Lectures on Meromorphic Functions

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Part I

Basic Theory

1.1 1

We shall develop in this course Nevanlinna's theory of meromorphic functions. This theory has proved a tool of unparallelled precision for the study of the roots of equations $f(z) = a$, $f^{(1)}(z) = b$, etc. whether single or multiple and their relative frequency. Basic to this study is the Nevanlinna Characteristic $T(r)$ which is an indication of the growth of the function $f(z)$. We shall see in Theorem 2 that for every a $T(r)$ is, apart from a bounded term, the sum of two components $m(r, a) + N(r, a)$ of which the second measures the number of roots of the equation $f(z) = a$ in $|z| < r$ and the first the average closeness of $f(z)$ to a on $|z| = r$. The second fundamental theorem shows that in general the second term dominates and many applications giving well beyond Picard's theorem result.

1.2 The Poisson-Jensen Formula

We shall start with Poisson-Jensen formula which plays a fundamental role in our study.

Theorem 1. *If $f(z)$ is meromorphic in $|z| \leq R$ and has there zeros a_p and poles b_ν and if $\zeta = re^{i}$, $f(\zeta) \neq 0$, then for $0 \leq r \leq R$ we have*

$$\log f(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\log |(Re^{i\phi})|(R^2 - r^2)dr}{R^2 - 2Rr\cos(\phi - \theta) + r^2} + \sum_\mu \log \left| \frac{R(\zeta - a_\mu)}{R^2 - \overline{a}_\mu \zeta} \right|$$

$$(1.1) \qquad -\sum_{\nu} \log \left| \frac{R(\zeta - b_\nu)}{R^2 - \overline{b}_\nu \zeta} \right|$$

2 *Proof.* Let $f(z) \neq 0$, in $|z| \leq 1$. Then, since we can define an analytic branch of $\log f(z)$ in $|z| \leq 1$, we have by the residue theorem

$$\frac{1}{2\pi i} \int\limits_{|z|=1} \log f(z) \frac{dz}{z} = \log f(0)$$

By change of variable,

$$\frac{1}{2\pi} \int\limits_{0}^{2\pi} \log f(e^{i\phi}) d\phi = \log f(0)$$

and now taking real part on both sides

$$\frac{1}{2\pi} \int\limits_{0}^{2\pi} \log |f(e^{i\phi})| d\phi = \log |f(0)|$$

For any ζ with $|\zeta| < 1$, we effect the conformal transformation $w = \dfrac{z - \zeta}{1 - \overline{\zeta} z}$ for the integral $\int\limits_{|z|=1} \log f(z) \dfrac{dz}{z}$. This in turn becomes,

$$\frac{1}{2\pi i} \int\limits_{|w|=1} \log \varphi(w) \frac{dw}{w} = \log f(\zeta) \quad \text{where} \quad \varphi(w) = f\{z(w)\}$$

so that $\varphi(0) = f(\zeta)$. Substituting in the integral $z = e^{i\phi}$ and taking real part we get,

$$\frac{1}{2\pi} \int\limits_{0}^{2\pi} \frac{\log |f(e^{i\phi})|}{1 - 2r\cos(\phi - \theta) + r^2} (1 - r^2)\, d\phi = \log |f(\zeta)|, \zeta = re^{i\theta}$$

□

Now for the function $f(z)$ with poles b_ν and zeros a_μ none of them being on $|z| = 1$, let us define

$$\psi(z) = f(z) \frac{\pi \dfrac{(z - b_\nu)}{(1 - \overline{b}_\nu z)}}{\pi \dfrac{(z - a_\mu)}{(1 - \overline{a}_\mu z)}}$$

On $|z| = 1$, $|\psi(z)| = |f(z)|$ and the function has no zeros or poles in $|z| \leq 1$. **3**
By the above result,

$$\frac{1}{2\pi} \int_0^{2\pi} \log |\psi(e^{i\theta})| \frac{(1 - r^2)}{1 - 2r\cos(\phi - \theta) + r^2} d\phi = \log |\psi(\zeta)|$$

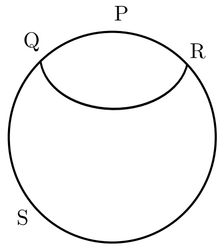
$$\zeta = re^{i\theta}, r < 1$$

Substitution for ψ gives the theorem for $R = 1$. In the case when there are poles and zeros on the circumference of the unit circle we proceed as follows. We have only to show that if $f(z)$ has no zeros or poles in $|z| < 1$, but has poles and zeros on $|z| = 1$, then

$$\frac{1}{2\pi i} \int\limits_{|z|=1} \log f(z) \frac{dz}{z} = \log f(0)$$

For if $f(z)$ has zeros and poles in $|z| < 1$ we can consider $\psi(z)$, in place of $f(z)$. Further we can assume that there is only one zero (the case of pole being treated in the same manner) on $|z| = 1$. For the case when $f(z)$ has a finite number (it can have at most only a finite number) of zeros (poles) can be treated similarly.

Let therefore $z = a$, $|a| = 1$ be a zero of $f(z)$ on $|z| = 1$. Let P be the point $z = a$ and consider a circle of radius $\rho < 1$ about P, ρ being small. Consider the contour SQR (fig.) Inside it $f(z)$ has no zeros or poles. Hence by the residue theorem, $\int \log f(z)\, dz = \log f(0)$. Thus it is enough to prove that $\int\limits_{QR} \log f(z)\, dz$ tends to zero as ρ tends to zero.



Let $z = a$ be zero of order k. Then $f(z) = (z-a)^k \lambda(z)$, $\lambda(a) \neq 0$, in
4 a certain neighbourhood of a; and we can assume the choice of ρ such that this expansion is valid within and on the circle of radius ρ about P.

$$\int_{QR} \log f(z) \frac{dz}{z} = k \int_{QR} \log(z-a) \frac{dz}{z} + \int \log \lambda(z) \frac{dz}{z}$$

Since $\lambda(z)$ remains bounded the second integral tends to zero. So we have only to prove that $\int_{QR} \log(z-a) \frac{dz}{z}$ tends to zero as $\rho \to 0$, Now,

$$\left| \int_{QR} \log(z-a) \frac{dz}{z} \right| \leq \max \left\{ \left| \frac{\log |(z-a)|}{|z|} \right| \right\}$$

$$\pi\rho \leq [\log(1/\rho) + 0(1)]\pi\rho \to 0 \text{ if } \rho \leq \frac{1}{2}$$

This proves the result, in the case when the function $f(z)$ has zeros or poles on the unit circle. In case $R \neq 1$, we consider the function $f(Rz)$ instead of $f(z)$ and arrive at the result. Hence the theorem is proved completely.

Corollary. *In the special case when $\zeta = 0$ we get the Jensen's formula*

$$(1.2) \quad \log |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\phi})| d\phi + \sum_\mu \log \left| \frac{a_\mu}{R} \right| - \sum_\nu \log \left| \frac{b_\nu}{R} \right|$$

the summation ranging over poles and zeros of $f(z)$ in $|z| \leq R$. The above formula does not hold if zero is a pole or a zero of $f(z)$. If $f(0) = 0$ or ∞

and $f(z)$ is not identically constant then $f(z) = C_\lambda z^\lambda + \cdots + \cdots$. Consider $f(z)/z^{\Upsilon}$. This has neither zero nor pole at zero. Hence we get,

$$\begin{aligned}
\log |C_\lambda| &= \frac{1}{2\pi} \int\limits_0^{2\pi} \log \left| \frac{f(Re^{i\phi})}{R^\lambda} \right| d\phi + \sum \log \frac{|a_\mu|}{R} - \sum \log \frac{|b_\nu|}{R} \\
&= \frac{1}{2\pi} \int\limits_0^{2\pi} \log |f(Re^{i\phi})| d\phi + \sum \log \frac{|a_\mu|}{R} - \sum \log \frac{|b_\nu|}{R} - \lambda \log R
\end{aligned}$$

where sums are taken over zeros and poles of $f(z)$ in $0 \leq |z| \leq R$. **5**

1.3 The Characteristic Function

Set for x, real and positive,

$$\begin{aligned}
\log^+ x &= \log x \quad \text{if} \quad x > 1, \\
\log^+ x &= 0 \qquad\; \text{if} \quad x \leq 1,
\end{aligned}$$

Then clearly $\log x = \log^+ x - \log^+(1/x)$. So

$$\int\limits_0^{2\pi} \log |f(Re^{i\phi})| d\phi = \int\limits_0^{2\pi} \log^+ |f(Re^{i\phi})| d\phi - \int\limits_0^{2\pi} \log^+ \frac{1}{|f(Re^{i\phi})|} d\phi$$

We note that the first term represents the contribution when f is large and the second term when f is small.

Let $0 < r_1 \leq r_2 \leq \ldots \leq r_n \leq R$ be the moduli of the poles in the order of increasing magnitude. Let $n(r)$ denote the number of poles in $|z| < r$ of $f(z)$. Then the Riemann-Stieljes' integral,

$$\int\limits_0^R \log \frac{R}{t} dn(t) = \sum_\nu \log \frac{1}{|b_\nu|}$$

given on integrating by parts,

$$n(t) \log \frac{R}{t} \Big]_0^R + \int\limits_0^R n(t) \frac{dt}{t} = \sum_\nu \log(R/|b_\nu|)$$

The first term is zero, in consequence of the fact $n(t) = 0$, near zero.

We write $n(r, f)$ for the number of poles of $f(z)$ in $|z| \leq r$, so that $n(r, 1/f)$ is equal to the number of zeros of $f(z)$ in $|z| \leq r$. We define $N(r, f)$ to be

$$\int_0^r n(t, f)\frac{dt}{t} \tag{3}$$

6 If $f(0) = \infty$ we define $N(r, f) = \int_0^r [n(t, f) - n(0, f)]\frac{dt}{t} + n(0, f)\log r$. Then the formula (1.2) becomes, for $f(0) \neq 0, \infty$

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{i\theta})|d\theta - \frac{1}{2\pi}\int_0^{2\pi} \log^+\frac{1}{|f(re^{i\theta})|}d\theta + N(r.f) \\ -N(r, 1/f).$$

We define,

$$T(r, f) = N(r, f) + m(r, f) \tag{1.4}$$

where,

$$m(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{i\theta})|d\theta \tag{1.5}$$

Again (1.2) takes the form, for $f(0) \neq 0, \infty$

$$T(r, f) = T(r, 1/f) + \log|f(0)| \tag{1.2$'$}$$

If $f(z) \sim C_\lambda Z^\lambda$ near $z = 0$, where $\lambda \neq 0$, then we obtain $T(r, f) = T(r, 1/f) + \log|C_\lambda|$. In future such modifications will be taken for granted.

The function $T(r, f)$ is called the *Characteristic Function* of $f(z)$.

This is the Nevanlinna characteristic function.

Theorem 2. *First fundamental theorem.*

For any finite complex a,

$$T(r, f) = T[r, 1/(f - a)] + \log|f(0) - a| + \epsilon(a)$$
$$\text{where } |\epsilon(a)| \leq \log^+ |a| + \log 2.$$

Proof. We note that $\log^+ |z_1 + z_2| \leq \log^+ z_1| + \log^+ |z_2| + \log 2$
and $\log^+ |z_1 - z_2| \geq \log^+ |z_1| - \log^+ |z_2| - \log 2$. Whence, **7**

$$\log^+ |f(z) - a| - \log^+ |f(z)| \leq \log 2 + \log^+ |a|$$

Integrating we get,

$$-\log 2 - \log^+ |a| + m(r, f - a) \leq m(r, f) \leq \log 2 + \log^+ |a| + m(r, f - a)$$

Since f and $f - a$ have the same poles,

$$N(r, f) = N(r, f - a).$$

Therefore,

$$T(r, f - a) - \log^+ |a| - \log 2 \leq T(r, f) \leq \log 2 + \log^+ |a| + T(r, f - a)$$

That is, $|T(r, f) - T(r, f - a)| \leq \log 2 + \log^+ |a|$

$$T(r, f) + T(r, f - a) + \epsilon(a), \quad \text{where } |\epsilon(a)| \leq \log 2 + \log^+ |a|$$

From (1.2′) we have,

$$T(r, f) = T\left(r, \frac{1}{f - a}\right) + \log|f(0) - a| + \epsilon(a)$$

where $|\epsilon(a)| \leq \log 2 + \log^+ |a|$. Hence the theorem is proved. □

If we write $m(r, a)$, $N(r, a)$ for $m\left(r, \frac{1}{f - a}\right)$, $N\left(r, \frac{1}{f - a}\right)$, then $m(r, a)$ represents the average degree of approximation of $f(z)$ to the value a on the circle $|z| = r$ and $N(r, a)$ the term involving the numbers of zeros of $f(z) - a$. Their sum can be regarded as the total affinity of $f(z)$ for the value a and we see than apart from a bounded term the total affinity for every value of a. However, the relative size of the two terms m, N remains in doubt. We shall see in the second fundamental theorem that in general it is $N(r, a)$ that is the larger component. The value of a for which this is not the case will be called exceptional.

Let us now consider some examples

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1. Let $f(z) = P(z)/Q(z)$, $P(z)$ and $Q(z)$ being polynomials of degrees m and n respectively, and prime to each other, that is have no common roots.

   Then for large r, $|f(z)| \sim Cr^{m-n}$.

   So if m is greater than n, $m(r, f) = (m - n)\log r + 0(1)$ and since

   $$\log^+ x = 0, \quad \text{for } 0 < x \leq 1, m(r, 1/f) = 0$$
   $$\text{and} \qquad m(r, 1/(f - a)) = 0. \tag{1.6}$$

   for large r and fixed a. Since $f(z) = \infty$ has n roots in the open plane, $n(t, f) = n$, for $t > t_0$ and $N(r, f) = n\log r + 0(1)$. Again $T(r, f) = m(r, f) + N(r, f)$ which is equal to $m\log r + 0(1)$ for $\underline{a}$ finite again by (1.6) and theorem 2, $N(r, 1/(f-a)) = m\log r + 0(1)$, $\underline{a}$ finite. Thus $\left[N\left(r, \dfrac{1}{f-a}\right)/T(r, f)\right] \to 1$ as $r \to \infty$. In this case $a = \infty$ is the only exceptional value.

   Similar conclusions follow if m is less than n by taking $1/f$ in place of f in the above discussion. In this case $a = 0 = f(\infty)$ is the only exceptional value.

   If m is equal to n, $f = c + 0(z^{-\lambda})$ (say). Or writing, $f - c \sim b(z^{-\lambda})$, we see that $m(r, 1/(f - c)) = \lambda\log r + 0(1)$

   $$m(r, 1/(f - a)) = 0(1), \quad \text{when } a \neq c$$
   $$m(r, f) = 0(1), N(r, f) = n\log r + 0(1)$$

   Thus $T(r, f) = n\log r + 0(1)$ and,

   $$N(r, a) = n\log r + 0(1), a \neq c$$
   $$N(r, c) = (n - \lambda)\log r + 0(1).$$

   Thus in any case

   $$T(r, f) = (\text{Max}.\ \ m, n)\log r + 0(1)$$
   $$\text{i.e., } T(r, f) \sim (\text{Max}.\ \ m, n)\log r.$$

9 Thus in the case of a rational function there is always only one exceptional value, viz., $f(\infty)$

2. Let $f(z) = e^z$. In this case the value of $m(r, f)$ is,

$$
\begin{aligned}
m(r, f) &= 1/2\pi \int_0^{2\pi} \log^+ e^{r\cos\theta} d\theta \\
&= 1/2\pi \int_{-\pi/2}^{\pi/2} \cos\theta \, d\theta \\
&= r/\pi
\end{aligned}
$$

because $\cos\theta \leq 0$ in $\pi/2 \leq \theta \leq \pi$, $-3\pi/2 \leq \theta \leq -\pi/2$ and so $\log^+ e^{r\cos\theta}$ is equal to zero.

Since e^z has no poles, $N(r, f) = 0$. Consequently $T(r, f)$ is equal to $m(r, f)$ which in turn is equal to r/π.

We employ the notation $m(r, a) = m(r, 1/(f - a))$ for finite $\underline{a}$ and $m(r, \infty) = m(r, f)$, and similarly with the functions n, N, and T. We have $|e^z - a| \geq ||e^z| - |a|| = |e^{r\cos\theta} - |a||$ if $z = re^{i\theta}$. If $a \neq 0$, we have for large $r\,\overline{e}^{r} < |a| < e^r$. Therefore $|a| = e^{r\cos\alpha}$ for $0 \leq \alpha \leq \pi$. Thus $2m(r, a) = \int_0^{2\pi} \log^+ \dfrac{1}{|e^z - a|} d\theta$

$$
\begin{aligned}
&\leq \int_0^{2} \log^+ \frac{1}{|e^{r\cos\theta} - e^{r\cos\alpha}|} \, d\theta \\
&= 2\pi \log^+ \frac{1}{e^{r\cos\alpha}} + 2\int_0^{2} \log^+ \frac{1}{|e^{r(\cos\theta - \cos\alpha)} - 1|} d\theta
\end{aligned}
$$

If $\cos\theta - \cos\alpha \geq 0$, we have $|e^{r(\cos\theta - \cos\alpha)} - 1|$ 10

$$
= r(\cos\theta - \cos\alpha) + \frac{r^2}{2}(\cos\theta - \cos\alpha)^2 + \cdots \geq r(\cos\theta - \cos\alpha)
$$

$$\geq (\cos\theta - \cos\alpha) \text{ if } r \geq 1.$$

If $\frac{1}{2} < |e^{r(\cos\theta-\cos\alpha)} - 1|$ and $\cos\alpha - \cos\theta > 0$ we have $e^{r(\cos\theta-\cos\alpha)} > \frac{1}{2}$ or $0 < r(\cos\alpha - \cos\theta) < 1$. Since $1 - e^{-x} = xe^{-x}$ we have

$$|e^{r(\cos\theta-\cos\alpha)} - 1| = 1 - e^{-r(\cos\alpha-\cos\theta)} \geq r(\cos\alpha - \cos\theta)e^{-r(\cos\alpha-\cos\theta)}$$
$$\geq \frac{r(\cos\alpha - \cos\theta)}{2} \geq \frac{\cos\alpha - \cos\theta}{2}$$

Thus if $|e^{r(\cos\theta-\cos\alpha)} - 1| < \frac{1}{2}$ we have

$$|e^{r(\cos\theta-\cos\alpha)} - 1| \geq |\frac{\cos\theta - \cos\alpha}{2}|.$$

Let E be the set of θ at which

$$|e^{r(\cos\theta-\cos\alpha)} - 1| \geq \frac{1}{2}.$$

Then

$$2\pi m(r,a) \leq 2\pi \log^+ \frac{1}{|a|} + 2\int_E \log^+ \frac{1}{|e^{r(\cos\theta-\cos\alpha)} - 1|} d\theta$$
$$+2\int_{[0,\pi]-E} \log^+ \frac{1}{|e^{r(\cos\theta-\cos\alpha)} - 1}| d\theta$$
$$\leq 2\log^+ \frac{1}{|a|} + 2\int_E \log^+ 2 d\theta + 2\int_{[0,\pi]-E} \log^+ \frac{2}{|\cos\theta - \cos\alpha|} d\theta.$$
$$\leq 2\log^+ \frac{1}{|a|} + 4\pi\log 2 + 2\int_0^\pi \log\frac{2}{|\cos\theta - \cos\alpha|} d\theta$$

11 Further $\int_0^\pi \log\frac{2}{|\cos\theta - \cos\alpha|} d\theta$ is a continuous function of α in $0 \leq \alpha \leq \pi$ and hence is bounded. Thus

$$m(r,a) = 0(1)$$

Hence

$$T(r,a) = T(r, f) + 0(1) = r/\pi + 0(1).$$
$$\text{i.e. } N(r,a) + m(v,a) = r/\pi + 0(1).$$
$$\text{i.e. } N(r,a) = r/\pi + 0(1).$$

This shows that $\dfrac{N(r,a)}{T(r)} \to 1.$

1.4 Some Inequalities

We have already seen that

$$\log^+ z_1 + z_2 \log^+ z_1 + \log^+ |z_2| + \log 2$$

More generally,

$$\log^+ \left|\sum_{\nu=1}^{\nu=n} z_\nu\right| \leq \log^+ |n \max |z_\nu|| \leq \log^+ n + \log^+ |\max |z_\nu||$$
$$\leq \log^+ n + \sum_{\nu=1}^{\nu=n} \log^+ |z_\nu|$$

Hence,

$$m\left(r, \sum_{n=1}^{N} f_n\right) \leq \sum_{n=1}^{n=N} m(r, f_n) + \log N$$

Now $F = \sum_{n=1}^{N} f_n(z)$ has poles only where the $f_n(z)$ have poles and the multiplicity of such pole is at most the maximal multiplicity of the poles of $f_n(z)$ which is not greater than the sum of the multiplicities. This gives

$$N(r,F) \leq \sum_{n=1}^{N} N(r, f_n)$$

and $$T\left(r, \sum_{n=1}^{N} f_n\right) \leq \sum_{n=1}^{N} T(r, f_n) + \log N$$

Again if $|a_1 a_2 \dots a_n| \leq 1$ **12**

$$\log^+ |a_1 a_2 \dots a_n| = 0 \leq \sum_{1}^{n} \log^+ |a_\nu|$$

If $|a_1 a_2 \dots a_n| > 1$, $\log^+ |a_1 a_2 \dots a_n| = \log |a_1 a_2 \dots a_n|$, which is in turn is equal to sum of $\log |a_i|$ $i = 1$ to m, and hence less than or equal to $\sum\limits_{i=1}^{n} \log^+ |a_i|$.

Therefore we get

$$\begin{aligned} m(r, \pi f_n) &\leq \sum m(r, f_n) \\ T(r, \pi f_n) &\leq \sum T(r, f_n) \\ \text{because} \qquad N(r, f_n) &\leq \sum N(r, f_n). \end{aligned}$$

We remark that in all cases equality is not excluded. For example take $a_1 = a_2 = \dots = a$ and $|a| > 1$, then $\log^+ |a_1 a_2 \dots a_n|$ is equal to $\sum\limits_{i=1}^{n} \log^+ |a_i|$.

Example

If a, b, c, d are constants with $ad - bc \neq 0$,

$$T(r, (af + b)/(cf + d)) = T(r, f) + 0(1)$$

For,

$$\begin{aligned} T(r, (af + b)/(cf + d)) &= T(r, a/c + bc - ad)/(cf + d)c) \\ &= T(r, k/(f + k')) + 0(1) \end{aligned}$$

and by the first fundamental theorem,

$$\begin{aligned} &= T(r, (f + k')/k) + 0(1) \\ &\leq T(r, f + k') + 0(1) \\ &\leq T(r, f) + 0(1) \end{aligned}$$

Similarly we get $T(r, (af + b)/(cf + d)) \geq T(r, f) + 0(1)$.

1.5

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Theorem 3 (H. Cartan)**.** *We have* $T(r, f) = (1/2\pi)\int_0^{2\pi} N(r, e^{i\theta})d\theta +$ $\log^+|f(0)|$ *that is,* $T(r, f)$ *apart from a constant term is the average of* $N(r, a)$*, for* $\underline{a}$ *on the circle* $|a| = 1$.

For the proof we require the following lemma.

Lemma 1. *If a is complex,*

$$1/2\pi \int_0^{2\pi} \log|a - e^{i\theta}|d\theta = \log^+|a|$$

This can be proved in various ways. If $|a| \geq 1$, $z - a$ has no zeroes in $|z| < 1$, and apply Poisson formula with $f(a) = z - a$, $R = 1$.

Then

$$\frac{1}{2\pi}\int_0^{2\pi} \log|e^{i\theta} - a|d\theta = \log|a| = \log^+|a|,\ |a| \geq 1$$

If

$$|a| < 1 \text{ write, } |a - e^{i\theta}| = |a||1 - e^{i\theta}/a| = |a||1 - e^{-i\theta}/\overline{a}| = |a||e^{i\theta} - i/\overline{a}|$$

and we get by the first part, since $\left|\frac{1}{a}\right| > 1$,

$$\frac{1}{2\pi}\int_0^{2\pi} \log|e^{i\theta} - a|d\theta = \log|a| + \log\left|\frac{1}{a}\right| = 0 = \log^+|a|.$$

To prove the theorem consider now Jensen's formula applied to $f(z) - e^{i\theta}$,

$$\log|f(0) - e^{i\theta}| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\phi}) - e^{i\theta}|d\phi + N(r, f) - N(r, 1/(f - e^{i\theta}))$$

since $N(r, f) = N(r, f - e^{i\theta})$ (f and $f - e^{i\theta}$ have the same poles). Now integrate both sides with respect to θ and invert the order of integration in the first term in the right hand side which we can do by Fubini's
14 Theorem, because the integrand is bounded above by $\log^+[|f(re^{i\phi})| + 1]$ and this is positive, integrable and independent of θ.

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(0) - e^{i\theta}|d\theta = \frac{1}{2\pi}\int_0^{2\pi} d\theta \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\phi}) - e^{i\theta}|d\phi$$
$$+\frac{1}{2\pi}\int_0^{2\pi} N(r, f)d\theta - \frac{1}{2\pi}\int_0^{2\pi} N(r, e^{i\theta})d\theta$$

and by the previous lemma,

$$\log^+|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(e^{i\phi})|d\phi + N(r, f) - \frac{1}{2\pi}\int_0^{2\pi} N(r, e^{i\theta})d\theta$$
$$= T(r, f) - \frac{1}{2\pi}\int_0^{2\pi} N(r, e^{i\theta})d\theta$$

which gives the result.

Corollary 1. *$T(r, f)$ is an increasing convex function of $\log r$. It is also easily seen that $m(r, f)$ need not be either an increasing or a convex function of $\log r$ (e.g. by considering rational functions). In fact from theorem 3,*

$$\frac{d(T(r))}{d\log r} = \frac{d}{d\log r}\frac{1}{2\pi}\int_0^{2\pi} d\theta \int_0^{r} n(t, e^{i\theta})\frac{dt}{t} = \frac{1}{2\pi}\int_0^{2\pi} n(r, e^{i\theta})d\theta$$

and the right hand side is non-negative and non-decreasing with r.

Corollary 2. $\frac{1}{2\pi}\int_0^{2\pi} m(r, e^{i\theta})d\theta \le \log 2.$

In fact by theorem 2,

$$
\begin{aligned}
m(r, e^{i\theta}) + N(r, e^{i\theta}) &= T\left(r, \frac{1}{f - e^{i\theta}}\right) \\
&= T(r, f) - \log|f(0) - e^{i\theta})| + < \log 2 >
\end{aligned}
$$

where $< \epsilon >$ *denotes any quantity that is less than* ϵ *in modulus.* **15**
Integrating,

$$
\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} m(r, e^{i\theta})d\theta + \frac{1}{2\pi}\int_0^{2\pi} N(r, e^{i\theta})d\theta \\
&\qquad = T(r, f) - \log^+|f(0)| + < \log 2 >
\end{aligned}
$$

by the lemma.

So

$$
\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} m(r, e^{i\theta})d\theta + T(r, f) - \log^+|f(0)| \\
&\qquad = T(r, f) + < \log 2 > - \log^+|f(0)|
\end{aligned}
$$

which is the required result. i.e.

$$
\frac{1}{2\pi}\int_0^{2\pi} m(r, e^{i\theta})d\theta = < \log 2 >
$$

1.6 The Ahlfors-Shimizu Characteristic:

We have defined the Nevalinna characteristic function of a meromorphic function $f(z)$. We now proceed to define the characteristic function after Ahlfors [1] and Shimizu [1]. Prior to that let us prove the following lemma.

Lemma 2 (Spencer)**.** *Suppose that D is a bounded domain in the complex plane, whose boundary is composed of a finite number of analytic* *curves, γ, and that $G(R)$ is twice continuously differentiable on the set of values R assumed by $|f(z)|$ in D and on γ, $f(z)$ being regular in D and on γ. Then*

$$\int_{\gamma} \frac{\partial}{\partial n} G(|f(z)|) ds = \iint_{D} g(|f(z)|) |f'(z)|^2 dx\, dy$$

16 *where $g(R) = G''(R) + (1/R)G'(R)$ and we differentiate out of the domain along the normal in $\dfrac{\partial}{\partial n}$.*

Let us make use of Green's formula which under the hypothesis of the lemma gives,

$$\int_{\gamma} \frac{\partial}{\partial n} G(|f|) ds = \iint_{D} \nabla^2 [G(|f|)] dx\, dy \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Suppose first that $f(z)$ has no zeros for z in D. Put $u = \log|f|$, in order to facilitate the easy calculation of $\nabla^2 G(|f|)$. u is a harmonic function. Therefore, $\nabla^2 u = 0$. Now $|f| = e^u$, $G(|f|) = G(e^u)$. Calculating the partial derivatives with respect to x and y we see that,

$$\frac{\partial^2}{\partial x^2}[G(e^u)] = \left(\frac{\partial u}{\partial x}\right)^2 \left\{e^u G'(e^u) + e^{2u} G''(e^u)\right\} + e^u G'(e^u) \frac{\partial^2 u}{\partial x^2}$$

and

$$\frac{\partial^2}{\partial y^2}(G(e^u)) = \left(\frac{\partial u}{\partial y}\right)^2 \left\{e^u G'(e^u) + e^{2u} G''(e^u)\right\} + G'(e^u) e^u \frac{\partial^2 u}{\partial y^2}$$

And $u = Rl. \log f(z)$ and $\dfrac{\partial u}{\partial y} - i \dfrac{\partial u}{\partial y} = \dfrac{d}{dz}(\log f) = f'/f$.

Hence on addition we get,

$$\nabla^2 [G(e^u)] = \frac{|f'(z)|^2}{|f(z)|^2} e^u \left\{G'(e^u) + e^u G''(e^u)\right\}$$

since $\nabla^2 u = 0$.

