Lectures on
Lie Groups and Representations of Locally Compact Groups

By

F. Bruhat

Tata Institute of Fundamental Research, Bombay
1958
(Reissued 1968)

Lectures on
Lie Groups and Representations
of Locally Compact Groups

By

F. Bruhat

Notes by

S. Ramanan

Tata Institute of Fundamental Research, Bombay
1958
(Reissued 1968)

Introduction

We shall consider some heterogeneous topics relating to Lie groups and the general theory of representations of locally compact groups. The first part exclusively deals with some elementary facts about Lie groups and the last two parts are entirely independent of the material contained in the first. We have rigidly adhered to the analytic approach in establishing the relations between Lie groups and Lie algebras. Thus we do not need the theory of distributions on a manifold or the existence of integral manifolds for an involutory distribution.

The second part concerns itself only with the general theory of measures on a locally compact group and representations in general. Only a passing reference is made to distributions (in the sense of L. Schwartz), and induced representations are not treated in detail.

In the third part, we first construct the continuous sum ('the direct integral') of Hilbert spaces and then decompose a unitary representations into a continuous sum of irreducible representations. We derive the Plancherel formula for a separable unimodular group in terms of factorial representations and derive the classical formula in the abelian case.

Contents

Part I

Lie Groups

Chapter 1

Topological groups

1.1

We here assemble some results on topological groups which we need in **1**
the sequel.

Definition. *A* topological group G *is a topological space with a composition law* $G \times G \to G$, $(x, y) \to xy$ *which is*

(a) *a group law, and*

(b) *such that the map* $G \times G \to G$ *defined by* $(x, y) \mapsto x^{-1}y$ *is continuous.*

The condition (b) is clearly equivalent to the requirement that the maps $G \times G \to G$ defined by $(x, y) \to xy$ and $G \to G$ defined by $x \to x^{-1}$ be continuous.

It is obvious that the translations to the right: $x \to xy$ are homeomorphisms of G. A similar statement is true for left translations also. We denote by A^{-1}, AB the subsets $\{a^{-1} : a \in A\}$, $\{ab : a \in A, b \in B\}$ respectively. It is immediate from the definition that the neighbourhoods of the identity element e satisfy the following conditions:

(V_1) For every neighbourhood V of e, there exists a neighbourhood W of e such that $W^{-1}W \subset V$. (This follows from the fact that $(x, y) \to x^{-1}y$ is continuous).

(V_2) For every neighbourhood V of e and for every $y \in G$, there exists a neighbourhood W of e such that $yWy^{-1} \subset V$. (This is because $x \to yxy^{-1}$ is continuous).

These conditions are also sufficient to determine the topology of the group. More precisely,

2 **Proposition 1.** *Let $\mathscr{V}$ be a family of subsets of a groups G, such that*

(a) *For every $V \in \mathscr{V}$, $e \in V$.*

(b) *Any finite intersection of elements of $\mathscr{V}$ is still in $\mathscr{V}$.*

(c) *For every $V \in \mathscr{V}$, there exists $W \in \mathscr{V}$ such that $W^{-1}W \subset V$.*

(d) *For every $V \in \mathscr{V}$, and for every $y \in G$ there exists $W \in \mathscr{V}$ such that $yWy^{-1} \subset V$.*

Then G can be provided with a unique topology Φ compatible with the group structure such that the family $\mathscr{V}$ is a fundamental system of neighbourhoods at e.

Supposing that such a topology exists, a fundamental system of neighbourhoods at y is given by either $\mathscr{V}y$ or $y\mathscr{V}$. It is, therefore, a natural requirement that these two families generate the same filter. It is this that necessitates the condition (d).

We may now take the filter generated by $y\mathscr{V}$ and $\mathscr{V}y$ as the neighbourhood system at y. This can be verified to satisfy the neighbourhood axioms for a topology. It remains to show that $(xy) \to x^{-1}y$ is continuous.

Let $Vx_0^{-1}y_0$ be any neighbourhood of $x_0^{-1}y_0$. By (c), (d), there exist $W, W_1 \in \mathscr{V}$ such that $W^{-1}W \subset V$ and $x_0^{-1}y_0W_1y_0^{-1}x_0 \subset W$. If we take $x \in x_0W$, $y \in y_0W_1$, we have $x^{-1}y \in W^{-1}x_0^{-1}y_0W_1 \subset W^{-1}Wx_0^{-1}y_0 \subset Vx_0^{-1}y_0$. This shows that $(x,y) \to x^{-1}y$ is continuous.

Examples of topological groups.

(1) Any group G with the discrete topology.

(2) The additive group of real numbers R or the multiplicative group of non-zero real numbers R^* with the 'usual topology'.

(3) The direct product of two topological groups with the product topology. 3

(4) The general linear group $GL(n, R)$ with the topology induced by that of R^{n^2}.

(5) Let X be a locally compact topological space, and G a group of homeomorphisms of X onto itself. This group G is a topological group with the 'compact open topology'. (The compact-open topology is one in which the fundamental system of neighbourhoods of the identity is given by finite intersections of the sets $u(K, U) = \{f \in G : f(x) \in U, f^{-1}(x) \in U$ for every $x \in K\}$, K being compact and U an open set containing K).

1.2 Topological subgroups.

Let G be a topological group and g a subgroup in the algebraic sense. g with the induced topology is a topological group which we shall call a *topological subgroup* of G. We see immediately that the closure $\bar{g}$ is again a subgroup.

If g is a normal subgroup, so is $\bar{g}$. Moreover, an open subgroup is also closed. In fact, if g is open, $G - g = \bigcup_{x \notin g} xg$, which is open as left translations are homeomorphisms. Hence g is closed.

Let now V be a neighbourhood of e and g the subgroup generated by the elements of V. g is open containing as it does a neighbourhood of every element belonging to it. Consequently it is also closed. So that we have

Proposition 2. *If G is a connected topological group, any neighbourhood of e generates G.*

On the contrary, if G is not connected, the connected component G_0 4
of e is a closed normal subgroup of G. Since $G_0 x G_0 \to G_0^{-1} G_0$ is continuous and $G_0 x G_0$ connected, $G_0^{-1} G_0$ is also connected, and hence $\subset G_0$.

This shows that G_0 is a subgroup. As $x \to yxy^{-1}$ is continuous, yG_0y^{-1} is a connected set containing e and consequently $\subset G_0$. Therefore, G_0 is a normal subgroup.

Proposition 3. *Every locally compact topological group is paracompact.*

In fact, let V be a relatively compact open symmetric neighbourhood of e. $G' = \bigcup\limits_{n=1}^{\infty} V^n$ is an open and hence closed subgroup of G. G' is countable at ∞ and therefore paracompact. Since G is the topological union of left cosets modulo G', G is also paracompact.

1.3 Factor groups.

Let g be a subgroup of a topological group G. We shall denote by $\dot{x}$ the right coset gx containing x. On this set, we already have the quotient topology. Then the canonical map $\pi : G \to G/g$ is open and continuous. For, if $\pi(U)$ is the image of an open set U in G, it is also the image of gU which is an open saturated set. Hence $\pi(U)$ is also open. But this canonical map is not, in general, closed. The space G/g is called a *homogeneous space*. If g is a normal subgroup, G/g is a group and is a topological group with the above topology. This is the *factor group* of G by g.

1.4 Separation axiom.

Theorem 1. *The homogeneous space G/g is Hausdorff if and only if the subgroup g is closed.*

5 If G/g is Hausdorff, $g = \pi^{-1}(\pi(e))$ is closed, since $\pi(e)$ is closed. Conversely, let g be closed. Let $x, y \in G/g$ and $x \neq y$. Since $xy^{-1} \notin g$ and g is closed, there exists a symmetric neighbourhood V of the identity such that $xy^{-1}V \cap g = \phi$. Hence $xy^{-1} \notin gV$. Now choose a neighbourhood W of e such that $WW^{-1} \subset y^{-1}Vy$. We assert that gxW and gyW are disjoint. For, if they were not, $\gamma_1, \gamma_2 \in g$, $w_1, w_2 \in W$ exist such that

$$\gamma_1 x w_1 = \gamma_2 y w_2; \quad \text{i.e.} \quad \gamma_2^{-1}\gamma_1 x = y w_2 w_1^{-1} \in yWW^{-1} \subset Vy.$$

Hence $\gamma_2^{-1}\gamma_1 xy^{-1} \in V$, or $xy^{-1} \in \gamma_1^{-1}\gamma_2 V \subset gV$.

This being contradictory to the choice of V, gxW, gyW are disjoint or $\pi(gxW) \cap \pi(gyW) = \phi$. Hence G/g is Hausdorff.

In particular, if $g = \{e\}$, G is Hausdorff if and only if $\{e\}$ is closed. On the other hand, if $\{e\}$ is not closed, $\overline{\{e\}}$ is a normal subgroup and $G/\overline{\{e\}}$ is a Hausdorff topological group. We shall hereafter restrict ourselves to the consideration of groups which satisfy Hausdorff's axiom.

1.5 Representations and homomorphisms.

Definition . *A* representation *h of a topological group G into a topological group H is a continuous map : $G \to H$ which is an algebraic representation. In other words, $h(xy) = h(x) \cdot h(y)$ for every $x, y \in G$.*

Obviously the image of G by h is a subgroup of H and the kernel N of h is a closed normal subgroup of G. The canonical map $\{\overline{h}\} : G/N \to H$ is a representation and is one-one.

Definition . *A representation h is said to be a* homomorphism *if the induced map $\overline{h}$ is a homeomorphism.*

Proposition 4. *Let G and H be two locally compact groups, the former being countable at ∞. Then every representation h of G* onto *H is a homomorphism.* **6**

It is enough to show that for every neighbourhood V of e in G/N, $\overline{h}(V)$ is a neighbourhood of $h(e)$ in H. Choose a relatively compact open neighbourhood W of e such that $\overline{W}\,\overline{W}^{-1} \subset V$. G is a countable union of compact sets and $\bigcup_{x\in G} Wx = G$. Therefore, one can find a sequence $\{x_j\}$ of points such that $G = \bigcup_j Wx_j$. Since h is onto, $H = \bigcup_j h(Wx_j) = \bigcup_j h(W)h(x_j)$. H is a locally compact space and hence a Baire space (Bourbaki, Topologie générale, Ch. 9). There exists, therefore, an integer j such that $\overline{h(Wx_j)}$ has an interior point. $h(\overline{W})$ being compact, $h(\overline{W}) = \overline{h(W)}$. Consequently, $h(\overline{W})h(x_j)$ and hence $h(\overline{W})$ has an interior point y. There exists a neighbourhood U of e such that $h(\overline{W}) \supset Uy$. Now $h(V) \supset h(\overline{W}\,\overline{W}^{-1}) = h((\overline{W}) \cdot h(\overline{W})^{-1} \supset Uy(y^{-1}U^{-1}) = UU^{-1}$. $h(V)$ is therefore a neighbourhood of $h(e)$, which completes the proof of proposition 4.

Chapter 2

Local study of Lie groups

2.1

7

Definition. A Lie group *G is a real analytic manifold with a composition law* $(x, y) \to xy$ *which is*

(a) *a group law, and*

(b) *such that the map* $(x, y) \to x^{-1}y$ *is analytic.*

(b) is equivalent to the analyticity of the maps $(x, y) \to xy$ *and* $x \to x^{-1}$.

Remarks. (1) A Lie group is trivially a topological group.

(2) We may replace 'real' by 'complex and define the notion of a complex Lie group. We shall not have occasion to study complex Lie groups in what follows, though most of the theorems we prove remain valid for them.

(3) It is natural to inquire whether every topological group with the structure of a topological manifold is a Lie group. This problem (Hilbert's fifth problem) has been recently solved by Gleason [18] who has proved that a topological group G which is locally compact, locally connected, metrisable and of finite dimension, is a Lie group.

Examples of Lie groups.

(1) R - real numbers, C - complex numbers,

T - the one-dimensional torus and R^n, C^n and T^n in the usual notation are all Lie groups.

8 (2) Product of Lie groups with the product manifold structure is a Lie group.

(3) $GL(n, R)$ - the general linear group.

2.2 Local study of Lie groups.

We shall assume that V is a sufficiently small neighbourhood of e in which a suitably chosen coordinate system, which taken e into the origin, is defined.

The following notations will be adhered to throughout these lectures:

If $a \in V$, $(a_1, \ldots, a_n)$ will denote the coordinate of a. $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index with α_i, non-negative integers.

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$$

[i] will stand for α with $\alpha_i = 1$ and $\alpha_j = 0$ for $j \neq i$.

$$\alpha! = \alpha_1! \ldots \alpha_n!$$
$$x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$$
$$\frac{\partial^\alpha}{\partial x^\alpha} = \frac{\partial^{\alpha_1+\cdots+\alpha_n}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

Let $x, y \in V$ be such that $xy \in V$. $(xy)_i$ are analytic functions of the coordinate of x and y.

(1) $(xy)_i = \varphi_i(x_1, \ldots, x_n, y_1, \ldots, y_n)$ where the φ_i are analytic functions of the $2n$ variables x, y in a neighbourhood of 0 in R^{2n}. These φ_i cannot be arbitrary functions, as they are connected by the group relations. These are reflected in the following equations:

$$(x\,e)_i = (e\,x)_i = x_i, \quad \text{or}$$

(2) $\varphi_i(x, e) = \varphi_i(e, x) = x_i$. **9**

The φ_i are analytic functions and so are of the form

$$\varphi_i(x, y) = \sum_{\alpha,\beta} \lambda^i_{\alpha_\beta} x^\alpha y^\beta, i = 1, 2, \ldots, n$$

By equation (2), this can be written

(3) $\varphi_i(x, y) = x_i + y_i + \sum_{\substack{|\alpha|\geq 1\\|\beta|\geq 1}} \lambda^i_{\alpha,\beta} x^\alpha y^\beta$.

By associativity,

$$\varphi_i(xy, z) = \varphi_i(x, yz), \quad \text{or}$$
$$\varphi_i(\varphi_1(x, y), \ldots, \varphi_n(x, y), z) = \varphi_i(x, \varphi_1(y, z), \ldots, \varphi_n(y, z)).$$

These may also be written

(4) $\varphi_i(\varphi(x, y), z) = \varphi_i(x, \varphi(y, z))$.

One is tempted to expect another equation in φ_i, due to the existence of the inverse of every element. However, these two equations are sufficient to characterise locally the Lie group, and the existence of the inverse is, in a certain sense, a consequence of the associative law and the existence of the identity. To be more precise,

Proposition 1. *Let G be the semigroup with an identity element e. If it can be provided with the structure of an analytic manifold such that the map $(x, y) \to xy$ of $GxG \to G$ is analytic, then there exists an open neighbourhood of e which is a Lie group.*

In fact, the existence of the inverse element of x depends upon the existence of the solution for y of $\varphi_i(x, y) = 0, 1 = 1, \ldots, n$. Now $\dfrac{\partial\varphi_i}{\partial x_j} = \delta_{ij} +$ terms containing positive powers of the y_i. If we put $y = e$, the latter terms vanish and

$$\left(\frac{\partial\varphi_i(x, y)}{\partial x_j}\right)_{y=e} = \delta_{ij}.$$

Hence 10

$$J = \det \left(\frac{\partial \varphi_i(x, y)}{\partial x_j} \right)_{y=e} = 1$$

J being a continuous function of x and y, $J \neq 0$ in some neighbourhood V' of e. Therefore there exists a neighbourhood V of e every element of which has an inverse. Then the neighbourhood $W = \bigcup_{n=1}^{\infty} (V \cap V^{-1})^n$ is a group compatible with the manifold structure.

2.3 Formal Lie groups.

Definition. *A* formal Lie group *over a commutative ring A with unit elements, is a system of n formal series φ_i in $2n$ variables with coefficients in A such that*

$$\varphi_i(x_1, \ldots, x_n, 0, 0, \ldots) = x_i = \varphi_i(0, 0, \ldots, x_1, \ldots, x_n)$$

and

$$\varphi_i(\varphi_1(x, y), \ldots, \varphi_n(x, y), z) = \varphi_i(x, \varphi_1(y, z), \ldots, \varphi_n(y, z)).$$

Almost all that we prove in the next few lectures will be valid for formal Lie groups over a field of characteristic zero also. For a study of formal Lie groups over a field of characteristic $p \neq 0$, one may see, for instance, [9], [10].

2.4 Taylor's formula.

Let f be a function on an open neighbourhood of e, and let τ_y, σ_z denote respectively the right and left translates of f defined by $\tau_y f(x) = f(xy)$; $\sigma_z f(x) = f(z^{-1}x)$ for sufficiently small y and z. These two operators commute,

i.e.

$$\tau_y(\sigma_z f) = \sigma_z(\tau_y f)$$

If f is analytic in V, $\tau_y f$ is (for $y \in W$) analytic in W, where W is a neighbourhood of e such that $W^2 \subset V$.

11 Now,

$$\varphi_i f(x) = f(\varphi_1(x,y),\dots,\varphi_n(x,y))$$

with

$$\tau_i(x,y) = x_i + y_i + \sum_{\substack{|\alpha|\geq 1\\|\beta|\geq 1}} \lambda^i_{\alpha,\beta} x^\alpha y^\beta.$$

If we set

$$u_i = y_i + \sum_{\substack{|\alpha|\geq 1\\|\beta|\geq 1}} \lambda^i_{\alpha,\beta} x^\alpha y^\phi$$

$\tau_y f(x) = f(x+u)$ in the usual notation. This can be expanded as a Taylor series

$$\tau_y f(x) = \sum_\alpha \frac{1}{\alpha!} u^\alpha \frac{\partial^\alpha f}{\partial x^\alpha}(x).$$

We may now substitute for the u_i in this convergent series.

$$\begin{aligned} u^\alpha &= u_1^\alpha \cdots u_n^{\alpha_n} \\ &= \Big(y_1 + \sum_{\substack{|\gamma|\geq 1\\|\delta|\geq 1}} \lambda^1_{\gamma,\delta} x^\gamma y^\delta\Big)^{\alpha_1} \dots \\ &= y^\alpha + \sum_{|\beta|\geq|\alpha|} g^\alpha_\beta(x) y^\beta \end{aligned}$$

where the coefficient of powers of y are analytic functions of x and $g^\alpha_\beta(e) = 0$. If here we take $\alpha = 0$, $u^\alpha = 1$, and hence $g^0_\beta = 0$ for every β. Thus

$$\tau_y f(x) = \sum_\alpha \frac{1}{\alpha!} \frac{\partial^\alpha f}{\partial x^\alpha}(x)(y^\alpha + \sum_{|\beta|\geq|\alpha|} g^\alpha_\beta(x) y^\beta).$$

These are uniformly absolutely convergent in a suitable neighbourhood of e, on the explicit choice of which we shall not meticulously insist. Hence the above formula can be written as

$$\tau_y f(x) = \sum_\alpha y^\alpha (\frac{1}{\alpha!} \frac{\partial^\alpha f}{\partial x^\alpha}(x) + \sum_{|\beta|\leq|\alpha|} g^\beta_\alpha(x) \frac{1}{\beta!} \frac{\partial^\beta f}{\partial x^\beta}(x)).$$

This we shall denote by

$$\tau_y f(x) = \sum_\alpha \frac{1}{\alpha!} y^\alpha \Delta_\alpha f(x)$$

12 where Δ_α is a differential operator not depending on f. This formula is the generalised Taylor's formula we sought establish.

2.5 Study of the operator Δ_α.

Now, $\Delta_\alpha = \dfrac{\partial^\alpha}{\partial x^\alpha} + \sum_{|\beta| \leq |\alpha|} g^\beta_\alpha(x) \dfrac{\partial^\beta}{\partial x^\beta} \dfrac{\alpha!}{\beta!}$ with $g^\beta_\alpha(e) = 0$. Since at $x = e$, $\Delta_\alpha = \dfrac{\partial^\alpha}{\partial x^\alpha}$, the Δ_α are linearly independent at the origin. At $\alpha = 0$, Δ_α is the identity operator and if $\alpha \neq 0$, Δ_α is without constant term as $g^0_\alpha = 0$.

Let us denote $\Delta_{[i]}$ by X_i. Then $X_i = \dfrac{\partial}{\partial x_1} + \sum_j a_{ij}(x) \dfrac{\partial}{\partial x_j}$, with $a_{ij}(e) = 0$. These are vector fields in a neighbourhood of e. Now we shall use the fact that, for every $y, z \in G$, the operators σ_z and τ_y commute. We have

$$\sigma_z(\tau_y f) = \sigma_z(\sum \frac{1}{\alpha!} y^\alpha \Delta_\alpha f) = \sum \frac{1}{\alpha!} y^\alpha (\sigma_z \circ \Delta_\alpha)(f)$$

and, on the other hand,

$$\tau_y(\sigma_z f) = \sum \frac{1}{\alpha!} y^\alpha \Delta_\alpha(\sigma_z f).$$

Therefore, by the uniqueness of the expansion in power-series in y of $\sigma_z(\tau_y f) = \tau_y(\sigma_z f)$, we have $\sigma_z \circ \Delta_\alpha = \Delta_\alpha \circ \sigma_z$, for every z in a sufficiently small neighbourhood of e. Otherwise stated, Δ_α is left invariant in this neighbourhood. This enables us to define Δ_α at every point z in the Lie group by setting $\Delta_\alpha f(z) = \sigma_{z^{-1}} \Delta_\alpha \sigma_z f(z)$, so that the extended operator remains left invariant.

Theorem 1. *The linear differential operators Δ_α form a basis for the algebra of left invariant differential operators.*

We have already remarked that the Δ_α are linearly independent at e.
13 Let D be any left invariant linear differential operator on G. Then $D = \sum_{|\alpha|\leq r} b_\alpha(x)\dfrac{\partial^\alpha}{\partial x^\alpha}$, r being some positive integer. Define $\Delta = \sum_{|\alpha|\leq r} b_\alpha(e) \dfrac{\partial^\alpha}{\partial x^\alpha}$. Obviously, $\Delta \circ \sigma_z = \sigma_z \circ \Delta$. At e, $D = \Delta$, and by the left invariance of both D and Δ, $D = \Delta$ everywhere. This proves our contention that the Δ_α form a basis of the algebra of left invariant differential operators.

We shall hereafter denote this algebra by $\mathcal{U}(G)$.

2.6 The Lie algebra of a Lie group G.

Let $\mathscr{G}$ be the subspace of $\mathcal{U}(G)$ generated by the X_i. This is the same as the subspace composed of vector fields which are left invariant. This is obviously isomorphic as a vector space to the tangent space at e. If there are two vector fields X, Y, then XY is an operator of order 2, as also YX. But $XY - YX$ is a left invariant vector field, as can be easily verified. Lef $[X, Y]$ stand for this composition law. It is not hard to see that this bracket operation satisfies

$$[X, Y] = 0, \quad \text{and} \quad [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

This leads us to the following

Definition. *A* Lie algebra $\mathfrak{g}$ *over a field, is a vector space with a composition law* $[X, Y]$ *which is a bilinear map* $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *satisfying* $[X, X] = 0$, *and the Jacobi's identity, viz.*

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

Example. (1) The left invariant vector fields of a Lie group form a Lie algebra.

(2) Any vector space $\mathcal{U}$ with the composition Law $[X, Y] = 0$ for every $X, Y \in \mathcal{U}$ is a Lie algebra.

Such an algebra is called an *abelian Lie algebra*. 14

(3) Any associative algebra with the bracket operation

$$[X, Y] = XY - YX$$

is a Lie algebra.

In particular, the matrix algebra $\mathscr{M}_n(K)$ over a field K and the space of endomorphisms of a vector space $\mathcal{U}$ are Lie algebras.

Definition. *A subspace* $\mathscr{J}$ *of a Lie algebra* $\mathfrak{g}$ *is a* subalgebra *if for every* $x, y \in \mathscr{J}$, $[x, y] \in \mathscr{J}$.

A subspace $\mathcal{U}$ of a Lie algebra $\mathfrak{g}$ is an *ideal*, if for every $x \in \mathcal{U}$, $y \in \mathfrak{g}$, $[x, y] \in \mathcal{U}$.

Example. (1) The set of all matrices in $\mathscr{M}_n(K)$ whose traces are zero is an ideal of $\mathscr{M}_n(K)$.

(2) The set of all elements $z \in \mathfrak{g}$ such that $[z, x] = 0$ for every $x \in \mathfrak{g}$ is an ideal of $\mathfrak{g}$, called *centre* of $\mathfrak{g}$.

If $\mathcal{U}$ is an ideal in $\mathfrak{g}$, the quotient space $\mathfrak{g}/_{\mathcal{U}}$ can be provided with the structure of a Lie algebra by defining $[(x+\mathcal{U}), (y+\mathcal{U})] = [x, y]+\mathcal{U}$. This is called the *factor algebra of* $\mathfrak{g}$ *by* $\mathcal{U}$.

2.7 Representations of a Lie algebra.

Definition. *A* representation *of a Lie algebra* $\mathfrak{g}$ *in another Lie algebra* $\mathfrak{g}'$ *is a linear map* f *such that* $f([x, y]) = [f(x), f(y)]$ *for every* $x, y \in \mathfrak{g}$.

It can be verified that the image of $\mathfrak{g}$ by f and the kernel of f are subalgebras of $\mathfrak{g}'$ and $\mathfrak{g}$ respectively. The latter is, in fact, an ideal and $\mathfrak{g}/\ker f$ is isomorphic to $f(\mathfrak{g})$.

15 In particular, $\mathfrak{g}'$ may be taken to the the space of endomorphisms of a vector space V, leading us to the definition of a linear representation of $\mathfrak{g}$.

Definition. *A* linear representation *of a Lie algebra* $\mathfrak{g}$ *in a vector space* V *is a representation of* $\mathfrak{g}$ *into the Lie algebra of endomorphisms of* V.

2.8 Adjoint representation.

Let $\mathfrak{g}$ be a Lie algebra and $x \in \mathfrak{g}$. The map $\mathfrak{g} \to \mathfrak{g}$ defined by $y \longrightarrow [x, y]$ is a linear map of $\mathfrak{g}$ into itself. This map is denoted adx. Thus, adx $(y) = [x, y]$.

Remarks. (1) Ad x is a derivation of the Lie algebra. We recall here the definition of a derivation in an algebra $\mathfrak{g}$, associative or not. A linear map D of $\mathfrak{g}$ into itself is a *derivation* if for any two elements $x, y \in \mathfrak{g}$, we have $D(xy) = x(Dy) + (Dx)y$. In a Lie algebra derivations of the type ad x are called *inner derivations*.

(2) $x \to ad\ x$ is a linear map of $\mathfrak{g}$ into the Lie algebra of endomorphisms of the vector space $\mathfrak{g}$.

This is, moreover, a linear representation of $\mathfrak{g}$ in $\mathfrak{g}$. The verification of the relation $ad[x, y] = [adx, ady]$ is an immediate consequence of Jacobi's identity.

This linear representation will henceforth be referred to as the *adjoint representation* of $\mathfrak{g}$.

Chapter 3

Relations between Lie groups and Lie algebras - I

3.1 Differential of an analytic representation.

16

Definition. *An* analytic representation *of a Lie group G into a Lie group G' is an algebraic representation which is an analytic map.*

Remark. It is true, as we shall see later (Cor. to Th. 4, Ch. 4.5) that any representation of the underlying topological group G in G' is itself a representation in the above sense.

We now seek to establish a correspondence between analytic representations of Lie groups and algebraic representations of their Lie algebras. As a first step, we prove the following

Proposition 1. *To every analytic representation $h : G \longrightarrow G'$ there corresponds a map $dh : \mathcal{U}(G) \to \mathcal{U}(G')$ which is a representation of algebras such that $\Delta(f \circ h) = (dh(\Delta) f) \circ h$.*

Let $y \in G$ and $y' = h(y)$. If f is an analytic function on G', we have

$$\tau_y(f \circ h)(x) = f(h(xy)) = f(h(x)h(y)) = (\tau_{y'} f) \circ h(x).$$

We may now write down the Taylor formula for both sides of the equation and equate the coefficients of powers of y (by the uniqueness of development in power series). We have

$$\sum_{\alpha} \frac{1}{\alpha!} y^{\alpha} \Delta_{\alpha}(f \circ h) = (\sum_{\alpha'} \frac{1}{\alpha'!} y'^{\alpha'} \Delta'_{\alpha'} f) \circ h.$$

But h being an analytic map, $(h(y))_i = \sum_{\alpha'} \nu^i_{\alpha} y^{\alpha}$. There is no constant term in this summation since $(h(y))_i = 0$ at $y = e$. Hence $y'^{\alpha'} =$
17 $\sum_{|\alpha| \geq |\alpha'|} \mu^{\alpha'}_{\alpha} y^{\alpha}$, where $\mu^{\alpha'}_{\alpha}$ are constants, and on substitution in the above equation, we obtain

$$\begin{aligned} \sum_{\alpha} \frac{1}{\alpha!} y^{\alpha} \Delta_{\alpha}(f \circ h) &= \sum_{\alpha'} (\frac{1}{\alpha'!} \sum_{|\alpha| \geq |\alpha'|} \mu^{\alpha'}_{\alpha} y^{\alpha} \Delta'_{\alpha'} f) \circ h \\ &= \sum_{\alpha} y^{\alpha} (\sum_{|\alpha'| \leq |\alpha'|} \mu^{\alpha}_{\alpha} \frac{1}{\alpha'!} \Delta'_{\alpha'} f) \circ h. \end{aligned}$$

the series being uniformly absolutely convergent. If D'_{α} denotes $\sum_{|\alpha'| \leq |\alpha'|} \mu^{\alpha}_{\alpha} \dfrac{\alpha!}{\alpha'!} \Delta'_{\alpha'}$, which is a left invariant differential operator, then $\Delta_{\alpha}(f \circ h) = (D'_{\alpha} f) \circ h$. Moreover, this equation completely determines D'_{α} since its value at e given by $D_{\alpha} f(e') = \Delta_{\alpha}(f \circ h)(e)$. As the Δ_{α} form a basis for $\mathcal{U}(G)$ in G, we may define a linear map $dh : \mathcal{U}(G) \to \mathcal{U}(G')$ by setting $dh(\Delta_{\alpha}) = D'_{\alpha}$. It is obvious that $\Delta(f \circ h) = (dh(\Delta) f) \circ h$ for any $\Delta \in \mu(G)$. To complete the proof of proposition 1, one has only to show that $dh(\Delta_1 \Delta_2) = dh(\Delta_1) dh(\Delta_2)$. But this is obvious since

$$\begin{aligned} (dh(\Delta_1 \Delta_2) f) \circ h &= \Delta_1 \Delta_2 (f \circ h) \\ &= \Delta_1 (dh(\Delta_2) f \circ h) \\ &= (dh(\Delta_1 dh(\Delta_2) f) \circ h. \end{aligned}$$

Now, $dh(\Delta_{\alpha}) = \sum_{|\alpha'| \leq |\alpha|} \mu^{\alpha'}_{\alpha} \dfrac{\alpha!}{\alpha'!} \Delta'_{\alpha'}$ is of order less than or equal to that of Δ_{α}. By linearity, the same is also true of any operator $\Delta \in \mathcal{U}(G)$. Also, dh preserves constant terms. The image of $\mathfrak{g}$ is in $\mathfrak{g}'$, and by Proposition 1, dh restricted to $\mathfrak{g}$ is a Lie algebra representation. This is said to be the *differential* of the map h.

Remarks. (1) If we have another representation $h' : G' \to G''$, it is obvious that $d(h' \circ h) = dh' \cdot dh$.

(2) If h is an analytic map of a manifold V into a manifold W, then **18**
we can define the differential of h at x, viz., $dh_x : T_x \to T_{h(x)}$ where T_x, $T_{h(x)}$ are tangent spaces at the respective points. This map makes correspond to a tangent vector X at x, the vector X' at $h(x)$ such that $X' f = X(f \circ h)$ for every function f analytic at $h(x)$. However, we cannot, in general, define the image a vector field. As we have seen, in the case of Lie groups, as long as one is considering only left invariant vector fields, one can talk of an image vector field. Thus we now have, corresponding to an analytic representation of a Lie group G into another Lie group G', two notions of a differential map: the linear map of the tangent space at a into that at $h(e) = e'$, and the representation of the Lie algebra of G in that of G'. These two notions are essentially the same in the following sense. Let φ, φ' be the canonical vector space isomorphisms of $\mathfrak{g}$, $\mathfrak{g}'$ with T_e, $T_{e'}$ respectively. Then the diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\varphi"] \arrow[d, "dh"] & T_e \arrow[d, "dh_e"] \\ \mathfrak{g}' \arrow[r, "\varphi'"] & T_{e'} \end{tikzcd}$$

is commutative.

