# Lectures on Elliptic Partial Differential Equations

By

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**Tata Institute of Fundamental Research, Bombay**
**1957**

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# Introduction

In these lectures we study the boundary value problems associated with elliptic equation by using essentially $L^2$ *estimates* (or abstract analogues of such estimates. We consider only linear problem, and we do not study the Schauder estimates.

We give first a general theory of "weak" boundary value problems for elliptic operators. (We do not study the *non-continuous* sesquilinear forms; of. Visik [17], Lions [7], Visik-Ladyzeuskaya [19]).

We study then the *problems of regularity*-firstly regularity in the interior, and secondly the more difficult question of regularity at the boundary. We use the Nirenberg method for Dirichlet and Neumann problems and for more general cases we use an additional idea of Aronsazajn-Smith.

These results are applied to the study of new boundary problems: *the problems of Visik-Soboleff.* These problems are related and generalize the problems of the Italian School (cf. Magenes [11]).

We conclude with the study of the Green's kernels, some indications on unsolved problems and short study of systems. Due to lack of time we have not studied the work of Schechter [15] nor the work of Morrey-Niren-berg [13] which rots essentially on $L^2$ estimates.

# Contents

# Lecture 1

## 1 Spaces $H(A;\Omega)$

### 1.1 General notations

We shall recall some standard definition and fix some usual notation. $R^n(x = (x_1, \ldots, x_m))$ will denote the $n$-dimensional Euclidean space, $\Omega$ an open set of $R^n$, $\mathscr{D}(\Omega)$ will be the space of all indefinitely differentiable functions (written sometimes $C^\infty$ functions) with compact support in $\Omega$ with the usual topology of Schwartz. $\mathscr{D}'(\Omega)$ will be the space of distributions over $\Omega$. $L^2(\Omega)$ will be the space of all square integrable functions on $\Omega$. The norm of a function $\mathscr{F} \in L^2(\Omega)$ will be denoted by $||\mathscr{F}||_o$. *Derivatives of functions on $\Omega$ will always be taken in sense of distributions*; more precisely $D^p$ will stand for $\dfrac{\partial^{|p|}}{\partial x_1^p, \ldots \partial x_n^{p_n}}$ where $p = (p_1, \ldots, p_n)$ with $p_i$ non-negative integers and $|p| = p_1 + \cdots + p_n$ is the order of $D^p$. If $T \in \mathscr{D}'(\Omega), \langle D^p T, \varphi\rangle = (-1)^{|p|}\langle T, D^p\varphi\rangle$. **1**

### 1.2 Spaces $H(A;\Omega)$

A defferential operator with constant coefficients $\underline{\mathrm{A}}$ is an expression of the for $A = \sum\limits_{|p|\leq m} a_p D^p$ where $a_p$ are all constants. The highest integer $m$ for which there exists an $a_p \neq 0$ for $|p| = m$ will be called the order of the operator $A$.

**Definition 1.1.** *Let $A = \{A_1, \ldots, A_\nu\}$ be a system of differential operator with constants coefficients. By $H(A;\Omega)$ we shall denote the space of $u \in L^2(\Omega)$ for which $A_i u \in L^2(\Omega)$.*

Evidently $\mathscr{D}(\Omega) \subset H(A;\Omega)$. On $H(A;\Omega)$ we define a sesquilinear form by

$$(u,v)_{H(A;\Omega)} = (u,v)_o + \sum_{i=1}^{\gamma}(A_i u, A_i v)_o. \tag{1}$$

**Theorem 1.1.** *With the norm defined by (1), $H(A;\Omega)$ is a Hilbert space.* 2

*Proof.* It is evident from the expression (1) that $(u,v) = \overline{(v,u)}(u,u) \geq 0$, and that $(u,u) = 0$ if and only $u = 0$. So it remains to verify that under the topology defined by the norm, $H(A;\Omega)$ is complete. If $\{u_n\}$ is any Cauchy sequences in $H(A;\Omega)$, from (1) it follows that $\{u_n\}$ and $\{A_i u_n\}$ are Cauchy sequences in $L^2(\Omega)$. Hence $\{u_n\}$ and $\{A_i u_n\}$ converge to $u$ and $v_i$ respectively, say, in $L^2(\Omega)$. Since the convergence in $L^2(\Omega)$ implies the convergence in $\mathscr{D}'(\Omega), \{u_n\}$ and $\{A_i u_n\}$ converge to $u$ and $v_i$ in $\mathscr{D}'(\Omega)$ respectively. Since $A_i$ are continuous on $\mathscr{D}'(\Omega), A_i(u_u) \to A_i(u)$ in $\mathscr{D}'(\Omega)$. Hence $A_i(u) = v_i$ which proves that $u \in H(A;\Omega)$. □

**Proposition 1.1.** *If $W \subset \Omega$ and for $u \in H(A;\Omega), u_w$ denotes the restriction of $u$ to $W$, then $(a) u_w \in H(A;W)$, and $(b)$ the mapping $u \to u_w$ is continuous mapping of $H(A;\Omega) \to H(A;W)$.*

## 1.3 The space $H_0(A;\Omega)$.

**Definition 1.2.** *$H_0(A;\Omega)$ will be the closure of $\mathscr{D}(\Omega)$ in $H(A;\Omega).H(A;\Omega)$ will be the "orthogonal complement" of $H_0(A;\Omega)$ in $H_0(A;\Omega)$.*

The following question then naturally arises:

**Problem 1.1.** When is $H_0(A;\Omega) = H(A;\Omega)$?

If A differential operator, let $A^*$ denote the differential operator defined by $\langle A^*T,\varphi\rangle = \langle T,\overline{A\varphi}\rangle$. If $A = \sum a_p D^p$, then it is easily verified that $A^* = \sum(-1)^p \overline{a}_p D^p$.

**Proposition 1.2.** *$H_0^{\perp}(A;\Omega)$ is the space of solution in $H(A;\Omega)$ of* $(1 + \sum_{i=1}^{\gamma} A_i^* A_i)$ 3
$T = 0$.

*Proof.* $T \in H_0^1(A;\Omega)$ if and only $T$ is orthogonal to every $\varphi \in \mathscr{D}(\Omega)$, i.e., if and only if □

$$(T,\varphi)_o + \sum_{i=1}^{\gamma}(A_i T, A_i\varphi) = 0$$

for all $\varphi \in \mathscr{D}(\Omega)$, which is equivalent to say that

$$\langle T + \sum A_i^* A_i T, \varphi\rangle = 0 \text{ for all } \varphi \in \mathscr{D}(\Omega),$$

or that

$$(1 + \sum A_i^* A_i)T = 0.$$

## Some examples.

1) If there is no differential operators, $H(A;\Omega) = L^2(\Omega) = H_o(\Lambda;\Omega)$.

2) Let $\Omega = ]0,1[, x = x_1, A = \dfrac{d}{dx}; H(A;\Omega) = \{u/u \in L^2$ and only $u' \in L^2$. Then $T \in H_o^\perp(A;\Omega)$ if and if $T - T'' = 0$, i.e.,$T = \lambda e^x + \mu e^{-x}$. Hence $H_o^\perp(A;\Omega)$ is space of dimension 2.

3) Let $\Omega = ]0,+\infty[, x = x_1, A = \dfrac{d}{dx}.T = \lambda e^x + \mu e^{-x}$ is in $L^2(\Omega)$ is $\lambda = 0$. Hence $H_o^\perp(A;\Omega)$ is of dimension 1.

4) Let $\Omega = ]0,+\infty[, x = x_1, A = \dfrac{d}{dx} H_o^\perp(A;\Omega) = \{\underline{0}\}$, i.e.,

$$H_o(A;\Omega) = H(A;\Omega).$$

In general it can be proved that if $\Omega = ]0,1[A = \dfrac{d^m}{dx^m}, H_o^\perp(A,\Omega)$ is 2m-dimensional.

## 1.4

We recall some properties of Fourier transformations of distributions. Let $\mathscr{S}$ be the space of fastly decreasing functions in $R^n, \mathscr{S}'$ be the dual of $\mathscr{S}$ consisting of tempered distributions. For $T \in \mathscr{S}'$ we shall denote the Fourier Transform 4
of $T$ by $\underline{\mathscr{F}T = \hat{T}}$. We know that $L^2(R^n) \subset \mathscr{S}'$ and that $|\hat{T}|_o = |T|_o$ if $T \in L^2(R^n)$ (Plancherel's formula). Also $\mathscr{F}(D^p T) = (2\pi i\xi)^p \hat{T}$, where $\xi = (\xi_1, \ldots, \xi_n)$ and $\xi^p = \xi_n^{p_1} \ldots \xi_n^{p_n}$. Since $\mathscr{F}$ is linear, it follows that if $A = \sum_{|p|\le m} D^p$ is any differential operator with consists coefficients

$$\begin{aligned}
\mathscr{F}(AT) &= A(2\pi i\xi)\hat{T} \text{ where}\\
A_j(2\pi i\xi) &= \sum_{|p|\le m} a_p \xi_1^{p_1} \cdots\cdots \xi_n^{p_n} (2\pi i)^{|p|}\\
&= \sum a_p (2\pi i\xi)^p.
\end{aligned}$$

**Proposition 1.3.** *$u \in H(A, R^n)$ if and only if $\hat{u} \in L^2(R^n)$ and $A_j(2\pi i\xi)\hat{u} \in L^2(R^n)$, $j = 1, \ldots, N$.*

This is immediate as $u \in L^2 \Leftrightarrow \hat{u} \in L^2(R^n)$ and $A_j u \in L^2 \Leftrightarrow A_j(2\pi i\xi)\hat{u} \in L^2(R^n)$.

**Proposition 1.4.** *$H_o(A, R^n) = H(A, R^n)$, for any $A = \{A_1, \ldots, A_\nu\}$ with constant coefficients.*

From Proposition 1.2 we have $T \in H_o^{\perp}(A, R^n)$ if and only if $T \in L^2, A_j T \in L^2$ and $(1 + \sum\limits_{j=1}^{\nu} A_j^* A_j)T = 0$. By taking Fourier transforms, it follows that $T \in H_o^{\perp}(A, R^n)$ if and only if $\hat{T} \in L^2 A_j \hat{T} \in L^2$ and $(1 + \sum |A_j(2\pi i\xi)|^2)\hat{T} = 0$.

But since $(1 + \sum_j |A_j(2\pi i\xi)|^2) \neq 0, \hat{T} = 0$ a.e., and hence $T = 0$ which proves the proposition.

## 1.5 Extension of functions in $H_o(A;\Omega)$ to $R^n$.

**Theorem 1.2.** *There exists one and only one continuous linear mapping $u \to \tilde{u}$ of $H_o(A;\Omega)$ into $H(A, R^n)$ such that if $u \in \mathscr{D}(\Omega), \tilde{u} = u$. a.e. in $\Omega$.*

For $\varphi \in \mathscr{D}(\Omega)$, define $\tilde{\varphi} = \begin{cases} \varphi(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \notin \Omega. \end{cases}$ 5

Then $\tilde{\varphi} \in \mathscr{D}(R^n)$ and $|\tilde{\varphi}|_{H(A,R^n)} = |\varphi|_{H(A,\Omega)}$. Hence $\varphi \to \tilde{\varphi}$ is a continuous mapping of $\mathscr{D}(\Omega)$ with the topology induced by $H(A;\Omega)$ into $H(A, R^n)$. This proves the theorem

**Definition 1.3.** *If $u \in H_o(A;\Omega), \tilde{u}$ is called the extension of $u$ to $R^n$.*

**Remark.** If $u \in H(A;\Omega)$ and we put $\tilde{u}(x) = \begin{cases} u(x), & x \in \Omega \\ 0, & x \notin \Omega \end{cases}$ it is not true that $\tilde{u} \in H(A, R^n)$. What that theorem 1.2 says is that if $u \in H_o(A;\Omega)$, then $\tilde{u} \in H_o(A, R^n)$. Thus if $A\dfrac{d}{dx}, \Omega = ]0, 1[$ then for $u = 1$, $\dfrac{d}{dx}\tilde{u}$ is not in $L^2(R)$.

# Lecture 2

## 1.6

Let $H_o'(A;\Omega)$ be the dual of $H_o(A;\Omega)$. **6**

**Theorem 1.3.** *a)* $H_o'(A;\Omega)$ *is space of distributions.*

*b) If* $T \in H_o'(A;\Omega)$*, then there exists a unique* $g \in H_o(A;\Omega)$ *such that* $T = (1 + \sum A_i^* A_j)g$.

*c) The correspondence* $T \to g$ *is a topological isomorphism of* $H_o(A;\Omega)$ *onto* $H_o'(A;\Omega)$.

*Proof.* Let $u \to L(u)$ be a continuous linear form on $H_o(A;\Omega)$. The restrictions of $L$ to $\mathscr{D}(\Omega)$ is continuous on $\mathscr{D}(\Omega)$ with its usual topology. Hence it define a distribution $T_L$ so that $L(\varphi) = \langle T_L, \varphi\rangle$ for all $\varphi \in \mathscr{D}(\Omega)$. If $T_L = 0$, then $\langle T_L, \varphi\rangle = L(\varphi) = 0$ for all $\varphi \in \mathscr{D}(\Omega)$. Since $\mathscr{D}$ is dense in $H_O(A;\Omega), L = 0$. This proves $H_o'(A;\Omega) \subset \mathscr{D}(\Omega)$. Now, if $L \in H_o'(A;\Omega)$ by Riesz's theorem, there exists $g_L \in H_o(A;\Omega)$ such that $L(\overline{u}) = (g_L, u)_{(H_o(A;\Omega))}$. Hence for every $\varphi \in \mathscr{D}(\Omega)$, ◻

$$
\begin{aligned}
L(\varphi) = \langle T_L, \varphi\rangle (g_L, \overline{\varphi}) = (g_L, \overline{\varphi})_o + \sum_{i=1}^{\gamma} (A_i g, A_i)_o \\
= \langle (1 + \sum A_i^* A_i) g, \varphi\rangle.
\end{aligned}
$$

Hence $T = (1 + \sum A_i^* A_i)g$ and $T \to 0$ in $H_o'(A;\Omega)$ if and only if $g \to 0$ in $H_o(A;\Omega)$.

**Remark.** As we shall see later on this theorem constitutes the solution of certain (weak) Dirichlet's problem.

**Proposition 1.5.** *Every distribution $T \in H'_o(A;\Omega)$ can be written in the form $T = g + \sum A^*_i f_i$ with $f_i \in L^2(\Omega)$ and $g \in H_o(A;\Omega)$ and conversely any distribution of the above form is in $H'_o(A;\Omega)$.*

Since by theorem 1.3, any $T \in H'_o(A;\Omega)$ is of the form $T = g + \sum A^*_i A_i g$, **7**
putting $A_i g = f_i$ we obtain the first part. Conversely if $s = g + \sum A^*_i f_i$, we have for any $\varphi$ in $\mathscr{D}_\nu(\Omega)$,

$$\langle s, \overline{\varphi}\rangle = \langle g, \overline{\varphi}\rangle + \sum \langle A^*_i f_i \overline{\varphi}\rangle = (g,\varphi)_o + \sum_{i=1}^{\nu} (f_i, A_i\varphi)_o.$$

Hence $\varphi \to (s,\varphi)$ is a continuous semi-linear functional on $\mathscr{D}(\Omega)$ with the topology induced by $H_o(A,\Omega)$, for, if $\varphi \to 0$ in $L^2$ and $A_i\varphi \to 0$ in $L^2$, then $\langle S, \overline{\varphi}\rangle \to 0$. Hence $S \in H'_o(A;\Omega)$.

Notice that the above representation $S = g + \sum A^*_i f_i$ is *not unique*.

**Corollary.** *$A^*_i$ is a continuous mapping of $L^2$ into $H'_o(A;\Omega)$.*

## 1.7 Regularization

When $\Omega = R^n$, we write simply $H(A)$, $\mathscr{D}$ instead of $H(A;\Omega)\mathscr{D}(\Omega)$, etc.

Let $\rho_k$ be a sequence in $\mathscr{D}$ such that

1) $\rho_k \geq 0$,

2) $\int\limits_{R^n} \rho_k(x)dx = 1$

3) Support of $\rho_k \subset B_{r_k}, r_k \to 0$, $B_{r_k}$ is the ball of radius $r_k$.

Such a sequence exists; for let $\rho \in \mathscr{D}$ be such that $\rho \geq 0, \int \rho dx = 1$ and the support of $\rho$ is contained in the ball $|x| < 1$. We obtain such a function by considering $\begin{cases} ae^{-\frac{1}{1-|x|^2}} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$ with suitable $a$ to make the integral equal to 1. Let $\rho'_k = \rho(kx).\rho_k$ have their support in the balls $|x| < \dfrac{1}{k}$. Let $\int\limits_{\sigma} \rho'_k dx = \alpha_k$. Then $\rho_k(x) = \alpha_k.\rho'(kx)$ is a sequence of the required type.

Such a sequence is called a regularization sequence. **8**

**Theorem 1.4.** *1) If $u \in H(A)$, then $u^*\varphi \in H(A)$, for $\varphi \in \mathscr{D}$, where $*$ denotes the convolution product.*

*2)* $u^*\rho_k \to u$ *in* $H(A)$*, where* $\rho_k$ *is a regularization sequence.*

*Proof.* 1) Since $u \in L^2$ and $A_i u \in L^2$ for $\varphi \in \mathscr{D}, u^*\varphi \in L^2$ and $A_i(u * \varphi) = u * A_i\varphi \in L^2$. Hence $u * \varphi H(A)$.

2) $u * \rho_k \to u$ and $A_i(u * \rho_k) = A_i(u) * \rho_k \to A_i u$ in $L^2$. Hence $u * \rho_K$ tends to $u$ in $H(A)$.

□

## 1.8 Problem of local type.

*In general if* $uH(A;\Omega)$ *and* $\varphi \in \mathscr{D}(\Omega)$*, it is not true* that $\varphi u$ is in $H(A;\Omega)$. The problem of determining sufficient conditions in order that $\varphi u$ should be in $H(A;\Omega)$ is the problem of local type.

## 1.9 Some generalizations.

Beside considering operators $A_i$ with constant coefficients, we could consider the case of operators with variable coefficients $A = \sum a_p(x).D^p, a_p(x) \in \xi(\Omega)$. (It is also possible, of course, to consider operator with not "smooth" coefficients). We could define as above $H(A;\Omega)$ to be the space of $u \in L^2(\Omega)$ such that $A_i u \in L^2(\Omega)$. Similarly as before we can prove that $H(A;\Omega)$ is a Hilbert space. We can consider also then the problem of determining $H'_o(A;\Omega)$. However, if $A_i$ are of variable coefficients $A_i(\rho_k * u) \neq (A_i u) * \rho_k$ so that theorem 1.4 is no longer true.

We could replace $L^2(\Omega)$ by any normal space of distributions $E$, i.e., a space $E$ such that $\mathscr{D}(\Omega), \subset E \subset \mathscr{D}'(\Omega)$ the inclusion being continuous, and $\mathscr{D}$ being everywhere dense in $E$. $H(A, E, \Omega)$ will be the space of $u \in E$ for which $A_i u \in E$. We topologize $H(A, E, \Omega)$ in such a way that the mapping $\underset{u \to u}{u \to A_i u}$ are continuous from $H(A, E, \Omega)$ to $E$. If, for instant, $E$ is a Frechet space, then $H(A, E, \Omega)$ also is a Frechet space. **9**

# Lecture 3

## 2 Spaces $H^m$.

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### 2.1

**Definition 2.1.** *$u \in H^m(\Omega) \Leftrightarrow D^p u \in L^2(\Omega)$ for $|p| \leq m[D^o u = u]$. Hence $H^m(\Omega) = H(A; \Omega)$, where $A = \{D^p, |p| \leq m\}$. If we write $|u|_k^2 = \sum\limits_{|p|=k} |D^p u|_o^2$ and $\| u \|_m^2 = \sum\limits_{k \leq m} |u|_k^2$, then the norm in $H^m(\Omega)$ is $\| u \|_m$.*

By theorem 1.1 and propositions 1.5 and 1.3, we have

**Theorem 2.1.** *$H^m(\Omega)$ is a Hilbert space. In order that a distribution $T$ on $\Omega$ belongs to $H_o'^m(\Omega)$ it is necessary and sufficient that $T = \sum\limits_{|p| \leq m} D^p f_p$ for $f_p \in L^2(\Omega)$.*

We shall write $H_o^m(\Omega) = H^{-m}(\Omega)$.