Writing $|f(z)| = R$,

$$\nabla^2[G(R)] = |f'(z)|^2[G'(R) + RG''(R)]/R$$
$$= g(R)|f'(z)|^2$$

where $g(R) = (1/R)G'(R) + G''(R)$. **17**

This establishes the result provided that $f(z) \neq 0$ for z in D. Otherwise, exclude the zeros (which are necessarily finite in number) of $f(z)$ in D, by circles of small radii, and in these $\dfrac{\partial}{\partial n}G(|f|)$ is bounded, since by hypothesis $G(R)$ is continuously differentiable near $R = 0$. Hence the contribution to the left hand side of the formula tends to zero with the radii of the circles, and so the lemma is proved.

Let us now specialise with $G(R) = \frac{1}{2}\log(1 + R^2)$ and $f(z)$ a meromorphic function in $|z|$, r with no poles on $|z| = r$. Exclude the poles in $|z| < r$ by circles of radii ρ (small). Let us apply the lemma to the region D consisting of $|z| < r$ with the poles excluded. We find in this case,

$$g(R) = (1/R)G'(R) + G''(R) = 2/(1 + R^2)^2, \quad \text{and}$$
$$\int_\gamma \frac{\partial}{\partial n}G(|f|dx = \iint_D g(|f|)|f'(z)|^2 dx\, dy.$$

γ consists of the circumference of $|z| = r$ and the smaller circles. Near a pole z_0 of order k, of $f(z)$

$$|f(z)| \sim |0|/\lambda^k \quad |z - z_0| = \lambda$$

and

$$G(|f|) = \frac{1}{2}\log(1 + |f|^2) = \log|f| + 0(1).$$

that is,

$$G(R) = k\log\gamma_\lambda + 0(1)$$

Therefore,

$$\frac{\partial}{\partial n}[G(R)] = -\frac{\partial}{\partial\rho}\left(k\log\frac{1}{\rho}\right) + 0(1)$$

since in the lemma the derivative $\frac{\partial}{\partial n}$ is along the normal directed out of the region and hence here into the circle (isolating z_0). **18**

$$\frac{\partial}{\partial n}[G(R)] = k/\rho + 0(1).$$

Hence the contribution to the left hand side in the lemma is $2\pi\rho[k/\rho + 0(1)]$, that is, $2\pi k + 0(\rho)$. Adding over all the circles and letting ρ tend to zero,

$$2\pi n(r, f) + \int\limits_{|z|=r} \frac{\partial}{\partial n}[\log(1 + |f|^2)^{\frac{1}{2}}]ds$$
$$= \iint\limits_{|z|<r} \frac{2|f'(z)|^2}{[1 + |f(z)|^2]^2} dx\, dy$$

that is again,

$$n(r, f) + \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial r}\left[\log \sqrt{1 + |f(re^{i\theta})|^2}\right] r\, d\theta$$
$$= \frac{1}{\pi} \iint\limits_{|z|<r} \frac{|f'(z)|^2}{[1 + f(z)^2]^2} dx\, dy.$$

We call the right hand side $A(r)$.

Integrate both sides from 0 to 'r' with respect to r after dividing by r throughout to get,

$$N(r, f) + \frac{1}{2\pi} \int_0^{2\pi} \log(1 + |f(re^{i})|^2)^{\frac{1}{2}} d\theta - \log(1 + |f(0)|^2)^{\frac{1}{2}}$$

$$= \int_0^r \frac{A(t)dt}{t}. \tag{1.7}$$

The integration is justified since both sides are continuous and have equal derivatives except for the isolated values of r for which $|z| = r$

contains poles of the function $f(z)$. Now $\log^+ R \leq \log(1+R^2)^{\frac{1}{2}} \leq$
19 $\log R + \frac{1}{2}\log 2$, if $R \geq 1$ so that $\log(1+R^2)^{\frac{1}{2}} \leq \log^+ R + < \frac{1}{2}\log 2 >$ and if $R \leq 1$, $0 \leq \log(1+R^2)^{\frac{1}{2}} \leq \frac{1}{2}\log 2 = \frac{1}{2}\log 2 + \log^+ R$ so that in all cases $\log(1+R^2)^{\frac{1}{2}} \leq \log^+ R + < \frac{1}{2}\log 2 >$. Hence $\log[1+|f(re^{i\theta})|^2]^{\frac{1}{2}} = < \frac{1}{2}\log 2 > + \log^+ |f(re^{i\theta})|$ substituting,

$$\begin{aligned}\int_0^r A(t)\frac{dt}{t} &= \frac{1}{2\pi}\int_0^{2\pi} \log^+(f(re^{i\theta})|d\theta + N(r,f) + < \frac{1}{2}\log 2 > - \log^+ |f(0)| \\ &= m(r,f) + N(r,f) + < \frac{1}{2}\log 2 > - \log^+ |f(0)| \\ &= T(r,f) + < \frac{1}{2}\log 2 > - \log^+ |f(0)|\end{aligned}$$

Now if we put,

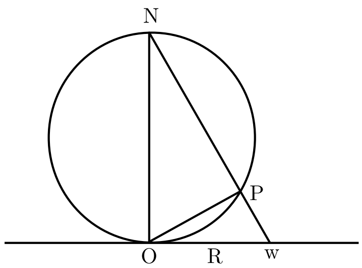
$$T_0(r,f) = \int_0^r A(t)\frac{dt}{t}$$

We get

$$T_0(r,f) = T(r,f) + < \frac{1}{2}\log 2 > - \log^+ |f(0)|$$

Definition. $T_0(r,f)$ *is called Ahlfors-Shimizu Characteristic function of* $f(z)$.

Interpretation of (1.7)



Consider the Riemann Sphere of diameter one lying over the w-plane and touching it at $w = 0$. To every point w corresponds the point $P(w)$ in which the line Nw (fig) intersects the sphere. From the figure $NP = \cos\theta = 1/(1+R^2)^{\frac{1}{2}}$, $R = |w|$. NP is called the chordal distance between w and ∞ and is denoted by $[w, \infty]$.

20 Then

$$\frac{1}{2\pi}\int_0^{2\pi} \log[1 + |f(re^{i\theta})|^2]^{\frac{1}{2}} d\theta = \frac{1}{2\pi}\int_0^{2\pi} \log\frac{1}{[f(z), \infty]} d\theta$$

and so is the average of the chordal distances between $f(z)$ and infinity, when z runs over $|z| = r$. The left hand side of the above equation is an alternative for $m(r, f)$ in the sense that it differs from $m(r, f)$ by less than $\frac{1}{2}\log 2$.

It is easy to see that if ds is an element of length in the plane and $d\sigma$ the corresponding element on the Riemann-sphere then $d\sigma = ds/(1+R^2)$. Hence if $du\, dv$ is an element of area in the w-plane, then the corresponding element of area on the sphere is $du\, dv/(1+R^2)^2$. If $dx\, dy$ is an element of area in z-circle, $|z| < r$, its image in the w-plane is $|f'(z)|^2 dx\, dy$, since the element of length is multiplied by $|f'(z)|$. Thus the element of area corresponding to $dx\, dy$ on the Riemann-w-sphere is precisely $|f'(z)|^2 dx\, dy/[1 + |f(z)|^2]^2$. Therefore, we interpret $\pi A(r)$ as the area on the Riemann-w-sphere of the image, with due count of multiplicity (the map $w = f(z)$ may not be one-one, and more than one lement of arc x in the z-plane may go into the same element of arc x in the w-plane) of $|z| < r$ by $w = f(z)$. Since the area of the sphere is π, $A(r)$ itself being the area of the image divided by the area of the sphere may be interpreted as the average number of roots in $|z| < r$ of the equation $f(z) = w$, as w moves over the Riemann sphere.

21

We have seen above that $A(r) = \dfrac{1}{\pi}$ area of image of $|z| < r$ by $f(z)$ on the Riemann Sphere. So that $A(r)$ can be interpreted as the average value of the number $n(r, a)$ of roots of $f(z) = a$, in $|z| < r$ as $\underline{a}$ runs over the complex plane.

Let us now rotate the sphere. This corresponds to a one-one conformal map of the closed plane onto itself and in fact to a bilinear map of the type $w' = e^{i\lambda}(1+\overline{a}w)/(w-a)$ (for the proof refer Carathéodery, [1]). Further under this rotation $A(r)$ will remain unaltered.

Set $\varphi(z) = e^{i\lambda}[1 + \overline{a}f(z)]/[f(z) - a]$, and observe that $[\varphi(z), \infty] = [f(z), a]$, where w, a is the chordal distance between the points corresponding to w and a on the Riemann Sphere. In particular $[w, \infty] = \{1 + |w|^2\}^{\frac{1}{2}}$. If we apply our previous result to $\varphi(z)$, we get

$$\int_0^r A(t)\frac{dt}{t} = N(r,\varphi) + \frac{1}{2\pi}\int_0^{2\pi} \log\frac{1}{[\varphi(re^{i\theta}), \infty]}d\theta - \log\frac{1}{[\varphi(0), \infty]}$$

$$= N(r,a) + \frac{1}{2\pi}\int_0^{2\pi} \log\frac{1}{[f(re^{i\theta}), a]}d\theta - \log\frac{1}{[f(0), a]}.$$

Thus we have proved the theorem.

Theorem. For every complex a and $a = \infty$

$$\int_0^r A(t)\frac{dt}{t} = N(r,a) + \frac{1}{2\pi}\int_0^{2\pi} \log\frac{1}{[f(re^{i\theta}), a]}d\theta - \log\frac{1}{[f(0), a]}.$$

The new chordal distance is **22**

$$[w, a] = \frac{1}{\left[1 + \left|\dfrac{1+\overline{a}w}{w-a}\right|^2\right]^{\frac{1}{2}}} = \frac{|w-a|}{[(1+|a|^2)(1+|w|^2)]^{\frac{1}{2}}}$$

Thus in this above theorem, the expression $m_0(r, a)$ replaces $m(r, a)$ of the Theorem 2, where,

$$m_0(r,a) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\log[(1+|a|^2)(1+|f(re^{i\theta})|^2)]^{\frac{1}{2}}}{|f(re^{i\theta}) - a|}d\theta$$

Unlike theorem 2 theorem 2’ is exact. Notice that $m_0(r, a)$ is always nonnegative because the chordal distance between any two points is always is less or equal to one.

Corollary. *If $f(z)$ is meromorphic in the plane then*

$$\int_1^r n(t,a)\frac{dt}{t} \leq \int_1^r A(t)\frac{dt}{t} + C.$$

for all a where C is independent of a and $r(>1)$ but may depend on f.

Proof.

$$\int_0^r A(t)\frac{dt}{t} = N(r,a) + m_0(r,a) - m_0(0,a)$$

$$\int_0^1 A(t)\frac{dt}{t} = N(1,a) + m_0(1,a) - m_0(0,a)$$

subtracting, since $N(r,a) = \int\limits_0^r n(t,a)\dfrac{dt}{t}$,

$$\begin{aligned}\int_1^r n(t,a)\frac{dt}{t} &= \int_1^r A(t)\frac{dt}{t} + m_0(1,a) - m_0(r,a)\\ &\leq \int_1^r A(t)\frac{dt}{t} + \max_a m_0(1,a)\end{aligned}$$

23 since $m_0(r,a)$ is greater or equal to zero.

The maximum on the right is finite. In fact, for variable $\underline{a}$ and fixed r $m_0(r,a)$ is continuous in $\underline{a}$ and is bounded as $\underline{a}$ moves over the Riemann sphere. This is evident if the function $f(z) \neq \underline{a}$ on $|z| = r$. If $f(z) = a$, at a finite number of points the argument is similar to that in the Poisson-Jensen formula. Putting $C = \text{Max}_a\, m_0(1,a)$ the result is obtained. □

Remark. By an inequality for real positive functions due to W.K. Hayman and F.M. Stewart [1] and using corollary 1 we deduce that if $f(z)$ is meromorphic and not constant in the plane and

$$n(r) = \sup_a \cdot n(r,a) \quad \text{and} \quad \epsilon > 0,$$

then

$$n(r) < (e + \epsilon)A(r) \ldots \quad \ldots \quad \ldots \tag{1.8}$$

on a set E of r having the following property. If $E(r)$ is the part of E in $[1, r]$ then $\int\limits_{E(r)} \dfrac{dt}{t} \geq c(\epsilon) \log r$ for all large r, where $c(\epsilon)$ depends only on ϵ.

The following are some open problems:

(a) Can e be replaced by a smaller constant and in particular by one in (1.8)?

(b) Does (1.8) hold for all large r; or for all large r except a small set? **24**

(c) Can we assert

$$\int\limits_1^r n(t)\frac{dt}{t} < \text{ (constant). } A(r)$$

for some arbitrarily large r?

1.7 Functions in the plane **25**

Let $S(r)$ be a real function ≥ 0, and increasing for $0 \leq r \leq \infty$. The *order* k and the *lower order* λ of the function $S(r)$ are defined as

$$\left.\begin{matrix} k \\ \lambda \end{matrix}\right\} = \overline{\underline{\lim}}_{r\to\infty} [\log S(r)]/\log r.$$

The order and the lower orders of the function always satisfy the relation, $0 \leq \lambda \leq k \leq \infty$.

If $0 < k < \infty$, we distinguish the following possibilities,

(a) $S(r)$ is maximal type if

$$C = \overline{\lim} S(R)/R^k = \text{ infinity.}$$

(b) $S(r)$ is of mean type of order k if $0 < C < \infty$.

(c) $S(r)$ is of minimal type if $C = 0$.

(d) $S(r)$ is of convergence class if,

$$\int_1^\infty S(t)dt/t^{k+1} \quad \text{converges}.$$

Note that if $S(r)$ is of order k and $\epsilon > 0$,

$$S(r) < r^{k+\epsilon} \text{ for all large } r.$$

and

$$S(r) > r^{k-\epsilon} \text{ for some large } r.$$

(This follows from the definition of order of $S(r)$.)

It can be seen that if $S(r)$ is of order k and of convergence type i.e., (d) then it is of minimal type, (c). In fact in this case

$$\int_{r_0}^\infty \frac{S(r)}{r^{k+1}} dr < \epsilon \quad \text{if} \quad r_0 > t(\epsilon)$$

26 Then

$$\int_{r_0}^{2r_0} \frac{S(r)}{r^{k+1}} dr < \epsilon$$

and since $S(r)$ increases with r,

$$\int_{r_0}^{2r_0} \frac{S(r)}{r^{k+1}} dr \geq \frac{S(r_0)}{(2r_0)^{k+1}} r_0$$

and

$$S(r_0) 2^{-(k+1)}/r_0 k < \epsilon \quad \text{for} \quad r_0 > t(\epsilon)$$

that is,

$$\varlimsup_{r\to\infty} S(r)/r^k = 0.$$

So if (d) holds $S(r)$ is of minimal type. We also note that if k' is greater than the order of $S(r)$ then,

$$\int_1^\infty S(r)dr/r^{k'+1} \quad \text{converges.}$$

and if k' is less than k $\int\limits_1^\infty \frac{S(r)}{r^{k'+1}}dr$ diverges.

We have next

Theorem 4. *If $f(z)$ is a regular function for $|z| \leq R$ and*

$$M(r, f) = \text{Max}_{|z|=r}\, f(z) \quad \textit{then,}$$
$$T(r, f) \leq \log^+ M(r, f) \leq \frac{r+R}{R-r}T(R, f) \quad \textit{for} \quad 0 < r < R.$$

Proof. Since $f(z)$ is regular in $|z| \leq R$ and $r < R$ we have,

$$\begin{aligned} T(r, f) = T(r) = m(r, f) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|d\theta \\ &\leq \frac{1}{2\pi}\,\text{Max}_{|z|=r} \log^+ |f(re^{i\theta})|2\pi \\ &= \log^+ M(r, f) \end{aligned}$$

To prove the other side of the above inequality of the theorem we distinguish between the two cases (i) $M(r) < 1$ or (ii) $M(r) \geq 1$. In the case (i), $\log^+ M(r) = 0$ and $T(r, f) \times (R + r)/(R - r)$ being non-negative the inequality holds good. Hence we can now suppose that $M(r) \geq 1$, and in which case $\log^+ M(r) = \log M(r)$. **27**

The Poisson-Jensen formula gives, for $z = re^{i\theta}$

$$\log |f(z)| = \frac{1}{2\pi}\int_0^{2\pi} \log |f(Re^{i\phi})| \frac{(R^2 - r^2)}{R^2 + 2Rr\cos(\phi - \theta) + r^2}d\phi.$$

$$- \sum \log^+ \left| \frac{(R^2 - \overline{a}_\mu z)}{R(z - a_\mu)} \right|$$

in the above equality a_μ runs through all the zeros of $f(z)$. For, as regards the zeros with modulus less or equal to R

$$\log^+ \left| \frac{(R^2 - \overline{a}_\mu z)}{R(z - a_\mu)} \right| = \log \left| \frac{(R^2 - \overline{a}_\mu z)}{R(z - a_\mu)} \right| \quad \text{since} \quad \left| \frac{(R^2 - \overline{a}_\mu z)}{(z - a_\mu)R} \right| \geq \underline{1}.$$

and for the zeros $a\mu$ with modulus greater than R,

$$\log^+ \left| \frac{(R^2 - \overline{a}_\mu z)}{R(z - a_\mu)} \right| = 0$$

and Poisson's formula is unaffected. □

Clearly,

$$\begin{aligned} \log |f| \frac{(R^2 - r^2)}{R^2 - 2Rr\cos(\phi - \theta) + r^2} &\leq \log^+ |f| \frac{(R^2 - r^2)}{R^2 - 2Rr\cos(\phi - \theta) + r^2} \\ &\leq \frac{R + r}{R - r} \log^+ f \; - \; \frac{R + r}{R - r} \log^+ f \end{aligned}$$

28

$$\begin{aligned} \log |f(z)| &\leq \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2)}{(r^2 + R^2 - 2Rr)} \log^+ |f(Re^{i\phi})| d\phi \\ &= (1/2\pi) \int_0^{2\pi} (R + r/(R - r)) \log^+ |f(Re^{i\phi})| d\phi \\ \log |f(z)| &\leq (R + r/(R - r)) T(R, f). \end{aligned}$$

This being true for all z, $|z| = r$, holds good for the z at which $f(z)$ takes the maximum $M(r, f)$.

Hence,

$$\log M(r, f) \leq (\overline{R + r}/\overline{R - r}) T(R, f).$$

Corollary. *If $f(z)$ is an integral function, the order and type (a, b, c or d defined in the last article) of $T(r)$ and $\log^+ M(r)$ are the same.*

From the left hand side of the inequality of Theorem 4, it follows that the order and type of $T(r, f)$ are not bigger than those of $\log^+ M(r)$. In order to prove the reverse, let us substitute $R = 2r$ in the right side of the theorem. We get,

$$\log^+ M(r, f) \leq 3T(2r, f)$$

Suppose k is the order of $T(r, f)$, then given $\epsilon > 0$, $T(r, f) < \epsilon r^{k+\epsilon}$ for all $r > r_0$ by the definition of order. Hence combining the two inequalities,

$$\log^+ M(r, f) \leq 3T(2r, f) < 3\epsilon(2r)^{k+\epsilon} \quad \text{for} \quad r > r_0.$$

This implies that the order of $\log^+ M(r, f)$ is less or equal to k. If $T(r)$ has mean type, we may take $\epsilon = 0$. If $T(r)$ has minimal type, we may 29
in addition take c arbitrarily small. Hence $T(r, f)$ and $\log^+ M(r, f)$ both have same order and belong to the same order type a, b or c. In order to complete the corollary, we have to consider the case when $T(r, f)$ is of (d) i.e., convergence type. Consider,

$$\int_{r_0}^{\infty} \frac{\log^+ M(r)}{r^{k+1}} dr$$

$$\int_{r_0}^{\infty} \frac{\log^+ M(r)}{r^{k+1}} dr \leq \int_{r_0}^{\infty} \frac{3T(2r)}{r^{k+1}} dr$$

k being the order of $T(r, f)$.

$$= \int_{2r_0}^{\infty} 3.2^k \frac{T(r)dr}{r^{k+1}}$$

by change of variable.

Since $T(r)$ is of convergence type, it follows from the above inequality that $\log^+ M(r)$ also belongs to the same class. It is clear that this relation holds good in the reverse direction.

We now proceed to define the order and the order type of a function $f(z)$ meromorphic in the plane on the above analogy.

Definition. *The* order *and* type *of a function* $f(z)$ meromorphic *in the plane are* defined *as the* order *and* type of $T(r, f)$.

We note that this coincides with that of order and type of $\log^+ M(r, f)$ if $f(z)$ is an entire function.

30 **Example.** (1) If the function $f(z) = e^z$, $\log M(r) = r$ and $T(r) = r/\pi$. The function has order one, mean type. In this case $\log^+ M(r)$ and $T(r)$ are of the same type, viz., mean type, though the values of the type are different. More precisely, for $T(r)$, $\overline{\lim}_{r\to\infty} T(r)/r = 1/\pi$ (the order being 1) and for $\log^+ M(r)$, $\overline{\lim}_{r\to\infty} (1/r) \log^+ M(r) = 1$.

In the above example the ratio of the two super. limites that is,

$$\frac{\overline{\lim}_{r\to\infty} \dfrac{T(r)}{r^k}}{\overline{\lim}_{r\to\infty} \dfrac{\log^+ M(r)}{r^k}} = 1$$

In general case for a function of order k and mean type this ratio is bounded by 1. But it is still an open problem to find the best possible lower bound. It can be shown easily from theorem 4, taking $R = r[1 + (1/k)]$, that a lower bound is $1/e(2k + 1)$. In this connection P.B. Kennedy [1] has given a counter example of a function of order k mean type in $|\arg z| < \pi/2k$, and bounded for $\pi/2k \le \arg z \le \pi$, provided $k > \frac{1}{2}$, such that

$$\log^+ |f(re^{i\theta})| \sim r^k \cos k\theta \quad \text{for} \quad |\theta| < \pi/2k$$
$$\log^+ |f(re^{i\theta})| = 0(1) \quad \frac{\pi}{2k} \le |\theta| < \pi$$

Hence $T(r, f) \sim r^k/k$, $\log M(r) \sim r^k$. Thus $e(2k + 1)$ cannot be replaced by any constant less than $k\pi$ if $k > \frac{1}{2}$.

Remark. For functions of infinite order $T(r)$ and $\log^+ M(r)$ no longer have necessarily the same order of magnitude. Actually it has been shown by W.K. Hayman and F.M. Stewart [1] that if $\epsilon > 0$, $\log M(r) < (2e+\epsilon)[T(r)+A(r)]$ for some large r. As an example consider $f(z) = e^{e^z}$ 31

$$\log M(r) = e^r$$

and

$$T(r) \sim e^r (2\pi^3)^{\frac{1}{2}} / r^{\frac{1}{2}}.$$

1.8 Representation of a function in terms of its zeros and poles 32

Definition. *For any complex number z, and any integer $q > 0$,*

$$E(z,q) = (1-z)e^{z+\frac{1}{2}z^2+\cdots+\frac{1}{q}z^q}.$$

Theorem 5 (Nevanlinna)**.** *If $f(z)$ is meromorphic in the plane with zeros a_μ and poles b_ν and $f(z)$ being of order at most q of the minimal type then,*

$$f(z) = z^p e^{P_{q-1}(z)} \lim_{R\to\infty} \frac{\prod\limits_{1a_\mu|<R} E\left(\dfrac{z}{a_\mu}, q-1\right)}{\prod\limits_{1b_\mu|<R} E\left(\dfrac{z}{b_\nu}, q-1\right)}$$

where p is a suitable integer and $P_{q-1}(z)$ is a polynomial of degree at most $q-1$.

Note that the theorem only asserts that the limit on the right exists, but it does not indicate whether the products are convergent or divergent.

In fact if $f(z)$ only satisfies the weaker condition that lower limit as r tends to infinity of $T(r)/r^q$ is equal to zero (instead of the condition assumed in the theorem viz., upper limit of the same is zero.), the result still holds provided that R is allowed to tend to infinity through a suitable sequence of values, instead of all values.

To start with let us assume that the function $f(z)$ has no zero or pole 33 at zero. The Poisson-Jensen formula gives,

$$\log|f(z)| = \frac{1}{2}\pi \int_0^{2\pi} \frac{(R^2 - r^2)\log|f(Re^{i\phi})|}{R^2 - 2Rr\cos(\phi-\theta) + r^2} d\phi + \sum_{|a_\mu|<R} \log\left|\frac{R(z-a_\mu)}{(R^2 - \overline{a}_\mu z)}\right|$$

$$z = re^{i\theta} \qquad\qquad + \sum_{|b_\nu|<R} \log\left|\frac{R^2 - \overline{b}_\nu z}{R(z-b_\nu)}\right|$$

Both the sides are equal harmonic functions $v(z)$ (say) of z near the point $z = re^{i\theta}$ where $f(re^{i\theta}) \neq 0, \infty$. Let us operate $\frac{\partial v}{\partial x} - i\frac{\partial v}{\partial y}$ on both the sides. We assume that R is such that $f(z) \neq 0, \infty$ on $|z| = R$. Differentiating under the integral sign and observing that

$$\text{Real}\left(\frac{Re^{i\phi} + z}{Re^{i\phi} - z}\right) = \frac{(R^2 - r^2)}{R^2 - 2Rr\cos(\phi-\theta) + r^2}$$

We deduce,

$$\begin{aligned}\frac{f'(z)}{f(z)} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\phi})| \frac{2Re^{i\phi}}{(Re^{i\phi} - z)} d\phi \\ - \sum_{|a_\mu|<R}\left[\frac{1}{(a_\mu - z)} - \frac{\overline{a}_\mu}{(R^2 - \overline{a}_\mu z)}\right] \\ + \sum_{|b_\nu|<R}\left[\frac{1}{(b_\nu - z)} - \frac{\overline{b}_\nu}{(R^2 - \overline{b}_\nu z)}\right]\end{aligned}$$

34 provided that there are no zeros or poles on $|z| = R$. Differentiating this $q - 1$ times,

$$\begin{aligned}\left(\frac{d}{dz}\right)^{q-1}\left(\frac{f'(z)}{f(z)}\right) = \frac{1}{\pi}\int_0^{2\pi} \frac{\log|f(Re^{i\phi})|Re^{i\phi}}{(Re^{i\phi} - z)^{q+1}} \\ + (q-1)\sum_{|b_\nu|<R}\left(\frac{1}{(b_\nu - z)^q} - \frac{\overline{b}_\nu^q}{(R^2 - \overline{b}_\nu z)^q}\right)\end{aligned}$$

$$+ (q-1) \sum_{|a_\mu|<R} \left(\frac{1}{(a_\mu - z)^q} - \frac{\overline{a}_\mu^q}{(R^2 - z\overline{a}_\mu)^q} \right)$$

Suppose now that $T(2r)/(r)^q$ tends to zero as r tends to ∞ either through all values or through a suitable sequence of values, say R_k (which tends to ∞ with k). Such a sequence of values exists by our assumptions. Also by decreasing R_k slightly if necessary we may assume that $f(z) \neq 0, \infty$ on $z = R_k$, we take $R = R_k$ in (1).

Then since $m(r, f) \geq 0$,

$$T(2R_k, f) \geq N(2R_k, f) \geq \int_{R_k}^{2R_k} n(t, f)\frac{dt}{t} \geq n(R_k, f) \log 2.$$

Thus $n(R_k, f)R_k^q$ tends to zero as k tends to ∞. Similarly, $n(R_k, 1/f)/R_k^q \to 0$, as $k \to \infty$) since, $T(R_k, 1/f) = T(R_k, f) + 0(1)$ and $T(R_k, f)/R_k^q \to 0$. Our aim now is to show that some of the terms on the right of the equation 11.9 including the integral tend to zero uniformly for z on any bounded set as k tends to infinity. Then letting k tend to infinity, we get a modified equation integrating which q times we will get the result.

Now suppose that $|z| < \frac{1}{2}R_k$. Then, $|\overline{b}_\nu z| < \frac{1}{2}R_k \cdot R_k$ for $|b_\nu| < R_k$. **35**
and

$$|R_k^2 - \overline{b}_\nu z| \geq R_k^2 - |b_\nu z| \geq \frac{1}{2}R_k^2$$

Hence,

$$\left| \frac{\overline{b}_\nu^q}{(R_k^2 - \overline{b}_\nu z)} \right| < \frac{R_k^q}{(\frac{1}{2}R_k^2)^q} = \frac{2^q}{R_k^q}$$

this inequality being true for all poles b_ν with $|b_\nu| < R_k$. Therefore, summing up for all poles b_ν, $|b_\nu| < R_k$

$$\left| \sum \frac{\overline{b}_\nu^q}{(R_k^2 - \overline{b}_\nu z)} \right| \leq \sum \left| \frac{\overline{b}_\nu^q}{(R_k^2 - \overline{b}_\nu z)^q} \right| < \frac{n(R_k, f)2^q}{R_k^q}$$

and hence the right hand side tends to zero uniformly as k tends to infinity for z in any bounded set. A similar result holds good in the case of the zeros of the function.