Proposition 2. *Let G and G' be two connected Lie groups. An analytic representation of $G \to G'$ is surjective if and only if the differential map is surjective.*

Proposition 3. *A representation h of a Lie group G in another Lie group G' is* locally injective *(i.e. there axises a neighbourhood of e on which h is injective) if and only if dh is injective.*

These two propositions are consequences of the corresponding prop- **19**
erties of manifolds, the proofs of which we omit.

3.2 Subgroups of a Lie group.

Definition. *An analytic map f of a manifold U into another manifold V is said to be* regular *at a point x in U if the differential map df_x is injective.*

Definition. *A* submanifold *of an analytic manifold U is a pair (V, π) consisting of a manifold V which is countable at ∞ and an injective analytic map π of V into U which is everywhere regular.*

Remarks. (1) The topology on $\pi(V)$ is not that induced from the topology of U in general. For instance, if T^2 is the two - dimensional torus, V the space of real numbers, and π the map $t \to (t, \alpha t)$ of V into T^2, where α is irrational, it is easy to see that π is an injective, analytic, regular map. But π cannot be a homeomorphism of V into T^2. For, every neighbourhood of $(0, 0)$ in T^2 contains points $(t, \alpha t)$ with arbitrarily large values of t. Hence, the inverse image of this neighbourhood in $\pi(U)$ with the induced topology can never be contained in a given neighbourhood of 0 in R.

(2) Nevertheless, it is true that locally, for every point x of V, there exist neighbourhoods W *in* V and W^1 in U which satisfy the following : A coordinate system $(x_1, \ldots, x_n)$ can be defined in W^1 such that W is defined by the annihilation of certain coordinates.

20 **Definition.** *A Lie subgroup of a Lie group G is a submanifold (H, π), $\pi(H)$ being a subgroup of G.*

We define on H the group structure obtained by requiring that π be a monomorphism. Since the map π of H in G is regular, locally the analytic structure of H is induced form that of G. Hence the group operations in H are analytic in H, as they are analytic in G. H is therefore a Lie group.

Proposition 4. *The Lie algebra of a Lie subgroup of a Lie group G can be identified with a subalgebra of the Lie algebra of G.*

In fact, if (H, π) is the subgroup, π is a representation of H in G and $d\pi$ is injective since π is regular. We identify the Lie algebra $\mathscr{S}$ of H with the subalgebra $d\pi(\mathscr{S})$ or $\mathfrak{g}$.

3.3 One-parameter subgroups.

Definition. *An analytic representation ρ of R into G is said to be a* one-parameter subgroup *of G.*

We know that the representation ρ gives rise to a differential map $d\rho$ of the Lie algebra of R (spanned by $\dfrac{d}{dt}$) into the Lie algebra of G. Let $d\rho(\dfrac{d}{dt}) = X = \sum \lambda_j X_j$. We now form the differential equations satisfied by the function ρ.

Let $(x_1, x_2, \ldots x_n)$ be a coordinate system in a neighbourhood of e in G and Let ρ_i denote $x_i \circ \rho$. Now

$$\rho_i(t + t') = \varphi_1(\rho(t), \rho(t'))$$
$$\frac{\partial}{\partial t'}(\rho_i(t + t')) = \sum_k \frac{\partial \varphi_i}{\partial y_1}(\rho(t), \rho(t')).\frac{d\rho_k}{dt}(t)$$

Putting $t' = 0$, we get

$$\frac{d\rho_i}{dt}(t) = \sum_k \frac{d\rho_k}{dt}(0)\frac{\partial \varphi_i}{\partial y_k}(\rho(t), e).$$

21

If $X_i = \Delta_{[i]}$, we have

$$X_i = \frac{\partial}{\partial x_i} + \sum_j a_{ij}(x)\frac{\partial}{\partial x_j} \quad \text{with} \quad a_{ij}(e) = 0.$$

Since $(Xf) \circ \rho = \frac{d}{dt}(f \circ \rho)$ for every function analytic at e, we get

$$\frac{d\rho_i}{dt} = (Xx_i) \circ \rho \quad \text{by setting } f = x_i.$$

Hence

$$\frac{d\rho_i}{dt}(0) = (Xx_i) \circ \rho(0)$$

$$\begin{aligned} &= Xx_i(e) \\ &= \sum (\lambda_j X_j) x_i(e) \\ &= \lambda_i. \end{aligned}$$

To sum up, ρ satisfies the system of differential equations

(A) $\dfrac{d\rho_i}{dt} = \sum_k \lambda_k \dfrac{\partial \varphi_i}{\partial y_k} (\rho(t), e)$

with the initial condition

(B) $\rho_i(0) = 0$.

(A) implies $\dfrac{d\rho_i}{dt}(0) = \lambda_i$.

Now, conversely if are given the system of differential equations (A) with the initial condition (B), then by Cauchy's theorem on the existence and uniqueness of solutions of differential equations, there exists one and only one solution $t \to \rho(t, \lambda)$ which is analytic in t and λ in a neighbourhood of $(0, \lambda)$. We shall now show that $\rho(t + u) = \rho(t) \cdot \rho(u)$ for sufficiently small values of t and u.

22 Let

$$\begin{aligned} \sigma_i'(t) &= \varphi_i(\rho(u), \rho(t)). \quad \text{Then} \\ \frac{d\sigma_i'}{dt} &= \sum_l \frac{\partial \varphi_i}{\partial y_i} (\rho(u), \rho(t)) \frac{d\rho_l}{dt}(t) \\ &= \sum_{k,j} \lambda_k \frac{\partial \varphi_i}{\partial y_l} (\rho(u), \rho(t)) \frac{\partial \varphi_l}{\partial y_k} (\rho(t), e) \end{aligned}$$

since the ρ_i are solutions of (A). On the other hand, we have

$$\begin{aligned} \varphi_i(\rho(u)\rho(t), y) &= \varphi_i(\rho(u), \rho(t)y) \\ &= \varphi_i(\rho(u), \varphi(\rho(t), y)) \\ \frac{\partial \varphi_i}{\partial y_k}(\rho(u)\rho(t), y) &= \sum_{l=1}^{n} \frac{\partial \varphi_i}{\partial y_l} (\rho(u), \rho(t)y) \frac{\partial \varphi_l}{\partial y_k} (\rho(t), y). \end{aligned}$$

Hence,

$$\frac{d\sigma_i'}{dt} = \sum_k \lambda_k \frac{\partial \varphi_i}{\partial y_k}(\sigma'(t), e).$$

i.e. σ_i' is a solution of (A) with the initial condition (C):

$$\sigma_i'(0) = \varphi_i(e, \rho(u)) = \rho_i(u).$$

Also, $t \to \sigma(t) = \rho(t+u)$ is a solution of (A) since the differential equation (A) is a invariant for translations of it. Also $\sigma(0) = \rho(u)$.

Hence σ_i' and σ_i are two sets of solutions of (A) with the same initial conditions, and therefore

$$\sigma_i'(t) = \rho(t+u), \quad \text{i.e.} \quad \varphi_i(\rho(t), \rho(u)) = \rho_i(t+u).$$

or $\rho(t)\rho(u) = \rho(t+u)$ for sufficiently small values of t and u. Also this map $t \to \rho(t, X)$ is analytic. We assume the following

Lemma 1. *Let H be a connected, locally connected and simply connected topological and f a local homomorphism of $H \to G$ (i.e. a continuous map of a neighbourhood of e into H such that $f(xy) = f(x)f(y)$ for all x, y such that x, y, $xy \in V$). Then there exists one and only one representation $\tilde{f}$ of H in G which coincides with f on V.* **23**

We immediately obtain (since R is simply connected), the

Theorem 1. *For every $X \in \mathfrak{g}$, there exists one and only one one - parameter subgroup $\rho(t, X)$ such that $d\rho\dfrac{(d)}{dt} = X$. The function $\rho(t, X)$ is analytic in t and X.*

One can assign to any finite dimensional vector space over the real number field a manifold structure which is induced by that of the real numbers. In particular, The Lie algebra of a Lie group also has an analytic structure. Whenever we talk of an analytic map into of from a Lie algebra, it is to this analytic structure that we refer.

Proof of the Lemma Consider the Cartesian set product $\hat{H} = H \times G$. We provide $\hat{H}$ with a topology by defining the neighbourhood system at each point (x, y) in the following way:

Let W be a neighbourhood of e in $H \subset V$, where V can be assumed to be connected since H is locally connected. The fundamental system of neighbourhoods at (x, y) is given by $N(W, x, y) = \{(x', y') : x' \in xW, y' = yf(x^{-1}x')\}$. It is easily verified that this satisfies the neighbourhood axioms for a topology, and that $\hat{H}$ with the usual projection $\pi : H \times G \to H$ is a covering space of H. Let H_1 be the connected component of (e, e) in $\hat{H}$. Then H_1 is a connected, covering space of H and since H is simply connected,π is a homeomorphism of H_1 onto H.
24 Let η be its inverse. Define $\tilde{f}(x) = \pi_2 \circ \eta(x)$ for every x in H where $\pi_2 : H \times G \to G$ is the second projection. $N(W, x, y)$ is mapped homeomorphically by π onto xW. Hence $N(W, x, y) \subset H_1$ if W is connected and $(x, y) \in H_1$. It follows that $\tilde{f}$ is a representation which extends f.

3.4 The exponential map.

We shall denote $\rho(t, X)$ by $\exp(tX)$.

But such a notation involves the tacit assumption that $\rho(t, X)$ depends only on tX. In other words, one has to make sure that $\rho(1, sX) = \rho(s, X)$ before such a notation becomes permissible. But this is obvious in as much as $t \to \rho(st, X)$ is a one -parameter subgroup with $d\rho\dfrac{(d)}{d(st)} = X$ or $d\rho\dfrac{(d)}{dt} = sX$. The one-parameter subgroup such that $d\rho\dfrac{(d)}{dt} = sX$ is, by definition, $t \to \rho(t, sX)$. By uniqueness of the one-parameter subgroups, $\rho(st, X) = \rho(t, sX)$, or in particular, $\rho(s, X) = \rho(1, sX)$. It is easy to see that $\exp(tX)\exp(t'X) = \exp(t + t')X$ and $\exp(-X) = (\exp X)^{-1}$. But, in general, $\exp Y \cdot \exp Y' \neq \exp(Y + Y')$.

Theorem 2. *The map $h : X \to \exp X$ of $\mathfrak{g}$ into G is an analytic isomorphism of a neighbourhood of 0 in $\mathfrak{g}$ onto a neighbourhood of e in G.*

In fact, since h is an analytic map, it is enough to show that the Jacobian of the map $h \neq 0$ in a neighbourhood of the origin. $(X_1, \ldots, X_n)$ form a basis for $\mathfrak{g}$, where $X_i = \Delta_{[i]}$.

$$h\left(\sum_i y_i X_i\right) = \exp(\sum_i y_i X_i)$$

$$\frac{\partial h_j}{\partial y_k}(0) = \frac{d}{dt}(\exp tX_k)_j(t=0) = (X_k x_j)_{x=e} = \delta_{jk}.$$

i.e. the jacobian = 1 at e. By continuity, the Jacobian does not vanish in a neighbourhood of e. 25

Now, let $X_1, \ldots, X_n$ be an arbitrary basis of $\mathfrak{g}$. This can be transported into a system of coordinates in G by means of the above map. For every $x \in G$ sufficiently near e, there exists one and only one system $(x_1, \ldots, x_n)$ near 0 such that $x = \exp(\sum_i x_i X_i)$.

This system of coordinates is called the *canonical system of coordinates* with respect to any given basis. Hereafter, we will almost always operate only with a canonical system of coordinates.

Remark. Let $x \in V$, V being a neighbourhood of e in which a canonical coordinate system exists and x is sufficiently near e. Now, if

$$x = \exp(\sum_i x_i X_i),$$
$$x^p = \exp(\sum_i (px_i) X_i),$$

i.e. the coordinates of x^p are $(px_1, \ldots, px_n)$.

Proposition 5. *Let h be a representation of G in H, and $dh : \mathfrak{g} \to \mathcal{F}$ its differential. Then the diagram*

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "dh"] \arrow[d, "\exp"'] & \mathscr{S} \arrow[d, "\exp"] \\ G \arrow[r, "h"'] & H \end{tikzcd}$$

is commutative.

Consider $t \xrightarrow{\rho'} h(\exp tX)$.

This is obviously a one-parameter subgroup, and $d\rho' = dh \circ d\rho$. Therefore

$$h(\exp X) = \exp(d\rho'(\frac{d}{dt}))$$

$$= \exp(dh(X)).$$

It follows, therefore, that if $h(G) = (e)$, dh is the map $\mathfrak{g} \to (0)$.

26 Conversely, if $dh = C$, and G connected, $h(\exp X) = e$ for every element in a neighbourhood of e, and $h = e$. Again, if G is connected and two representations h_1, h_2 of G in H are such that $dh_1 = dh_2$, then $h_1 = h_2$.

Proposition 6. *For every analytic function f on a neighbourhood of e, we have*

$$f(\exp tX) = \sum_{n=0}^{\infty} \frac{t^n}{n!}(X^n f)(e).$$

In fact,

$$\frac{d}{dt} f(\exp tX) = (Xf)(\exp tX).$$

By induction on n, we have

$$\frac{d^n}{dt^n} f(\exp tX) = (X^n f)(\exp tX)$$

or

$$\left\{\frac{d^n}{dt^n} f(\exp tX)\right\}_{t=0} = X^n f(e).$$

Now, $f(\exp tX)$ is an analytic function of t and by Taylor's formula, we have

$$f(\exp tX) = \sum_{n=0}^{\infty} \frac{t^n}{n!}(X^n f)(e)$$

Theorem 3. *In canonical coordinates, we have $\Delta_\alpha = \dfrac{\alpha!}{|\alpha|!} S_\alpha$ where S_α is the coefficient of t^α in the expansion of $(\sum_{i=1}^n t_i X_i)^{|\alpha|}$ and $S_\alpha \in \mathcal{U}(G)$.*

In fact, it is enough to prove the equality of Δ_α and $\dfrac{\alpha!}{|\alpha|!} S_\alpha$ at e since both Δ_α and S_α are invariant. Now,

$$f(y) = \tau_y f(e) = \sum_\alpha \frac{1}{\alpha!} y^\alpha \Delta_\alpha f(e) \text{ with } \Delta_\alpha f(e) = \left\{\frac{\partial^\alpha}{\partial y^\alpha} f(y)\right\}_{y=e}$$

$y = \exp(\sum y_i X_i)$ where y_i are the canonical coordinates of y.

27 Therefore

$$\begin{aligned} f(y) &= f(\exp(\sum y_i X_i)) \\ &= \sum_{p=0}^{\infty} \frac{1}{p!} \left\{ (\sum y_i X_i) p_f \right\} (e) \end{aligned}$$

by Prop. 6, chapter 3, 5.

Taking partial derivatives at $y = e$, we have

$$\frac{\partial^\alpha}{\partial y^\alpha} f(y) \quad \text{at} \quad y = e \quad \text{is} \quad \frac{\alpha!}{|\alpha|!} (S_\alpha f)(e).$$

Hence, $Delta_\alpha f(e) = \dfrac{\alpha!}{|\alpha|!} S_\alpha f(e)$ which is what we wanted to prove.

Chapter 4

Relation between Lie groups and Lie algebras - II

4.1 The enveloping algebras.

Let $\mathfrak{g}$ be a Lie algebra, and T the tensor algebra of the underlying vector 28
space of $\mathfrak{g}$. Consider the two-sided ideal I generated in T By the elements of the form $x \otimes y - y \otimes x - [x, y]$. Then the associative algebra T/I is said to be the *universal enveloping algebra* of the Lie algebra $\mathfrak{g}$.

Definition. *A linear map h of al Lie algebra $\mathfrak{g}$ into as associative algebra A is said to be a* linearisation *if $h([x, y]) = h(x)h(y) - h(y)h(x)$ for every $x, y \in \mathfrak{g}$.*

We have obviously a canonical map of $\mathfrak{g}$ in to T/I, which we shall denote by j.

Proposition 1. *To any linearisation f of $\mathfrak{g}$ in an associative algebra A, there corresponds one and only one representation $\bar{f}$ of T/I in such that $\bar{f} \circ j = f$.*

In fact, f being a linear map, it can be lifted uniquely into a representation $\hat{f}$ of the tensor algebra T in A. Obviously, the kernel of $\hat{f}$ contains elements of the form $x \otimes y - y \otimes x - [x, y]$ and hence contains I. Hence this gives rise to a map $\bar{f}$ with the required property.

4.2 The Birkhoff-Witt theorem.

With reference to the universal enveloping algebra of a Lie algebra of a Lie group, we have the following

29 **Theorem 1.** *The algebra of left invariant differential operators $\mathcal{U}$ is canonically isomorphic to the universal enveloping algebra of the Lie algebra.*

In fact, the inclusion map of the Lie algebra $\mathfrak{g}$ into $\mathcal{U}$ can be lifted to a representation of the enveloping algebra $\mathcal{U}'$ of $\mathfrak{g}$ in $\mathcal{U}$ by proposition 1. It only remains to show that this map $h: \mathcal{U}' \to \mathcal{U}$ is an isomorphic. If S_α is the coefficient of t^α in the expansion of $(\sum\limits_{i=1}^{n} t_i X_i)^{|\alpha|}$, it has been proved (Th. 3, Ch. 3.4) that S_α form a basis for $\mathcal{U}$. S_α is an operators of the form $\sum(\ldots)X_{i_1}\cdots X_{i_{|\alpha|}}$, where $X_{i_1}\cdots X_{i_{|\alpha|}}$ are obtained by certain permutations of $x_1^{\alpha_i}\cdots X_n^{\alpha_n}$. Let S'_α denote the element of the form $\sum(\ldots)X_{i_1}\otimes\ldots\otimes X_{i_{|\alpha|}}$, where $S_\alpha = \sum(\ldots)X_{i_1}\ldots x_{i_{|\alpha|}}$. By definition of h, we have $h(S'_\alpha) = S_\alpha$. To prove that h is an isomorphism, it is therefore sufficient to show that the S'_α generate $\mathcal{U}'$. We shall do this showing that the S'_α for $|\alpha| \leq r$ generate the space T_r of tensors of order $\leq r$ modulo I. The statement being trivially true for $r = 0$, we shall assume it verified for $(r-1)$ and prove it for r. Again, it is enough to prove that S'_α for $|\alpha| = r$ generate the space T'_r of tensors of order $= r$, modulo $I + T_{r-1}$. Let $X_{i_1}\otimes\cdots\otimes X_{i_r}$ be an element $\in T'_r$. Then

$$X_{i_1}\otimes X_{i_2}\otimes\ldots\otimes X_{i_r} \equiv X_{i_2}\otimes X_{i_1}\otimes X_{i_3}\otimes\ldots + [X_{i_1}, X_{i_2}]\otimes X_{i_3}\otimes\ldots \mod I.$$

Hence, if σ is a permutation of $(1, 2, \ldots, r)$,

$$X_{i_1}\otimes X_{i_2}\otimes\ldots\otimes X_{i_r} \equiv X_{i_{\sigma(1)}}\ldots X_{i_{\sigma(r)}} \mod (T_{r-1} + I)$$

30 by successive transpositions. Now, $S'_\alpha = \sum_\sigma \mathcal{U}_\sigma X_{i_{\sigma(1)}}\ldots x_{i_{\sigma(r)}}$, where $\mathcal{U}_\sigma$ are positive integers. Therefore

$$(\sum_\sigma \mathcal{U}_\sigma)X_{i_1}\otimes\ldots\otimes X_{i_r} \equiv S'_\alpha \mod (T_{r-1} + I).$$

Since

$$(\sum_\sigma \mathcal{U}_\sigma) \neq 0, X_{i_1}\otimes\ldots\otimes X_{i_r} \equiv kS_\alpha \mod (T_{r-1} + I),$$

and hence the theorem is completely proved.

Incidentally we have proved that the Lie algebra $\mathfrak{g}$ can be embedded in its universal enveloping algebra by the natural map h. This is known as the Birkhoff-Witt theorem, and it true in the more general case when the Lie algebra is over a principal ideal ring.

4.3 Group law in terms of structural constants.

We now show that the Lie algebra of a Lie group completely characterises the group locally. In other words, the group laws of the Lie group can be expressed in terms of the *structural constants* of its Lie algebra. (If $(X-\alpha)_{\alpha\in A}$ be a basis of the Lie algebra, and $[X_i, X_j] = \sum_k C^k_{i,j} X_k, C^k_{i,j}$ are called the structural constants of the Lie algebra).

Theorem 2. *Lie groups having isomorphic Lie algebras are locally isomorphic. If they are connected and simply connected, they are isomorphic.*

Choose a basis $X_1, \ldots, X_n$ of the Lie algebra $\mathfrak{g}$. If $\theta_i(x)$ be the i^{th} coordinates of x in the canonical of coordinates with respect to the above basis we have.

$$\varphi_i(x,y) = \theta_i(xy) = \tau_y\theta_i(x) = \sum \frac{1}{\alpha!} y^\alpha \Delta_\alpha \theta_i(x)$$

$$\text{But}\quad \Delta_\alpha\theta_i(x) = \tau_x(\Delta_\alpha\theta_i)(e)$$
$$= \sum_\beta \frac{1}{\beta!} x^\beta \Delta_\beta(\Delta_\alpha\theta_i)(e).$$

Hence 31

$$\varphi_i(x,y) = \sum_{\alpha,\beta} \frac{1}{\alpha!}\frac{1}{\beta!} y^\alpha x^\beta (\Delta_\beta\Delta_\alpha\beta_i)(e).$$

If

$$\Delta_\beta\Delta_\alpha = \sum_\gamma d^\gamma_{\beta,\alpha}\Delta_\gamma = d^\gamma_{\beta,\alpha}$$

are completely known, once the Lie algebra $\mathfrak{g}$ is given, because $\Delta_\alpha = \frac{\alpha!}{|\alpha|!} S_\alpha$. Since

$$\Delta_r\theta_i(e) = \left(\frac{\partial^r}{\partial x^r xi}\right)_{x=e} = \begin{smallmatrix} 1 \text{ if } r=[i] \\ 0 \text{ if } r\neq[i] \end{smallmatrix}$$

$$\text{we have} \quad \varphi_i(x,y) = \sum_{\alpha,\beta} \frac{d^{[i]}_{\beta,\alpha}}{\alpha!\beta!} x^\beta y^\alpha.$$

Thus the group law is completely determined by the constants $d^{[i]}_{\beta,\alpha}$. If two Lie groups have isomorphic Lie algebras, the constants $d^{[i]}_{\beta,\alpha}$ are the same for both, and the group operation is given locally by the above formula, which is to say the groups are locally isomorphic. By Lemma 1, Ch. 3.3, if the groups are connected and simply connected, they are isomorphic.

We can compute the constants $d^{[i]}_{\beta,\alpha}$ in terms of the structural constants of the Lie algebra and obtain a universal formula (i.e. a formula which is the same for all Lie groups - the Campbell–Hausdorff formula). For instance, if δ is a multi-index of order 2 with 1 in the j^{th} and k^{th} indices and 0 elsewhere, it can easily be seen that $\Delta_\delta = \frac{1}{2}S_\delta = \frac{1}{2}(X_jX_k + X_kX_j)$ and if $C^i_{j,k}$ are the structural constant of the Lie algebra,

$$\begin{aligned} X_jX_k &= \frac{1}{2}[X_j, X_k] + \frac{1}{2}(X_jX_k + X_kX_j) \\ &= \frac{1}{2}\sum_i C^i_{j,k}X_i + \frac{1}{2}(X_jX_k + X_kX_j) \end{aligned}$$

$$\text{and hence} \quad d^{[i]}_{[j],[k]} = \frac{1}{2}C^i_{j,k}.$$

We have therefore

$$\varphi_i(x,y) = x_i + y_i + \frac{1}{2}\sum C^i_{j,k}x_jx_k + \text{ terms of order } \geq 3.$$

32 Again, if $x = \exp X$, $y = \exp Y$ (x_i, y_i begin canonical coordinates) and $xy = \exp Z$, we have

$$Z = X + Y + \frac{1}{2}[X,Y] + \text{ terms of order } \geq 3.$$

4.4

We have proved (Prop. 4, Ch. 3.2) that the Lie algebra of a Lie subgroup can be identified with a subalgebra of the Lie algebra. We now establish the converse by proving the following

Theorem 3. *To every subalgebra $\mathscr{J}$ of a Lie algebra $\mathfrak{g}$ of a Lie group G, there corresponds one and only one connected Lie subgroup H, having it for its Lie algebra.*

Let $X_1, \ldots, X_n$ be a basis of the Lie algebra $\mathfrak{g}$ such that $X_1, \ldots, X_r$ is a basis of $\mathscr{J}$. Let ϑ be the subalgebra generated by X_i with $i \leq r$ in $\mathscr{U}(G)$. We assert that the subspace ϑ of $\mathscr{U}(G)$ generated by S_α with $\alpha = (\alpha_1, \ldots, \alpha_n \circ \cdots)$ is the same as ϑ. By definition of S_α, it is evident that $\mathscr{V}' \subset \mathscr{V}$. It is enough to show that elements of the form $X_{i_1} \cdots X_{i_s} \in \mathscr{U}'$ if $1 \leq i_k \leq r$. We prove this by induction on the length of the product. Now,

$$X_{i_1} X_{i_2} \ldots = X_{i_2} X_{i_1} \ldots + [X_{i_1}, X_{i_2}] X_{i_3} \ldots$$

and since $[X_{i_1}, X_{i_2}] \in \vartheta$, $\mathscr{J}$ being a subalgebra, we have, by induction assumption

$$X_{i_1} X_{i_2} \ldots \equiv X_{i_2} X_{i_1} \ldots \mod \vartheta'.$$

If σ is any permutation of $(1, 2, \ldots, s)$, we have

$$X_{i_1} \ldots X_{i_s} \equiv X_{i_{\sigma(1)}} \ldots x_{i_{\sigma(s)}} (\mod \vartheta').$$

It follows (as in Th. 1), that $\vartheta = \vartheta'$.

Now, let U be a symmetric neighbourhood of e in which the system **33**
of canonical coordinates with respect to $X_1, \ldots, X_n$ is valid. Let N denote the subset of U consisting of points for which $x_{r+1} = \cdots = x_n = 0$. N is obviously a closed submanifold of U. Let $x, y \in N$ sufficiently near e.

$$(xy)_i = \sum_{\alpha,\beta} \frac{1}{\alpha!} \frac{1}{\beta!} x^\beta y^\alpha d^{[i]}_{\beta,\alpha}$$

We now show that $(xy)_i = 0$ for $i > r$. In the summation, unless both α and β are off the form $(\alpha_1, \ldots, \alpha_r, 0 \ldots) x^\beta y^\alpha = 0$. If both α and β are of the above form, $\Delta_\alpha, \Delta_\beta \in \vartheta$ and ϑ being generated by Δ_γ, γ of the same form $d^{[i]}_{\beta,\alpha} = 0$ for $i > r$. Hence $(xy)_i = 0$ for $i > r$. Thus, $x, y \in N \Rightarrow xy \in N$ and $x \in \Rightarrow x^{-1} \in N$ for x, y sufficiently near e.

Finally, let H be the subgroup algebraically generated by the connected component N^1 of e in N. Then H can be provided with an analytic structure such that the map $H \to G$ is everywhere regular. We define neighbourhoods of e in H by intersecting neighbourhood of e in G with N^1. This system can easily be seen to satisfy the neighbourhood axioms for a topological group (Prop 1, Ch. 1.1). For every $x \in H$, the neighbourhood xN^1 of x can be provided with an analytic structure induced by that of G, since xN is a closed submanifold of xU. For $x, y \in H$, these analytic structure agree an $xN^1 \cap yN^1$ because those of xU and yU agree on $xU \cap yU$. H of course has $\mathscr{J}$ as its Lie algebra.

We now prove the uniqueness of such a group. Let H^1 be another connected Lie subgroup having $\mathscr{J}$ for its Lie algebra. Exp $\mathscr{J}$ is open
34 in H^1, as the map $h \to \exp h$ is open (Th. 2, Ch. 3.4). But $\exp \mathscr{J} \subset H$. Hence H is open in H^1. As H is open, it is also closed (Ch. 1.2) and therefore $=H^1$. This completes the demonstration of Theorem 3.

Remark. We have incidentally proved that if a Lie subgroup has $\mathscr{J}$ for its Lie algebra, it contains H as an open subgroup.

It has already been proved (Prop. 1, Ch. 3.1) that if $f : G \to H$ is a representation of Lie groups, there exists a representation $df : \mathfrak{g} \to \mathscr{J}$ of Lie algebras. Now, we establish the converse in the form of a

Corollary. *Let G and H be two Lie groups having $\mathfrak{g}$ and $\mathscr{J}$ as their Lie algebras. If G is connected and simply connected, to every representation π of $\mathfrak{g}$ in $\mathscr{J}$, there corresponds one and only one representation f of $G \to H$ such that $df = \pi$.*

If there exists one such representation, by Prop. 5, Ch. 3.4, it is unique. We shall now prove the existence of such an f.

We first remark that if f is a representation of G in H, K the graph of f in $GxHviz.$ the set $\{(x, f(x)), x \in G\}$, and λ the restriction to K of the projection of $GxH \to G$, then λ is an analytic isomorphism. Conversely, to every subgroup of GxH the first projection from which is an isomorphism to G, there corresponds one and only one representation of G in H. $\mathfrak{g}x\mathscr{J}$ is evidently the Lie algebra of GxH.

Now, Let π be a representation of $\mathfrak{g}$ in $\mathscr{J}$. Let $\mathcal{K}$ be the subset $\{(x, \pi(x)), x \in \mathfrak{g}\}$ of $\mathfrak{g} \times \mathscr{J}$. It can easily be seen that $\mathcal{K}$ is a subalgebra. Then there exists (Th. 3) a connected Lie subgroup K of GxH whose
35 Lie algebra is isomorphic to $\mathcal{K}$. Let λ be the restriction to K of the

projection of $G \times H \rightarrow G$. Then $d\lambda$ is the map $\mathcal{K} \rightarrow \mathfrak{g}$ defined by $d\lambda(x, \pi(x)) = x$. Obviously $d\pi$ is an isomorphism. Hence λ is a local isomorphism of K in G. K is therefore a covering space of G and G being simply connected, λ is actually an isomorphism. To this there corresponds (by our remark above) a representation f of G in H,the graph of whose differential is $\mathcal{K}$, i.e. $df = \pi$.

4.5

Theorem 4 (E. Cartan)**.** *Every closed subgroup of a Lie group is a Lie subgroup.*

For proving this, we require the following

Lemma 1. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}$ be the direct sum of vector subspaces $\wp, \delta$. Then the map $f : (A, B) \rightarrow \exp A \exp B$ of $\mathfrak{g} \rightarrow G$ is a local isomorphism.*

It is obvious that f is an analytic map. To prove that it is a local isomorphism, it is enough to show that the Jacobian of the map $\neq 0$ in a neighbourhood of $(0, 0)$. Let $(X_1, \ldots, X_r)$ be a basis of $\mathcal{U}$ and $X_{r+1}, \ldots, X_n$, a basis of δ. Let $(y_1, \ldots, y_n)$ be the canonical coordinate system with respect to $(X_1 \ldots, X_n)$ and $y_i \circ = f_i$. Then, if $X = \sum\limits_{i=1}^{n} y_i X_i$,

$$f\{X\} = \exp\,(\sum_{i=1}^{r} y_i X_i) \exp\,(\sum_{j=r+1}^{n} y_j X_j)$$

$$\frac{\partial f_k}{\partial y_l}(0) = \frac{d}{dt}(\exp\ t\, X_l)_k\ (t = 0) = (X_l . x_k)_{x=e} = \delta_{k,l}.$$

Hence Jacobian $\neq 0$ at $(0, 0)$ and by continuity, $\neq 0$ in a neighbourhood of the origin. This completes the proof of the lemma.