**Proposition 2.1.** *$u \in H^m(R^n)$, if and only if $\hat{u} \in L^2$ and $\xi^p \hat{u} \in L^2$ for $|p| \leq m$. Or equivalently, if and only if $(1 + |\xi|^m)\hat{u} \in L^2$ where $|\xi|^2 = \xi_1^2 + \cdots + \xi_n^2$.*

Regarding the local nature of $H^m(\Omega)$, we have the

**Proposition 2.2.** *Let $u \in H^m(\Omega)$ (respectively $H_o^m(\Omega)$) and $\varphi \in \mathscr{D}(\Omega)$. Then $(i)\varphi u \in H^m(\Omega)$ (respectively $H_o^m(\Omega)$), $(ii) u \to \varphi.u$ is a continuous mapping from $H^m(\Omega)$ to $H^m(\Omega)$ (respectively $H_o^m(\Omega)$ to $H_o^m(\Omega)$).*

This theorem holds actually with $\varphi \in L^\infty(\Omega)$ such that $D^p\varphi \in L^\infty$ for $|p| \leq m$.

Let $X$ be a closed set in $R^n$. Write $H_X^{-m} = \{T \in H^{-m}(R^n)$ such that the support of $T \subset X\}$.

**Definition 2.2.** *$X$ is said to be m-polar if $H_X^{-m} = 0$, i.e., if they only distribution* 11
*of $H^{-m}(R^n)$ with support in $X$ is 0.*

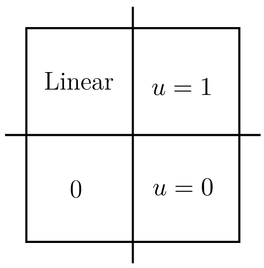
We shall see later that if $2m > n, X$ is void. We shall admit, without proof,

**Theorem 2.2.** *$H^m(\Omega) = H_o^m(\Omega)$ if and only if [$\Omega$ is m- polar.*

## 2.2 Extension of functions in $H_o(A;\Omega)$ to $R^n$.

**Definition 2.3.** *An open set $\Omega \subset R^n$ is said satisfy $m$-*extension property *if we can find a continuous linear mapping $\pi$ of $H^m(\Omega)$ to $H^m(R^n)$ such that $\pi u = u$ a.e. in $\Omega$.*

There are examples to show that not all $\Omega$ posses this property. For example in the case $m = 1, n = 2$ take the domain in the figure, which is an open square with the thickened line removed. Let u be a function as indicated in the figure. Let $\varphi$ be a $C^\infty$ functions which vanishes outside the unit circle, is 1 within a smaller circle and $0 \leq \varphi \leq 1$ elsewhere. Then $v = \varphi u$ is zero on the boundary of the given square. We now prove that it is impossible to find to find $V$ such that $V = va.e.$ on $\Omega$. For, if $V = v$, a.e. on $\Omega$, then




$$\frac{\partial v}{\partial y} = \begin{cases} \text{sometimes outside } \Omega. \\ \frac{\partial \varphi}{\partial y} u + \varphi \frac{\partial u}{\partial y} \text{ in } \Omega. \end{cases}$$

Now $\dfrac{\partial u}{\partial y}$ is a measure supported by th thickened line. Hence $\dfrac{\partial V}{\partial y}$ is not a function.

However, in the following two theorems, it will be proved that some usual domains posses the $m$-extension property.

**Theorem 2.3.** *Let $\Omega = \{x_n > 0\} = R_n^+$. Then $\Omega$ has m-extension property for* 12
*any m.*

Let $\mathscr{D}(\bar{\Omega})$ be the restrictions of functions of $\mathscr{D}(R^n)$ to $\Omega$. We require the following

**Lemma.** *$H^m(\Omega) \cup \mathscr{D}(\bar{\Omega})$ is dense in $H^m(\Omega)$.*

Assume for the time being this lemma, we shall first complete the proof of the theorem 2.3. It is enough to show that $\pi$ can be defined continuously on $H^m(\Omega) \cap \mathscr{D}(\bar{\Omega})$. We do this explicitly as follows: For $u \in H^m(\Omega) \cap \mathscr{D}(\bar{\Omega})$, put

$$\pi(u(x)) \qquad = \begin{cases} u(x) & \text{if } x_n \geq 0. \\ \lambda_1 & u(x', -x_n) + \cdots + \lambda_m u(x', -\frac{x_n}{m}) = v(x) \\ & \text{if } x_n < 0. \end{cases}$$

where $x' = (x_1, \ldots, x_{n-1})$.

We determine $\lambda_i$ in order to ensure that $\pi(u(x)) \in H^m(R^n)$. For that we need verify

$$(v(x', 0) = u(x', 0), \qquad \text{i.e.} \lambda_1 + \cdots + \lambda_m = 1.$$

$$\vdots$$

$$\frac{\partial^{m-1} v}{\partial^{m-1} x_n}(x', 0) = \frac{\partial^{m-1} u}{\partial^{m-1} x_n}, \text{ i.e., } (-1)^{m-1}(\lambda_1 + \cdots \frac{\lambda m}{m^{m-1}}) = 1.$$

These equations determine $\lambda_1' s$ and it is at once seen that $D^p(\pi(u)) = D^p u$ for $|p| \leq m$, a.e, on $\Omega$ and that the mapping $\pi$ is continuous.

Now we prove the lemma.

Let $u \in H^m(\Omega)$; for every $\in> 0$, define $u_\epsilon(x) = u(x', x_n + \in)$. Let $v_\epsilon$ be the restriction of $u_\epsilon$ to. It is easy to see that $u_\in \to u$ in $H^m(\Omega)$ as $\in\to 0$, and so we need prove only that $v_\epsilon$ for every fixed $\epsilon > 0$ can be approximated by functions of $H^m(\Omega) \cap \mathscr{D}(\bar{\Omega})$, i.e., we have to prove that given a function $w \in H^m(\Omega_\alpha)$, where $\omega_\alpha$ is the domain $\{x_n > -\alpha\}, w$ can be approximated on $\bar{\Omega}$ by functions of $H^m(\Omega) \cap \mathscr{D}(\bar{\Omega})$. Let $\cup(x_n)$ be a $C^\infty$ function defined as follow: $\theta = 0$ for $x_n < -\alpha, 1$ for $x_n > 0, 0 < \theta < 1$, elsewhere. Now $\theta w \in H^m(R^n)$ and $\theta w = v$ a.e. in $\Omega$. However, $\mathscr{D}(R^n)$ is dense in $H^m(R^n)$. Hence there exists a sequences $\phi_k \in \mathscr{D}(R^n)$ such that $\phi \underline{\to} \theta w$ in $H^m(R^n)$. Let $\varphi_k$ be the restriction of $\phi_k$ to $\Omega$. Then $\varphi_k \in H^m(\Omega) \cap \mathscr{D}(\bar{\Omega})$ and $\varphi_k \to \theta w = w$ in $H^m(\Omega)$. 13

**Remark 1.** If $\Omega$ has $m$-extension property, then the above lemma holds, i.e., $H^m(\Omega) \cap \mathscr{D}(\bar{\Omega})$ is dense in $H^m(\Omega)$. For, since $\mathscr{D}$ is dense in $H^m(R^n)$, and since there exists a continuous mapping $\pi$ of $H^m(\Omega)$ in $H^m(R^n)$, the restrictions of functions of $\mathscr{D}$ to $\Omega$ are dense in $H^m(\Omega)$.

**Remark 2.** This lemma holds also, for instance, for star-shaped domains.

## 2.3

**Theorem 2.4.** *Let $\Omega$ be an open bounded set such that the boundary of $\Omega$ is an $(n-1)$ dimensional $C^m$ manifold $\Gamma\Omega$ lying on one side of $\Gamma$. Then $\Omega$ has the* *m-extension property.*

*Proof.* Let $\mathbb{Z}^n$ be the $m$-dimensional Euclidean space with coordinates $\xi, \ldots, \xi_n$ and let $W$ be the open rectangle define by $\begin{cases} 0 < \xi_i < 1 \\ -1 < \xi_n < 1 \end{cases}$ $i = 1, \ldots, n-1$. Let $W_+, W_-, W_o$ denote the subsets of $W$ determined by $\xi > 0, \xi_n < 0, \xi_n = 0$, respectively. □

On account of the hypothesis on $\Gamma$, there exists a finite open covering $O', O_i$ of $\bar{\Omega}$ and $m$-times continuously differentiable functions $\psi_i$ of $W$ to $O_i$ such that $\psi_i$ maps $W_-$ onto $O_i \cap \Gamma$, $W_+$ onto $O_i \cap [\bar{\Omega}$ and $W_o$ onto $O_i \cap \Gamma$, and further, $O_i \cap \Gamma$ cover $\Gamma$ and Let $(a', a_i)$ be a $C^m$ partition of unity subordinate to this covering. If $u \in H^m(\Omega)$, then $u = a'u + \sum a_i u$ and $a_i u$ have their supports in $O_i$ respectively. Now $\psi_i$ defines an isomorphism of $H^m(O_i)$ onto $H^m(W)$ and of $H^m(0_i \cap \Omega)$ onto $H^m(W_-)$, which we still denote by $\psi_i$. Hence $v_i = \psi_i^{-1}(a_i u) \in H^m(W_-)$ and $v_i = 0$ near the part of the boundary of $W_-$ which is not contained in $\xi_n = 0$. Hence $v_i$ can be extended to $(\xi_n < 0)$ by putting it equal to zero outside $W_-$. By theorem 2.3, there exists $\pi v_i \in H^m(\mathbb{Z}^n)$ such that $\pi v_i = v_i, a.e.$ on $\mathbb{Z}^n$. Let $\theta(x_n)$ be a $C^\infty$ function which is 0 for $\xi_n > \dfrac{2}{3}$ and 1 for $\xi < \dfrac{1}{3}, 0 < \theta < 1$ elsewhere. Let $p(\xi) = \theta\pi(v_i)$. $P_i(\xi)$ has its support in $W$ and is zero near the boundary of $W$. Let $\varphi_i(x) = \psi_i(P)$. Then $\varphi(x) \in H^m(O_i)$ and is zero near the boundary of $O_i$. Hence $\pi(u) = a'_u + \sum \tilde{\varphi}_i(x)$ answers the theorem. 14

# Lecture 4

## 2.4 The mapping $\gamma$.

Let $\mathscr{H}^m(\Omega) = H^m\Omega \cap \mathscr{D}(\bar{\Omega})$. For the function $f \in \mathscr{H}^m(\Omega)$, the restriction of $f$ to the boundary $\Gamma$ of $\Omega$ defines a function $\gamma f$ on $\Gamma$. We wish to know for what for what spaces $X(\Gamma)$ of function on $\Gamma$, this mapping $\gamma$ of $\mathscr{H}^L(\Omega)$ to $X(\Gamma)$ is continuous. If $\gamma$ is continuous, we can extend $\nu$ to $H^1(\Omega)$ if, for example, $\Omega$ has 1-extension property, and $\gamma u$ for $u \in H'(\Omega)$ may be considered as generalized boundary value of $u$. 15

**Theorem 2.5.** *Let $\Omega = \{x_n > 0\}$ so that $\Gamma = \{x_n = 0\}$. Let $X(\Gamma) = L^2(\Gamma) = L^2(R^{n-1})$. Then $u \to \gamma u$ is a continuous mapping of $\mathscr{H}^1(\Omega) \to L^2(\Gamma)$, i.e., there exists a unique mapping $\gamma : H^1(\Omega) \to L^2(\Gamma)$ which on $\mathscr{H}^1(\Omega)$ is restriction.*

*Proof.* Let $x' = (x_1, \ldots, x_{n-1})$. Let $\mathscr{O}(x_n)$ be a function defined for $x_n > 0$, zero for $x_n > 1$, and $0 < \mathscr{O}(x_n) < 1$ in $(0, 1)$. We have $|u(x', 0)|^2 = -\int_0^\infty \frac{\partial}{\partial x_n}(u(x)\bar{u}(x)\theta(x_n))dx$. Hence □

$$\begin{aligned}\int\limits_{R^{n-1}} |u(x', 0)|^2 dx' &= -\int\limits_{\Omega} \frac{\partial}{\partial x_n}(\theta(x_n)u(x)\bar{u}(x))dx \\ &= \int\limits_{\Omega} \theta'|u|^2 dx - \int\limits_{\Omega} \theta(\frac{\partial u}{\partial x_n}\bar{u} + \frac{\partial \bar{u}}{\partial x_n})dx.\end{aligned}$$

So by Schwartz's inequality,

$$\int_\Gamma |u(x', 0)|^2 dx' \leq C(\int_\Omega |u|^2 dx + \int\limits_\Omega |\frac{\partial u}{\partial x_n}|^2 dx),$$

$$\leq C \parallel u \parallel_1^2 .$$

This means $\gamma$ is continuous.

**Remark.** Let $A = \dfrac{\partial}{\partial x_n}$. Then $H(A;\Omega) \cap \mathscr{D}(\overline{\Omega})$ is dense in $H(A;\Omega)$, and by the same method as above, $u \to \gamma u$ is continuous from $H(A;\Omega)$ to $L^2(\Gamma)$. **16**

In the text few propositions, we are going to determine the image and the kernel of $\gamma$.

We have seen that $u \in H^m(R^n)$ if and only if $u \in L^2$ and $(1 + |\xi|^m)\hat{u} \in L^2$. Generalizing we define $H^\alpha(R^n)$ for non-integer $\alpha > 0$.

**Definition 2.4.** *$u \in H^\alpha(R^n)$ if and only if $u \in L^2$ and $(1 + | \in |^\alpha)\tilde{u} \in L^2$. On $H^\alpha(R^n)$, we put the topology defined by the norm*

$$(u, u)_{H^\alpha(R^n)} = ((1 + |\xi|^\alpha)\hat{u})_{L^2(\mathbb{R}^n)}.$$

**Theorem 2.6.** *Let $\Omega = \{x_n > 0\}$. For $u \in H^1(\Omega)$, we have*

1) $\gamma u \in H^{\frac{1}{2}}(\Gamma)$, and

2) $u \to \gamma u$ is continuous mapping of $H^1(\Omega)$ onto $H^{\frac{1}{2}}(\Gamma)$.

*Proof.* Let $\xi = (\xi_1, \ldots, \xi_{n-1})_{n-1}$ and $\hat{u}(\xi', x_n) = \int\limits_{R^{n-1}} e^{-2\pi i x'.\xi'} u(x', x_n) dx'$ be truncated Fourier transform of $u(x)$. Since ◻

$$\mathscr{F}\left(\frac{\partial u}{\partial x_i}\right) = \xi_i \hat{u} \xi L^2(\xi', x_n), i = 1, \ldots, n-1,$$

we have

1) $(1 + |\xi'|)u \in L^2(\xi', x_n)$.

Further, since $\dfrac{\partial \hat{u}}{\partial x_n} = \dfrac{\partial \hat{u}}{\partial x_n}$ we have

2) $\dfrac{\partial \hat{u}}{\partial x_n} \in L^2(\xi', x_n)$.

Now, as in theorem 2.5, $|\hat{u}(\xi', 0)|^2 = -\int_o^\infty (\hat{u}\overline{\hat{u}}\theta) dx_n$. Hence $\int\limits_\Gamma (1 + |\xi|)|\hat{u}(\xi', 0)|^2 d\xi' = -\int\limits_{R_n^+} (1 + |\xi'|) \dfrac{\partial}{\partial x_n} (\hat{u}\overline{\hat{u}}\theta) dx < \infty$ by Schwartz's inequality and (1) and (2).

Hence $\gamma\, u\ \in\ H^{\frac{1}{2}}(\Gamma)$.

We now prove the second point. $f\ \in\ H^{\frac{1}{2}}(\Gamma)$ of and only if $(1+|\xi|^{\frac{1}{2}})\hat{f}\ \in$ **17**
$L^2(\mathbb{Z}^n)$. We have to look for a function $u \in H^1(\Omega)$ such that $\gamma u = f$. Let

$$U(\xi', x_n) = \exp(-(+|\xi'|)x_n)\hat{f}(\xi') \text{ for } x_n > 0,$$

and $u = \mathscr{F}_\xi(U(\xi', x_n))$. We prove that $u\xi H^1$ and $\gamma u = f$. The only not completely trivial point is to verify that $\dfrac{\partial u}{\partial x_n}\xi L^2$.

$$\frac{\partial u}{\partial x_n} = -(1+|\xi|)\exp(-(1+|\xi|x_n)\hat{f}(\xi').$$

Hence

$$\begin{aligned}\int |\frac{\partial u}{\partial x_n}|^2 dx'_n &= (1+|\xi|^2|\hat{f}(\xi')|^2 \int_o^\infty \exp(-(1+|\xi|)x_n)dx_n\\ &= (1+|\xi||\hat{f}(\xi')|^2.\end{aligned}$$

Since $f \in H^{\frac{1}{2}}(\Gamma)$, we have $\int\limits_\Omega |\dfrac{\partial u}{\partial x_n}|^2 dx$ is finite.

**Theorem 2.7.** $\gamma\, u = 0$ *if and only if* $u\ \in\ H^1_o(\Omega)$.

*Proof.* (a) $u\ \in\ H^1_o(\Omega) \implies u = 0$ for we have $u = \lim \varphi_k$ in $H^1(\Omega)\varphi_k \in \mathscr{D}(\Omega).\gamma u = \lim \gamma(\varphi_k)$ in $L^2(\Gamma) = 0$.

(b) Conversely to prove that $\gamma\, u = 0$ implies $u \in\ H^1_O(\Omega)$ we require the

□

**Lemma.** *Let* $\Omega = \{x_n > 0\}, u \in H^1_O(\Omega), \phi \in \mathscr{D}(\overline{\Omega})$. *Then* $\phi u \in H^1(\Omega)$, *and* $\gamma(\phi u) = \gamma(\phi)\gamma(u)$.