Now coming to the integral,

$$|R_k e^{i\phi} - z)| \geq \frac{1}{2} R_k \quad \text{for} \quad |z| < \frac{1}{2} R_k.$$

Hence the modulus of the integral on the right of (1.9) is at most,

$$\begin{aligned}
&(q!)\frac{R_k}{\pi}\frac{2^{q+1}}{R_k^{q+1}}\int_0^{2\pi}\left|\log|f(R_k e^{i\phi})|\right| d\phi\\
&\frac{(q!)}{\pi}\frac{2^{q+1}}{R_k^q}\left[\int_0^{2\pi}\log^+|f(Re^{i\phi})|d\phi + \int_0^{2\pi}\log^+\frac{1}{|f(Re^{i\phi})|}d\phi\right]\\
&< (q!)\frac{2^{q+2}}{R_k^q}[m(R_k, f) + m(R_k, 1/f)]\\
&< (\text{Const.})\frac{T(R_k, f)}{R_k^q} < (\text{Const.})\frac{T(2R_k, f)}{R_k^q} \to 0\\
&\qquad\qquad\qquad\qquad \text{as } k \to \infty.
\end{aligned}$$

36 Now the equation (1.9) takes the form,

$$\left(\frac{d}{dz}\right)^{q-1} \frac{f'(z)}{f(z)} = \lim_{k\to\infty} S_k(z)$$

Where

$$S_k(z) = (q-1)!\left\{\sum_{|b_\nu|<R_k}\frac{1}{(b_\nu - z)}q - \sum_{|a_\mu|<R_k}\frac{1}{(a_\mu - z)}q\right\}$$

the convergence being uniform for any bounded set of values of z not containing any of the zeros or poles of $f(z)$.

By the uniform convergence we may therefore, integrate both sides $(q-1)$ times along a suitable path from 0 to z to get,

$$\frac{f'(z)}{f(z)} = \lim_{k\to\infty}\left[\sum_{|b_\nu|<R_k}\left\{\frac{1}{(b_\nu - z)} - \frac{1}{b_\nu} - \frac{z}{b_\nu^2} - \cdots - \frac{z^{q-2}}{b_\nu^{q-1}}\right\}\right.$$

$$- \sum_{|a_\mu|<R_k} \left\{ \frac{1}{(a_\mu - z)} - \frac{1}{a_\mu} - \cdots - \frac{z^{q-2}}{a_\mu^{q-1}} \right\} \Bigg] + p_{q-2}(z)$$

where $P_{q-2}(z)$ is a polynomial of degree at most $q-2$. Now integrate both the sides once more from 0 to z, and take exponentials,

$$f(z) = P_{q-1}(z) \lim_{k\to\infty} \frac{\prod\limits_{|a_\mu|<R_k} \left(1 - \dfrac{z}{a_\mu}\right) e^{\frac{z}{a_\mu} + \frac{z^2}{2a_\mu^2} + \cdots + \frac{z^{q-1}}{(q-1)a_\mu^{q-1}}}}{\prod\limits_{|b_\nu|<R_k} \left(1 - \dfrac{z}{b_\nu}\right) e^{\frac{z}{b_\nu} + \cdots \frac{1}{q-1}\left(\frac{z}{b_\nu}\right)^{q-1}}}$$

Hence the result in the case when $f(0) \neq 0, \infty$. In case zero is a pole or **37**
zero of the function of order p, consider the function $f(z)/z^p$ and apply the result just obtained to get the theorem in its final form.

1.9 Convergence of Weierstrass products **38**

Let $a_1, a_2, \ldots a_n \ldots$ be a sequence of complex numbers (none 0) with moduli $r_1, r_2, \ldots, r_n, \ldots$ in the increasing order of magnitude. Let $n(r)$ be defined as

$$n(r) = \operatorname*{Sup.}_{r_k<r} k.$$

Then follows the result:

Lemma. *If* $N(r) = \int\limits_0^r \frac{n(t)}{t} dt$ *then* $N(r)$ *and* $n(r)$ *have the same order and type; and for any* k, *such that* $0 < k < \infty$, *the series* $\sum 1/r_n^k$ *and the integrals* $\int\limits_0^\infty \frac{n(r)}{r^{k+1}} dr$, *and* $\int\limits_0^\infty N(r) \frac{dr}{r^{k+1}}$ *converge or diverge together.*

Proof. By Riemann-Stieltjes' integrals,

$$\sum_{r_n<R} 1/r_n^k = \int_0^R \frac{1}{t^k} dn(t)$$

$$= k \int_0^R n(t) \frac{dt}{t^{k+1}} + n(R)/R^k \text{ because } n(t) = 0$$

near 0. Suppose now $I_1 = \int_0^\infty n(R) \frac{dR}{R^{k+1}}$ is less than infinity. $n(R)$ has at most convergence type of order k. (Hence it is also of minimal type). Therefore upper limit of $n(r)/r^k$ as r tends to infinity = zero, and $\sum 1/r_n^k$ converges to $k(I_1)$. The convergence of the series implies the convergence of the integral, because $n(R)/R^k \geq 0$. □

39 Now consider the integral,

$$\int_0^R N(r) \frac{dr}{r^{k+1}} = N(R)/(-k)R^k + \int_0^R dN(r)/kr^k$$

$$= -N(R)/kR^k + \int_0^R \frac{n(r)}{k} dr/r^{k+1}$$

From this inequality we get that the integrals $\int_0^\infty N(r)dr/r^{k+1}$ and $\int_0^\infty n(r)dr/r^{k+1}$ will converge or diverge together, due to the following reason. The convergence of $\int_0^\infty N(r)\frac{dr}{r^{k+1}}$ implies that $\overline{\lim\limits_{R\to\infty}} N(R)/R^k = 0$, hence the convergence of $\int_0^\infty n(r)\frac{dr}{r^{k+1}}$ follows. On the other hand if $\int_0^\infty n(r)dr/r^{k+1}$ converges $\int_0^R N(r)\frac{dr}{r^{k+1}}$ at once by comparison.

Therefore it remains to be proved that $n(r)$ and $N(r)$ have the same order and type. Suppose $n(r)$ has order k, then given $\delta > 0$, $n(r) < cr^{k+\delta}$ for r greater than r_0

$$N(r) = \int_0^r n(t) \frac{dt}{t}$$

$$= \int_0^{r_0} n(t)\frac{dt}{t} + \int_{r_0}^{r} n(t)\frac{dt}{t} \quad \text{for} \quad r > r_0$$

$$N(r) \leq \int_0^{r_0} n(t)\frac{dt}{t} + \int_{r_0}^{r} cr^{k-1^+}\, dr \;\; \text{for} \;\; r > r_0$$

$$\leq \;\; \text{constant} \;\; + cr^{k+\delta}/(k+\delta) \;\; \text{for} \;\; r > r_0.$$

This implies that order of $N(r)$ is not greater than that of $n(r)$, and if $n(r)$ has got mean or minimal type so has $N(R)$. The result for convergence or divergence class follows from earlier inequalities. The estimate **40** for $n(r)$ in terms of $N(r)$ follows from

$$n(r) \leq (1/\log 2)\int_r^{2r} n(t)\frac{dt}{t} \leq N(2r)/\log 2.$$

From this inequality it can be derived in the same way as before, that the order of $n(r)$ is not greater than that of $N(r)$ etc.

Definition . *The order and type of $n(r)$ or $N(r)$ (being the same) are called the order and type of the sequence (a_n). The order of $n(r)$ is also sometimes called* exponent of convergence *of the sequence.*

We note that $\sum 1/r_n^k$ converges if and only if $n(r)$ has order less than k, or order k of convergence type. (This is a consequence of lemma 3)

Next let us state the theorem,

Theorem 6. *Suppose that a_n is a sequence having at most order $q+1$ (a positive integer) convergence class. Then the product $\pi(z) = \prod_{n=1}^{\infty} E(z/a_n, q)$ converges absolutely and uniformly in any bounded region, and for $|z| = r$,*

$$\log|\pi(z)| < A(q)\left\{ r^q \int_0^r n(t)\frac{dt}{t^{q+1}} + r^{q+1}\int_0^{\infty} n(t)\frac{dt}{t^{q+2}} \right\}$$

Proof.

$$E(u,q) = (1-u)e^{u+\frac{1}{2}u^2+\cdots+\frac{u^q}{q}}$$

$$\log\ E(u,q) = \log(1-u) + u + \frac{1}{2}u^2 + \cdots + \frac{u^q}{q}$$

$$= -\sum_{q+1} u^k/k$$

41 so that $|\log E(u,q)| - \sum\limits_{q+1} \dfrac{|u|^k}{k} \leq 2|u|^{q+1}$ if $|u| \leq \dfrac{1}{2}$ since $\log|E(u,q)|$ is the real part of $\log E(u,q)$ we get $\log|E(u,q)| \leq q+1$ if $|u| \leq \dfrac{1}{2}$. Suppose $\dfrac{1}{2} \leq |u| \leq 1$. Then

$$\log|E(u,q)| \leq \log|(1-u)| + |u| + \frac{1}{2}|u|^2 + \cdots + \frac{|u|^q}{q}$$

$$\leq |u| + |u| + \frac{1}{2}|u|^2 + \cdots + \frac{|u|^q}{q}.$$

Also $|u| \geq \dfrac{1}{2}$, $|u|^{q-1} \geq 1/2^{q-1}$, $2^{q-1}|u|^{q-1} \geq |u|$. So that since $|u| \leq 1$, $|u|^q \leq |u| \leq 2^{q-1}|u|^q$ and

$$\log|E(u,q)| \leq 2^{q-1}|u|^q + q2^{q-1}|u|^q = A(q)|u|^q < 2A(q).$$

$A(q)$ being a constant depending only on q.

Thus for $|u| \leq 1$, $\log|E(u,q)| \leq A(q)|u|^{q+1}$. □

Let

$$|u| \geq 1, \quad \text{then} \quad \log|E(u,q)| \leq |u| + |u| + \frac{1}{2}|u|^2 + \cdots + \frac{u^q}{q}$$

$$\leq (q+1)|u|^q$$

Now $u = z/z_n$. Let $|z| = r$, $|z_n| = r_n$ and N the least integer for which $r_n \geq r$. Then $|u| \geq 1$. Thus

$$\sum_{n=1}^{N-1} \log|E(z/z_n,q)| \leq A(q)r^q \sum_{1}^{N-1} r_n^{-q}$$

and

$$\sum_{n+1}^{\infty} \log E(z/z_n, q) \leq B(q) r^{q+1} \sum_{n+1}^{\infty} r_n^{-q-1}$$

Thus

$$\begin{aligned}
\sum_{1}^{\infty} \log |E(z/z_n, q)| &\leq C \left[r^q \sum_{1}^{N-1} r_n^{-q} + r^{q+1} \sum_{N}^{\infty} r_n^{-q-1} \right] \\
&= C \left[r^q \int_0^r \frac{1}{t^q} dn(t) + r^{q+1} \int_r^{\infty} \frac{1}{t^{q+1}} dn(t) \right] \\
&= K \left[r^q \int_0^r \frac{1}{t^{q+1}} n(t) dt + r^{q+1} \int_r^{\infty} \frac{1}{t^{q+2}} n(t) dt \right]
\end{aligned}$$

K depending only on q. This proves the theorem. 42

We also see that for large n,

$$\left| \log E\left(\frac{z}{z_n}, q\right) \right| < A(q) \left(\frac{r}{r_n}\right)^{q+1},$$

and, the product converges since $\sum r_n^{-(q+1)}$ converges. Our Theorem 6 shows that if in Theorem 5 $f(z)$ has at most order q convergence class, then the two products converge separately, uniformly and absolutely on every bounded set.

As a consequence of the theorem we have

Theorem 7. *If a sequence a_n defined as in the last theorem, has order ρ, $q - 1 \leq \rho < q$, q an integer, then $\pi_a(z) = \prod_{n=1}^{n=\infty} E(z/a_n, q - 1)$ has order ρ and further if ρ is not an integer, $\prod_a(z)$ has the same type class as (a_n).*

Hence if $f(z)$ is meromorphic of finite non-integral order, then the roots of $f(z) = a$, have the same order and type class as $f(z)$ except for at most one value of $\underline{a}$ on the Riemann sphere.

Proof. If $n(t) < ct^{\rho+\epsilon}$ for $t > t_0$ where $\epsilon > 0$

$$r^{q-1}\int_0^r n(t)\frac{dt}{t^q} \leq r^{q-1}\int_0^{t_0} n(t)dt/t^q + r^{q-1}\int_{t_0}^r ct^{\rho+\epsilon}dt/t^q$$
$$-0(r^{q-1}) + C\frac{r^{\rho+1-q+\epsilon}}{\rho+1+-q+\epsilon}r^{q-1} = 0\left(r^{q-1} + \frac{Cr^{\rho+\epsilon}}{\rho+\epsilon+1-q}\right)$$

43 and the result regarding the order of the first integral follows. The second integral is treated similarly and then the first part follows from Theorem 6.

If $\rho > q-1$ and $f(z)$ is of mean type or minimal type of order ρ we can take $\epsilon = 0$, and in the case of minimal type c small and the results for the type at once follow. Suppose for instance that $n(t)$ has order less than q so that $n(t) < cr^{\rho+\epsilon}$ for $t > t_0$. Then for large r,

$$r\int_r^\infty \frac{n(t)dt}{t^{q+1}} < cr^q\int_r^\infty \frac{t^{\rho+\epsilon}}{t^{q+1}}dt = \frac{cr^{\rho+\epsilon}}{(q-\rho-\epsilon)} \quad \text{if} \quad \epsilon < q-\rho.$$

and the integral is $0(r^{\rho+\epsilon})$ and is $0(r^\rho)$ if $n(r) = 0(r^\rho)$ as required. □

Suppose now that $f(z)$ is a meromorphic function of finite non-integral order ρ and q such that $q-1 < \rho < q$. Then,

$$f(z) = e^{P_{q-1}(z)}\Pi_1(z)/\Pi_2(z)$$

where $\Pi_1(z)$ and $\Pi_2(z)$ are the products over the zeros and poles respectively. Now if both have a smaller order and type than $f(z)$, so does their
44 ratio. since

$$\begin{aligned} T(r, \Pi_1/\Pi_2) &\leq T(r, \Pi_1) + T(r, 1/\Pi_2) \\ &= T(r, \Pi_1) + T(r, \Pi_2) + 0(1). \end{aligned}$$

and $e^{P_{q-1}(z)}$ has order $q-1 < \rho$, so we should get a contradiction. Hence at least one of the $\Pi_1(z)$ or $\Pi_2(z)$ and so either zeros or the poles have the same order and type as the function $f(z)$.

If for instance the poles do not have the order and type of $f(z)$ then since $f(z) - a$ has the same poles as $f(z)$ (for every finite a), the roots of $f(z) = a$ that is the zeros of $f(z) - a$ have the order and type of $f(z)$.

This kind of argument was used to show that Riemann-Zeta function has an infinity of zeros.

We now illustrate the above by means of an example.

If $f(z)$ is meromorphic in the plane and $\varliminf_{\mu\to\infty} T(r, f)/\log r^{<+\infty}$ then $f(z)$ is rational.

In this case $f(z)$ has lower order zero, and lower limit of $N(r, f)/\log r$ as r tends to ∞ is less than ∞. $[T(r, f) \geq N(r, f)]$.

Similarly,

$$\varliminf_{\nu\to\infty} \frac{N(r, 1/f)}{\log r} < \infty$$

further,

$$N(R, f) \geq \int_r^R n(t, f)\frac{dt}{t} \geq n(r, f)\log\frac{R}{r}$$

so that $n(r, f) \leq \dfrac{N(r^2, f)}{\frac{1}{2}\log r} = 0(1)$ for a sequence of $n \to \infty$. So $f(z)$ has only a finite number of zeros and poles. Hence from theorem 5 **45**
$f(z) = z^p(\Pi(1 - z/a_\mu)/(1 - z/b_\nu)$ where both the products are finite. Thus for all transcendental meromorphic functions $T(r, f)/\log r$ tends to infinity as r tends to infinity.

Part II

Nevanlinna's Second Fundamental Theorem

2.1 46

As the name indicates this theorem is most fundamental in the study of meromorphic functions. It is an extension of Picard's Theorem, but goes much farther. We develop the theorem in theorems 8 and 9 of this chapter, and then proceed systematically to explore some of its consequences.

Theorem 8. *Suppose that $f(z)$ is a non-constant meromorphic function in $|z| \leq r$. Let $a_1, a_2, \ldots, a_q$ be distinct finite complex numbers, $\delta > 0$ such that $|a_\mu - a_\nu| \geq \delta$ for $1 \leq \mu \leq \nu \leq q$. Then,*

$$m(r, f) + \sum_{\nu=1}^{\nu=q} m(r, a_\nu) \leq 2T(r, f) - N_1(r) + S(r)$$

where $N_1(r)$ is positive and is given by

$$N_1(r) = N(r, 1/f') + 2N(r, f) - N(r, f')$$

and

$$S(r) = m(r, f'/f) + m\left(r, \sum_{\nu=1}^{\nu q} f'/(f - a_\nu)\right)$$

$$+ q\log^+\left(\frac{3q}{\delta}\right) + \log 2 + \log^+ 1/|f'(0)|$$

with modifications if $f(0) = 0$ *or* ∞, $f'(0) = 0$.

Proof. Let a new function F be defined as follows

$$F(z) = \sum_{\nu=1}^{\nu=q} 1/[f(z) - a_\nu]$$

and suppose that for some ν, $|f(z) - a_\nu| < \delta/3q$. Then for $\mu \neq \nu$,

$$\begin{aligned} |f(z) - a_\mu| &\geq |a_\mu - a_\nu| - |f(z) - a_\nu| \\ &\geq \delta - \delta/3q \\ &> 2\delta/3. \qquad (q \geq 1) \end{aligned}$$

Therefore,

$$\begin{aligned} 1/|f(z) - a_\mu| &< 3/2\delta \quad \text{for} \quad \mu \neq \nu \\ &< (1/2q)[1/|f(z) - a_\nu|] \end{aligned}$$

47 Again,

$$\begin{aligned} |F(z)| &\geq 1/|f(z) - a_\nu| - \sum_{\mu\neq\nu} 1/f(z) - a_\mu| \\ &\geq [1/|f(z) - a_\nu|][1 - (q-1)/2q] \\ &\geq 1/2|f(z) - a_\nu| \end{aligned}$$

Hence

$$\log^+|F(z)| \geq \log^+ 1/|f(z) - a_\nu| - \log 2.$$

In this case,

$$\begin{aligned} \log^+|F(z)| &\geq \sum_{\mu=1}^{q} \log^+ 1/|f(z) - a_\mu| - q\log^+ 2/\delta - \log 2 \\ &\geq \sum_{\mu=1}^{q} \log^+ \frac{1}{|f(z) - a_\mu|} - q\log^+ 3q - \log 2. \end{aligned}$$

since all the term for $\mu \neq \nu$ are at most $\log^+ 2/\delta$. This is true if $|f(z) - a_\nu| < \delta/3q$ for some $\nu \leq q$. This inequality is true evidently for at most one ν (with the condition $|a_\mu - a_\nu| \geq \delta$). If it is not true for any value then we have trivially,

$$\log^+ |F(z)| \geq \sum_{\nu=1}^{q} \log^+ 1/f(z) - a_\nu| - q \log^+ 3q/\delta - \log 2$$

(because, $\log^+ |F(z)| \geq 0$). So the last relationship holds good in all cases. □

Taking integrals we deduce,

$$m(r, F) \geq \sum_{\nu=1}^{q} m(r, 1/f - a_\nu) - q \log^+ \frac{3q}{\delta} - \log 2 \tag{2.1}$$

Again to get an inequality in the other direction,

$$m(r, F) = m\left(r, \frac{1}{f}\frac{f}{f'}f'F\right)$$

By equation (1.2′) of 1.2 i.e. ,

$$\begin{aligned} T(r, f) &= T(r, 1/f) + \log |f(0)| \\ m(r, f/f') &= m(r, f'/f) + N(r, f'/f) - N(r, f/f') + \log[|f(0)|/|f'(0)|] \\ m(r, 1/f) &= T(r, f) - N(r, 1/f) + \log 1/|f(0)| \end{aligned}$$

Hence we get finally, 48

$$\begin{aligned} m(r, F) \leq{} & T(r, f) - N(r, 1/f) + \log 1/|f(0)| + m(r, f'/f) \\ & + N(r, f'/f) - N(r, f/f') + \log |f(0)|/|f'(0)| + m(r, f'F) \end{aligned}$$

The above inequality, combined with (2.1) gives $\sum\limits_{\nu=1}^{q} m(r, a_\nu) - q \log^+ 3q/\delta - \log 2 \leq$ Right hand side of the above inequality. Add to both the sides $m(r, f)$, we get the inequality,

$$m(r, f) + \sum_{\nu=1}^{q} m(r, a_\nu) \leq T(r, f) + [m(r, f) + N(r, f)] - N(r, f)$$

$$
\begin{aligned}
&+ m(r, f'/f) + \log 1/|f'(0)| + \log 2\\
&+ m\left[r, \sum_{\nu=1}^{q} f'/(f-a_\nu)\right] + q\log^+ 3q/\delta - N(r, 1/f)\\
&+ N(r, f'/f) - N(r, f/f').\\
&= 2T(r, f) - N_1(r) + m(r, f'/f) + \log \frac{2}{|f'(0)|}\\
&+ q\log^+ \frac{3q}{\delta} + m\left[r, \sum_{\nu=1}^{q} \frac{f'}{(f, a_\nu)}\right]
\end{aligned}
$$

where, $N_1(r) = N(r, f) + N(r, 1/f) + N(r, f/f') - N(r, f'/f)$. Now for any two functions $f(z)$ and $g(z)$ Jensen's formula gives

$$
\begin{aligned}
N(r, f/g) - N(r, g/f) &= \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{g(re^{i\theta})}{f(re^{i\theta})}\right| d\theta - \log|g(0)/f(0)|\\
&= \frac{1}{2\pi}\int_0^{2\pi} \log|g(re^{i\theta})| d\theta - \log|g(0)|\\
&\quad + \frac{1}{2\pi}\int_0^{2\pi} \log\frac{1}{|f(re^{i\theta})|} d\theta + \log|f(0)|\\
&= N(r, 1/g) - N(r, g) + N(r, f) - N(r, 1/f)
\end{aligned}
$$

49 Thus

$$
\begin{aligned}
N_1(r) &= N(r, f) + N(r, 1/f) + N(r, 1/f')\\
&\quad + N(r, f) - N(r, f') - N(r, 1/f)\\
&= 2N(r, f) + N(r, 1/f') - N(r, f')
\end{aligned}
$$

as required.

2.2 Estimation of the error term

We shall firstly prove some lemmas.

Lemma 1. *Suppose that $f(z)$ is meromorphic in the $|z| \leq R$ and that $0 < r < R$. Let $\rho = \frac{1}{2}(r + R)$ and $\delta(z)$ the distance of z from the nearest pole or zero of $f(z)$ in $|z| < \rho$. Then,*

$$m(r, f'/f) < \log^+ T(R, f) + 2\log^+[1/(R - r)] + 2\log^+ R + \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{\delta(re^{i\theta})} d\theta + 0(1)$$

$0(1)$ *depending only on the behaviour of $f(z)$ at $z = 0$.*

Proof. We have by differentiation of Poisson formula as in theorem 5,

$$f'(z)/f(z) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(\rho e^{i\phi})| \frac{2\rho e^{i\phi}}{(\rho e^{i\phi} - z)^2} d\phi + \sum_\mu \left[\frac{\overline{a}_\mu}{(\rho^2 - \overline{a}_\mu z)} - \frac{1}{(a_\mu - z)}\right] + \sum_\nu \left[\frac{1}{(b_\nu - z)} - \frac{\overline{b}_\nu}{(\rho^2 - \overline{b}_\nu z)}\right]$$

where the sums as usual run over the zeros a_μ and poles b_ν in $|z| < \rho$. From $|\rho^2 - \overline{a}_\mu z| \geq \rho^2 - \rho|z| = \rho^2 - r\rho = (\rho - r)\rho$ for $|z| = r$ we get $\dfrac{|\overline{a}_\mu|}{|\rho^2 - \overline{a}_\mu z|} \leq \dfrac{\rho}{\rho^2 - \rho r} = \dfrac{1}{(\rho - r)}$ and by definition of $\delta(z)$

$$\left|\frac{1}{(a_\mu - z)}\right| \leq \frac{1}{\delta(z)}, \left|\frac{1}{b_\nu - z}\right| \leq \frac{1}{\delta(z)}$$

Hence 50

$$\left|\sum \left[\frac{\overline{a}_\mu}{(\rho^2 - \overline{a}_\mu z)} - \frac{1}{(a_\mu - z)}\right] + \sum \left[\frac{1}{(b_\nu - z)} - \frac{\overline{b}_\nu}{\rho^2 - b_\nu z}\right]\right| \leq [n(\rho, f) + n(\rho, 1/f)]\left[\frac{1}{\delta(z)} + \frac{1}{(\rho - r)}\right]$$

We now estimate $n(\rho, f)$ and $n(\rho, 1/f)$. For this we have,

$$\int_\rho^R n(t, f)dt/t \leq N(R, f) \leq T(R, f)$$

and since $n(t, f)/t \geq n(\rho, f)/R$ for t greater than ρ, $\int\limits_{\rho}^{R} n(t, f)dt/t \geq n(\rho, f)(R - f)/R$. Therefore $[n(\rho, f)](R - \rho)/R \leq T(R, f)$ or $n(\rho, f) \leq RT(R, f)/(R - \rho) = 2RT(R, f)/(R - r)$ since $2\rho = (R + r)$.

Similarly $n(\rho, 1/f) \leq 2RT(R, 1/f)/(R - r) = 2R[T(R, f) + 0(1)]/(R - r)$. 0(1) depends only on the behaviour of $f(z)$ at $z = 0$. Thus

$$n(\rho, f) + n(\rho, 1/f) \leq \frac{4R}{(R - r)}[T(R, f) + 0(1)]$$

and so

$$\begin{aligned}\left|\sum\left[\frac{\overline{a}_\mu}{\rho^2 - \overline{a}_\mu z} - \frac{1}{(a_\mu - z)}\right] + \sum\left[\frac{1}{(b_\nu - z)} - \frac{\overline{b}_\nu}{(\rho^2 - b_\nu z)}\right]\right| \\ \leq \frac{4R}{(R - r)}[T(R, f) + 0(1)]\left[\frac{1}{\delta(z)} + \frac{2}{(R - r)}\right]\end{aligned}$$

Further,

$$\begin{aligned}\frac{1}{2\pi}\left|\int\limits_0^{2\pi} \log|f(\rho e^{i\phi})|\frac{2\rho e^{i\phi}}{(\rho e^{i\phi} - z)^2}d\phi\right| \\ \leq \frac{1}{2\pi}\frac{2\rho}{(\rho - r)^2}\int\limits_0^{2\pi} \log|f(\rho e^{i\phi})|d\phi\end{aligned}$$

51 for,

$$\begin{aligned}|\rho e^{i\phi} - z| &\geq ||\rho e^{i\phi}| - |z|| = (\rho - r) \\ &= \frac{1}{\pi}\frac{4\rho}{(R - r)^2}\left[\int\limits_0^{2\pi} \log^+|f(\rho e^{i\phi})d\phi + \int\limits_0^{2\pi} \log^+\frac{1}{|f(e^{i\phi}\rho)|}d\phi\right] \\ &= \frac{8\rho}{(R - r)^2}[m(\rho, f) + m(\rho, 1/f)] \leq \frac{8\rho}{(R - r)^2}[2T(\rho, f) + 0(1)] \\ &= \frac{16\rho}{(R - r)^2}[T(\rho, f) + 0(1)] \leq \frac{16R}{(R - r)^2}[T(R, f) + 0(1)]\end{aligned}$$

0(1) depending only on the behaviour of $f(2)$ at $z = 0$. Thus from the above inequalities and the equation for $f'(z)/f(z)$ we get finally,

$$
\begin{aligned}
|f'(z)/f(z)| &\leq \frac{4R}{(R-r)}[T(R,f)+0(1)]\left[\frac{2}{(R-r)}+\frac{1}{\delta(z)}\right]\\
&\quad + \frac{16R}{(R-r)^2}[T(R,f)+0(1)]\\
&= \frac{4R}{(R-r)^2}\left[6T(R,f)+T(R,f)\frac{(R-r)}{\delta(z)}+0(1)+0(1)\frac{R-r}{\delta(z)}\right]\\
&\leq \frac{4R}{(R-r)^2}\left[6+\frac{R-r}{\delta(z)}\right][T(R,f)+0(1)]
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\log^+|f'(z)/f(z)| \leq \log^+\frac{4R}{(R-r)^2}+\log^+\left(6+\frac{R-r}{\delta(z)}\right)+\log^+T(R,f)\\
&+0(1) \leq \log^+ R\, 2\log^+\frac{1}{(R-r)}+\log^+T(R,f)\log^+\frac{(R-r)}{\delta(z)}+0(1)\\
&\leq 2\log^+ R+\log^+\frac{1}{\delta(z)}+2\log^+\frac{1}{(R-r)}+\log^+T(R,f)+0(1)
\end{aligned}
$$

0(1) depending only on the behaviour of $f(z)$ at $z = 0$. Integrating the above inequality on the circle $|z| = r$, **52**

$$
\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi}\log|f'(re^{i\theta})/f(re^{i\theta})|d\theta &\leq 2\log^+ R+\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{\delta(re^i\theta)}d\theta\\
&+2\log^+\frac{1}{(R-r)}+\log^+T(R,f)+0(1)
\end{aligned}
$$

which gives the lemma. □

Lemma 2. *Let z be any complex number and $0 < r < \infty$. Let E_k be the set of all θ such that $|z - re^{i\theta}| < kr$ where $0 < k < 1$. Then*

$$\int_{E_k}\log[r/|(z-re^{i\theta})|]d\theta < \pi\left[\log\left(\frac{1}{k}\right)+1\right]$$

Proof. We may after rotation assume z real and positive. For if $z = 0$ E_k is obviously void and there is nothing to prove. So let $z > 0$. Then for θ in E_k $|z - re^{i\theta}| \geq |\operatorname{Im}(z - re^{i\theta})| = r\sin\theta$ and E_k is contained in an interval of the form $|\theta| < \theta_0$ where $r\sin\theta_0 \leq kr$, that is, $\sin\theta_0 \leq k$. So,

$$\int\limits_{E_k} \log[r/|re^{i\theta} - z|]d\theta \leq 2\int\limits_{\theta}^{\theta_0} \log^+ \frac{1}{\operatorname{Sin}\theta}d\theta$$

53 Now, $\theta < \dfrac{\pi}{2}$ for when $\dfrac{\pi}{2} \leq |\theta| \leq \pi$, $|z - re^{i\theta}| > |re^{i\theta}| = r$. Thus

$$\frac{\sin\theta}{\theta} \geq \frac{2}{\pi}|\theta| < \theta_0 \text{ we get } \int\limits_{E_k} \log[r/|re^{i\theta} - z|]d\theta \leq 2\int\limits_{0}^{\theta_0} \log\frac{\pi}{2\theta}d\theta$$

$$= 2\int\limits_{0}^{\theta_0} \log\frac{\pi}{2\theta}d\theta$$

because $\theta_0 < \dfrac{\pi}{2}$ or $\dfrac{\pi}{2\theta} > 1$.

$$\int\limits_{E_k} \log[r/|(re^{i\theta} - z)|]d\theta \leq 2\int\limits_{0}^{\theta_0} \log\frac{\pi}{2}d\theta - 2\int\limits_{0}^{\theta_0} \log\theta d\theta$$

$$= 2\theta_0\log\frac{\pi}{2} - 2\theta_0\log\theta_0 + 2\theta_0$$

□

Lemma 3. *With the hypothesis of lemma 1,*

$m(r, f'/f) \leq 3\log^+ T(R, f) + 4\log^+ R + 4\log^+[1/(R-r)] + \log^+(1/r) + 0(1)$

Proof. Notations being the same as in the proof of lemma 1, write, $[1/\delta(z)] = [r/\delta(z)] \cdot (1/r)$. Then $\log\dfrac{1}{\delta(z)} \leq \log\dfrac{1}{r} - \log\dfrac{r}{\delta(z)}$. So

$$\frac{1}{2\pi}\int\limits_{0}^{2\pi} \log^+ \frac{1}{\delta(re^{i\theta})}d\theta \leq \log^+\frac{1}{r} + \frac{1}{2\pi}\int\limits_{0}^{2\pi} \log^+ \frac{r}{\delta(re^{i\theta})}d\theta$$

Let E denote the set of θ in $(0, 2\pi)$ for which $\delta(re^{i\theta}) < \dfrac{r}{n^2}$ where $n = n(\rho, f) + n(\rho, 1/f)$. (If $n = 0$ so that there are no zeros and poles we can put $\delta(z) = +\infty$ and there is nothing to prove. So assume $n \geq 1$). For each point θ in E there is a zero or pole z_ν such that $\delta(re^{i\theta}) = |z_\nu - re^{i\theta}| < r/n^2$ **54**
and then

$$\log^+[r/\delta(z)] = \log^+[r/|re^{i\theta} - z_\nu|]$$

Now write $\log_0 x = \log x$ if $x \geq n^2$ and $\log_0 x = 0$ otherwise. Then since $n \geq 1$, $\log_0 x \geq 0$ always.

