscr{C}(V)$ of differentiable vector fields on V
99 with values in the module $\varepsilon(V)$ of differentiable sections of the bundle E.

For every n-tuple $d_1x, d_2x, \ldots d_nx$ of vectors at $x \in V$, the value $\alpha(d_1x, d_2x, \ldots d_nx)$ of α belongs to E_x. The $\mathscr{U}(V)$-module of forms of degree n on V with values in E will be denoted by $\varepsilon^n(V)$.

Let P be a differentiable principal bundle over V, with group G and projection p. Assume E to be associated to P, with typical fibre L and let q be the map $P \times L \to E$. For every integer $n \geq 0$, we shall define an isomorphism λ of $\varepsilon^n(V)$ into the space $\mathscr{L}^n(P)$ of differential forms of degree n on P with values in the vector space L. For $n = 0, \lambda$ has been defined in 5.6. Assume $n > 0$. For every $\alpha \in \varepsilon^n(V)$, we define a form $\tilde{\alpha} = \lambda\alpha$ on P with values in L by the condition

$$q(\xi, \tilde{\alpha}(d_1\xi, d_2\xi, \ldots d_n\xi)) = \alpha(pd_1\xi, pd_2\xi, \ldots pd_n\xi)$$

100 for every sequence $d_i\xi$ of n vectors with origin at $\xi \in P$. If $X_1, X_2, \ldots X_n$ are n projectable vector fields on P, then

$$(\lambda\alpha)(X_1, X_2, \ldots X_n) = \lambda(\alpha(pX_1, pX_2, \ldots pX_n))$$

Therefore, $\lambda\alpha$ is differentiable and belongs to $\mathscr{L}^n(P)$. It is immediately seen that λ is an injective homomorphism of $\varepsilon^n(V)$ into $\mathscr{L}^n(P)$ regarded as $\mathscr{U}(V)$-modules. Moreover, if $\tilde{\alpha}$ belongs to the image of λ in $\mathscr{L}^n(P)$, then :

1) $\tilde{\alpha}(d_1\xi s, d_2\xi s, \ldots d_n\xi s) = s^{-1}\tilde{\alpha}(d_1\xi, d_2\xi, \ldots d_n\xi)s$

   for every sequence $d_i\xi$ of n vectors with origin at $\xi \in P$ and every $s \in G$,

2) $$\tilde{\alpha}(d_1\xi, d_2\xi, \ldots d_n\xi) = 0$$

   for every sequence $d_i\xi$ of n vectors with origin at $\xi \in P$ such that one of the $d_i\xi$ has projection 0.

Definition 6. A form $\alpha \in \mathscr{L}^n(P)$ satisfying the conditions 1) and 2) is said to be a *G- form of degree n* on P with values in L.

Let $\mathscr{L}^n_G(P)$ be the set of G-forms of degree n. It is easy to see that $\mathscr{L}^n_G(P)$ is a submodule of $\mathscr{L}(P)$ over $\mathscr{U}(V)$. Moreover $\mathscr{L}^n_G(P)$ is the image of the homomorphism $\lambda : \varepsilon^n(V) \to \mathscr{L}^n(P)$.

5.9 Examples

1. Let γ, γ' be two connection forms on the principal bundle P over V. Let $\beta = \gamma - \gamma'$. Then we have $\beta(d\xi\, s) = s^{-1}\beta(d\xi)s$; and $\beta(d\xi) = \gamma(d\xi) - \gamma'(d\xi) = 0$ if $pd\xi = 0$. In other words, $\beta \in \mathscr{L}^1_G(P)$ with respect to the adjoint representation of G in $\mathscr{Y}$. Hence $\beta = \tilde{\alpha}$ where α is a differential form of degree 1 on V with values in the adjoint bundle of P which is a vector bundle associated to P with typical fibre $\mathscr{Y}$. This gives a method of finding all connection forms from a given one. This is particularly useful when G is abelian, in which case the adjoint representation of G in $\mathscr{Y}$ is trivial and consequently α may be considered as a differential form on V with values in $\mathscr{Y}$

2. Let K be the curvature form of the connection form γ on a principal bundle P. Then **101**

$$K(d_1\xi s, d_2\xi s) = s^{-1}K(d_1\xi, d_2\xi)s$$

and
$$K(d_1\xi, d_2\xi) = 0 \text{ if either } d_1\xi \textit{ or } d_2\xi \in \mathfrak{N}_\xi$$

Therefore K is an alternate G-form of degree 2 on P with values in $\mathscr{Y}$ and corresponds to a form of degree 2 on V with values in the associated adjoint bundle.

5.10 Linear connections and geodesics

Let C be the vector bundle of tangent vectors on V. We consider C as a vector bundle associated to the principal bundle P of tangent frames on V with typical fibre R^n. There exists an one-one correspondence between connections on P (which are called *linear connections* on V) and derivations laws in the module $\mathscr{C}(V)$ of vector fields on V. The torsion form is an alternate form of degree 2 with values in the tangent bundle C.

Given any linear connection on V, we have the notion of *geodesics* on V. (In Ch.5.10 to Ch.5.12, we consider paths with arbitrary intervals of definition). In fact, if φ is a differentiable path in V, then there exists a *canonical lift* of φ in C defined by $t \to \varphi'(t)$. We shall denote this lift by φ'.

Definition 7. A path φ in V is said to be a *geodesic* if the canonical lift
102 φ' is integral. (In other words, the vector $\varphi''(t)$ at $\varphi'(t) \in C$ should be horizontal for the given linear connection).

Lemma 3. *Let φ be a path in V and $\hat{\varphi}$ an integral lift of φ in P. Then φ is a geodesic if and only if there exists $v \in R^n$ such that $q(\hat{\varphi}(t), v) = \varphi'(t)$ for every $t \in I$.*

The proof is trivial.

It will be noted that the notion of a geodesic depends essentially on the parametrisation of the path. However, a geodesic remains a geodesic for *linear* change of parameters.