Let H be a closed subgroup of a Lie group G. We first construct a subgroup $\mathscr{J}$ of $\mathscr{G}$ and prove that the Lie subgroup H^1 of G with $\mathscr{J}$ as its Lie algebra is relatively open in H. Then H is the topological union of cosets of H modulo H^1 and is hence a closed submanifold of G. **36**

Let $\mathscr{J}$ be the set $\{X \in \mathfrak{g} : \exp tX \in H$ for every $t \in R\}$. We assert that $\mathscr{J}$ is a Lie subalgebra of $\mathfrak{g}$. To prove this, we have to verify

(i) $X \in \mathscr{J} \Rightarrow \alpha X \in \mathscr{J}$ for every $\alpha \in R$

(ii) $X, Y \in \mathscr{J} \Rightarrow X + Y \in \mathscr{J}$

(iii) $X, Y \in \mathscr{J} \Rightarrow [X, Y] \in \mathscr{J}$.

(a) is a trivial consequence of the definition.

(b) Let now $X, Y \in \mathscr{J}$. We have seen that (Ch. 4.3)

$$\exp X \exp Y = \exp(X + Y + \frac{1}{2}[X, Y] + \cdots)$$

Hence $(\exp \frac{tX}{n} \exp \frac{tY}{n})^n = (\exp\{t\frac{X+Y}{n} + \frac{1}{2}\frac{t^2}{n^2}[X, Y] + 0(\frac{1}{n^2}\})^n$

$$= \exp\{t(X + Y) + \frac{t^2}{2n}[X, Y] + 0(\frac{1}{n}).\}$$

But $\exp \frac{tX}{n}, \exp \frac{tY}{n} \in H$ and H is a subgroup. Therefore $(\exp \frac{tX}{n} \exp \frac{tY}{n})^n \in H$ and since H is closed, $\lim_{n\to\infty} (\exp \frac{tX}{n} \exp \frac{tY}{n})^n = \exp t(X + Y)$ (by the above formula) $\in H$. Hence $X + Y \in \mathscr{J}$.

(iv) As before,

$$X, Y \in \mathscr{J} \Rightarrow \lim_{n\to\infty} (\exp \frac{tX}{n} \exp \frac{tY}{n} \exp \frac{-tY}{n} \exp \frac{-tY}{n})^{n^2} \in H.$$

The right hand side in this case tends to $\exp t^2[X, Y]$ as $n \to \infty$.

Hence $\exp t[X, Y] \in H$ for positive values of t, and since $\exp(-t [X, Y]) = (\exp t[X, Y])^{-1}$ for all values of t, i.e. $[x, y] \in \mathscr{J}$.

37 Let K be the connected Lie subgroup of G having $\mathscr{J}$ for its Lie algebra (Th. 3, Ch. 4.4). We now show that K is open in H. It is obviously sufficient to prove that K contains a neighbourhood of e in H. If $\mathcal{U}$ is a vector subspace of $\mathfrak{g}$ supplementary to $\mathscr{J}$, by Lemma 1, there exists a neighbourhood V of 0 in $\mathfrak{g}$ such that the map $\lambda : (X, A) \to \exp X \exp A$, $X \in \mathscr{J}$, $A \in \mathcal{U}$ is an isomorphism of V onto $\lambda(V) = W$. Suppose that K does not contain any neighborhood of e in H. Then, we can find a sequence of points $a_n \in H \cap W$ which are not in K and which tend to e. There is no loss of generality in assuming a_n to be of the form $\exp A_n, A_n \in \mathcal{U} \cap V$, $A_n \neq 0$. For, If $a_n = \exp X_n \exp A_n$, then $(\exp X_n)^{-1} a_n = \exp A_n \notin K$. Let V^1 be a compact neighborhood of 0 in $\mathfrak{g} \subset V/2$. For sufficiently large n, $A_n \in V^1$. Let r_n be the largest integer for which $r_n A_n \in V^1$. i.e. $(r_n + 1)A_n \notin V^1$.

But

$$(r_n + 1)A_n = r_n(A_n) + A_n \in V/2 + V/2 = V \cdot a_n^{r_n} \in W^1 = \lambda(V^1)$$

and $a_n^{r_n+1} \notin W^1$ but $\in W$. Since W^1 is compact, we may assume (by taking a suitable subsequence) that $a_n^{r_n}$ converges to an $a \in W^1$. Now, we assert that $a \neq e$. For, if $a = e, a_n^{r_n+1} = a_n{}^{r_n} a_n \to e$. But $a_n^{r_n}$ cannot tend to e. Hence $a \neq e \in W^1$. Therefore $a = \lim a_n r_n = \exp A$ with $A \neq 0$ and $A \in \mathcal{U} \cap V^1$.

We shall now show that $A \in \mathscr{J}$, which will imply that $\mathscr{J} \cap \mathcal{U} \neq (0)$ and hence will give the contradiction we were seeking. It is enough to show that $\exp p/qA \in H$ for every rational number p/q. Now let $pr_n/q = s_n + t_n/q$, s_n an integer and $0 \leq t_n \leq q$.

$$\begin{aligned} \exp \frac{pA}{q} &= \lim_{n\to\infty} \exp(\frac{pr_n}{q} A_n) \\ &= \lim_{n\to\infty} \exp s_n A_n \exp(\frac{t_n A_n}{q}) \end{aligned}$$

Now, $\exp \dfrac{t_n}{q} A_n \to e$ as $n \to \infty$, and $\lim\limits_{n\to\infty} \exp s_n A_n = \lim\limits_{n\to\infty} a_n{}^{s_n} \in H$ **38**
as H is closed. Hence $A \in \mathscr{J}$, and Theorem 4 is completely proved.

Remark. The theorem is not true in the case of complex Lie groups. For instance, the space of real numbers is a closed subgroup of the complex plane, but is not a complex Lie group.

Corollary 1. *Every continuous representation f of the underlying topological group of a Lie group G into that of another Lie group H is an analytic representation.*

In fact, the graph K of f is a closed subgroup of the Lie group GxH, and hence is a Lie subgroup. Then f is the composite of the maps $G \to K$, and $K \to H$, and both of them can be seen to be analytic. As an immediate consequence, we have the following

Corollary 2. *Lie groups with isomorphic underlying topological group structures are analytically isomorphic.*

4.6 Some examples.

We have seen that the general linear group $GL(n, R)$ is a Lie group, and by Theorem 4, every closed subgroup, and in particular, the orthogonal and symmetric groups are Lie groups. A group matrices defined by some polynomial identities in the coefficients of the matrices is a Lie group.

39 We proceed to study $GL(n, R)$ in greater detail. If $x \in GL(n, R)$ is the matrix (a_{ij}), $x_{i,j} = a_{i,j} - \delta_{i,j}$ is a coordinate system which takes the unit matrix to origin in the space R^{n^2}. Now,

$$\varphi_{i,j}(x, y) = x_{i,j} + y_{i,j} + \sum_k x_{i,k} y_{k,j}$$

Setting $u_{i,j} = y_{i,j} + \sum_k x_{i,k} y_{k,j}$, we have

$$\begin{aligned}\tau_y f(x) &= f(x + u)\\ &= f(x) + \sum_{i,j} y_{i,j}\Big(\frac{\partial f}{\partial x_{i,j}} + \sum_k^{x_{k,i}} \frac{\partial f}{\partial x_{k,j}}\Big)\end{aligned}$$

The left invariant differential operators of order 1 are therefore generated by $x_{i,j} = \sum\limits_{k=1}^{n} a_{k.i} \dfrac{\partial}{\partial x_{k,j}}$. The $X_{i,j}$ form a basis of the Lie algebra of $GL(n, R)$.$Y = \sum_{i,j}^{\lambda} X_{i,j}$ is a generic element of the Lie algebra. We associate the matrix $\hat{Y} = (\lambda_{i,j})$ with this element Y. we now have the

Proposition 2. *The map $Y \to \hat{Y}$ of the Lie algebra of $GL(n, R)$ into the algebra of all n-square matrices $\mathscr{M}_n(R)$ is a Lie algebra isomorphism.*

Let Y be an element of the Lie algebra of $GL(n, R)$. We show that the map $t \to \exp tY$ assigns to t the usual exponential matrix $\exp tY$. It has been proved (Ch. 3.3) that $x = \exp tY$ satisfies

$$\frac{\partial x_{i.j}}{\partial t} = \sum_{k,l} \lambda_{k,l} \frac{\partial \varphi_{i,j}}{\partial y_{k,l}}(x(t), e)$$

$$= \sum_{k,l} \lambda_{k.l}\delta_{j,l}(\delta_{i,k} + x_{i,k})$$
$$= \sum_{k} \lambda_{k,j}(\delta_{i,k} + x_{i,k})$$

i.e. x is a matrix satisfying $\frac{d}{dt}x(t) = x(t)\hat{Y}$ with $x(O) = I$. These two conditions can easily be seen to be satisfied by $\exp t\hat{Y}$. By the uniqueness theorem on differential equations, $\exp tY = \exp t\hat{Y}$, where the latter **40**
exponential is in the sense of the exponential matrix. Let $Y, Z \in \mathfrak{g}$ the Lie algebra of $GL(n, R)$. The map $X \to \hat{X}$ is trivially a vector space isomorphism of $\mathfrak{g}$ onto $\mathscr{M}_n(R)$. Now,

$$\exp Y \exp Z = \exp(Y + Z + \frac{1}{2}[Y, Z] + \cdots) \quad \text{by Ch. 4.3.}$$

But
$$\exp Y \exp Z = \Big(\sum \frac{\hat{Y}^n}{n!}\Big)\Big(\sum \frac{\hat{Z}^m}{m!}\Big)$$
$$= 1 + \hat{Y} + \hat{Z} + \frac{\hat{Y}^2}{2!} + \cdots$$

and

$$\exp(Y + Z + \frac{1}{2}[Y, Z] + \cdots) = 1 + \hat{Y} + \hat{Z} + \frac{1}{2}(\hat{Y}^2 + \hat{y}\hat{Z} + \hat{Z}\hat{Y} + \hat{Z}^2) + \cdots$$

Comparing the coefficients, we get

$$\widehat{[Y, Z]} = [\hat{Y}, \hat{Z}].$$

Remarks. (1) $Y = \frac{d}{dt}(\exp tY)_{t=0}$.

(2) The Lie algebra of a closed subgroup H of $GL(n, R)$ is simply the Lie algebra of matrices Y such that $\exp tY \in H$ for every $t \in R$ (by Theorem 4, Ch. 4.5).

(3) Let $B(a, b)$ be a bilinear form on R^n.

Then, the set of all regular matrices x which leave $B(a, b)$ invariant is a Lie subgroup H of $GL(n, R)$. Then the Lie algebra of this Lie Group consists of matrices Y such that $B(Ya, b) + B(a, Yb) = O$ for every $a, b \in R^n$. In fact, if Y is in the Lie algebra, $\exp tY \in H$, B being invariant under H, B $(\exp tYa, \exp tYb) = B(a, b)$. But

$$B(\exp tYa, \exp tYb) = \sum_{p,Q} \frac{t^{p+q}}{p!q!} B(Y^p a, Y^q b).$$

Hence

$$B(Ya, b) + B(a, Yb) = 0.$$

41 Conversely, if Y satisfies this condition, by induction it can be seen that $\sum\limits_{p+q=n} \dfrac{t^{p+q}}{p!q!} B(Y^P a \cdot Y^q b) = 0$, which proves that $B(\exp tYa, \exp tYb) = B(a, b)$, i.e. $\exp tY \in H$ for every $t \in R$. This proves that Y is in the Lie algebra of H.

4.7 Group of automorphisms.

Let G be a connected Lie group. Then the set of automorphisms of G (continuous representations of G onto itself), form a group. We shall denote this group by $\operatorname{Aut} G$.

Let $\alpha \in \operatorname{Aut} G$. This gives rise to a map $d\alpha : \mathfrak{g} \to \mathfrak{g}$ where $d\alpha$ is an automorphism of the Lie algebra. We thus have a map; $\operatorname{Aut} G \to \operatorname{Aut} \mathfrak{g}$. This map is one-one, and, if G is simply connected, onto. We have, in this case, an isomorphism of $\operatorname{Aut} G \to \operatorname{Aut} \mathfrak{g}$, for $d(\alpha_1, \alpha_2) = d\alpha_1 \circ d\alpha_2$ and $d\alpha^{-1} = (d\alpha)^{-1}$. Now, $\operatorname{Aut} \mathfrak{g} \subset GL(\mathfrak{g})$. $\operatorname{Aut} \mathfrak{g}$ is actually a *closed subgroup* of $GL(\mathfrak{g})$. For, if $C^k_{i,j}$ be the structural constants of $\mathfrak{g}$, $A \in \operatorname{Aut} \mathfrak{g} \Leftrightarrow [A(x_i), A(x_j)] = \sum\limits_k C^k_{i,j} A(x_k)$ for every i, j and $A \in GL(\mathfrak{g})$, $\{x_i\}$ being a basis of $\mathfrak{g}$. Since $\operatorname{Aut} \mathfrak{g}$ is determined by these n^2 equations, it is a closed subgroup of $GL(\mathfrak{g})$. Hence, $\operatorname{Aut} \mathfrak{g}$ is a Lie group.

Proposition 3. *Let* Γ *be the Lie algebra of* $\operatorname{Aut} \mathfrak{g}$. *Then* $X \in \operatorname{Hom}(\mathfrak{g}, \mathfrak{g})$ *(which is the Lie algebra of* $GL(\mathfrak{g})$*) is in* Γ *if and only if* $\exp tX \in \operatorname{Aut} \mathfrak{g}$ *for every* t *in* R. *This is obvious from the proof of Theorem 4, Ch. 4.5.*

Proposition 4. $X \in \operatorname{Hom}(\mathfrak{g}, \mathfrak{g})$ *is in* Γ *if and only if* X *is a derivation.*

42 By Proposition 3,

$$(\exp tX)[Y,Z] = [\exp tX \cdot Y, \exp tX \cdot Z]$$

$$\text{i.e.,} \qquad \sum \frac{t^n X^n}{n!}([Y,Z]) = \sum_{p,q} \frac{t^{p+q}}{p!q!}[x^p Y, X^q Z]$$

$$X^n[Y,Z] = \sum_{p+q=n} \frac{n!}{p!q!}[X^p Y, X^q Z]$$

for every n. In particular

$$X[Y,Z] = [XY,Z] + [Y,XZ]$$

X is therefore a derivation. Conversely,

$$X[Y,Z] = [XY,Z] + [Y,XZ] \Rightarrow X^n[Y,Z] = \sum_{p+q=n} \frac{n!}{p!q!}[X^p Y, X^q Z]$$

by induction on $n \Rightarrow \exp tX[Y,Z] = [\exp tX \cdot Y, \exp tX \cdot Z]$.

Hence $X \in \Gamma$.

In other words, the Lie algebra of Aut $\mathfrak{g}$ is only the *Lie algebra of derivations of* $\mathfrak{g}$ (it is a trivial verification to see that the derivations of $\mathfrak{g}$ form a Lie algebra and the set of *inner derivations* (Remark 1, Ch. 2.8) form an ideal in that algebra).

Now, corresponding to every $y \in G$, there exists an inner automorphism $\rho_y : x \to yxy^{-1}$ of G. Obviously $y \to \rho_y$ is an algebraic representation of G in Aut G. ρ_y induces an automorphism $d\rho_y$ of $\mathfrak{g}$. We denote this by ady.

$y \to ady$ is an algebraic representation of G in Aut $\mathfrak{g}$. We now show that this is an analytic representation. By Corollary to Theorem 4, Chapter 4.5, it is enough to show that this is continuous, i.e. if $y \to e$ then $adyX \to X$ for every $X \in \mathfrak{g}$. Since G and $\mathfrak{g}$ are locally isomorphic, it suffices to prove that as $y \to e$, $y \exp Xy^{-1} \to \exp X$ but this is obvious. 43
This analytic representation of G in Aut G = Aut $\mathfrak{g}$ is called the *adjoint representation* of G. Let θ be the differential of the representation $y \to ady$. We now show that this is actually the *adjoint representation* (Ch. 2.8) of the Lie algebra $\mathfrak{g}$.

Theorem 5. *$\theta(X) = ad\ X$ for every $X \in \mathfrak{g}$.*

By Remark 1, Prop. 2, Ch. 4.6,

$$\begin{aligned}\theta(X) &= \{\frac{d}{dt}(\exp\ t\,\theta(X))\}_{t=0}\\ &= \frac{d}{dt}\,(ad\ \exp\ t\ x)_{t=0}\end{aligned}$$

by definition of exponential. We have now to show that $\theta(X)Y = [X, Y]$ for every $Y \in \mathfrak{g}$. Let $x = \exp tX \cdot \theta(X)Y = \frac{d}{dt}(ad\ Y)_{t=0}$. But $(adx\ Y\ f) \circ \rho_x = Y(f \circ \rho_x)$ where f is any analytic function on G. It follows that

$$\begin{aligned}\sigma_{x^{-1}} \circ \tau_{x^{-1}}\ adx\ Yf &= Y\sigma_{x^{-1}} \circ \tau_{x^{-1}} \circ f\\ or \qquad (adx\ Y) \circ \tau_x &= \tau_x \circ \sigma_x Y \sigma_{x^{-1}}\\ &= \tau_x Y\end{aligned}$$

since Y is left invariant

$$adx\ Y = \tau_x Y \tau_{x^{-1}}.$$

But

$$\tau_x\ f(e) \ = \ f(\exp tx) = \sum_n \frac{t^n X^n}{n!} f(e) \quad \text{(Prop. 6, Ch. 3.4)}$$

$$\begin{aligned}\text{Hence} \quad adx\ Y &= \sum_{m,n} \frac{t^m X^m}{m!} \frac{(-t)^n X^n}{n!}\\ &= \sum_{m,n} (-1)^n \frac{t^{m+n}}{m!n!} X^m Y X^n\end{aligned}$$

Therefore

$$\theta(X)Y = \{\frac{d}{dt}(adx\ Y)\}_{t=0} = XY - YX = [X, Y].$$

Corollary. *Let H be a connected Lie subgroup of G. H is a normal subgroup if and only if its Lie algebra $\mathscr{J}$ is an ideal in $\mathfrak{g}$.*

44 In fact, if H is a normal subgroup, $\rho_y(H) \subset H$ for every $y \in G$, i.e. $ady \mathscr{J} \subset \mathscr{J}$ for every $y \in G$. Let $X \in \mathfrak{g}$. Then ad $\exp tX \mathscr{J} \subset \mathscr{J}$.

$$\frac{d}{dt}(ad\ \exp tX)_{t=0}\mathscr{J} = ad\ X \mathscr{J} \subset \mathscr{J}$$

i.e. $\mathscr{J}$ is an ideal.

Reciprocally, let $\mathscr{J}$ be an ideal. ad $X\mathscr{J} \subset \mathscr{J} \cdot (\sum \dfrac{t^n adX^n}{n!})\mathscr{J} \subset \mathscr{J}$. But $\exp t\ ad\ X = ad\ \exp tX \cdot (ad \exp tX)H \subset H$ for every X in a neighborhood of e. Since G is connected, H is normal.

4.8 Factor groups.

Theorem 6. *Let H be a closed subgroup of a Lie group G. The homogeneous space G/H is an analytic manifold in a canonical way. The operations by G on G/H are isomorphisms. If H is a normal subgroup, is a Lie group, and its Lie algebra is isomorphic to $\mathfrak{g}/\mathscr{J}$.*

Let $\mathscr{J}$ be the Lie algebra of H, and let $\mathcal{U}$ be a vector subspace of $\mathfrak{g}$ supplementary to $\mathscr{J}$. We have seen in the proof of Theorem 4, Ch. 4.5, that there exists a neighborhood V^1 of $(0,0)$ in $\mathscr{J} \times \mathcal{U}$ such that the map $\lambda : (X, A) \to \exp X \exp A$ is an isomorphism of V^1 onto a neighborhood W^1 of e in G. Let U and V be neighborhoods of 0 in $\mathscr{J}$ and $\mathcal{U}$ respectively such that $U \times V \subset V^1$ and $WW^{-1} \subset W^1$ with $W = \lambda(U \times V)$. We now show that $L = \exp V$ is a cross-section of the canonical map $\eta : G \to G/H$ in the neighborhood $\dot{W} = \eta(W)$ of $\eta(e)$. In other words, $L \cap Hx$ contains one and only one element for every $x \in W$. For, we have $x = \exp X \exp A$ with $X \in U$ and $A \in V$ and $\exp A \in L \cap Hx$. On the other hand, if $\exp A_1$ and $\exp A_2$ belong to Hx
(with $A_1, A_2 \in V$), then $\exp A_1 (\exp A_2)^{-1} \in H \cap W^1$; hence there exists 45
an $X \in V^1 \cap \mathscr{J}$ such that $\exp A_1 = \exp X \exp A_2$ and this implies $X = 0$, $A_1 = A_2$ because λ is an isomorphism from V^1 onto W^1.

We can, therefore, provide $\dot{W}$ with a manifold structure induced from that of $\mathcal{U}$. This can be extended globally by translating that on W. It is easily seen that on the overlaps $\dot{x}\dot{W}$, $\dot{y}\dot{W}$, the analytic structures agree because the analytic structure on $\dot{W}$ is induced from that of $\mathcal{U}$. By the definition of the manifold G/H, it is obvious that the operations by G on G/H are analytic isomorphisms.

If H is normal subgroup, G/H has also a group structure and is a Lie group with the above manifold. By Theorem 6, Ch. 4.8, $\mathscr{J}$ ia an ideal of $\mathscr{J}$ and if $\mathcal{K}$ is the Lie algebra of G/H, the map $\eta :\to G/H$ gives rise to a representation $d\eta : \mathscr{J} \to \mathcal{K}$. The kernal of this map is $\mathscr{J}$ since $\mathcal{K}$ is isomorphic as a vector space to the tangent space at e of L which is $\mathcal{U}$. Hence $\mathscr{H}/\mathscr{J}$ is isomorphic to $\mathcal{K}$ as a Lie algebra also, i.e. G/H has its Lie algebra isomorphic to $\mathfrak{g}/\mathscr{J}$.

Corollary. *Let f be a representation of a Lie group G which is countable at ∞ in another Lie group H. Then the image $f(G)$ is a Lie subgroup. If N is the kernel of f, then f can be factored into $G \xrightarrow{\pi} G/N \xrightarrow{\bar{f}} H$ where π is the canonical map and $\bar{f}$ an injective regular map.*

The proof is an immediate consequence of the isomorphism theorem on Lie algebras and Theorem 6.

Part II

General Theory of Representations

Chapter 5

Measures on locally compact spaces

1.1 Definition of a measure.

In this chapter and the following, we shall give a brief summary of certain results on measure theory, a knowledge of which is essential in what follows. 46

Let X be a locally compact topological space and $\mathscr{C}_X$ the algebra of continuous complex-valued functions on X with compact support. Let K be a compact subset of X and $\mathscr{C}_K$ the subset $\{f\colon \text{support of } f \subset K\}$ of $\mathscr{C}_X$. Then $\mathscr{C}_K$ is a Banach space under the norm $\|f\| = \sup\limits_{x\in K} |f(x)|$.

Definition. *A* measure *on X is a linear form on $\mathscr{C}_X$ such that the restriction to $\mathscr{C}_K$ is continuous for every compact subset K of X. A measure μ is said to be* positive *if $\mu(f) \geq 0$ for every $f \geq 0$.*

Proposition 1. *Every positive linear form on $\mathscr{C}_X$ is a measure on X.*

In fact, if K is any compact subset of X, there exists a continuous function f on X which $= 1$ in K, $= 0$ outside a compact neighbourhood of K and $0 \leq f \leq 1$. If g is a function belonging to $\mathscr{C}_K$, obviously $-\|g\| f \leq g \leq \|g\| f$ and hence

$$-\|g\|\mu(f) \leq \mu(g) \leq \|g\|\mu(f).$$

i.e.,

$$|\mu(g)| \leq \|g\|\mu(f),$$

which shows that μ is continuous when restricted to $\mathscr{C}_K$.

On the other hand, if μ is any real measure (i.e. a measure μ such that $\mu(f)$ is real whenever f is real) it can be expressed as the difference of
47 two positive measures. Moreover, a complex measure μ can be uniquely decomposed into $\nu + i\nu'$ where ν and ν' are real. Hence a measure can alternatively be defined as a linear combination of positive linear forms on $\mathscr{C}_X$. For a real measure ν, there exists a unique 'minimal' decomposition $\mu_1 - \mu_2$ (μ_1, μ_2 positive) in the sense that if $\nu = \mu'_1 - \mu'_2$ be any other decomposition, we have $\mu'_1 = \mu_1 + \pi$, $\mu'_2 = \mu_2 + \pi$ with π positive.

1.2 Topology on $\mathscr{C}_X$.

The space $\mathscr{C}_X = \bigcup\limits_K \mathscr{C}_K$ where K runs through all the compact subsets of X can be provided with the topology obtained by taking the direct limit of the topologies on $\mathscr{C}_K$. This topology makes of $\mathscr{C}_X$ a locally convex topological vector space. The fundamental property of this space is that a linear map of $\mathscr{C}_X$ in a locally convex space is continuous if and only if its restriction to each $\mathscr{C}_K$ is continuous. (Bourbaki, Espaces vectoriels topologiques, Chapter 6). A measure is, by definition, a continuous linear form on $\mathscr{C}_X$ with its topology of direct limit. The space $\mathscr{M}_X$ of measures is none other than the dual of $\mathscr{C}_X$. One can provide $\mathscr{M}_X$ with several topologies, as for instance, the weak topology in which $\mu \to 0 \Leftrightarrow$ for every $f \in \mathscr{C}_X$, $\mu(f) \to 0$.

1.3 Support of a measure.

Definition. *The* support *of a measure μ is the smallest closed set S such that for every function $f \in \mathscr{C}_X$ whose support is contained in $X - S$, $\mu(f) = 0$.*

Let $\mathscr{M}^c$ be the space of measures with compact support. If f is a continuous function on X, for every $\mu \in \mathscr{M}^c$, we can define $\mu(f) =$
48 $\mu(\alpha f)$ where α is a function 1 on a neighbourhood of the support K of μ, and 0 outside a compact neighbourhood of K. It is obvious that the

value of $\mu(f)$ does not depend on α. We shall denote by $\mathscr{E}^{\circ}_X$ the space of continuous function on X. $\mathscr{E}^{\circ}_X$ with the topology of compact convergence is a locally convex topological vector space. μ defined on $\mathscr{E}^{\circ}_X$ in the above manner is continuous with respect to this topology. Conversely, let μ be a continuous form on $\mathscr{E}^{\circ}_X$. The topology on $\mathscr{C}_X$ is finer than that induced from the topology of $\mathscr{E}^{\circ}_X$. Hence μ restricted to $\mathscr{C}_X$ is again continuous and is consequently a measure. We now show that this has compact support. Since μ is a continuous function on $\mathscr{E}^{\circ}_X$, we can find a neighbourhood V of 0 such that $|\mu(f)| < 1$ for every $f \in V$. V may be taken to be of the form $\{f : |f| < \epsilon \text{ on } K\}$ as the topology on $\mathscr{E}^{\circ}_X$ is the topology of compact convergence. Let $g \in \mathscr{C}_X$ be a function 0 on K. Then $|\mu(g)| < 1$. If λ is any complex number, $\mu(\lambda g) = \lambda\mu(g)$, and $|\mu(\lambda g)| < 1$. Hence $\mu(g) = 0$, i.e. the support of μ is contained in K.

1.4 Bounded measures.

Let μ ba a measure $\in \mathscr{M}_X$. We define a positive measure $|\mu|$ in the following way:

$|\mu| f = \sup\limits_{0 \leq |g| \leq f} |\mu(g)|$ for every positive function f and extend it by linearity to all functions $\in \mathscr{C}_X$. If μ is a real measure with the minimal decomposition (Ch. 1.1) $\mu = \mu_1 - \mu_2$, then $|\mu| = \mu_1 + \mu_2$.

Definition . *A measure μ is* bounded *if and only if there exists a real number k such that $|\mu(f)| \leq k\|f\|$ with $\|f\| = \sup\limits_{x \in X} |f(x)|$.*

Obviously μ is bounded if and only if $|\mu|$ is bounded, μ is continuous 49
for this norm and can be extended to the completion $\overline{\mathscr{C}}_X$ (which is only the space of continuous functions tending to zero at ∞). $\overline{\mathscr{C}}_X$ is actually the adherence of $\mathscr{C}_X$ in the space of all continuous bounded functions. The space of bounded measures is a Banach space under the norm $\|\mu\| = \sup\limits_{f \in \mathscr{C}_X} \dfrac{|\mu(f)|}{\|f\|}$. $\|\mu\|$ is the smallest number k such that $|\mu(f)| \leq k\|f\|$. It can proved that every bounded continuous function is integrable with respect to a bounded measure, and we have still the inequality $|\mu(f)| \leq \|\mu\|\,\|f\|$ for bounded continuous functions f.

1.5 Integration of vector valued functions.

We introduce here the notion of integration of a vector valued function with respect to a scalar measure, a use of which we will have frequent occasions to resort to in the sequel. Let K be a compact space and E a locally convex quasi-complete topological vector space (i.e. every closed bounded subset is complete). We shall provide the space $\mathscr{C}(K,E)$ of continuous functions of K into E with the topology of uniform convergence.

Theorem 1. *Corresponding to every measure μ on K, there exists one and only one continuous linear map $\tilde{\mu}$ of $\mathscr{C}(K,E)$ in E such that $\tilde{\mu}(f\cdot a) = \mu(f)\cdot a$ for every continuous complex valued function f on K and $a \in E$.*

μ can obviously be lifted to a linear map $\bar{\mu}$ of $\mathscr{C} \otimes E$ in E by setting $\bar{\mu}(c \otimes e) = \mu(c)e$ and extending by linearity. Also, $\mathscr{C} \otimes E$ can be identified
50 with a subset of the space $\mathscr{C}(K,E)$ of continuous functions of K into E. We will now show that $\bar{\mu}$ is continuous with respect to the induced topology on $\mathscr{C} \otimes E$ and that $\mathscr{C} \otimes E$ is dense in $\mathscr{C}(K,E)$. We will in fact prove more generally that every function $f \in \mathscr{C}(K,E)$ is adherent to a bounded subset of $\mathscr{C} \otimes E$.

Let V be a convex neighbourhood of 0 in E. Then there exists a neighbourhood A_x of each point $x \in K$ such that $f(y) - f(x) \in V$ for every $y \in A_x$. Now the A_x cover the compact space K and let $A_{x_1}, \ldots, A_{x_n}$ be a finite cover extracted from it. Let φ_i be positive continuous functions on K such that $\sum\limits_{i=1}^{n} \varphi_i = 1$ and the support of $\varphi \subset A_{x_i}$. If $g = \sum \varphi_i f(x_i)$, then $g \in \mathscr{C} \otimes E$, and we have

$$\begin{aligned} g(y) - f(y) &= \sum \varphi_i(y) f(x_i) - \sum \varphi_i(y) f(y) \\ &= \sum \varphi_i(y)[f(x_i) - f(y)] \\ &\in V \quad \text{since } V \text{ is convex.} \end{aligned}$$

By allowing V to describe fundamental system of neighbourhoods of 0, we see that f is adherent to the set of such functions g. This set is a bounded subset of $\mathscr{C}(K,E)$. For, $\sum \varphi_i(y) f(x_i)$ is in the convex envelope of $f(K)$ for every $y \in K$ and the convex envelope of a compact set is bounded. It follows that $\sum \varphi_i f(x_i)$ are uniformly bounded and hence form a bounded subset of $\mathscr{C}(K, E)$. This proves, in particular,that $\mathscr{C} \otimes E$ is dense in $\mathscr{C}(K, E)$.