Also for θ in E $\log^+[r/\delta(z)] = \log^+[r/|(re^{i\theta} - z_\nu)|]$ for some ν Since, $1 < n^2 < [r/|re^{i\theta} - z_\nu|]$,

$$\log^+ \frac{r}{|re^{i\theta} - z_\nu|} = \log_0 \frac{r}{|re^{i\theta} - z_\nu|} \leq \sum_\mu \log_0 \frac{r}{|re^{i\theta} - z|}$$

where the sum is taken over all zeros and poles z_μ in $|z| < \rho$. Thus

$$\begin{aligned}\int_E \log^+ \frac{r}{\delta(re^{i\theta})} d\theta &\leq \sum_\mu \int_E \log_0 \frac{r}{|re^{i\theta} - z_\mu|} d\theta \\ &\leq \sum_\mu \frac{\pi}{n^2}[\log n^2 + 1] = \frac{\pi}{n}[2\log n + 1] \leq A.\end{aligned}$$

by lemma 2 with $k = 1/n^2$, and where A is an absolute constant.

Also on the complement of E, $\delta(re^{i\theta}) \geq r/n^2$ and,

$$\int_{\text{compl. of } E} \log^+ \frac{r}{\delta(re^{i\theta})} d\theta \leq \int_0^{2\pi} \log n^2 d\theta = 2\pi \log n^2$$

Adding we obtain

$$\frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{r}{\delta(re^{i\theta})} d\theta \leq [2\log n + A]$$

$$\frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{|\delta(re^{i\theta})|} d\theta = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{r} \frac{r}{|\delta(re^{i\theta})|} d\theta$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} \log \frac{r}{|\delta(re^{i\theta})|} d\theta + \log^+ \frac{1}{r}$$
$$\leq 2 \log^+ n + \log^+ 1/r + A.$$

55 where $n = n(\rho, f) + n(\rho, 1/f)$.

We have from (2.2)

$$n = n(\rho, f) + n(\rho, 1/f) \leq 4R[T(R, f) + 0(1)]/(R - r).$$

Hence,

$$\log^+ n \leq \log^+ R + \log^+ 1/(R - r) + \log^+ T(R, f) + 0(1),$$

giving

$$\frac{1}{2\pi} \int_0^{2\pi} \log \frac{1}{|\delta(re^{i\theta})|} d\theta \leq \log^+ \frac{1}{r} + 2 \log^+ \frac{1}{R - r} + 2 \log^+ R + 2 \log^+ T(R, f) + 0(1)$$

From this and lemma 1, lemma (3) follows. That is,

$$m(r, f'/f) < 3 \log^+ T(R, f) + 4 \log^+ 1/(R - r) + 4 \log^+ R + \log^+ 1/r\, 0(1).$$

□

Lemma 4 (Borel). (i) *Suppose $T(r)$ is continuous, increasing and $T(r) \geq 1$ for $r_0 \leq r < \infty$. Then we have*

(2.3) $$T[r + 1/T(r)] < 2T(r)$$

outside a set E of r which has length (that is) linear measure at most 2.

56 (ii) *If $T(r)$ is continuous and increasing for $r_0 \leq r < 1$ and $T(r) \geq 1$ then we have*

(2.4) $$T[r + (1 - r)/(eT(r))] < 2T(r)$$

outside a set E of r such that $\int_E dr/(1-r) \leq 2$. In particular $T[r+(1-r)/eT(r)] < 2T(r)$ for some r such that $\rho < r < \rho'$ if $r_0 < \rho < 1$ and $1-\rho' < (1-\rho)/e^2$.

Proof. We prove first (i) [that is in the plane]. Let r_1 be the lower bound of all r for which (2.3) is false. If there are no such r there is nothing to prove. We now define by induction a sequence of numbers r_n. Suppose that r_n has been defined and write $r'_n = r_n + 1/T(r_n)$. Define then r_{n+1} as the lower bound of all $r \geq r'_n$ for which (2.3) is false. We have already defined r_1 and so we obtain the sequence (r_n). Note that by continuity of $T(r)$ (2.3) is false for $r = r_n$, for $n = 1, 2, 3, \ldots$ that is r_n belongs to E_0, E_0 being the exceptional set. From the definition of r_{n+1} is follows that there are no points of E in (r'_n, r_{n+1}) and that the set of closed intervals $[r_n, r'_n]$ contains E_0. If there are an infinity of r_n, (r_n) cannot tend to a finite limit r. For then since $r_n < r'_n \leq r_{n+1}$, r'_n tends to r also. But $r'_n - r_n = 1/T(r_n)$ which is greater or equal to $1/T(r) > 0$, since $T(r)$ is increasing, for all m which is a contradiction. □

It remains to be shown that $\sum(r'_n - r_n) \leq 2$. Now $T(r'_n) = T[r_n + 1/T(r_n)] \geq 2T(r_n)$ since r_n belongs to E. And so $T(r_{n+1}) \geq T(r'_n) \geq 2T(r_n)$. Therefore, $T(r_{n+1}) \geq 2T(r_n) \geq \ldots \geq 2^n T(r_1) \geq 2^n$ since $T(r) \geq$ **57**
1. Thus $T(r_n) \geq 2^{n-1}$. Now $\sum(r'_n - r_n) = \sum 1/T(r_n) \leq \sum 2^{1-n} \leq 2$.

To prove part (ii) of the theorem, set $R \log[1/(1-r)]$ getting $r = 1 - e^{-R}$ and put $T(r) = \varphi(R) \cdot \varphi(R)$ then is continuous and increasing for $R_0 = \log[1/(1-r_0)] \leq R < \infty$ and $\varphi(R) \geq 1$. Apply then the first part to $\varphi(R)$. Then we have $\varphi[R + 1/\varphi(R)] < 2\varphi(R)$ for $R > \log[1/(1-r_0)]$ outside a set E of R such that $2 \geq \sum(R'_n - R_n) = \int_E dR = \int dr/(1-r)$. Translating back to r, $R' = R + 1/\varphi(R)$ becomes $\log[1/(1-r')] = \log[1/(1-r)] + 1/T(r)$. That is $(1-r') = (1-r)e^{-\frac{1}{T(r)}}$ and $T(r') < 2T(r)$. By the first mean value theorem $f(b) = f(a) + (b-a)f'(x)$, where $a \leq x \leq b$. Since $T(r)$ increases with r, $T[r + (1-r)/eT(r)] < 2T(r)$ outside the exceptional set E of r for which $\int_E dr/(1-r) \leq 2$. If E contains the whole of the interval $\rho < r < \rho'$ then $\int_\rho^{\rho'} dr/(1-r) \leq \int_E dr/(1-r) \leq 2$, that is $\log(1-\rho)/(1-\rho') \leq 2$, and so $(1-\rho)/(1-\rho') \leq e^2$ as required.

2.3

Theorem 9. *If $f(z)$ is meromorphic and non-constant in the plane and $S(r, f)$ denotes the error term in Theorem 8 then we have*

$$S(r, f) = O[\log T(r, f)] + O(\log r) \tag{2.5}$$

as r tends to infinity through all values if $f(z)$ has finite order, and through all values outside a set of finite linear measure otherwise.

(ii) *If $f(z)$ is meromorphic (non-constant) in $|z| < 1$ and the $\overline{\lim\limits_{r\to 1}}\{T(r, f)/\log[1/(1-r)]\} = \infty$, then we have $S(r, f) = O[T(r, f)]$ as r tends to one on a set E such that $\int\limits_E dr/(1-r) = \infty$.*

Proof. If $\varphi(z) = \prod\limits_{\nu=1}^{q}[f(z) - a_\nu]$ then

$$S(r, f) = m(r, f'/f) + m[r, \varphi'/\varphi] + O(1)$$

58 because $S(r, f) = m(r, f'/f) + m(r, \sum\limits_{\nu=1}^{q} f'/(f - a_\nu)) + O(1)$ and $\sum\limits_{\nu=1}^{q} f'/(f - a_\nu) = \dfrac{\varphi'}{\varphi}$ by logarithmic differentiation. Therefore from the lemma 3, for any $R > r$ we have,

$$\begin{aligned} S(r, f) \leq{}& 3\log^+ T(R, f) + 4\log^+ R + 4\log^+[1/(R-r)] \\ &+ \log^+(1/r) + 3\log^+ T(R, \varphi) \\ &4\log^+ R + 4\log^+ 1/(R-r) + \log^+(1/r) + 0(1) \end{aligned}$$

Also

$$T(r, \varphi) \leq \sum_{\nu=1}^{q} T(r, f - a_\nu) \leq T(r, f) + O(1), \ldots \tag{2.6}$$

Thus

$$\begin{aligned} S(r, f) \leq{}& 3(1+q)\log^+ T(R, f) + 8\log^+ R + 8\log^+[1/(R-r)] \\ &+2\log^+(1/r) + O(1) \end{aligned}$$

If r is greater than 1 (we are considering $S(r, f)$ for large r) $\log^+(1/r) = 0$. Now suppose that $f(z)$ is meromorphic of finite order so that $T(r, f) < r^K$ for r greater than r_0. Also choose $R = r^2$ and $r \geq 2$ so that $R - r > 1$.

Then $\log^+\left(\dfrac{1}{R-r}\right) = 0$. We get

$$\begin{gathered} S(r, f) \leq 3(1+q)\log^+ T(R, f) + 8\log^+ R + O(1), \\ \log^+ T(R, f) < k\log^+ R < 2k\log^+ r = 2k\log r \end{gathered}$$

since $R = r^2$ and $r > 1$. We thus have finally, $S(r, f) \leq 6(1+q)K\log r + 16\log r + O(1)$ showing that $S(r, f) = 0(\log r)$ which gives (2.5) since $\log^+ T(r, f) = O(\log r)$. $\square$

Note that by our examples $T(r, f)/\log r$ tends to infinity unless $f(z)$ is a rational function in which case $S(r, f)$ is bounded because $f'/f \to 0$ as $z \to \infty$ for any polynomial and hence for any rational function. Thus if $f(z)$ has finite order $S(r, f)/T(r, f) \to 0$ as $r \to \infty$. 59

If $f(z)$ has infinite order take $R = [r + 1/T(r)]$, then $\log^+ R \sim \log r$[since $T(r) \to \infty$] $\cdot \log^+ 1/(R-r) = \log^+ T(r)$ and $\log 1/r = 0$ finally. Outside the exceptional set of lemma 4, $T(R, f) < 2T(r, f)$ and so, $\log^+ T(R, f) \leq \log^+ T(r, f) + \log 2$. Hence again we have (2.5) outside the exceptional set. This completely proves i) In order to prove (ii) denote by r_n a sequence such that $T(r_n, f)/\log\left(\dfrac{1}{1-r_n}\right) \to \infty$ as $n \to \infty$ and by taking a sub sequence if necessary we can assume that $1 - r_{n+1} < \dfrac{1-r_n}{10}$. Then let r_n be defined by $1 - r'_n = (1-r_n)/10$ so that $r_n < r'_n < r_{n+1} < 1$. Then since $\displaystyle\int_{r_n}^{r'_n} \frac{dr}{1-r} = \log(1-r_n)/(1-r_n) = \log 10 > 2$ the union E_1 of all the intervals (r_n, r'_n) is such that $\displaystyle\int_{E_1} \frac{dr}{1-r} = +\infty$.

Further each such interval contains a point not in the exceptional set E, for $T(r, f)$ because by lemma 4 $\displaystyle\int_E dr/(1-r) \leq 2$, provided only that $T(r_1) > 1$. For a not exceptional point r of E_1 take

$R = r + (1-r)/eT(r)$, then

$$\log^+ \frac{1}{R-r} = \log^+ \frac{eT(r)}{1-r} < \log^+ T(r) + \log^+ \frac{1}{1-r} + 1$$

60 and $\log^+ T(R) < \log^+ T(r) + \log 2$, Thus by (2.6)

$$\begin{aligned} S(r, f) &< 3(1+q)\log^+ T(R, f) + 8\log^+ R + 8\log^+ \frac{1}{(R-r)} + O(1) \\ &< (3 + 3q + 8)\log^+ T(r) + 8\log^+ \frac{1}{(1-r)} + O(1) \end{aligned}$$

Also, $\log^+[1/(1-r)] < \log^+ 1/(1-r_n') + \log^+ \frac{10}{(1-r_n)} < \log^+ \frac{1}{1-r_n} + 0(1) = O[T(r_n)] + O(1) = O[T(r)]$.

So since $\log T(r, f) = O\ T(r)$ we get $S(r, f) = O\ T(r)$. This proves (ii) for a set E_1 of r such that $\int\limits_{E_1} \frac{dr}{1-r} = +\infty$ and containing at least one point in each interval $r_n < r < r_n'$. In face E comprises all the r in the sequence of intervals $[r_n, r_n']$, $n > 1$ except possibly a set E_0 such that $\int\limits_{E_0} \frac{dr}{1-r} \leq 2$.

2.4 Applications

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Definition. *Let $n(t, a)$ denote the number of roots of $f(z) = a$ in $|z| \leq t$, the multiple roots being counted according to their multiplicity and $\overline{n}(t, a)$ the number of roots of $f(z) = a$ in $|z| < t$ with the multiple roots counted simply. Further let $\overline{N}(t, a) = \int\limits_0^r \overline{n}(t, a) - \overline{n}(0, a)dt/t$. [$\overline{n}(0, a)$ is equal to one if $f(0) = a$ and zero otherwise]; and $N(r, a)$ as before with $n(t, a)$ instead of $\overline{n}(t, a)$. Let now the function $f(z)$ be meromorphic, and non-constant in $|z| < R$, $0 < R \leq \infty$ and suppose that $f(z)$ satisfies the hypotheses of theorem 9, so that $\varliminf\limits_{r \to R}[S(r, f)/T(r)] = 0$ and $T(r)$ tends to infinity as r tends to R.*

Now write $\delta(a) = \varliminf\limits_{r\to R}[m(r, a)/T(r)] = 1 - \varlimsup\limits_{r \to R}[N(r, a)/T(r)]$ because

$$[m(r, a) + N(r, a)]/T(r) = [T(r) + 0(1)]/T(r) \to 1;$$

thus

$$\varliminf_{r\to R}[m(r,a)/T(r)] = \varliminf_{r\to R} 1 + \frac{-N(r,a)}{T(r)} = 1 + \varliminf_{r\to R} -N(r,a)/T(r)$$
$$= 1 - \varlimsup_{r\to R} N(r,a)/T(r)$$

Again write,

$$\theta(a) = \varliminf_{r\to R}[N(r,a) - \overline{N}(r,a)]/T(r)$$

and

$$\Theta(a) = 1 - \varlimsup_{r\to R} \overline{N}(r,a)/T(r) = \varliminf_{r\to R}[1 - \overline{N}(r,a)/T(r)].$$

Clearly, $\delta(a)$, $\theta(a)$ and $\Theta(a)$ lie in the closed interval $[0,1]$. Also

$$1 - \frac{\overline{N}}{T} = 1 - \frac{N}{T} + \frac{N - \overline{N}}{T}$$
$$\varliminf_{r\to R} 1 - \frac{\overline{N}}{T} \geq \varliminf_{r\to R}\left(1 - \frac{N}{T}\right) + \varliminf_{r\to R}\left(\frac{N - \overline{N}}{T}\right)$$

i.e. $\Theta(a) \geq \delta(a) + \theta(a)$. **62**

The quantity $\delta(a)$ is called the *defect of a* and $\theta(a)$ the *Branching index* (Verzweigungsindex) of a. Now we have the defect relation of Nevanlinna. This is the second fundamental theorem in its most effective form and is very important in the theory.

Theorem 10. *If $f(z)$ satisfies the hypotheses of the Theorem 9, then $\Theta(a) = 0$ except possibly for a finite or countable sequence a_ν of values of a and for these $\sum \Theta(a_\nu) \leq 2$.*

Proof. From theorem 8, for any q distinct values a_ν, of a including $a_1 = \infty$

$$\sum_{\nu=1}^{q} m(r,a_\nu) < 2T(r,f) - N_1(r) + S(r)$$

and adding $\sum\limits_{\nu=1}^{q} N(r,a_\nu)$ to both sides and using first fundamental theorem $T(r,a) = T(r,f) + O(1)$, we get

$$qT(r,f) < 2T(r,f) - N_1(r) + \sum_{\nu=1}^{q} N(r,a_\nu) + S(r,f) + 0(1)$$

$$(q-2)T(r,f) < \sum_{\nu=1}^{q} N(r,a_\nu) - N_1(r) + S(r) + 0(1).$$

Now, $N_1(r) = 2N(r,f) - N(r,f') + N(r,1/f')$, and since by definition $N(r,f) = N(r,\infty)$, $N(r,\infty) - N_1(r) = N(r,a_1) - N_1(r) = N(r,f') - N(r,f) - N(r,1/f')$.

63 So,

$$(q-2)T(r,f) < \sum_{\nu=1}^{q} N(r,a_\nu) + N(r,f') - N(r,f) - N(r,1/f') + S(r) + 0(1)$$

If $f(z)$ has a pole of order p at z_0, $f'(z)$ has a pole of order $p+1$ at z_0 so that $n(t,f') - n(t,f) = \overline{n}(t,f)$ and so $N(r,f') - N(r,f) = \overline{N}(r,f)$,. Similarly if a is anyone of $a_2, a_3, \ldots, a_q$ (finite) and $f(z) = a$ has a root of multiplicity p, $f'(z)$ has there a zero of order $p-1$. Thus

$$\sum_{\nu=2}^{q} N(r,a_\nu) - N(r,1/f') = \sum_{\nu=2}^{q} \overline{N}(r,a_\nu) - N_0(r,1/f')$$

where $N_0(r,1/f')$ refers to zeros of $f'(z)$ at points other than the roots of $f(z) = a$.

Hence we get,

$$(q-2)T(r,f) < \sum_{\nu=2}^{q} \overline{N}(r,a_\nu) - N_0(r,1/f') + S(r) + \overline{N}(r,\infty) + 0(1)$$

i.e. $$(q-2)T(r,f) < \sum_{\nu=1}^{q} \overline{N}(r,a_\nu) - N_0(r,1/f') + S(r,f) + 0(1)$$

i.e. $$\sum_{\nu=1}^{q} \frac{N(r,a_\nu)}{T(r)} \geq (q-2) - \frac{S(r,f)+0(1)}{T(r)} \quad \text{since} \quad N_0(r,1/f') \geq 0$$

since $\underline{\lim}_{r\to R} S(r,f)/T(r) = 0$ and $T(r,f) \to \infty$ as $r \to R$

$$\overline{\lim_{r\to R}} \left[-\frac{O(1)+S(r)}{T(r)} \right] = 0. \quad \text{Thus} \quad \overline{\lim_{r\to R}} \sum_{\nu=1}^{q} \frac{N(r,a_\nu)}{T(r)} \geq (q-2)$$

and afortiori

$$\sum_{\nu=1}^{q} \overline{\lim_{r\to R}} \frac{N(r, a_\nu)}{T(r)} \geq q - 2$$

that is, **64**

$$\sum_{\nu=1}^{q} [1 - \Theta(a_\nu)] \geq (q-2) \quad \text{as required.}$$

This shows that $\Theta(a) > (1/n)$ at most for $2n$ values of a and so $\Theta(a) > 0$ for at most a countable number of a. If these a's are arranged in a sequence $\sum_{r=1}^{q} \Theta(a_r) \leq 2$ for any finite q, and so to infinity. □

Consequences

(1) Since $\delta(a) \leq \Theta(a)$ we have $\sum \delta(a_\nu) \leq 2$ and thus there exists at most two values of a for which $\delta(a) = 1$, or more generally $\delta(a) > \frac{2}{3}$. Thus if the equation $f(z) = a$ has only a finite number of roots in the plane, $N(r, a) = 0(\log r)$ as r tends to infinity, and we should have

$$\overline{\lim_{r\to\infty}} \log[r/T(r)] > 0, \text{ i.e. } \underline{\lim_{r\to\infty}} \frac{T(r)}{\log r} < \infty.$$

i.e., $f(z)$ is rational. Thus if $f(z)$ is transcendental in the plane $\delta(a) < 1$ the equation $f(z) = a$ has infinitely many roots. The same is true in all cases if R is finite, since otherwise $N(r, a) = 0(1)$ and so $\underline{\lim}_{r\to R}\, T(r) < +\infty$ that is $T(r) = 0(1)$ as r tends to R, since $T(r)$ is increasing. This would contradict

$$\overline{\lim_{r\to R}} T(r) / \log\left(\frac{1}{1-r}\right) = +\infty.$$

This result thus contains Picard's theorem as a special case.

(2) Θ *in relation to N and $\overline{N}$.* Suppose that the function $f(z) = a$ has only multiple roots of multiplicity $k \geq 2$. Then **65**

$$\overline{N}(r, a) \leq (1/k) N(r, a) \leq (1/k)[T(r, f) + 0(1)]$$

Hence in this case,

$$\overline{\lim_{r\to R}}\,\overline{N}(r,a)/T(r) \leq (1/k)$$

$$\Theta(a) = 1 - \overline{\lim_{r\to R}}\,\overline{N}(r,a)/T(r) \geq 1 - (1/k) \geq \frac{1}{2}$$

If a_1, $a_2, \ldots, a_q$ are q such values with $k = k_\nu$ for a_ν, we have since $\sum\limits_\nu \Theta(a_\nu) \leq 2$, $\sum\limits_\nu [1 - (1/k_\nu)] \leq 2$.

In particular there can be at the most four such values a_ν of a for $[1 - (1/k_\nu)] \geq \frac{1}{2}$.

If $f(z)$ is regular $m(r, f) = T(r, f)$ and $\delta(\infty) = \underline{\lim}\, m(r, f)/T(r, f) = 1$. and so $\Theta(\infty) = 1$ because $\Theta(\infty) \leq \delta(\infty) = 1$. So, $\sum\limits_\nu \Theta(a_\nu) \leq 1$ for any finite number of finite a_ν's. So that there can be only two such values a_ν which are taken multiply. Such values are called fully branched (Vollständig Verzweight). These results are best possible, for $\sin z$ and $\cos z$, have the values ± 1 "fully branched". Again for the Wierstrassian elliptic function $P(z)$ which satisfies the differential equation

$$[P'(z)]^2 = (P(z) - e_1)(P(z) - e_2)(P(z) - e_3)$$

where e_1, e_2, e_3 are distinct finite numbers the values e_1 e_2 and e_3 are evidently fully branched and so is infinity. If $P(z)$ has a pole of order k,
66 $P(z) \sim A(z-\zeta)^{-k}$ and $[P'(z)]^2 \sim P(z)^3 \sim A^3(z-\zeta)^{-3k}$ from the differential equation.

But

$$[\mathscr{P}'(z)]^2 \sim A' z^{-2k-2} \quad \text{i.e.,} \quad -2k - 2 = -3k$$

Hence we have $k = 2$ so all the poles are double and infinity is fully branched.

We also note that the equation $w^2 = \prod_{\nu=1}^q (z - a_\nu)$ can have no parametric solution $z = \rho(t)$, $w = \psi(t)$ which are integral functions of t if $q \geq 3$ or which are meromorphic if $q \geq 5$. Because if $\varphi(t_0) = a_1$ for instance then $w^2 = \psi^2(t)$ has a zero at t_0 and so $\psi(t)$ has a zero and $\psi^2(t)$ has at least a double zero at t_0. Hence also $\varphi(t)^{-a_1}$ has at least a double zero at t_0 and all roots of $\varphi(t) = a_\nu$ therefore will be multiple and $\sum\limits_{\nu=1}^{q} \Theta(a_\nu) \geq 2$ for $\varphi(t)$, a contradiction.

We just remark that this result extends to a general equation $g(z, w) = 0$ of genus greater than 1.

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Theorem 11. *Suppose $f(z)$ is meromorphic of finite order in the plane and $\Theta(a_1) = \Theta(a_2) = 1$, $a_1 \neq a_2$. Then if a is not equal to a_1, a_2, $\overline{N}(r, a) \sim T(r)$ as r tends to infinity.*

Proof. In fact we have if $f(z)$ is not rational

$$T(r, f) < \overline{N}(r, a_1) + \overline{N}(r, a_2) + \overline{N}(r, a) + S(r, f) + O(1)$$

and in this case $S(r, f) = O[T(r)]$, (even $S(r) = 0(\log r)$). By hypothesis since $\Theta(a_i) = 1 - \overline{\lim}[\overline{N}(r, a)/T(r)]$, $N(r, a_i) = 0[T(r)]\, i = 1, 2$. Therefore $[1 + 0(1)]T(r, f) < \overline{N}(r, a)$ as $r \to \infty$.

That is $\underline{\lim}_{r\to\infty}[\overline{N}(r, a)/T(r)] \geq 1$, and evidently $\overline{\lim}_{r\to\infty} \dfrac{\overline{N}(r, a)}{T(r)} \leq 1$ that is, $\overline{N}(r, a)/T(r) \to 1$, as r tends to infinity. Similarly since $N(r, a) \geq \overline{N}(r, a)$, $\underline{\lim}_{r\to\infty} N(r, a)/T(r) \geq 1$ and again $N(r, a) \sim T(r)$. If $f(z)$ is rational these results follow by elementary methods; in fact in this case there is at most one $\underline{a}$ namely $a = f(\infty)$ for which $N(r, a) = OT(r)$ unless f is a constant. □

Theorem 12. *If $a_\nu(z)$ for $\nu = 1, 2, 3$ are three functions satisfying $T(r, a_\nu) = O[T(r, f)]$ as $r \to R$, and $f(z)$ is as in theorem 10, then we have*

$$[1 + O(1)]T(r, f) < \sum_{i=1}^{3} N\left(r, \frac{1}{f - a_i(z)}\right)$$

as r tends to R through a suitable sequence of values.