Let $Y \in C$ be a vector at a point $x \in V$ and θ_Y the horizontal vector at Y whose projection is Y. Then θ is a vector field on C called the *geodesic* *vector field*. The geodesic vector field is *differentiable* according to the following computation of θ in local coordinates.

5.11 Geodesic vector field in local coordinates

Let C be the tangent bundle of a manifold V with a derivation law D. Let p be the projection $C \to V$. If Y is a vector at $x = pY \in V$ and U a neighbourhood of pY wherein a coordinate system $(x^1, \dots x^n)$ is defined, we denote the vector fields $\dfrac{\partial}{\partial x^i}$ on U by P_i. Let $y^i = dx^i$ be a family of differential forms on U dual to P_i, i.e., $y^j(P_i) = \delta_i^j$. The y^j may also be regarded as scalar functions on $p^{-1}(U) \subset C$. Thus for $Y \in P, (y^1, y^2, \dots y^n, x^1, x^2, \dots x^n)$ form a coordinate system in $p^{-1}(U)$. Set $Q_i = \partial/\partial y^i$. Then we assert that the geodesic vector field θ is given in $p^{-1}(U)$ by

$$\theta = \sum_i y^i P_i - \sum_{i,j,k} \Gamma^k_{i,j} y^i y^j Q_K$$

where the $\Gamma^k_{i,j}$ are defined by **103**

$$D_{P_i} P_j = \sum_k \Gamma^k_{i,j} P_k.$$

In fact, using Th.5, Ch.5.7, we see that the value of the differential forms $dy^j + \sum\limits_{i,k} y^k \Gamma^j_{i,k}(x) dx^i$ on the vector is zero. It is easy to observe that $\theta_Y = \sum\limits_i y^i (P_i)_Y - \sum\limits_{i,j,k} \Gamma^k_{i,j} y^i y^j Q_k$ is horizontal and that $p\theta_Y = Y$. This shows that θ as defined above is the geodesic vector field.

5.12 Geodesic paths and geodesic vector field

Proposition 5. *φ is geodesic path in V if and only if the canonical lift φ' of φ is integral for θ, i.e., $\varphi''(t) = \theta_{\varphi'(t)}$ for every t.*

In fact, if φ is a geodesic, then we have $p\varphi'(t) = \varphi(t)$ and $p\varphi''(t) = \varphi(t)$ for every $t \in I$. Hence $\varphi''(t) = \theta_{\varphi''(t)}$. The converse is trivial in as much every path integral for θ is also integral for the connection.

Proposition 6. *If ψ is an integral path for θ then $\varphi = p\psi$ is a geodesic $\varphi'(t) = \psi(t)$*

By assumption $\psi'(t) = \theta_{\psi(t)}$ and if $\varphi(t) = p\psi(t)$, we have $\varphi'(t) = p\psi'(t) = p\theta_{\psi(t)}$, i.e., ψ is the canonical lift of φ and is integral.

104 The geodesic vector field θ on C generates a local one-parameter group of local automorphisms of C. We say that the linear connection on V is *complete* if θ generates a one-parameter group of global automorphisms of V. It is known that in a compact manifold, any vector field generates such a one-parameter group of automorphisms. On other hand, if the local one-parameter group generated by θ in C is represented locally by $2n$ functions $\varphi_i(y_1, \ldots y_n, x_1, \ldots x_n, t)$ of $(2n + 1)$ variables, then it is easy to observe that the first n of these functions are linear in $y_1, \ldots, y_n$. From this it is immediate that any linear connection on a compact manifolds is complete.

The nomenclature 'complete' is due to the fact that a Riemannian connection is complete if and only if the Riemannian metric is complete ([27]).

Let Γ be a complete linear connection and θ the geodesic vector field. Let t_θ be the automorphism of C corresponding to the parameter t in the one-parameter group generated by θ. Paths which are integral for θ are of the form $t \to t_\theta Y$ (i.e., the orbit of Y under t_θ). Given any vector Y at a point x on the manifold $V, \varphi(t) = p(t_\theta Y)$ is the *geodesic curve* defined by Y. For every $x \in V$, the map $\rho : C_x \to V$ defined by setting $\rho(Y) = p(1_\theta Y)$ is a differentiable map of maximal rank at 0_x and therefore defines a diffeomorphism of an open neighbourhood of 0 in the vector space C_x onto an open neighbourhood of x in V([26]).

Chapter 6

Holomorphic Connections

6.1 Complex vector bundles

Definition 1. A *complex vector bundle* is a differentiable vector bundle E over a manifold V with a differentiable automorphism $J : E \to E$ such that 105

1) $JE_x \subset E_x$ for every $x \in V$;

2) $J^2 Y = -Y$ for every $Y \in E$.

Let P be a differentiable principal bundle over V with group G. Let us assume given a left representation $s \to s_L$ of G in a complex vector space L, such that each s_L is a complex automorphism of L. Then a vector bundle E associated to P with typical fibre L can be made into a complex vector bundle by setting $J_q(\xi, v) = q(\xi, \sqrt{-1}v)$ for every $\xi \in P$ and $v \in L, q$ being the usual projection $P \times L \to E$. Conversely any complex vector bundle E can be obtained in the above way. In fact, for $x \in V$ we define P_x to be the vector space of all linear isomorphisms $\alpha : \mathbb{C}^n \to E_x$ such that $\alpha(\sqrt{-1}v) = J\alpha(v)$ for $v \in \mathbb{C}^n$. Then as in Ch. 4, one can provide $P = \cup P_x$ with the structure true of a principal bundle over V with group $GL(n, \mathbb{C}).E$ is easily seen to be a complex bundle associated to P with fibre $\mathbb{C}^n$ with respect to the obvious representation of $GL(n, \mathbb{C})$ in $\mathbb{C}^n$.