*Proof.* We know $\mathscr{H}^1(\Omega)$ is dense in $H^1(\Omega)$. Hence there exists $u_k \in \mathscr{H}^1(\Omega)$, such that $u_k \to u$ in $H^1(\Omega)$. Now, $\phi u_k \to \phi u$ in $H^1(\Omega)$ and since $\gamma(\phi u_k) = \gamma(\phi)\gamma(u_k)$, we have $\gamma(\phi u) = \gamma(\phi)\gamma(u)$. □

Coming back to the proof of the theorem, let $a(x)$ be a $C^\infty$ function 1 on **18**
the unit ball, 0 outside another ball, and $0 \leq a(x) \leq 1$ else-where. Then if we define $a_j(x) = a(x/j)$, we have $a_j u \to u$ in $H^1(\Omega)$. Hence if we prove that $a_j u \in H^1_0(\Omega)$, we shall have proved that $u \in H^1_o(\Omega)$. Since $a_j u$ has compact support, and since $\gamma\langle a_j u\rangle = a_j \gamma u = 0$, this means, we may assume, that in

addition to $\gamma u = 0, u$ has compact support in $\bar{\Omega}$. Let $\mathscr{O}_k(x_n)$ be a function defined for $x_n > 0$,

$$\mathscr{O}_k(x) = \begin{cases} 0 & \text{for } 0 < x_n < 1/k \\ \text{linear} & \text{for } 1/k \le x_n \le 2/k \\ 1 & \text{for } x_n > 2/k. \end{cases}$$

Then $\widetilde{O_k u} \in H^1(R^n)$. By regularization, we may assume that $\mathscr{O}_k u \in H^1_0(\Omega)$. We now prove $\mathscr{O}_k u \to u$ in $H^1(\Omega)$. We have

$$\theta_k \frac{\partial u}{\partial x_i} = \frac{\partial}{\partial x_i}(\theta_k(u)) \to \frac{\partial u}{\partial x_i} \text{ for } 1 \le i \le n-1.$$

We have to prove then that $\theta'_k(x_n)u + \theta_k \dfrac{\partial u}{\partial x_n} \to \dfrac{\partial u}{\partial x_n}$; that is to say, $\theta'_k(x_n)u \to 0$. Now

$$\theta'_k(x_n) = \begin{cases} 0 & \text{for } x_n < 1/k \text{ and } x_n > 2/k \\ k & \text{for } 1/k < x_n < 2/k. \end{cases}$$

Further $u(x', x_n) = -\int_0^{x_n} \dfrac{\partial u}{\partial t}(x', t)$ dt. Hence

$$\begin{aligned} |u(x', x_n)|^2 &\le x_n \int_0^{x_n} |\frac{\partial u}{\partial t}(x', t)|^2 dt \\ &\le 2/k \int_0^{x_n} |\frac{\partial u}{\partial t}(x', t)|^2 dt \text{ if } x_n > 2/k \\ & \quad 2/k.C \qquad\qquad \text{if } x_n > 2/k \end{aligned} \tag{1}$$

Also 19

$$\begin{aligned} \int |\theta'_k(x_n)|^2 |u(x', x_n)|^2 dx_n &= \int_0^{2/k} |\theta'_k u|^2 dx \\ &\le 2k \int_0^{2/k} dx_n \int_0^{x_n} \left| \frac{\partial u}{\partial t}(x', t) \right|^2 dx' \\ &= 2k \int_0^{2/k} \left| \frac{\partial u}{\partial t}(x', t) \right|^2 dt \int_t^{2/k} dx_n \end{aligned}$$

(by changing the order of integration)

$$\leq 4 \int_0^{2/k} |\frac{\partial u}{\partial x_n}|^2 dx_n.$$

Hence $\int |\theta'_k u|^2 dx \to 0$ by (1), which completes the proof.

Thus we see that $H'_o(\Omega)$ is the space of functions which are weakly zero on the boundary.

The above results can be generalized to spaces $H^m(\Omega)$. Let $\Omega = \{x_n > 0\}$; $u \in H^m(\Omega)$, if and only if $D^p(u) \in H^1(\Omega)$, for $|p| \leq m-1$. Hence $\gamma D^p u$ can be defined as above for $|p| \leq m-1$. we have the

**Theorem 2.8.** *$u \in H^m_o(\Omega)$ if and only if $\gamma(D^p u) = 0$ for $|p| \leq m-1$.*

In fact, we can say something more,

**Exercise.** Let $u \in H^m_o(\Omega)$ and $u \in D^p_x$, with $|q|$ arbitrary.

Then

$$\gamma(D^q_{x'}, u) = D^q_{x'}, (\gamma u).$$

## 2.5

Let $\Omega$ be an open set of $R^n$ such that $(a)\Omega$ has 1-extension property, and $(b)$ the boundary $\Gamma$ of $\Omega$ is an $(n-1)$ dimensional $C^1$ manifold. In the case $\Omega$ is bounded $(b)$ implies $(a)$. On $\Gamma$ we have an intrinsic measure. We denote by $L^2_{\text{Loc}}(\Gamma)$ the space of square summable functions on every compact of $\Gamma$ with respect to this measure.

**Theorem 2.9.** *Under the above hypothesis on $\Omega$ (i.e.)* **20**

a) *$\Omega$ possesses 1 extension property*

b) *$\Gamma$ is an $(n-1)$ dimensional $C^1$ manifold,*

*there exists a unique continuous map $\gamma : H^1(\Omega) \to L^2_{loc}(\Gamma)$ which on functions of $\mathscr{H}^1(\Omega)$ coincides with the restriction to $\Gamma$.*

*Proof.* Form (a) it follows that $\mathscr{H}'(\Omega)$ is dense in $\mathscr{H}'(\Omega)$ and hence the uniqueness will follow from the existences. Let $\Gamma_2$ be any compact of $\Gamma$. We observe that by using a $C^1$ partition of unity the problem is reduced to a local one, that is to say, we may assume that the support of $\varphi \in \mathscr{H}^1(\Omega)$ is contained in an open set 0 and that there exists a homeomorphism $\psi$ as in theorem 2.4. Further we may assume, without loss of generality, that $\Gamma_2 \subset 0$. Let $a$ be a $C^\infty$ function in $R^n$ with compact support in 0 and which is 1 on $\gamma_2$. Then

$au\psi^{-1} \in H(\mathbb{Z}^n)$ (in the notation of theorem 2.4), and $\gamma au\psi^{-1} \in L^2(W_o)$. Since $\psi$ defines an isomorphism of $L^2(\Gamma_2)$ into $L^2(W_o), \psi(\gamma au\psi^{-1}) \in L^2(\Gamma_2)$. Now on $\Gamma_2$ we have $\gamma u = \psi(\gamma au\psi^{-1})$ which proves that $\gamma$ is continuous mapping of $\mathscr{H}^1(\Omega)$ into $L^2(\Gamma_2)$ which completes the proof. $\square$

**Remark.** A complete generalization of theorem 2.7 is due to N. Aronszajn [1].

# Lecture 5

## 3 General Elliptic Boundary Value Problems

### 3.1 General theory.

We formulate at the beginning certain problems on topological vector spaces **21**
and solve them. Later on we shall show how these answers will help us in solving many of the classical boundary value problems for elliptic differential equations.

As a matter of notation, we shall write $A \subset B$, where $A$ and $B$ are two topological vector spaces to mean the injection $i : A \to B$ is continuous or that the topology $A$ is finer than the topology induced by $B$.

Let $V$ be a Hilbert space over complex numbers. We shall denote by $|u|_V$ the norm in $V$. Let $Q$ be a locally convex topological vector space such that

1) $V \subset Q$ and $V$ is dense in $Q$;

2) On $Q$ an involution (i.e., an anti-linear isomorphism of order two) $f \to \overline{f}$ is given which leaves $V$ invariant;

3) Let $V$ be given a continuous sesquilinear form $a(u, v)$(i.e., $a(\lambda u, v) = \lambda a$ $(u, v)$, and $a(u, \lambda v) = \overline{\lambda} a(u, v)$. Let $Q'$ be the dual space of $Q$. On $Q'$ an involution is induced by the given one in $Q$ by the following formula $< \overline{f}, g > = < f, \overline{g} >$.

   We raise now the

   **Problem 3.1.** Give $f \in Q'$ does there exist a $u \in V$ such that

4) a $(u, v) = < f, \overline{v} >$ for all $v \in V$.

We shall show later that large classes of elliptic problems can be put in this form.

**Definition 3.1.** *The space $\underline{N}$ will consists of all $u \in V$ such that the mapping $v \to a(u, v)$ is continuous on $V$ with the* topology of $Q$. 22

*Since $V$ is dense in $Q$ we can extend this mapping to $Q$. Hence for every $u \in N$ we have an $Au \in Q'$ such that*

5) $a(u, v) =< Au, \overline{v} >$.

The mapping $A : N \to Q'$ is linear. On $N$ we introduce the upper bound topology to make the mapping $N \to V$ and $A : N \to Q'$ continuous. We ask now the

**Problem 3.2.** Is the mapping $A$ onto $Q'$?

**Lemma 3.1.** *Problem 1 is equivalent to problem 2.*

*Proof.* Let $f \in Q'$ and let $u$ be a solution of problem 1, i.e., $a(u, v) =< f, \overline{v} >$. Hence the mapping $v \to a(u, v) =< f, \overline{v} >$ is continuous on $V$ with the topology of $Q$. Hence $u \in N$. Further $< Au, \overline{v} >= a(u, v) =< f, \overline{v} >$ for all $v \in V$, and since $V$ is dense in $Q, Au = f$. Conversely, let $f \in Q'$ be given and $u \in N$ be such that $Au = f$. Then $a(u, v) =< Au, \overline{v} >=< f, \overline{v} >$, for all $v \in V$, i.e., $u$ is a solution of problem 1. □

## 3.2

We now consider certain sufficient handy condition so that $A$ should be an isomorphism of $N$ onto $Q'$

**Definition 3.2.** *We shall say that the sesquilinear $a(u, v)$ is elliptic on $V$, or is $V$-elliptic, if there exists an $\alpha > 0$ such that*

$$\mathrm{Re}(a(u, u)) \geq \alpha |u|_V^2 \text{ for all } u \in V.$$

**Theorem 3.1.** *Let $V, Q, a(u, v)$ be as given in § 3.1. If $a(u, v)$ is* V-elliptic*, then $A$ is an* isomorphism *of $N$ onto $Q'$.*

*Proof.* Let 23

$$a_1(u, v) = \frac{1}{2}[a(u, v) + i\, \overline{a(v, u)}]$$
$$\text{and} \quad a_2(u, v) = \frac{1}{2} i[a(u, v) - \overline{a(v, u)}].$$

Then $a_1(u,v)$ and $a_2(u,v)$ are hermitian and

$$a(u,v) = a_1(u,v) + ia_2(u,v).$$
$$\textit{Put} \quad [u,v] = a_1(u,v).$$

Since $|a(u,v)| \leq C|u|_V|v|_V$, it follows that $[u,u] \leq C|u|_V^2$. On account of the V-ellipticity, $[u,u] = \operatorname{Re} a(u,u) \geq \alpha|u|_V^2$. Hence the form $[u,v]$ defines on $V$ an Hilbertian structure equivalent to the one defined by $(u,v)_V$. □

Now, any $f \in Q'$ defines a continuous semi-linear function on $V$ and hence there exists $Kf$ such that

$$< f, \overline{v} >= [Kf, v], \quad K \in \mathscr{L}(Q', V).$$

For a fixed $u \in V$, the mapping $v \to a_2(u,v)$ is a semi-linear continuous mapping on $V$, hence

$$a_2(u,v) = [Hu, v].$$

Further $H$ is hermitian for the scalar product defined by $[u,v]$. For $[Hu,v] = a_2(u,v) = \overline{a_2(v,u)} = [\overline{Hv,u}] = [u, Hv]$.

Hence $\qquad a(u,v) = [u,v] + i[Hu,v],$

and we have to solve $\qquad a(u,v) =< f, \overline{v} >= [Kf, v],$

i.e., $\qquad (1+iH)u = Kf.$

From Hilbert space theory, we know that if $H$ is hermitian $(1+iH)$ is non-singular. Hence

$$u = (1+iH)^{-1}Kf,$$

which proves that $A$ is an isomorphism.

# Lecture 6

## 3.3 Examples: $\Delta + \lambda, \lambda > 0, \Delta = \sum\limits_{i=1}^{n} \dfrac{\partial^2 u}{\partial x_i^2}$.

Let $\Omega$ be an open set in $R^n$ and $H^1(\Omega), H_0^1(\Omega)$ be as defined before. Let $V$ be a 24
closed subspace of $H^1$ such that $H_0^1 \subset V \subset H^1$. The metric on $V$ is one induced by $H^1 : (u, v)_V = (u, v)_1$. Let $Q$ be $L^2(\Omega)$ with the involution $f \to \overline{f}$. Then $V \subset Q$ and is dense in $Q$. On $V$ consider the sesquilinear form

$$a(u, v) = (u, v)_1 + (u, v), \lambda > 0.$$

Then $a(u, v)$ is continuous on $V \times V$ and

$$\begin{aligned}
\operatorname{Re}(a(u, u)) &= |u|_0^2 + \lambda |u|_1^2 \geq \min(1, \lambda)(|u|_0^2 + |u|_1^2) \\
&= \alpha ||u||_1^2, \quad \alpha > 0.
\end{aligned}$$

Hence $a(u, v)$ is V-elliptic. Hence, for a given $f \in L^2(\Omega) = Q'$ we have $u \in V$ such that $a(u, v) =< f, \overline{v} >$ for all $v \in V$. We determine $N$ and $A$ explicitly in this case.

**Proposition 3.1.**

1) $A = -\Delta u + \lambda u$ for $u \in N \lambda > 0$

2) $u \in N \Leftrightarrow \begin{cases} u \in V, \Delta u \in L^2 & \text{and} \\ (-\Delta u, v)_0 = (u, v)_1 & \text{for all} \quad v \in V. \end{cases}$

*Proof.* We know $u \in N$ if and only if $u \in V$ and the mapping $v \to a(u, v)$ is continuous on $V$ with the topology of $Q$. Further since $a(u, v)$ is V-elliptic, for $f \in L^2$ there exists $u \in N$ such that $a(u, v) =< Au, \overline{v} >=< f, \overline{v} >$. Let $v = \varphi \in \mathscr{D}(\Omega)$. Then

$$(u, \varphi)_1 + \lambda(u, \varphi)_0 = (Au, \varphi)_0.$$

$$\text{Now,} \qquad (u, \varphi)_1 = \Sigma\left(\frac{\partial u}{\partial x_i}, \frac{\partial \varphi}{\partial x_i}\right)_0 =< -\Sigma \frac{\partial^2 u}{\partial x_i^2}, \varphi >$$
$$=< -\Delta u, \varphi > .$$

Hence $< -\Delta u, \overline{\varphi} > +\lambda < u, \overline{\varphi} >=< Au, \overline{\varphi} >=< f, \overline{\varphi} >$ for all $\varphi \in \mathscr{D}(\Omega)$. **25**
This means $A = -\Delta + \lambda$ and $-\Delta u + \lambda u = f$. Since $f \in L^2$ and $u \in L^2$ we have $\Delta u \in L^2$. Further, if $u \in N, a(u, v) =< Au, \overline{v} >$ and hence

$$(u, v)_1 + \lambda(u, v)_0 = (-\Delta u, v)_0 + \lambda(u, v)_0$$
$$\text{which gives} \qquad (u, v)_1 = (-\Delta u, v)_0.$$

Conversely if $u$ satisfies the above conditions, since $-\Delta u + \lambda u \in L^2$ the mapping $v \to a(u, v) = (u, v)_1 + \lambda(u, v)_0 = (-\Delta u + \lambda u, v)_0$, is continuous on $V$ in the topology induced by $Q$. Hence $u \in N$. ◻

Now we give a *formal* interpretation of $u \in N$. The correct meaning of this interpretation will be brought out later on. Assuming the boundary $\Gamma$ of $\Omega$ to be smooth, we have, by a formal Green's formula,

$$(-\Delta u, v)_0 = -\int_\Omega \Delta u.\overline{v} dx = \int_\Gamma \frac{\partial u}{\partial n} \overline{v} d\sigma + (u, v)_1$$

where $\frac{\partial u}{\partial n}$ is the normal derivative. However if $u \in N$, by proposition 3.1, we have

$$(-\Delta u, v)_0 = (u, v)_1.$$

Hence $u \in N$ implies $\int \frac{\partial u}{\partial n} \overline{v} d\sigma = 0$.

We now take particular cases of $V$ and interpret this formal result.

1) Let $V = H^1$. $u \in H^1$ is not a boundary condition, neither is $\Delta u \in L^2$. However, $\int\limits_\Gamma \frac{\partial u}{\partial n} \overline{v} d = 0$ for every $v \in H^1$, is a boundary condition, and formally this means $\frac{\partial u}{\partial n} = 0$, i.e., $u \in N$ implies the normal derivative vanishes.

2) Let $V = H_0^1 . u \in H_0^1$ implies $u = 0$ on the boundary, and hence is a boundary condition. $\Delta u \in L^2$ is not a boundary condition and $\int\limits_\Gamma \frac{\partial u}{\partial n}\bar{v} d\sigma$ is always zero for $v \in H_0^1$. **26**

3) Let $\Gamma_1$ be the subset of $\Gamma$ and define $V$ to consist of function $u \in H^1$ such that $\gamma u = 0$ on $\Gamma_1 . V$ is a closed subspace of $H^1$. $u \in N$ if and only if $u \in V$, that is to say, $\gamma u = 0$ on $\Gamma_1$; this is a boundary condition, $\Delta u \in L^2$ which is not a boundary condition, and $\int\limits_\Gamma \frac{\partial u}{\partial n}\bar{v} d\sigma = 0$ for $v \in V$, but since $\gamma v = 0$ on $\Gamma_1$, this means $\int\limits_{\Gamma - Q} \frac{\partial u}{\partial n}\bar{v} d\sigma = 0$ for all $v \in V$. This means again formally $\frac{\partial u}{\partial n} = 0$ on $\Gamma - \Gamma_1$. So formally the condition is $u = 0$ on $\Gamma_1$ and $\frac{\partial u}{\partial n} = 0$ on $\Gamma - \Gamma_1$.

We call 1), 2) and 3) weak homogeneous, Neumann, Dirichlet and mixed Dirichlet-Neumann problems respectively.

We may state the above results in the

**Theorem 3.2.** *If $\lambda > 0, \Omega$ an arbitrary open set in $R^n$, then the equation $-\Delta u + \lambda u = f$ with $f \in L^2(\Omega)$ has a unique solution with homogeneous boundary data.*

**Remarks.** *Non-homogeneous problems:* Corresponding to the homogeneous problems considered above, we may consider non-homogeneous ones in which not necessarily vanishing boundary values are prescribed. We shall show formally that this can be reduced to a homogeneous case together with a problem of first order partial differential equation.

**Problem 3.3.** Given $F \in L^2$ and $G \in V$ such that $\Delta G \in L^2$ determine $u$ such that $-\Delta U + \lambda U = F$ and $U - G \in N$. **27**

**Theorem 3.3.** *Problem 3.3 admits a unique solution for $\lambda > 0$.*

*Proof.* Put $u = U - G$. Then we have to seek $u$ such that

$$-\Delta u + \lambda u = F - (-\Delta + \lambda)G = f, \text{ say .}$$

Since $f \in L^2$, there exists unique $u \in N$ by theorem 3.2. □

In the case $V$ is as in examples 1), 2) and 3) respectively, this means $\frac{\partial u}{\partial n}$ on $\Gamma, G = U$ on $\Gamma$ and $U = G$ on $\Gamma_1$ and $\frac{\partial U}{\partial n} = \frac{\partial G}{\partial n}$ on $\Gamma - \Gamma_1$ respectively. The above solution of problem 3.3 implies then that if we wish to determine $U$ with $\dfrac{\partial U}{\partial n}, U$, given on the boundary, we have only to determine $G \in L^2$ satisfying $\Delta G \in L^2$ and $\dfrac{\partial U}{\partial n} = \dfrac{\partial G}{\partial n}$ and $U = G$ on the respective parts of the boundary.