Now let $g = \sum g_i a_i$ be a function $\in \mathscr{C} \otimes E$ tending to zero in the topology of $\mathscr{C}(K, E)$. Then $\langle \sum g_i a_i, a' \rangle \to 0$ uniformly on the compact set K and on any equicontinuous subset H of the dual of E.

$$\begin{aligned}
\mu\langle g, a' \rangle &= \mu\langle \sum g_i a_i, a' \rangle \\
&= \mu(\sum g_i \langle a_i, a' \rangle) = \sum \mu(g_i)\langle a_i, a' \rangle \\
&= \langle \sum \mu(g_i) a_i, a' \rangle = \langle \overline{\mu} g, a' \rangle
\end{aligned}$$

51

Since $\langle g, a' \rangle \to 0$ uniformly on $K \times H$, $\mu\langle g, a' \rangle \to 0$ and hence $\langle \overline{\mu}_g, a' \rangle \to 0$ uniformly on any equicontinuous subset H of E. Consequently $\overline{\mu} g \to 0$. This shows that $\overline{\mu}$ is continuous on $\mathscr{C} \otimes E$.

Therefore $\overline{\mu}$ can be extended uniquely to a continuous linear map of $\mathscr{C}(K, E)$ in the completion $\hat{E}$ of E. But if $f \in \mathscr{C}(K, E)$, it is adherent to a bounded set B and $\mu(B)$ is also bounded in E. By the quasi-completeness of E, the closure of $\mu(B)$ in $\hat{E}$ and E are the same. Hence μ(f)$\in \mu\ (\overline{B})$ $\subset \overline{\mu(B)} \subset E$. Thus we have extended μ to a continuous linear map $\tilde{\mu}$ of $\mathscr{C}(K, E) \to E$ and it is obvious this is unique. Now by Theorem 1, if G be any locally compact space and μ a measure on G, we can define $\int f(x) d\mu = \tilde{\mu}(f)$ for every continuous function f from G to E with compact support.

Remark. The measure with this extended meaning is factorial in character in the following sense: Let E and F be two locally convex spaces and f a continuous map of a compact space K into E. If A is a continuous map of E in F, we have $Af \in \mathscr{C}(K, F)$ and μ satisfies $\mu(Af) = A\mu(f)$.

Chapter 6

Convolution of measures

2.1 Image of a measure

Definition. *Let X, Y be two locally compact topological spaces and π a map $X \to Y$. Let μ be a positive measure on X. Then Π is said to be $\mu-$* proper *if for every function $f \in \mathscr{C}_Y$, $f \circ \pi$ is integrable with respect to μ. The value $\mu(f \circ \pi)$ depends linearly on f and is therefore a linear form on $\mathscr{C}_Y$. In other words, $\mu(f \circ \pi)$ defines a positive measure on Y, which we denote by $\pi(\mu)$. We have, by definition, $\int_Y f(y) d\pi(\mu)(y) = \int_X f \circ \pi(x) d\mu(x)$.* 52

If μ is not positive, but is equal to $(\mu_1 - \mu_2) + i(\mu_3 - \mu_4)$, μ_1, μ_2, μ_3, μ_4 positive, and if π is $|\mu|$ - proper, we can define the image measure

$$\pi(\mu) = \pi(\mu_1) - \pi(\mu_2) + i\pi(\mu_3) - i\pi(\mu_4).$$

Examples.

(1) A continuous *proper* map of $X \to Y$ (i.e. a map such that inverse image of every compact set is compact) is μ - proper for every μ.

In fact, $f \in \mathscr{C}_K \to f \circ \pi \in \mathscr{C}_{\pi-1(K)}$ and $\pi^{-1}(K)$ is compact.

(2) Let π be a continuous map $G \to H$ and let μ have compact support K. Then π is μ-proper; the support of $\pi(\mu) \subset \pi(K)$ and is hence compact.

If f is a continuous function on H with compact support K, $f \circ \pi$ is continuous and hence μ-integrable (Ch. 1.3). This shows that π is μ-proper and if $f = 0$ on $\pi(K)$, then $f \circ \pi = 0$ on K and therefore $\mu(f \circ \pi) = 0$, i.e. support of $\pi(\mu) \subset \pi(K)$.

53 (3) More generally, when π is continuous and μ bounded, π is μ-proper. Also $\pi(\mu)$ is bounded and $||\pi(\mu)|| \leq ||\mu||$.

In fact, $f \circ \pi$ is bounded and in view of the remark in Ch. 1.4, $f \circ \pi$ is integrable with respect to μ. Moreover,

$$||\pi(\mu)|| = \sup_{g \in \mathscr{C}_Y} \frac{|\pi\mu(g)|}{||g||} = \sup_{g \in \mathscr{C}_Y} \frac{|\mu(g \circ \pi)|}{||g \circ \pi||} \leq ||\mu||$$

2.2 Convolution of two measures.

Let G, H be two locally compact topological spaces and μ, ν measures on G, H respectively. Then there exists one and only measure λ on $G \times H$ such that if f, g be functions with compact support respectively on G, H we have

$$\int f(x)g(y)d\lambda(x,y) = (\int f(x)d\mu(x))(\int g(y)d\nu(y)).$$

λ shall be called the product measure of μ and ν.

If μ and ν are two measures on a locally compact group G, we denote the product measure by $\mu \otimes \nu$ and, if the group operation $\pi : G \times G \to G$ defined by $(x, y) \to xy$ is $\mu \otimes \nu$ - proper, its image in G by $\mu * \nu$. The latter is said to be the *convolution product* of μ and ν. The most general class of measures for which convolution product can be defined are those for which $f(xy)$ is integrable with respect to the product measure for every function $f \in \mathscr{C}_G$. The following cases are the particular interest to us:

(1) If μ and ν are bounded, the convolution product exists and is bounded.

This is almost obvious, π being continuous and $\mu \otimes \nu$ bounded (Example 3, Ch. 2.1).

54 (2) If μ and ν are measures on G with compacts K, K' respectively, $\mu * \nu$ exists and has compact support. In fact, $\mu \otimes \nu$ has support $\subset K \times K'$. Hence, convolution product exists, and has support $\subset KK'$.

(3) If either μ or ν has compact support, $\mu * \nu$ exists. Let f be a continuous function on G with compact support K and let μ have compact support K'. Obviously

$$\iint f(xy)d\mu(x)d\nu(y) = \int_{K'} d\mu(x) \int_{KK'^{-1}} f(xy)d\nu(y)$$

Hence $f(xy)$ is integrable with respect to $\mu \otimes \nu$. Consequently, $\mu * \nu$ exists.

We denote as usual by $\mathscr{M}^1$, $\mathscr{M}^c$, $\mathscr{E}^0_G$ the spaces of bounded measures, the space of measures with compact support and the space of all continuous functions on G respectively. Let λ, μ, ν be three measures on G such that either all three are bounded or two of them have compact support. In any case the function $(x, y, z) \to f(xyz)$ is integrable with respect to $\lambda \otimes \mu \otimes \nu$ and hence Fubini's theorem can be applied.

$$\begin{aligned}
\iiint f(xyz)d\lambda(x)d\mu(y)d\nu(z) &= \int d\nu(z) \int \int f(xyz)d\lambda(x)d\mu(y) \\
&= \int d\nu(z) \int f(tz)d(\lambda * \mu)(t) \\
&= \iint f(tz)d(\lambda * \mu)(t)d\nu(z) \\
&= \{(\lambda * \mu) * \nu\} f \\
&= \{\lambda * (\mu * \nu)\} f
\end{aligned}$$

by a similar computation. This shows that m^1 with the convolution product is an associative algebra and that $\mathscr{M}^c$ acts on $\mathscr{M}$ on both sides and 55
makes it a two-sided module. Moreover, $\mathscr{M}^1$ is actually a Banach algebra under the usual norm, since we have $||\mu * \nu|| \leq ||\mu||\,||\nu||$.

Remarks. (1) It is good to point out here that the associativity does not hold in general. Take,for instance, R to be the locally compact group and λ the Lebesgue measure. Let μ be $\epsilon_1 - \epsilon_0$ (ϵ_a begin the Dirac measure at a - see Ch. 2.5 and $\nu = \varphi(x)dx$ where φ is the Heaviside function viz. $\varphi = 0$ for $x < 0$ and $=1$ for $x \geq 1$. Then $(\lambda * \mu) * \nu = 0$ and $\lambda * (\mu * \nu) = dx$. Again $\lambda * (\mu * \nu)$ may exist without $\lambda * \mu$ begin well defined. Let R be the locally compact group, λ and μ Lebesgue measures on R and $\nu = \epsilon_1 - \epsilon_0$. Then $\lambda * (\mu * \nu) = 0$ but $\lambda * \mu$ is not defined. However, when $f(xyz)$ is integrable with respect to $\lambda \otimes \mu \otimes \nu$ then the convolution product is associative.

(2) The formula for the integration of functions with respect to the convolution of two measures is valid also for vector-valued function. Thus we have

$$\int_G f(x)d(\mu * \nu) = \iint f(xy)d\mu(x)d\nu(y).$$

2.3 Continuity of the convolution product.

That the convolution product is continuous in $\mathscr{M}^1$ is trivial in virtue of our remark that it is a Banach algebra. Regarding the continuity of the convolution product in the other cases, we have the following

Lemma 1. *Let f be a continuous function and μ a measure on G, one of them having compact support. Then* (i) *the function $g(x) = \int f(xy)d\mu(y)$*
56 *is continuous;* (ii) *the map $f \to g$ is a continuous linear map of $\mathscr{C}_G$ in $\mathscr{E}^0_G$ (with the usual topologies);* (iii) *if μ has compact support, then the above map $f \to \int f(xy)d\mu(y)$ is also continuous from $\mathscr{E}^\circ_G \to \mathscr{E}^\circ_G$ and from $\mathscr{C}_G \to \mathscr{C}_G$.*

(1) Let H and K be two compact subsets of G. Then $H \times K$ is also compact and $f(xy)$ is uniformly continuous on $H \times K$. For every $\epsilon > 0$, and for every $x \in H$, there exists a neighbourhood U of x such that $|f(x'y) - f(xy)| < \epsilon$ for every $x' \in U \cap H$ and $y \in K$. If f has compact support S, we choose a compact neighbourhood H of x and K such that $HS^{-1} \subset K$. If $y \notin K$, then $xy, x'y \notin S$.

Hence

$$\int_{G-K} \left\{ f(x'y) - f(xy) \right\} d\mu(y) = 0.$$

So, we have

$$|g(x') - g(x)| \leq \int_K \left\{ f(x'y) - f(xy) \right\} d|\mu|(y) \leqslant \epsilon|\mu|(K).$$

This shows that g is continuous. If however μ has compact support C, we take $K = C$ and the same inequality as above results.

(2) Again, as in (i) if we assume that f has compact support S ans $HS^{-1} \subset K$, it is immediate that

$$\begin{aligned} |g(x)| &\leq \int_K |f(xy)| d|\mu|(y) \text{ for every } x \in H \\ &\leq \int_K |f(y)| d|\mu|(x^{-1}y) \\ &\leq \sup |f||\mu|(H^{-1}K). \end{aligned}$$

Hence we have $\sup_{x \in H} |g(x)| \leq \sup |f||\mu|(H^{-1}K)$.

It follows that whenever $f \to 0$ on $\mathscr{C}_S$, $g(x) \to 0$ uniformly on the compact set H. A similar proof holds when μ has compact support.

(3) Let now C be the support of μ, and f has compact port K; obviously $g \in \mathscr{C}_{KC} - 1$. Since the map $\mathscr{C}_G \to \mathscr{E}_G^\circ$ is continuous, so also is the map $\mathscr{C}_K \to \mathscr{C}_{KC^{-1}}$ and by the properly of the direct limit topology, $\mathscr{C}_G \to \mathscr{C}_G$ is continuous. An analogous proof holds for the other part. 57

2.4 Duality and convolution products

Let E be a locally convex topological vector space an E' its dual. Then E' can be provided with several interesting topologies (Bourbaki, Espaces vectoriels topologiques, Chapter 8). The following three are of fundamental importance:

(i) The *weak topology*, in which $x' \in E' \to 0$ if and only if $\langle x', x\rangle \to 0$ for every $x \in E$.

(ii) The *convex compact topology*, in which $x' \in E' \to 0$ if and only if $\langle x', x\rangle \to 0$ uniformly on every convex compact subset, and

(iii) The *strong topology*, where $x' \in E' \to 0$ if and only if $\langle x', x\rangle \to 0$ uniformly on every bounded set.

In general, these topologies are distinct. If E is a Banach space, the usual dual is the E' with the strong topology. However, the convex compact topology is often the most useful, in as much as it shares the 'good' properties of both the weak and the strong topologies. To mention but one such, $(E')' = E$ is true for the weak, but not for the strong, topology. The convex compact topology possesses this property. We shall almost always restrict ourselves to the consideration of this topology.

58 In particular, the spaces $\mathscr{M}$, $\mathscr{M}^c$ being duals of $\mathscr{C}_G$ and $\mathscr{E}_G^\circ$ respectively, they can be provided with the convex compact topology. With reference to the convolution map we have the

Proposition 1. *The convolution map* $(\mu, \nu) \to \mu * \nu$ *is continuous in each variable separately in the following situations:*

$$\mathscr{M}^c \times \mathscr{M}^c \to \mathscr{M}^c; \mathscr{M}^c \times \mathscr{M} \to \mathscr{M}; \mathscr{M} \times \mathscr{M}^c \to \mathscr{M}.$$

In fact, let μ be fixed in $\mathscr{M}^c$ and $\nu \to 0$ in $\mathscr{M}$. Then

$$\begin{aligned} \mu * \nu(f) &= \iint f(xy) d\mu(x) d\nu(y) \\ &= \int d\nu(y) \int f(xy) d\mu(x). \end{aligned}$$

Denoting by $f_0(y)$ the function $\int f(xy)d\mu(x)$ the map $f \to f_0$ is continuous from $\mathscr{C}_G \to \mathscr{C}_G$ (Lemma 1, Ch. 2.3). The image of a convex compact subset being again a convex compact subset, $\mu * \nu \to 0$ uniformly on a convex compact subset. All other assertions in the proposition can be demonstrated in an exactly similar manner.

2.5 Convolution with the Dirac measure

If x is a point of G, the Dirac measure ϵ_x is defined by $\epsilon_x(f) = f(x)$. This is trivially a measure with compact support. Let ν be any arbitrary measure. Then

$$\begin{aligned} \epsilon_x * \nu(f) &= \iint f(yz) d\epsilon_x(y) d\nu(z) \\ &= \int f(xz) d\nu(z) \\ &= \nu(\sigma_x - 1 f). \end{aligned}$$

In a similar manner, $\nu * \epsilon_x(f) = \nu(\tau_x f)$. We may define left and right translation of a measure by setting $d(\tau_x \nu)(y) = d\nu(yx)$ and $d(\sigma_x \nu)(y) = d\nu(x^{-1}y)$. It requires a trivial verification to establish that $\tau_x \nu(f) =$ 59
$\nu(\tau_{x^{-1}} f)$ and $\sigma_x \nu(f) = \nu(\sigma_{x^{-1}} f)$. Hence we have

$$\epsilon_x * \nu = \sigma_x \nu, \quad \text{and} \quad \nu * \epsilon_x = \tau_x^{-1} \nu.$$

In particular, $\epsilon_x * \epsilon_y = \sigma_x \epsilon_y = \epsilon_{xy}$. In other words, the map $x \to \epsilon_x$ is a representation in the algebraic sense of the group G into the algebra $\mathscr{M}^c$ or $\mathscr{M}^1$. As a matter of fact, this can be proved to be a topological isomorphism (Bourbaki, Intération, Chapter 7).

Chapter 7

Invariant measures

3.1 Modular function on a group. 60

We assume the fundamental theorem relating to measures on locally compact groups, namely the existence and uniqueness (upto a positive constant factor) of a right invariant positive measure. If μ is such a measure, we have

$$(\epsilon_y * \mu) * \epsilon_x = \epsilon_y * (\mu * \epsilon_x) = \epsilon_y * \mu.$$

Hence $\epsilon_y * \mu$ is also a right invariant positive measure. By our remark above, $\epsilon_y * \mu = k\mu$ where, of course, k depends on y. We shall denote k by $\Delta(y)^{-1}$ where $\Delta(y)$ is a positive real number. It is immediate that $\Delta(yz) = \Delta(y)\Delta(z)$. Δ is therefore a representation of G in the multiplicative group of R^+. In fact, the continuity of $\Delta(y) = \dfrac{\int f(y^{-1}x)d\mu(x)}{\int f(x)d\mu(x)}$ follows at once from that of $\int f(y^{-1}x)d\mu(x)$ (Lemma 1, Ch. 2.3). This representation Δ of a locally compact group is said to be its *modular function*.

Proposition 1. *If a right invariant positive measure on G is denoted by* dx*, then the following identity holds:* $dx^{-1} = \Delta(x^{-1})dx$

In fact, if $d\mu$ stands for $\Delta(x^{-1})dx$, we have

$$d\mu(yx) = \Delta(x^{-1}y^{-1})d(yx) = \Delta(x^{-1})dx = d\mu.$$

Hence $d\mu$ is left invariant. So also is dx^{-1} for,

$$\int f(yx)dx^{-1} = \int f(x)d(y^{-1}x)^{-1} = \int f(x)d(x^{-1}).$$

So $kdx^{-1} = \Delta(x^{-1})dx$ where k is a constant. We now prove that $k = 1$. When x is near e, $\Delta(x^{-1})$ is arbitrarily near 1 and if we take $g(x) =$
61 $f(x) + f(x^{-1})$, f being a positive continuous function with sufficiently small support, we have

$$\int g(x)dx = \int g(x)dx^{-1} = \frac{1}{k}\int g(x)\Delta(x^{-1})dx$$

k being a fixed number and $\Delta(x^{-1})$ arbitrarily near 1; it follows that $k = 1$.

Definition. *A locally compact group G is said to be* unimodular *if its modular function is a trivial map which maps G onto the unit element of R.*

The group of triangular matrices of the type $\begin{pmatrix} a_{11}\ldots 0 \\ \ldots\ldots \\ \ldots a_{nn} \end{pmatrix}$ can be proved to be non-unimodular.

Examples of unimodular groups.

(1) A trivial example of unimodular groups is that of commutative groups.

(2) Compact groups are unimodular. This is due to the fact that $\Delta(G)$ is a compact subgroup of R^+ which cannot but be (1).

(3) If in a group the commutator subgroup is everywhere dense, then the group is unimodular. This again is trivial as Δ maps the commutator subgroup and consequently the whole group onto 1.

(4) A connected semi-simple Lie group is unimodular. (A Lie group G is said to be *semi-simple* if its Lie algebra $\mathfrak{g}$ has no proper abelian ideals. Consequently, it dose not have proper ideals such that the quotient is abelian). The kernel $\mathscr{N}$ of the representation

$d\Delta$ of the Lie algebra into the real number is an ideal in $\mathfrak{g}$ such that $\mathfrak{g}/\mathscr{N}$ is abelian and is therefore the whole Lie algebra. It follows that the map $d\Delta$ maps the Lie algebra onto (0). This shows that group is unimodular.

3.2 Haar measure on a Lie group.

62

Let G be a Lie group with a coordinate system $(x_1, \ldots, x_n)$ in a neighbourhood of e. We now investigate the form of the right invariant Haar measure on the Lie group. By invariance of a measure μ here we mean that $\int f(xy^{-1})d\mu(x) = \int f(x)d\mu(x)$ for y which are sufficiently near e and for f whose supports are sufficiently small. We set $d\mu(x) = \lambda(x)dx_1 \wedge \ldots \wedge dx_n$ and enquire if integration with respect to this measure is invariant under right translations. For invariance, we require

$$\int f(x')\lambda(x'y)\left|\det \frac{\partial \varphi_i(x', y)}{\partial x'_j}\right| x'_1 \wedge \ldots \wedge dx'_n$$

$$= \int f(x)\lambda(x)dx_1 \wedge \ldots \wedge dx_n,$$

or still $\quad \lambda(x) = \lambda(xy)J(x, y)$ with $J(x, y) = \left|\det \dfrac{\partial \varphi_i}{\partial x_j}(x, y)\right|.$

For this it is obviously necessary and sufficient to take $\lambda(x) = J^{-1}(e, x)$. This gives an explicit construction of the Haar measure in the case of Lie groups.

3.3 Measure on homogeneous spaces.

If G/H is the quotient homogeneous space of a locally compact group G by a closed subgroup H, we denote the elements of G by letters $x, y, \ldots$ those of H by $\xi, \eta, \ldots$ the respective Haar measure by $dx, dy, \ldots d\xi$, $d\eta \ldots$ and the respective modular functions by Δ, δ. Also π is the canonical map $G \to G/H$. Let f be a continuous function on G with compact support K. Then $f^o(x) = \int_H f(\xi x)d\xi$ is a continuous function on G (as in lemma 1, Ch. 2.3) and we have $f^o(\zeta x) = f^o(x)$ for every $\xi \in H$. Therefore f^o may be considered as a continuous function on G/H. Obviously it has support $\pi(K)$ which is again compact.

63 **Proposition 2.** *The map $f \to f^0$ is a homomorphism (in the sense of N. Bourbaki) of $\mathscr{C}_G$ onto $\mathscr{C}_{G/H}$.*

We prove this with the help of

Lemma 1. *There exists a positive continuous function f on G such that for every compact subset K of G the intersection of HK and the support S of f is compact and such that $\int_H f(\xi x) d\xi = 1$ for every $x \in G$.*

A locally compact group is always paracompact (Prop. 3, Ch. 1.2. part I) and using the fact that the canonical map is open and continuous, we see that G/H is also paracompact. Let U be an open relatively compact neighbourhood of e in G. $\pi(Ux)$ is a family of open subsets covering G/H. Let (V_i), (V'_i) be two locally finite open refinements of this covering such that $\overline{V_i} \subset V'_i$. Then there exist open relatively compact subsets $(W_i), (W'_i)$ such that $\overline{W_i} \subset W'_i$ and $\pi(W_i) = V_i$. In other words these are families of subsets such that each point in G has a saturated neighbourhood which intersects only a finite number of the subsets. We can moreover say that for every compact subset K of G, HK intersects only a finite number of W'_i. Now, let us define continuous functions g_i such that $g_i = 1$ on W_i and 0 outside W'_i, and set $g = \sum\limits_i g_i$. This last summation has a sense as the summation is only over a finite indexing set at each point.This is continuous, as every point in G has a neighbourhood in which g is the sum of a finite number of continuous functions. Let S be the support of g and K any compact subset of G. Then HK$\cap S$ is the union of a finite number of $\overline{W_i}$ and is hence compact.

Now let $g^0 = \int_H g(\xi x) d\xi > 0$.

64 This inequality is strict as at each point x, xH intersects some W_i. Obviously $f = g/g^0$ is a continuous function of G with S as its support. Trivially, $f^0 = 1$ and the proof of the lemma is complete.

Proof of the Proposition 2: That the map $f \to f^0$ is continuous from $\mathscr{C}_G \to \mathscr{E}^0_G$ has already been proved (Lemma 1, Ch. 2.3) and it is easy to see that this implies that the map $\varphi : f \to f^0$ of $\mathscr{C}_G \to \mathscr{C}_{G/H}$ is also continuous. We now exhibit a continuous map $\psi : \mathscr{C}_{G/H} \to \mathscr{C}_G$ such that $\varphi \circ \psi =$ Identity. For this one has only to define for every $g \in \mathscr{C}_{GH}$, $\psi(g)$ to be $\psi(g)(x) = g(\pi(x)) f(x)$ where f is the function constructed in the lemma. This has support $HK \cap S$ where K is a compact subset of G canonical image in G/H is the support of g.

$$\begin{aligned}(\psi(g))^0(x) &= \int_H g(\pi(\xi x)) f(\xi x) d\xi \\ &= g(\pi(x)) \int_H f(\xi x) d\xi \\ &= g(\pi(x))\end{aligned}$$

by the construction of f. Hence $\varphi(\psi(g)) = g \cdot \psi$ is of course continuous.

Every measure ν on G/H gives rise to a measure ν^0 on G in the following way $\nu^0(f) = V(f^0)$ for every continuous function f on G with compact support.

Corollary to Proposition 2: The image of $\mathscr{M}_{G/H}$ under the map $\nu \to \nu^0$ is precisely the set of all measure on G which vanish on the kernel $\mathscr{N}$ of the map $f \to f^0$ of $\mathscr{C}_G \to \mathscr{C}_{G/H}$.

This is an immediate consequence of the proposition.

Proposition 3. *A measure μ on G is zero on $\mathscr{N}$ if and only if $d\mu(\xi x) = \delta(\xi) d\mu(x)$ for every $\xi \in H$.*

By the above corollary μ is of the form ν° where ν is a measure on G/H. 65

Hence

$$\begin{aligned}\int_G f(\xi^{-1}x) d\nu^0(x) &= \int\limits_{G/H} f^0(\xi^{-1}x) d\nu(x) \\ &= \int\limits_{G/H} \int\limits_{H} f(\xi^{-1}\eta x) d\eta d\nu(x) \\ &= \int\limits_{G/H} \int\limits_{H} \delta(\xi) f(\eta x) d\eta d\nu(x) \\ &= \int\limits_{G/H} \delta(\xi) f^0(x) d\nu(x) = \int_G \delta(\xi) f(x) d\nu^0(x)\end{aligned}$$

It follows that $d\mu(\xi x) = \delta(\xi) d\mu(x)$.

Conversely let $d\mu(\xi x) = \delta(\xi) d\mu(x)$.

Let f, g be any two continuous functions on G with compact support. Then

$$\begin{aligned}
\mu(g^0 f) &= \int_G f(x) d\mu(x) \int_H g(\xi x) d\xi \\
&= \iint f(\xi^{-1} x) g(x) d\mu(\xi^{-1} x) d\xi \\
&= \iint f(\xi^{-1} x) g(x) \delta(\xi^{-1}) d\mu(x) d\xi \\
&= \iint f(\xi x) g(x) \delta(\xi) d\mu(x) \delta(\xi^{-1}) d\xi \\
&\qquad\qquad \text{(by Prop. 1, Ch. 3.1)} \\
&= \mu(f^\circ g).
\end{aligned}$$

If f is in $\mathscr{N}$, one can choose g such that $g^0 = 1$ on the support of f. Then $\mu(f) = \mu(fg^0) = \mu(f^0 g) = 0$. Hence $\mu = 0$ on $\mathscr{N}$.

If there exists an invariant measure ν on G/H, then ν^0 must be the Haar measure and conversely if the Haar measure is of the form ν^0 then ν is an *invariant* measure on G/H. Hence $\delta(\xi) = \Delta(\xi)$ is a necessary and sufficient condition for the existence of a right invariant measure on G/H.

3.4 Quasi-invariant measures.

66

Definition. *Let* Γ *be a transformation group acting on a locally compact space* E. *We say that a positive measure* μ *on* E *is* quasi-invariant *by* Γ *if the transform of* μ *by every* $\gamma \in \Gamma$ *is* equivalent *to* μ *in the sense that there exists a positive function* $\lambda(x, \gamma)$ *on* $E \times \Gamma$ *which is bounded on every compact subset and measurable for each* γ *such that* $d\mu(\gamma, x) = \lambda(x, \gamma) d\mu(x)$.

If under the above conditions $\lambda(x, \gamma)$ is independent of x, the measure is said to be *relatively invariant*.

Proposition 4. *There always exists a quasi-invariant measure on the homogeneous space G/H.*

We prove this by making use of

Lemma 2. *There exists a strictly positive continuous function ρ on G such that $\rho(\xi x) = \delta(\xi)/\Delta(\xi)\rho(x)$ for every $x \in G$ and $\xi \in H$.*

Let f be the function on G constructed in Lemma 1, Ch. 3.3. Define $\rho(x) = \int_H \left(\dfrac{\delta(\xi)}{\Delta(\xi)}\right)^{-1} f(\xi x) d\xi$.

Then it is an immediate verification to see that $\rho(\eta x) = \dfrac{\delta(\eta)}{\Delta(\eta)}\rho(x)$ and that ρ is positive continuous.

Proof of Proposition 4: Let μ be the measure $\rho(x)dx$ on G, where ρ is the function of Lemma 2. Then

$$\begin{aligned} d\mu(\xi x) &= \frac{\delta(\xi)}{\Delta(\xi)}\rho(x)\Delta(\xi)dx \\ &= \delta(\xi)\rho(x)dx = \delta(\xi)d\mu(x). \end{aligned}$$

By proposition 1, Ch. 2.4, there exists a measure ν on G/H such that

$$d\mu(x) = d\nu^0(x).$$

Now 67

$$d\mu(xy) = \frac{\rho(xy)}{\rho(x)} d\mu(x)$$

and hence we get

$$d\nu(\pi(xy)) = \frac{\rho(xy)}{\rho(x)} d\nu(\pi(x)),$$

$\dfrac{\rho(xy)}{\rho(x)}$ depending only on the coset of x modulo H. ν is therefore quasi-invariant.

This incidentally gives also the following relation between the Haar measure on G and the quasi-invariant measure on G/H, viz.

$$\int_G f(x)\rho(x)dx = \int\limits_{G/H} d\nu(\pi(x)) \int\limits_H f(\xi x)d\xi.$$

If we relax the condition of continuity on ρ, then we can assert that all the quasi invariant measures on G/H can be obtained this way [6]. So ν is relatively invariant if and only if there exists a positive function ρ on G Such that $\rho(xy)/\rho(x) = \rho(y)/\rho(e)$. If we take $\rho(e) = 1$, we have $\rho(xy) = \rho(x)\cdot\rho(y)$ with $\rho(\xi) = \delta(\xi)/\Delta(\xi)$ for every $\xi \in H$. In other words, the one dimensional representation $\xi \to \delta(\xi)/\Delta(\xi)$ of H can be extended globally to a representation of G.

3.5 Some applications.

Let G be the group product of two closed subgroups A and B such that the map $(a, b) \to ab$ of $A \times B \to G$ is a homeomorphism. Then the homogeneous space G/A is homeomorphic to B. We define a function on G by setting $\rho(ab) = \delta(a)/\Delta(a)$. To this function, there corresponds a quasi-invariant measure on G/A such that

$$\int_G f(ab)\delta(a)/\Delta(a)dx = \int_B d\mu(b)\int_A f(ab)da.$$

68 If $x = ab'$, we have $\rho(xb)/\rho(x) = \rho(ab'b)/_{\rho(ab')} = 1$ by definition. Hence $d\mu(b)$ is right invariant, and $d\mu(b) = db$.

Let $d_r x$, $d_l x$ denote respectively the right and left Haar measures. Then $\int_G f(ab)\delta(a)/_{\Delta(a)}d_r x = \int d_r b\int f(ab)d_r a$, or again

$$\begin{aligned}\int_G f(x)d_r x &= \iint\limits_{AxB} f(ab)\frac{\Delta(a)}{\delta(a)}d_r a d_r b\\ &= \iint\limits_{AxB} f(ab)\Delta(a)d_l a d_r b\end{aligned}$$

Thus we have got the right Haar measure on G in terms of the Left and right Haar measures on A, B respectively and the modular function on G. This dependence on the modular function can be done away with if we restrict ourselves to unimodular groups.Thus in the case of a unimodular group G, we have the simple formula

$$\int_G f(x)d_r x = \iint\limits_{AxB} f(ab)d_l a d_r b$$

Again when A is a normal subgroup, we have $\delta(a) = \Delta(a)$ or A and hence

$$\int_G f(x)d_r x = \iint\limits_{AxB} f(ab)d_r a d_r b.$$

3.6 Convolution of functions

Definition. *A function f is said to be* locally summable *with respect to a measure μ if for every continuous function φ with compact support, φf is μ-integrable.*

This has the property that for compact set $K, \chi(K)f$ is μ-integrable.