If E is a complex vector bundle, the $\mathscr{U}(V)-$ module $\varepsilon(V)$ of differ- 106 entiable sections of E can be provided with the structure of an $\mathscr{U}_{\mathbb{C}}(V)-$ module $\mathscr{U}_{\mathbb{C}}(V)$ being the algebra of complex-valued differentiable functions on V) by setting $(f+ig)\sigma = f\sigma + Jg\sigma$ for $f, g \in \mathscr{U}(V)$. Conversely if I is an endomorphism of the $\mathscr{U}(V)-$ module of sections of a differentiable vector bundle E over V such that $I^2\sigma = -\sigma$ for every section σ, then there exists one and only one automorphism J of E such that $J(\sigma x) = (I\sigma)x$ (for every section σ and $x \in V$) and J makes of E a complex vector bundle.

6.2 Almost complex manifolds

Definition 2. An *almost complex manifold* is a differentiable manifold V for which the tangent bundle has the structure of a complex vector bundle, i.e., there exists an $\mathscr{U}(V)-$ endomorphism I of $\mathscr{C}(V)$ such that $I^2 = -$ (Identity).

The $\mathscr{U}_{\mathbb{C}}(V)$- module $\mathscr{C}_{\mathbb{C}}(V)$ of complex vector fields (defined as formal sums $X_1 + iX_2, X_1, X_2 \in \mathscr{C}(V)$) can be identified with the module of derivations of $\mathscr{U}_{\mathbb{C}}(V)$ over $\mathbb{C}$. The $\mathscr{U}(V)-$ endomorphism I of $\mathscr{C}(V)$ can then be extended to an $\mathscr{U}_{\mathbb{C}}(V)$- endo-morphism of $\mathscr{C}_{\mathbb{C}}(V)$ by setting $I(X_1 + iX_2) = IX_1 + iIX_2$ for $X_1, X_2 \in \mathscr{C}(V)$.

Definition 3. A vector field $X \in \mathscr{C}_{\mathbb{C}}(V)$ is said to be of *type* $(1, 0)$ (resp. . of *type* $(0, 1)$) if $IX - iX$ (resp. . $IX = -iX$).

We shall denote by $\mathscr{C}_{(1,0)}(V), \mathscr{C}_{(0,1)}(V)$ the $\mathscr{U}_{\mathbb{C}}(V)$- modules of vector fields of type (1, 0) and type (0, 1) respectively.

107 Clearly one has $\mathscr{C}_{\mathbb{C}}(V) = \mathscr{C}_{(1,0)}(V) \oplus \mathscr{C}_{(0,1)}(V)$. Moreover, any vector field X of type $(1, 0)$ can be expressed in the form $X_1 - iIX_1$ with $X_1 \in \mathscr{C}(V)$. On the other hand, $X - iIx$ is of type $(1, 0)$ for every $X \in \mathscr{C}_{\mathbb{C}}(V)$. Similarly every vector field of type (0, 1) can be expressed in the form $X_1 + iIX_1$ with $X_1 \in \mathscr{C}(V)$ and $X + iIX$ is of type (0, 1) for every $X \in \mathscr{U}_{\mathbb{C}}(V)$.

Let V be an almost complex manifold and E a complex vector bundle over V. A *complex form of degree p on V with values in E* is a multilinear form of degree p on the $\mathscr{U}_{\mathbb{C}}(V)$ - module $\mathscr{C}_{\mathbb{C}}(V)$ with values in $\varepsilon(V)$. Any real form can be extended in one and only one way to a complex form. On the other hand, every complex form is an extension of a real form.

Every holomorphic manifold V carries with it a canonical almost complex structure. With reference to this a vector field on V is of type (1, 0) if and only if its expression in terms of any system $(Z_1, \dots, Z_n)$ of local coordinates does not involve $\dfrac{\partial}{\partial \overline{Z}_1}, \dots, \dfrac{\partial}{\partial \overline{Z}_n}$. Hence it follows that $[\mathscr{C}_{(1,0)}(V), \mathscr{C}_{(1,0)}\ (V)] \subset \mathscr{C}_{(1,0)}(V)$. Conversely an almost complex structure on V comes from a holomorphic structure on V if $[\mathscr{C}_{(1,0)}(V), \mathscr{C}_{(1,0)}$ $(V)] \subset \mathscr{C}_{(1,0)}(V)$ cf. Newlander and Nirendberg, Annals of Math. 1957. Let $X, Y \in \mathscr{C}_{\mathbb{C}}(V)$. Then $X - iIX, Y - iIY \in \mathscr{C}_{(1,0)}(V)$ and the above condition requires that $[X - iIX, Y - iIY] \in \mathscr{C}_{(1,0)}(V)$. In other words,

$$[X, Y] - [IX, IY] = -I[IX, Y] - I[X, IY].$$

$$F(X, Y) = [X, Y] + I[IX, Y] + I[X, IY] - [IX, IY]$$

is easily seen to be $\mathscr{C}_{\mathbb{C}}(V)-$ bilinear. Hence we have: **108**

An almost complex structure on V is induced by a holomorphic structure if and only if $F(X, Y) = 0$ for every $X, Y \in \mathscr{C}_{\mathbb{C}}(V)$.

6.3 Derivation law in the complex case

We wish to study derivation laws in the module of sections of a complex vector bundle over an almost complex manifold (mostly holomorphic manifold). This can be done in the algebraic set-up of Ch.1.

Let A be a commutative algebra over the field of real numbers and $A_{\mathbb{C}}$ its complexification. Let C be the $A_{\mathbb{C}}$ module of derivations of $A_{\mathbb{C}}$. We assume given on C and $A_{\mathbb{C}}$ - endomorphism I such that $I^2 = -$ (Identity). If we define $\overline{X} \in C$ by $\overline{X}a = \overline{X\overline{a}}$ for $a \in A$, then we also assume that I satisfies $I\overline{X} = \overline{IX}$. $X \in C$ is said to be of type $(1, 0)$ (resp. . type $(0, 1)$) if $IX = iX$ (resp. . $IX = -iX$). Let M be an A- module. A multilinear form α of degree p on C with values in M is said to be of type (r, s) if $r + s = p$ and $\alpha(X_1, \dots X_p) = 0$ whenever either more than $rX_i's$ are of type $(1, 0)$ or more than $sX_i's$ are of type $(0, 1)$. We shall denote the submodule of $\mathscr{F}^p(C, M)$ consisting of all forms of type (r, s)

by $\mathscr{F}^{r,s}(C,M)$. Then it is easy to see that $\mathscr{F}^p(C,M) = \sum\limits_{r+s=p} \mathscr{F}^{r,s}(C,M)$, the sum being direct. Let us denote the module of alternate forms of type (r,s) on C with values in M by $\mathscr{U}^{r,s}(C,M)$. Let M_1, M_2, M_3 be three $A_{\mathbb{C}}$- modules with a bilinear product $M_1 \times M_2 \to M_3$. It is then
109 easy to see that if $\alpha \in \mathscr{U}^{r,s}(C,M_1), \beta \in \mathscr{U}^{r',s'}(C,M_2)$, then $\alpha \wedge \beta \in \mathscr{U}^{r+r',s+s'}(C,M_3)$.