**Remark.** If we take $V = H_0^1, Q = H^{-1}$ so that $Q' = H^{-1}$ we have

**Theorem 3.4.** *Given a distribution $T \in H^{-1}$, there exists a unique solution $U \in H_0^1$ such that $-\Delta u + \lambda u = T$.*

**Remark.** Roughly speaking we may say that the boundary conditions are introduced by means of the following two conditions :$(a) u \in V, (b) (-\Delta u, v)_0 = (u, v)_1$. The two extreme cases are $H_0^1$ (Dirichlet) and $H^1$ (Neumann) wherein, in the first case, only $u \in V$ is the boundary condition, and in the second one, $(-\Delta u, v)_0 = (u, v)_1$ is the boundary condition. The condition $u \in V$ may be considered to be *stable* and the other one *unstable*. Heuristically this may be justified as follows: if we consider smooth functions in $H^1(\Omega)$ such that $\dfrac{\partial u}{\partial n} = 0$, on completion this property no longer holds, so we may say this condition is unstable, while in the second case, the completion of smooth functions 28
vanishing on boundary still possesses this property in a weaker sense.

**Exercise 1.** With the hypothesis as in theorem 3.1, if $a(u, v)$ is V-elliptic, the existence of $u \in V$ such that $a(u, v) =< f, \overline{v} >$ for all $v \in V$ and any $f \in Q'$ can be carried on the following lines: Since the mappings $v \to< f, \overline{v} >$ and $v \to a(u, v)$ are continuous on $V$ with the topology induced by $Q$, there exists $Kf$ and $\tilde{A}f$ in $V$ such that

$$a(u, v) = (\tilde{A}u, v)_V < f, \overline{v} >= (Kf, v)_V.$$

Hence to solve the problem we require $\tilde{A}u = Kf$. This is proved if we prove $\tilde{A}$ is an isomorphism of $V$ onto $V$.

**Exercise 2.** The same results as in theorem 3.1 is true on a weaker assumption that

$$|a(u, v)| \geq \alpha |u|_V^2, \ \alpha > 0.$$

# Lecture 7

## 3.4

Hitherto we considered the particular case where $H_0^1 \subset V \subset H^1$. Now we shall consider a more general case in which $\mathscr{D} \subset V \subset Q \subset \mathscr{D}'$, $\mathscr{D}$ being *dense* in $Q$, but not necessarily in $V$. Involution in $Q$ is as before, viz. $f \to \overline{f}$. **29**

Let $a(u, v)$ be a continuous sesquilinear form on $V$. In this situation the operator $A$ and the space $N$ associated with $a(u, v)$ can be characterized in another way as follows. For a fixed $u \in V$, the mapping $\varphi \to a(u, \varphi)$ for $\varphi \in \mathscr{D}$ is a continuous semi-linear form on $\mathscr{D}(\Omega)$ and hence defines an element $\mathscr{A}u \in \mathscr{D}'(\Omega)$ so that $< \mathscr{A}u, \overline{\varphi} >= a(u, \varphi)$.

This defines a mapping $\mathscr{A} : V \to \mathscr{D}'(\Omega)$. Let $\eta$ be the space of $u \in V$ such that $(a)\mathscr{A}u \in Q'$ and $(b)$ $< \mathscr{A}u, \overline{\varphi} >= a(u, v)$ for all $v \in V$. On $\eta$ we introduce the topology so as to make both the injection $\eta \to V$ and the mapping $\mathscr{A} : \eta \to Q'$ continuous.

**Theorem 3.5.** *$\eta = N$ and for $u \in' N, \mathscr{A}u = Au$.*

*Proof.* 1) Let $u \in N$. Then $v \to a(u, v)$ is a continuous semilinear form on $V$ with the topology induced by $Q$ and $a(u, v) =< Au, \overline{v} >$ with $Au \in Q'$. This holds in particular if $v = \varphi \in \mathscr{D}(\Omega)$. Hence $a(u, \varphi) =< Au, \overline{\varphi} >=< \mathscr{A}u, \overline{\varphi} >$ for all $\varphi \in \mathscr{D}(\Omega)$. This means $\mathscr{A}u = Au$ and that $\mathscr{A}u \in Q'$. Hence $< \mathscr{A}u, \overline{v} >=< Au, \overline{v} >= a(u, v)$ for all $v \in V$ and so $u \in \mathcal{M}$.

2) Conversely, let $u \in \mathcal{M}$. Then $a(u, v) =< \mathscr{A}u, \overline{v} >$ for all $v \in V$ and $\mathscr{A}u = f \in Q'$. Hence $a(u, v) =< f, \overline{v} >$ so that the mapping $v \to a(u, v)$ is continuous on $V$ with the topology induced by $Q$. Hence $u \in N$ and $\mathscr{A}u = f = Au$. **30**

□

**Remark.** In practice it is the operator $\mathscr{A}$ that is known a priori and $A$ is the restriction of $\mathscr{A}$ to $N$. We agree however to denote $\mathscr{A}$ by $A$ itself.

## Generalizations.

Let $\nu$ be an integer. If $E$ is a topological vector space, let $E^\nu$ be $E \times \ldots \times E$, the topology on $E^\nu$ being the product topology. Let $V, Q$ be such that $\mathscr{D}(\Omega)^\nu \subset V \subset Q\mathscr{D}'(\Omega)^\nu$. Let $a(u, v)$ be a continuous sesquilinear form on $V$. As in before, we can define the operator $\mathscr{A} \in \mathscr{L}(V, \mathscr{D}'^\nu)$. The operator $\mathscr{A}$ on $\mathscr{D}(\Omega)$ may be considered to be a generalisation of differential systems.

## General examples:

a) An interesting example of the above kind would be where $V$ is the set of functions continuous on a given discrete set in $R^n$. The solutions of this problem may be considered to be finite difference approximation to boundary value problems.

b) Let $\Omega$ be an open set in $R^n$ and $A_1, \ldots, A_\nu$ be differential operators with constant coefficients. Let $V$ be such that $H^0(A, \Omega) \subset V \subset H^1(A, \Omega)$. Let $Q = L^2(\Omega)$. Then $\mathscr{D}(\Omega) \subset V \subset Q \subset \mathscr{D}'(\Omega)$ and $\mathscr{D}(\Omega)$ is dense in $Q$. Let

$$a(u, v) = \sum_{j,i=1}^{\nu} \int_\Omega g_{ij}(x) A_j(u) \overline{A_i(v)} dx + \int_\Omega g_0(x) u \bar{v} dx$$

with $g_o, g_{ij} \in L^\infty(\Omega)$. $a(u, v)$ is a continuous sesquilinear form on $V$. The corresponding operator $\mathscr{A} = \sum A_i^*(g_{ij} A_j) + g_0$. 31

## 3.5 Green's kernel.

We have proved that in the case $a(u, v)$ is $V$-elliptic, the operator $A$ is an isomorphism of $N$ onto $Q'$. Let $G$ be the *inverse operator* of $A$. $G$ is then an isomorphism of $Q'$ onto $N$. The restriction of $G$ to $\mathscr{D}(\Omega)$ is then a continuous mapping of $\mathscr{D}(\Omega)$ into $\mathscr{D}'(\Omega)$ and conversely the restriction of $G$ to $\mathscr{D}(\Omega)$ defines $G$ uniquely $\mathscr{D}$ is dense in $Q'$.

Now, $L$. Schwartz's kernel Theorem [3] states that any continuous mapping of $\mathscr{D}$ into $\mathscr{D}'$ is defined by an element of $\mathscr{D}'(\Omega_x \times \Omega_y)$, the space of distributions on $\Omega_x \times \Omega_y$.

Thus $G$ defines an element $G_{x,y} \in \mathscr{D}'(\Omega_x \times \Omega_y)$.

**Definition 3.3.** *$G_{x,y}$ defined above is called the Green's kernel of the form $a(u, v)$ on $V$.*

## 3.6 Relations with unbounded operators.

Let $\Omega$ be an open set in $R^n.V, Q$ be two vector spaces not necessarily of distributions, $Q$ being a Hilbert space and $V \subset Q$. Let $a(u, v)$ be a continuous sesquilinear form on $V$. As we have seen already in (§ 3.1), this defines a space $N$ and an operator $A : N \to Q$ by identifying $Q'$ and $Q$. This operator *in the topology induced on N by Q* is an unbounded operator.

Let $a^*(u, v) = \overline{a(v, u)}$. On $V, a^*(u, v)$ is a continuous sesquilinear form. Let the spaces $N$ and operator $A$ associated with $a^*(u, v)$ be denoted by $N^*$ and $A^*$, **32**

i.e.,
$$u \in N^* \Leftrightarrow v \to A^*(u, v)$$

is continuous on $V$ with the topology induced by $Q$ and

$$a^*(u, v) =< A^*u, \overline{v} >= (A^*u, v)_Q.$$

We shall give a theorem establishing relationships between usual concepts associated with the unbounded operators and $N, A$ and $A^*$.

**Theorem 3.6.** *Suppose there exists $\lambda > 0$ such that*

$$\operatorname{Re} a(u, v) + \lambda |v|_Q^2 \geq \alpha |u|_v^2 \text{ for all } \quad u \in V.$$

Then

(1) $N$ is dense in $Q$.

(2) $A$ is closed. | (definitions will be recalled

(3) $A^*$ is the adjoint of $A$. | in the course of proof)

*Proof.* We first prove that $A$ is closed. We have to prove that if $u_n \in D_A$ (the domain of definition of $A$) and if $u_n \to u$ in $Q$ and $Au_n \to f$ in $Q$, then $u \in D_A$ and $Au = f$. □

$a(u, v) + \lambda(u, v)$ is a continuous sesquilinear form on $V$ and the space and the operator associated with it are $N$ and $A + \lambda$ respectively. By assumption this form is V-elliptic and hence by theorem 3.1, $A + \lambda$ is an isomorphism of $N$ **33**
onto $Q' = Q$.

Now, $(A + \lambda)u_n \to f + \lambda\, u$ *in* $Q$ and hence

$$u_n = (A + \lambda)^{-1}(A + \lambda)u_n \to (A + \lambda)^{-1}(f + \lambda_u) \text{ in } N.$$

Hence $u_n \to (A + \lambda)^{-1}(f + \lambda u)$ in $Q$ also and so $u = (A + \lambda)^{-1}(f + \lambda u)$, and $u \in N$. Further $Au_n \to Au$ in $Q$ and so $Au = f$. Hence $A$ is closed.

Now we prove that $N$ is dense in $Q$. We need prove if $f \in Q$ and $(u, f)_Q = 0$ for all $u \in N$. Then $f = 0$. Since $(A + \lambda)$ is an isomorphism of $N$ onto $Q$, there exists $w \in N$ such that $(A+\lambda)w = f$. Hence $((A+\lambda)w, u)_Q = 0$ for all $u \in N$. But

$$((A+\lambda)w, u)_Q = (Aw, u)Q + \lambda(w, u)Q = a(w, u) + \lambda(w, u)_Q.$$

Taking $u = w$ in particular, we get

$$0 = \text{Re } a(w, w) + \lambda|w|_Q^2 \geq \alpha|w|^2 Q.$$

Hence $w = 0$ and so $f = 0$.

Now we prove that the adjoint of $A$ is $A^*$. The domain of the adjoint $\tilde{A}$ of $A$ consists of $u \in Q$ such that the mapping $v \to (Av, u)_Q$ is continuous on $N$ with the topology induced by $Q$. Since $N$ is dense in $Q$, this mapping can be extended to a linear form on $Q$ and hence by Riesz'z theorem we have $\tilde{A}u \in Q$ such that

$$(Av, u)_Q = (v, \tilde{A}u)_Q \quad \text{for} \quad v \in D_A \quad \text{and} \quad u \in D_{\tilde{A}}.$$

This defines $\tilde{A}$ on $D_{\tilde{A}}$.

Since

$$\begin{aligned}(Av, u)_Q = a(v, u) &= \overline{a^*(u, v)} = \overline{(A^*u, v)_Q} \\ &= (v, A^*u)_Q \quad \text{for} \quad v \in N \quad \text{and} \quad u \in N^*,\end{aligned} \tag{1}$$

we have $N^* \in D_{\tilde{A}}$, and $\tilde{A} = A^*$ on $N^*$. We need only prove now $D_{\tilde{A}} \subset N^*$.

Let $u \in D_{\tilde{A}}$, then there exists $u_0 \in N^*$ such that $(A^*+\lambda)u_o = (\tilde{A}+\lambda)u$, since $A^*+\lambda$ is an isomorphism of $N^*$ onto $Q$ on account of V-ellipticity of $a^*(u, v)$. Now, for all $v \in N$

$$\begin{aligned}((A+\lambda)v, u)_Q &= (v, (\tilde{A}+\lambda)u)_Q = (v, (A*+\lambda)u_0)_Q \\ &= a(v, u_o) + \lambda(v, u_o)_Q (by (1)) \\ &= (Av, u_o) + \lambda(v, u_o)_Q \text{ since } v \in N \text{ and by definition of } A.\end{aligned}$$

Hence for all $v \in N$, $((A+\lambda)v, u-u_o)_Q = 0$. Since $(A+\lambda)$ is an isomorphism of $N$ onto $Q'$, this means $u - u_0 = 0$, i.e., $u \in N^*$, which completes the proof.

# Lecture 8

## 4 Complements on $H^m(\Omega)$

### 4.1 Estimates on $H^m_o(\Omega)$.

**Theorem 4.1.** *Let $\Omega$ be a bounded open set in $R^n$. Then there exists a $c > 0$ such that $|u|_0 \leq c|u|_1$ for all $u\epsilon H^1_0(\Omega)$.* 34

*Proof.* Since $\mathscr{D}(\Omega)$ is dense in $H^1_o$ we need prove the inequality for $u = \varphi\epsilon\mathscr{D}(\Omega)$. Let $\tilde{\varphi} = \varphi$ on $\Omega$ and 0 outside $\Omega$ in $R^n$.

Since $\Omega$ is bounded, we have

$$\tilde{\varphi}(x) = \int\limits_{-\infty}^{x_1} \frac{\partial}{\partial x_1}\varphi(t, x_2, \ldots, x_n)dt = \int\limits_{a}^{x_1} \frac{\partial\varphi}{\partial x_1}(t, x_2, \ldots, x_n)dt$$

where $a$ and $b$ are such that $\Omega$ is contained in the region determined by $]a, b[xR^{n-1}$. □

By Schwartz's inequality

$$|\tilde{\varphi}(x)|^2 \leq (x_1 - a)\int\limits_a^b |\frac{\partial\varphi}{\partial x_1}(t, x_2, \ldots, x_n)|^2 dt$$
$$\leq (b - a)\int\limits_a^b |\frac{\partial\varphi}{\partial x_1}|(t, x_2, \ldots, x_n)|^2 dt.$$

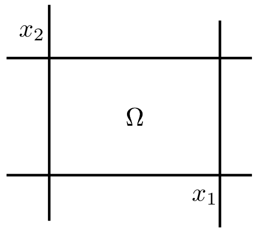
Hence $\int|\tilde{\varphi}(x)|^2 dx \leq (b-a)^2 \int_a^b |\frac{\partial\varphi}{\partial x_1}|^2 dx$, and so $|\varphi|_0 \leq (b-a)|\frac{\partial\varphi}{\partial x_1}| \leq c|\varphi|_1$ as it was to be proved.

**Remarks.**

(1) From the above proof it is seen that the theorem remains true even if $\Omega$ is bounded only in any one direction.

(2) The theorem is not true for $H^1(\Omega)$. Thus, for instance, if we take $u = 1$, then $u \epsilon H^1(\Omega)$ and $|u|_0 =$ measure of and $|u|_1 = 0$, so there does not exists $c$ such that $|u|_0 \leq c|u|_1$.

(3) The theorem may remain true however for some spaces $V$ such that $H_o^1 \subset$ **35**
$V \subset H^1$. Thus if $\Omega$ is as shown in the figure and $V = u \epsilon H^1$ such that $u(0, x_2) = 0$, then $|u|_0 \leq c|u|_1$.




(4) If $u \in H_0^m(\Omega)$, then $|u|_0 \leq c|u|_m, |u|_k \leq c|u|_m$, for $k \leq m - 1$.

**Applications:** We have already proved that for Dirichlet problems ($V = H_0^1(\Omega)$) the operator $-\Delta + \lambda$ associated with the form $a(u, v) = (u, v)_1 + \lambda(u, v)_o$ is an isomorphism of $H_o^1$ onto $H_o^{-1}$ for $\lambda > 0$. We now prove the

**Theorem 4.2.** *If $\Omega$ is bounded, then $-\Delta + \lambda$ is an isomorphism of $H_o^1$ onto $H_o^{-1}$ for $\lambda > -\alpha$ for certain $\alpha > 0$.*

*Proof.* We look for values of $\lambda$ for which $a(u, v)$ is $V$-elliptic,

$$\text{i.e.,} \qquad \operatorname{Re} a(u, u) = |u|_1^2 + \lambda|u|_0^2 \geq \gamma|u|_1^2.$$

Since $\Omega$ is bounded $|u|_1^2 \geq \dfrac{1}{c^2}|u|_1^2$ for some $c > 0$, and

$$\begin{aligned} |u|_1^2 + \lambda|u|_0^2 &= |u|_1^2 + (\lambda - \epsilon)|u|_0^2 + \epsilon|u|_0^2 \\ &\geq (1 + (\lambda - \epsilon)c^2)|u|_1^2 + \epsilon|u|_0^2 \\ &\geq \gamma||u||_1^2 \text{ for positive } \gamma if 1 + (\lambda - \epsilon)c^2 > 0, \end{aligned}$$

i.e., $\lambda > \dfrac{-1 + \epsilon c^2}{2}$. Choosing $\epsilon$ sufficiently small, we have $\alpha = \dfrac{1 - c^2}{c^2}$ such that for $\lambda > -\alpha, a(u, v)$ is V-elliptic and thus the theorem is proved. □

**Theorem 4.3.** *For every $\epsilon > 0$, there exists $c(\epsilon)$ such that $|u|_k^2 \leq \epsilon|u|_m^2 + c(\epsilon)|u|_0^2$ for all $u\epsilon H_0^m(\Omega)$ and 0) $\leq k \leq m-1$.*

*Proof.* Let $\tilde{u}$ be the function defined in $R^n$ which is equal to $u$ on $\Omega$ an 0 elsewhere. We have then $|\tilde{u}|_m = |u|_m$. Let $\tilde{u} = \mathscr{F}(\tilde{u})$ be the Fourier transform of $\tilde{u}$. By Plancherel's theorem **36**

$$|\tilde{u}|_k^2 = \sum_{p=k}(2\pi)^{2k}\int\limits_{\mathbb{Z}^n}|\xi|^{2k}|\tilde{u}|^2 d\xi = |\tilde{u}|_k^2 = |u|_k^2.$$

To verify the stated inequality it is enough to prove that for $\epsilon > 0$ there exists $c(\epsilon)$ such that

$$(2\pi)^{2k}\int|\xi|^{2k}|\hat{u}|^2 d\xi \leq (2\pi)^{2m}\epsilon\int|\xi|^{2m}|\hat{u}|^2 d\xi + c(\epsilon)\int|\hat{u}|^2 d\xi,$$

$$\text{i.e.,}\qquad \int|\xi|^{2k}|\hat{u}|^2 d\xi \leq \int(\epsilon(2\pi)^{2m-2k}|\xi|^{2m} + \frac{c(\epsilon)}{(2\pi)^{2k}})|\hat{u}|^2 d\xi.$$

This will be true if

$$|\xi|^{2k} \leq \epsilon_1|\xi|^{2m} + c_1(\epsilon) \quad for \quad k \leq m-1.$$

Since $k \leq m-1$, for any $\epsilon_1 > 0, |\xi|^{2k} \leq \epsilon_1|\xi|^{2m}$ for large values of $\xi$ and for remaining necessarily bounded values of $\xi, |\xi|^{2k} - \epsilon_1|\xi|^{2m}$ is bounded by $C_1(\epsilon)$ say. ◻

**Remark.** The status of this theorem is different from that of the theorem 4.1: for it may be sometime true for $H^m(\Omega)$ also. As we shall see later, this is connected with the problem of m-regularity. For example, if $\Omega = ]0,1[$, theorem 4.3 holds for $u\epsilon H^m(\Omega)$.