If f is locally summable, the map $\varphi \to \int \varphi f d\mu$ is a continuous linear form on $\mathscr{C}_G$ and hence defines a measure denoted by μ_f. Let now f be locally summable with respect to the Haar measure on G and ν another measure on G. We shall assume that $\mu_f * \nu$ exists. Then for 69
every continuous function g with compact Support, we have

$$\begin{aligned}\mu_f * \nu_{(g)} &= \iint g(xy)f(x)dxd\nu(y)\\ &= \int d\nu(y) \int g(xy)f(x)dx\\ &= \int d\nu(y) \int g(x)f(xy^{-1})dx\end{aligned}$$

Now the map $(x, y) \to (xy^{-1}, y)$ obviously preserves the product measure $dxd\nu(y)$ on GxG because for continuous functions u with compact support we have

$$\iint u(xy^{-1}, y)\,\mathrm{dxd}\,\nu(y) = \int\int u(x, y)dxd\nu(y).$$

Hence $\iint f(xy^{-1}g(x)\,\mathrm{dxd}\,\nu(y)$ exists and the theorem of Lebesgue-Fubini can be applied. It therefore results that $\int f(xy^{-1})d\nu(y)$ exists for almost every x and $g(x) \int f(xy^{-1})d\nu(y)$ is integrable, In other words, $\int f(xy^{-1})d\nu(y)$ is locally summable. If we denote by $h(x), \int f(xy^{-1})d\nu$

(y), this can be expressed by $\mu_f * \nu = \mu_h$. We can now define the convolution of a measure and a locally summable function f by putting $h(x) = f * \nu(x)$.One can similarly define a convolution $\nu * \mu_f = \mu_k$ where $k(x) = \nu * f(x) = \int f(y^{-1}x)\Delta(y^{-1})d\nu(y)$. This is unsatisfactory in as much as it is necessary to choose between left and right before one can identify functions with measures. Thus the notion of convolution of a function and a measure is not very useful in groups are not unimodular.

Let now f, g be two locally summable functions on G. Then we can define convolution of f and g such that $\mu_f * \mu_g = \mu_{f*g}$ by setting

$$\begin{aligned} f * g(x) &= \int f(xy^{-1})g(y)dy \\ &= \int f(y)g(y^{-1}x)\Delta(y^{-1})dy \end{aligned}$$

But we have in Prop. 1, Ch. 3.1 that $\Delta(y^{-1})dy = dy^{-1}$.

70 Thus the convolution of f and g can be satisfactorily defined even if the group G is not unimodular.

Note that the convolutions of two measures and of a measure and a function are uniquely defined, whereas the convolution of two functions is defined only upto a constant factor, as it depends on the particular Haar measure we consider.

If f and g are integrable, μ_f, μ_g are bounded and so is μ_{f*g}. Consequently $f * g$ is also integrable. Thus the map $f \to \mu_f$ is an imbedding of L^1 in $\mathscr{M}^1$ as a closed subspace. It is linear and one-one and also preserves metric, for,

$$\|f\|_1 = \int |f(x)|dx = \int |d\mu_f| \leq \|\mu_f\| \quad \text{and,}$$

on the other hand, $\|\mu_f\| \leq \|f\|_1$, trivially. Actually L^1 is a Banach subalgebra of $\mathscr{M}^1$. In $\mathscr{M}^1$, the Dirac measure at the unit element acts as the unit element of the algebra but L^1 does not possess any unit element, unless the group is discrete.

3.7 Convolutions of distributions

We close this chapter with a brief discussion of convolutions of distributions on a Lie group. A detailed account of distributions may be found in Schwartz's 'Théorie des Distributions' and de Rham's 'Variétés différentiables'.

Let G be Lie group and $\mathcal{D}_G$ the space of indefinitely differentiable functions on G which have compact support. Let $\mathcal{D}_K$ be the subset of $\mathcal{D}_G$ consist ring of functions whose supports are contained in the compact set K. One can provide $\mathcal{D}_K$ with the topology of uniform convergence of each derivative, and $\mathcal{D}_G$ with the topology of direct limit of those on $\mathcal{D}_K$. The topology on $\mathcal{D}_K$can be characterised by the fact that 71
$f \to 0$ on $\mathcal{D}_K$ if for every differential operator D on G with continuous coefficients, $Df \to 0$ uniformly on K. It is enough to consider only the left invariant differential operators, or still the Δ_α alone (Ch. 2.4, Part I). This topology makes of $\mathcal{D}_K$ a *Fréchet space* (i.e. a locally convex topological vector space which is metrisable and complete).

Definition. *A distribution on G is a continuous linear form on $\mathcal{D}_G$.*

As in the case of measures, one can define the notion of the support of a distribution, distributions with compact support, etc. Let T, S be two distributions on G. If one of them has compact support we define the convolution product as for measures:$T * S(\varphi) = \iint \varphi(xy) dT(x) dS(y)$. Let ξ' be the space of distributions with compact support. It is an algebra with convolution as product and the space of distributions is a module over ξ'_G. We denote by ξ'_e the space of distributions with support $= \{e\}$. Let $(x_1, \ldots x_n)$ be a coordinate system at e. Then $T \in \xi'_e$ implies the existence of $\lambda_\alpha \in C$ such that $T(\varphi) = \sum \lambda_\alpha \dfrac{\partial^\alpha \varphi}{\partial x^\alpha}(e)$, λ_α being zero except for a finite number of terms. If f is a locally summable function, we can identify f with the distribution $f(x)dx$ and we can define the notion of convolution $f * T$ of f and distribution T under some assumptions on f and T. But this product even when it is defined, is not in general a distribution of the form $g(x)dx$; however, if f is *indefinitely differentiable* with compact support (or if f is indefinitely differentiable and T has compact support) $f * T$ is a distribution of the form $g(x)dx$ where

$g(x)$ is an indefinitely differentiable function. This function is, in virtue of the above identification, given by $g(x) = f * T(x) = \int f(xy^{-1})dT(y)$.

72 If $T = \sum_\alpha \lambda_\alpha \frac{\partial^\alpha}{\partial x^\alpha}(e)$, $f * T(x) = \sum_\alpha \lambda_\alpha \{\frac{\partial^\alpha}{\partial y^\alpha} f(xy^{-1})\}y = e$.

The map $f \to f * T$ is a differential operator which is left invariant. We have already seen that $\epsilon_y * \mu = \sigma_y(\mu)$ (Ch. 2.5). Hence

$$\begin{aligned}(\sigma_y f) * T &= \epsilon_y * f * T\\ &= \epsilon_y * (f * T)\\ &= \sigma_y(f * T)\end{aligned}$$

In other words, T commutes with the left translation. At $x = e$ we have $f * T(e) = \sum \alpha\lambda_\alpha \{\frac{\partial^\alpha}{\partial y^\alpha} f(y^{-1})\}_{y=e}$ i.e. every left invariant differential operator is obtained in the above manner. This leads us to the

Proposition 5. *The algebra $\mathcal{U}(G)$ is canonically isomorphic to the algebra of distributions with support $\{e\}$ with convolution as multiplication.*

Chapter 8

Regular Representations

4.1 General notions 73

Let G be a locally compact group and E a locally convex topological vector space.

Definition. *A* continuous representation *of G in E is a map $x \to U_x$ of G into* $\mathrm{Hom}(E, E)$ *such that this is a representation in the algebraic sense (i.e. $U_{xy} = U_x U_y$ and U_e = identity) and such that the map* $(a, x) \to U_x a$ *of $E \times G \to E$ is continuous.*

The latter of these conditions, which we denote by R, is equivalent to the following:

R'_1: For every compact subset K of G, the set $\{U_x : x \in K\}$ is equicontinuous, and R'_2: for every $a \in E$, the map $x \to U_x a$ of $G \to E$ is continuous.

In fact, $R \Rightarrow R'_2$ trivially. Let V be a neighbourhood of 0 in E. For every $x \in K$, there exists an neighbourhood A_x of x in G and W_x of 0 in E such that for every $y \in A_x$ and $a \in W_x$, $U_y\, a \in V$. Since K is compact, we can choose $A_{x_1}, \ldots A_{x_n}$ which cover K and let $W = \bigcap W_{x_j}$. Now, $x \in K$, $b \in W \Rightarrow U_x b \in V$. Hence the set $\{U_x\}$ is equicontinuous. We proceed to prove the converse; in fact, we show more generally that R'_1 with the following condition R''_2: There exists a dense subset F of E such that for every $a \in F$, the map $x \to U_x a$ of $G \to E$ is continuous, implies R. It is required to show that the map $(x, a) \to U_x a$ is continuous

at any point (x, a). Let V be any convert neighbourhood of 0 in E. We seek a neighbourhood W of 0 in E and a neighbourhood A of x in G such that $b \in a + W$, $y \in A \Rightarrow U_y b \in U_x\ a + V$. Let K be a compact neighbourhood of x in G. Then there exists a neighbourhood V_1 of 0 such that $y \in K$, $b \in V_1 \Rightarrow U_y b \in V/4$ (by R_1'). Let $b \in (a + V_1) \bigcap F$.
74 Then we can find a neighbourhood A of x in C contained in K such that $U_y b - U_x b \in V/4$ for every $y \in A$ (by R_2''). Now, if $c \in a + V_1$, $y \in A$, we have

$$U_y c - U_x a = U_y(c - a) + (U_y - U_x)b + (U_y - U_x)(a - b) \in V.$$

This completes the proof of the equivalence.

Moreover, if E is a barrelled space (or in particular a Banach space) then axiom R_2' itself $\Rightarrow R$. For, the map $x \to U_x$ is continuous from G to $\text{Hom}(E, E)$ with the topology of simple convergence and hence the image of a compact subset is again compact and, E being barrelled, equicontinuous.

4.2 Examples of representations

(i) Unitary representations. Let U be a representation of G in a Hilbert space H such that U_x is a unitary operator, i.e. $U_x - 1 = U_x^*$ for every $x \in G$. Then V is called *a unitary representation.*

(ii) Bounded representations. A representation U of G in a Banach space B is said to be *bounded* if there exists M such that $\|U_x\| < M$ for every $x \in G$. It should be remarked here that in general representations in Banach spaces are not bounded as for instance the representation $x \to e^x$. Id of R in itself. However, such representations are bounded on every compact subset.

(iii) Regular representations. Left and right translations in G, as we have seen before, give rise to representations of G in the space $\mathscr{C}_G, L^p$ etc. In fact they give rise to representations of G in any function space connected with G with a reasonably good definition and a convenient topology.

The space $\mathscr{C}_G$ with the usual topology can easily be seen to be barrelled. Therefore, in order to verify that τ is a continuous representation, we have only to prove that $y \to \tau_y f$ is continuous. 75
It is again sufficient to establish continuity at the point $y = e$. The function f is uniformly continuous and hence when $y \to e$, $\tau_y f \to f$ uniformly having its support contained in a fixed compact set.

In the case of $L^p(1 \leq p \leq \infty)$ with respect to the right Haar measure, $\|\tau_y\| = 1$ by the invariance of the measure and hence the τ_y form an equicontinuous set for $y \in G$. Also the map $y \to \tau_y f$ is continuous for the topology on $\mathscr{C}_G$ which is finer than that of L^p and since $\mathscr{C}_G$ is dense in L^p is a representation of G in L^p. On the other hand, if we consider σ_y, we have $\|\sigma_y f\| = (\int |f(y^{-1}x)|^p dx)^{1/p} = (\Delta(y))^{1/p}\|f\|_p$ and the continuity of $y \to \sigma_y f$ follows as a consequence of the continuity of $\Delta(y)$. Note that the proof is not valid when p is infinite as $\mathscr{C}_G$ is not dense in L_∞. In fact the map $y \to \tau_y f$ of $G \to L_\infty$ is continuous if and only if f is uniformly continuous on G.

(iv) Induced representations. Let H be a closed subgroup of a topological group G, and L a continuous representation of H in a locally convex space E. Let $\mathscr{C}^L$ be the space of functions on G with values in E which are continuous with compact support modulo H (i.e. their supports are contained in the saturation of a compact set), and which satisfy the following equality:

$f(\xi x) = \left(\dfrac{\delta(\xi)}{\Delta(\xi)}\right)^{\frac{1}{2}} L_\xi f(x)$ for every $x \in G$ and $\xi \in H$. The factor $\left(\dfrac{\delta(\xi)}{\Delta(\xi)}\right)^{\frac{1}{2}}$, it will be noted, occurs purely for technical reasons and can without much trouble be done away with. The above equality, in essence, expresses only the condition of covariance of f with respect to left translations by ξ. On this space $\mathscr{C}^L$ we can, as we have more than once done before, introduce the topology of direct limit of those on $\mathscr{C}^L_K$, the latter being the space of functions 76
in $\mathscr{C}^L$ whose supports are contained in HK, with the topology of

uniform convergence on K. This is again a locally convex space and right translations by elements of G give rise to a (regular) representation of G in $\mathscr{C}^L$. This is called the *representation* of G *induced* by L.

(v) Let us again assume L to be a unitary representation of H in a Hilbert space E. Let $\mathscr{C}^L$ be the space of continuous functions on G with values in E having compact support modulo H such that $f(\xi x) = \left(\dfrac{\delta(\xi)}{\Delta(\xi)}\right)^{\frac{1}{2}} L_\xi f(x)$. Naturally one tries to introduce a scalar product in $\mathscr{C}^L$ in the usual way, but the possibility of $f(x)$ not being square integrable (which it is not in general) foils the attempt. However, though $f(x)$ may not be square integrable, we are in a position to assert that $\int_{G/H} ||f(x)||^2(\rho(x))^{-1}dx < \infty$ (ρ of course is the function defined in Lemma 2, Ch. 3.4 and dx the quasiinvariant measure on G/H). In fact, the function $||f(x)||^2(\rho(x))^{-1}$ is invariant modulo H and consequently can be considered as a continuous function on G/H with compact support. Hence we can define $||f||^2_L = \int_{G/H} ||f(x)||^2\,((\rho(x)))^{-1}\,dx$. Let $\mathscr{H}^L$ be the completion of $\mathscr{C}^L$ under this norm. As usual, $\mathscr{H}^L$ is the space of measurable functions f which satisfy the condition of covariance and are such that $\int_{G/H} ||f(x)||^2(\rho(x))^{-1}dx < \infty$. This is a Hilbert space in which the scalar product is given by $\langle f, g\rangle = \int\limits_{G/H} \langle f(x), g(x)\rangle(\rho(x))^{-1}dx$.

The right regular representation of G in $\mathscr{H}^L$ is unitary. For,

$$\begin{aligned}
||\tau_y f||^2 &= \int_{G/H} ||f(xy)||^2\,((\rho(x)))^{-1}\,d\dot{x}\\
&= \int_{G/H} ||f(x)||^2 \left((\rho(xy^{-1}))\right)^{-1} \frac{\rho(xy^{-1})}{\rho(x)} dx \quad \text{by quasi invariance}\\
&= \int_{G/H} ||f(xy)||^2\,((\rho(x)))^{-1}\,d\dot{x}\\
&= ||f||^2_L \quad \text{for every} \quad f \in \mathscr{H}^L
\end{aligned}$$

77 The same proof as in (iv) gives the continuity of the representation.

Thus one can define induced representations in many ways, in each case the representation space being so chosen as to reflect the particular properties of the representation one wishes to study. We shall not dwell on induced representations any longer, but only give the following general definition which, it is needles to say, includes the last two cases.

Definition. *A representation U of G in F is said to be* induced *by representation L of H in E if there exists a linear continuous map η of $\mathscr{C}^L$ into F such that*

(i) η is injective with its image in F everywhere dense, and

(ii) η commutes with the representation in the sense that $U_x \circ \eta = \eta \circ \tau_x$ for every $x \in G$.

4.3 Contragradient representation

Let U be a continuous representation of G in a locally convex space E. For every $x \in G$ consider $t_{U_x} \in \mathrm{Hom}(E', E')$ which is continuous for any 'good' topology on E' (weak, strong of convex-compact). We denote by $\check{U}$ the map $x \to {}^tU_x^{-1}$. Regarding this map we have the following

Proposition 1. *If U is a continuous representation of G in a quasi complete locally convex space E, then $\check{U}$ is also a continuous representation of G in E'_C (convex compact topology). We need here the following formulation of* Ascoli's theorem. *Let X be a locally compact topological space, F a uniform Hausdorff space and $\mathscr{C}(X, F)$ the space of continuous functions from $X \to F$. Let Λ be an equicontinuous subset of $\mathscr{C}(X, F)$ such that the set $\{\lambda(x) : \lambda \in \Lambda\}$ is relatively compact in F for every $x \in X$. Then (i) Λ is relatively compact in $\mathscr{C}(X, F)$ with the topology of compact convergence, and (ii) on Λ the topology of com-* 78
pact convergence coincides with every Hausdorff weaker topology (in particular, with the topology of simple convergence).

Proof of proposition 1. Let K be a compact of G. The set $\{U_x : x \in K\}$ is equicontinuous and for every $a \in E$, $\{U_x a : x \in K\}$ is compact as the map $x \to U_x a$ is continuous. Hence by Ascoli's theorem, the topology of simple convergence and the topology of compact convergence are one and the same on this subset of $\mathscr{C}(E,E)$ i.e. $x \to e \Rightarrow U_x a \to a$ uniformly for a in a compact subset H of E.

Let a' be an element of E'_C. We wish to prove that $x \to \check{U}_x a'$ is continuous. It is enough to prove the continuity at the unit element e. Let $x \to e$. Then $U_{x^{-1}} a \to a$ uniformly on a compact set H and hence $\langle U_{x^{-1}} a, a' \rangle = \langle a, \check{U}_x a' \rangle \to \langle a, a' \rangle$ uniformly on H. This shows that $x \to \check{U}_x a'$ is continuous for the convex-compact topology on E'. We have to show moreover that the set$\{\check{U}_x : x \in K\}$ is equicontinuous. If H be a convex compact subset of E, we seek to prove the existence of another convex compact set H' such that $a' \in (H')^0 \Rightarrow \check{U}_x a' \in H^0$ for every $x \in K$ where A^0 denotes the polar of A. But $\check{U}_x a' \in H^0$ for every $x \in K$ if and only if $\left|\langle U_{x^{-1}} a, a' \rangle\right| \leq 1$ for every $x \in K$ and $a \in H$. Let H' be the closed convex envelope of the compact set description by $U_{x^{-1}} a$. It is obvious that $a^1 \in (H')^0 \Rightarrow \left|\langle b, a' \rangle\right| \leq 1$ for every $b \in H' \Rightarrow \left|\langle U_{x^{-1}} a, a' \rangle\right| \leq 1$ for every $x \in K$ and $a \in H$. It only remains to show that H' is compact. But H' is precompact, and being a closed bounded set, also complete. This shows that H' is compact.

This representation in E' is called the *contragradient* of U.

Remark. It will be noted that we have used the quasi completeness of
79 the space E only to prove that the closed convex envelope of a compact set is also compact. Hence the proposition is valid for the more general class of locally convex spaces which satisfy the above condition.

Example. We have seen that the right and left translations give representations of G in the function spaces $\mathscr{C}_G$, $\mathscr{E}^0$, $\overline{\mathscr{C}_G}$, etc. By our proposition above,we see that σ is continuous in $\mathscr{M}$, $\mathscr{M}^C$, $\mathscr{M}^1$ (with the convex compact topology).

Remark. The regular representation of G in $\mathscr{M}^1$ is *not continuous* with respect to the *strong* topology. For, $\tau_x \epsilon_e = \epsilon_x$ and $\|\epsilon_x - \epsilon_e\| = 2$ if $x \neq e$. Hence as $x \to e$, ϵ_x does not tend to ϵ_e.

4.4 Extension of a representation to $\mathscr{M}^C$

Let U be a continuous representation of G in a locally convex quasi complete space E. Let μ be any measure on G with compact support. Then we write $U_\mu a = \int_G U_x a d\mu(x)$. (The function $x \to U_x a$ is a vector-valued function and $\int U_x a d\mu(x)$ has been defined in Ch. 1.5).

Theorem 1. (1) *U_μ is a linear continuous function of E in itself.*

(2) *$\mu \to U_\mu$ is an algebraic representation of $\mathscr{M}^C$ in E.*

(3) *If U is a bounded representation in a Banach space, then this representation can be extended to a continuous representation of the Banach algebra $\mathscr{M}^1$ in to the Banach algebra* $\mathrm{Hom}(E, E)$.

(1) In fact, as $a \to 0$, $U_x a \to 0$ uniformly on the compact support of μ and hence U_μ is continuous. Its linearity is trivial.

(2) Again the linearity of the map $\mu \to U_\mu$ is obvious.

$$\begin{aligned} U_{\mu * \nu} a &= \int U_x a d(\mu * \nu)(x) \\ &= \iint U_{xy} a d\mu(x) d\nu(y) \quad \text{(see Remark 2, Ch. 2.2)} \\ &= \int d\mu(x) U_x (U_\nu a) \text{(Remark, Ch. 1.5)} \\ &= U_\mu U_\nu a. \end{aligned}$$

80

(3) If $\mu \in \mathscr{M}^C$, we have

$$\begin{aligned} \|U_\mu a\| &= \|\int U_x a d_\mu(x)\| \\ &\leq \int \|U_x a\| d|\mu| \\ &\leq k\|a\| \, \|\mu\| \end{aligned}$$

This proves that the map $(a, \mu) \to U_\mu a$ of $E \times \mathscr{M}^C \to E$ is continuous. Since $\mathscr{M}^1$ is only the completion of $\mathscr{M}^C$ with the topology of the norm, this map can be extended to a continuous map $E \times \mathscr{M}^1 \to E$, which proves all that was asserted.

4.5 Convolution of measures

We can get, in particular,representations of the space $\mathscr{M}$ of measures on a group G by considering regular representations of G. We define $\sigma_\mu \nu = \int \sigma_x(\nu) d\mu(x)$. $\mathscr{M}$ is the dual of the barrelled space $\mathscr{C}_G$ and is hence quasi complete. We therefore have

$$\begin{aligned}
\langle f, \sigma\mu(\nu)\rangle &= \int \langle f, \sigma_x(\nu)\rangle d\mu(x)\\
&= \int d\mu(x) \int f(y) d\nu(x^{-1}y)\\
&= \iint f(xy) d\mu(x) d\nu(y)\\
&= \langle f, \mu * \nu\rangle
\end{aligned}$$

In other words, $\sigma_\mu(\nu) = \mu * \nu$ and $\tau_\mu(\nu) = \nu * \check{\mu}$ where $d\check{\mu}(x) = d\mu(x^{-1})$. We have not imposed any conditions on ν, and μ has been assumed to have compact support. Thus convolution of two measures could have been defined as $\sigma_\mu(\nu) = \int \sigma_x(\nu) d\mu(x)$ straightaway.

4.6

81

Proposition 2. *Let E be a subspace of $\mathscr{M}$ with a finer topology such that*

(a) *E is invariant by τ;*

(b) *τ restricted to E is continuous;*

(c) *E is quasi complete.*

Then for every $\mu \in \mathscr{M}^C$ and $a \in E$, we have $a*\mu \in E$ and $a*\mu = \tau_{\check{\mu}} a$. If moreover τ is bonded on E, then this true for $\mu \in \mathscr{M}^1$.

The proposition is immediate in view of our remarks in Ch. 4.5.

In particular, if $a \in L^P(p < \infty)$, $\mu \in \mathscr{M}^1$, then $a * \mu \in L^p$, and $\|a*\mu\|_p \leq \|a\|_p\|\mu\|$. Again, we may take an integrable function f instead of μ and get $a * f \in L^p$ and $\|a * f\|_p \leq \|a\|_p\|f\|_1$. Otherwise stated, L^1 is represented as an algebra of operators in the Banach space $L^p(p < \infty)$. Another case is that of $E = \mathscr{C}_G$. If f is a function and μ a measure both with compact supports, then $f * \mu \in \mathscr{C}_G$.

4.7 Process of regularisation.

Let V be any neighbourhood of e in G. Let A_V be the set $\{f \in \mathscr{C}_v : f \geq 0$ and $\int f(x)dx = 1\}$. As V describes the neighbourhood filter at e, A_V also describes a filter Φ in the function space $\mathscr{C}_G$.

Proposition 3. *If U is a continuous representation of G in a quasi complete space E, then $U_f a \to a$ following Φ for every $a \in E$.*

In fact $U_f a = \int U_x a f(x)dx$. If W is a closed convex neighbourhood of 0 in E, then by the continuity of $U_f a$ one can find a neighbourhood V of e such that $U_x a \in a + W$ for every $x \in V$. Now $U_f a - a = \int(U_x a - a)f(x)dx \in W$ whenever $f \in A_V$ by convexity of W.

Remark. $U_f a$ has certain properties of continuity stronger than that of **82**
a. For instance if we take for U the regular representation of G in $\mathscr{M}$, $\tau_f\mu \in \mathscr{E}^0$. When $\Phi \to e$, $\tau_f\mu \to \mu$. This is a process of approximation of a measure, as it were by continuous functions. If G satisfies the first axiom of countability, we can find a sequence $\{f_n\}$ of continuous functions generating the filter Φ. In particular, if G is a Lie group, we have thus an approximation of measures by sequences of continuous functions. Finally we remark in passing that the same procedure can be adopted in the case of Lie groups for distributions instead of measures. Thus a distribution on a Lie group can be approximated by a sequence of indefinitely differentiable functions.

Chapter 9

General theory of representations

5.1 Equivalence of representations

83

Definition. *A representation U of a topological group G in locally convex space E is said to be* equivalent *to another representation U' in E' if there exists an isomorphism T of E onto E' such that $TU_x = U'_xT$ for every $x \in G$.*

This is evidently a very strong requirement which fails to characterise as equivalent certain representations which are equivalent in the intuitive sense. However, we are interested in the case of unitary representations in Hilbert spaces and the definition is good enough for our purposes.

Definition. *Two representations U in H, U' in H' are* unitarily equivalent *if there exists a unitary isomorphism $T : H \to H'$ such that $TU_x = U'_xT$ for every $x \in G$.*

Proposition 1. *Two equivalent unitary representations are unitarily equivalent. In fact, $TT^*U'_x = TU_xT^* = U'_xTT^*$ i.e. U_x commutes with the positive Hermitian operator TT^* and hence also with $H = \sqrt{TT^*}$. It can be easily seen that $H^{-1}T$ is a unitary operator which transforms U into U'.*

5.2 Irreducibility of representations

Definition 1 (algebraic irreducibility)**.** *A representations U of a group G in a vector space E is said to be* algebraically irreducible *if there exists no proper invariant subspace of E.*

Definition 2 (topological irreducibility)**.** *A representation U of a topological group G in a locally convex space E is said to be* topologically irreducible *if there exists no proper* closed *invariant subspace.*

84 **Definition 3** (complete irreducibility)**.** *A representation U of a topological group G in a locally convex space E is said to be* completely irreducible *if any operator in* $\mathrm{Hom}(E, E)$ *(with the topology of simple convergence) can be approximated by finite linear combinations of the U_x.*

It is at once obvious that (i) $\Rightarrow$ (ii) and that (iii) $\Rightarrow$ (ii). It can be proved that when E is a Banach space, (i) $\Rightarrow$ (iii) (Proof can be found in Annals of Mathematics, 1954, Godement). For unitary representations, (ii) and (iii) are equivalent (due to von Neumann's density theorem, Th. 2. Ch. 5.6). Finally, all the three definitions are equivalent for finite dimensional representations ((ii) $\Rightarrow$ (iii) due to Burnside's theorem, Th. 1, Ch. 5.5).

5.3 Direct sum of representations

Definition. *A representation U of G in E is said to be the* direct sum *of representations U_i of G in E_i if E_i are invariant closed subspaces of E such that the sum $\sum E_i$ is direct and is everywhere dense in E, and if U_i is the restriction of U to E_i. Moreover, if U is a unitary representation in Hilbert space, U is said to be the* Hilbertian direct sum *of the U_i if E_i is orthogonal to E_j whenever $i \neq j$.*

Definition. *A representation is* completely reducible *if it can be expressed as a direct sum of irreducible representations.*

5.4 Schur's lemma*

We give here two formulations (Prop. 2 and 3) of Schur's lemma, the first being trivial and the second more suited to our purposes.

Proposition 2. *Let U and U' be two algebraically irreducible representations in E, E' respectively. If T is a linear map: $E \to E'$ such that $TU_x = U'_x T$ for every $x \in G$, then either $T = 0$ or an algebraic* **85**
isomorphism.

From this, we immediately deduce the following

Corollary. *Let U be an algebraically irreducible finite - dimensional representation of a group G in E. The only endomorphisms of E which commute with all the U_x are scalar multiples of the identity.*

In fact, if λ is an eigenvalue of T, $T - \lambda I$ is not an isomorphism and is, by Schur's lemma, $= 0$.

Proposition 3. *Let U, U' be two unitary topologically irreducible representations in H, H' respectively. If T is a continuous linear map $H \to H'$ such that $TU_x = U'_x T$ for every $x \in G$, then either $T = 0$ or an isomorphism of Hilbert spaces.*

In fact, T^* is a continuous operator with $UT^* = T^*U'$. $H = T^*T$ is a Hermitian operator commuting with every U_x. Hence U_x commutes with every E_λ in the spectral decomposition $H = \int \lambda dE_\lambda$ and consequently leaves every spectral subspace invariant. Therefore, the spectral sub-spaces reduce to $\{0\}$ or E. i.e. H is a scalar $= \lambda I$. Similarly $H' = TT^* = \lambda I$. Hence T is either 0 or an isometry up to a constant.

The proof of Prop. 3 implicitly contains the following

Corollary. *Let U be a unitary topologically irreducible representation of a group G in a Hilbert space E. The only operators of E which commute with all the U_x are scalar multiples of the identity. This is immediate since any operator can be expressed as a sum of Hermitian operators for which the corollary has been proved in prop. 3.*

5.5 Burnside's theorem

Theorem 1 (Burnside)**.** *Let U be an algebraically irreducible representation of G into E of finite dimension. Then every operator in E is a linear combination of the U_x.*

86

Consider the algebra $\mathscr{A}$ of finite linear combinations of U_x. Let $\mathscr{B}$ be the subset of $\mathrm{Hom}(E,E)$ consisting of elements B such that $\mathrm{Tr}(AB) = 0$ for every $A \in \mathscr{A}$. Obviously it is enough to show that $\mathscr{B} = \{0\}$. Now we define a representation V of G in $\mathscr{B}$ by defining $V_x(B) = U_x \circ B$ for every $B \in \mathscr{B}$. This is not in general an irreducible representation. However, if $\mathscr{B} \neq \{0\}$, we can find a non-zero irreducible subspace $\mathscr{C}$ of $\mathscr{B}$. Now the map $\lambda_a : B \to B_a$ of $\mathscr{C}$ into E is a linear map which transforms the representation V into U. For,

$$\lambda_a \circ V_x(B) = \lambda_a \circ U_x \circ B = U_x \circ Ba = U_x \circ \lambda B \text{ for every } B \in \mathscr{C}.$$

Hence by Schur's lemma (prop. 2, Ch. 5.4), $\lambda_a = 0$ or is an isomorphism. If $\lambda_a = 0$ for every $a \in E$, $B = 0$ for every $B \in \mathscr{C}$ and hence $\mathscr{C} = \{0\}$. But this is contradictory to our assumption that $\mathscr{C}$ is non-zero. Therefore, there exists $a_1 \in E$ such that $\lambda_{a_1} \neq 0$. So, λ_{a_1} is an isomorphism of $\mathscr{C}$ onto E. Let $(a_1, a_2 \ldots)$ be a basis of E. Then $\lambda_{a_1}^{-1} \circ \lambda_{a_2}$ is an operator on $\mathscr{C}$. This obviously commutes with every V_x. Hence by cor. to prop. 2, Ch. 5.4 $\lambda_{a_1}^{-1} \circ \lambda_{a_2} = \mu_2 I$ where μ_2 is a scalar. We shall thus write $\lambda_{a_j} = \mu_j \lambda_{a_1}$ with $\mu_1 = 1$. Now, one can introduce a scalar product in E such that $\mathrm{Tr}(U_x B) = \sum_j \langle U_x B a_j, a_j \rangle = 0$ for every $x \in E$. But

$$\begin{aligned}\sum_j \langle U_x B a_j, a_j \rangle &= \sum_j \mu_j \langle U_x B a_1, a_j \rangle \\ &= \langle U_x B a_1, \sum_j \mu_j a_j \rangle = 0\end{aligned}$$

Since the $U_x B a_1$ generate E, $\sum_j \mu_j a_j = 0$ or again $\mu_j = 0$ for every
87 j. But $\mu_1 = 1$. This gives us a contradiction and it follows that $B = \{0\}$.