Proof. Set 68

$$\varphi(z) = \frac{f(z) - a_1(z)}{f(z) - a_3(z)}\ \frac{a_2(z) - a_3(z)}{a_2(z) - a_1(z)}$$

It is easy to see using $T(r, a_i) = O[T(r, f)]$ that $T(r,\ ) = T(r, f) = [1 + O(1)]T(r, f)$, also $\varphi = 0, 1, \infty$, only if $f - a_1(z)$, $f - a_2(z)$, $f - a_3(z)$

are zero or if two suitable a's are equal.

$$N\left[r, \frac{1}{a_2 - a_3}\right] \leq T[r, 1/(a_2 - a_3)] = T[r, a_2 - a_3] + O(1)$$
$$\leq T(r, a_2) + T(r, a_3) + O(1) = O[T(r)]$$

So, $N(r, 1/\varphi) \leq N\left(r, \dfrac{1}{f - a_1}\right) + O[T(r, f)]$ etc. by Jensen's formula and hypothesis. □

In the course of the proof of theorem 10 we obtained,

$$(q-2)T(r, f) < \sum_{i=1}^{q} N(r, a_i) - N_0(r, 1/f') + S(r, f) + O(1)$$
$$< \sum_{i=1}^{q} N(r, a_i) + S(r, f) + O(1).$$

Take $f = \varphi$, and $a_1 = \infty$, $a_2 = 0$, $a_3 = 1$ to get

$$T(r, \varphi) < N(r, \varphi) + N\left(r, \frac{1}{\varphi}\right) + N\left(r, \frac{1}{\varphi - 1}\right) + O[T(r, \varphi)]$$

That is,

$$[1 + O(1)]T(r, \varphi) < N(r, \varphi) + N\left(r, \frac{1}{\varphi}\right) + N\left(r, \frac{1}{\varphi - 1}\right)$$

for a suitable sequence of r tending to R. $\varphi = 0$, only if either $f = a_1$,
69 or $a_2 = a_3$. So $N\left(r, \dfrac{1}{\varphi}\right) \leq N\left(r, \dfrac{1}{f - a_1}\right) + N\left(r, \dfrac{1}{a_2 - a_3}\right)$ which is equal to $N\left(r, \dfrac{1}{f - a_1}\right) + O[T(r)]$. Similar reasoning gives $N\left(r, \dfrac{1}{\varphi}\right) + N(r, \varphi) + N\left(r, \dfrac{1}{\varphi - 1}\right) < \sum\limits_{i=1}^{3} N\left(r, \dfrac{1}{f - a_i}\right) + O[T(r)]$. Thus since $T(r, \varphi) = [1 + O(1)]T(r)$ the result follows.

Remark. Note that the same reasoning cannot be applied to more than three functions. In fact it is not known whether the analogous result is still true if we take more than three functions.

2.5 Picard values of meromorphic functions and their derivatives:

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A function $f(z)$ will be called admissible if it satisfies the hypothesis of theorem 9 in a circle $|z| < R$ and also $f(z)$ is transcendental if $R = \infty$.

Note that if $f^{(l)}(z)$ lth derivative of $f(z)$, all the poles of $f^{(1)}(z)$ have multiplicity at least $l + 1$. Therefore, $\overline{N}[r, f^{(l)}] \leq N(r, f^{(l)})/(l + 1)$ and

$$\Theta(\infty) = \underline{\lim}\left(1 - \frac{N(r, f^{(l)})}{T(r, f^{(l)})} \geq \underline{\lim}\right) 1 - \frac{1}{(l+1)} \frac{N(r, f^{(l)})}{T(r, f^{(l)})} \geq \frac{l}{(l+1)}$$

We obtain for $f^{(l)}(z)$, since $\sum\limits_{0}^{q} \Theta(a_\nu) \leq 2$, if $a_1, \ldots, a_q$, are distinct and finite, $\sum\limits_{1}^{q} \Theta(a_\nu) \leq 1 + \dfrac{1}{l+1}$. Thus there can be at most one finite value which is taken only a finite number of times or more generally for which $\Theta(a) > \dfrac{3}{4}$.

Write now $\psi(z) = a_0(z) + f(z) + \cdots + a_l(z) f^l(z) a_i(z)$ being functions satisfying $T[r, a_i(z)] = OT(r, f)$ and we assume $\psi(z)$ is not identically constant. Then we have the following sharpened form of a theorem of Milloux.

Theorem 13. *If $f(z)$ is admissible in $|z| < R$ then,*

$$T(r, f) < \overline{N}(r, f) + N(r, 1/f) + \overline{N}\left[r, \frac{1}{\psi - 1}\right] - N_0(r, 1/\psi') + S_1(r, f)$$

where $N_0(r, 1/\psi')$ indicates that zeros of ψ' corresponding to the repeated roots of $\psi = 1$ are to be omitted, and $\underline{\lim}_{r \to R} S_1(r, f)/T(r, f) = 0$. Note that this reduces to the second fundamental theorem for $q = 3$ when $\psi = f$.

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Firstly let us prove some lemmas.

Lemma 5. *If l is a positive integer and $f(z)$ admissible in $|z| < R$, such $0 < r < \rho < R$ then,*

$$m\left[r, \frac{f^{(l)}}{f}\right] < A(l)\left[\log^+ T(\rho, f) + \log^+ \rho + \log^+ \frac{1}{\rho - r} + \log^+(1/r) + O(1)\right]$$

$A(l)$ being a constant depending only on l.

Proof. The proof is by induction. The result is true for $l = 1$, by Lemma 3 of section 2.1. Take $\rho_1 = \frac{1}{2}(\rho + r)$ and assume that result for l. Now,

$$\begin{aligned} T[\rho_l, f^1(z)] &= m[\rho_1, f^l(z)] + N[\rho_1, f^l(z)] \\ &\leq m[\rho_1, f] + m(\rho_1, f^{(l)}/f) + (l+1)N(\rho_1, f) \end{aligned}$$

since at a pole of f of order k, $f^{(l)}$ has a pole of order $k + l \leq k(l+1)$. By our induction hypothesis the right hand side is less than

$$\begin{aligned} &(l+1)T(\rho_1, f) + A(l)\Bigg[\log^+ T(\rho, f) + \log^+ \rho \\ &\qquad + \log^+ \frac{1}{\rho - \rho_1} + \log^+ \frac{1}{\rho_1} + O(1)\Bigg] \\ &< [l + 1 + A(l)]T(\rho, f) + A(l)\left[\log^+ \rho + \log^+ \frac{1}{\rho - \rho_1} + \log^+ \frac{1}{\rho_1} + O(1)\right] \end{aligned}$$

So,

$$\begin{aligned} \log^+ T[\rho_1, f^{(l)}] &< \log^+ T(\rho, f) + \log^+(\log^+ \rho) + \log^+\left(\log^+ \frac{1}{\rho - \rho_1}\right) \\ &\qquad + \log^+\left(\log^+ \frac{1}{\rho_1}\right) + O(1) \\ &< \log^+ T(\rho, f) + \log^+ \rho + \log^+ \frac{1}{\rho - \rho_1} + \log^+ \frac{1}{\rho_1} + O(1) \end{aligned}$$

72 Also by Lemma 3 applied to $f^{(l)}$,

$$\begin{aligned} m\left(r, \frac{f^{l+1}}{f^l}\right) &< 3\log^+ T\left[\rho_1, f^{(l)}\right] + 4\log^+ \rho_1 + \log^+ \frac{1}{r} \\ &\qquad + 4\log^+ \frac{1}{\rho_1 - r} + O(1) \end{aligned}$$

$$< 3\log^+ T(\rho, f) + 7\log^+ \rho + 7\log^+ \frac{1}{\rho - r} + 2\log^+ \frac{1}{r} + O(1)$$
$$< A\left[\log^+ T(\rho, f) + \log^+ \rho + \log^+ \frac{1}{\rho - r} + \log^+ \frac{1}{r} + O(1)\right]$$

because,

$$\rho_1 - r = \rho - \rho_1 = \frac{1}{2}(\rho - r)(\rho > \rho_1 > r)$$

Therefore, since

$$m\left[r, \frac{f^{l+1}}{f}\right] < m\left[r, \frac{f^{l+1}}{f^{(l)}}\right] + m\left[r, f^{(l)}/f\right].$$

$m\left(r, \dfrac{f^{l+1}}{f^l}\right)$ is less than

$$[A + A(l)]\left[\log^+ T(\rho, f) + \log^+ \frac{1}{\rho - r} + \log^+(1/r) + O(1) + \log^+ \rho\right]$$

completing the inductive proof. □

Lemma 6. *If $\psi(z)$ is defined as in theorem 13, and $0 < r < \rho < R$, then* $m(r, \psi/f) < O[T(r, f)] + A(l)[\log^+ T(\rho, f) + \log^+ \dfrac{1}{\rho - r} + \log^+ \rho] + O(1)$ *as $r \to R$ in any manner.*

Proof.

$$m(r, \psi/f) \leq \sum_{i=0}^{l} m\left[r, a_i \frac{(z)^{f^{(i)}}}{f}\right] + \log(l + 1).$$
$$\leq \sum_{i=0}^{l} m[r, a_i(z)] + \sum_{i=0}^{l} m[r, f^{(i)}/f] + \log(l + 1) + (l + 1)\log 2.$$
$$= O[T(r)] + \sum_{i=0}^{l} m[r, f^{(i)}/f] + O(1)$$
$$< O[T(r)] + A(l)\left[\log^+ T(\rho, f) + \log^+ \frac{1}{\rho - r} + \log^+ \rho\right] + O(1)$$

from the lemma 5 and because $m[r, a_i(z)] \leq T[r, a_i(z)] = O[T(r)]$, and **73**
$\log^+(1/r)$ remains bounded as r tends to R. □

Lemma 7. *If $\psi(z)$ is defined as above then as $r \to R$ in any manner while $0 < r < \rho < R$,*

$$T(r,\psi) < [l + 1 + O(1)]T(r, f) + A(l)$$
$$\left[\log^+ T(\rho, f) + \log^+\left(\frac{1}{\rho - r}\right) + \log^+ \rho + O(1)\right].$$

Proof. $m(r,\psi) \leq m(r,\psi/f) + m(r, f)$. If f has a pole of order K at a point and $a_\nu(z)$ a pole of order K_ν then $a_\nu(z) f^{(\nu)}$ has a pole of order $\nu + K + K_\nu$, and so $\psi(z)$ has a pole or of order Max . $(\nu + K + K_\nu) \leq (l + 1)K + \sum K_\nu$. This gives

$$N(r,\psi) \leq (l + 1)N(r, f) + \sum_\nu N(r, a_\nu)$$
$$\leq (l + 1)N(r, f) + O[T(r, f)]$$

Adding the above two inequalities,

$$T(r,\psi) \leq [l + 1 + O(1)]T(r, f) + m(r,\psi/f),$$

and now the lemma follows from the previous lemma. □

Lemma 8. *If $S(r,\psi)$ is defined as in Theorem 8 with ψ instead of f then if $0 < r < \rho < R$ and r tends to R,*

$$S(r,\psi) < A\left[\log^+ T(\rho, f) + \log^+\left(\frac{1}{\rho - r}\right) + \log^+ \rho + O(1)\right].$$

Proof. Let $\rho_1 = \frac{1}{2}(\rho + r)$. Lemma 3 gives then

$$S(r,\psi) < A\left[\log^+ T(\rho_1,\psi) + \log^+\left(\frac{1}{\rho_1 - r}\right) + \log^+ \rho_1 + O(1)\right].$$

By lemma 7 since $\log^+ x \leq x$,

$$T(\rho_1,\psi) < A(l)\left[T(\rho, f) + \left(\frac{1}{\rho - \rho_1}\right) + \rho + O(1)\right]$$

74 that is, $\log^+ T(\rho_1,\psi) < \log^+ T(\rho, f) + \log^+ \dfrac{1}{\rho - \rho_1} + \log^+ \rho + O(1)$.

Substituting this and remembering that $r < \rho_1 < \rho, \rho - \rho_1 = \rho_1 - r = \frac{1}{2}(\rho - r)$ we get the result. □

Now we are ready to prove theorem 13.

Proof of theorem 2.13. We have from theorem 8,

$$m(r,\psi) + m\left(r, \frac{1}{\psi}\right) + m\left(r, \frac{1}{\psi - 1}\right) \leq 2T(r,\psi) - N_1(r,\psi) + S(r,\psi)$$

i.e.

$$T\left(r, \frac{1}{\psi}\right) + T\left(r, \frac{1}{\psi - 1}\right) \leq T(r,\psi) - N_1(r,\psi) + S(r,\psi)$$
$$+ N(r,\psi) + N\left(r, \frac{1}{\psi}\right) + N\left(r, \frac{1}{\psi - 1}\right)$$

i.e.

$$T(r,\psi) \leq N(r,\psi) - N_1(r,\psi) + N\left(r, \frac{1}{\psi}\right) + N\left(r, \frac{1}{\psi - 1}\right) + S(r,\psi) + O(1)$$

also

$$N(r,\psi) - N_1(r,\psi) = \overline{N}(r,\psi) - N\left(r, \frac{1}{\psi'}\right) \quad \text{and}$$
$$N\left(r, \frac{1}{\psi - 1}\right) - N\left(r, \frac{1}{\psi'}\right) = \overline{N}\left(r, \frac{1}{\psi - 1}\right) - N_0\left(r, \frac{1}{\psi'}\right). \quad \text{Hence}$$

Thus,

$$T(r,\psi)\overline{N}(r, \ ) + N(r, 1/ \ ) + \overline{N}\left(r, \frac{1}{-1}\right) - N_0(r, 1/1) + S(r, \ ) + O(1)$$

where $N_0(r, 1/\psi')$ denotes the fact that zeros of ψ' at multiple roots of $\psi - 1$ are omitted. Thus, since $T(r,\psi) = m(r, 1/\psi) + N(r, 1/\psi) + O(1) m(r, 1/\psi) \leq \overline{N}(r,\psi) + \overline{N}\left(r, \dfrac{1}{\psi - 1}\right) - N_0(r, 1/\psi') + S(r,\psi) + O(1)$.
Note again that poles of ψ occur only at poles of f or of $a_\nu(z)$, and in $\overline{N}$ **75**
each pole is counted only once. Then

$$\overline{N}(r,\psi) \leq \overline{N}(r, f) + \sum_{\nu} N(r, a_\nu)$$

$$\leq \overline{N}(r, f) + OT(r, f).$$

Again,

$$\begin{aligned} T(r, f) &= m(r, 1/f) + N(r, 1/f) + O(1) \\ &\leq m(r, \psi/f) + m(r, 1/\psi) + N(r, 1/f) + O(1). \end{aligned}$$

Substituting we obtain,

$$[O(1) + 1]T(r, f) \leq \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{\psi - 1}\right) + N(r, 1/f) - N_0(r, 1/\psi') \\ +S(r, \psi) + m(r, \psi/f),$$

since $f(z)$ being admissible $O(1) = O[T(r)]$. Now if $0 < r < \rho < R$ lemmas 6 and 8 give

$$m(r, \psi/f) + S(r, \psi) < A(l)\left[\log^+ T(\rho, f) + \log^+ \frac{1}{\rho - r} + \log^+ \rho + O(1)\right].$$

and now the result is completed just as in theorem 9 by means of lemma 4. Hence theorem 13.

Consequences.

Theorem 14 (Milloux)**.** *If $f(z)$ is admissible in $|z| < R$ and is regular there then either $f(z)$ assumes every finite value infinitely often or every derivative of $f(z)$ assumes every finite value except possibly zero, infinitely often.*

Proof. Suppose $f(z) = a$, $f^{(l)}(z) = b$ have only a finite number of roots

76 where $b \neq 0$. Choose $g(z) = f(z) - a$, $\psi(z) = \dfrac{g^{(l)}(z)}{b} = \dfrac{f^{(l)}(z)}{b}$ in theorem 13. Then since $\overline{N}(r, g) = 0$, g being regular

$$[1 + 0(1)]T(r, g) < \overline{N}\left[r, \frac{1}{g^{(l)}(z) - b}\right] + N(r, 1/g) + S_1(r)$$

where $\varliminf\limits_{r \to R} S_1(r)/T(r, g) = 0$.

If $R = \infty$ this gives $\varliminf\limits_{r\to\infty} T(r,g)/\log r < +\infty$ by assumptions that $n\left[t, \dfrac{1}{f^{(l)}(z)-b}\right]$ and $n(t,1/g)$ are finite as r tends to infinity so that $g(z)$ is rational and so is $f(z)$. That is $f(z)$ is a polynomial. □

If R is less than infinity we obtain, $\varliminf\limits_{r\to R} T(r,g) < \infty$, and so since $T(r,g)$ increases with $r \lim T(r,g) < \infty$ and the same result applies to $f(z)$ giving a contradiction to admissibility.

Theorem 15 (Saxer)**.** *If $f(z)$ is meromorphic in the plane and f, f', f'' have only a finite number of zeros and poles then $f(z) = P_1(z)/P_2(z)$ $e^{P_3(z)}$ P_1, P_2, P_3 being polynomials. If f, f', f'' have no zeros and poles then $f(z) = e^{a+bz}$ where a, b are constants.*

Proof. Set $g(z) = f(z)/f'(z)$. Then $g'(z) = 1 - [f(z)f''(z)/f'^2(z)]$. Suppose that $g(z)$ is transcendental and so admissible in the plane. Now $g = 0, \infty$ only when $f' = 0$, on $f = 0, \infty$ that is a finite number of times and so, $N(r,g) + N(r,1/g) = O(\log r)$. Next $g'(z) = 1$ only for $f = 0$ or $f'' = 0$ that is $N\left(r, \dfrac{1}{g'-1}\right) = O(\log r)$ by hypothesis. Now theorem 13 applied to $g(z)$ gives for a sequence of r tending to infinity, taking $\psi = g'$,

$$[1 + O(1)]T(r,g) = O(\log r).$$

This implies $g(z)$ is rational, i.e. a contradiction. Hence $g(z)$ is rational 77
so that $f'/f = g$ is rational. Now f'/f has simple poles with integer residues at the poles and zeros of $f(z)$. Since f'/f is rational by expanding it in partial fractions we get,

$$f'/f = \sum_r k_r/(z - z_r) + P(z)$$

with k_ν integers and $P(z)$ a polynomial. Integrating the above,

$$f(z) = \prod_r (z - z_r)^k \int_e P(z)dz$$

This proves the first part and if $f(z)$ has no zeros or poles the product term disappears and $f = e^{P(z)}$, $f'(z) = P'(z)e^{P(z)}$ and $f'(z) \neq 0$ implies $P'(z) \neq 0$, which gives $P(z) = a + bz$. □

Remark. Note that we cannot do the same thing with $f(z)$ and $f'(z)$. For if $f(z) = e^{g(z)}$, $f'(z) = g'(z)e^{g(z)}$ and if we put $g'(z) = e^{h(z)}$ where $h(z)$ is an arbitrary integral function, so that $g(z) = \int e^{h(z)}dz$ and $f(z) = \exp\cdot\left[\int e^{h(z)}dz\right]$. Then f is an integral function with $f \neq 0$, and $f' \neq 0$. So in theorem 15 we cannot leave out the restriction on f''. Further $F(z) = \int f(z)dz$ is a function for which $F' \neq 0$, $F'' \neq 0$ and so we cannot leave the restriction on f.

But we will show that we can leave out the restriction on f'. This is precisely Theorem 19.

78 In connection with theorem 15 we also quote the following extension by Csillag [1].

Theorem 16. *If l and m are different positive integers and $f(z)$ an integral function such that $f(z) \neq 0$, $f^{(l)}(z) \neq 0$ and $f^{(m)}(z) \neq 0$, then $f(z)$ is equal to e^{az+b}.*

2.6 Elimination of $N(r, f)$

We shall prove the following theorem

Theorem 17. *If $f(z)$ is admissible in $|z| < R$, and $l \geq 1$ then,*

$$T(r, f) < [2 + (1/l)]N(r, 1/f) + 2[1 + (1/l)]\overline{N}\left(r, \frac{1}{f^{(l)} - 1}\right) + S_2(r, f),$$

where $\varliminf_{r\to R} S_2(r, f)/T(r) = 0.$

Let us set $\psi(z) = f^{(l)}(z)$ in theorem 13 (Th. of Milloux), to get

(2.7)
$$T(r, f) < \overline{N}(r, f) + N(r, 1/f) + \overline{N}\left(r, \frac{1}{f^{(l)} - 1}\right) - N_0\left(r, \frac{1}{f^{(l+1)}}\right) + S_1(r, f)$$

where in $N_0\left(r, \dfrac{1}{f^{(l+1)}}\right)$ zeros of $f^{(l+1)}$ at multiple roots of $f^{(l)}(z) = 1$ are to be omitted.

Further we need,

Lemma 9. *If* $g(z) = \left[f^{(l+1)}(z)\right]^{l+1} / [1 - f^{(l)}(z)]^{l+2}$ *then*

$$lN_1(r, f) \leq \overline{N}_2(r, f) + \overline{N}\left(r, \frac{1}{f^{(l)} - 1}\right) + N_0\left(r, \frac{1}{f^{(l+1)}}\right) + m(r, g'/g) + \log|g(0)/g'(0)| \tag{2.8}$$

where $N_1(r, f)$ *stands for the* N*-sum over the simple poles of* $f(z)$*;* 79
$\overline{N}_2(r, f)$ *for the N sum over multiple poles of* $f(z)$*, with each pole counted only once. Thus* $\overline{N}(r, f) = N_1(r, f) + N_2(r, f)$*.*

Now at a simple pole z_0 of $f(z)$, $f(z) = O(1) + [a/(z - z_0)]$ where a is not equal to zero.

Differentiating l times,

$$1 - f^{(l)}(z) = \frac{al!(-1)^{l+1}}{(z - z_0)^{l+1}} + O(1)$$

This can be written as

$$1 - f^{(l)}(z) = \frac{al!(-1)^{l+1}}{(z - z_0)^{l+1}}[1 + O\{(z - z_0)^{l+1}\}]$$

The differentiation of both the sides again gives,

$$f^{(l+1)}(z) = \left[1 + O\{(z - z_0)^{l+2}\}\right] \frac{a(l + 1)!(-1)^{l+1}}{(z - z_0)l + 2}$$

Hence,

$$g(z) = \frac{(-1)^{l+1}(l + 1)^{l+1}}{a(l)!}[1 + O\{(z - z_0)^{l+1}\}]$$

So, $g(z_0) \neq 0, \infty$ but $g'(z)$ has a zero of order at least $\underline{l}$ at $z = z_0$. Now we have

$$N(r, g/g') - N(r, g'/g) = m(r, g'/g) - m(r, g/g') + \log|g(0)/g'(0)|$$

from Jensen's formula. As we saw at the end of section 2 the left hand 80
side is

$$N(r, g) + N(r, 1/g') - N(r, g') - N(r, 1/g)$$

$$= N(r, 1/g') - N(r, 1/g) - \overline{N}(r, g)$$
$$= N_0(r, 1/g') - \overline{N}(r, 1/g) - \overline{N}(r, g)$$

where in $N_0(r, 1/g')$ only zeros of g' which are not zeros of g are to be considered. By our above analysis,

$$lN_1(r, f) \leq N_0(r, 1/g')$$
$$\overline{N}(r, 1/g) + \overline{N}(r, g) + m(r, g'/g) + \log |g(0)/g'(0)|$$

Then note that $g = 0, \infty$ only at poles of $f(z)$ which must be multiple, at zeros of $f^{(l)}(z) - 1$, and zeros of $f^{(l+1)}(z)$ which are not zeros of $f^{(l)}(z) - 1$. This gives lemma 9.

Now (2.7) gives on writing $T(r, f) = m(r, f) + N(r, f)$

(2.9)
$$N(r, f) - \overline{N}(r, f) < N(r, 1/f) + \overline{N}\left(r, \frac{1}{f^{(l+1)}}\right) - N_0\left(r, \frac{1}{f^{(l+1)}}\right) + S_1(r, f)$$

On the left the contribution of each multiple pole to the sum being counted once for $\overline{N}$, but at least twice for N. So,

$$N_2(r, f) \leq N(r, f) - \overline{N}(r, f) \tag{2.10}$$

Also $\overline{N}(r, f) = \overline{N}_2(r, f) + N_1(r, f)$. Hence it follows from lemma 9 that,

$$N(r, f) \leq [1 + (1/l)]N_2(r, f)$$
$$+ \frac{1}{l}\left[\overline{N}\left(r, \frac{1}{f^{(l)} - 1}\right) + N_0\left(r, \frac{1}{f^{(l+1)}}\right) + m(r, g'/g)\right] + \frac{1}{l} \log \frac{|g(0)|}{|g'(0)|}$$

81 By the inequalities (2.9) and (2.10) it is at the most,

$$\left(1 + \frac{1}{l}\right)\left[N(r, 1/f) + \overline{N}(r, 1/(f^{(l)} - 1)) - N_0\left(r, \frac{1}{f^{(l+1)}}\right) + S_1(r, f)\right]$$
$$+ (1/l)\left[\overline{N}\left(r, \frac{1}{f^{(l)} - 1}\right) + N_0\left(r, \frac{1}{f^{(l+1)}}\right) + m(r, g'/g) + \log |g(0)/g'(0)|\right]$$
$$= (1 + 1/l)N(r, 1/f) + (1 + 2/l)\overline{N}(r, 1/(f^{(l)} - 1))$$
$$- N_0\left(r, \frac{1}{f^{(l+1)}}\right) + S_3(r)$$

where

$$S_3(r) = [1 + (1/l)]S_1(r, f) + (1/l)[m(r, g'/g) + \log|g(0)/g'(0)|].$$

Substituting for $\overline{N}(r, f)$ in (2.7) we obtain,

$$T(r, f) < [2 + (1/l)]N(r, 1/f) + 2[1 + (1/l)]$$
$$\overline{N}\left(r, \frac{1}{f^{(l)} - 1}\right) - 2N_0\left(r, \frac{1}{f^{(l+1)}}\right) + S_1(r, f) + S_3(r, f)$$

We observe that the above gives the result of theorem 17, provided we pose

$$\begin{aligned} S_2(r, f) &= S_1(r, f) + S_3(r, f) \\ &= [2 + (1/l)]S_1(r, f) + (1/l)[m(r, g'/g) + \log|g(0)/g'(0)|] \end{aligned}$$

and in order to complete the proof it is sufficient to prove that if $r < \rho < R$, and r tends to R,

$$m(r, g'/g) < A[\log^+ T(\rho, f) + \log^+ 1/(\rho - r) + \log^+ \rho + 0(1)]$$

The above inequality is true for,

$$\begin{aligned} \log g(z) &= (l + 1)\log f^{(l+1)}(z) - (l + 2)\log[1 - f^{(l)}(z)] \\ g'/g &= (l + 1)\left[f^{(l+2)}/f^{(l+1)}\right] + (l + 2)f^{(l+1)}/(1 - f^{(l)}) \\ m(r, g'/g) &\leq m(r, f^{(l+2)}/f^{(l+1)}) + m\left(r, f^{(l+1)}/(f^{(l)} - 1)\right) + O(1) \end{aligned}$$

Now the result follows from Lemma 8, with $\psi = f^{(l)} - 1$ or $f^{(l+1)}$. Hence **82**
the proof of theorem 17 is complete.

2.7 Consequences

Theorem 18. *The result of theorem 14 extends to meromorphic functions without any further hypothesis. To be precise, if $f(z)$ is admissible in $|z| < R$ and is meromorphic there, then either $f(z)$ assumes every finite value infinitely often or every derivative of $f(z)$ assumes every finite value except possibly zero infinitely often.*

The proof is as before by the consideration of $g(z) = [f(z) - a]/b$ instead of $f(z)$; if the equations $f(z) = a$ and $f^{(l)} = b$ have only a finite number of roots. For then $g = 0$ and $g^{(l)} = 1$ have only finite number of roots and we can use Theorem 17.

Theorem 19. *Theorem 15 (Saxer's theorem) still holds if the hypothesis of $f'(z)$ is omitted. That is, if $f(z)$ is meromorphic in the plane and f, and f'' have only a finite number of zeros and poles then,*

$$f(z) = P_1(z)/P_2(z)e^{P_3(z)}$$

P_1, P_2, P_3 being polynomials. If f'', f have no zeros and poles then $f(z) = e^{az+b}$, a and b being constants.

83 *Proof.* If $f(z)$ and $f''(z)$ have only a finite number of zeros and poles, put $g(z) = f(z)/f'(z)$. Then $g'(z) = 1 - \{f(z)f''/[f'(z)]^2\}g(z)$ has only finite number of zeros and so does $g'(z) - 1 = ff''/f'^2$ namely at the poles of f and at the zeros of f and f''. Hence by Theorem 18, $g(z)$ is rational and now the proof is completed as in Saxer's theorem. If $f(z)$ and $f''(z)$ have no zeros or poles, $f(z) = e^{P(z)}$ where $P(z)$ is a polynomial. Therefore, $f''(z)$ equals $[P''(z) + P'(z)^2]e^{P(z)}$, and if $P'(z)$ has degree n greater than or equal to 1, then $P'(z)^2$, $P''(z)$ have degree $2n$, $n - 1$, respectively, so that $f''(z)$ has $2n$ zeros. Thus $P'(z) = a =$ const. $P(z) = az + b$. $\square$

A slightly more delicate analysis shows that if $f(z)$, $f''(z)$ have no zeros but $f(z)$ may have a finite number of poles, then $f(z) = e^{az+b}$ or $(az + b)^{-n}$, where n is a positive integer.

Part III

Univalent Functions

3.1 Schlicht functions 84

Definition. *A function $f(z)$ regular in a domain D is said to be* univalent *(Schlicht, Simple) if $f(z)$ takes different values at different points of D. Then $f(z)$ maps D, one-one conformally into a domain Δ.*

Since $f(z)$ takes no value more than once, if $f(z)$ is Schlicht in the plane, the function $A(r)$ (area on the Riemann sphere) is less or equal to one and $T_0(r) \leq \log r$. So $f(z)$ is rational and is in fact a polynomial which must be linear. We consider functions $f(z)$ univalent in $|z| < 1$. If $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$, $(f(z) - a_0)/a_1$ is also univalent. In fact a_1 cannot be zero, for otherwise $f(z)$ would assume values at least twice near $z = 0$, $a_1 \neq 0$. Hence we may assume $f(z)$ to be of the form

$$z + \sum_{n=2}^{\infty} a_n z^n \tag{2}$$

The class of functions, univalent in $|z| < 1$ with the expansion (2) is called S.