## 4.2 Regularity of the function in $H^m(\Omega)$.

**Theorem 4.4.** *If $2m > n, H^m(\Omega) \subset \mathscr{E}^o(\Omega)$ algebraically and topologically.*

*Proof.* Let $u\epsilon H^m(\Omega)$. We need prove that for every $\varphi\epsilon\mathscr{D}(\Omega), v = u\varphi\epsilon\mathscr{E}^o(\Omega)$. Since $v$ vanishes near the boundary of $\Omega$, the function $\tilde{v}$ is in $H_o^m(R^n)$. Let $\tilde{v}$ be the Fourier transform of $\tilde{v}$, ◻

then $(1 + |\xi|^m)\hat{v}\epsilon L^2$. Now, **37**

$$\hat{v} = (1 + |\xi|^m)\hat{v}.\frac{1}{1 + |\xi|^m}.$$

Since $2m > n$,

$$\int_{\mathbb{Z}} |\frac{1}{1+|\xi|^m}|^2 d\xi = O(\int_o^\infty \frac{r^{n-1}}{1+r^n} dr) < \infty.$$

Hence $\hat{u}\epsilon L^1$. That is to say $v$ is continuous.
If now $u \to 0$ in $H'(\Omega)$ we have $\hat{v} \to 0$ in $L^1$.
Hence $v \to 0$ in $\xi^o(\Omega)$ and so $u \to 0$ in $\xi^o$

**Remark.** Better results valid for more general classes of domains are due to Soboleff. A typical result is if $n \geq 3$, then $u\epsilon H'(\Omega) \Longrightarrow u \in L^q(\Omega), \dfrac{1}{q} = \dfrac{1}{2} - \dfrac{1}{n}$, for certain $\Omega$. (viz., Deny-Lions [7] and also Schwartz [1]).

## 4.3 Reproducing kernels.

Let $v$ be a closed subspace of $H^m(\Omega)$ such that $H^m_o(\Omega) \subset V \subset H^m(\Omega)$ and $Q = L^2(\Omega)$. Let $a(u, v)$ be a continuous sesquilinear from on $V$. Assume now $2m > n$. Hence in each class of functions $v\epsilon V$, there exists a *unique* continuous function $v_o$(say). Then, for fixed $y\epsilon\Omega$, the mapping $v \to \overline{v_o(y)}$ is a continuous semilinear form on $V$. Hence by Riesz's theorem, there exists $k(y)\epsilon V$ such that $\overline{v_o(y)} = (k(y), v)_V$. The mapping $y \to k(y)$ is weakly continuous mapping of $\Omega$ into $V$.

**Definition 4.1.** *$k(y)$ is called reproducing kernel in V (Aronszajn [2]).*

If $a(u, v)$ is V-elliptic we have by theorem 3.1

**Lemma 4.1.** *For every $y\epsilon\Omega$ there exists unique $g(y)\epsilon V$ such that $a(g(y), v) = (k(y), v)_V$ and the mapping $y \to g(y)$ of $\Omega \to V$ is weakly continuous.*

We now relate the $V$ valued function $g(y)$ with the Green's operator $a(u, v)$ in $V$. For every $v\epsilon V$, we have 38

$$a(g(y), v) = \overline{v(y)}.$$

Hence, for any $\varphi\epsilon\mathscr{D}(\Omega), a(\varphi(y)g(y), v) = \varphi(y)\overline{v(y)}$.
Integrating over $\Omega, \int_\Omega a(\varphi(y)g(y), v) = (\varphi, v)_o$. Hence

$$a\left(\int_\Omega g(y)\varphi(y)dy, v\right) = (\varphi, v)_o$$

where $\int\limits_\Omega g(y)\varphi(y)$ by is a weak integral. Now since $a(u, v)$ is $V$-elliptic, given $\varphi\epsilon\mathscr{D}(\Omega)$, there exists $u\epsilon V$ such that $Au = \varphi$, $a(u, v) = (\varphi, v)_o$ for all $v\epsilon V$, and $u = G\varphi$. Hence

$$G\varphi = u = \int\limits_\Omega g(y)\varphi(y)dy.$$

**Theorem 4.5.** *Let $\Omega$ be an open set in $R^n$ and $2m > n$. Let $V, Q, a(u, v)$ be as above. Then $G\varphi = \int\limits_\Omega g(y)\varphi(y)\, dy$ where $g(y)V$, and is given by $a(g(y), v) = \overline{v(y)}$.*

This is a particular case of Schwartz's kernel theorem.

The kernel $G_{x,y}$ defined by the operator $G$ in 3.5 is $g(y)(x)$.

There is yet another way of defining the $V$-valued function $g(y)$. Let $Q = L^2(\Omega) \cap \varepsilon^o(\Omega)$. On $Q$ we put the upper bound topology of $L^2$ and $\varepsilon^o$. Since $2m > n$ any $V$ such that $H^m_o(\Omega) \subset V \subset H^m(\Omega)$ is contained in $\varepsilon^o(\Omega)$, and hence in $Q$. Further since $\mathscr{D}(\Omega)$ is hence in $Q$. If $a(u, v)$ is a continuous sesquilinear $V$-elliptic from on $V$, from theorem 3.1 it follows that there exists a space $N \subset V$ and an operator $A$, such that $A$ is an isomorphism of $N$ onto $Q'$.

Now **39**

$$\begin{aligned} Q' &= (L^2(\Omega))' + (\varepsilon^o(\Omega)) \\ &= L^2(\Omega) + \varepsilon'^o(\Omega) \end{aligned}$$

where $\varepsilon'^o(\Omega)$ is the space of measures with compact support.

Let $G$ be the inverse operators of $A; G$ is an isomorphism of $Q'$ onto $N$. Then $g(y) = G(\delta_y)$.

**Remark.** $G$ as defined here, has slightly different meaning from the one defined previously, but the abuse of language is justified since both of these are inverse of the restriction of the same operator $\mathscr{A} : V \to \mathscr{D}'$, see § 3.4.

# Lecture 9

## 5 Complete Continuity.

### 5.1

We recall the definition of a completely continuous operator. Let $E$ and $F$ be two Hilbert spaces, then a continuous linear mapping $U$ of $E$ into $F$ is said to be completely continuous if for any sequence $u_n \to 0$ weakly in $E, U(u_n) \to 0$ strongly in $F$ or equivalently for bounded set $B$ in $E$, $U(B)$ relatively compact. 40

**Theorem 5.1.** *Let $\Omega$ be a bounded open set in $R^n$. Then the injection $H^1_o(\Omega) \to L^2(\Omega)$ is completely continuous.*

*Proof.* We have to prove that if $u_k \to 0$ in $H^1_o(\Omega)$ weakly, then $u_k \to 0$ strongly in $L^2(\Omega)$. Let $\tilde{u}_k$ be the extension of $u_k$ to $R^n$ equal to $u_k$ on $\Omega$ and 0 elsewhere. Then $u_k \to 0$ weakly in $H^1_o(R^n)$ and hence weakly in $L^2(R^n)$. Let $\hat{u}_k$ be the Fourier transform of $u_k$, i.e., $\hat{u}_k(\xi) = (u_k, e^{2\pi ix\xi})_o$. Since $\Omega$ is bounded for every $\xi, e^{2\pi ix\xi} \in L^2(\Omega)$ and hence for fixed $\xi, \hat{u}_k(\xi) \to 0$. Further $u_k$ is weakly bounded in $H^1_o(\Omega)$ and hence bounded in $H^1_o(\Omega)$. So $|u_k|_o \leq c_o, |u_k|_1 \leq c_1$. Hence, by Schwartz's inequality $|\hat{u}_k(\xi)| \leq c_2$. □

To prove $u_k \to 0$ strongly in $L^2$ we need prove $\int |\hat{u}_k(\xi)|^2 d\xi \to 0$. Now

$$\int |\hat{u}_k(\xi)|^2 d\xi = \int\limits_{|\xi|<R} |\hat{u}_k(\xi)|^2 d\xi + \int\limits_{|\xi|\geq R} |\hat{u}_k(\xi)|^2 d\xi.$$

Given any $\in > 0$ we shall prove that we can choose $R$ so large that the second term is less than $\in /2$, and then that we can choose $k_o$ such that for $k = k_o$, the first term is less than $\in /2$. This will complete the proof. Now 41

$$\int\limits_{|\xi|\geq R} |\hat{u}_k(\xi)|^2 = \int\limits_{|\xi|\geq R} (1+|\xi|^2)|\hat{u}_k(\xi)|^2 . \frac{1}{1+|\xi|^2} d\xi$$
$$\leq \frac{1}{1+R^2} \int\limits_{|\xi|\geq R} (1+|\xi|^2)|\hat{u}_k(\xi)|^2 d\xi$$
$$\leq \frac{||u_k||_1}{1+R^2} \leq \frac{c_3}{1+R^2}.$$

We choose $R$ so that $\dfrac{c_3}{1+R^2} < \in /2$.

Next since we have proved above that $|\hat{u}_k(\xi)| < c_2$ and that for every $\xi$, $\hat{u}_k(\xi) \to 0$, observing that $c_2$ is integrable on $|\xi| < R$, by Lebesgue bounded convergence theorem, it follows that

$$\int |\hat{u}_k(\xi)|^2 d\xi \to 0$$
$$|\xi| < R.$$

## 5.2

We have seen that if $\Omega$ is bounded, the injection of $H^1_o(\Omega)$ into $L^2(\Omega)$ is completely continuous. It is not true that the injection of $H^1(\Omega)$ into $L^2(\Omega)$ is always completely continuous. (For a necessary and sufficient condition, see Deny-Lions [7]).

However we have the

**Theorem 5.2.** *If $\Omega$ is bounded and has 1-extension property, then the injection $H^1(\Omega) \to L^2(\Omega)$ is completely continuous.*

*Proof.* Let 0 be a relatively compact open set containing $\Omega$. Let $u_k$ be a sequence weakly converging to 0 in $H^1(\Omega)$ and $\pi u_k$ be extensions of $u_k$ to $R^n$. Since $\pi$ is continuous from $H^1(\Omega)$ to $H^1(R^n)$, $\pi(u_k)$ converge to 0 weakly in $H^1(R^n)$, and hence the restrictions of $\pi(u_k)$ to 0 also converge to 0 weakly in $H^1(0)$. ◻

Let $\Theta$ be a function in $\mathscr{D}(0)$ which is 1 on $\bar{\Omega}$. Then $\Theta u_k \in H^1_o(0)$. Since 0 is bounded by theorem 5.1 $\Theta\pi(u_k) \to 0$ strongly in $L^2(0)$, and hence $u_k \to 0$ strongly in $L^2(\Omega)$. **42**

**Corollary.** *If $\Omega$ is bounded and has m-extension property, then the injection of $H^m(\Omega)$ into $L^2(\Omega)$ is completely continuous.*

## 5.3 Applications

Let $V$ be such that $H^1_o(\Omega) \subset V \subset H^1(\Omega)$ and $a(u,v) = (u,v)_1$. The operator $A$ associated with $a(u,v)$ is $-\Delta$. We wish to show how when $\Omega$ is bounded and has 1-extension property, Fredholm theory can be applied to consider the solutions of $(A-\lambda)u = f$ for $f \in L^2(\Omega)$.

We recall the Riesz- Fredholm theorem for completely continuous operator.

Let $H$ be a Hilbert space and $A$ be a Hermitian and a completely continuous operator of $H$ into $H$. Then

1) $A - \mu I$ is an isomorphism of $H$ onto itself except for countable values of $\mu$, say $\mu_o \geq \mu_1 \geq \cdots$ such that $\mu_n \to 0$. $\mu_n$ are called eigenvalues of $A$.

2) The kernel of $A - \mu_n$ is finite dimensional. It is called the eigenspace corresponding to $\mu_n$ and its dimension is called the multiplicity of $\mu_n$.

3) If $w_{n_1}, \ldots w_{n_m}$ is an orthonormal base for the eigenspace then $(w_n)$ from an orthonormal system and any $y \in H$ can be written as $y = h + \Sigma(y,w_n)w_n$, where $h$ is a solution of $Ah = 0$.

   Hence if we assume that $Ah = 0$ implies $h = 0$, we have 43

4) $(w_n)$ forms a complete orthonormal system and

$$Ay = \Sigma \mu_n (y, w_n) w_n.$$

Hence $(A-\mu)x = y$ has a unique solution for all $\mu$ except those which are eigenvalues and the solution is given by

$$x = \sum \frac{(y,w_n)}{\mu_n - \mu} w_n \text{ for } \mu \neq \mu_n$$

and if $\mu = \mu_n x = \sum\limits_{n \neq m} \dfrac{(y,w_m)}{\mu_m - \mu_n} w_m + h_n$ where $h_n$ is such that $(A-\mu_n)h_n = 0$.

We know that the problem of finding $u \in N$ such that $(-\Delta - \lambda)u = f$ for $f \in L^2(\Omega)$ is to find $u \in N$ such that $(u,v)_1 - \lambda(u,v)_o = (f,v)_o$ for all $v \in V$.

Let $[u,v] = (u,v)_1 + (u,v)_o$ so that we have to consider $[u,v] - (\lambda+1)(u,v)_o$ for all $v \in V$. Now the semilinear mapping $v \to (f,v)_o$ is continuous on $V$, hence there exists $Jf \in V$ such that $[Jf,v] = (f,v)_o$. $J$ is then a continuous mapping of $L^2 \to V$. Let $J_1$ be the restriction $J$ to $V$. We have to consider then

$$[u,v] - (\lambda+1)[J_1 u, v] = [Jf, v] \text{ for all } v,$$

i.e., $$(J_1 - \mu)_u = -\frac{g}{\lambda+1} \text{ where } = \frac{1}{\lambda+1}.$$

**Lemma.** *$J_1$ is a completely continuous mapping of $V$ into $V$.*

*Proof.* $J_1$ is the composite of $V \to L^2 \xrightarrow{J} L^2$. Since $\Omega$ is bounded and has 1-extension property, the injection $V \to L^2$ is completely continuous. Hence $J_1$ is completely continuous. ◻

Further $(J_1 u, v) = (u, v)_o$. Hence $J_1 u = 0$ implies $u = 0$, and trivially $J_1$ is Hermitian. 44

Applying the theorem of Riesz-Fredholm quoted above, $J_1 - \mu$ is an isomorphism of $V$ onto $V$ except for $\mu = \mu_1 \cdots \mu_1 \cdots$. Let $\lambda_n = -1 + \dfrac{1}{\mu_n}$. Let $w_n$ be orthonormal set of eigenvalues. We have proved then

**Theorem 5.3.**

(1) *$-\Delta - \lambda$ is an isomorphism of $N \to L^2$ expect for $\lambda = \lambda_1 \cdots \lambda_n \cdots$ such that $-1 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots, \lambda_n \to \infty$.*

(2) *$-\Delta w_n = \lambda w_n$ and $w_n$ is a complete orthonormal system in $V$ and complete orthogonal in $L^2$*

(3) *$\dfrac{w_n}{\sqrt{1+\lambda_n}}$ is complete orthonormal in $L^2 (\|w_n\|_1^2 = 1)$ and so $(1+\lambda_n)|w_n|^2 = 1$.*

(4) *$w_n$ is complete orthogonal in $N$.*

# Lecture 10

## 6 Operators of order 2

### 6.1

Hitherto we considered the problems in which the $V$-elliptic from $a(u, v)$ was given a priori and then we solved boundary value problems for the operator $A$ associated with the form $a(u, v)$. Now we with to consider the natural converse 45

**Problem.** Given a differential operator $A$, determine the spaces $V$ and $V$-elliptic forms $a(u, v)$ on $V$ such that

1) $\langle Au, \overline{\varphi}\rangle = a(u, \varphi)$ for all $u \in V$ and $\varphi \in \mathscr{D}(\Omega)$

2) $a(u, v)$ is V-elliptic.

Stated in this general from the problem has not been completely solved, even in the case of differential operator of order 2; however, several results, depending on the domain $\Omega$, coefficients of $A$, $V$ and $a(u, v)$ are know and we give some of these.

We shall always consider the case when $V \subset H'(\Omega)$. We take a second order differential operator $A$ in the form

$$A = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(g_{ij}(x)\frac{\partial}{\partial x_i}\right) + \sum g_i(x)\frac{\partial}{\partial x_i} + g_o(x), g_{ij}, g_i, g_o \text{ in } L^\infty(\Omega).$$

A more general form would be $\sum\limits_{|p|\leq 2} a_p(x)D^p$ which reduces to the above if $a_p(x)$ are regular enough.

We associate with $A$ the form

$$a(u,v)=\sum_{i,j=1}^{n}\int_{\Omega} g_{ij}\frac{\partial u}{\partial x_j}\frac{\partial \overline{v}}{\partial x_i}dx+\sum\int_{\Omega} g_i\frac{\partial u}{\partial x_i}\overline{v}+\int g_o u\overline{v}$$

and consider the ellipticity of this form. Another kind of sesquilinear from will be considered later. We observe that with the same operator several forms can 46
be associated in the above fashion, merely by rearranging the operator. For instance, let

$$A=-\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial x_1\partial x_2}-\frac{\partial^2}{\partial x_2^2}.$$

We may write

$$A=-\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial x_2^2}-\frac{\partial}{\partial x_2}\left(\frac{1}{2}+i\right)\frac{\partial}{\partial x_2}-\frac{\partial}{\partial x_1}\left(\frac{1}{2}-i\right)\frac{\partial}{\partial x_2}.$$

The associated forms are

$$a(u,v)=\left(\frac{\partial u}{\partial x_1},\frac{\partial v}{\partial x_1}\right)_o+\left(\frac{\partial u}{\partial x_2},\frac{\partial v}{\partial x_2}\right)_o+\left(\frac{\partial u}{\partial x_2},\frac{\partial v}{\partial x_1}\right)_o, \text{ and}$$

$$a(u,v)=\left(\frac{\partial u}{\partial x_1},\frac{\partial v}{\partial x_1}\right)_o+\left(\frac{\partial u}{\partial x_2},\frac{\partial v}{\partial x_2}\right)_o+\left(\frac{1}{2}+i\right)\left(\frac{\partial u}{\partial x_1},\frac{\partial v}{\partial x_2}\right)_o + \left(\frac{1}{2}-i\right)\left(\frac{\partial u}{\partial x_1},\frac{\partial v}{\partial x_2}\right)$$

which are different.

Let $(u,v)_g$ be the leading part of $a(u,v)$,

$$(u,v)_g=\sum_{i,j=1}^{n}\int g_{ij}\frac{\partial u}{\partial x_j}\frac{\partial \overline{v}}{\partial x_i}dx.$$

To determine when $a(u,v)$ is elliptic, we have to investigate when $\mathrm{Re}(u,u)_g\geq \alpha|u|_1^2$ for all $u\in V$ and for some $\alpha>0$.