We shall now make the additional assumption that $F[X,Y] = [X,Y] + I[IX,Y] + I[X,IY] - [IX,IY] = 0$ for every $X, Y \in C$. (This corresponds to the case when the base manifold V is holomorphic). A derivation law D in an $A_{\mathbb{C}}$ module M gives rise to an exterior derivation d in $\mathscr{U}^p(C,M)$. Using the explicit formula for d (Ch.1.7) and the fact that $F \equiv 0$, it is easily proved that for $\alpha \in \mathscr{U}^{r,s}(C,M), d\alpha$ is the sum of a form $d'\alpha$ of type $(r+1,s)$ and a form $d''\alpha$ of type $(r,s+1)$. The curvature form $K \in \mathscr{U}^2(C, \mathrm{Hom}_{A_{\mathbb{C}}}(M,M)$ is a sum of three components $K^{2,0}, K^{1,1}$ and $K^{0,1}$. For $\alpha \in \mathscr{U}^{p,q}(C.M)$ we have $d^2\alpha = K \wedge \alpha$ (Lemma 5, Ch. 1.9) and calculating the components of type $(p,q+2)$ we obtain

$$d''^2\alpha = K^{0,2} \wedge \alpha.$$

Now we make the further assumption that $K^{0,2} = 0$. It follows that $d''^2 = 0$. For every integer p, form the sum $\sum\limits_q \mathscr{U}^{p,q}(C,M) = \mathscr{T}_p$. Then we have a complex

$$\mathscr{U}^{p,0}(C,M) \xrightarrow{d''} \mathscr{U}^{p,1}(C,M) \xrightarrow{d''} \cdots \mathscr{U}^{p,q}(C,M) \xrightarrow{d''} \cdots$$

The complex $\mathscr{T}_p$ of course depends on the derivation law D. Let $\hat{D}$ be another derivation law in M. Then $\hat{D} = D + \omega$ where $\omega \in$
110 $\mathrm{Hom}_{\mathbb{C}}(C, \mathrm{Hom}_{A\mathbb{C}}(M,M))$ (see Ch. 1.10). We shall say that two derivation laws $D, \hat{D}$ are *equivalent* if ω is of type $(1,0)$. The boundary operator d'' in the complex $\mathscr{S}_p$ remains the same when D is replaced by an equivalent derivation law $\hat{D}$. For, if $D = \hat{D} + \omega$, we have (Ch.1.10) $\hat{d}\alpha = d\alpha + \omega \wedge \alpha$ for $\alpha \in \mathscr{U}^{p,q}(C,M)$. Hence on computing the components of type $(p,q+1)$, one obtains $\hat{d}''\alpha = d''\alpha$. Thus we have associated to every equivalence class of derivation laws, a complex $\mathscr{T}_p$ (for every integer p).

A derivation law D in M induces a derivation law $\underline{D}$ in $\mathrm{Hom}_{A_{\mathbb{C}}}(M, M) = \mathrm{End}\, M$(Ch.1.3). Then it is easy to see that

$$\underline{K}(X, Y)\rho = K(X, Y) \circ \rho - \rho \circ K(X, Y)$$

for $X, Y \in C, \rho \in$ end M. Hence if $K^{0,2} = 0, \underline{K}^{0,2}$ is also $= 0$. Moreover, if $D, \hat{D}$ are equivalent, so are $\underline{D}$, $\underline{\hat{D}}$. Hence to an equivalence class of derivation laws in M corresponds a complex $\underline{\mathscr{T}}_p$ (for every integer p):

$$\mathscr{U}^{p,o}(C,\ \mathrm{End}\ M) \xrightarrow{d''} \mathscr{U}^{p,1}(C,\ \mathrm{End}\ M) \xrightarrow{d''} \cdots \mathscr{U}^{p,q}(C,\ \mathrm{End}\ M) \xrightarrow{d''} \cdots$$

Now $K^{1,1}$ is an element of $\mathscr{U}^{1,1}(C, EndM)$. Since we have $dK = 0$ (Lemma 6, Ch.1.9) and $K^{0,2} = 0$, we get $d''K^{1,1} = 0$, i.e., $K^{1,1}$ is a cocycle of the complex $\underline{\mathscr{T}}$. If D is replaced by an equivalent derivation law $\hat{D}$, we have (Ch.1.10) $\hat{K} = K + d\omega + \omega \wedge \omega$ where $\hat{D} = D + \omega$. **111**
But $\omega \in \mathscr{U}^{1,0}(C, EndM)$. Hence $\hat{K}^{1,1} = K^{1,1} + d''\omega$. In other word, the cohomology class of $K^{1,1}$ in the above complex $\underline{\mathscr{T}}_1$ is uniquely fixed.