5.6 Density theorem of von Neumann

Let U be a unitary representation of a topological group G in a Hilbert space H. We denote by $\mathcal{A}$.the subset of the set $\mathrm{Hom}(H, H)$ of operators on H consisting of finite linear combinations of the U_x. This is a self adjoint subalgebra of $\mathrm{Hom}(H, H)$, but is not in general closed in it. Let $\mathcal{A}'$ be the set of operators which commute with every element of $\mathcal{A}$. Obviously $\mathcal{A}'$ is also self-adjoint.

Proposition 4. *$\mathcal{A}'$ is* weakly *closed in* $\mathrm{Hom}(H, H)$.

In fact, if $A = \lim A_i$ (in the weak topology) with $A_i \in \mathcal{A}'$, then

$$\begin{aligned}\langle BAx, y\rangle &= \lim\langle BA_i x, y\rangle = \lim\langle A_i Bx, y\rangle \\ &= \langle ABx, y\rangle \text{ for every } B \in \mathcal{A}.\end{aligned}$$

Hence $A \in \mathcal{A}'$.

Let $\mathcal{A}''$ be the commutator of the algebra $\mathcal{A}'$. Then $\mathcal{A}''$ is a weakly closed self-adjoint subalgebra containing $\mathcal{A}$ and hence it contains the weak closure of $\mathcal{A}$. We can in fact assert

Theorem 2 (von Neumann)**.** *$\mathcal{A}''$ = weak closure of $\mathcal{A}$.*

We actually prove a stronger assertion, viz. Let (x_n) be a sequence of elements in E such that $\sum \|x_n\|^2 < \infty$ and T an operator in $\mathcal{A}''$. Then for every $\epsilon > 0$, there exists $A \in \mathcal{A}$ such that $\sum \|Tx_n - Ax_n\|^2 < \epsilon$. (This in particular implies that $\mathcal{A}''$ = strong closure of $\mathcal{A}$ or even the strongest closure of $\mathcal{A}$, in the sense of von Neumann).

We first show that for every $x \in E$, Tx is in the closure of $\{Ax : A \in \mathcal{A}\}$. In fact, $F = \overline{\{Ax\}}$ is a closed invariant subspace of E and let $F^\perp$ be **88**
its orthogonal complement. $F^\perp$ is also invariant under the self adjoint algebra $\mathcal{A}$ and hence the orthogonal projection P of E onto F commutes with every element of $\mathcal{A}$. P therefore belongs to $\mathcal{A}'$ and T leaves F invariant. Since $x \in F$, Tx is also in F.

Now consider the space $E_1 = E \oplus E \oplus \cdots\cdots$ (Hilbertian sum). Every element $x \in E_1$ is of the form $(x_1, \ldots x_n \cdots)$ with $\sum \|x_n\|^2 < \infty$. Let $\tilde{A}$ be the operator on E_1 defined by $Ax = (Ax_1, \ldots Ax_n, \ldots)$. The map $A \to \tilde{A}$ is an isomorphism of $\mathrm{Hom}(H, H)$ into $\mathrm{Hom}(H_1, H_1)$. Denote the image of $\mathscr{A}$ by $\tilde{\mathscr{A}}$. If B is any operator on E_1, it can be expressed in the form $B(x_1, \ldots x_n \ldots) = (y_1, \ldots y_n \ldots)$ where $y_n = \sum\limits_p b_{n,p} x_p$ and $b_{n,p}$ is an operator on E. We now show that $B \in (\tilde{\mathscr{A}})'$ if and only if $b_{n,p} \in \mathscr{A}'$ for every n and p. For, $B\tilde{A}(x_1, \ldots x_n, \ldots) = \tilde{A}B(x_1, \ldots x_n, \ldots)$ for every $x \in E_1$ implies that $b_{n,p}A = Ab_{n,p}$ by taking all x_n, $n \neq p$ to be zero. Conversely, if this is satisfied, $\tilde{A}B(x_1, ..x_n..)$

$$\begin{aligned} &= \tilde{A}(\ldots, \sum_p b_{n,p} x_p, \ldots) = (\ldots, \sum_p b_{n,p} A x_p, \ldots) \\ &= B(Ax_1, \ldots Ax_n, \ldots) = B\tilde{A}(x_1, \ldots x_n, \ldots) \end{aligned}$$

Again, $B \in (\tilde{A})''$ if and only if the diagonal elements of the infinite matrix $b_{n,p}$ are equal and in $\mathscr{A}''$, and the rest of the elements are zero. In fact, if $C \in (\tilde{A})''$, we have $(BC)_{m,n} = (CB)_{m,n}$ for every $B \in \mathscr{A}'$, or $\sum\limits_p b_{m,p} c_{p,n} = \sum\limits_q c_{m,q} b_{q,n}$ for every $b_{i,j} \in \mathscr{A}'$. Putting $b_{i,j} = \delta_{n,i}\delta_{n,j}$,
89 we get $c_{i,j} = 0$ if $i \neq j$ and $c_{m,m} = c_{n,n}$ for every m and n. So we have $(\tilde{A})'' = (\widetilde{A''})$. Therefore there exists $A \in \mathscr{A}$ such that $\sum \|Tx_n - Ax_n\|^2 < \epsilon$.

Moreover, in theorem 2, if T is Hermitian we can find a *Hermitian* operator A such that $\sum \|Tx_n - Ax_n\|^2 < \epsilon$. As before it is enough to prove this for one vector x. In other words, we have to show the existence of Hermitian operator A such that $\|Tx - Ax\| < \epsilon$. We know that T is the strong limit of $A \in \mathscr{A}$. Hence $T = T^*$ is the *weak* (and not strong, in general) limit of A^* or again the weak limit of $\dfrac{A + A^*}{2}$. Now $\dfrac{A + A^*}{2}$ is a Hermitian operator in $\mathscr{A}$. Hence T is weakly adherent to this convex set. In this case, weak adherence is the same as the weak adherence in the sense of topological vector spaces, but weak and strong topologies are the same in a convex space.

In particular, if we have a unitary topologically irreducible representation, by Schur's lemma (Prop. 3, Ch. 5.4) $\mathscr{A}' = \{\lambda I\}$ and hence $\mathscr{A}'' = \mathrm{Hom}(E, E)$. Therefore every operator is strongly adherent to $\mathscr{A}$. This is the analogue of Burnside's theorem (Th. 1, Ch. 5.5) for unitary representations.

Part III

Continuous sum of Hilbert Spaces

Chapter 10

Continuous sum of Hilbert Spaces-I

1.1 Introduction **90**

In the general theory of unitary representations of a locally compact group, two main problems are (i) to determine all the irreducible unitary representations of a group, and (ii) to decompose a given unitary representation into irreducible ones. The first of these has been completely solved in certain cases (e.g. abelian groups, compact, certain semisimple Lie groups), but it is to the latter that we address ourselves in the following pages. We start by giving some Examples.

Let U be the regular representation of the one dimensional torus T^1 in the space L^2 of square summable functions. If f belongs to L^2, it can be expressed in Fourier series $\sum a_n e^{inz}$. If $x = e^{it}$, then $\sigma_x f = \sum\limits_n \sigma_x a_n e^{inz} = \sum\limits_n (a_n e^{\int}) e^{inz}$ and we have decomposed a unitary representation into a direct sum of irreducible representations.

If we take R instead of T^1 and $F \in L^2$, then $\hat{f}(y) = \int f(x) e^{ixy} dx$ also belongs to L^2. By the inversion formula, when f is sufficiently regular (we do not enter into these details) we have $f(x) = \dfrac{1}{2\pi} \int \hat{f}(y) e^{-ixy} dy$. Hence

$$\sigma_z f = \frac{1}{2\pi} \int \hat{f}(y) e^{-i(x+z)y} dy$$

$$= \frac{1}{2\pi}\int \hat{f}(y)e^{izy}e^{-ixy}dy$$
$$= \frac{1}{2\pi}\int (\hat{\sigma f})(y)e^{-ixy}dy.$$

Therefore $(\widehat{\sigma_z f})(y) = e^{izy}\hat{f}(y)$. The regular representation has again been decomposed into one-dimensional representations but this is not a discrete sum but a 'continuous sum' - a concept which we shall define presently.

Before proceeding with the formal definitions, we give one more example which is more akin to the theory we are to develop. Let E
91 be a locally compact space with a positive measure μ and $\mathscr{H}$ a Hilbert space. In the space $\mathscr{C}(E,\mathscr{H})$ of continuous functions $f : E \to \mathscr{H}$ with compact support, we introduce a semi-norm

$$\|f\| = \left(\int \|f(z)\|^2 d\mu(z)\right)^{\frac{1}{2}} < \infty.$$

Let $\mathscr{L}^2(\mathscr{H})$ be the completion of the Hausdorff space associated with $\mathscr{C}(E,\mathscr{H})$. We have also a scalar product in this space given by $\langle f,g\rangle = \int\langle f(z),g(z)\rangle d\mu(z)$.

If μ is discrete (i.e. is a linear combination of Dirac measures at certain points), $\mathscr{L}^2(\mathscr{H})$ becomes a discrete sum of Hilbert spaces associated to each of those points, all the Hilbert spaces being isomorphic to $\mathscr{H}$.

These considerations motivate some kind of a continuous sum of Hilbert spaces indexed by points of a locally compact space. We therefore assume the following data to start with:

(1) $\mathcal{Z}$, a locally compact space (which will be assumed for simplicity to be countable at ∞) with a positive measure μ;

(2) For every $z \in \mathcal{Z}$, a Hilbert space $\mathscr{H}(z)$. In other words, we assume given at each point a 'tangent space' which is a Hilbert space. For instance, in a Riemannian manifold, the metric assigns a scalar product to the tangent space at each point of $\mathcal{Z}$. Of course in this case the spaces are of finite dimension. Having in mind the

example above, we seek to find an analogue of the concept of functions on $\mathcal{Z}$. This is served by the notion of a *vector field* (in exactly the same sense as in manifolds) which is an assignment to each point, of an element of the associated Hilbert space. We would like to have a notion analogous to that of continuous functions in our example. To this end, we introduce a *fundamental family* of vector fields with reference to which the continuity of **92**
an arbitrary vector field will defined. Thus we suppose given

(3) *A fundamental family* $\wedge$ of vector fields which satisfies the following conditions:

(a) $\wedge$ forms a vector space under the 'usual' operations.

(b) For every vector field $X \in \wedge$, the real valued function $\|X(z)\|$ is continuous. This in particular implies that the map $z \to \langle X(z), Y(z)\rangle$ is continuous for every $X, Y \in \wedge$.

(c) For every z, the vectors $X(z)$ for $X \in \wedge$ are everywhere dense in $\mathscr{H}(z)$. This only ensures that the system $\wedge$ is sufficiently large. Sometimes the fundamental family $\wedge$ will be supposed to satisfy the following stronger condition:

(c′) There exists a countable subset $\Lambda_0 = \{X_n\}$ of Λ such that for every $z \in \mathcal{Z}$, $X_n(z)$ are dense in $\mathscr{H}(z)$. In particular, this implies that all the Hilbert spaces $\mathscr{H}(z)$ are separable. (We will always assume that the stronger condition (c′) is valid though some of the results remain true without this supposition).

1.2 Notion of continuity

We proceed to construct the continuous sum of the spaces $\mathscr{H}(z)$. In our axioms relating to the fundamental family, we have not imposed any restrictions on its behaviour at ∞. Consequently it cannot be asserted that the $\|X(z)\|$ are square summable. Moreover, the class Λ may be too small (as they will be if we take them to be constants in our example) to be dense in $\mathscr{L}^2(\mathscr{H})$. This necessitates the extension of this family to the class of *continuous* vector fields by means of the

93 **Definition.** *A vector field X is continuous at a point ζ_0 if given an $\epsilon > 0$ there exists $Y \in \Lambda$ and a neighbourhood V of ζ_0 such that $\|X(\zeta) - Y(\zeta)\| < \epsilon$ for every $\zeta \in V$.*

Remark. When we take Λ to be constants in the example, this corresponds to the usual continuity of functions.

Proposition 1. *A vector field X is continuous if an only if $\|X\|$ is continuous and $\langle X, X_n\rangle$ is continuous for every $X_n \in \Lambda_0$.*

If X is continuous, trivially $\|X\|$ is continuous. Also if X and X' are continuous, then $\langle X, X'\rangle$ is continuous: in fact, for every $\epsilon > 0$ and every $\zeta \in \mathcal{Z}$, there exist a neighbourhood V of ζ and Y, $Y' \in \Lambda$ such that $\|X - Y\| < \epsilon$, $\|X' - Y'\| < \epsilon$ in V.

Hence

$$|\langle X, X'\rangle - \langle Y, Y'\rangle| \leq |\langle X - Y, X'\rangle| + |\langle Y, Y' - X'\rangle|$$

$\leq M\epsilon$ in V, where M is some constant. As $\langle Y, Y'\rangle$ is continuous, it follows that $\langle X, X'\rangle$ is also continuous. To prove the converse, it is enough to show that $\|X(\zeta) - X_n(\zeta)\|$ is continuous. But $\|X(\zeta) - X_n(\zeta)\|^2 = \|X\|^2 - 2Rl\langle X, X_n\rangle + \|X_n\|^2$, all continuous by our assumption.

A continuous vector field can be multiplied by a scalar continuous function without affecting its continuity.

Proposition 2. *The vector fields $\sum \varphi_i(\zeta) Y_i(\zeta)$ with $\varphi_i \in \mathscr{C}(\mathcal{Z})$ and $Y_i \in \Lambda$ are dense in the space of continuous vector fields with the topology of uniform convergence on compact sets.*

At each point x in a compact set K, there exists a neighbourhood A_x in which $\|X - Y_x\| < \epsilon$ for some $Y_x \in \Lambda$. We can extract a finite cover
94 $\{A_{x_i}\}$ from $\{A_x\}$ and take the partition of unity with respect to this cover. Hence there exist continuous functions ϕ_i such that $\|X - \sum \phi_i Y_{x_i}\| < \epsilon$ on k.

1.3 The space $L^2_\wedge$

Our next step is to construct the space $L^2_\wedge$ of square summable vector fields. We shall say that X belongs to $L^2_\wedge$ if given an $\epsilon > 0$ there exists a continuous vector field Y with compact support such that

$$\int^{*} \|X(\zeta) - Y(\zeta)\|^2 d\mu(\zeta) < \epsilon.$$

(We do not know a priori whether $\|X(\zeta) - Y(\zeta)\|^2$ is measurable or not and hence we can only consider the upper integral). In this space, we can define

$$\|X\|^2 = \int \|X(\zeta)\|^2 d\mu(\zeta) \text{ and}$$

$$\langle X, X' \rangle = \int \langle X(\zeta), X'(\zeta) \rangle d\mu(\zeta).$$

By passing to the quotient space modulo vector fields of norm 0, we get a Hilbert space (which again we denote by $L^2_\wedge$). As in the case of the theory of integration, we have of course to prove the completeness of $L^2_\wedge$, but there is no trouble in imitating the proof of the Riesz-Fisher theorem in this case.

$L^2_\wedge$ is the *continuous sum* of the $\mathscr{H}(\zeta)$ that we wished to construct.

1.4 Measurablility of vector fields

Definition. *A vector field X is said to be* measurable *if for every compact K and positive ϵ, there exists a set $K_1 \subset K$ such that $\mu(k - k_1) < \epsilon$ and X is continuous on K_1.*

Proposition 3. *A vector field X is measurable if and only if $\langle X, X_n \rangle$ is measurable for every $X_n \in \Lambda_0$.*

If X is measurable, $\langle X, X_n \rangle$ is continuous on K_1 and hence $\langle X, X_n \rangle$ **95**
is measurable. Conversely, let $\langle X, X_n \rangle$ be measurable. Then $\|X\| = \sup \dfrac{|\langle X, X_n \rangle|}{\|X_n\|}$ is also measurable. (Here we put $\dfrac{\langle X, X_n \rangle}{\|X_n\|} = 0$ if $\|X_n(\zeta)\| = 0$). But $\langle X, X_n \rangle$ are continuous outside a set $\{K - K_n\}$ of measure $< \epsilon/2^n$. If $K_\infty = \cap K_n$, it is obvious that $\mu(K - K_\infty) < \epsilon$ and all the functions $\langle X, X_n \rangle$ are continuous on K_∞. Hence $\|X\|$ is continuous on K_∞. By prop. 1, Ch. 1.2, prop. 3 follows.

In particular, this shows that strong measurablility and weak measurablility are the same in a separable Hilbert space.

Proposition 4. *A vector field X belongs to $L^2_\wedge$ if and only if X is measurable and $\int \|X(\zeta)\|^2 d\mu(\zeta) < \infty$.*

The proof is exactly similar to that for ordinary integration of scalar functions.

1.5 Orthogonal basis

We now find an orthogonal basis for the space of vector fields by Schmidt's orthogonalisation process. We can of course define

$$e_1 = \frac{X_1}{\|X_1\|}, \quad e_2 = \frac{X_2 - \langle X_2, e_1\rangle e_1}{\|X_2 - \langle X_2, e_1\rangle e_1\|}, \ldots$$

where we put $e_1 = 0$ whenever the numerator is zero [20]. However, the process is defective as the basic elements are not continuous, and the following orthogonalisation seems preferable:

Put

$$e_1 = X_1; \quad e_2 = X_2 - \langle X_2, e_1\rangle e_1,$$

$e_n = x_n$ - orthogonal projection of X_n on the space generated by

$$e_1, \ldots e_{n-1}$$

96 These are of course continuous vector fields. At each point ζ, the nonzero $e(\zeta)$ from an orthogonal basic for $\mathscr{H}(\zeta)$.

1.6 Operator fields

Let X be a square summable vector field and $A(\zeta)$ an operator on $\mathscr{H}(\zeta)$. Then we may define $(AX)(\zeta) = A(\zeta)X(\zeta)$ and get another vector field AX. The assignment to each ζ of an operator of $\mathscr{H}(\zeta)$ is called an *operator field*. However, in order that A may act as operator on $L^2_\wedge$, we have to make sure that AX is also square summable. Obviously some restrictions an $A(\zeta)$ will be necessary to achieve this. In the particular case when all the spaces $\mathscr{H}(\zeta)$ are the isomorphic, $A(\zeta)$ is a map of $\mathcal{Z}$ into the set of operators of the Hilbert space $\mathscr{H}$. Now, Hom $(\mathscr{H}, \mathscr{H})$ can be provided with uniform, strong or weak topologies and we may restrict A to be continuous for one of these topologies. However, the first topology is too strong and consequently the space 'good operator fields' will become too restricted to be of any utility. Therefore, in this particular case, we may require the map A to be continuous in the weak or strong topology as the case may be. To transport this definition to the general case, we have to reformulate this in a suitable way. Take for instance the requirement of strong continuity. This is equivalent to requiring that (a) $A(\zeta)$ is locally bounded (i.e. bounded on compact sets) and (b) for every continuous function $f(\zeta)$ of $\mathcal{Z}$ with values in $\mathscr{H}$, $A(\zeta)f(\zeta)$ is continuous.

This motivates the following

Definition. *An operator field $\zeta \to A(\zeta)$ is said to be* strongly continuous *if (a) $A(\zeta)$ is locally bounded and (b) for every continuous vector field $X(\zeta)$, $A(\zeta)X(\zeta)$ is also continuous.* **97**

Similar considerations for weak continuity give us the following

Definition. *An operator field A is said to be* weakly continuous *if (a) it is locally bounded, and (b) for any two continuous vector fields $X(\zeta), Y(\zeta)$, the map $\zeta \to \langle A(\zeta)X(\zeta), Y(\zeta)\rangle$ is continuous.*

Proposition 5. *An operator field A is* strongly continuous *if and only if A is locally bounded and $\zeta \to A(\zeta)X_n(\zeta)$ is continuous for every $X_n \in \wedge_0$.*

This follows straight from the definition.

The strongly continuous operator fields form an algebra which is not however self adjoint, while the weakly continuous operator fields do not even form an algebra. In order to ensure that our definitions are good enough, we should know if there exist sufficiently many non-scalar continuous operator fields. This is answered by the following

Theorem 1. *Let K be a compact subset of $\mathcal{Z}$ and $Y_1, \ldots Y_n, Z_1, \ldots Z_n$, $2n$ continuous vector fields such that $Y_i(\zeta)$* are linearly independent for every $\zeta \in K$. *Then there exists a continuous operator field A such that $A(\zeta)Y_j(\zeta) = Z_j(\zeta)$ for every $\zeta \in K$ and such that $A(\zeta)^*$ is also continuous.*

We first remark that we can as well assume that $K = \mathcal{Z}$. For, $Y_i(\zeta)$ are linearly independent if and only if $\Delta = \det\langle Y_i(\zeta), Y_j(\zeta)\rangle \neq 0$. This being a continuous function of K, there exists a compact neighborhood V of K such that $|\Delta| \geq \alpha > 0$ on V. If the theorem were true for a compact space, there exists an operator $A_\circ$ on V satisfying the conditions
98 above. Set $A = \varphi A_0$ where φ is a continuous function 1 on K, 0 outside V; then A verifies all the conditions.

Now let $P(\zeta)$ be the space spanned by $Y_i(\zeta)$. We can then define $A(\zeta) = 0$ on the orthogonal complement of $P(\zeta)$ and $A(\zeta)Y_i(\zeta) = Z_i(\zeta)$ and extend A by linearity. If π is the projection of $\mathscr{H}(\zeta)$ onto $P(\zeta)$, we have $\langle X, Y_j\rangle = \langle \pi X, Y_j\rangle$ and if $\pi X = \sum \xi_k Y_k$, then $\langle X, Y_j\rangle = \sum \xi_k \langle Y_k, Y_j\rangle$ with $|\det\langle Y_j, Y_k\rangle| = |\Delta| \geq \alpha > 0$. On solving the linear equations for ξ_k, we get $\xi_k = \Delta_k/\Delta$. Since the functions occurring in the linear equations are continuous, ξ_k are continuous functions and we have $|\xi_k| = |\dfrac{\Delta_k}{\Delta}| \leq M\|X\|$ where M is a constant. Hence $A(\zeta)X(\zeta) = \sum \xi_k(\zeta) Z_k(\zeta)$ is continuous for every continuous vector field X. A is locally bounded by virtue of the above remark and hence A is a continuous operator field.

It is obvious that A^* is also locally bounded. Now, A^* maps the whole of $\mathscr{H}(\zeta)$ onto $P(\zeta)$ and hence $A^*(\zeta)X(\zeta) = \sum \eta_k(\zeta) Y_k(\zeta)$ where the $\eta(\zeta)$ are given by

$$\begin{aligned}
\sum \eta_k(\zeta)\langle Y_k, Y_j\rangle &= \langle \sum \eta_k(\zeta) Y_k, Y_j\rangle \\
&= \langle A^*(\zeta)X(\zeta), Y_j(\zeta)\rangle \\
&= \langle X(\zeta), A(\zeta)Y_j(\zeta)\rangle \\
&= \langle X(\zeta), Z_j(\zeta)\rangle
\end{aligned}$$

The Gram determinant in this case also is Δ. Hence by the same argument as before, A^* is a continuous operator field.

1.7 Measurablility of operator fields

99

Definition. *An operator field A is said to be* measurable *if (a) it is almost everywhere locally bounded, and (b) for every compact K and positive ϵ, there exists $K_1 \subset K$ such that $\mu(K - K_1) < \epsilon$ and $A(\zeta)$ is continuous on K_1.*

If (e_p) form an orthogonal basis (Ch. 1.5), then any operator can be expressed by means of a matrix with respect to this base. The matrix coefficients are only $\langle Ae_p, e_q\rangle$. We then have

Proposition 6. *A locally bounded operator field A is measurable if and only if the matrix coefficients of $A(\zeta)$ are measurable functions on $\mathcal{Z}$.*

That a measurable operator field satisfies the above condition is a trivial consequence of the definition. Conversely, if $\langle Ae_p, e_q\rangle$ is measurable for every q, by prop. 3, ch. 1.4, Ae_p is measurable for every p. As in Prop. 3, Ch. 1.4, we can find a compact set K_1 such that the Ae_p are continuous on k_1 and $\mu(K - K_1) < \epsilon$.

1.8 Decomposed operators

Let $A(\zeta)$ be a measurable operator field bounded almost everywhere. For every $X(\zeta) \in L^2_\wedge$, $A(\zeta)X(\zeta)$ is also a measurable vector field and $\|A(\zeta)X(\zeta)\| \leq \|A(\zeta)\|_\infty \|X(\zeta)\|$. Hence $\int \|A(\zeta)X(\zeta)\|^2 d\mu$ exists and we have $\|AX\|_{L^2_\wedge} \leq \|A(\zeta)\|_\infty \|X\|_{L^2_\wedge}$. In other words, A is a continuous operator on $L^2_\wedge$ and $\|A\| \leq \|A(\zeta)\|_\infty$.

If A is an operator in $L^2_\wedge$ which arises from an operator field, we say that it is a *decomposed operator* and write $A \sim \int_{\mathcal{Z}} A(\zeta)$. In particular if we take $A(\zeta) = f(\zeta)$. Identity where $f \in L^\infty(\mu)$ we obtain a decomposed operator $M_f \sim \int f(\zeta)\,\mathrm{Id}$. This is called a *scalar decomposed operator* 100
on $L^2_\wedge$. We denote the space of all such operators by $\mathscr{M}$. This is a self adjoint subalgebra of $\mathrm{Hom}(L^2_\wedge, L^2_\wedge)$. For,

$$\begin{aligned}\langle M_f X, Y\rangle &= \int \langle f(\zeta)X(\zeta), Y(\zeta)\rangle d\mu(\zeta)\\ &= \int \langle X(\zeta), \overline{f(\zeta)}Y(\zeta)\rangle d\mu(\zeta)\\ &= \langle X, M_{\bar{f}}Y\rangle\end{aligned}$$

Hence $M_f^* = M_{\bar{f}}$.

We now give a characterisation of decomposed operators in terms of this algebra $\mathscr{M}$ by means of

Theorem 2. *The set of decomposed operators is precisely the commutator of $\mathscr{M}$. If $A \sim \int A(\zeta)$, then $\|A\| = \|A(\zeta)\|_\infty$.*

In fact, if A is a decomposed operator, it commutes with all the elements of $\mathscr{M}$ and hence belongs to $\mathscr{M}'$.

Conversely let A be an operator in $L^2_\wedge$ which commutes with M_f for every $f \in L^\infty(\mu)$. Let K be a compact subset of $\mathcal{Z}$. Then $\chi_K e_n(\zeta) \in L^2_\wedge$, $\{e_n\}$ being the orthogonal basis (Ch. 1.5). Let H be a compact nighbourhood of K. Then

$$\begin{aligned} A(\chi_K e_n) &= A(\chi_K \chi_H e_n) \\ &= AM_{\chi_K}(\chi_H e_n) \\ &= M_{\chi_K} A(\chi_H e_n) \text{ by assumption} \\ &= \chi_K A(\chi_H e_n) \text{ almost everywhere.} \end{aligned}$$

It is obvious that at the intersection of any two compact sets K, K^1, the vector fields $A(\chi_K e_n)$ and $A(\chi_{K^1} e_n)$ coincide almost everywhere. Hence there exists a vector field $A(e_n)$ such that $\chi_K(e_n)A(e_n)(\zeta) = A(\chi_K e_n)(\zeta)$ almost everywhere. Thus we have a countable family of relations and hence there exists a set N of measure 0 such that $\chi_K A(e_n)(\zeta) =$
101 $A(\chi_K e_n)(\zeta)$ for every $\zeta \notin N$. Since the X_n in the fundamental sequence $\wedge_0$ are finite linear combinations of the e_n we have $\chi_K A(X_n)(\zeta) = A(\chi_K X_n)(\zeta)$ for every $\zeta \notin N$. Also $\int_K \|A(X_n(\zeta))\|^2 d\mu(\zeta) = \|A(\chi_K X)\|^2 \leq \|A\|^2 \int_K \|X_n\|^2 \mu(\zeta)$ for every compact set K. Hence the set of elements ζ such that $\|A(X_n)(\zeta)\|$ is strictly greater than $\|A\|\,\|X_n(\zeta)\|$ is of measure zero.

We have thus proved that the algebra of all decomposed operators is $\mathscr{M}'$ and we know that $\mathscr{M} \subset \mathscr{M}''$. We can moreover assert

Theorem 3. *$\mathscr{M}$ is a weakly closed algebra.*

In fact, let (e_n) be an orthogonal basis of $L^2_\wedge$. Then we define for any two integers p, q, a decomposed operator $H_{p,q}$ by setting

$$\begin{aligned} H_{p,q}(\zeta)e_n(\zeta) &= 0 \text{ if } n \neq p \text{ or } q \\ H_{p,q}(\zeta)e_p(\zeta) &= \|e_p(\zeta)\|^2 e_q(\zeta). \end{aligned}$$

$$\text{and } H_{p_q}(\zeta)e_q(\zeta) = \|e_q(\zeta)\|^2 e_p(\zeta).$$

If B commutes with every $A \in \mathscr{M}'$, in particular, it commutes with all the $H_{p,q}$ therefore $B(\zeta)$ commutes with $H_{p,q}(\zeta)$ for every fixed p, q and almost every ζ. Since the $H_{p,q}$ are countable, $B(\zeta)$ commutes with $H_{p,q}(\zeta)$ for almost every (ζ) and for all p, q. Hence outside a set of measure zero, we have

$$\begin{aligned}
\langle B(\zeta)e_p, e_p\rangle &= \frac{\langle Be_p, H_{n,q}, e_n\rangle}{\|e_n\|^2} \quad \text{for every } \ p, q. \\
&= \langle BH_{n,q}\ e_p,\ \frac{e_n}{\|e_n\|^2}\rangle \quad \text{(since the } \ H_{n,q} \ \text{ are Hermitian)} \\
&= 0 \ \text{ if } \ p \neq q.
\end{aligned}$$

On the other hand, **102**

$$\begin{aligned}
\langle B(\zeta)e_p, e_p\rangle &= \langle BH_{n,p}e_p, \frac{e_n}{\|e_n\|^2}\rangle \\
&= \frac{\|e_p\|^2}{\|e_n\|^2}\langle Be_n, e_n\rangle
\end{aligned}$$

This shows that $B(\zeta)$ is a scalar operator for almost every (ζ). Hence $B = \int B(\zeta)d(\zeta)$ is a scalar decomposed operator and hence $\epsilon\mathscr{M}$. This shows that $\mathscr{M} = \mathscr{M}''$ and by Th. 2, Ch. 5.6, Part II, is weakly closed.

Chapter 11

Continuous sum of Hilbert spaces - II

2.1

103

In the last chapter, given a family $\mathscr{H}(\zeta)$ of Hilbert spaces indexed by elements ζ of a locally compact space $\mathcal{Z}$, we constructed the continuous sum $\mathscr{H} = L^2_\wedge$. Now we shall decompose a Hilbert space $\mathscr{H}$ into a continuous sum with reference to a given comutative, weakly closed *-subalgebra m of Hom $(\mathscr{H}, \mathscr{H})$.

$\mathscr{M}$ satisfies Gelfand's conditions and is hence isomorphic and isometric to the space $\mathscr{C}(\Omega)$ of continuous complex valued functions on a compact space Ω which is called the *spectrum* of $\mathscr{M}$. By this isomorphism, every continuous linear form on $\mathscr{M}$ is transformed into a continuous liner form on $\mathscr{C}(\Omega)$ or, what is the same, a measure on the space Ω. In particular, the continuous linear form $\langle Mx, y\rangle$ where $x, y \in \mathscr{H}$ gives rise to a measure which we shall denote by $d\mu_{x,y}$ i.e. we have $\langle Mx, y\rangle = \int_\Omega \hat{M}(\chi) d\mu_{x,y}(\chi)$. This measure is called the *spectral measure* associated to x, y. This depends linearly on x and anti linearly on y.