## 6.2

**Theorem 6.1.** *Let $\Omega$ be a bounded open set in $R^n, g_{ij}$ be constants and $V=H^1(\Omega)$. A necessary and sufficient condition that*

$$\mathrm{Re}(u,u)_g\geq\alpha|u|_1^2 \text{ for all } u\in H^1(\Omega) \tag{1}$$

*is that*

$$\sum(g_{ij}+\overline{g}_{ij})p_i\overline{p}_i \text{ for all complex } (p_i). \tag{2}$$

*Proof.*

(a) *Necessity*. Let $u(x) = \sum\limits_{i=1}^{n} p_i x_i$. Because $\Omega$ is bounded $u(x) \in H^1(\Omega)$. Hence by (1)

$$\mathrm{Re}\left(\sum \int_{\Omega} g_{ij} p_i \overline{p}_i dx\right) \geq \alpha \sum |p_i|^2 \int_{\Omega} dx,$$

i.e., $$\mathrm{Re}\left(\sum g_{ij} p_j \overline{p_i}\right) \geq \alpha \sum |p_i|^2 \qquad \text{which is (2).}$$

(b) *Sufficiency*. From (2) we have 47

$$\sum (g_{ij} + \overline{g}_{ji}) \frac{\partial u}{\partial x_j}(x) \frac{\partial \overline{v}}{\partial x_i}(x) \geq \alpha \sum |\frac{\partial u}{\partial x_i}(x)|^2 \text{ a.e.}$$

Integrating over

$$\sum \int_{\Omega} (g_{ij} + \overline{g}_{ji}) \frac{\partial u}{\partial x_j} \frac{\partial \overline{u}}{\partial x_i} dx \geq \alpha |u|_1^2$$

i.e., $$\mathrm{Re}(u, u)_g \geq \alpha |u|_1^2.$$ □

**Theorem 6.2.** *Let $\Omega = R^n$, and $g_{ij}$ be constant. Then a necessary and sufficient condition in order that (1) holds is that*

$$\mathrm{Re}\left(\sum g_{ij} \xi_i \xi_j\right) \geq \alpha \sum \xi_i^2 \textit{ for real } \xi_i \textit{ and for some } \alpha > 0 \tag{3}$$

(We observe (2) $\Longrightarrow$ (3), but converse is not true, e.g., the example quoted above).

*Proof.* By Fourier transform

$$\begin{aligned}(u, u)_g &= \sum g_{ij} \int 2\pi i \xi_i \hat{u} . \overline{2\pi i \xi_j \hat{u} d\xi} \\ &= 4\pi^2 \int \sum g_{ij} \xi_i \xi_j |\hat{u}|^2 d\xi.\end{aligned}$$

□

Hence (1) is equivalent to

$$\mathrm{Re}\left(\int \sum_{ij} \xi_i \xi_j |\hat{u}(\xi)|^2 d\xi\right) \geq \alpha |\xi|^2 |\hat{u}(\xi)|^2 d\xi \text{ for all } u \in H^1 \tag{4}$$

Let $\quad p(\xi) = \mathrm{Re}(\sum g_{ij}\xi_i\xi_j) - \alpha|\xi|^2$.
Form (4), (1) is equivalent to

$$\int P(\xi)|\hat{u}(\xi)|^2 d\xi \geq 0 \tag{5}$$

We have to prove (5) holds if and only if $P(\xi) \geq 0$.

Sufficiency is trivial. To see the necessity if $P(\xi_o) < 0, P(\xi) < 0$ in a certain neighbourhood and then to obtain a contradiction we need take $u$ the Fourier 48 transform of which has support in this neighbourhood.

The following problem however is not answered: If $x_i \in H^1(\Omega)$, $(\Omega)$ of capacity $> 0$ is (2) necessary in order that (1) holds for $u \in H^1(\Omega)$.

## 6.3 $V = H^1_o(\Omega), g_{ij}$ constant.

**Theorem 6.3.** *Let $V = H^1_o(\Omega)$ and $g_{ij}$ be constant. A necessary and sufficient condition in order that*

$$\mathrm{Re}(u, u)_g \geq \alpha|u|_1^2 \text{ for all } u \in H^1_o(\Omega) \text{ for some } \alpha > 0 \tag{6}$$

is that

$$\mathrm{Re}\left(\sum g_{ij}\xi_i\xi_j\right) \geq \alpha \sum |\xi|^2 \text{ for all } \ \xi_i \in \mathbb{Z}^n. \tag{7}$$

*Proof.* In order to apply theorem 6.2 we prove that (6) implies that (1) holds for $u \in H^1_o(R^n) = H^1(R^n)$. We require a lemma. We may assume without loss of generality that the origin is in $\Omega$. Further, we observe $\cup\lambda\Omega = R^n$. □

**Lemma 6.1.** *(6) holds if and only if* $\mathrm{Re}(u, u)_g \geq \partial|u|_1^2$ *for all $u \in H^1_0(\Omega)$ for all $\lambda$.*

*Proof.* Let $u \in H^1_0(\Omega)$. Define $u_\lambda(x) = u(\lambda x)$ for $x \in \Omega$. □

It is easily seen that $u_\lambda \in H^1_o$. From (6) we get

$$\mathrm{Re}\left(\sum \int_\Omega g_{ij} \frac{\partial u_\lambda}{\partial x_j} \frac{\partial \overline{u}_\lambda}{\partial x_i} dx\right) \geq \alpha \sum \int_\Omega \left|\frac{\partial u_\lambda}{\partial x_i}\right|^2 dx \tag{8}$$

Since $\dfrac{\partial u_\lambda}{\partial x_i}(x) = \lambda \dfrac{\partial u(\lambda x)}{\partial x_i}$, from (8) we get

$$\mathrm{Re}\left(\sum \int_\Omega g_{ij} \frac{\partial u(\lambda x)}{\partial x_i} \overline{\frac{\partial u(\lambda x)}{\partial x_i}} dx\right) \geq \alpha \sum \int \left|\frac{\partial u(\lambda x)}{\partial x_i}\right|^2 dx.$$

Putting $\lambda x = y$, we get the required inequality and lemma (6.1) is proved. 49

Returning to the proof of theorem, let $\varphi \in \mathscr{D}(R^n)$. There exists $\lambda$ such that $K \subset \lambda\Omega$. Then $\varphi \in \mathscr{D}(R^n)$. and hence $\varphi \in H_o^1(\lambda\Omega)$. This means

$$\text{Re } (u,u)_g \geq \alpha|u|^2, \text{ for all } \varphi \in \mathscr{D}(R^n).$$

Since $\mathscr{D}(R^n)$ is dense in $H_o^1(R^n)$, we have proved

$$\text{Re } (u,u)_g \geq \alpha|u|_1^2 \text{ for all } u \in H_o^1(R^n).$$

Theorem 6.2 then gives (7).

## 6.4

Some problems with variable coefficients : $V = H_0^1(\Omega)$.

**Theorem 6.4.** *Let $\Omega$ be any open set in $R^n$ and $g_{ij}$ be continuous.*

*If*

$$\text{Re}(u,u)_g \geq \alpha|u|_1^2 \text{ for all } u \in H_0^1(\Omega), \tag{9}$$

*then*

$$\text{Re} \sum g_{ij}(x_o)\xi_i\xi_j \geq \alpha \sum_{i=1}^{n} \left|\xi_i\right| \text{ for all } (\xi_i) \in R^n.$$

*Proof.* Given any $\in > 0$, let $B$ be a neighbourhood of $x_o$ such that

$$|(u,u)_{g(x_o)} - (u,u)_g| \leq \in |u|_1^2 \text{ for all } u \in H_o^1(B).$$

We need choose $B_\epsilon$ such that $\left|g_{ij}(x) - g_{ij}(x_o)\right|$ are sufficiently small. (9) gives then

$$\text{Re}(u,u)_g \geq (\alpha - \in |u|_1^2) \text{ for all } u \in H_0^1(\beta_\in).$$

From theorem 6.3, it follows that

$$\text{Re} \sum g_{ij}(x_o)\xi_i\xi_j \geq (\alpha - \xi) \sum |\xi|_i^2.$$

Since this is true for arbitrarily small $\in$, we have

$$\text{Re} \sum g_{ij}(x_o)\xi_i\xi_j \geq (\alpha - \xi) \sum |\xi_i|^2.$$

Regarding the sufficiency of the above condition, we have □

**Theorem 6.5.** *Garding's inequality. If* $\mathrm{Re}\,\Sigma g_{ij}\xi_i\xi_j \geq \alpha\Sigma|\xi_i|^2$ *for some* $\alpha > 0$ *for all* $x \in \overline{\Omega}$ *and* $\Omega$ *is bounded then there exists* $\lambda > 0$ *such that* 50

$$\mathrm{Re}(u,u)_g + \lambda|u|_0^2 \geq \alpha|u|_1^2 \text{ for all } u \in H_o^1(\Omega).$$

*We do not prove this. For a proof, see Yosida [21].*

We have a general sufficient condition

**Theorem 6.6.** *If* $\Sigma(g_{ij} + \overline{g}_{ji})p_j\overline{p}_i \geq \alpha\Sigma|p_i|^2)$ *for some* $\alpha > 0$ *and* $p_i$*complex a.e. in* $\Omega$*, then*

$$\mathrm{Re}(u,u)_g \geq \alpha|u|_1^2 \text{ for all } u \in H^1(\Omega).$$

Having seen some cases when $\mathrm{Re}(a(u,u)_g) \geq \alpha|u|_1^2$ we see now some examples when different forms $a(u,v)$ giving rise to the same operator $A$ are $V$-elliptic.

1) Let $a(u,v) = (u,v)_g + (g_o u, v)_o$ with $g_o(x) \geq \beta > 0$.

   Then
   $$\mathrm{Re}(a(u,u)) \geq \alpha|u|_1^2 + \beta|u_0|^2 \geq min(\alpha,\beta)||u||_1^2.$$

   Hence $a(u,v)$ is $V$-elliptic for any $V$ such that $H_o^1 \subset V \subset H^1$.

2) Let $a(u,v) = (u,v)_g + (g_o u,v)_o + \Sigma\left(g_i\dfrac{\partial u}{\partial x_i}, v\right)_o$, $g_i$ real constants, $g_o(x) \geq \beta > 0$. Let $V = H_0^1(\Omega)$. Let $V = H_0^1(\Omega)$. We first observe that for $u \in H_0^1(\Omega)\,\mathrm{Re}\left(u, \dfrac{\partial u}{\partial x_i}\right) = 0$. For, if $\varphi \in \mathscr{D}(\Omega)$, by integration by parts $\left(\dfrac{\partial\varphi}{\partial x_i}, \varphi\right)_o = \left(\varphi\dfrac{\partial\varphi}{\partial x_i}\right)_o$ and since

   $$\mathrm{Re}\left(\frac{\partial\varphi}{\partial x_i}, \varphi\right)_o = \left(\frac{\partial\varphi}{\partial x_i}, \varphi\right)_o + \left(\varphi, \frac{\partial\varphi}{\partial x_i}\right)_o$$

   we have $\mathrm{Re}(\varphi, \dfrac{\partial\varphi}{\partial x_i}) = 0$ for all $\varphi \in \mathscr{D}(\Omega)$. Since $\mathscr{D}(\Omega)$ is dense in $H_o^1(\Omega)$we have the result for $u \in H_o^1(\Omega)$. Hence $\mathrm{Re}(a(u,u)) = \mathrm{Re}(u,u)_g + \Re(g_o u, u)$. Hence $a(u,v)$ is $H_o^1(\Omega)$ elliptic.

# Lecture 11

## 6.5

We now consider another kind of sesquilinear forms giving rise to the same operator $A = \sum\limits_{i,j=1}^{n} \dfrac{\partial}{\partial x_j}(g_{ij}(x)\dfrac{\partial}{\partial x_i}) + g_i(x)\dfrac{\partial}{\partial x_j} + g_i\dfrac{\partial}{\partial x_i} + g_o(x)$. 51

Let $\Omega$ be an open set with the boundary $\Gamma$ having a $C^1(n-1)$ dimensional piece $\Sigma$. Let $\gamma u$ be the extension of functions in $H^1(\Omega)$ to $\sum$ as defined in § 2.4.

On $H^1(\Omega)$ consider the sesquilinear form

$$a(u,v) = (u,v)_g + \sum\left(g_i\frac{\partial u}{\partial x_i}, v\right)_o + \sum(g_o u, v)_o + \int\limits_{\Sigma} \gamma\, u\overline{\gamma\, u} d\sigma,$$

where $d\sigma$ is the intrinsic measure on $\Sigma$. The operator associated with it is the same $A$ as before. To consider the ellipticity of this from we require some definitions.

**Definition 6.1.** *Let $\Omega$ be a bounded* connected *open set; we shall say that $\Omega$ is of Nykodym type if there exists a constants $P(\Omega) > 0$ such that the following inequality holds for all $u \in H^1(\Omega)$.*

$$\int\limits_{\Omega} |u|^2 dx - \frac{1}{mes\Omega}\Big| \int udx P(\Omega)|u|_1^2. \tag{1}$$

The inequality (1) is called Poincare inequality. We admit without proof the

**Theorem 6.7.** *Every $\Omega$ with "smooth boundary" is of Nykodym type. (For proof, see Deny [7]).*

Another interpretation of the inequality (1 ) is obtained by observing that

$$\int |u|^2 dx - \frac{1}{\text{mes}\,\Omega}\left|\int u dx\right|^2$$

is the minimum of $\left| + c_o\right|_0$ for all constants $c$.

For 52

$$\begin{aligned}\left|u+c\right|_0^2 &= |u|_o^2 + \overline{c}\int {}_u dx + c\int \overline{u} dx + |c|^2 \,\text{mes}\,\Omega \\ &= \frac{1}{\text{mes}\,\Omega}\left(c+\int {}_u dx\right)\left(\overline{c}+\int \overline{u} dx\right) + |u|_o^2 - \frac{1}{\text{mes}\,\Omega}\left|\int u dx\right|^2\end{aligned}$$

Thus (1) means Inf $|u+c|_o^2 \leq P(\Omega)|u|_1^2$.

**Theorem 6.8.** *Let $\Omega$ be a domain of Nykodym type with the boundary $\Gamma a(n-1)$ dimensional $C^1$ manifold. Then the form*

$$a(u,v) = (u,v)_g + \int\limits_{\Gamma} \gamma\, u \overline{\gamma\, v}\; d\,\sigma$$

*is $V$-elliptic on $H^1(\Omega)$.*

*Proof.* Since $\text{Re}(a(u,u)) = \text{Re}(a(u,u))_g + \int |\gamma u|^2 d\sigma \geq \alpha|u|_1^2 + \int\limits_{\Gamma} |\gamma u|^2 d\sigma$ to prove the $V$-ellipticity of $a(u,v)$ it is enough to prove that there exists a $\beta > 0$ such that

$$\alpha|u|_1^2 + \int |\gamma u|^2 d\sigma \leq \beta||u||_1^2,$$

or that

$$\int |\gamma u|^2 d\sigma + |u|_1^2 \geq \beta_1||u||_1^2.$$

Let $[u,v] = (u,v)_1 + \int \gamma u\overline{\gamma v} d\sigma$. $[u,v]$ is a continuous sesquilinear form on $H^1(\Omega)$ since $[u,u] = 0$ implies $|u|_1^2 = 0$ and $\int |\gamma\, u|^2 d\sigma = 0$, we have $u = c$, a constant for $|u|_1^2 = 0$ and $c = 0$ for $\int |\gamma u|^2 = 0$. That is to say $[u,u] = 0$ implies $u = 0$. In fact, we have the □

**Lemma.** $[u,v]$ *defines a Hilbertian structure on $H^1(\Omega)$.*

Assuming the lemma for a moment, we see that on account of the closed graph theorem, the two norms $\sqrt{[u,v]}$ and $\sqrt{(u,v)_V}$ are equivalent. Hence $[u,u] \geq \beta||u||_1^2$ which was to be proved.

To complete the proof we have to prove the lemma, i.e., that under the scalar product [ ], $H^1(\Omega)$ is complete.

Let $u_k$ be a Cauchy sequence for the scalar product [ ]. Then $\dfrac{\partial u_k}{\partial x_i}, i =$ 53
$1, \ldots, n$, and $\gamma u_k$ are Cauchy sequences in $L^2(\Omega)$, and $L^2(\Gamma)$ respectively. Hence $\dfrac{\partial u}{\partial x_i} \to f_i, i = 1, \ldots, n$ in $L^2(\Omega)$ and $\gamma u_k \to g$ in $L^2(\Gamma)$. Since $\Omega$ is of Nykodym type from (1), we have

$$\int\limits_{\Omega} |u_k - \frac{1}{\text{mes}\,\Omega}| \int u\, dx|^2|^2\, dx \leq P|u|_1^2$$

i.e., $$\int\limits_{\Omega} |\, u_k - c_k|^2 \leq P|u_k|_1^2 \text{ where } c_k = \frac{1}{\text{mes }\Omega} \int u_k\, dx.$$

Since $u_k$ is a Cauchy sequence in $L^2(\Omega), u_k - c_k$ is a Cauchy sequence in $L^2(\Omega)$. Hence $u_k - c_k \to v$ in $L^2(\Omega)$ and $\dfrac{\partial v}{\partial x_i} = \lim \dfrac{\partial u_k}{\partial x_i} = f_i$. Hence $u_k - c_k \to v$ in $H^1(\Omega)$ and so $\gamma(u_k - c_k) \to v$ in $L^2(\Gamma)$. Since $\gamma u_k \to g$ in $L^2(\Gamma), c_k \to c$. However $u_k = (u_k - c_k) + c_k$. Hence $u_k \to v + c$ in $H^1(\Omega)$ under the norm [ ], which proves the lemma.

## 6.6 Formal interpretation:

If $\Omega$ is of Nykodym type with a smooth $(n-1)$ dimensional boundary $\Gamma$, we have just proved that the form $a(u, v) = (u, v)_g + (\gamma u, \gamma v)_0$ is elliptic on $H^1(\Omega)$. The operator $A$ that it defines is $A = -\sum \dfrac{\partial}{\partial\, x_i}\left(g_{ij}(x)\dfrac{\partial}{\partial\, x_j}\right)$ and $u \in N$ implies $a(u, v) = (Au, v)_o$ for all $v \in V$. Now formally,

$$\int\limits_{\Omega} A\, u\, \overline{v}\, d\, x = a(u, v) + \int \frac{\partial u}{\partial \eta_A} \overline{v}\, d\, \sigma$$

where $\dfrac{\partial u}{\partial \eta_A} = \sum g_{ij} \dfrac{\partial u}{\partial x_j} \cos(n, x_i), (n, x_i)$ being the angle between the outer normal and $x_i$. Thus $u \in N$ implies formally $\dfrac{\partial u}{\partial \eta_A} = 0$.

## 6.7 Complementary results.

*Boundary value problems of oblique type for* $\Omega = \{x_n > 0\}$. For general theory, 54 see Lions [4]. Let $\Gamma$ be the boundary of $\Omega : \{x_n = 0\}$. We recall the definition of the spaces $H^\alpha(\Gamma)$ for $\alpha$ real defined in § 2.5. $H^\alpha(\Omega) = \{f\ L^2(\Gamma)\}$ such that $(1 + |\xi|^\alpha)\hat{f} \in L^2(\Gamma)$, where $\hat{f}$ is the Fourier transform of $f$. We have proved in 2.4, that there exists a unique mapping $\gamma : H^1(\Omega) \to H^{\frac{1}{2}}(\Gamma)$ which on $\mathscr{D}(\overline{\Omega})$ is the restriction to $\Gamma$ and this mapping is *onto*.

**Theorem 6.9.** *The dual of $H^\alpha(\Gamma)$ is $H^{-\alpha}(\Gamma)$.*

*Proof.* Let $\mathscr{F}(H^\alpha(\Gamma))$ be the space of Fourier transforms of $H^\alpha(\Gamma)$. $\mathscr{F}(H^\alpha(\Gamma))$ consists of functions $\hat{f} \in L^2(\Gamma)$ such that $(1 + |\xi|^\alpha)\hat{f} \in L^2(\Gamma)$. Hence its dual consists of functions $\hat{g} \in L^2(\Gamma)$ such that $\dfrac{1}{1 + |\xi|^\alpha}\ \hat{g} \in L^2(\Gamma)$, i.e., $(1 + |\xi|^{-\alpha})\ \hat{g} \in L^2(\Omega)$. Hence the dual of $\mathscr{F}(H^{-\alpha}(\Gamma))$ is $\mathscr{F}(H^{-\alpha}(\Gamma))$ which proves the theorem. $\square$

Let $\Lambda = \sum_{i=1}^{n-1} \alpha_i \dfrac{\partial}{\partial x_i}$ with $\alpha_i$ *real constants*. We call $\Lambda$ a *tangential operator*.