6.4 Connections and almost complex structures

Let P be a differentiable principal bundle over an almost complex manifold V with complex Lie group G. Let $\mathscr{Y}$ be the Lie algebra of G **112**
identified with the space of real tangent vectors at e. We denote by I_G the endomorphism $\mathscr{Y} \to \mathscr{Y}$ given by the complex structure of G and by I the almost complex structure of V. Obviously we have

$$s(I_G a)^{-1} s = I_G(sas^{-1}) \text{ for } s \in G, a \in \mathscr{Y}.$$

Let $\xi \in P$ and $p\xi = x \in V$. Then we have the exact sequence

$$0 \to \mathfrak{N}_\xi \to T_\xi \to T_x \to 0$$

where $\mathfrak{N}_\xi$ is the space of vectors at ξ tangential to the fibre. If a connection is given on P, then we may define as almost complex structure on P by carrying over the complex structure on T_x to S_ξ (the space of horizontal vectors at ξ) and that of $\mathscr{Y}$ to $\mathfrak{N}_\xi$ (by virtue of the isomorphism $a \to \xi a$). This is easily seen to define an almost complex structure on P so that we have the

Proposition 1. *Let γ be a connection form on a differentiable principal bundle P over an almost complex manifold V with complex Lie group G. Then there exists one and only one almost complex structure on the manifold P such that*

1) $\gamma(I_p d\xi) = I_G\gamma(d\xi)$

2) $pI_p d\xi = Ipd\xi$ *for* $\xi \in P$.

In virtue of the general procedure given in Ch.6.2, γ is defined on complex vector fields by setting $\gamma(iX) = I_G\gamma(X)$ for $X \in \mathscr{C}(P)$. The connection form is obviously of type $(1, 0)$ *with respect to the induced almost complex structure on P. Moreover, the $P \times G \to P$ given by the action of G on P is almost complex. We shall call a principal bundle P over an almost complex manifold V with complex Lie group G an almost complex principal bundle if P has an almost complex structure such that the projection p and the map $(\xi, s) \to \xi s$ of $P \times G \to P$ are both almost complex (i.e., compatible with the almost complex structure). We have seen above that P with the almost complex structure induced by a connection form γ on P is an almost complex principal bundle. Conversely, let P be an almost complex principal bundle. If $\gamma = \gamma^{1,0} + \gamma^{0,1}$ is a connection form on P then $\gamma^{1,0}$ is again a connection form. In fact,*

$$\begin{aligned}
\gamma^{1,0}(\xi a) &= \frac{1}{2}\gamma^{1,0}((\xi a + iI_p\xi a) + (\xi a - iI_p\xi a)) \\
&= \frac{1}{2}\gamma(\xi a - iI_p\xi a) \\
&= \frac{1}{2}(a - I_G\gamma(I_P\xi a)) = \frac{1}{2}(a - I_G\gamma(\xi I_G a)) \\
&= a
\end{aligned}$$

113 Similarly it may be shown that $\gamma^{1,0}(d\xi s) = s^{-1}\gamma^{1,0}(d\xi)s$. Moreover we have

$$\begin{aligned}
\gamma^{1,0}(I_P d\xi) &= \frac{1}{2}\gamma^{1,0}(I_P d\xi + id\xi + I_P d\xi - id\xi) \\
&= \frac{1}{2}\gamma(I_P d\xi + id\xi)
\end{aligned}$$

$$= I_G\gamma^{1,0}(d\xi).$$

On the other hand, $pI_Pd\xi = I_pd\xi$ by assumption. Hence by prop.1, Ch.6.4, $\gamma^{1,0}$ induces the given almost complex structure on P.

Let $\gamma, \hat{\gamma}$ be two connection forms on an almost complex principal bundle P. Then $\gamma - \hat{\gamma}$ is a G-form (Ch.5) of degree 1 on P with values in $\mathscr{Y}$. It may be identified with a differential form of degree 1 on V with values in the adjoint bundle. Using the fact that every vector field on V of type (1,0) is the projection of a vector field on P of type (1,0) it is easy to see that this identification is type-preserving. In other words, the type of G-forms on P with values in $\mathscr{Y}$ depends only on the almost complex structure on V. We say that $\gamma, \hat{\gamma}$ are *equivalent* if $\gamma - \hat{\gamma}$ is of type (1,0). The following proposition in then immediate.

Proposition 2. *Two connections $\gamma, \hat{\gamma}$ on P induce the same almost complex structure on P if and only if γ and $\hat{\gamma}$ are equivalent.*

On the other hand we have seen that every almost complex structure
on P which makes of it an almost complex principal bundle is induced **114**
by a connection of type $(1, 0)$. We have thus set up one-one correspondence between almost complex bundle structure on P and equivalence classes of connections.

We will hereafter assume that the base manifold V is holomorphic and investigate when a connection form γ induces a holomorphic structure on P. Then curvature form K of γ is a G-form of degree 2 with values in $\mathscr{Y}$. Let $K = K^{2,0} + K^{1,1} + K^{0,2}$ be its decomposition.

Proposition 3. *A connection form γ on P induces a holomorphic structure on P if and only if $K^{0,2} = 0$.*

In fact, if γ induces a holomorphic structure on P, then $d''\gamma = 0$. Since $K = d\gamma + [\gamma, \gamma]$ and γ is of type $(1, 0)$ for the induced complex structure, $K^{0,2} = 0$. Conversely, let $K^{0,2} = 0$. We have then to show that

$$F_P(X, Y) = [X, Y] + I_P[_PX, Y] + I_P[X, I_PY] - [I_PX, I_PY] = 0$$

for any two vector fields X, Y on P. Since F_P is a tensor, it is sufficient to prove that $F_P(X, Y) = 0$ for projectable vector fields X, Y. Then I_PX, I_PY

are also projectable, and we have $P^F P(X, Y) = F_V(P^X, P^Y) = 0$ since V is holomorphic. On the other hand,

$$
\begin{aligned}
\gamma(F(X, Y)) &= \gamma([X, Y] - [I_P X, I_P Y]) + I_G \gamma([I_P X, Y] + [X, I_P Y]) \\
&= \gamma([X, Y] - [I_P X, I_P Y] + [iI_P X, Y] + [X, iI_P Y]) \\
&= \gamma([X + iI_P X, Y + iI_P Y]).
\end{aligned}
$$

115 But since $K^{0,2} = 0, K(X + iI_P X, Y + iI_P Y) = 0$.

i.e., $(X + iI_P X)\gamma(Y + iI_P Y) - (Y + iI_P Y)\gamma(X + iI_P X) - \gamma([X + iI_P X, Y + iI_P Y]) = 0$.