Let $\mathscr{M}'$ be the commutator of $\mathscr{M}$. We now assume that there exists an element $a \in \mathscr{H}$ such that the set $\{Aa : A \in \mathscr{M}'\}$ is dense in $\mathscr{H}$. This assumption however is *not* a real restriction on our theory. For otherwise, we can decompose $\mathscr{H}$ into a discrete sum of Hilbert spaces

each of which satisfies the above condition. Let a_1 be any element of $\mathscr{H}$ and $\mathscr{H}_1$ the closed subspace generated by Aa_1, $A \in \mathscr{M}'$. $\mathscr{H}_1$ is
104 invariant under both $\mathscr{M}$ and $\mathscr{M}'$. If $\mathscr{H}_1^{\perp}$ is the orthogonal complement of $\mathscr{H}_1$, we can carry out the same process for $\mathscr{H}_1^{\perp}$, and so on. Thus $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2 \oplus ..$ where all the $\mathscr{H}_i$ satisfy the above condition.

Now we show that the spectral measure $d\mu_{a,a}$ has Ω as its support. In fact, if f is a positive continuous function on Ω such that its integral is 0, then $f = 0$. We can write f as $|\hat{M}|^2$ and we have $\int |\hat{M}|^2 d\mu_{(a,a)} = \langle M * Ma, a \rangle = \|Ma\|^2$. Hence if $\int |\hat{M}|^2 d\mu_{(a,a)} = 0$, $Ma = 0$ or $AMa = 0$ for every $A \in \mathscr{M}'$, or again $M(Aa) = 0$. Since the As are dense in $\mathscr{H}$, it follows that $M = 0$. Moreover, this shows that the measure $d\mu_{a,a}$ is positive.

Proposition 1. *Corresponding to any operator $A \in \mathscr{M}'$ there exits one and only one continuous function φ_A on Ω such that $d\mu_{(Aa,a)} = \varphi_A d\mu_{a,a}$ and we have $\|\varphi_A\| \leq \|A\|$.*

In fact, it is enough to prove the proposition when A is a positive Hermitian operator since any operator is a finite linear combination of them. Under this assumption we have

$$\begin{aligned} \int |\hat{M}|^2 d\mu_{Aa,a} &= \langle M^* MAa, a \rangle \\ &= \langle AMa, Ma \rangle \\ &=\leq \|A\| \langle Ma, Ma \rangle \\ &= \|A\| \int |\widehat{M}|^2 d\mu_a, a \end{aligned}$$

Now $d\mu_{Aa,a}$ is positive and $d\mu_{Aa,a} \leq k\, d\mu_{a,a}$. Therefore, by Lebesgue - Nikodym theorem, there exist a measurable function $\psi \in L^{\infty}(\mu_{a,a})$ such that $d\mu_{Aa,a} = \psi_A d\mu_{a,a}$. We also have $\|\psi_A\|_{\infty} \leq \|A\|$.

The proof is complete if we prove the

105 **Lemma .** *For every bounded measurable function ψ on Ω there exists one and only one continuous function φ such that $\varphi = \psi$ a.e.*

By the definition of $\mu_{x,y}$, we have $\|\mu_{x,y}\| \leq \|x\|\ \|y\|$. Therefore $\int \psi(\chi) d\mu_{x,y}(\chi) \leq \|\psi\|_{\infty} \|x\|\ \|y\|$.

Thus $\int \psi(\chi) d\mu_{x,y}(\chi)$ is a sesquilinear map which is continuous in each of the variables x, y. As a consequence of Riesz representation theorem, there exits a linear operator T such that $\int \psi(\chi) d\mu_{x,y}(\chi) = \langle Tx, y\rangle$. We now show that T commutes with every element of $\mathscr{M}'$. For, if $M \in \mathscr{M}'$, we have

$$\begin{aligned}\langle MTx, y\rangle &= \langle Tx, M^* y\rangle \\ &= \int \psi(\chi) d\mu_{X,M^*y}(\chi) \\ &= \int \psi(\chi) d\mu_{Mx,y}(\chi) (\text{ since } M \in \mathscr{M}') \\ &= \langle TMx, y\rangle\end{aligned}$$

Since $\mathscr{M}$ is weakly closed, $T \in \mathscr{M}$.

We have $\int \psi(\chi)\hat{M} d\mu_{a,a} = \int \hat{T}\hat{M} d\mu_{a,a}$ and hence $\hat{T} = \psi(\chi) a.e$ - This proves the lemma.

The function $\varphi_A \in \mathscr{C}(\Omega)$ thus constructed satisfies the following properties:

(a) $\varphi_{\lambda A+\mu B} = \lambda\varphi_A + \mu\varphi_B$.

For,

$$\begin{aligned}\int \varphi_{\lambda A+\mu B} d\mu_{a,a} &= \langle(\lambda A + \mu B)a, a\rangle = \lambda\langle Aa, a\rangle + \mu\langle Ba, a\rangle \\ &= \int (\lambda\varphi_A + \mu\varphi_B) d\mu_{a,a}\end{aligned}$$

(b) $\varphi_{A^*} = \overline{\varphi}_A$.

For,

$$\begin{aligned}\int \varphi_{A^*} d\mu_{a,a} &= \langle A^* a, a\rangle = \overline{\langle Aa, a\rangle} \\ &= \int \overline{\varphi}_A d\mu_{a,a}.\end{aligned}$$

(c) $\varphi_{MA} = \hat{M}\varphi_A$ **106**

$$\int \varphi_{MA} d\mu_{a,a} = \langle MAa, a\rangle$$
$$= \int \hat{M} d\mu_{Aa,a} = \int \hat{M}\varphi_A d\mu_{a,a}$$

(d) $\varphi_{A^*A} \geq 0$

In fact, if f is a positive function, we can express it as $|\hat{M}|^2$.

$$\int \hat{M}^2 \varphi_{A^*A} d\mu_{a,a} = ||AMa||^2 \geq 0.$$

(e) $|\varphi_A| \leq ||A||$

This has been proved in the Prop. 1, Ch. 2.1.

2.2

In order to get a decomposition of $\mathscr{H}$ into a continuous sum, we need the following

Lemma . *Let $\mathcal{B}$ be any *-subalgebra of* Hom$(\mathscr{H}, \mathscr{H})$ *which is uniformly closed. Let φ be a positive continuous linear form on $\mathcal{B}$ such that $\varphi(A^*) = \overline{\varphi(A)}, \varphi(A^*A) \geq 0$ and $|\varphi(A)| \leq k||A||$. To φ we can make correspond a canonical unitary representation of the algebra $\mathcal{B}$.*

In fact, because of the conditions we have imposed on φ, $\varphi(B^*A)$ is a positive Hermitian form on $\mathscr{B}$. Hence we have by the Cauchy-Schwarz inequality $|\varphi(B^*A)||^2 \leq \varphi(B^*B)\varphi(A^*A)$. Therefore, $\varphi(B^*B) = 0$ if and only if $\varphi(B^*A) = 0$ for every $A \in$ Hom$(\mathscr{H}, \mathscr{H})$. Hence $\varphi(B^*B) = 0 \Rightarrow \varphi((AB)^*AB) = 0$. It follows that the set N of elements B such that $\varphi(B^*B) = 0$ is a left ideal. On the space B/N, φ is transformed into a positive definite Hermitian from and consequently φ gives rise to a scalar product. The completion of this space under this norm shall be denoted
107 $\mathscr{H}_\varphi$. The canonical map $\mathscr{B} \to \mathscr{H}_\varphi$ is continuous since $\varphi(B^*B) \leq k||B||^2$. On the other hand we have also a map $f : \mathscr{B} \to$ Hom$(\mathscr{H}_\varphi, \mathscr{H}_\varphi)$ defined by $f(A) = U_A$ where $U_A(\dot{B}) = \widehat{AB}$. We show that this as also continuous. Consider $B^*||A||^2B - B^*A^*AB = B^*(||A||^2 - A^*A)B$; $(||A||^2 - A^*A)$ is positive

Hermitian and hence so is $H = B^*||A||^2B - B^*A^*AB$. $H^{\frac{1}{2}}$ is the uniform limit of polynomials in H and since $\mathscr{B}$ is uniformly closed, $H^{\frac{1}{2}} \in \mathscr{B}$. Therefore we have $\varphi_H = \varphi_{H^{\frac{1}{2}}H^{\frac{1}{2}}} \geq 0$ by assumption. We have now proved that $\varphi(B^*A^*AB) \leq ||A||^2\varphi(B^*B)$. This shows that the map U_A is continuous. This can be extended to an operator of $\mathscr{H}$. We have also shown that $||U_A|| \leq ||A||$. It remains to prove that this is a unitary representation. Let $A, B, C \in \mathscr{B}$. Then $\langle U_A\dot{B}, \dot{C}\rangle = \varphi((A^*C)^*B) = \langle \dot{B}, U_A * C\rangle$. Hence $U_A^* = U_A$ which shows that this is unitary.

In fact all the above considerations hold for a Banach algebra with involution.

2.3

After this lemma in the general set-up, we revert to our decomposition of $\mathscr{H}$ into a continuous sum. For every fixed $\chi \in \Omega$, $\varphi_A(\chi)$ is a positive continuous linear from on $\mathscr{M}'$ which satisfies all the conditions of the lemma. Hence we have a unitary representation of $\mathscr{M}'$ in the Hilbert space $\mathscr{H}_\chi = \dfrac{\mathscr{M}'}{N_\chi}$ where $N_\chi = \{A : \varphi_{A^*A}(\chi) = 0\}$. In other words, to each point $\chi \in \Omega$ we have assigned a Hilbert space $\mathscr{H}_\chi$. If $M \in \mathscr{M}$, since $\varphi_{MA} = \hat{M}\varphi_A$, we have $U_M(\chi) = \hat{M}(\chi)$ identity. For, $\langle U_M(\chi)\dot{B}, \dot{C}\rangle = \varphi_{C^*MB}(\chi) = \hat{M}(\chi)\varphi_{C^*B}(\chi) = \hat{M}(\chi)\langle\dot{B}, \dot{C}\rangle$. We have now all the data necessary for the construction of a continuous sum except **108**
the fundamental family of vector fields. We have so far operated with $\mathscr{M}'$, but, in practice, $\mathscr{M}'$ is very large. For instance it is not in general separable in the norm. So we assume given a subalgebra $\mathscr{A}$ of $\mathscr{M}'$ such that

(a) $\mathscr{A}$ is uniformly closed.

(b) There exists a sequence $A_n \in \mathscr{A}$ such that $\mathscr{A}$ is generated by the A_n and $\mathscr{A} \cap \mathscr{M}$.

(c) There exists $a \in \mathscr{H}$ such that $\{Aa : a \in \mathscr{H}\}$ is dense in $\mathscr{H}$. It is actually this algebra $\mathscr{A}$ which is in general given and the problem will then be to find an $\mathscr{M} \subset \mathcal{A}'$ such that the above conditions are satisfied.

We have a map $\mathcal{A} \to \mathrm{Hom}(\mathscr{H}_\chi, \mathscr{H}_\chi)$. However it is possible that there are more functions φ than would be absolutely necessary. That is, there may exist elements χ, χ' in Ω such that $\varphi_A(\chi) = \varphi_A(\chi')$ for every $A \in \mathcal{A}$. In this case, we have two points χ, χ' in the base space Ω which are in some sense equivalent with respect to $\mathcal{A}$. Therefore we introduce an equivalence relation R in Ω by setting $\chi \sim \chi'$ whenever $\varphi_A(\chi) = \varphi_A(\chi')$ for every $A \in \mathcal{A}$. This is a closed equivalence relation and $\Omega/R = \mathcal{Z}$ is a compact Hausdorff space. The same procedure for $\mathcal{Z}$ and $\mathcal{A}$ as for Ω and $\mathscr{M}'$ gives a Hilbert space $\mathcal{H}_\zeta$ at each point $\zeta \in \mathcal{Z}$ and a continuous representation of the algebra $\mathcal{A}$ in $\mathrm{Hom}(\mathscr{H}_\zeta, \mathscr{H}_\zeta)$. The image of the measure $d\mu_{a,a}$ by the canonical map $\Omega \to \mathcal{Z}$ is denoted by μ. At each point ζ we have a map $\mathcal{A} \to \mathcal{H}_\zeta$ and hence for a fixed $A \in \mathcal{A}$ we obtain a vector field. This family of vector fields is the *fundamental family* we sought to construct. In fact,

(a) They constitute a vector space, since $\mathcal{A}$ is an algebra.

109 (b) $\|X_A(\zeta)\| = \varphi_{A*A}(\zeta)^{\frac{1}{2}}$ and hence $\|X_A(\zeta)\|$ is continuous.

(c) For each ζ, $X_A(\zeta)$ in everywhere dense since $\mathscr{H}_\zeta$ is only the completion of the space of $X_A(\zeta)$.

(c′) Consider $\sum M_i B_i$ where $M \in \mathcal{A} \cap \mathscr{M}$ and B is a finite product $Ai_1 Ai_2 \ldots Ai_p$ of the A_n. Then $X_{\sum M_i B_i} = \sum U_{M_i} X_{B_i}$ with U_M being scalars. $\{X_{B_i}\}$ is only a countable family and the vectors $X_{\sum M_i B_i}(\zeta)$ are dense in $\mathscr{H}(\zeta)$. Consequently the countable family $X_{\sum \alpha_i B_i}(\zeta)$ where the α_i are complex numbers with real and imaginary parts rational is also dense in $\mathscr{H}(\zeta)$.

Thus we have now all the data for the construction of a continuous sum $\mathscr{L}^2_\wedge$. of course we have still to establish that $L^2_\wedge = \mathscr{H}$. In fact, since $\mathcal{Z}$ is compact, every continuous vector field is square summable. Therefore, we have

$$\|X_A\|^2 = \int \|X_A(\zeta)\|^2 d\mu(\zeta) = \int \varphi_{A*A}(\chi) d\mu_{a,a}(\chi) = \|Aa\|^2$$

Also $Aa = 0$ implies $X_A = 0$. Therefore, the map $Aa \to X_A$ is an isometry of a dense subspace of $\mathscr{H}$ and hence can be extended to an isometry $J : \mathscr{H} \to L^2_\wedge$. It remains to prove that this map is surjective. We have seen (Prop. 2, Ch. 1.2) that vector fields of the form $\sum \varphi_i(\zeta) Y_i(\zeta)$ where $\varphi_i(\zeta)$ are continuous functions of ζ are dense in $L^2_\wedge$. Therefore it is enough to prove that $J(\mathscr{H})$ contains all such elements (the image of $J(\mathscr{H})$ being closed in $L^2_\wedge$). Let $\mathscr{M}_0$ be the self adjoint subalgebra of $\mathscr{M}$ consisting of elements M such that $\hat{M}$ is constant on cosets modulo the equivalence relation R. Consequently, $\hat{M}$ may be considered as a continuous map on $\mathcal{Z}$. But we have $\| \sum M_i A^a_i \|^2 = \int_{\mathcal{Z}} \| \sum \hat{M}_i X_A \|^2 d\mu$ and therefore the map $J^1 : \sum M_i A^a_i \to \sum \hat{M}_i X_{A_i}$ is an isometry which coincides with J on the elements Aa. This shows that $\sum \hat{M}_i X_{A_i} \in J(\mathscr{H})$ **110**
and hence $L^2_\wedge = J(\mathscr{H})$.

Hereafter we shall identify $L^2_\wedge$ with $\mathscr{H}$. We now assert that $\mathcal{A}$ is contained in the space of decomposed operators on $\mathscr{H}$. In fact, we will show that $A = \int U_A(\zeta)$. We have already proved (Ch. 2.2) that $\|U_A(\zeta)\|| \leq \|A\|$ and $U_A(\zeta)$ is hence bounded. Also $U_A(\zeta) X_B(\zeta) = X_{AB}(\zeta)$ is again a continuous vector field and by Prop. 5, Ch. 1.6, $U_A(\zeta)$ is a continuous operator field. If now $\bar{A} = \int U_A(\zeta)$, then $\bar{A}Ba = U_A(\zeta) X_B(\zeta) = X_{AB}(\zeta) = ABa$ by our identification for every $B \in \mathcal{A}$. Since $\{Ba : B \in \mathcal{A}\}$ is dense in $\mathscr{H}$, we have $\bar{A} = A$. In other words, every operator $A \in \mathcal{A}$. is decomposable into a *continuous* operator field.

Now, we have another algebra $\mathscr{M}$ of operators on $\mathscr{H}$. It is natural to expect then $\mathscr{M}$ consists of scalar decomposed operators. It is of course true, but the proof is not obvious. As before, let $\mathscr{M}_0$ be the subalgebra of $\mathscr{M}$ composed of elements M such that $\hat{M} \in \mathscr{C}(\mathcal{Z})$. We first prove that $\mathscr{M}_0$ consists of scalar decomposed operators. Let $B, C \in \mathcal{A}$. Then

$$\begin{aligned}\langle MBa, Ca\rangle = \langle MC^*Ba, a\rangle &= \int_\Omega \bar{M}(\chi)d\mu_{C^*Ba,a}(\chi)\\ &= \int \hat{M}(\chi)\varphi_{C^*B}(\chi)d\mu_{a,a}(\chi)\\ &= \int_{\mathcal{Z}} \hat{M}(\zeta)\varphi_{C^*B}(\zeta)d\mu(\zeta) \quad \text{(since the continuous}\\ &\qquad\qquad \text{functions are constant}\\ &\qquad\qquad \text{on the equivalence classes)}\\ &= \int'_{\mathcal{Z}} \hat{M}(\zeta)\langle X_B(\zeta), X_C(\zeta)\rangle d\mu(\zeta) \ \text{ by definition of the norm.}\end{aligned}$$

111 That is to say that $M = \int_{\mathcal{Z}} \hat{M}(\zeta)$. Identity. We now extend this result to every element $M \in \mathscr{M}$. Since we know that the space of scalar operators is weakly closed, it suffices to prove that $\mathscr{M} \subset \mathscr{M}_0''$. Again by the Hahn-Banach theorem, it is enough to show that any weakly continuous linear form which is zero on $\mathscr{M}_0$ (and hence on $\mathscr{M}_0''$) is also zero on $\mathscr{M}$. But any weakly continuous linear form on $\mathrm{Hom}_S(\mathscr{H}, \mathscr{H}_W)$ is of the form $U \to \sum_{i=1}^{n}\langle UX_i, Y_i\rangle$. if $\sum\langle MX_i, Y_i\rangle = 0$ for every $M \in \mathscr{M}_0$, then $\sum\int_{\mathcal{Z}} \hat{M}(\zeta)\langle X_i(\zeta), Y_i(\zeta)\rangle d\mu(\zeta) = 0$. Hence $\sum\langle X_i(\zeta), Y_i(\zeta)\rangle = 0$ for almost every ζ in $\mathcal{Z}$, or again $\sum\langle X_i(\pi(\chi)), Y_i(\pi(\chi))\rangle = 0 a.e.$ on Ω where π is the canonical map $\Omega \to \mathcal{Z}$. Therefore, $\sum\langle MX_i, Y_i\rangle = \int_\Omega \hat{M}(X)\sum\langle X_i(x), Y_i(x)\rangle d\mu_{a,a} = 0$ for every $M \in \mu$. This completes the proof of our assertion.

2.4 Irreducibility of the components - Mautner's theorem

Finally it remains to show that the unitary representations of the algebra $\mathscr{A}$ in the $\mathscr{H}_\zeta$ are irreducible. The algebra $\mathscr{M}$ is at our choice and we are interested in taking it as large as possible. Thus we assume that $\mathscr{M}$ is a maximal commutative subalgebra of $\mathscr{A}'$ and obtain the

Theorem 1 (Mautner)**.** *Let $\mathscr{A}$ be any uniformly closed *-subalgebra of* $\mathrm{Hom}\,(\mathscr{H}, \mathscr{H})$ *such that*

(a) *there exists a sequence A_n which generates $\mathcal{A}$;*

(b) *there exists an element $a \in \mathscr{H}$ such that the set $\{Aa : a \in \mathcal{A}\}$ is dense in $\mathscr{H}$.*

*Let $\mathscr{M}$ be any maximal commutative *-subalgebra of $\mathcal{A}'$. Then in the decomposition of $\mathscr{H}$ into the continuous sum of the $\mathscr{H}_\zeta$ with respect to $\mathscr{M}$ and $\mathcal{A}$, almost every representation $U_A(\zeta)$ of $\mathcal{A}$ in $\mathscr{H}(\zeta)$ is* **112**
irreducible.

Let $\{e_n\}$ be the orthogonal basics given in Ch. 1.5 with respect to the fundamental sequence $\wedge_0$ and let $\mathscr{M}_0$ be the subset $\{M \in \mathscr{M} : \hat{M} \in \mathscr{C}(\mathcal{Z})\}$ of $\mathscr{M}$. If $\mathscr{B}$ be the algebra generated by $\mathcal{A}$ and $\mathscr{M}_0$, then $\mathscr{M} = \mathscr{B}'$. In fact, we have seen that $\mathscr{M} \subset \mathscr{M}_0''$ (Ch. 2.3) and therefore $\mathscr{M}' = \mathscr{M}_0'$. Hence $\mathscr{B}' = \mathcal{A}' \cap \mathscr{M}_0' = \mathcal{A}' \cap \mathscr{M}'$ and $\mathcal{A}' \cap \mathscr{M}' = \mathscr{M}$, $\mathscr{M}$ being a maximal subalgebra. We define for any two integers p, q as in theorem 3, Ch. 1.8, Hermitian decomposed operators $H_{p,q}$ on $\mathscr{H}$ such that

$$\begin{aligned} &H_{p,q}(\zeta)e_n(\zeta) = 0 \ \text{ if } \ n \neq p \ \text{ or } \ q; \\ &H_{p,q}(\zeta)e_p(\zeta) = \|e_p(\zeta)\|^2 e_q(\zeta), \ \text{ and} \\ &H_{p,q}(\zeta)e_q(\zeta) = \|e_q(\zeta)\|^2 e_p(\zeta). \end{aligned}$$

We have already seen (Ch. 1.8) that any operator which commutes with all the $H_{p,q}(\zeta)$ is a scalar operator. Now, $H_{p,q}$ is bounded since we have $\|H_{p,q}(\zeta)\| \leq \|e_p(\zeta)\| \, \|e_q(\zeta)\| \leq \|e_p\| \, \|e_q\|$. $H_{p,q}$ is continuous since it transforms every vector filed of type $e_j(\zeta)$ into another continuous vector field (Prop. 5, Ch. 1.6). Now $H_{p,q} = \int_{\mathcal{Z}} H_{p,q}(\zeta) d\mu(\zeta)$ and this commutes with every element of $\mathscr{M}$. Therefore

$$H_{p,q} = \int_{\mathcal{Z}} H_{p,q}(\zeta) d\mu(\zeta) \in \mathscr{M}' = \mathscr{B}''.$$

Let $Y_n = e_n/\|e_n\|^{1/n}$. Then we have $\sum \|Y_n\|^2 < \infty$ and by theorem 2, Ch. 5.6 Part II, there exist Hermitian operators $B_k \in \mathscr{B}$ such that $\sum_n \|H_{p,q}Y_n - B_kY_n\|^2 < 1/k^2$. Therefore $\|H_{p,q}e_n - B_ke_n\| \leq \dfrac{\|e_n\|^n}{k} \to 0$
as $k \to \infty$., i.e., $\int \sum_n \|H_{p,q}(\zeta)Y_n(\zeta) - B_k(\zeta)Y_n(\zeta)\|^2 d\mu(\zeta) \to 0$ as $k \to \infty$. **113**

As in Riesz-Fisher theorem, we can find a subsequence B_{k_i} such that $H_{p,q}Y_n - B_{k_i}Y_n \to 0$ as $k_i \to \infty$ outside a set of measure zero. Since the $H_{p,q}$ are only countable in number, we can pass to the diagonal sequence and get a sequence B_{k_j} such that *for every* p, q, $H_{p,q}Y_n - B_{k_j}Y_n \to 0$ as $k_j \to \infty$ outside a set N measure zero.

Let $\zeta \notin N$ and L be a subspace invariant under $\mathscr{A}(\zeta)$ or again under $\mathscr{A}(\zeta)''$. Let S be any element of $\mathscr{A}(\zeta)'$. Then S commutes with every $U_B(\zeta)$ where $B \in \mathscr{B}$ (where $U_B(\zeta) = \sum_i M_i(\zeta)U_A(\zeta)$ whenever $B = \sum_i M_iA_i$).

So,

$$\begin{aligned}
\langle S H_{p,q}(\zeta)e_n(\zeta), e_m(\zeta)\rangle &= \lim\langle S\, U_{B_k}(\zeta), e_n(\zeta), e_m(\zeta)\rangle\\
&= \lim\langle U_{B_k}(\zeta), S\, e_n(\zeta), e_m(\zeta)\rangle\\
&= \lim\langle S_{\,e_n}(\zeta), U_{B_k}(\zeta), e_m(\zeta)\rangle\\
&= \langle S_{\,e_n}(\zeta), H_{p,q}(\zeta), e_m(\zeta)\rangle.
\end{aligned}$$

or $H_{p,q}(\zeta)$ commutes with every $S \in \mathscr{A}(\zeta)'$ for almost every ζ. Hence $\mathscr{A}(\zeta)'$ consists only of scalar operators for almost every ζ. Hence the only invariant subspaces of $\mathscr{A}(\zeta)''$ are the trivial ones and the representation $A \to U_A(\zeta)$ of $\mathscr{A}$ in $\mathscr{H}(\zeta)$ is irreducible.

The following corollary is more or less immediate:

Corollary. *Let U be unitary representation of a separable, locally compact group G in a Hilbert space $\mathscr{H}$ such that there exists $a \in \mathscr{H}$ with the minimal closed invariant subspace containing $a = \mathscr{H}$. Then U is a continuous sum of unitary representations which are* almost all *irreducible.*

114 In fact, in Mautner's theorem, we have only to take for $\mathscr{A}$ the uniform closure of the algebra generated by elements of the form U_x, $x \in G$.

2.5

We have already said that the space Ω could have been used in much the same way as the space $\mathcal{Z}$. We now give an illustration to explain our remark that Ω is too large for practical purposes and that in the decomposition with respect to Ω the same representations may repeat 'too often' (which is what we sought to avoid by our equivalence relation). Let $\mathscr{H}$ be the space $\mathscr{L}^2(R)$ and U the regular representation σ_x of R in $L^2(R)$. Let $\mathcal{A}$ be the algebra of operators σ_f for $f \in L^1$. It can easily be seen that $\mathcal{A}''$ contains all operators σ_μ on L^2 with μ, a bounded measure. $\mathcal{A}$ and $\mathcal{A}''$ are of course commutative. Therefore $\mathcal{A}' \supset \mathcal{A}''$. We say take m' to be $\mathcal{A}''$ itself. If Ω is the spectrum of $\mathscr{M}'$ it contains the spectrum of $\mathscr{M}^1$. This is the much larger than the spectrum of L^1, whereas a 'good' decomposition of $L^2(R)$ into a Fourier transform is given by $f \to \hat{f}(y) = \dfrac{1}{\sqrt{2\pi}} \int f(x) e^{ixy} dx$. The Fourier in version formula will be $f = \dfrac{1}{\sqrt{2\pi}} \int \hat{f}(y) e^{ixy} dy$. L^2 is the direct integral of Hilbert spacers of dimension 1. If $\mathscr{M}^1$ denotes the set of bounded measures, then L^1 is an ideal in $\mathscr{M}^1$ and is hence contained in a maximal ideal of $\mathscr{M}^1$. Thus two different characters of $\mathscr{M}^1$ give rise to the same character of L^1. Thus a decomposition with Ω consists of unnecessarily repeated representations while that with $\mathcal{Z}$ (spectrum of L^1 in our example) economises them and reduces the decomposed representations to a minimum.

2.6 Equivalence of representations

115

The decomposition into continuous sum is obviously not unique, because the process depends on the choice of $a \in \mathscr{H}$ such that $\{Aa : A \in \mathcal{A}\}$ is dense in $\mathscr{H}$ and on this choice of $\mathscr{M}$. The question therefore arises whether all these decompositions are equivalent in some sense.

Definition . *Two decompositions $L^2_{\wedge_1}, (\mathcal{Z}_1, \mu_1, \wedge_1)$ and $L^2_{\wedge_2}(\mathcal{Z}_2, \mu_2, \wedge_2)$ of $\mathscr{H}$ are said to be equivalent if there exists a measurable one-one map $t : \mathcal{Z}_1 \to \mathcal{Z}_2$ and a map U_ζ of $\mathscr{H}(\zeta)$ onto $\mathscr{H}(t(\zeta))$ such that, the correspondence to every vector field X on $\mathcal{Z}_1$ of a vector field on $\mathcal{Z}_2$ defined by $Y(t(\zeta)) = U_\zeta X(\zeta)$, is an isomorphism of $L^2_{\wedge_1}$ onto $L^2_{\wedge_2}$.*

It is almost immediate that if the vector a is changed, we get equivalent decompositions. However, it is not true that if $\mathscr{M}$ is chosen in different ways the corresponding decompositions are equivalent.

Chapter 12

The Plancherel formula

3.1 Unitary algebras

116

Consider the regular representation of a locally compact group G in the space L^2 and decompose this into a continuous sum of irreducible representations. Then we have an isometry $f \to X_f$ of L^2 onto $L^2_\wedge$. By the definition of the norm in $L^2_\wedge$, we have $\int_G |f|^2 d\lambda = \int_{\mathcal{Z}} \|X_f\|^2 d\mu(\mathcal{Z})$. The map $f \to X_f$ is in a sense the Fourier transform for G, and the above equality, the Plancherel formula. As a matter of fact, we do get the classical Plancherel formula from this as a particular case when G is commutative. However, we have had many choices to make in the decomposition and as such this definition of a Fourier transform is not sufficiently unique and consequently uninteresting. We now proceed to obtain a Plancherel formula which is unique.

Let G be a separable, locally compact, unimodular group. We have seen (Ch. 8, Part II) that the regular representation of G in L^2 gives rise to a representation of L^1 in L^2. In fact we have the formula for every $f \in L^1$ and $g \in L^2$.

$$\begin{aligned} g * f(x) &= \int g(xy^{-1}) f(y) dy \\ &= \tau y(a) \text{ where } \check{g}(x) = f(x^{-1}) \end{aligned}$$

If we take $x = e$, $g * f(e) = \int e(y) \check{f}(y) dy$ (Since the group is unimodular)

$$= \langle g, f^* \rangle \text{ where } f^*(x) = \overline{f(x^{-1})}$$

By associativity of convolution product, we see that

$$\begin{aligned}\langle g * f, h\rangle &= g * f * h^*(e)\\ &= g * (h * f^*)^*(e)\\ &= \langle g, h * f^*\rangle \text{ for every } f, g, h \in L^1 \cap L^2\end{aligned}$$

117

The $*$ operation we have defined is an involution. Moreover L^1 acts on L^2 or, what is the same, L^2 is a representations space for L^1. The mapping $f \to T_f$ where $T - f(g) = g * f$ is unitary. Thus L^1 is a self adjoint algebra of operator on $L^2 = \mathscr{H}$. $\mathscr{A} = L^1 \cap L^2$ is a subalgebra $\mathscr{H}$ with an involution $*$. This satisfies the following axioms:

(a) $\langle x, y\rangle = \langle y^*, x^*\rangle$ for every $x, y \in \mathscr{A}$

For,

$$\begin{aligned}\langle f, g\rangle &= \int f(x)\overline{g(x)}dx\\ &= \int \overline{g(x^{-1})} f(x^{-1})dx\\ &= \int g^*(x)\overline{f^*(x)}dx\\ &= \langle g^*, f^*\rangle \text{ for every } f, g, \in L^1 \cap L^2.\end{aligned}$$

(b) $\langle x, yz\rangle = \langle y^* x, z\rangle$, or equivalently

$\langle yx, z\rangle = \langle y, zx^*\rangle$ for every $x, y, z \in \mathscr{A}$.

(c) $\mathscr{A}$ is dense in $\mathscr{H}$.

As a consequence the operators $V_x(y) = yx$ on $\mathscr{A}$ can be extended to operators on $\mathscr{H}$.