**Lemma 6.2.** $\Lambda$ *is a continuous linear mapping of $H^{\frac{1}{2}}(\Gamma)$ into $H^{\frac{-1}{2}}(\Gamma)$.*

*Proof.* It is enough to prove that $\dfrac{\partial}{\partial x_i}$ is a continuous linear mapping from $H^{\frac{1}{2}}(\Gamma)$ into $H^{-\frac{1}{2}}(\Gamma)$ or that $\mathscr{F}\left(\dfrac{\partial}{\partial x_i}\right)$ is continuous from $\mathscr{F}(H^{\frac{1}{2}}(\Gamma))$ into $\mathscr{F}\left(H^{-\frac{1}{2}}(\Gamma)\right)$. Let $f \in H^{\frac{1}{2}}(\Gamma)$. Then $(1 + |\xi|^{\frac{1}{2}})\hat{f} \in L^2(\Gamma)$, and so $\mathscr{F}\left(\dfrac{\partial f}{\partial x_i}\right) = 2\pi\, i\, \xi_i\ \hat{f} \in H^{-\frac{1}{2}}(\Gamma)$. Since the mapping $g \to \xi_i g$ is continuous, from $\mathscr{F}(H^{\frac{1}{2}}(\Gamma))$ into $\mathscr{F}(H^{-\frac{1}{2}}(\Gamma))$ the proof is complete. $\square$

From lemma 6.2 we see that $\langle \Lambda\gamma u, \overline{\gamma v}\rangle$ is defined for all $u, v \in H^1(\Omega)$. Further we have the

**Lemma 6.3.** $\mathrm{Re}(\Lambda\gamma\, u, \gamma\, u) = 0$ *for all* $u \in H^1(\Omega)$. *For by Fourier transform* **55**
$\mathrm{Re}\ \langle \dfrac{\partial}{\partial x_i}\ \gamma u, \overline{\gamma u}\rangle = \mathrm{Re}\ \int 2\,\pi\, i\, \xi_i\, |\,\gamma\, \hat{u}\,|^2\, d\,\xi.$

Let $a(u, v) = (u, v)_1 + \lambda(u, v)_0 + \langle \Lambda\ \gamma\ u, \overline{\gamma v}\rangle$ for $u, v \in H^1(\Omega)$. From lemma 6.2, we see that $a(u, v)$ is a continuous sesquilinear form on $H^1(\Omega)$.

**Lemma 6.4.** *If $\lambda > 0$, $a(u, v)$ is $H^1(\Omega)$ elliptic. For* $\mathrm{Re}(a(u, u)) = |u|_1^2 + \lambda|u|_0^2 \geq \min(\lambda, 1)||u||_1$.

From theorem 3.1, we have the

**Theorem 6.10.** *The operator associated with $a(u, v)$ is $-\triangle + \lambda$ and $-\triangle + \lambda$ is an isomorphism from $N$ onto $L^2(\Omega)$.*

To get a formal interpretation of the problem, we have to see that $u \in N$ means. $u \in N$ if and only if

$$((-\triangle + \lambda)u, v)_0 = (u, v)_1 + (\Lambda\,\gamma\, u, \overline{\gamma u})_o + \lambda(u, v)_o.$$

By Green's formula, $(-\triangle\, u, v)_0 = (u, v)_1 + \int_\Gamma \frac{\partial u}{\partial x_n}\overline{v}$. Hence $u \in N$ implies *formally* $\frac{\partial\, u}{\partial x_n}(x_1, \ldots, x_n, 0) = \Lambda\gamma\, u$, a condition of oblique derivative.

# Lecture 12

## 6.8

Upto now we considered problems in which the space $V$ was a closed subspace of $H^1(\Omega)$. We wish to consider now some cases in which $V$ is not closed in $H^1(\Omega)$. **56**

Let $\Omega = \{x_n > 0\}$, $\ulcorner$ be the boundary of $\Omega$ and $\gamma$ be the mapping of $H^1(\Omega) \to H^{\frac{1}{2}}(\Gamma)$ as defined in § 2.4. Let $V = u \in H^1(\Omega)$ such that $\gamma\, u \in H^1(\Gamma)$. On $V$ we introduce the norm

$$| \, u \, |_V^2 = ||u||_1^2 + ||\gamma\, u||_{H^1(\Gamma)} \tag{1}$$

**Lemma 6.5.** *(1) defines on V a Hilbert structure.*

**Remark.** $V$ is not closed in $H^1(\Omega)$.

On $V$ consider the sesquilinear form

$$a(u, v) = (u, v)_1 + \lambda(u, v)_0 + (\gamma u, \gamma v)_1, \lambda > 0.$$

**Lemma 6.6.** *$a(u, v)$ is continuous on $V$ and is elliptic for $\lambda > 0$. Let $Q = L^2(\Omega)$. Then by theorem 3.1, $a(u, v)$ determines a space $N$ and an operator $A$ which is an isomorphism of $N$ onto $L^2$. To see what $A$ is we observe $a(u, v) = \langle Au, \overline{v}\rangle$ for all $\varphi = v \in \mathscr{D}(\Omega)$. Then $a(u, \varphi) = (-\triangle\, u + \lambda u, \varphi)_o$. Hence $A = -\triangle + \lambda$. Further $u \in N$ if and only if $u \in V, -\triangle\, u \in L^2(\Omega)$ and $(Au, v)_o = a(u, v)$ for all $v \in V$.*

To interpret *formally* $u \in N$ we see that from above we have

$$(-\triangle\, u, v)_o + \lambda(u, v)_o = (u, v)_1 + \lambda(u, v)_o + (\gamma\, u, \gamma\, v)$$

for all $v \in V$. Applying Green's formula

$$\int_{\Gamma} \frac{\partial u}{\partial x_n}(x', 0)\, \overline{\gamma\, v}\, dx' = -\sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2}\, \gamma\, u\, \overline{v}\, dx', \text{ where } x' = x_1, \ldots, x_{n-1}),$$

for all $v \in V$. Hence $u \in N$ if and only if $\dfrac{\partial u}{\partial x_n} = -\triangle_{x'} u(x', 0)$.

Before leaving the study of second order equations, we allude to its connections with the theory of semi-groups and to mixed problems. **57**

a) Let $V$ be such that $H_0^1(\Omega) \subset V \subset H^1(\Omega)$ and $Q = L^2(\Omega)$. Let $a(u, v)$ be a continuous sesquilinear form. Then by theorem 3.1, a space $N \subset V$ and an operator $A \in \mathscr{L}(N, L^2)$ is defined. If on $N$ we consider the topology induced by $L^2(\Omega), A$ is an unbounded operator with domain $N$. If $a(u, v)$ is elliptic, it is easily proved that there exists $\xi$ so that $(A + \lambda)I$ has an inverse $(A + \lambda)^{-1}$ bounded in norm by $1/\lambda$ when $\lambda > \xi$, $A$ is an infinitesimal generator of a regular semi-group.

b) In mixed boundary value problems we have to consider the following problem: A family of sesquilinear forms

$$(a(u, v, t)) = \int \sum a_{ij}(x, t) \frac{\partial u}{\partial x_j}\, \frac{\partial \overline{v}}{\partial x_i}$$

are given where $a_{ij}(t)$ are continuous functions from $R$ to $L^\infty$ with the weak topology of dual. Let $V = H^1(\Omega)$ and $Q = L^2(\Omega)$ and let for every $t$, $a(u, v)$ be $V$-elliptic. Then for every $t$, a space $N(t)$ and an operator $A(t)$ is defined such that $A(t)$ is an isomorphism of $N(t)$ onto $L^2(\Omega)$. If $f \in L^2(\Omega)$ and $u(t) \in N$ such that $A_t u(t) = f$, then $u(t)$ is a *continuous* function from $R$ into $V$.

# 7 Operators of order 2m

## 7.1

**Definition 7.1.** *An operator* $A = \sum(-1)^{|p|} D^p (a_{pq}(x) D^q)$, $a_{pq} \in L^\infty(\Omega)$ *is called* uniformly elliptic *in* $\underline{\Omega}$ *if there exists an* $\alpha > 0$ *such that*

$$\operatorname{Re} \sum_{|p|,\ |q|=m} a_{pq}(x)\, \xi^{\,p} \xi^q \geq \alpha \left( \sum_{i=1}^{m} \xi_i^{\ 2} \right)^m \textit{ for all } x \in \overline{\Omega} \textit{ and } \xi \in R^n.$$

We admit without proof (for a proof, see Yosida [21]).

**Theorem 7.1.** *Garding's inequality.* **58**

If $\Omega$ is bounded and $A$ is uniformly elliptic, then there exists a $\lambda > 0$ such that

$$\text{Re } a(\varphi, \varphi) + \lambda|\varphi|_0^2 \geq \alpha||\varphi||_m^2 \text{ for all } \varphi \in \mathscr{D}(\Omega) \tag{2}$$

where
$$a(u, v) = \sum \int\limits_{\Omega} a_{pq}(x) D^q u D^{\overline{p_v}} \, dx \tag{3}$$

## 7.2 Applications to the Dirichlet's problem.

**Theorem 7.2.** *If $\Omega$ is bounded and $A$ is uniformly elliptic, then*

a) $(A + \lambda)$ *is an isomorphism of $H_o^m(\Omega)$ onto $H^{-m}(\Omega)$ for $\lambda$ large enough;*

b) $(A+\lambda)$ *is an isomorphism for all $\lambda$ except for a countable system $\lambda_1, \ldots, \lambda_n$; such that $\lambda_n \to 0$.*

*Proof.* $(A+\lambda)$ is the operator associated with $a(u, v)+\lambda(u, v)_o$ which on account of Garding's inequality is elliptic on $H_o^m(\Omega)$, for large $\lambda$. Hence by theorem 3.1, $A + \lambda$ is an isomorphism of $H_o^m(\Omega)$ onto $H^{-m}(\Omega)$. Further since the injection $H_o^m(\Omega) \to L^2$ is completely continuous, we have the second assertion. ◻

## 7.3

To consider other boundary value problems and specially the Neumann problem it is useful to introduce the motion of *m-regularity*.

Let $K^m(\Omega)$ be the space of all $u \in L^2(\Omega)$ such that $D^p u \in L^2(\ )$ *for* $|p| = m$. On $K^m(\Omega)$ we define the norm $|u|_{Km}^2 = |u|^2 + |u|_m^2 . K^m(\Omega)$ is a Hilbert space. Trivially $H^m(\Omega) \subset K^m(\ )$. However, the inclusion *can be strict*.

**Definition 7.2.** *$\Omega$ is said to be m-regular if $H^m(\Omega) = K^m(\Omega)$ algebraically.*

For instance, $\Omega = R^n$ is m-regular for $H^m(R^n) = K^m(R^n)$ as is seen easily by Fourier transformation.

**Theorem 7.3.** *If $\Omega$ is m-regular, then there exists a constant c such that* **59**

$$|u|_k^2 \leq c\,(|u|_1^2 + |u|_m^2) \text{ for } k = 1, \ldots, m-1. \tag{4}$$

*Proof.* The injection of $H^m(\Omega)$ into $K^m(\Omega)$ is onto and continuous. Hence by the closed graph theorem, it is an isomorphism. And so $||u||_m \leq c_1\ (|u|_0^2 + |u|_m^2)$, which implies the inequalities (4). □

Now the problem arises whether if (4) holds $\Omega$ is m-regular or not. If (4) holds the inclusion mapping is continuous, one to one, and its range is closed. We have to prove then that $H^m(\Omega)$ is dense in $K^m(\Omega)$. This is still an unsolved problem.

we admit following theorems without proof.

**Theorem 7.4.** *Every open set with smooth boundary is m-regular.*

**Theorem 7.5.** *If the injection $H^1(\Omega) \to L^2(\Omega)$ is completely continuous, then $\Omega$ is m-regular.*

**Definition 7.3.** *$\Omega$ is strongly m-regular, if (a) it is m-regular, and (b) for every $\in > 0$, there exists a $c(\in)$ such that*

$$|u|_k^2 \leq \in\ |u|_m^2 + c(\in)\ |u|_0^2 \text{ for } k = 1, \ldots, m-1 \tag{5}$$

*for all $u \in H^m(\Omega)$.*

**Proposition 7.1.** *$\Omega = R^n$ is strongly m-regular for every m.*

*Proof.* By Plancherele's theorem, we have to prove that given any $\in > 0$ there exists $c(\in)$ such that

$$|u|_k^2 \ \in \ \int |\hat{u}(\xi)|^2\ |\xi|^{2k} d\ \xi \leq \int\ (\in |\xi|^{2m} + c(\in))|\hat{u}|^2 d\xi$$

for $k = 1, \ldots, m-1$, i.e., $|\xi|^{2k} \leq \in |\xi|^{2m} + c(\in)$ for $k = 1, \ldots, m-1$, which follows from elementary considerations. □

We do not know however if there exists $m$-regular domain which are not strongly $m$-regular. **60**

**Theorem 7.6.** *If the injection $H^1(\Omega) \to L^2(\Omega)$ is completely continuous, then $\Omega$ is strongly m-regular.*

*Proof.* By theorem 7.5, we see that $\Omega$ is m-regular. We have now to prove the inequality (5). If it is not true there exists an $\in > 0$ and a sequence $u_i \in H^m(\Omega)$ and a sequence $c_i \to \infty$ such that

$$|u_i|_k^2 \geqq \in |u_i|_m^2 + c_i |u|_0^2.$$

Let $v_i = \dfrac{u_i}{(|u_i|_m^2 + |u_i|_o)^{\frac{1}{2}}}$. Then $v_i \in K^m(\Omega) = H^m(\Omega)$. □

Further

$$|v_i|_k^2 \geq \in +(c_i - \in)\frac{|u_i|_0^2}{|u_i|_m^2 + |u_i|^2} \text{ and } c'_i = c_i - \in \to \infty.$$

Hence

$$|v_i|^2 \geq \in +c'_i|v_i|_0^2. \tag{6}$$

Now $|v_i|^2 + |v_i|_m^2 = 1$, so that $v_i$ are bounded in $H^m(\Omega)$ and hence $|v_i|_k \leq C$. From (6) it follows that $|v_i|^2 \leq \dfrac{C - \in}{c'_i}$, and hence $v_i \to 0$ in $L^2(\Omega)$. Therefore there exists a sequence $v_\mu$ converging weakly to 0 in $H^{m-1}(\Omega)$. Since the injection of $H^1(\Omega)$ into $L^2(\Omega)$ is completely continuous $v_\mu \to 0$ strongly in $H^{m-1}(\Omega)$, i.e., $|v|_k \to 0$ which contradicts (6).

# Lecture 13

## 7.4 Applications

Let $a(u,v) = \sum_{|p|,|q| \leq m} \int a_{pq} D^q(u) \bar{D}^p\, v\, dx$ with $a_{pq} \in L^\infty$ 61
and

$$A(u,v) = \sum_{|p|,|q|=m} a_{pq} D^q\,(u)\,\bar{D}^p\, v\, dx$$

be the leading part of $a(u,v)$.

**Theorem 7.8.** *Let* $(a)\Omega$ *be strongly m-regular and* $(b)\, \mathrm{Re}\, A(u,u) \geq \alpha|u|_m^2$ *for some* $\alpha > 0$ *and for all* $u \in H^m(\Omega)$*. Then there exists* $\lambda$ *such that* $\mathrm{Re}\ a(u,u) + \lambda|u|_0^2 \geq \beta||u||_m^2$ *for some* $\beta > 0$*, and for all* $u \in H^m$*.*

*Proof.* We have

$$\mathrm{Re}\ a(u,v) = \mathrm{Re}\ A(u,u) + \mathrm{Re}\ \rho(u,u)$$

where

$$\rho(u,v) = \sum_{|p|\leq m,} \int a_{pq} D^q u D^p v\, dx\ |q| \leq m \text{ and } |p| + |q| \leq 2m-1.$$

□

Every term of $\rho(u,v)$ is majorized by $c||u||_m||u||_{m-1}$ and so $\mathrm{Re}\ \rho(u,u) \leq c_1||u|\ _m||u||_{m-1}$. Hence

$$\mathrm{Re}\ a(u,u) \geq \alpha|u|_m^2 - c_1\ ||u||_m\ ||u||_{m-1}.$$

We have then to prove that we can find $\lambda$ such that there exists $\beta$ satisfying

$$X = \ \alpha|u|_m^2 - c_1||u||_m\ ||u||_{m-1} + \lambda|u|_0^2 \geq \beta||u||_m^2 \tag{1}$$

Since $\Omega$ is strongly m-regular, using definition for any $\in > 0$, there exists $c(\in)$ such that $||u||_{m-1} \leq \in ||u||_m + c(\in)|u|_o$. Hence $c_1||u||_m||u||_{m-1} \leq c_1 \in ||u||_m^2 + c(\in)||u||_m|u|_o$. Since $\Omega$ is m-regular also $||u||_m$ is equivalent to $|u|_m + |u|_o$. Hence $c_1||u||_m||u||_{m-1} \leq c_2 \in (|u|_m + |u|_0^2) + c'(\in)(|u|_m|u|_o + |u|_o^1)$. So $X \geq \alpha|u|_m^2 - c_2 \in (|u|_m^2 + |u|_o^2) - c'(\in)(|u|_m|u|_o + |u|_o^2 + |u|_o^2$. Now $2|u|_m|u|_o \leq \in_1 |u|_m^2 + \dfrac{1}{\in_1}|u|_o^2$ for any $\in_1$. Hence 62

$$X \geq \left(\alpha - c_2 \in - \frac{\in_1 c'(\in)}{2}\right) |u|_m^2 + (\lambda - c''(\in) + \frac{1}{\in_1})|u|_o^2.$$

First we choose $\in$ so that $\alpha - c_2 \in = \frac{\alpha}{2}$. This determines $c(\in)$ and $c'(\in)$. Then we choose $\in_1$ so small that $\in_1 c'(\in) < \alpha/4$, and then $\lambda$ so large that $\lambda - c''(\in) + \dfrac{1}{\in_1} > 0$.

Then $X \geq \beta_1(|u|_m^2 + |u|_o^2)$ and by m-regularity of $\Omega$, $X \geq \beta||u||_m^2$ as it was required to be proved.

# 8 Regularity in the Interior

## 8.1

Having established the existence and uniqueness of weak solutions of certain elliptic differential equations, we turn now to consider their regularity problem, that is to say, to see whether in the equation $Au = f$ sufficient regularity of $f$ will imply some regularity of $u$. First, we shall investigate when $u$ is regular in the interior of the given domain $\Omega$ and next we shall consider when $u$ is regular in $\bar{\Omega}$ in some sense.

To formulate the problem of interior regularity, we shall require some definitions of new spaces.

We recall having defined in § 2.1, that $H^{-r}(\Omega) = (H^r(\Omega))'$, for positive $r$. If 0 is an open set in $\Omega$ and if $u$ is a function in $\Omega$, $u_0$ will denote the restriction of $u$ to 0.