Since γ is of type $(1, 0)$, we obtain $\gamma(F(X, Y)) = 0$ which gives $F(X, Y) = 0$.

Definition 4. A differentiable principal bundle P over a holomorphic manifold V with a complex Lie group G is said to be a *holomorphic principal* bundle if P is a holomorphic manifold such that the projection $p : P \to V$ and the map $(\xi, s) \to \xi s$ of $P \chi G \to P$ are holomorphic.

If a connection form γ exists on P with $K^{0,2} = 0$, then the induced almost complex structure on P is holomorphic and its is obvious that this makes of P a holomorphic principal bundles.

6.5 Connections in holomorphic bundles

Let P be a holomorphic principal bundle over V with group G. The definitions of holomorphic vector bundles with respect to a holomorphic representation of G in a complex vector space L are given in analogy with the differentiable case.

All the results of Chapter 2 can be carried over to holomorphic bundles with obvious modifications. Let $\mho(U)$ be the algebra of holomorphic functions on an open subset U of V and $\mho$ the sheaf of holomorphic functions on V. The notion of a sheaf of $\mho$- modules is defined as in Chapter 4. Moreover, there exists a functor from the category of holo-
116 morphic vector bundles over V to the category of locally free sheaves of $\mho$ modules which takes exact sequences to exact sequences, and vice versa.

Let G be a complex Lie group, P a holomorphic principal bundle with group G over a holomorphic manifold V, and E a holomorphic vector bundle over V associated to P with typical fibre L. Any connection form γ on P defines a derivation law in the module of sections of E (Ch.5).

Proposition 4. *If the connection form γ is of type $(1,0)$, then a differentiable section σ of E over an open subset U of V is holomorphic if and only if $d''\sigma = 0$.*

If $\tilde{\sigma}$ is the G-function on P with values in L corresponding to σ under our usual isomorphism, it is easy to prove (as in the differentiable case) that σ is holomorphic if and only if $\tilde{\sigma}$ is holomorphic. But this is equivalent to saying that $d''\tilde{\sigma} = 0$ for the canonical derivation law in the $\mathscr{U}_{\mathbb{C}}(P)$ module of L-valued functions on P. Since the derivation law D induced by γ in $\mathscr{L}(P)$ was defined by $D_X f = Xf + \gamma(X)_L f$ (in the usual notation) and γ is of type $(1,0)$, the d'' for D and the canonical derivation law are the same. Since the isomorphism $\alpha \to \tilde{\alpha}$ is type-preserving, we have $(d''\tilde{\alpha}) = \tilde{d}''\tilde{\alpha}$ where $\tilde{d}''$ corresponds to the derivation law D in $\mathscr{L}_G(P)$. Hence $d''\tilde{\sigma} = 0$.

Let U be an open subset of V over which E is trivial and let $\sigma_1, \ldots,$ σ_n be holomorphic sections on U which form a basis for the $\mathscr{U}(U)$–module $\varepsilon(U)$ of sections of E over U. Then for any section $\sigma = \sum f_i\sigma_i$ **117**
over U we have

$$\begin{aligned} d''\sigma &= \sum (d'' f_i) \wedge \sigma_i + \sum f_i \wedge d''\sigma_i \text{ (Ch.1.8)} \\ &= \sum d'' f_i \wedge \sigma_i \text{ Prop. 4; Ch. 6.5} \\ &= \sum (d'' f_i)\sigma_i \end{aligned}$$

In other words, d'' depends only on the manifold V and *not* on the principal bundle P. Moreover it is obvious that $d''^2\sigma = 0$.

However, this can be proved algebraically. In fact, if $\gamma, \hat{\gamma}$ are of type $(1,0)$, so is $\gamma - \hat{\gamma} = \omega$. From this it follows that that induced derivation laws are equivalent and the corresponding d'' are the same (Ch. 6.3). Furthermore, since $K^{0,2} = 0$ (which is a consequence of prop.

3. Ch.6.4 γ being of type (1, 0)) we have seen $d''^2 = 0$ in Ch.6.3. We have therefore a complex

$$\varepsilon^{p,0}(U) \xrightarrow{d''} \varepsilon^{p,1}(U) \xrightarrow{d''} \cdots \varepsilon^{p,q}(U) \xrightarrow{d''} \cdots$$

for every open subset U of V, where $\varepsilon^{p,q}(U)$ is the module of differential forms of type (p, q) on U with values in E. This defines a complex $S^P(E)$ of sheaves over V:

$$\varepsilon^{p,0} \xrightarrow{d''} \varepsilon^{p,1} \xrightarrow{d''} \cdots\cdots \varepsilon^{p,q} \xrightarrow{d''} \cdots\cdots$$

where $\varepsilon^{p,q}$ denotes the sheaf of differential forms of type (p, q) on V
118 with values in the vector bundle E. Let $\mathscr{U}^{p,q}$ be the sheaf of complex valued differentiable functions of type (p, q) on $V, \mho$ the sheaf of holomorphic functions on V, ε_h the sheaf of holomorphic sections of E. Then $\varepsilon^{p,q}$ is isomorphic to the sheaf $\mathscr{U}^{p,q} \underset{\mho}{\otimes} \varepsilon_h$ and therefore is a fine sheaf. Moreover if $\mho^p$ is the sheaf of holomorphic forms of degree p on V then the sequence

$$0 \to \mho^p \underset{\mho}{\otimes} \varepsilon_h \to \varepsilon^{p,0} \xrightarrow{d''} \varepsilon^{p,1} \to \cdots$$

is exact Dolbeault theorem [12]). Therefore the complex $\mathscr{T}^p(\varepsilon)$ is a fine resolution of the sheaf $\mho^p \underset{\mho}{\otimes} \varepsilon_h$ and if $S^p(\varepsilon)$ is the complex of sections

$$\varepsilon^{p,0}(V) \xrightarrow{d''} \varepsilon^{p,1}(V) \xrightarrow{d''} \cdots \varepsilon^{p,q}(V) \xrightarrow{d''} \cdots$$

Then

$$H^q(S^p(\varepsilon)) \simeq H^q(C, \mho^p \otimes \varepsilon_h)$$

for every pair of integers $p, q > 0$.