(d) The identity operator is the strong limit of that V_x.

This is an immediate consequence of prop. 3, Ch. 4.7, Part II,

Definition . *Let $\mathcal{A}$ be subspace of the Hilbert space $\mathscr{H}$. If $\mathcal{A}$ is an associative with involution satisfying conditions (a), (b), (c) and (d), $\mathcal{A}$ is said to be a* unitary algebra *(Godement). (Ambrose with a slightly different definition calls it an H^*- algebra).*

3.2

118

Associated with a given unitary algebra, we have a representation $U_x(y) = xy$ and an antirepresentation $V_x(y) = yx$. Axiom (b) asserts that these two representations are unitary. The map $x \longrightarrow x^*$ is an isometry by a (a) and consequently can be extended to a map $S : \mathscr{H} \to \mathscr{H}$. The U_x and the V_x are related by means of the relations $V_x = SU_x * S$ for every $x \in \mathcal{A}$. In fact, if $y, z \in \mathcal{A}$, we have

$$\begin{aligned}\langle V_x y, z\rangle &= \langle yx, z\rangle \\ &= \langle z^*, x^* y^*\rangle \\ &= \langle z^*, U_{x^*} y^*\rangle \\ &= \langle SU_{x*} Sy, z\rangle\end{aligned}$$

Hence $V_x = SU_x * S$ on $\mathcal{A}$ and hence on $\mathscr{H}$. We shall denote by $\mathcal{U}$, $\mathcal{V}$ the uniformly closed algebras generated by the U_x, V_x respectively. Let $\mathcal{R}$ be the uniformly closed algebra generated by both the U_x and the V_x.

Definition . *An element $a \in \mathscr{H}$ is said to be* bounded *if the the linear map $x \to V_x a$ of $\mathcal{A} \to \mathscr{H}$ is continuous.*

The mapping shall be denoted U_a, and the set of bounded elements $\mathscr{B}$.

Remark. To start with, one should have defined right -boundedness and left-boundedness of elements in $\mathscr{H}$. But if a is bounded in the above sense, a trivial computation shows that $U_a^* x = V_x S a$ for every $x \in \mathcal{A}$. So Sa is bounded and $U_{Sa} = U_a^*$. Now we have $U_x a = SV_{x*}S_a = SU_a^* S_x$ and the map $x \to U_x a$ is continuous and hence defines a continuous operator V_a and we have $V_a = SU_a^* S$.

Proposition 1. *$\mathscr{M} = \{U_a : a \in \mathscr{B}\}$ is a self adjoint ideal which is weakly dense is $\mathcal{V}'$.* 119

In fact, for every$x, y \in \mathcal{A}$, $U_a V_x y = U_a(yx) = V_{yx} a = V_x V_y a = V_x U_a y$; therefore U_a commutes with V_x; hence $U_a \in \mathcal{V}'$. Moreover if $T \in \mathcal{V}'$, we have $TU_a x = TV_x a = V_x Ta$; hence Ta is bounded and $U_{Ta} = TU_a$ and consequently $\mathscr{M}$ is an ideal in $\mathcal{V}'$. Since $U_a^* = U_{Sa}$, it is self-adjoint. Since $\mathcal{V}'$ is weakly closed, it only remains to show that $\mathcal{V}' \subset \mathscr{M}''$, or again that $T \in \mathcal{V}'$, $X \in \mathscr{M}'$ implies that $TX = XT$. But we have seen that, for $x \in \mathcal{A}$, $TU_x \in \mathscr{M}$. Hence $TU_x X = XTU_x$ and we may now allow U_x to tend to 1 in the strong topology to obtain $TX = XT$.

From this follows at once the

Theorem 1 (Godement-Segal)**.** *In the notations, $\mathcal{U}' = \mathcal{V}''$, or equivalently $\mathcal{V}' = \mathcal{U}''$.*

In fact, since $\mathcal{V} \subset \mathcal{U}'$, $\mathcal{V} \supset \mathcal{U}''$. We have only to show that $\mathcal{V}' \subset \mathcal{U}''$. In other words, we have to prove that every element of $\mathcal{V}'$ commutes with every element of $\mathcal{U}'$. Since the U_a, $a \in \mathscr{B}$ and similarly V_a, $a \in \mathscr{B}$ are dense in $\mathcal{V}'$, $\mathcal{U}'$ respectively, it suffices to establish the commutativity of V_a, V_b, $a, b \in \mathscr{B}$. First we assert that $U_c b = V_b c$ for every $c \in \mathscr{B}$.

For,

$$\begin{aligned}\langle U_c b, x\rangle = \langle b, U_{Sc} x\rangle &= \langle b, V_x Sc\rangle \\ &= \langle V_x^* b, Sc\rangle = \langle c, SV_x^* b\rangle \\ &= \langle c, U_x Sb\rangle = \langle c, V_{Sb} x\rangle = \langle V_b c, x\rangle\end{aligned}$$

Now, $U_a V_b x = U_a U_x b = U_{U_a x} b = V_b(U_a x)$ by the above calculation and the proof of theorem 1 is complete.

3.3 Factors

120

A weakly closed self-adjoint subalgebra of operators on a Hilbert space $\mathscr{H}$ is said to be a *factor* if its centre reduces to the scalar operators.

If in the above discussion we assume $\mathscr{R}$ to be irreducible, then $\mathscr{R}' = \mathcal{U} \cap \mathcal{V}' = \mathcal{U}' \cap \mathcal{U}''$ = Centre of $\mathcal{U}''$. Since $\mathscr{R}$ is irreducible, by cor. to Schur's lemma (Ch. 5.4, Part II), $\mathscr{R}'$ = scalar operators. Hence $\mathcal{U}''$ is a factor.

Example. (1) The set of all bounded operators on $\mathscr{H}$ is a factor.

(2) The set of bounded operators on $\mathscr{H}$ which is isomorphic to $\mathrm{Hom}(\mathscr{H}_1, \mathscr{H}_1)$ where $\mathscr{H}_1$ is another Hilbert space is also a factor. This is said to be a *Factor of type I*. If $\mathscr{H}_1$ is of dimension n, this is said oe be if type I_n.

3.4 Notion of a trace

If we consider only the operators on $\mathscr{H}$ which are of finite rank, then we have the notion of a trace defined by $\sum_n \langle Te_n, e_n\rangle$ where the e_n form an orthonormal basis. In the general case, we may define trace axiomatically in the following way:

Definition. *If $\mathscr{P}$ is the set of positive operators on $\mathscr{H}$,* trace *is a map of $\mathscr{P}$ into* $[0, \infty]$ *satisfying*

(a) $\mathrm{Tr}(UPU^{-1}) = \mathrm{Tr}\, P$ *for every unitary operator U, and*

(b) If P is a positive operator $= \sum T_\alpha$, where the T_α are also positive operators, and the series in strongly convergent, then $\mathrm{Tr}\, P = \sum \mathrm{Tr}\, T_\alpha$.

In particular, (b) implies that for every positive λ, $\mathrm{Tr}(\lambda P) = \lambda \,\mathrm{Tr}\, P$. It **121**
is obvious that this is true if λ is rational and since the rational numbers are dense in R, by (b), it is also true for all $\lambda \in R^+$. If A is any operator on $\mathscr{H}$ with a minimal decomposition into positive operators, then trace can be defined on A by extending by linearity.

Now, instead of $\mathrm{Hom}(\mathscr{H}, \mathscr{H})$, we may consider any $*$-subalgebra F of $\mathrm{Hom}(\mathscr{H}, \mathscr{H})$ and define the notion of a trace as above. However, for arbitrary $*$-subalgebras, neither the existence of a non-trivial trace nor its uniqueness is assured. For instance, if $\mathscr{H} = \mathscr{H}_1 + \mathscr{H}_2$ is the direct sum of the Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ and F is the subalgebra

$\mathrm{Hom}(\mathscr{H}_1, \mathscr{H}_1) + \mathrm{Hom}(\mathscr{H}_2, \mathscr{H}_2)$ of $\mathrm{Hom}(\mathscr{H}, \mathscr{H})$, then the function φ defined by $\varphi(T_1 + T_2) = \lambda_1 \operatorname{Tr} T_1 + \lambda_2 \operatorname{Tr} T_2$ (where λ_1 and λ_2 are arbitrary positive constants) is a trace. However, when F is a factor, the nontrivial trace, if it exists, is unique. Those factors which do not possess a nontrivial trace are said to be of type III.

A nontrivial trace on F can be proved to have the following properties:

1. If $H \in F \cap \mathscr{P}$, $TrH = 0$ of and only if $H = 0$; and

2. For every positive $H \in F$, there exists $H^1 \in F$ such that $0 < H^1 \leq H$ and $\operatorname{Tr} H^1 < \infty$.

Definition. *An element A of a$*$-algebra F operators on $\mathscr{H}$, is said to be* normed *(or of* Hilbert-Schmidt type*) with respect to a trace on F, if* $\operatorname{Tr}(A^* A) < \infty$.

122 Let F_0 be the set of operators of finite trace and F_1 the set of normed operators in F. Than if $A, B \in F_1$, we have $B^* A \in F_0$ and $\operatorname{Tr}(B * A)$ is a scalar product on F_1.

In the case of the algebra $\mathcal{U}''$, one can prove

Theorem 2. *There exists on $\mathcal{U}''$ one and only one trace such that*

(a) *$A \in \mathcal{U}''$ is normed if and only if $A = U_a$ for some $a \in \mathscr{B}$.*

(b) *If $A = U_a$ and $B = U_b$ are normed, then* $\operatorname{Tr}(B * A) = \langle a, b\rangle$.

The proof may be found in [23] or [12], Ch. I, § 6, n° 2.

In particular, if R is irreducible, then the factor $\mathcal{U}''$ is *not* of type III.

3.5

We now assume that two more conditions are satisfied by $\mathcal{A}$, viz.

(1) $\mathcal{A}$ is separable i.e. there exists a $*$-subalgebra everywhere dense in $\mathcal{A}$, which has a countable basis (in the algebraic sense).

(2) There exists an element $e \in \mathcal{A}$ such that $e^* = e$ and $\mathscr{R}$ is dense in $\mathcal{A}$.

Condition (2), however, is, as in the case of general decomposition, not a real restriction, and if it is not satisfied, $\mathscr{A}$ can be split up into a discrete sum of algebras each of which satisfies this condition.

We shall now perform the decomposition of $\mathscr{H}$ into a continuous sum of Hilbert spaces with reference to the uniformly closed algebra $\mathscr{R}$ of operators on $\mathscr{H}$. Let $\mathscr{A}_1$ be the set $\{Re : R \in \mathscr{R}\}$.

The fundamental family of vector fields in $L^2_\wedge$ is given by $a = Re \to$ **123**
$\tilde{a}(\zeta) = X_R(\zeta) \in \mathscr{H}(\zeta)$. We have already seen (Ch. 2) that this map is an isometry. Consider for every ζ the set $\mathscr{A}(\zeta) = \{\tilde{a}(\zeta) : a \in \mathscr{A}_1\}$. $U_a, a \in \mathscr{A}$ is a decomposed operator $= \int_Z U_a(\zeta) d\mu(\zeta)$.

Our object now will be to put on $\mathscr{A}(\zeta)$ the structure of an algebra and an involution with respect to which $\mathscr{A}(\zeta)$ becomes a unitary algebra, To this end, we define for $\xi = \tilde{a}_1(\zeta)$ and $\eta = \tilde{a}_2(\zeta), a_1, a_2 \in \mathscr{A}_1$, $\xi \cdot \eta = U_{a_1}(\zeta)\tilde{a}_2(\zeta)$ and $\xi^* \tilde{a}_1^*(\zeta)$. Of course we have to prove that these definitions are independent of the particular a_1, a_2 we choose. In other words we have to verify that if $\tilde{a}_1(\zeta) = \widetilde{a'_1}(\zeta)$ and $\tilde{a}_2(\zeta) = \widetilde{a'_2}(\zeta)$, then $U_{a_1}(\zeta)\tilde{a}_2(\zeta) = U_{a_1}(\zeta)\widetilde{a'_2}$ and that $\tilde{a}_1^*(\zeta) = \tilde{a}'^*_1(\zeta)$. In order to prove the former, we show that $U_{a_1}(\zeta)\tilde{a}_2(\zeta) = V_{a_2}(\zeta)\tilde{a}_1(\zeta)$. If $a_1 = R_1 e$ and $a_2 = R_2 e$, we have

$$\begin{aligned} U_{a_1}(\zeta)\tilde{a}_2(\zeta) &= (\widetilde{U_{a_1} a_2})(\zeta) = X_{U_{a_1}} R_2(\zeta) \\ &= \tilde{b}(\zeta) \text{ where } b = U_{a_1} R_2 e \\ &= (R_1 e)(R_2 e) = V_{a_2}(R_1 e) \end{aligned}$$

Hence $\tilde{b}(\zeta) = X_{V a_2 R_1}(\zeta) = V_{a_2}(\zeta)\tilde{a}_1(\zeta)$. Therefore, we have proved that the vector fields $U_{a_1}(\zeta)\tilde{a}_2(\zeta)$ and $V_{a_2}(\zeta)\tilde{a}_1(\zeta)$ are equal and so we have $U_{a_1}(\zeta)\tilde{a}_2(\zeta) = V_{a_2}(\zeta)\tilde{a}_1(\zeta)$ for almost every ζ. But since $\mathscr{A}_1$ has a countable basis, we can find a set N of measure zero such that for *every* $a_1, a_2 \in \mathscr{A}_1$, we have $U_{a_1}(\zeta)\tilde{a}_2(\zeta) = V_{a_2}(\zeta)\tilde{a}_1(\zeta)$ for $\zeta \notin N$. Now the left hand side is unaltered if we replace a_2 by a'_2 while the right hand side remains the same if we replace a_1 by a'_1. **124**

It only remains to prove that ξ^* is well-defined for almost all ζ'. It is enough to prove that for every $a_1, a_2 \in \mathscr{A}_1$, the set of ζ such that $\widetilde{a_1}(\zeta) = \widetilde{a_2}(\zeta)$ and $\widetilde{a_1 *}(\zeta) \neq \widetilde{a_2 *}(\zeta)$ is of measure zero. Let $E = \{\zeta : \widetilde{a_1}(\zeta) =$

$\widetilde{a_2}(\zeta)\}$. The operator field A defined by

$$A(\zeta) = \begin{cases} \text{Identity if } \zeta \in E \\ 0 \text{ if } \zeta \notin E \end{cases}$$

is a Hermitian scalar decomposed operator in $\mathscr{H}$ and we have $Aa_1 = Aa_2$. Hence we have $Aa_1^* = Aa_2^*$, i.e. $A(\zeta)\,\widetilde{a_1^*}(\zeta) = A(\zeta)\widetilde{a_2^*}(\zeta)$ almost everywhere and $\widetilde{a_1^*}(\zeta) = \widetilde{a_2^*}(\zeta)$ for almost every $\zeta \in E$.

We shall now prove that with the above operations, $A(\zeta)$ is a unitary algebra.

(a) $\langle \xi, \eta \rangle = \langle \eta^*, \xi^* \rangle$

We have, in our usual notation, $\langle \xi, \eta \rangle = \varphi_{R_2^* R_1}(\zeta)$. $a_1^* = SR_1Se = SR_1Se$ (since $e^* = e$) and $a_2^* = SR_2Se$. We have now to prove that $\varphi_{R_2^* R_1}(\zeta) = \varphi_{(SR_1S)*(SR_2S)}(\zeta)$. We assert that $M^* = SMS$ for every $M \in \mathscr{R}$. In fact, if $x \in \mathscr{A}$, $U_xM = MU_x = U_{Mx}$ and $U_{M*Sx} = M^*U_{Sx} = U_{(Mx)^*} = U_{SMx}$. Since the map $x \to U_x$ is one-one, we have $M^*SX = SMx$ or $M^* = SMS$. Therefore

$$\begin{aligned}\langle M(SR_1) * (SR_2S)e, e\rangle &= \langle MSR_2Se, SR_1e\rangle \\ &= \langle R_1e, R_2SMSe\rangle \\ &= \langle R_1e, R_2M^*e\rangle.\end{aligned}$$

In other words,

$$\langle M(SR_1S) * (SR_2)e, e\rangle = \langle MR_2^*R_1e, e\rangle$$

for every $M \in A$. By the definition of the spectral measure,

$$d\mu_{(SR_1S)} * (SR_2S)e, e = d\mu_{R_2^*R_1e,e}$$

125 and hence $\varphi(SR_1S) * (SR_2S) = \varphi_{R_2^*R_1}$.

(b) $\langle \xi_1\xi_2, \xi_3\rangle = \langle \xi_1, \xi_3, \xi_2^*\rangle$.

We have to show that $\varphi_{R_3^*R_1R_2} = \varphi_{(R_3R_2^*)^*R_1}$ which is obvious.

(c) $A(\zeta)$ is dense in $\mathscr{H}(\zeta)$.

This is again evident.

(d) Regarding the existence of sufficiently many operators, we cannot assert that is true for all ζ. However, this is true for almost all ζ For this we need a

Lemma 1. *In a self adjoint algebra $\mathcal{A}$ of operators on $\mathscr{H}$, the set $\{Ax : A \in \mathcal{A}, x \in \mathscr{H}\}$ is dense in $\mathscr{H}$ if and only if the Identity is the strong limit of $A \in \mathcal{A}$.*

In fact, if x is an element of $\mathscr{H}$, we denote $\overline{\{Ax\}}$ by F. Let $F^{\perp}$ be its orthogonal complement. If x is not in F, let $x = x_1 + x_2$ with $x_1 \in F$, $x_2 \in F^{\perp}$. Then $Ax = Ax_1 + Ax_2$ with $Ax_1 \in F$, $Ax_2 \in F^{\perp}$ (since $\mathcal{A}$ is a self-adjoint algebra). But $Ax \in F$. Hence $Ax_2 \in f^{\perp}$ as well as F. Since the sum is direct, $Ax_2 = 0$ for every $A \in \mathcal{A}$. If $x_2 \neq 0$, this contradicts the assumption that $\{Ax : A \in \mathcal{A}, x \in \mathscr{H}\}$ is dense in $\mathscr{H}$. For, space $E = \{x : Ax = 0$ for every $A \in \mathcal{A}\}$is non-zero. E is invariant under $\mathcal{A}$ and therefore $E^{\perp}$ is invariant under $\mathcal{A}$. Consequently $\{\mathcal{A}a : a \in \mathcal{H}\}$ is contained in $E^{\perp}$. Now, $\{U_x y : x, y \in \mathcal{A}\}$ is dense in $\mathcal{H}$ and hence $\{xy : x, y \in \mathcal{A}_\infty\}$ is dense in $\mathcal{A}_1$. On the other hand, there exists a sequence Y_n is $\mathcal{A}_1$ such that the $\tilde{Y}_n(\zeta)$ are dense in $\mathcal{H}(\zeta)$ for every ζ. Each Y_n can be approximated by a sequence $X_{n,p}Y_{n,p}$ **126**
with $X_{n,p}, Y_{n,p} \in A_1$. Hence we have $\tilde{Y}_n(\zeta) = \lim\limits_{p\to\infty} U_{X_{n,p}}(\zeta)\widetilde{Y_{n,p}}(\zeta)$ for almost every ζ and for each n, since we have only a countable family Y_n. Therefore $\{\xi\eta : \xi, \eta \in A(\zeta)\}$ is dense in $\mathcal{A}(\zeta)$ for almost every ζ.

Thus we have shown that the algebra $\mathcal{A}(\zeta)$ is unitary for almost every ζ. By Mautner's theorem, almost all these algebras are irreducible. Thus $U(\zeta)''$ is a factor. We can apply Theorem 2 to this factor. Thus the scalar product in the space of bounded elements is given by a trace. More precisely, we have a trace function on $U(\zeta)''$ such that $\langle \tilde{a}_1(\zeta), \tilde{a}_2(\zeta)\rangle = Tr(U_{a_2} * (\zeta) U_{a_1})(\zeta))$. But we know that the correspondence $a \to \tilde{a}$ is an isometry and hence one gets

$$\langle a_1, a_2\rangle = \int_{\mathcal{Z}} Tr\Big[u^*_{a_2}(\zeta)U_{a_1}(\zeta)\Big] d\mu(\zeta)$$

This is the Plancherel formula which is in a certain sense unique. However this is obviously not absolutely unique as the trace function is unique only upto a constant multiple.

In the case of a locally compact, separable, unimodular group G, we take $\mathscr{A} = L^1 \cap L^2$ and $\mathscr{H} = L^2$. In this case Plancherel formula can be rewritten as

$$\int_G f(x)\overline{g(x)dx} = \int_{\mathcal{Z}} \mathrm{Tr}(U_g^*(\zeta)U_f(\zeta))d\mu(\zeta).$$

Or again $g^* * f(e) = \int_{\mathcal{Z}} \mathrm{Tr}(U_{g** f}(\zeta))d\mu(\zeta)$.
If we write $g^* * f = h$, we get

$$h(e) = \int_z \mathrm{Tr}(U_h(\zeta)d\mu(\zeta)$$

This is the generalisation of the *Fourier inversion formula*. At any point x, the value of $h(x)$ is given by

$$h(x), = \int_{\mathcal{Z}} \mathrm{Tr}(U_h(\zeta)U_x(\zeta))d\mu(\zeta)$$

127 This is of course true not for all functions, but only for function of the type $g^* * f$ with $g, f \in L^1 \cap L^2$ as in the classical case.

3.6 A particular case

We have obtained a Plancherel formula in terms of the factorial representations of the group G and it would be more desirable to have a formula in terms of the irreducible representations of the group. This is however possible only in the following particular case.

Definition. *A locally compact group is said to be* type I *if every factorial representation of the group is of type I.*

This definition implies that every factorial representation is a discrete multiple of an irreducible representation. In fact, if F is the factor corresponding to a factorial representation of G, then F is isomorphic (algebraically) to Hom$(\mathscr{H}_2, \mathscr{H}_2)$ where $\mathscr{H}_2$ is a Hilbert space. If F is

of type I, it can be proved (see, for instance [1], § 8, Ch. I) that F' is also of type I. by the isomorphism between Hom $(\mathscr{H}_1, \mathscr{H}_1)$ and F', we therefore get that for every projection P in F', there exists a minimal projection $< P$. in other words, every invariant subspace $\neq 0$ of $\mathscr{H}$ contains a *minimal* invariant subspace. The restriction of the operators of F to any such minimal invariant subspace gives rise to an irreducible unitary representation and since the minimal projections in a Hilbert space are conjugate by unitary isomorphisms, these irreducible unitary representations are *equivalent*. On the other hand, the family of invariant subspaces of $\mathscr{H}$ which are direct sums of minimal invariant subspace, partially ordered by inclusion, is obviously inductive. By Zorn's lemma, there exists in it a maximal element say $\mathscr{H}^1$. If $\mathscr{H}^1 \neq \mathscr{H}$, $\mathscr{H}^{1\perp}$ is **128**
nonempty and consequently contains a minimal invariant subspace $\mathscr{H}_1^1$. Then $\mathscr{H}^1 \oplus \mathscr{H}_1^1$ again belongs to the family, thereby contradicting the maximality of $\mathscr{H}^1$.

Thus if $x \in \mathscr{A}$, the operator of the factorial representation is decomposed into irreducible U_x^0 which are all equivalent. The map $U_x \to U_x^0$ is an isomorphism. Hence in a group of type 1, we have the formula

$$\int_G f(x)\overline{g(x)}dx = \int_{\mathcal{Z}} Tr(U_g^*(\zeta)U_f(\zeta))d\mu(\zeta)$$

where the $U(\zeta)$ are irreducible representations and *not* merely factorial representations, and the trace is the *usual* trace.

The definition of a group of type I seems a little inoccuous but is of importance since all semisimple lie groups are of type I. The problem remains however to give an explicit Plancherel measure, etc.

It is known in the case of complex semisimple lie groups (see [26]) and in the case of $SL(2, R)$ ([2], [25]), but not in the general case.

3.7 Plancherel formula for commutative groups

Let Ω be the spectrum of $\mathscr{R}'$ in this case. We have however to pass to a quotient $\mathcal{Z}$ by means of an equivalence relation. It can be proved that $\mathcal{Z}$ is actually the one point compactification of the spectrum of L^1. We now assert that every representation of L^1 in a space E arises from a representation of the group G. In fact if $a = U_f b$, with $b \in \mathscr{H}$, $f \in L^1$, we put $U_x a = U_{\epsilon_x * f} b$. It can be proved (see for instance [22], [6]) that the U_x are well defined.

129 This establishes a one-one correspondence between characters of G and one dimensional representation of L^1. Hence the spectrum of L^1 is only the character group of $GU\{0\}$ to compactify it. In this case, every factor consists only of scalar operators and hence any factorial representation is a discrete multiple of irreducible representations of dimension 1. If the character group of G is denoted by $\hat{G}$, we have, since $\mathrm{Tr}\chi(f) = \chi(f) = \int f(x)\chi(x)dx$,

$$\int_G \|f\|^2 dx = \int_{\hat{G}} |\chi(f)|^2 d\mu(\chi).$$

It only remains to prove that the Plancherel measure in this case is the Haar measure on $\hat{G}$. But this is obvious since each character is of norm 1 and multiplication of $\chi(f)$ by another character leaves the integral invariant. Thus in this case, we have the classical plancherel formula

$$\int_G \|f\|^2 dx = \int_{\hat{G}} \mid \chi(f) \mid^2 d\chi.$$

Again the Fourier inversion formula becomes in this case

$$f(x) = \int_{\hat{G}} \chi(f)\overline{\chi(X)}d\chi.$$

Bibliography

For Part I, see particularly [3], [8], [9], [10], [43], [49]; for Part II, [4], [5], [6], [7], [19], [22], [29], [31], [44], [49]; and Part III, [12], [20], [23], [33], [34], [36], [40], [46], [47], **130**

[1] W. Ambrose - The L^2-system of a unimodular group I. Trans. Am. Math. Soc. 65, 1949, p.27-48.

[2] V. Bargmann - Irrejucible unitary representations of the Lorentz group. Annals of Math, 48, 1947, p. 568-640.

[3] N. Bourbaki - Topologie Generale, Ch. III-IV, Paris 1951.

[4] N. Bourbaki - Espaces Vectoriels Topologiques, Ch.I-V Paris 1953 and 1955.

[5] N. Bourbaki - Integration, Ch.$I - V$, paris, 1952 and 1956.

[6] F. Bruhat - Sur Ies representations induites des groupes de Lie Bull. Soc. Math. France, 84, 1956, p. 97-205.

[7] H. Cartan and R. Godement - Theorie de la dualite et Analyse Harmonique dans les groupes abeliens localement compacts, Annales Scientifiques, E.N.S., 64, 1947, p. 79-99.

[8] C. Chevalley - Theory of Lie groups, Princeton 1946. **131**

[9] J. Dieudonné - Groupes de Lie et hyperalgèbres de Lie sur un corps de caractéristiqur $p > 0$, Comm. Math. Helv., 28, 1954, p. 87-118.

[10] J. Dieudonné - Lie groups and Lie hyperalgebras over a field of characteristic $p > 0 - II$. Am. Journ. Math., 77, 1955, p.218-244.

[11] J. Dixmier - Algebres quasi-unitaires. Comm. Math. Helv., 26, 1952, p. 275-322.

[12] J. Dixmier - Les algebres d'operateurs dans l'espace Hilbertien, Paris 1957.

[13] I. Gelfand - Normierte Ringe Rec. Math[Math.Sbornik] N.S. 9 [51], 3-24.

[14] I. Gilfand - M. Naimark - Unitary Representations of the group of the linear transformations of the straight line. Doklady Akad. Nauk S.S.S.R 63, 1948, p. 609-612.

[15] I. Gilfand - M. Naimark - Unit are Darstellungen der Klassischen gruppen, Berlin, 1957.

[16] I. Gilfand - M. Naimark - The analogue of Plancherel's Formula for Complex unimodular group. Doklady Akad. Nauk S.S.S.R 63, 1948, p. 609-612.

[17] I. Gelfand - D. Raikoy - Irreducible unitary representation of locally bicompact group.

132 [18] A.M. Gleason - Groups without small subgroups. Annals of Math., 56, 1952, p. 193-212.

[19] R. Godement - Les fonctions de type positif et le theorie des groupes. Trans. Am. Math. Soc., 63, 1948, p. 1-84.

[20] R. Godement - Sur la theorie des representations unitaires Annals of Math., 53 (1952), p. 68-124.

[21] R. Godement - Memoire sur la theorie des caracteres. Journal de Math.Pures et Appl. 30, 1951, p.1-110.

[22] R. Godement - A theory of spherical functions-I. Trans. Am. Math. Soc., 73, 1952, p. 496-556.

[23] R. Godement - Theorie des caracters I Algebres Unitaires. Annals of Math., 59, 1954, p. 54-62.

[24] R. Godement - Theorie des caracteres II. Definition et proprietes generales des caracteres. Annals of Math., 59, 1954, p. 63-85.

[25] Harishchandra - Plancherel formula for the 2×2 real uni - modular group. Proc.Nat.Acad. Sc. 38, p.337-342.

[26] Harishchandra - The Plancherel formula for complex semi - simple lie groups. Trans.Am.Math.Soc., 76, 1954, p. 485-528.

[27] I. Kaplansky - A theorem on rings of operators. Pacific Journ. of Math., 1, 1951, p. 227-232.

[28] G.W. Mackey - Induced representations of groups. Amer. Journ. of **133**
Math., 73, p. 576-592.

[29] G.W. Mackey - Induced representations of locally compact groups I. Annals of Math., 55, 1952, p. 101-140.

[30] G.W. Mackey - Induced representations of locally compact groups II. Annals of Math., 58, 1953, p. 193-221

[31] G.W. Mackey - Functions on locally compact groups. Bull. Amer. Math. Soc., 56, 1950, p. 385-412.

[32] F.I. Mautner - The completeness of the irreducible unitary representations of a locally compact group. Proc. Nat. Acad. Sc., 34, 1948, p. 52-54.

[33] F.I. Mautner - Unitary representations of locally compact groups I. Annals of Math., 51, 1950, p.1-25.

[34] F.I. Mautner - Unitary representations of locally compact groups II. Annals of Math., 52, 1950, p. 528-556.

[35] F.I. Mautner - On the decomposition of unitary representa- tions of Lie groups. Proc. Amer. Math. Soc., 2, 1951, p. 490-496.

[36] F.I. Mautner - Note on the Fourier inversion formula for groups. Trans. Am. Math. Soc., 78, 1955, p. 371-384.

134 [37] M.A. Naimark - Normed rings - Moscow 1956.

[38] M.A Naimark - S.V.Fomin - Continuous direct sum of Hilbert spaces and some of their applications. Uspehi Mat. Nauk., 10. 1955, p.111-142 and Transl. Amer. Math. Soc., 2, Vol. 5, 1957, p.35-65.

[39] J. von Neumann - On certain topology for rings of operators. Annals of Math., 37, 1936, p. 111-115.

[40] J. von Neumann - On rings of operators- Reduction Theory. Annals of Math., 50, 1949, p. 401-485.

[41] R. Pallu de la Barriere - Algebres unitaires et espaces D'Ambrose. Annals Scientifiques, E.N.S., 70, 1953, p. 381-401.

[42] L. Pontrjagin - Topolgical groups, Princeton, 1946.

[43] L. Pontrjagin - Continuous group, Moscow, 1954.

[44] L. Schwartz - Theorie des Distributions, Paris 1950 / 1951.

[45] I.E. Segal - The group algebras of a locally compact group. Trans. Amer. Math. Soc., 61, 1947, p. 69-105.

[46] I.E. Segal - An extension of Plancherel's formula to separable unimodular groups. Annals of Math., 52, 1950, p. 272-292.

[47] I.E. Segal - Decomposition of operator algebras. Memoirs of Amer. Math. Soc., 9, 1951.

135 [48] I.E. Segal- The two - sided regular representations of a unimodular locally compact group. Annals of Math., 51, 1950, p. 293-298.

[49] A. Weil - L' Integration dans les groupes topologiques et ses applications. Paris, 1940 and 1953.