The first two results are,

Theorem 1. *If $f(z) \in S$, $|a_2| \leq 2$, and equality is possible only for $f(z) = f_\theta(z) = \dfrac{z}{[1 - ze^{i\theta}]^2}$, and*

Theorem 2. *If $f(z) \in S$ and w is a value not taken by $f(z)$ then $|w| \geq \frac{1}{4}$, again equality is possible only for $f(z) = f_\theta(z)$.*

85 Both results in the above form are due to Biberbach, theorem 2 with a smaller constant than $\frac{1}{4}$ is due to Koebe though it seems we had the proof eventually form

Note that $f(z) \in S$, takes all values with modulus $< \frac{1}{4}$. We shall prove theorem 1 first and then deduce theorem 2 from it. For that we need

Lemma 1. *Suppose $f(z)$ is regular and univalent in the annulus $r_1 < |z| < r_2$ then the area of the image of the annulus is $\pi \sum\limits_{-\infty}^{\infty} n|a_n|^2(r_2^{2n} - r_1^{2n})$, where $f(z)$ has the expansion $\sum\limits_{-\infty}^{\infty} a_n z^n$ in the annulus.*

Proof. The area of the image is clearly $\int\limits_{r_1}^{r_2} r\,dr \int\limits_0^{2\pi} |f'(re^{i\theta})|^2 d\theta$. The integral

$$\int_0^{2\pi} |f'(re^{i\theta})|^2 d\theta = \int_0^{2\pi} f'(re^{i\theta})\overline{f(re^{i\theta})} d\theta$$
$$= \int_0^{2\pi} \left[\sum na_n r^{n-1} e^{i(n-1)\theta}\right]\left[\sum m\overline{a}_m r^{m-1} e^{-i(m-1)\theta}\right]$$

Since the multiplication of the two series and then term by term integrations are valid, and $\int\limits_0^{2\pi} e^{i(m-n)\theta} d\theta = \begin{cases} 0, & m \neq n, \\ 2\pi, & m = n. \end{cases}$

$$\int_0^{2\pi} |f'(re^{i\theta})|^2 d\theta = 2\pi \sum_{-\infty}^{\infty} n^2 |a_n|^2 r^{2n-2}$$

86 $\therefore$ Area of image

$$= \int_{r_1}^{r_2} r\, dr \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta$$

$$= 2\pi \int_{r_1}^{r_2} \sum_{-\infty}^{+\infty} n^2 |a_n|^2 r^{2n-1} dt.$$

Again since integration term by term is valid,

$$= \pi \sum_{-\infty}^{+\infty} n |a_n|^2 (r_2^{2n} - r_1^{2n})$$

Hence the lemma. □

We now proceed to prove the theorem.
Consider the function $f(z) \in S$.

$$f(z) = z + \sum_2^\infty a_n z^n, \quad \text{let} \quad F(z) = \left[f(z^2)\right]^{\frac{1}{2}} = z\left[\frac{f(z^2)}{z^2}\right]^{\frac{1}{2}}$$

$$= z\left[1 + \sum_2^\infty a_n z^{2n-2}\right]^{\frac{1}{2}}$$

Since $\dfrac{f(z)}{z}$ is not zero $\dfrac{f(z)}{z}$ has a one valued square root which is a power series in z and hence $\left[\dfrac{f(z^2)}{z^2}\right]^{\frac{1}{2}}$ is a power series in z^2.

$$F(z) = z + \frac{1}{2} a_2 z^3 + \cdots\cdots$$

$F(z)$ is odd. Also $F(z)$ is univalent. For if $F(z_1) = F(z_2)$, then

$$\left[f(z_1^2)\right]^{\frac{1}{2}} = \left[f(z_2^2)\right]^{\frac{1}{2}}$$

i.e.,

$$f(z_1^2) = f(z_2^2) \Longrightarrow z_1^2 = z_2^2.$$

$\therefore z_1 = \pm z_2$.

But $F(-z_1) = -F(z_1) \neq F(z_1)$ unless $z_1 = 0$. So $z_1 = z_2$. Now set

$$g(z) = \frac{1}{F(z)} = \left[f(z^2)\right]^{-\frac{1}{2}} = \frac{1}{2} + b_1 z + b_3 z^3 + \cdots\cdots$$
$$= \frac{1}{z} + \sum_1^\infty b_n z^n$$

where $b_1 = -\frac{1}{2}a_2$.

And $g(z)$ is univalent in $0 < |z| < 1$. Let $J(r)$ be the curve which is the image of $|z| = r$ by $g(z)$, $0 < r < 1$ and $A(r)$ the area inside it. Then for $0 < r_1 < r_2 < 1$ we have by the lemma

$$A(r_1) - A(r_2) = \pm\pi\left[\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right) + \sum_1^\infty n|b_n|^2(r_2^{2n} - r_1^{2n})\right.$$

Since $J(r_1)$ and $J(r_2)$ do not cross, the left hand side and hence the right hand side is different from zero. So $A(r)$ is monotonic. Further for small r, $A(r) \sim \dfrac{\pi}{r^2}$ which tends to ∞ as $r \to 0$. Hence $A(r)$ decreases. The left hand side is therefore positive and the quantity inside the brackets is positive and so we take the positive sign.

Set $S(r) = \dfrac{1}{r^2} - \sum\limits_1^\infty |b_n|^2 n r^{2n}$.

88 Then $A(r) = \pi S(r) + C$, C being a constant. We want to prove that $S(r) \geq 0$ for $0 < r < 1$.

Suppose now that b_1 i.e. a_2 is real Otherwise if $a_2 = |a_2|e^{i\theta}$ we consider $e^{i\theta} f(ze^{-i\theta})$ in place of $f(z)$ and then

$$e^{i\theta} f(ze^{-i\theta}) = z + e^{i\theta} a_2 e^{-2i\theta} z^2 + \cdots\cdots$$
$$= z + |a_2| z^2 + \cdots\cdots$$

Thus there is no loss of generality in assuming a_2 real and positive. Now

$$g(z) = \frac{1}{z} + b_1 z + b_3 z^3 + \cdots\cdots, \quad \text{so if, } z = re^{i\theta}$$
$$g(re^{i\theta}) = \left(\frac{1}{r} + b_1 r\right)\cos\theta + i\left(b_1 r - \frac{1}{r}\right)\sin\theta + O(r^3).$$

We have then

$$|\text{Rl.}\ g(re^{i\theta})| \leq \left|\frac{1}{r} + b_1 r\right| + O(r^3)$$

$$|\ \text{Im}\ .g(re^{i\theta})| \leq \left|\frac{1}{r} - b_1 r|\right| + O(r^3) = \left(\frac{1}{r} - b_1 r\right) + O(r^3)$$

(since $a_2 > 0$, $b_1 < 0$) so that the image of $|z| = r$ by $g(z)$ is contained in the ellipse of semi-axes $\left(\frac{1}{r} - b_1 r\right) + O(r^3)$, $|\frac{1}{r} + b_1 r| + O(r^3)$. Hence area inside the image $A(r)$ satisfies

$$A(r) \leq \pi\left(\frac{1}{r^2} - b_1^2 r^2\right) + O(r^2) = \frac{\pi}{r^2} + O(r^2).$$

Thus **89**

$$\pi S(r) + C \leq \frac{\pi}{r^2} + O(r^2).$$

But

$$S(r) = \frac{1}{r^2} + O(r^2)$$

so that

$$\frac{1}{r^2} + O(r)^2 + C \leq \frac{\pi}{r^2} + O(r^2)$$

or

$$C + O(r^2) \leq O(r^2)$$

letting $r \to 0$ we see that $C \leq 0$. This proves that $-C + A(r) = \pi S(r) \geq 0$ since, $-C \geq 0$, $A(r) \geq 0$. Thus for $0 < r < 1$ $S(r) \geq 0$. [Actually a little more refined argument shows that $A(r) = \pi S(r)$].

Therefore,

$$\frac{1}{r^2} \geq \sum_1^\infty |b_n|^2 n r^{2n}$$

and letting $r \to 1$, $1 \geq \sum\limits_1^\infty n|b_n|^2$. Thus

$$|b_1| \leq 1 \quad \text{or} \quad |a_2| \leq 2$$

Equality can hold only if $b_n = 0$ for $n > 1$ i.e. $g(z) = \dfrac{1}{z} - ze^{i\theta}$.

$$\text{i.e. } F(z) = \frac{z}{1 - z^2 e^{i\theta}} = [f(z^2)]^{\frac{1}{2}}$$

Thus

$$f(z) = \frac{z}{[1 - ze^{i\theta}]^2} = f_\theta(z).$$

This proves the theorem provided we show $f_\theta(z) \in S$. To deduce theorem 2, set $g(z) = \dfrac{wf(z)}{w - f(z)}$ where $f(z) \neq w$ for $|z| < 1$.

Then

$$g(z) = \frac{w(z + a_2 z^2 + \cdots\cdots)}{w - (z + a_2 z^2 + \cdots .)} = (z + a_2 z^2 + \cdots)\left(1 + \frac{z}{w} + \frac{a_2 z^2}{w} + \cdots\right)$$

90 by expansion

$$= z + \left(a_2 + \frac{1}{w}\right) z^2 + \text{ higer powers of } z.$$

Since the map is bi-linear, $g(z)$ is also univalent. In fact $g(z_1) = g(z_2)$ implies, $f(z_1) = f(z_2)$ from which it follows that $z_1 = z_2$. Further $g(z) \in S$. Hence by the above theorem,

$$\begin{aligned} \left|\left(\frac{1}{w} + a_2\right)\right| &\leq 2 \\ \left|\frac{1}{w}\right| &\leq 2 + |a_2| \\ &\leq 4 \end{aligned}$$

$\therefore$ $|w| \geq \dfrac{1}{4}$ and equality is possible only if $|a_2| = 2$, i.e. $f(z) = f_\theta(z)$.

Note that

$$f_0(z) = \frac{z}{(1 - z)^2} = \frac{1}{4}\left[\frac{(1 + z)^2}{(1 - z)^2} - 1\right]$$

if $\zeta = \dfrac{1 + z}{1 - z}$ then by this linear transformation $|z| < 1$ corresponds to real $\zeta > 0$ i.e., $Z = \zeta^2$ gives the plane cut along the negative axis. So

$f_0(z)$ is univalent and in fact $f_0(z)$ maps onto the plane cut from $-\dfrac{1}{4}$ to ∞ along the real axis. Hence $f_0(z)$ is univalent possessing such an expansion defined for elements of S and so does $f_\theta(z) = e^{-i\theta} f_0(ze^{i\theta})$. Also $f_\theta(z) \neq -\dfrac{1}{4}e^{-i}$ because $f_0(z) \neq -\dfrac{1}{4}$. Note also that

$$f_\theta(z) = z + \sum_{2}^{\infty} nz^n e^{i(n-1)\theta}$$

so that $|a_n| = n$ for all n.

Theorem 3. *Suppose $f(z) \in S$ and $|z| = r$, $0 < r < 1$ then the following* 91
inequalities hold.

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}$$
$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}$$
$$\frac{(1-r)}{r(1-r)} \leq \left|\frac{f'(z)}{f(z)}\right| \leq \frac{1+r}{r(1-r)}$$

where equality is possible only for functions $f_\theta(z)$ defined already.

Proof. Assume $|z_0| < 1$ and set

$$\varphi(z) = f\left[\frac{z_0 + z}{1 + \bar{z}_0 z}\right] = b_0 + b_1 z + \cdots\cdots$$

Since $\dfrac{z0 + z}{1 + \bar{z}_0 z} = w$ is a bi-linear map of the unit circle onto itself. $\varphi(z)$ is univalent in $|z| < 1$ and so $\dfrac{\varphi(z) - b_0}{b_1} \in S$.

Applying theorem 1 we deduce,

$$\left|\frac{b_2}{b_1}\right| \leq 2$$
$$|b_2| \leq 2|b_1|$$

we have

$$\begin{aligned}\varphi'(z) &= f'\left(\frac{z+z_0}{1+\overline{z}_0 z}\right)\left[\frac{1}{1+\overline{z}_0 z} - \frac{(z0+z)\overline{z}_0}{(1+\overline{z}_0 z)^2}\right]\\ &= f'\left[\frac{z+z_0}{1+\overline{z}_0 z}\right]\left\{\frac{1-(z_0)^2}{(1+z\overline{z}_0)^2}\right\}\end{aligned}$$

92 and

$$\begin{aligned}\varphi''(z) &= f''\left(\frac{z+z_0}{1+\overline{z}_0 z}\right)\left[\frac{1}{1+\overline{z}_0 z} - \frac{(z_0+z)\overline{z}_0}{(1+\overline{z}_0 z)^2}\right]^2\\ &\quad - 2f'\left(\frac{z+z_0}{1+\overline{z}_0 z}\right)\left[\frac{(1-|z_0|^2)\overline{z}_0}{(1+z\overline{z}_0)^2}\right]\end{aligned}$$

Thus

$$\begin{aligned}b_1 &= \varphi'(0) = (1-|z_0|^2)f'(z_0)\\ b_2 &= \frac{1}{2}\varphi''(0) = \frac{1}{2}[1-|z_0|^2]^2 f''(z_0) - f'(z_0)\overline{z}_0(1-|z_0|^2).\end{aligned}$$

We have seen $|b_2| \leq 2|b_1|$ i.e.

$$|f''(z_0)(1-|O|^2)^2 - 2\overline{z}_0 f'(z_0)(1-|z_0|^2)| \leq 4(1-|z_0|^2)|f'(z_0)| \tag{3.1}$$

If $z_0 = \rho e^{i\theta}$ this gives

$$\left|\frac{f''(z_0)}{f'(z_0)}z_0 - \frac{2\rho^2}{1-\rho^2}\right| \leq \frac{4\rho}{1-\rho^2}, \rho < 1$$

Now for any complex function $\dfrac{\partial w}{\partial r} = \dfrac{dw}{dz}e^{i\theta}$. If $z = re^{i\theta}$, and so $\dfrac{\partial}{\partial r}[\log f'(z)] = \dfrac{f''}{f'}e^{i\theta}$. Thus the above inequality is

$$\left|\rho\frac{\partial}{\partial\rho}\log f'(z) - \frac{2\rho^2}{1-\rho^2}\right| \leq \frac{4\rho}{1-\rho^2}, (z_0 e^{-i\theta} = \rho),$$

i.e.

$$\frac{2\rho-4}{1-\rho^2} \leq \frac{\partial}{\partial\rho}[\log|f'(\rho e^{i\theta})] \leq \frac{2\rho+4}{1-\rho^2}$$

because $\log|f'(z)| = \text{Rl. } \log f'(z)$.

Now integrate the above with regard to ρ,

$$\log\frac{1}{1-\rho^2} - 2\log\frac{1}{1-\rho} - 2\log(1+\rho) \le \log|f'(\rho e^{i\theta})|$$
$$\le \log\frac{1}{1-\rho^2} + 2\log\frac{1}{1-\rho} + 2\log(1+\rho).$$
$$\frac{1-\rho}{(1+\rho)^3} \le |f'(z)| \le \frac{1+\rho}{(1-\rho)^3}, z = \rho e^{i\theta}.$$

This gives bounds for $f'(z)$. **93**

Again,

$$|f(re^{i\theta})| = \left|\int_0^r f'(\rho e^{i\theta})d\rho\right| \le \int_0^r |f'(\rho e^{i\theta})|d\rho \le \int_0^r \frac{1+\rho}{(1-\rho)^3}d\rho = \frac{r}{(1-r)^2}$$

To get the lower bound for $w = f(\rho e^{i\theta})$, we suppose $|w| < \frac{1}{4}$, since for $|w| \ge \frac{1}{4}$ the result is trivial $\left(\frac{\rho}{(1+\rho)^2} < \frac{1}{4}\right)$. Thus by theorem 2 (Section 3) the line segment $[0, w]$ lies entirely in the image of $|z| < 1$ by $w = f(z)$. If λ is this line segment, γ the corresponding curve in the z-plane

$$|w| = \int_\lambda |dw| = \int_\gamma \left|\frac{dw}{dz}\right||dz| \ge \int_0^\rho \frac{1-\rho}{(1+\rho)^3}d\rho.$$
$$= \frac{\rho}{(\theta+\rho)^2}$$

because $\left|\frac{dw}{dz}\right||dz| \ge \frac{1-\rho}{(1+\rho)^3}d\rho$ if $d\rho$ is positive since $d\rho < |dz|$, if $d\rho < 0$ the result is even more evident.

Since $\frac{\varphi(z)-b_0}{b_1} \in S$, we obtain, **94**

$$\frac{\rho}{(1+\rho)^2} \le \left|\frac{\varphi(\rho e^{i\theta})-b_0}{b_1}\right| \le \frac{\rho}{(1-\rho)^2}.$$

Now $z_0 = \rho e^{i\theta}$.

Then $\varphi(z_0) = 0$, $b_0 = f(z_0)$, $b_1 = [1-|z_0|^2]f'(z_0)$

$$\frac{|z_0|}{1+|z_0|^2} \le \frac{|f(z_0)|}{[1-|z_0|^2]}|f'(z_0)| \le \frac{|z_0|}{1-|z_0|^2}$$

$$\text{i.e.} \quad \frac{r(1-r)}{1+r} \le \left|\frac{f(z)}{f'(z)}\right| \le \frac{r(1+r)}{1-r}.$$

□

This gives the bounds for $\left|\frac{f'}{f}\right|$ since equality holds in (3.1) only for $\varphi(z) = f_\theta(z)$ it can be shown that for all the inequalities of theorem 3 equality is possible only for this function.

Theorem 4 (Littlewood, Paley, Spencer)**.** *Suppose* $f(z) \in S$ *and for any* $\lambda \to 0$ *set*

$$I_\lambda(r, f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^\lambda d\theta \tag{3.2}$$

$S_\lambda(r) = r\frac{d}{dr}I_\lambda(r)$.

Then

$$S_\lambda(r) = \frac{\lambda^2}{2\pi}\int_0^r \rho d\rho \int_{-\pi}^{\pi} |f(\rho e^{i\theta})|^{\lambda-2}|f'(\rho e^{i\theta})|^2 d\theta \tag{3.3}$$

Thus

$$S_\lambda(r) \le \lambda M(r, f)^\lambda \le \frac{\lambda r^\lambda}{(1-r)^{2\lambda}}, \tag{3.4}$$

$$I_\lambda(r) = \int_0^r \frac{S_\lambda(\rho)d\rho}{\rho} \le \lambda \int_0^r M(\rho, f)^\lambda \frac{d\rho}{\rho}$$

$$\le \begin{cases} A(\lambda)[1-r]^{1-2\lambda}, & \lambda > \frac{1}{2} \\ A\log\dfrac{1}{1-r} & \lambda = \frac{1}{2} \\ A(\lambda) & \lambda < \dfrac{1}{2} \end{cases} \tag{3.5}$$

Proof. Suppose

$$f(z) = z\left[1 + \sum_{2}^{\infty} a_n z^{n-1}\right]$$

Set

$$\varphi(z) = [f(z)]^{\lambda/2} = z^{\frac{1}{2}\lambda}\left[1 + \sum_{1}^{\infty} b_n z^n\right]$$

This is possible since $\dfrac{f(z)}{z}$ is regular and non-zero in $|z| < 1$ and so has a $\dfrac{\lambda}{2}$th power which is also regular. For definiteness we take the principal value of $z^{\lambda/2}$, $|\arg z| < \pi$.

We have $\varphi'(z) = \sum\limits_{0}^{\infty}\left[n + \dfrac{\lambda}{2}\right] b_n z^{n+\frac{1}{2}\lambda-1}$.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |\varphi'(\rho e^{i\theta})|^2 d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{\varphi}'\varphi' d\theta = \sum_{n=0}^{\infty}\left[n + \frac{\lambda}{2}\right]^2 |b_n|^2 \rho^{2n+\lambda-2}$$

exactly as in lemma 1 (Section 3)

$$I_\lambda(r, f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\varphi(re^{i\theta})|^2 d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi\overline{\varphi} d\theta = \sum_{0}^{\infty} |b_n|^2 \rho^{2n+\lambda}$$

$$S_\lambda(r, f) = r\frac{d}{dr} I_\lambda(r, f) = \sum (2n + \lambda)|b_n|^2 r^{2n+\lambda}.$$

Further **96**

$$\int_{0}^{r} \rho d\rho \int_{-\pi}^{\pi} |\varphi'(\rho e^{i\theta})d\theta = \frac{\pi}{2}\sum (2n + \lambda)|b_n|^2 r^{2n+\lambda} = \frac{\pi}{2} S_\lambda(r, f).$$

Now $\varphi' = \dfrac{\lambda}{2} f' f^{\frac{\lambda-1}{2}}$

$$|\varphi'|^2 = \frac{\lambda^2}{4}|f'|^2|f^{\lambda-2}|$$

Thus

$$S_\lambda(r, f) = \frac{\lambda^2}{2\pi} \int\limits_0^r \rho d\rho \int\limits_{-\pi}^{\pi} |f'(\rho e^{i\theta})|^2 |f(\rho e^{i\theta})|^{\lambda-2} d\theta.$$

giving (3.3).

We now interpret the above formula. $\rho d\rho\ d\theta$ is an element of area in $|z| < 1$, $|f'|^2 \rho d\rho d\theta$ is the corresponding area in the image plane at $w = Re^{i\phi}$ and $|f|^{\lambda-2}|f'|^2 \rho d\rho\ d\theta$ the mass of the area, if we imagine a mass density $R^{\lambda-2}$ on the circle $|w| = R$ in the image plane. Then our integral is the total mass of the image. Since the image covers no point more than once and lies in $|w| < M(r, f) = M$ say, the mass of image $\leq$ total mass of the circle which is

$$\int\limits_0^M 2\pi R \cdot R^{\lambda-2} dR = \frac{2\pi}{\lambda} M^\lambda \text{ and so}$$

$$S_\lambda(r) \leq \frac{\lambda^2}{2\pi} \frac{2\pi}{\lambda} M^\lambda = \lambda M^\lambda \leq \lambda \frac{r^\lambda}{(1-r)^{2\lambda}}$$

This gives (3.4), because from theorem 3,

$$M(r, f) \leq \frac{r}{(1-r)^2}.$$

97 Thus

$$I_\lambda(r) \leq \lambda \int\limits_0^r \frac{\rho^{\lambda-1}}{(1-theta)^{2\lambda}} d\rho. \text{ If } \lambda \geq 1, \rho^{\lambda-1} \leq 1 \text{ and so}$$

$$I_\lambda(r) \leq \lambda \int\limits_0^r \frac{1}{(1-\rho)^{2\lambda}} d\rho \leq \frac{\lambda}{2\lambda-1} (1-r)^{1-2\lambda}.$$

Assume that $\lambda < 1$. Then

$$I_\lambda(r) \leq \lambda \int\limits_0^r \frac{\rho^{\lambda-1}}{(1-\rho)^2} d\rho = \frac{\rho^\lambda}{(1-\rho)^{2\lambda}} \Bigg]_0^r - 2\lambda \int\limits_0^r \frac{\rho^\lambda}{(1-\rho)^{2\lambda+1}} d\rho.$$

$$\leq \frac{r^\lambda}{(1-r)^{2\lambda}} - 2\lambda \int_0^r \frac{\rho}{(1-\rho)^{2\lambda+1}} d\rho \quad \text{since} \quad \rho \leq \rho^\lambda \quad \text{if} \quad 0 \leq \lambda \leq 1$$

$$= \frac{r^\lambda}{(1-r)^{2\lambda}} + 2\lambda \int_0^r \frac{1}{(1-\rho)^{2\lambda}} d\rho - 2\lambda \int_0^r \frac{1}{(1-\rho)^{2\lambda+1}} d\rho$$

$$= \frac{r^\lambda}{(1-r)^{2\lambda}} + \frac{2\lambda}{2\lambda-1}(1-r)^{1-2\lambda} - \frac{2\lambda}{2\lambda-1} - \frac{1}{(1-r)^{2\lambda}} + 1 \quad \text{if} \quad 2\lambda \neq 1.$$

$$\leq \frac{2\lambda}{2\lambda-1}(1-r)^{1-2\lambda} - \frac{2\lambda}{2\lambda-1} + 1 \leq \frac{2\lambda}{2\lambda-1}(1-r)^{1-2\lambda}$$

Hence if $1-2\lambda > 0$, $(1-r)^{1-2\lambda} \leq 1$, $I_\lambda(r) \leq \dfrac{2\lambda}{2\lambda-1}$.

If $2\lambda = 1$, $I_\lambda(r) \leq \frac{1}{2}\int_0^r \frac{1}{\rho^{\frac{1}{2}}(1-\rho)} d\rho = \int_0^r \frac{d\rho}{1-\rho^2}$ **98**

$$= \frac{1}{2}\log\frac{1+r}{1-r} = \frac{1}{2}\log\frac{1-r^2}{(1-r)^2} \leq \frac{1}{2}\log\frac{1}{(1-r)^2} = \log\frac{1}{(1-r)}$$

Thus

$$I_\lambda(r) \leq \begin{cases} \dfrac{2\lambda}{2\lambda-1}(1-r)^{1-2\lambda} & \text{if} \quad \lambda > \dfrac{1}{2} \\ \log\dfrac{1}{1-r} & \text{if} \quad \lambda = \dfrac{1}{2} \\ \dfrac{2\lambda}{2\lambda-1} & \text{if} \quad \lambda < \dfrac{1}{2} \end{cases}$$

This proves the theorem completely. □

Theorem 5 (Littlewood)**.** *If* $f(z) = z + \sum_2^\infty a_n z^n \in S$ *then* $|a_n| < e$ *for* $n \geq 2$.

Proof. We have $|a_n| = \left| \frac{1}{2\pi} \int_{|z|=\rho} \frac{f(z)dz}{z^{n+1}} \right|$

$$\leq \frac{1}{2\pi\rho^n} \int_{-\pi}^{\pi} |f(\rho e^{i\theta})| d\theta = \frac{I_1(\rho, f)}{\rho^n}$$

By inequality (3.5)

$$I_1(\rho, f) \leq \int_0^{\rho} \frac{dt}{(1-t)^2} = \frac{\rho}{1-\rho}.$$

Thus $|a_n| < \dfrac{1}{\rho^{n-1}(1-\rho)}$. We choose ρ so that $\dfrac{1}{\rho^{n-1}(1-\rho)}$ is minimal i.e. put $\rho = 1- = \dfrac{n-1}{n}$ to get $|a_n| < \left(\dfrac{n}{n-1}\right)^{n-1} n$

$$\text{i.e., } |a_n| < \left(1 + \frac{1}{n-1}\right)^{n-1} n < en$$

99 This completes theorem 5. □

Remark. Bazilevic [1] improved this to $I_1(\rho, f) < \dfrac{\rho}{1-\rho^2} + 0.55$ so that we get

$$|a_n| < \frac{1}{2} en + 1.51$$

Theorem 6. *Suppose $f(z) \in S$ and set*

$$\varphi(z) = [f(z)]^{\lambda} \quad \varphi(z) = z^{\lambda}\left[\sum_0^{\infty} a_{n,\lambda} z^n\right].$$

Then if $\lambda > \frac{1}{4}$, $|a_{n,\lambda}| < A(\lambda) n^{2\lambda-1}$. In particular if $f(z) = z + a_{k+1}z^{k+1} + \cdots + a_{kn+1}z^{kn+1} + \cdots \in S$, then $|a_{2n+1}| < A_1$ if $k = 2$ and $|a_{3n+1}| < A_2 n^{-1/3}$ if $k = 3$, A_1 and A_2 are absolute constants.

Proof. Now $(n+\lambda)|a_{n,\lambda}| = \left|\frac{1}{2\pi i} \int\limits_{|z|=\rho} \dfrac{\varphi'(z)dz}{z^{n+\lambda}}\right|$

$$\leq \frac{1}{\rho^{n+\lambda-1}} I_1(\rho, \varphi') \ldots \tag{3.6}$$

with the notation of theorem 4. □

Since $\varphi' = \lambda f^{\lambda-2} f'$

$$I_1(\rho, \varphi') = \frac{\lambda}{2\pi} \int_0^{2\pi} |f'(\rho e^{i\theta})||f(\rho e^{i\theta})|^{\lambda-1} d\theta$$

$$= \frac{\lambda}{2\pi} \int_0^{2\pi} |f'(\rho e^{i\theta})||f(\rho e^{i\theta})|^{t-1}|f(\rho e^{i\theta})|^{\lambda-t} d\theta$$

for any t

$$I_1(\rho, \varphi') \leq \left[\frac{1}{2\pi} \int_0^{2\pi} |f'(\rho e^{i\theta})|^2 |f(\rho e^{i\theta})|^{2t-2} d\theta\right]^{\frac{1}{2}} \times$$

$$\lambda \left[\frac{1}{2\pi} \int_0^{2\pi} |f(\rho e^{i\theta})|^{2\lambda-2t} d\theta\right]^{\frac{1}{2}} \tag{3.7}$$

by Schwarz inequality. 100

Since $\lambda > \frac{1}{4}$, we choose t sufficiently small but positive such that $2\lambda - 2t > \frac{1}{2}$; for example $t = \frac{1}{2}(\lambda - \frac{1}{4})$ can be a choice of t. By theorem 4

$$\int_{1-\frac{1}{n}}^{1-\frac{1}{2n}} \rho d\rho \int_0^{2\pi} |f'(\rho e^{i\theta})|^2 |f(\rho e^{i\theta})|^{2t-2} d\theta \leq \frac{\pi}{2t^2} S_{2t}\left(1 - \frac{1}{2n}, f\right)$$

$$\leq A(t) n^{4t}$$

Suppose

$$2\pi\mu = \min_{1-\frac{1}{n} \leq \rho \leq 1-\frac{1}{2n}} \int_0^{\pi} |f'(\rho e^{i\theta})|^2 |f(\rho e^{i\theta})|^{2t-2} d\theta$$

Hence

$$\mu \leq \frac{A(t)n^{4t}}{\int\limits_{1-\frac{1}{n}}^{1-\frac{1}{2n}} \rho \, d\rho} \leq A(t)n^{4t+1} \tag{3.8}$$

Also by theorem 4,

$$\int\limits_0^{2\pi} |f(\rho e^{i\theta})|^{2\lambda-2t} d\theta \leq A(\lambda, t)(1-\rho)^{-4\lambda+4t+1} \text{ since } 2\lambda - 2t \geq \frac{1}{2}$$

$$\leq A(\lambda)n^{4\lambda-4t-1} \tag{3.9}$$

101 since t depends upon λ.