Now assume E to be the adjoint bundle ad (P) to P. We shall construct a canonical cohomology class in $H^1(S^1(ad(P)))$. Let γ be a connection form of type (1, 0) on P and $K = K^{2,0} + K^{1,1}$ its curvature form. As a G-form on P with values in the Lie algebra $\mathscr{Y}$, K corresponds to an alternate form χ of degree 2 on V with values in $ad(P)$. Since $d''K^{1,1} = 0$

(Ch. 6.3), we have $d''\chi^{1,1} = 0$. In other words, $\chi^{1,1}$ as a 1-cocycle of the complex $S^1(ad(P))$.

119 Moreover if $\hat{\gamma}$ is another connection of type $(1,0), \hat{\chi}^{1,1}$ differs from $\chi^{1,1}$ by a coboundary of the complex $S^1(ad(P))$. We have therefore associated a unique 1-cohomology class in $S^1(ad(P))$ to the holomorphic bundle P. To the cohomology class of $\chi^{1,1}$ corresponds an element of $H^1(V, \mho^1 \underset{\mho}{\otimes} (ad(P))_h)$. This will be referred to as the *Atiyah class* of the principal bundle P.

Regarding the existence of a holomorphic connection on the bundle P (i.e., a connection such that γ is a holomorphic form) , we have the

Theorem 1. *There exists a holomorphic connection in a holomorphic bundle P if and only if the Atiyah class $a(P)$ of the bundle is zero.*

In fact, if γ is a connection form on P of type $(1,0), \gamma$ is holomorphic if and only if $d''\gamma = 0$. Since $K^{0,2} = 0$, we see that $d''\gamma = K^{1,1} = 0$ and hence $\chi^{1,1} = 0$.

Conversely, if $a(P) = 0$ and γ a connection form of type $(1,0)$ on P, we have $\chi^{1,1} \sim 0$. Hence there exists a form $\alpha \in (adP)^{1,0}(V)$ such that $d''\alpha = \chi^{1,1}$. Then $\hat{\gamma} = \gamma - \tilde{\alpha}$ is a connection form on P such that $d''\hat{\gamma} = 0$.

Corollary. *There always exists a holomorphic connection on a holomorphic bundle P over a Stein manifold V.*

In fact, the sheaf $\mho^1 \underset{\mho}{\otimes} (ad\mathscr{G})_n$ is a coherent sheaf and therefore $a(P) = 0$, since V is a Stein manifold. By Theorem 1, there exists a holomorphic connection on P.

6.7 Atiyah obstruction

Let $0 \to \mathscr{F}^1 \to \mathscr{F} \to \mathscr{F}'' \to 0$ be an exact sequence of locally free 120
sheaves of $\mho$ modules over a holomorphic manifold V. Since $\mathscr{F}'$ is locally free, the corresponding sequence

$$0 \to \mathrm{Hom}_{\mho}(\mathscr{F}'', \mathscr{F}') \to \mathrm{Hom}_{\mho}(\mathscr{F}, \mathscr{F}') \to \mathrm{Hom}_{\mho}(\mathscr{F}', \mathscr{F}') \to 0$$

is exact. This gives rise to the exact sequence

$$0 \to H^o(V, \mathrm{Hom}_{\mho}(\mathscr{F}'', \mathscr{F}')) \to \cdots \to H^o(V, \mathrm{Hom}_{\mho}\mathscr{F}', \mathscr{F}')) \\ \to H^1(V, \mathrm{Hom}_{\mho}(\mathscr{F}'', \mathscr{F}') \to \cdots$$

$H^o(V, \mathrm{Hom}_{\mho}(\mathscr{F}', \mathscr{F}'))$ is then the module of sections of the sheaf $\mathrm{Hom}_{\mho}$ $(\mathscr{F}', \mathscr{F}')$. The image of the identity section of $\mathrm{Hom}_{\mho}(\mathscr{F}'\mathscr{F}')$ is called the *obstruction* to the splitting of the given sequence.

Let $\mathcal{K}_h, \mathscr{T}_h, \mathscr{C}_h$ be the sheaves of holomorphic vector fields on P tangential to the fibres, of all holomorphic invariant vector fields on P and of all holomorphic vector fields on V respectively. Then we have the exact sequence

$$0 \to \mathcal{K}_h \to \mathscr{T}_h \to \mathscr{C}_n \to 0$$

Let $b(P)$ be the obstruction to the splitting of this exact sequence. $b(P)$ is then a class in $H^1(V, \mathrm{Hom}_{\mho}(\mathscr{C}_h, \mathcal{K}_h))$. We shall call this the *Atiyah obstruction class.*

121 **Theorem 2.** *The necessary and sufficient condition for a holomorphic connection to exist on P is that $b(P) = 0$.*

In fact, $b(P) = 0$ if and only if the above sequence splits and the rest of the proof is exactly as for the differentiable case.

Theorem 3. *There exists a canonical isomorphism ρ of the sheaf $\mathcal{R} = \mho^1 \underset{\mho}{\otimes} (adP)_h$ onto $\mathrm{Hom}_{\mho}(\mathscr{C}_h, \mathcal{K}_h)$ such that $\rho a(P) = -b(P)$.*

We shall define $\rho_U : \mathcal{R}(U) \to \mathrm{Hom}_{\mho(U)}(\mathscr{C}_h(U), \mathcal{K}_h(U))$ for every open subset U of V. Every $\omega \in \mathcal{R}(U)$ is a differential form on U with values i the adjoint bundle and $\tilde{\omega}$ is a G-form on $p^{-1}(U)$ with values in $\mathscr{Y}$. For every $X \in \mathscr{C}_h(U)$ and $\xi \in p^{-1}(U)$, we define $(\rho_U(\omega)(X))_\xi = \dfrac{1}{2}\xi(\omega X_\xi - iI_G \omega X_\xi)$. It is easily seen that ρ_U is an isomorphism and that it defines an isomorphism

$$\rho : \mathcal{R} \to \mathrm{Hom}_{\mho}(\mathscr{C}_h, \mathcal{K}_h)$$

Let $\{U_i\}$ be a covering of V by means of open sets U_i, over each of which P is holomorphically trivial. We shall compute $a(P), b(P)$ as cocycles of the above covering with values in the sheaves $\mathcal{R}, \mathrm{Hom}_{\mho}(\mathscr{C}_h, \mathcal{K}_h)$ with respect to the above covering. Let γ_i be holomorphic connections on $p^{-1}(U_i)$ and γ a differentiable connection on P. $K^{1,1}$ is the element $d''\gamma$ in $(adP)^{1,1}(V)$. If $\gamma = \gamma_i + \alpha_i$ over $p^{-1}(U_i)$ with $\alpha_i \in (adP)^{1,0}(V)$, then $K^{1,1}$ is the $d''-$ image of α_i on U_i. For, $d''\gamma = d''\tilde{\alpha}_i = d''\tilde{\alpha}_i$ (in our usual notation). Hence $a(P)$ is represented in $H^1(X, \mathcal{R})$ 122
with respect to the above covering by the cocycle $\alpha_i - \alpha_j = -(\gamma_i - \gamma_j)$ on $U_i \cap U_j$.

On the other hand,in the exact sequence

$$0 \to \mathrm{Hom}_{\mho}(\mathscr{C}_n, \mathcal{K}_h) \to \mathrm{Hom}_{\mho}(\mathscr{I}_h, \mathcal{K}_h) \to \mathrm{Hom}_{\mho}(\mathcal{K}_h, \mathcal{K}_h) \to 0$$

the identity section of $\mathrm{Hom}_{\mho}(\mathcal{K}_h, \mathcal{K}_h)$ can be lifted on U_i into the map $\Gamma_i \in \mathrm{Hom}_{\mho}(\mathscr{T}_h, \mathcal{K}_h)$ where $\Gamma_i(X)_\xi = \dfrac{1}{2}\xi(\gamma_i(X_\xi) - iI_G\gamma(X_\xi))$ for $X \in \mathscr{T}_h, \xi \in P$. Hence the obstruction class is represented by λ_{ij} on $U_i \cap U_j$ defined by

$$\lambda_{i,j}(pd\xi) = \frac{1}{2}\xi((\gamma_i - \gamma_j)(d\xi) - iI_G(\gamma_i - \gamma_j)(d\xi))$$

for every $d\xi \in T_\xi$. This represents the cocycle of the class $b(P)$ for the covering $\{U_i\}$. Obviously $\rho(a(P)) = -b(P)$.

6.8 Line bundles over compact Kahler manifolds

Let P be a holomorphic principal bundle over a manifold V with group $G = C^*(C^* = GL(1, C))$. We shall compute the Atiyah class $a(P)$ for such a bundle. Let $\{U_i\}$ be a covering of V over each of which P is trivial with holomorphic sections σ_i on U_i and a set of holomorphic transition functions $\{m_{ij}\}$. Since the adjoint representation is trivial, the adjoint bundle is also trivial and the G-functions and G-forms on P are respectively functions and forms on P which are constant on each fibre. Let γ be a connections form of type $(1, 0)$ on P. It is easy to see that 123
$d''(\tilde{\sigma}_i^*\gamma) = d''\gamma$. But $\sigma_j^*\gamma - \sigma_i^*\gamma = m_{ij}^{-1}dm_{ij} = d(\log m_{ij})$. Thus the

forms $d(\log m_{ij})$ form a cocycle representing the Atiyah class for the covering $\{U_i\}$.

Finally, when the base manifold is compact Kahler $H^1(V, \mathcal{K} = \mho^1)$ may be identified ([30]) with a subspace of $H^2(V, C)$ by Dolbeault's theorem. The sequences

$$0 \to Z \to \mho \xrightarrow{e^{2\pi i}} \mho^* \to 0 \tag{1}$$

and

$$0 \to \mathbb{C} \to \mho \xrightarrow{d} \mho^1 \to 0 \tag{2}$$

where $Z, \mathbb{C}$ are constant sheaves and $\mho^*$ is the sheaf of nonzero holomorphic functions, can be imbedded in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Z \arrow[r] \arrow[d, "j"'] & \mho \arrow[r] \arrow[d, "i"'] & \mho^* \arrow[r] \arrow[d, "\frac{1}{2\pi i} d\log"] & 0 \\
0 \arrow[r] & \mathbb{C} \arrow[r] & \mho \arrow[r] & \mho' \arrow[r] & 0
\end{tikzcd} \tag{3}$$

where j, i are respectively the inclusion and identity maps. We get consequently the commutative diagram

$$\begin{tikzcd}
H^1(V, \mho^*) \arrow[r, "\alpha"] \arrow[d, "\delta"'] & H^2(V, Z) \arrow[d, "\beta"] \\
H^1(V, \mho') \arrow[r, "\gamma"] & H^2(V, C)
\end{tikzcd}$$

124 where α is the connecting homomorphism of the exact sequence (1), β, δ maps induced by diagram (3), and γ the injection given by Dolbeault's theorem ([29]). If $\{m_{ij}\}$ are the multiplicators for a covering (U_i) of $V, \delta(m_{ij})$ is given by $\dfrac{1}{2\pi i} d\log(m_{ij})$. The bundle P may be regarded as an element of $H^1(V, \mho^*)$ and its image by α is the first integral Chern class of P and its image by β is the Chern class $C(P)$ with complex coefficients. By the commutativity of the diagram, we have $a(P) = 2\pi i C(P)$.

In particular, we obtain the result that if there exists a holomorphic connection in a line bundle over a compact Kahler manifold, then its Chern class with complex coefficients = 0. It has been proved under more general assumptions on the group of the bundle that the existence of holomorphic connections implies the vanishing of all its Chern classes [2].

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