Hence for this value ρ from (3.7),

$$I_1(\rho, \varphi') \leq A(\lambda)n^{(4t+1+4\lambda-4t-1)\frac{1}{2}} = A(\lambda)n^{2\lambda}$$

Therefore from (3.6) it follows

$$\begin{aligned}(n+\lambda)|a_{n,\lambda}| \leq \frac{A(\lambda)n^{2\lambda}}{\rho^{n+\lambda-1}} &< \left(1 - \frac{1}{n}\right)^{-n-\lambda} A(\lambda)n^{2\lambda} \\ &\leq A(\lambda)n^{2\lambda}\end{aligned}$$

$A(\lambda)$ just standing for a constant depending upon λ. Hence the result $|a_{n,\lambda}| < A(\lambda)n^{2\lambda-1}$ follows.

Suppose now that $g(z) = z + \sum\limits_{n=1}^{\infty} a_{kn+1}z^{kn+1} \in S$. Then so does $f(z) = [g(z^{1/k})]^k = z\left(1 + \sum\limits_{n=1}^{\infty} a_{kn+1}z^n\right)^k$. For clearly $f(z)$ is regular in $|z| < 1$. $f(0) = 0$, $f'(0) = 1$. Suppose $f(z_1) = f(z_2)$. Then

$$\left[g\left(z_1^{\frac{1}{k}}\right)\right]^k = \left[g\left(z_2^{\frac{1}{k}}\right)\right]^k \Longrightarrow g\left(z_1^{\frac{1}{k}}\right) = \omega g\left(z_2^{\frac{1}{k}}\right) = g\left(z_2^{\frac{1}{k}}\omega\right)$$

where ω is a kth root of unity. But $g(z) \in S$, and we have $z_1^{\frac{1}{k}} = \omega z_2^{\frac{1}{k}} \Longrightarrow z_1 = z_2$.

Thus

$$g\left(z^{\frac{1}{k}}\right) = [f(z)]^{1/k}$$
$$= z^{1/k}\left(1 + \sum_{n=1}^{\infty} a_{kn+1} z^n\right)$$

Hence applying the first part, with $\lambda = 1/k$ (provided $k = 1, 2, 3$) 102

$$\begin{aligned} & |a_{kn+1}| < A(k)n \quad \text{as required} \\ \text{i.e.} \quad & |a_{n+1}| < A_1 n \quad \text{if} \quad k = 1 \\ & |a_{2n+1}| < A_2 \quad \text{if} \quad k = 2 \\ & |a_{3n+1}| < A_3 n^{-1/3} \quad \text{if} \quad k = 3 \end{aligned}$$

This inequality is false for large k as was shown by Little wood [1] even if $f(z)$ is continuous in $|z| \leq 1$. We do not know whether it is true for any $k \geq 4$. If $f(z) = z + \sum\limits_{2}^{\infty} a_n z^n$ is bounded and univalent then the area of the image of $|z| < 1$ is at most πM^2 where M is the least upper bound of $|f(z)|$. Hence $\sum n|a_n|^2 \leq M^2$ and it follows that $|a_n| = O(n^{-\frac{1}{2}})$ as $n \to \infty$. Nothing stronger than this is known. For further discussion see Hayman Chapter 3. This is the correct order for mean-valent functions and probably for $f(z) \in S$ also. The best example due to Clunie (unpublished) gives $|a_n| > n^{-13/14}$ for some large n, so that there is a gap between $\dfrac{1}{2}$ and $13/14$. It can be shown that in theorem 6 the conclusion that $|a_n| = O(1)$ obtains for all n if $|a_n| = O(1)$ for some sequence of n with constant common difference. We do not know if this conclusion is till true i.e. if $|a_n| = O(1)$ for a sequence $n = n_k$ such that $n_{k+1} - n_k <$ constant.

In this connection Biernacki [1] has shown that for every $f(z) \in S$,

$$||a_{n+1}| - |a_n|| < A[\log(n)]^{3/2} \quad \text{for} \quad n \geq 2.$$

Hence if $a_n = 0$ for a sequence $n = n_k$, with $n_{k+1} - n_k <$ const. $|a_n| =$ 103
$O(\log n)^{3/2}$ for the intermediate coefficients. It is still to be found out whether $O(1)$ can replace $O(\log n)^{3/2}$.

Examples for theorem 6.

$$f(z) = \frac{z}{(1-z)^2}$$
$$f_k(z) = [f(z^k)]^{1/k} = \frac{z}{(1-z^k)^{2/k}} = \sum a_{n,k} z^{kn+1}$$

where

$$\begin{aligned} a_{n,k} &= \frac{\frac{2}{k}\left(\frac{2}{k}+1\right)\ldots\ldots\left(\frac{2}{k}+n-1\right)}{1\cdot 2\ldots\ldots n} \\ &= \frac{\Gamma\left(n+\frac{2}{k}\right)}{\Gamma(n+1)\Gamma(2/k)} \sim \frac{n^{\frac{2}{k}-1}}{\Gamma(2/k)} \end{aligned}$$

so that theorem 6 is best possible.

Theorem 7 (Rogosinski, Deudonne', Szasz)**.** *If $f(z) = z + \sum\limits_2^\infty a_n z^n$ belongs to S and has real coefficients then $|a_n| \leq n$.*

Proof. Let $f = u + iv$. We have since a_n are real $f(\bar{z}) = \overline{f(z)}$. $z = x + iy$, $y \neq 0$ implies $v \neq 0$ for otherwise if $v = 0$ for $z = x + iy$, $y \neq 0$ then $f(\bar{z}) = f(z) = u$ which contradicts uni valency. Further we assert that for z such that $y > 0$ v must have constant sign. For if instead we have $v(z_1)$
104 positive and $v(z_2)$ negative z_1 and z_2 having the imaginary part positive, $v(z)$ would be zero somewhere on the line joining z_1 and z_2, which is not possible. Clearly by considering $f(z)/z$ for small z $v > 0$ for z with $\operatorname{Im} z > 0$. The proof of the theorem essentially depends on this nature of the sign of v. □

Now $v(re^{i\theta}) = \sum\limits_1^\infty a_n r^n \sin n\theta$ since a_n are real where $a_1 = 1$.

Hence we have on integration, $\int\limits_0^\pi v(re^{i\theta}) \sin\theta n \, d\theta = \dfrac{\pi}{2} r^n a_n$ since the other terms vanish.

Also $|\sin n\theta| \leq n \sin\theta$ for $0 \leq \theta \leq \pi$.

It is enough to show this inequality for $0 \leq \theta \leq \dfrac{\pi}{2}$ since $|\sin n(\pi - \theta)| = |\sin n\theta|$, $\sin(\pi - \theta) = \sin\theta$.

$\dfrac{\operatorname{Sin}\theta}{\theta}$ decreases in $0 < \theta \leq \dfrac{\pi}{2}$.

Hence if $n\theta \leq \dfrac{\pi}{2}$ i.e. $\theta \leq \dfrac{\pi}{2n}$

$$\frac{\sin n\theta}{n\theta} \leq \frac{\sin\theta}{\theta}$$

which gives the inequality for $\theta \leq \dfrac{\pi}{2n}$.

In particular when

$$n\theta = \frac{\pi}{2}\frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}} \leq \frac{\sin\frac{\pi}{2n}}{\frac{\pi}{2n}}$$

or $\sin\dfrac{\pi}{2n} \geq \frac{1}{n}$.

Hence if $\dfrac{\pi}{2n} \leq \theta \leq \dfrac{\pi}{2}$. **105**

$$|\sin n\theta| \leq 1 \leq n\sin\frac{\pi}{2n} \leq n\sin\theta$$

Hence follows the inequality $|\sin n\theta| \leq n\sin\theta \quad 0 \leq \theta \leq \pi$.

$$|a_n| = \left|\frac{2}{\pi r^n}\int_0^\pi v(re^{i\theta})\sin n\theta\, d\theta\right|$$

Now making use of the fact that $v > 0$ for the range of θ

$$\begin{aligned}
|a_n| &\leq \frac{2}{\pi r^n}\int_0^\pi v(re^{i\theta})|\sin\theta n|d\theta \\
&\leq \frac{2}{\pi r^n}\int_0^\pi v(re^{i\theta})n\sin\theta\, d\theta \\
&= \frac{n}{r^{n-1}}a_1 = \frac{n}{r^{n-1}}
\end{aligned}$$

Let us make $r \to 1$,

$$|a_n| \leq n \quad \text{as required.}$$

The function $\dfrac{z}{(1-z)^2} \in S$ has real coefficients, and satisfies $a_n = n$. Thus our inequality is sharp.

We next proceed to prove another special case of Bieberbach's conjecture. For this we need the definition:

Definition. *A domain D is said to be star-like w.r.t. the origin O if for any point $P \in D$, OP lies entirely in D.*

106 **Theorem 8** (Nevanlinna). *If $w = f(z) = z + \sum\limits_{2}^{\infty} a_n z^n \in S$ and maps $|z| < 1$ onto a domain D star-like with respect to $w = 0$ then $|a_n| \leq n$. Equality is possible only when $f(z) = f_\theta(z)$ defined already.*

Before proceeding with the proof of the theorem let us prove the following lemma due to Borel.

Lemma 2. *If $\varphi(z) = 1 + \sum\limits_{n=1}^{\infty} b_n z^n$ satisfies $Rl\varphi(z) \geq 0$ for $z \geq 0$ then $|b_n| \leq 2$, equality holds only for $\varphi(z) = \dfrac{1+z}{1-z} = 1 + \sum\limits_{1}^{\infty} 2z^n$.*

Consider $\psi(z) = \dfrac{\varphi(z)-1}{\varphi(z)+1}$. By hypothesis $\varphi(z)$ has real part +ve which clearly implies $|\varphi(z)-1| < |\varphi(z)+1|$ and so $|\varphi(z)| < 1$ and further $\psi(0) = 0$.

Hence now applying Schwarz's lemma

$$|\psi'(0)| \leq 1. \text{ Equality holds only if } \psi(z) = ze^{i\theta}.$$

Near

$$z = 0 \quad \psi(z) = \frac{\left(\sum\limits_{1}^{\infty} b_n z^n\right)}{2 + \sum\limits_{n=1}^{\infty} b_n z^n} = \frac{b_1}{2} z + \cdots\cdots$$

which gives that $|b_1| \leq 2$. In order to deduce the inequality for b_n, consider

$$\varphi_k(z) = 1 + b_k z + b_{2k} z^2 + \cdots\cdots + b_{nk} z^n + \cdots\cdots$$

$$= \frac{1}{k}\sum_{r=1}^{k}\varphi(w_r z^{1/k})$$

w_r running through the kth roots of unity and for a fixed value of $z^{1/k}$. For $w_r = w^r$ where $w = e^{\frac{2\pi i}{k}}$. If we set

$$\begin{aligned}
\zeta = z^{1/k},\quad \frac{1}{k}\sum_{r=1}^{k}(w^r z^{1/k}) &= 1 + \frac{1}{k}\sum_{n=1}^{\infty} b_n\zeta^n[w^n + \cdots + w^{kn}]\\
&= 1 + \sum_{n=1}^{\infty} b_{nk}\zeta^{nk}\\
&= 1 + \sum_{1}^{\infty} b_{nk} z^n\\
\text{as } \ w^n + \cdots\cdots + w^{kn} &= 0 \ \text{ if } \ k \ \text{ does not divide } \ n\\
&= k \ \text{ if } \ k \ \text{ divides } \ n.
\end{aligned}$$

Clearly $Rl\,\varphi_k(z) \geq 0$ since $Rl\varphi(w_r z^{1/k}) \geq 0$ for every. Hence applying **107**
previous part to $\varphi_k(z)$, $|b_k| \leq 2$.

Proof of theorem 3.8. Note first that if G_r denotes the image of $|z| < r$ by $f(z)$ then if G_1 is star-like so is G_r for $0 < r \leq 1$.

Consider $\zeta = \varphi(z) = f^{-1}[tf(z)]$ $0 < t < 1$. Then since G_1 is star-like $t\,f(z)$ lies in the image of $|z| < 1$ by $f(z)$ and so $f^{-1}[tf(z)]$ is a well defined point in $|\zeta| < 1$, and clearly $\zeta = 0$ corresponds to $z = 0$. Thus $|\varphi(z)| < 1$ $\varphi(0) = 0$ and hence by Schwarz's lemma

$$|\varphi(z)| \leq |z|$$

$tf(z) = f(\zeta)$ where $|\zeta| \leq |z|$ and so if $|z| < r$, $f(\zeta) \in G_r$ i.e. $tf(z) \in G_r$. This being true for any t, $0 < t < 1$, G_r is star-like.

Since G_r is star-like its boundary Γ_r meets any ray from 0 to ∞ in **108**
only one point. The limiting case when Γ_r contains a line segment in such a ray is excluded since Γ_r is a simple closed analytic curve. Thus $\arg.f(re^{i\theta})$ increases with θ for any fixed r, $0 < r < 1$. Let

$$\log f(re^{i\theta}) = u + iv = \log|f(re^{i\theta})| + i\arg.f(re^{i\theta})$$

and arg . $f(re^{i\theta})$ is increasing for any fixed r implies $\dfrac{\partial v}{\partial\theta} \geq 0$.

$$\begin{aligned} \text{i.e.,}\quad & \operatorname{Im}.\frac{\partial}{\partial\theta}[\log f(re^{i\theta})] \geq 0 \\ & \operatorname{Im}.\left[ire^{i\theta}\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right] \geq 0 \\ \text{i.e.,}\quad & Rl.\frac{re^{i\theta}f'(re^{i\theta})}{f(re^{i\theta})} \geq 0 \end{aligned}$$

In other words,

$$Rl.\frac{zf'(z)}{f(z)} \geq 0, \quad 0 \leq |z| < 1$$

Now applying the lemma with $\varphi(z) = zf'(z)/f(z)$ and observing, that this function satisfies the hypothesis of the lemma, (for $\varphi(z) = 1 + O(z)$ near $z = 0$) $Rl.\varphi(z) \geq 0$ and so if

$$\varphi(z) = \frac{zf'(z)}{f(z)} = 1 + \sum_1^\infty b_n z^n$$

then $|b_n| \leq 2$.

Again

$$zf'(z) = \sum_1^\infty na_n z^n = \varphi(z)f(z) = \left(\sum_1^\infty a_n z^n\right)\left(1 + \sum_{n=1}^\infty b_n z^n\right)$$

with $a_1 = 1$.

109 Equating coefficient of z^n on either side,

$$\begin{aligned} & na_n = a_n + b_1a_{n-1} + b_2a_{n-2} + \cdots\cdots + b_{n-1}a_1 \\ \text{i.e.}\quad & (n-1)a_n = b_1a_{n-1} + b_2a_{n-2} + \cdots\cdots + b_{n-1}a_1 \\ & (n-1)a_n \leq 2[1 + |a_2| + |a_3| + \cdots\cdots + |a_{n-1}|] \text{ by the lemma.} \end{aligned}$$

Suppose we now assume that $|a_\nu| \leq \nu$ for $\nu \leq n-1$ then we have,

$$\begin{aligned} (n-1)|a_n| &\leq 2[1 + 2 + \cdots\cdots + n - 1] \\ &\leq n(n-1) \end{aligned}$$

$$|a_n| \leq n.$$

and by theorem 1, $|a_2| \leq 2$. Hence the proof is complete by induction on n. Further equality is possible for $n \geq 2$ only if $|a_2| = 2$ i.e. for functions $f_\theta(z) = \frac{z}{(1-ze^{i\theta})^2}$.

For extensions of this result to a more general class of image domains see Kaplan, W. [1].

3.2 Asymptotic behaviour

Theorem 9. *If $f(z) \in S$ and unless $f(z) \equiv f_\theta(z)$ we have $\dfrac{(1-r)^2}{r} M(r, f)$ decreasing steadily with increasing r and so tends to α where $0 \leq \alpha < 1$ as $r \to 1$.*

To prove this we require

Lemma 3. *Suppose $f(z) \in S$ and for fixed θ $f(re^{i\theta}) = R(r)e^{i\lambda(r)}$. Suppose further that $0 < r_1 < r_2 < 1$ and $r = r_1, r_2$ correspond to $R = R_1, R_2$. Then* 110

$$\log \frac{R_2(1-r_2)^2}{r_2} \leq \log \frac{R_1(1-r_1)^2}{r_1} - \frac{r_1}{4} \int_{r_1}^{r_2} (1-r)[\lambda'(r)]^2 dr.$$

Proof. We have

$$\frac{d}{dr} \log(Re^{i\lambda}) = \frac{d}{dr} \log f(re^{i\theta}) = e^{i\theta} \frac{f'(re^{i\theta})}{f(re^{i\theta})}.$$

On the other hand

$$\frac{d}{dr} \log(Re^{i\lambda}) = \frac{1}{R} \frac{dR}{dr} + i\lambda'(r)$$

By theorem 3

$$\left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right| \leq \frac{1+r}{r(1-r)} \quad \text{so that}$$

$$\left[\frac{1}{r}\frac{dR}{dr}\right]^2 + [\lambda'(r)]^2 \leq \frac{(1-r)^2}{r^2(1-r)^2},$$

say $a^2 + b^2 \leq c^2$ i.e. $|a|^2 + |b|^2 \leq c^2$.

This can be written as

$$(|c| - |a|)(|c| + |a|) \leq b^2$$

$$\text{or} \qquad |c| - |a| \geq \frac{b^2}{|c| + |a|} \geq \frac{b^2}{2|c|}$$

since $|c| \geq |a|$. So $|a| \leq |c| - \dfrac{b^2}{2|c|}$. Since $a \leq |a|$ we get

$$\begin{aligned}\frac{1}{R}\frac{dR}{dr} &\leq \frac{1-r}{r(1-r)} - \frac{r(1-r)}{2(1+r)}[\lambda'(r)]^2. \\ &\leq \frac{1+r}{r(1-r)} - \frac{r_1}{4}(1-r)[\lambda'(r)]^2\end{aligned}$$

111 since $r_1 \leq r$ for the range $r_1 \leq r \leq r_2$ and $1 + r < 2$. Integrating from r_1 to r_2

$$\begin{aligned}\log R_2 - \log R_1 &\leq \int_{r_1}^{r_2} \frac{1+r}{r(1-r)}dr - \frac{r_1}{4}\int_{r_1}^{r_2}(1-r)[\lambda'(r)]^2. \\ &= \int_{r_1}^{r_2} \frac{1}{r(1-r)}dr + \int_{r_1}^{r_2} dr - \frac{r_1}{4}\int_{r_1}^{r_2}(1-r)[\lambda'(r)]^2 dr. \\ &= \log\frac{r_2}{(1-r_2)^2} - \log\frac{r_1}{(1-r_1)^2} - \frac{r_1}{4}\int_{r_1}^{r_2}(1-r)[\lambda'(r)]^2 dr.\end{aligned}$$

Thus

$$\log\frac{R_2(1-r_2)^2}{r_2} \leq \log\frac{R_1(1-r_1)^2}{r_1} - \frac{r_1}{4}\int_{r_1}^{r_2}(1-r)[\lambda'(r)]^2 dr$$

giving the lemma □

Proof of theorem 3.9. Set $\psi(r) = \dfrac{(1-r)^2}{r} M(r, f)$. Suppose that θ is so chosen that $f(r_2 e^{i\theta}) = M(r_2, f) = R_2$. Then lemma 3 gives

$$\log \psi(r_2) \leq \log \frac{(1-r_1)^2 R_1}{r_1} - \frac{r_1}{4} \int_{r_1}^{r_2} (1-r)[\lambda'(r)]^2 \, dr.$$

$$\leq \log \psi(r_1) - \frac{r_1}{4} \int_{r_1}^{r_2} (1-r)[\lambda'(r)]^2 \, dr.$$

since $R_1 \leq M(r_1, f)$. **112**

Hence $\psi(r_2) \leq (r_1)$ showing that $\psi(r)$ decreases (weakly) with r. If $\psi(r_1) = \psi(r_2)$ then $\psi(r) = a$ constant for $r_1 \leq r \leq r_2$ and

$$\int_{r_1}^{r_2} (1-r)[\lambda'(r)]^2 dr = 0.$$

Since $1 - r > 0$ therefore $\lambda'(r) = 0$ or $\lambda(r) =$ constant for $r_1 \leq r \leq r_2$. Also $\psi(r) =$ constant gives $\dfrac{(1-r)^2}{r} |f(re^{i\theta})| =$ constant $= \alpha$ (say) for $r_1 \leq r \leq r_2$. Because from the lemma $\dfrac{(1-r)^2}{r} |f(re^{i\theta})|$ decreases with increasing r for fixed θ. Therefore if θ_1 is such that $f(r_2 e^{i\theta_1}) = M(r_2, f)$ we see that

$$\frac{(1-r_2)^2}{r_2} M(r_2, f) = \frac{(1-r_2)^2}{r_2} |f(r_2 e^{i\theta_1})|$$

$$\leq \frac{(1-r_1)^2}{r_1} |f(r_1 e^{i\theta_1})| \leq \frac{(1-r_1)^2}{r_1} M(r_1, f)$$

$$= \frac{(1-r_2)^2}{r_2} M(r_2, f) \quad \text{because } \psi(r) = \text{ constant.}$$

Hence $\dfrac{(1-r)^2}{r} |f(re^{i\theta})|$ is constant for $r_1 \leq r \leq r_2$.

Now

$$f(re^{i\theta}) = Re^{i\lambda} = \frac{\alpha r}{(1-r)^2} e^{i\lambda} = \frac{\alpha(re^{i\theta}) e^{i(\lambda-\theta)}}{[1 - e^{i\theta} re^{-i\theta}]^2}$$

113 Or $f(z) = \dfrac{\alpha e^{i(\lambda-\theta)}z}{(1-ze^{-i\theta})^2}$ for $z = re^{i\theta}$, $r_1 < r < r_2$.

Hence by analytic continuation this equation holds in $|z| < 1$ and since $f(z) \in S$, $\alpha e^{i(\lambda-\theta)} = 1$ and so $f(z) \equiv f_{-\theta}(z)$ as required.

In all other cases $\psi(r)$ decreases strictly with increasing r and

$$\lim_{r\to 0} \psi(r) = 1 = \lim \frac{M(r, f)}{r}.$$

Thus $\psi(r) < 1$, $0 < r < 1$ and since $\psi(r)$ decreases $\alpha = \lim\limits_{r\to 1} \psi(r)$ exists and $\alpha \leq \psi\left(\dfrac{1}{2}\right) < 1$.

Theorem 10. *If $f(z) \in S$ and $\alpha = 0$ in theorem 9 then with the notation of theorem 6 we have for $\lambda > \dfrac{1}{4} a_{n,\lambda} = O(n^{2\lambda-1})$ as $n \to \infty$ and in particular if a_n are the coefficients of f, $\dfrac{a_n}{n} \to 0$ as n tends to infinity.*

Proof. We recall the proof of theorem 6 and the notations of that theorem. We proved

$$(n+\lambda)|a_{n,\lambda}| \leq \frac{1}{n+\lambda-1} I_1(\rho, \varphi')$$

$$I_1(\rho, \varphi') \leq \left[\frac{\lambda}{2\pi} \int\limits_{-\pi}^{\pi} |f'(\rho e^{i\theta})|^2 |f(\rho e^{i\theta})|^{2t-2}\right]^{\frac{1}{2}} \frac{\lambda}{2\pi} \int\limits_{-\pi}^{\pi} f(\rho e^{i\theta})^{2\lambda-2t} d\theta \;]^{\frac{1}{2}}$$

We can find ρ such that $1 - \dfrac{1}{n} \leq \rho \leq 1 - \frac{1}{2n}$ and

$$\frac{1}{2\pi} \int\limits_{-\pi}^{\pi} |f'(\rho e^{i\theta})|^2 |f(\rho e^{i\theta})|^{2t-2} d\theta \leq \frac{S_{2t}\left(1 - \frac{1}{2n}, f\right)}{\frac{1}{2}\left[\left(1 - \frac{1}{2n}\right)^2 - \left(1 - \frac{1}{n}\right)^2\right]} \frac{1}{4t^2}$$

114 By theorem 4

$$S_{2t}(r, f) \leq 2tM(r, f)^{2t} = O[(1-r)]^{-4t}$$

as $r \to 1$. So in the above inequality the right hand side is $n\left(O[n^{4t}]\right) = O[n^{4t+1}]$ instead of $O(n^{4t+1})$.

Just as before (Ch. (3.9) of section 3.1)

$$\int_{-\pi}^{\pi} |f(\rho e^{i\theta})|^{2\lambda-2t} d\theta = O\left[n^{-1+4\lambda-4t}\right]$$

and so we now get

$$\begin{aligned} n|a_{n,\lambda}| \leq (n+\lambda)|a_{n,\lambda}| &= O\left[n^{+\frac{1}{2}(1+4t+4t+4\lambda-1)}\right] \\ &= O(n^{2\lambda}) \quad \text{as required.} \end{aligned}$$

□

The case $\alpha > 0$.

Suppose then from now on $\alpha > 0$ in theorem 9 and we shall develop a series of asymptotic formulae by a form of the Hardy-Little wood method, culminating in a formula for the $a_{n,\lambda}$. We have first

Theorem 11. *There exists θ_0 in $0 \leq \theta_0 \leq 2\pi$ such that $(1-r)^2|f(re^{i\theta_0})| \to \alpha$ as $r \to 1$.*

Proof. Set $r_n = 1 - \dfrac{1}{n}$ and choose θ_n so that $f(r_n e^{i\theta_n}) = M(r_n, f)$.

Then from lemma 3 we have for $0 < r < r_n$

$$\frac{(1-r)^2}{r}|f(re^{i\theta_n})| \geq \frac{(1-r_n)^2}{r_n}|f(r_n e^{i\theta_n})| = \mathscr{B}_n \quad \text{(say)}$$

Let now θ_0 be a limit point of the θ_n and choose a fixed r in $0 < r < 1$. 115
Then

$$\frac{(1-r)^2}{r}|f(re^{i\theta_0})| = \lim \frac{(1-r)^2}{r}|f(re^{i\theta_n}) \geq \lim \mathscr{B}_n$$

as n tends to infinity through a suitable sequence for which $\theta_n \to \theta_0$, since after some n, $r < r_n$ and $\dfrac{(1-r)^2}{r}|f(re^{i\theta_n})| \geq \mathscr{B}_n$. Also $\lim \mathscr{B}_n = \alpha$ since $\dfrac{(1-r)^2}{r}M(r, f) \to \alpha$ as $r \to 1$. □

Hence $|f(re^{i\theta_0})| \geq \dfrac{r\alpha}{(1-r)^2}$ and this is true for $0 < r < 1$. Thus

$$\varliminf_{r\to 1}(1-r)^2|f(re^{i\theta_0})| \geq \alpha$$

On the other hand $\varlimsup_{r\to 1}(1-r)^2|f(re^{i\theta_0})| \leq \varlimsup_{\longrightarrow}(1-r)^2 M(r,f) = \alpha$.

Thus $\lim_{r\to 1}(1-r)^2|f(re^{i\theta_0})| = \alpha$ as required.

Theorem 12. *If θ_0 is as in theorem 11 and $f(re^{i\theta_0}) = R(r)e^{i\lambda(r)}$ then*

$$\int_0^1 (1-r)[\lambda'(r)]^2 dr \quad \text{\textit{converges.}}$$

Hence if $r \to 1$ and $\rho \to 1$ while $r < \rho < \dfrac{1}{2}(1+r)$ we have uniformly

$$\frac{(1-r)^2}{(1-\rho)^2}\frac{f(re^{i\theta_0})}{f(\rho e^{i\theta_0})} \to 1.$$

Proof. We have by lemma 3 for $0 < r_1 < r_2 < 1$

$$\frac{r_1}{4}\int_{r_1}^{r_2}(1-r)[\lambda'(r)]^2 dr \leq \log\frac{R_1(1-r_1)^2}{r_1} - \log\frac{R_2(1-r_2)^2}{r_2}$$

116 Letting $r_2 \to 1$ and by theorem 11 we get

$$\frac{r_1}{4}\int_{r_1}^{1}(1-r)[\lambda'(r)]dr \leq \log\frac{R_1(1-r_1)^2}{r_1\alpha}$$

proving convergence, since $\lambda'(r)$ is bounded in $0 \leq r \leq r_1$. Now suppose that r is so near 1 that

$$\int_r^1 (1-t)[\lambda'(t)]^2 dt \leq \epsilon \quad \text{and} \quad r < \rho < \frac{1}{2}(r+1)$$

Then

$$|\lambda(\rho)-\lambda(r)| = \left|\int_r^\rho \lambda'(t)dt\right| \le \left\{\int_r^\rho (1-t)[\lambda'(t)]^2 dt\right\}^{\frac{1}{2}} \times \left[\int_r^\rho \frac{dt}{1-t}\right]^{\frac{1}{2}}$$

(by Schwarz inequality).

$$|\lambda(\rho)-\lambda(r)| \le \left[\epsilon \log \frac{1-r}{1-\rho}\right]^{\frac{1}{2}} \le (\epsilon \log 2)^{\frac{1}{2}}$$

So $|\lambda(\rho)-\lambda(r)| \to 0$ if $r, \rho \to 1$ being related as in the theorem. Thus

$$\arg \frac{(1-r)^2}{(1-\rho)^2} \frac{f(re^{i\theta_0})}{f(\rho e^{i\theta_0})} = \lambda(r)-\lambda(\rho) \to 0$$

and

$$\left|\frac{(1-r)^2}{(1-\rho)^2} \frac{f(re^{i\theta_0})}{f(\rho e^{i\theta_0})}\right| \to \frac{\alpha}{\alpha} = 1.$$

by the previous theorem. The result follows. □ **117**

Theorem 13. *Suppose $f(z) \in S$. [By the hypothesis of theorem 9, $\alpha = \lim\limits_{r\to 1} \dfrac{(1-r)^2}{r} M(r,f)$ and]. Suppose $\alpha > 0$ and θ_0 as defined in theorem 11, [θ_0 such that $0 \le \theta_0 \le 2\pi$ and $(1-r)^2|f(re^{i\theta_0})| \to \alpha$ as $r \to 1$] and $\varphi(z) = f(z)^\lambda = z^\lambda \sum\limits_{n=0}^{\infty} a_{n,\alpha} z^n$. Then if $\lambda > \dfrac{1}{4}$*

$$na_{n,\lambda} e^{i(\lambda+n)\theta_0} \sim \frac{\varphi\left[\left(1-\frac{1}{n}\right)e^{i\theta_0}\right]}{\Gamma(2\lambda)} \quad as \quad n \to \infty.$$

Assuming the theorem let us first deduce some corollaries.

Corollary 1. *We have $|a_{n,\lambda}| \sim \dfrac{\alpha^\lambda n^{2\lambda-1}}{\Gamma'(2\lambda)}$ as $n \to \infty$ where $\alpha \le 1$. Equality is possible only in the case when $f(z) = f_\theta(z)$ defined in theorem 1.*

Proof. Taking moduli for the result of the theorem,

$$n|a_{n,\lambda}| \sim \frac{|\varphi\left[\left(1-\frac{1}{n}\right)e^{i\theta_0}\right]|}{\Gamma(2\lambda)} \quad \text{as} \quad n\to\infty$$
$$= \frac{|f\left[\left(1-\frac{1}{n}\right)e^{i\theta_0}\right]|^{\lambda}}{\Gamma(2\lambda)} \sim \frac{(\alpha n^2)^{\lambda}}{\Gamma(2\lambda)}$$

after theorem 11. □

For, the choice of θ_0 is such that $(1-r)^2\ f(re^{i\theta_0}) \to \alpha$ as $r\to 1$. Precisely we get $\left(\frac{1}{n}\right)^2 \left|f\left[\left(1-\frac{1}{n}\right)e^{i\theta_0}\right]\right| \to \alpha$ as $n\to\infty$.

Hence we get $|a_{n,\alpha}| \sim \dfrac{n^{2\lambda-1}}{\Gamma(2\lambda)}$. This is corollary 1.

Corollary 2. *If* $f_k(z) = z + \sum\limits_{n=1}^{\infty} a_{kn+1}z^{kn+1}$ *belongs to* S, *and if* $k=1$, 2 *or* 3 *then,*

$$|a_{kn+1}| \le \frac{\Gamma(n+2/k)}{\Gamma(n+1)\Gamma(2/k)} \quad \textit{for} \quad n\ge n_0.$$

118 *Equality holds for* $f(z) = \dfrac{z}{(1-z^k e^{i\theta})^{2/k}}$ *and all* n, *otherwise strict inequality holds for* $n > n_0(f)$.

To prove this note that

$$f(z) = [f_k(z^{1/k})]^k \in S$$

and

$$f_k(z^{1/k}) = z^{1/k}(\sum_0^{\infty} a_{kn+1}z^n) = [f(z)]^{1/k}.$$

Now either $f(z) = \dfrac{z}{(1-ze^{i\theta})^2}$ and so

$$= \frac{z}{(1-z^k e^1)^{2/k}} = \sum A_{kn+1}z^{kn+1}e^{in\theta} = f_k(z)$$

where

$$
\begin{aligned}
A_{kn+1} &= \frac{\frac{2}{k}\left(\frac{2}{k}+1\right)\ldots\ldots\left(\frac{2}{k}+n-1\right)}{1\cdot 2\ldots\ldots n} \\
A_{kn+1} &= \frac{\Gamma\left(\frac{2}{k}+n\right)}{\Gamma(2/k)\Gamma(n+1)} \\
&\sim \frac{n^{\frac{2}{k}-1}}{\Gamma(2/k)}
\end{aligned}
$$

Alternatively since in every other case $\alpha < 1$ for $f(z)$ so by corollary 1,

$$|a_{kn+1}| \sim \frac{\alpha^{\frac{1}{k}} n^{\frac{2}{k}-1}}{\Gamma(2/k)}$$

and since $\alpha < 1$,

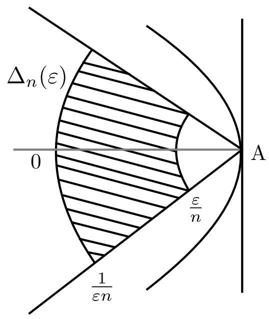
$$|a_{kn+1}| < A_{kn+1} \quad \text{for large} \quad n.$$

Note that if $k = 1$ we have $|a_{n+1}| < n+1$ finally and if $k = 2$, $|a_{2n+1}| < 1$. **119**

Let us now go to the proof of theorem 13. We suppose without loss of generality that $\theta_0 = (0)$ in theorem 13 for otherwise we can consider $e^{-i\theta_0} f(ze^{i\theta_0})$ instead of $f(z)$ and $e^{i\lambda\theta_0}[f(ze^{i\theta_0})]^{\lambda}$ instead of $\varphi(z)$.

Set $r_n = \left(1 - \dfrac{1}{n}\right)$. Given $\epsilon > 0$ we define a domain $\Delta_n = \Delta_n(\epsilon) = \{z : \dfrac{\epsilon}{n} < |1-z| < \dfrac{1}{n\epsilon} |\arg(1-z)| < \left(\frac{\pi}{2} - \epsilon\right)$ and $\alpha_n = \dfrac{f(r_n)}{n^2}$ so that $|\alpha_n| \to \alpha$ as $n \to \infty$ by theorem 11.

Suppose now $f_n(z) = \dfrac{\alpha_n}{(1-z)^2}$



Lemma 4. *We have as $n \to \infty$ uniformly for $z \in \Delta_n(\epsilon)$, $f(z) \sim f_n(z)$ and $f'(z) \sim f'_n(z)$.*

Proof. Put $z = l_n(w) = r_n + \frac{1}{n}w$, so that $(1 - z) = \frac{1}{n}(1 - w)$.

Then by this $\Delta_n(\epsilon)$ in the z-plane corresponds to $\Delta_1(\epsilon)$ in the w plane. Define a function $g_n(z)$ as

$$g_n(w) = (1 - w)^2 \frac{f[l_n(w)]}{f(r_n)}$$

Note that for a fixed ϵ, $\Delta_n(\epsilon)$ lies in $|z| < 1$ for large n. Hence $g_n(w)$ is defined in $\Delta_1(\epsilon)$ for sufficiently large n. □

Further $|g_n(w)| = O(1)$ for $w \in \Delta_1(\epsilon)$ and $n > n_0$. In fact if $\Delta_n(\epsilon)$,
120 $\Delta_n(\frac{1}{2}\epsilon)$, $\Delta_n(\frac{1}{4}\epsilon)$ are denoted by Δ_n, Δ'_n and $\underline{d}$ is the distance from Δ'_1 to the outside of Δ''_1, the distance between Δ'_n and the exterior of Δ''_n is exactly d/n.

So if Δ''_n lies in $|z| < 1$. Δ'_n lies in $|z| < 1 - \frac{d}{n}$ and Δ''_n lies in $|z| < 1$ for all large n. Now $|f(z)| \leq \frac{r}{(1 - r)^2}$; $|z| = r$ and so if $|z| < 1 - \frac{d}{n} < 1$; $1 - |z| \geq \frac{d}{n}$.

Hence in $\Delta'_n |f(z)| = O\left(\frac{n^2}{d^2}\right) = O(n^2)$ as $n \to \infty$ and $f(r_n) = f\left(1 - \frac{1}{n}\right) \sim \alpha n^2$ by theorem 11 as $n \to \infty$ and hence it follows that $\frac{f[l_n(w)]}{f(r_n)} = \frac{O(n^2)}{n^2} = O(1)$ as $n \to \infty$ uniformly for w in Δ'_1. i.e., $g_n(w)$ is bounded in Δ'_1 for large n.

Next choose $-1 < w < 0$ so that w is real and $r_n + \frac{1}{n}w = 1 - \frac{1}{n}(1 - w) = l_n(w)$. After theorem 12.

$$\frac{f[l_n(w)]}{f(r_n)} \sim \frac{(1 - r_n)^2}{\frac{1}{n^2}(1 - w)^2} = \frac{1}{(1 - w)^2} \quad \text{as} \quad n \to \infty$$

for the hypothesis of theorem 12 is satisfied because,

$$|z| = \left|1 - \frac{1}{n} + \frac{1}{n}w\right| < r_n < \frac{1}{2}|(z + 1)|, r_n \to 1 \quad \text{as} \quad n \to \infty$$

in this manner ensuring the above asymptotic equality.

Therefore it follows on bringing $(1-w)^2$ to the left hand side $g_n(w) = (1-w)^2 \dfrac{f[l_n(w)]}{f(r_n)}$ tends to one as $n \to \infty$, for real w satisfying $0 > w > -1$.

Thus we have obtained $g_n(w) = O(1)$ as w ranges in Δ_1' and $g_n(w) \to$ **121**
1 as $n \to \infty$ on a set of w having a limit in Δ_1'. Hence by Vitalis convergence theorem (see Titchmarsh p.p. 168) $g_n(w) \to 1$, $g_n'(w) \to 0$ uniformly in Δ_1 which is contained in Δ_1' and satisfies the condition that it is bounded by a contour that is interior to Δ_1'.

Now translating back into z, $n^2(1-z)^2 \dfrac{f(z)}{f(r_n)}$ tends to 1 as n tends to infinity for $z \in \Delta_n(\epsilon)$.

i.e. by the definition of $f_n(z)$

$$\frac{f(z)}{f_n(z)} \to 1 \quad \text{for} \quad z \in \Delta_n(\epsilon) \quad \text{as} \quad n \to \infty.$$

Also

$$\frac{d}{dw}(1-w)^2 \frac{f[l_n(w)]}{f(r_n)} \to 0$$

since $n(1-z) = (1-w)$; $\frac{1}{n}\dfrac{d}{dz} n^2(1-z)^2 \dfrac{f(z)}{f(r_n)} \to 0$

$$n\frac{d}{dz}(1-z)^2 \frac{f(z)}{f(r_n)} \to 0$$

$$\frac{d}{dz}(1-z)^2 f(z) = O(n) \ \text{ since } \ f(r_n) \sim \alpha n^2 \ \text{ as } \ n \to \infty.$$

for z in $\Delta_n(\epsilon)$.

Note that in $\Delta_n(\epsilon)\dfrac{1}{n}$ and $(1-z)$ have the same order of magnitude.

Thus, $(1-z)^2 f'(z) - 2(1-z) f(z) = 0(n)$, $z \in \Delta_n(\epsilon)$ as $n \to \infty$ **122**

$$\begin{aligned} f'(z) &= \frac{2f(z)}{(1-z)} + O(n^3), \quad z \in \Delta_n(\epsilon) \ \text{ as } \ n \to \infty. \\ &= \frac{2f_n(z)}{(1-z)} + O(n^3) \end{aligned}$$

$$= f_n'(z) + O(n^3) = f_n'(z)[1 + O(1)]$$

because

$$f_n'(z) = \frac{2\alpha n}{(1-z)} = O(n^3).$$

Therefore, $f'(z) \sim f_n'(z)$ uniformly for $z \in \Delta_n(\epsilon)$ as $n \to \infty$. This completes the proof of the lemma.

Lemma 5. *With the notation of theorem 4, for $\lambda > 0$,*

$$S_\lambda(r, f) = \alpha^\lambda S_\lambda[r, (1-z)^{-2}] + O(1-r)^{-2\lambda} \quad as \quad r \to 1$$

and if $\lambda > \frac{1}{2}$

$$I_\lambda(r, f) = \alpha^\lambda I_\lambda[r, (1-z)^{-2}] + O(1-r)^{1-2\lambda} \quad as \quad r \to 1$$

Proof. Let $R_n = |\alpha_n| \frac{n^2}{\epsilon^2}$, $\varphi_n(z) = f_n(z)^\lambda = \alpha_n^\lambda (1-z)^{-2\lambda}$ and $\Delta_n(\epsilon)$ as
123 defined in the previous lemma. Note that $\varphi_n(z)$ maps $\Delta_n(\epsilon)$ onto the sector

$$\epsilon^{4\lambda} R_n^\lambda < |w| < R_n^\lambda, |\arg(w) - \lambda \arg \alpha_n| < (\pi - 2\epsilon)$$

The area of the image $= \iint\limits_{\Delta_n(\epsilon)} |\varphi_n'(z)|^2 \, dx \, dy$

$$= \lambda(\pi - 2\epsilon) R_n^{2\lambda} [1 - \epsilon^{8\lambda}] \tag{3.2.1}$$

Also in $\Delta_n(\epsilon)$, $f_n(z) \sim f(z)$ and $f_n'(z) \sim f'(z)$ and so

$$\varphi_n'(z) = \lambda f_n'(z)[f_n(z)]^{\lambda-1} \sim \lambda f'(z)[f(z)]^{\lambda-1} = \varphi'(z)$$

So we have

$$\iint\limits_{\Delta_n(\epsilon)} |\phi_n'(z)|^2 dx \, dy \sim \lambda(\pi - 2\epsilon) R_n^{2\lambda} (1 - \epsilon^{8\lambda}) \tag{3.2.2}$$

Also since $|\varphi_n(z)| < R_n^\lambda$ in $\Delta_n(\epsilon)$ we have by lemma 4,

$$|\varphi(z)| < R_n^\lambda (1 + \epsilon) \quad \text{there for large } n.$$

Now choose n so that $R_{n-1} < M(r, f) < R_n$. This is possible at least if r is sufficiently near 1, since $R_n \to \infty$ with n. □

Let E_0 be the set of points of $|z| < r$ outside $\Delta_n(\epsilon)$. Then in E_0, $|\varphi(z)| < R_n^\lambda$, and in $\Delta_n(\epsilon)$, $|\varphi(z)| < (1+\epsilon)R_n^\lambda$. The image of $|z| < 1$ by $f(z)$ contains no point more than once, and so the length of the image over any circle $|w| = \rho$ is at the most $2\pi\rho$.

If we cut the unit circle along the negative real axis, then the image by $f(z)^\lambda$ covers any circle, $|w| = \rho$, with length at the most $\lambda 2\pi\rho$. So the area of the image of the union of E_0 and $\Delta_n(\epsilon)$ by $\varphi(z)$ is at most $\pi\lambda R_n^{2\lambda}(1+\epsilon)^2$.

That gives, **124**

$$\iint\limits_{E_0\cup\Delta_n(\epsilon)} |\varphi'(z)|^2 dx\, dy \leq \pi\lambda R_n^{2\lambda}(1+\epsilon)^2$$

Also by (3.2.2), we have for large n

$$\iint\limits_{\Delta_n(\epsilon)} |\varphi'(z)|^2 dx\, dy > \lambda\pi(1-\epsilon)(1-\epsilon^{8\lambda})R_n^{2\lambda}$$

[Note the ϵ terms of r.h.s. of this inequality and (3.2.2)]

So

$$\begin{aligned}\iint\limits_{E_0} |\varphi'(z)|^2 dx\, dy &< R_n^{2\lambda}\pi\lambda\left[(1+\epsilon)^2 - (1-\epsilon)(1-\epsilon^{8\lambda})\right]\\ &< \epsilon^1 R_n^{2\lambda}\end{aligned}$$

where ϵ^1 is small if ϵ is small.

Similarly,

$$\iint\limits_{E_0} |\varphi_n'(z)|^2 dx\, dy < \epsilon'' R_n^{2\lambda}$$

By theorem 4

$$\begin{aligned}S_{2\lambda}(r, f) &= \frac{\lambda^2}{2\pi}\int\limits_0^r \rho d\rho \int\limits_{-\pi}^{\pi} |f(\rho e^{i\theta})|^{2\lambda-2}|f'(\rho e^{i\theta})|^2 d\theta\\ &= \frac{\lambda^2}{2\pi}\iint\limits_{|z|<r} |f(\rho e^{i\theta})|^{2\lambda-2}|f'(\rho e^{i\theta})|^2 dx\, dy\end{aligned}$$

$$= \frac{1}{2\pi} \iint\limits_{|z|<r} |\varphi'(z)|^2 dx\, dy.$$

125 Hence

$$\begin{aligned} S_{2\lambda}(r, f) - S_{2\lambda}(r, f_n) &= \frac{2}{\pi} \iint\limits_{|z|<r} \left(|\varphi'(z)|^2 - |\varphi'_n(z)|^2\right) dx\, dy \\ &= \frac{2}{\pi}\left[\int\limits_{E_1} + \int\limits_{E_0}\right] \end{aligned}$$

where E_1 is the part of $|z| < r$ within $\Delta_n(\epsilon)$. Since in $E_1 \varphi'_n(z) \sim \varphi'(z)$

$$\begin{aligned} \int\limits_{E_1} \left(|\varphi'(z)|^2 - |\varphi'_n(z)|^2\right) dx\, dy &= O \int\limits_{E_1} |\varphi'_n(z)|^2 dx\, dy \\ &= O \int\limits_{\Delta_n(\epsilon)} |\varphi'_n(z)|^2 dx\, dy \\ &= O(R_n^2) \quad \text{by (3.2.1)} \end{aligned}$$

Also

$$\begin{aligned} \left|\int\limits_{E_0} \left(|\varphi'(z)|^2 - |\varphi'_n(z)|^2\right) dx\, dy\right| &\le \int\limits_{E_0} |\varphi'_n(z)|^2\, dx\, dy + \int\limits_{E_0} |\varphi'(z)|^2\, dx\, dy \\ &\le (\epsilon'' + \epsilon')R_n^2. \end{aligned}$$

Since ϵ'' and ϵ' can be made as small as we please by suitable choice of ϵ,

$$\begin{aligned} S_{2\lambda}(r, f) &= S_{2\lambda}(r, f_n) + O(R_n^{2\lambda}) \\ &= S_{2\lambda}(r, f_n) + O(R_{n-1}^{2\lambda}) \quad \text{as} \quad R_n \sim R_{n-1} \\ &= S_{2\lambda}(r, f_n) + OM(r, f)^{2\lambda}. \end{aligned}$$

126 i.e. $S_{2\lambda}(r, f) = S_{2\lambda}(r, f_n) + O(1 - r)^{-4\lambda}$ as $r \to 1$ (by theorem 3).

Hence we get

$$S_{2\lambda}(r,f) = |\alpha_n^{2\lambda}|S_{2\lambda}[r,(1-z)^{-2}] + O(1-r)^{-4\lambda} \quad \text{as} \quad r \to 1.$$

which is the first result. Integrating the above from 0 to r, w.r.t. r after dividing by r, we get

$$I_{2\lambda}(r,f) = |\alpha|^{2\lambda} I_\lambda[r,(1-z)^{-2}] + O(1-r)^{1-4\lambda} \ \text{ as } \ r \to 1, \ \text{ if } \ \lambda > \frac{1}{4}.$$

Hence we get the lemma [replacing 2λ by λ]

$$I_\lambda(r,f) = \alpha^\lambda I_\lambda[r,(1-z)^{-2}] + O(1-r)^{1-2\lambda} \ \text{ as } \ r \to 1$$

if $\lambda > \dfrac{1}{2}$.

Lemma 6. *If* $\eta > 0$ *and* $\lambda > \dfrac{1}{4}$ *we can choose* $k > 0$ *so that if* $r_0 < r < 1$, $1 - \dfrac{1}{n} \le r \le 1 - \dfrac{1}{2n}$

$$\int\limits_{k(1-r)\le|\theta|\le\pi} \left(|f(re^{i\theta})|^2 + |f_n(re^{i\theta})|^2\right) d\theta < \eta(1-r)^{1-4\lambda}$$

Let γ and γ' be the arcs $|\theta| \le k(1-r)$ and $k(1-r) \le |\theta| \le \pi$ on $|z| = r$. If k is any fixed number, we can choose ϵ so small that the arc γ 127
lies in $\Delta_n(\varepsilon)$ for $1 - \dfrac{1}{n} \le r \le 1 - \dfrac{1}{2n}$ for large n (see Hayman [1]).

Hence

$$\int\limits_\gamma |f_n(z)|^{2\lambda} d\theta \sim \int\limits_\gamma |f(z)|^{2\lambda} d\theta$$

$$\begin{aligned}\int\limits_\gamma |f_n|^{2\lambda} d\theta - \int\limits_\gamma |f|^{2\lambda} d\theta &= O\int |f|^{2\lambda} d\theta\\ &= O\{I_{2\lambda}(r,f)\}\\ &= O(1-r)^{1-4\lambda}\end{aligned}$$

by the application of theorem 4, as $2\lambda > \dfrac{1}{2}$ and also $I_{2\lambda}(r, f) - I_{2\lambda}(r, f_n) = O[(1-r)^{1-4\lambda}]$ by the previous lemma.

Hence subtraction gives,

$$\int\limits_{\gamma'} |f|^{2\lambda} d\theta - \int\limits_{\gamma'} |f_n|^{2\lambda} d\theta = O[(1-r)^{1-4\lambda}] \tag{3.2.3}$$

Now the integral

$$\begin{aligned}
\int\limits_{\gamma'} |f_n|^{2\lambda} d\theta & \int\limits_{k(1-r)}^{\pi} \frac{|\alpha_n|^{2\lambda} d\theta}{(1 - 2r\cos\theta + r^2)^{2\lambda}} \\
& \leq \int\limits_{k(1-r)}^{\pi} \frac{d\theta}{(r\sin\theta/2)^{4\lambda}} \\
& A(\lambda) \int\limits_{k(1-r)}^{\infty} \frac{d\theta}{\theta^{4\lambda}}
\end{aligned}$$

$\dfrac{1}{(1 - 2r\cos\theta + r^2)} \leq \dfrac{1}{(r\sin^2\theta/2)^2} \sin\theta/2 \geq A\theta$ in that range.

$$\begin{aligned}
\int\limits_{\gamma'} |f_n|^{2\lambda} d\theta & \leq A(\lambda)[k(1-r)]^{1-4\lambda} \\
& < \frac{1}{3}\eta(1-r)^{1-4\lambda}
\end{aligned} \tag{4}$$

128 by properly choosing k sufficiently large. Hence in virtue of the relation (2) we deduce same for $f(z)$ and hence the lemma.

We now proceed to prove theorem 13. We have as in lemma 5,

$$\varphi_n(z) = f_n(z)^{\lambda} = \alpha_n^{\lambda}(1-z)^{-2\lambda} = \alpha_n^{\lambda} \sum_{0}^{\infty} b_m z^m$$

$$b_m = \frac{2\lambda(2\lambda+1)\ldots\ldots(2\lambda+m-1)}{m!}$$

$$= \frac{\Gamma(m+1+2\lambda)}{\Gamma(2\lambda)\Gamma(m+1)} \sim \frac{m^{2\lambda-1}}{\Gamma(2\lambda)}$$

and as already defined

$$\varphi(z) = z^{\lambda}\left[1 + \sum_{m=1}^{\infty} a_{m,\lambda} z^{m}\right]$$

Hence by considering the coefficients of $\varphi'(z)$ and $\varphi'_n(z)$,

$$(n+\lambda)a_{n,\lambda} - nb_n\alpha_n^{\lambda} = \frac{1}{2\Pi}\int\limits_{|z|=r} \frac{\varphi'(z)}{z^{n+\lambda}} - \frac{\varphi'_n(z)}{z^{n}} dz$$

$$\left|(n+\lambda)a_{n,\lambda} - nb_n\alpha_n^{\lambda}\right| \leq \frac{1}{2\pi}\frac{1}{r^{n+\lambda-1}}\left|\int_{0}^{2\pi}\left(\varphi'(re^{i\theta}) - re^{\lambda i\lambda\theta}\varphi'_n(re^{i\theta})\right)d\theta\right.$$

We shall choose r in the range $1 - \dfrac{1}{n} \leq r \leq 1 - \dfrac{1}{2n}$ as usual. For a k to be determined, let γ and γ' be the arcs $|\theta| \leq k(1-r)$ and $k(1-r) \leq |\theta| \leq \pi$ respectively on $|z| = r$.

$$\gamma, \varphi' \sim \varphi'_n \circ (\varphi'_n)(r^{\lambda}e^{i\lambda\theta}) = O(1-r)^{-1-2\lambda} \text{ as } r \to 1.$$

and so **129**

$$\varphi' - (\varphi'_n)(r^{\lambda}e^{i\lambda\theta}) = O(1-r)^{-1-2\lambda} = O(n^{2\lambda+1}) \text{ as } r \to 1 \text{ or as } n \to \infty$$

The length of the path of integration of $\gamma = O(1-r) = O\left(\dfrac{1}{n}\right)$

$$\int \left(\varphi'(re^{i\theta}) - r^{\lambda}e^{i\lambda\theta}\varphi'_n(re^{i\theta})\right)d\theta = O(n^{2\lambda})$$

this being true for any fixed k.

Again,

$$\left|\int\limits_{\gamma'} \left(\varphi'(re^{i\theta}) - r^{\lambda}e^{i\lambda\theta}\varphi'_n(re^{i\theta})\right)d\theta\right|$$

$$\leq \int_{\gamma'} |\varphi'(re^{i\theta})| d\theta + \int_{\gamma'} r^{\lambda} |\varphi'_n(re^{i\theta})| d\theta$$

We now proceed as in theorem 6, choose t such that $2\lambda - 2t > \dfrac{1}{2}$ and using Schwarz's Inequality.

$$\int_{\gamma'} |\varphi'(re^{i\theta})| d\theta = \lambda \int_{\gamma'} |f'(re^{i\theta})||f(re^{i\theta})|^{\lambda-1} d\theta$$

$$\leq \lambda \left[\int_{\gamma} |f'(re^{i\theta})|^2 |f(re^{i\theta})|^{2t-2\lambda} d\theta \right]^{\frac{1}{2}}$$

$$\left[\int_{\gamma'} |f(re^{i\theta})|^{2\lambda-2t} d\theta \right]^{\frac{1}{2}}$$

By the same method as of theorem 6, we can choose r such that the first integral is $O(n^{4t+1})$.

By what we have just seen in lemma 6 the second integral can be made less that $\delta\eta(1-r)^{1-4\lambda+4t}$ i.e. (const.) $\eta n^{4\lambda-4t-1}$ and η can be made as small as we please. So

$$\int |\varphi'(re^{i\theta})| d\theta < \eta' n^{2\lambda}$$

130 using the inequality (8) of theorem 6.

η' can be made as small as we please by choosing k large enough $\int_{\gamma'} \varphi'_n(re^{i\theta}) d\theta$ can be dealt with similarly.

So finally we see that

$$\int_0^{2\pi} |[\varphi'(re^{i\theta}) - r^{\lambda} e^{i\lambda\theta} \varphi'_n(re^{i\theta})] d\theta \leq [2\eta' + O(1)] n^{2\lambda}$$

and since η' can be made as small as we please the right hand side is $O(n^{2\lambda})$.

That gives

$$\begin{aligned} a_{n,\lambda} &= \frac{nb_n\alpha_n^\lambda}{n+\lambda} + O(n^{2\lambda-1}) \\ &= \frac{n}{n+\lambda}\frac{f(r_n)^\lambda}{n^{2\lambda}}\frac{n^{2\lambda-1}}{\Gamma(2\lambda)} + O(n^{2\lambda-1}) \end{aligned}$$

Therefore,

$$na_{n,\lambda} \sim \frac{f(r_n)^\lambda}{\Gamma(2\lambda)} \text{ as required.}$$

This completes the proof of theorem 13. For an extension of the result to p-valent functions and further results see for example W.K. Hayman [1], [2].

Bibliography

[1] Ahlfors, L. [1]: Beitrage Zur Theorie der meromorphen Funktionen 7, Congr. Math. Scand. Oslo 1929.

[2] Bazilevič, I.E. [1]: On distortion theorem and coefficients of univalent functions. Mat. Sbornik N.S. 28 (70), 283-292 (1951).

[3] Biernacki, M. [1]: Bull. Acad. Polon. Sci. Cl.III 4 (1956) 5-8

[4] Carathéodory, C. [2]: "Theory of Functions of one variable" Vol.I Translated by Steinhardt

[5] Csillag, P. [1]: Math. Annalen 110 (1935) pp. 745-'52.

[6] Hayman, W.K. [1]: Proc. L.M.S. 1955 p.p. 257-284.

[7] Hayman, W.K. [2]: "Multivalent Functions" Cambridge University Press.

[8] Hayman, W.K. and Stewart, F.M. [1]: Proc. Camb. Phil. Soc. 1954.

[9] Kaplan, W. [1]: ('close to convex Schlicht functions') Michigan Math. J. 1(1953) 169-'85.

[10] Kennedy, P.B. [1]: Proc. L.M.S. Vol. 6 1956.

[11] Little wood, D. [1]: Q.J. of Maths. Vol.9 (1938)

[12] Shimizu, T. [1]: On the Theory of Meromorphic Functions Jap. J. Maths. 6 (1929).