**Definition 8.1.** *Let $\Omega$ be an open set in $R^n$.$\mathscr{L}^r = H^r_{loc}(\Omega)$ for any integer $r$, consists of functions $u$ which for any relatively compact $0 \subset \Omega$ are such that $u_0 \in H^r(0)$, $r$ integer $\geq 0$ or $< 0$.* 63

Let $K_n$ be an increasing sequence of closures of relatively compact open sets $0_n$ covering $\Omega$. Let $p_n = ||u_{0_n}||_r$ be the norms in $H^r_{0_n}$ of $u_{0_n}$. $p_n'$s are semi-norms in $\mathscr{L}^r$. On $\mathscr{L}^r$ we put the locally compact topology determined by the semi-norms $p_n$.

**Definition 8.2.** *$K^r$ $r$, any integer will denote the space of $u \in H^r$ with compact support.*

On $K^r$ we put the natural inductive limit topology. A sequence $u_n$ converges in $K^r$ if all $u_n$ have their supports in a fixed compact $A$ and all $u_n \to 0$ in $H^r(A)$ . We see easily $(Z^r)' = K^{-r}$.

**Proposition 8.1.** *$\mathscr{E}'(\Omega) = \bigcup\limits_{r \in Z} K^r(\Omega)$ algebraically.*

*Proof.* By definition $\cup K^r(\Omega) \subset \mathscr{E}'(\Omega)$. We have to prove only that if $T \in \mathscr{E}'$ then $T \in K^r$ for some $r$. Now by a theorem of Schwartz, $T \in \mathscr{E}'(\Omega)$ implies $T = \sum\limits_{|p| \leq \mu} D^p f_p$ where $f_p$ are continuous and have a compact support. Hence $f_p \in L^2(\Omega)$ and by theorem 2.1, $T \in H^{-\mu}(\Omega)$. This means that $T \in K^r(\Omega)$, where $r = -\mu$. ◻

**Proposition 8.2.** *Let $B = \sum\limits_{|p| \leq \mu} b_p(x) D^p$ with $b_p \in \mathscr{E}$. Then $B$ is a continuous linear mapping of $\mathscr{D}, \mathscr{E}, \mathscr{D}', \mathscr{E}'$ into itself and also a continuous linear mapping of $\mathscr{L}^r(\Omega)$ into $\mathscr{L}^{r-\mu}(\Omega)$ and $K^r(\Omega)$ into $K^{r-\mu}(\Omega)$.*

**Remark.** It is not true, however, that $B$ is continuous from $H^r$ to $H^{r-\mu}$.

*Proof.* The first assertion is trivial and the last one follows if we prove the middle one. Let $f \in \mathscr{L}^r(\Omega)$. Since $b_p \in \mathscr{E}$ on $\Omega$, $b_p'$s and their derivatives are bounded on 0 so that it is enough to prove that $D^\mu f \in H^{r-\mu}(0)$. We may assume 0 to be an open set with smooth boundary. If $\mu < r$ and $r > 0$ we have the result from the definition. If $\mu > r$, then by integration by parts for $g \in H^{\mu-r}(0)$. 64

$$\langle D^\mu f, g\rangle = (-1)^{\mu-r}\langle D^\mu f, D^{\mu-r} g\rangle$$

exists and hence $D^m f$ is a continuous linear function on $H^{\mu-r}(0)$, i.e., $D^\mu f \in H^{r-\mu}(0)$. ◻

## 8.2 Statements of theorems

Let

$$A = \sum_{|p|,|q| \leq m} (-1)^p D^p(a_{pq}(x) D^q), a_{pq} \in \mathscr{E}(\Omega) \tag{1}$$

**Definition 8.3.** *$A$ is* uniformly elliptic *in $\Omega$ if given any compact $K \subset \Omega$ we have an $\alpha_K > 0$ such that*

$$\text{Re}\left(\sum a_{pq}(x)\xi^p \xi^q\right) \geq \alpha_K |\xi|^{2m} \text{ for all } x \in K \text{ and all } \xi = (\xi_1, \ldots, \xi_n) \in R^n. \tag{2}$$

**Remark.** If $A$ is uniformly elliptic, Garding's inequality (Theorem 7.1) is true on every compact $K$.

Let

$$B = \sum_{|p|\leq\mu} b_p(x)\, D^p, b_p \in \mathscr{E}(\Omega) \tag{3}$$

**Definition 8.4.** *$B$ is elliptic in $\Omega$ if $\sum b_p(x)\xi^p = 0$ with $\xi \in R^n$, implies $\xi = 0$.*

We see at once that a uniformly elliptic operator is elliptic. The converse, however, is inexact. For example, in the case $n = 2$, $B = \dfrac{\partial}{\partial x_1} + i\dfrac{\partial}{\partial x_2}$ is elliptic, but evidently is not uniformly elliptic being not of even order.

In this and the next lecture, we shall prove the following two theorems on the regularity in the interior. 65

**Theorem 8.1.** *Let $A$ be a uniformly elliptic operator of order $2m$ in $\Omega$. If for some $T \in \mathscr{D}'(\Omega), A\,T \in \mathscr{L}^r(\Omega)$ for some fixed $r$, then $T \in \mathscr{L}^{\,r+2m}(\Omega)$.*

**Theorem 8.2.** *Let $B$ be an elliptic differential operator of order $\mu$ in $\Omega$. If for some $T \in \mathscr{D}'(\Omega), B\,T \in \mathscr{L}^r(\Omega)$ for some fixed $r$, then $T \in \mathscr{L}^{r+\mu}(\Omega)$.*

From these theorems, the regularity in the classical sense will follows by the

**Corollary.** *Let $B$ be an elliptic operator of order $\mu$. If for some $T \in \mathscr{D}'$, $B\,T$‘ $\in \mathscr{E}$, then $T \in \mathscr{E}$.*

For $B\,T \in \mathscr{E}$ means $B\,T \in \mathscr{L}^r$ for all $\mu$. Hence by the theorems $T \in \mathscr{L}^{r+\mu}$ for every $r$. Hence all the derivatives of $T$ will be functions which means $T \in \mathscr{E}$.

Before proving these theorems, we shall establish some connections between elliptic and uniformly elliptic operators. Using these, we shall prove that theorem 8.1 implies theorem 8.2, and then we shall occupy ourselves in the proof of theorem 8.1.

**Proposition 8.3.** *Let $n \geq 3$. If $B$ is elliptic, then $B$ is of even order (See Schechter [15]).*

Let $x \in \Omega$ and $\sum\limits_{|p|=\mu} b_p(x)\xi^p = Q(\xi)$, That $B$ is elliptic at $x$ means that only real zero of $Q(\xi)$ is $\xi = 0$. We prove $Q(\xi)$ must be of even degree. By a non-singular linear transformation, if necessary, we may assume $\xi_n$ has degree $\mu$.

Let $\xi' = (\xi_1, \ldots, \xi_{n-1}) \neq 0$ be a point in $R^{n-1} \subset \mathscr{C}^n$. Let $Q(\xi_n)$ be the 66 polynomial in $\xi_n$ obtained by substituting $(\xi_1, \ldots, \xi_{n-1})$ by $\xi'$. Then $Q(\xi_n)$ has $n$ complex roots all of which have imaginary part $\neq 0$, for otherwise $(\xi', \xi_n)$ would be a real non-trivial zero of $Q(\xi)$. Let $\pi_+$ and $\pi_-$ be number of roots of $Q(\xi_n)$, with positive and negative imaginary parts respectively. On account of homogeneity of $Q(\xi)$ if we put $\xi'' = -\xi'$, the number of positive roots of $Q(\xi'', \xi_n)$ will be $\pi_-$ and negative roots will be $\pi_+$. Let $\xi', \xi''$ be joined by an are not passing through the origin which is possible because $n - 1 \geq 2$. From a classical theorem the roots of $Q(\xi_n)$ can be continued from $\xi'$ to $\xi''$ continuously. Now at no point on the are $\xi', \xi''$ can $\xi_n$ be real on account of ellipticity. So the positive roots at $\xi'$ are continued into positive roots at $\xi''$. Hence $\pi_+ = \pi_-$ and $\mu = \pi_+ + \pi_- = 2\pi_+$ is even.

**Proposition 8.4.** *Let $B$ be an elliptic differential operator with real coefficients. Then $B$ is uniformly elliptic in $\Omega$.*

*Proof.* Let $\sum\limits_{|p|=\mu} b_p(x)\xi^p = P(x, \xi)$. Since $B$ is elliptic, $P(x, \xi) = 0$ implies $\xi = 0$. Hence on $|\xi| = 1, P(x, \xi)$ for fixed $x$ keeps same sign which we may assume $> 0$. Hence $\dfrac{P(x, \xi)}{|\xi|^\mu} \geq \alpha$ for fixed $x$. If now $K$ is any compact, $P(x, \xi)$ is continuous on the compact $K \times\ |\xi| = 1$ and hence $\dfrac{P(x, \xi)}{|\xi|^\mu} > \alpha$ for all $x \in K$ and all $\xi \neq 0$. If we put $-\xi$ for $\xi$ we get the inequality multiplied by $(-1)^\mu$ hence $\mu$ is even. □

**Proposition 8.5.** *Theorem 8.1 $\Rightarrow$ Theorem 8.2* **67**

*Proof.* Let $B = \sum\limits_{|p|\leq\mu} b_p(x)D^p$. Put $\bar{B} = \sum\limits_{|p|\leq\mu} \overline{b_p(x)} D^p$. Let

$$A = \bar{B}B = \sum_{|p|=|q|=\mu} b_p(x)\, \overline{b_p(x)}\, D^p\, D^q + \cdots$$

□

$A$ is of even order and if $P(x, \xi)$ is its associated form, then $P(x, \xi) = |\Sigma b_p(x)\xi^{\ p}|^2$. Further since $B$ is elliptic, $A$ also is. By proposition 8.4, $A$ is then uniformly elliptic. Let now $T \in \mathscr{D}'$ such that $BT \in \mathscr{L}^r$. Hence $A\,T = \bar{B}B\,T \in \mathscr{L}^{r-\mu}$. If theorem 8.1 is true, then $T \in \mathscr{L}^{r+\mu}$ proving theorem 8.2.

# Lecture 14

## 8.3

We now proceed to prove the theorem 8.1. First we prove a lemma of fundamental character which will help to establish an inductive procedure to prove the theorem. 68

**Lemma 8.1.** *Let A be a uniformly elliptic differential operator of order* $2m$*. Let* $u \in \mathscr{L}^m(\Omega)$ *and let* $Au \in \mathscr{L}^{-m+1}$ *(usually* $Au \in \mathscr{L}^{-m}$ *only). Then* $u \in \mathscr{L}^{m+1}$*.*

*Proof.* We prove the lemma in two steps. In the first one it will be shown that it is enough to prove the lemma assuming $A$ and $Au$ to have compact support, for which the assertion will be proved in the second step. □

**Step 1.** *The lemma is equivalent to "if* $u \in K^m$ *and* $Au \in K^{-m+1}$*, then* $u \in K^{m+1}$*".*

The direct part is evident. To prove the converse, let $u \in \mathscr{L}^m$ be such that $Au \in \mathscr{L}^{-m+1}$. For any $\varphi \in \mathscr{D}(\Omega), v = \varphi\, u \in \mathscr{L}^m$. Now $Au = A(\varphi\, u) = A\, u + \sum_{|p| \leq 2m,\ |q| \leq 2m-1} D^p\, D^q\, u$.

Since for $|q| \leq 2m-1, D^q u \in \mathscr{L}^{-m+1}$ and by assumption, $Au \in \mathscr{L}^{-m+1}$, it follows that $Av \in \mathscr{L}^{-m+1}$. Since $\varphi$ has compact support, $v$ and $Av$ are in $K^{-m+1}$. Hence $v \in K^{m+1}$. Since this is true for every $\varphi \in \mathscr{D}(\Omega), v \in \mathscr{L}^{-m+1}$.

Now we prove the

**Step 2.** *If* $u \in K^m$ *and* $Au \in K^{-m+1}$*, then* $u \in K^{m+1}$*.*

We have to prove that $\dfrac{\partial u}{\partial x_i} \in H^m(\Omega)$. A general method to prove this, here and in later occasions, will be to estimate the difference quotients of $u$. Let 69

$h = (h, 0, \ldots, 0)$ and $u^h(x) = \dfrac{1}{h}\, u(x+h) - u(x)$ which exists if $h$ is small enough. Now we establish the following:

a) $A(u^h) - (Au)^h = \sum (-1)^{\,p} D^p\, (a^h_{pq}\, D^q u(x+h))$.

b) $A(u^h)$ is bounded in $K^{-m}$.

c) $u^h$ is bounded in $H^m$.

Assuming for a moment that *a*), *b*), *c*) are proved, we complete the proof of the lemma. Since $u^h$ is bounded in $H^m$, it is a weakly compact and hence there exists $h_i \to 0$ such that $u^{h_i} \to g$ weakly in $H^m(\Omega)$. On the other hand, $u^{h_i} \to \dfrac{\partial u}{\partial x_i}$ in $\mathscr{D}'$. Hence $\dfrac{\partial u}{\partial x_i} = g \in H^m$, i.e., $u \in H^{m+1}$. Since $u$ has compact support, $u \in K^{m+1}$.

Now the prove *a*), *b*), *c*).

a) We verify easily that $(\alpha f)^h - \alpha f^h = \alpha^h\, f(x+h)$. Applying this term by term in $(Au)^h - A(u^h)$ we obtain (*a*).

b) On account of (*a*), to prove that $A(u^h)$ is bounded in $K^{-m}$ it is enough to prove that $(Au)^h$ and each of $D^p(a^h_{pq} D^q u(x+h))$ are bounded in $H^{-m}$. Since $Au = g \in K^{-m+1}$, $\dfrac{\partial g}{\partial x_1} \in K^{-m}$ and since $(Au)^h \to \dfrac{\partial g}{\partial x_1}$, $(Au)^h$ is a convergent sequence in $K^{-m}$ and so is bounded, Further, since $a^h_{pq} \in C^\infty$ as $h \to 0$, $a^h_{pq} \to \dfrac{\partial}{\partial x_1} a_{pq}(x) \in C^\infty$ uniformly on every compact set. Also $D^q u(x+h) \to D^q u(x)$ in $L^2$. Hence $a^h_{pq} D^q u(x+h)$ converge in $L^2$. Since $D^p$ are derivatives of order than or equal to $m$, $D^p(a^h_{pq} D^q u(x+h))$ converge in $H^{-m}$, and hence in $K^{-m}$. This proves (*b*).

c) Since by (*b*), $A(u^h)$ is bounded, we have

$$\begin{aligned}\langle A\, u^h, u^{-h}\rangle &\le \|A(u^h)\|_{H^{-m}} \|u^h\|_m \\ &\le c_1 \|u^h\|_m\end{aligned}$$

On account of Garding's inequality, we have **70**

$$\text{Re }\ a(u^h, u^h) + \lambda |u^h|_0^2 \ge \alpha \|u^h\|_m^2$$

on every compact set. As $h \to 0$, we may assume that all $u^h$ have their support in a fixed compact set. Hence

$$\alpha \|u^h\|_m^2 \le \lambda |u^h|_0^2 + c_1 \|u^h\|_m.$$

Since $u \in K^m, u^h \to \dfrac{\partial u}{\partial x_i}$ in $L^2$ and so $|u^h|_o$ is bounded. Further we have

$$c_1 \|u^h\|_m \leq \frac{c_1^2}{2} + \frac{\alpha}{2}\|u^h\|_m^2.$$

Hence $\dfrac{\alpha}{2}\|u^h\|_m^2 \leq c_3$ which proves $u_m^h$ is bounded, and this completes the proof of lemma 8.1.

**Lemma 8.2.** *Let $u \in \mathscr{L}^m$, and $Au \in \mathscr{L}^{-m+1+j}$. Then $u \in \mathscr{L}^{m+j+1}$ for every non-negative integer $j$.*

*Proof.* Lemma 8.1 proves the lemma for the case $j = 0$; assuming it proved for integers upto $j = 1$, we prove it for $j$. Since $\mathscr{L}^{-m+j+1} \subset \mathscr{L}^{-m+j}, Au \in \mathscr{L}^{-m+j+1}$ implies that $Au \in \mathscr{L}^{-m+j}$ and hence by induction hypothesis that $u \in \mathscr{L}^{m+j}$. Now $D\,Au - A\,Du = A'u$ where $A'$ is a differential operator of order $2m$. Since $Au \in \mathscr{L}^{-m+j+1}, D\,Au \in \mathscr{L}^{-m+j}$ and since $u \in \mathscr{L}^{m+j}, Au \in \mathscr{L}^{m+j}$. Hence $A(Du) \in \mathscr{L}^{-m+j}$. But $Du \in \mathscr{L}^m$ as $u \in \mathscr{L}^{m+j}$, $j \geq 1$. Hence by lemma 8.1, $Du \in \mathscr{L}^{m+j}$, i.e., $u \in \mathscr{L}^{m+j+1}$. ◻

Lemma 8.2 can be put in a slightly better form of

**Lemma 8.2′.** *Let $u \in \mathscr{L}^m$ and $Au \in \mathscr{L}^r$, then $u \in \mathscr{L}^{r+2m}$.*

For, if $r \leq -m$, the lemma is trivial and if $r > -m$ we have $r = -m + j$ and lemma 8.2′ follows at once from lemma 8.2.

Now we complete the proof of theorem 8.1. We have to prove that if $T \in \mathscr{D}'$ **71**
and $A\,T \in \mathscr{L}^r$, then $T \in \mathscr{L}^{r+2m}$. Let $O, O_1$ be two relatively compact open sets such that $O \subset O_1 \subset \Omega$. On account of a theorem of Schwartz, $T_0 = \sum D^p f_p$ where $f_p$ are continuous in with support contained in $O_1$. By theorem 2.1, $T_0 \in H^{-\beta}(O)$. Now $\triangle^{m+\beta}$ where $\triangle$ is the Laplacian is on account of theorem 1.3 is an isomorphism of $H^{m+\beta}(O)$ onto $H_o^{-m+\beta}$. Hence there exists $u \in H^{m+\beta}(0)$ such that $\triangle^{m+\beta}\, u = T_0$. Applying lemma 8.2′ to $\triangle^{m+\beta}$, we have $u \in \mathscr{L}^{2m+\beta}(0)$ as the order of $\triangle$ is $2(m+\beta), T_0 \in \mathscr{L}^{-\beta}$, and $u \in \mathscr{L}^{m+\beta}(0)$. Now $(A\,T_0) = (A \triangle^{m+\beta}\, u) \in \mathscr{L}^r(0)$. The order of $B = A \triangle^{m+\beta}$ is $4m + 2\beta$ and $B$ is uniformly elliptic. As $u \in Z^{2m+\beta}$ and $Bu \in \mathscr{L}^r$, applying lemma 8.2, we have $u \in \mathscr{L}^{r+4m+2\beta}$. Hence $T = \triangle^{m+\beta}\, u \in \mathscr{L}^{r+2m}$.

## 8.4 Some remarks.

We remark that theorem 8.2 implies theorem 8.1 trivially though in the course of the proof, we proved theorem 8.1, before proving theorem 8.2. This raises a vague question what properties which are true for uniformly elliptic differential equations can be upheld for the elliptic ones. For instance, we know for Dirichlet's problem for bounded domains with smooth boundary Fredholm's alternative holds if the operator is uniformly elliptic. In the case $n = 2$, we have the following counter example of Bicadze [4].

Consider the Dirichlet problem in the unit circle for the operator $A = \frac{1}{4}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)^2$. A is elliptic but is not uniformly elliptic, for the associated form has $\xi^2 - \eta^2$, as its real part. We prove that the space of $u$ such that $Au = 0, u = 0$ on the boundary is not finite dimensional and hence that Fred- 72
holm alternative does not hold. $Au = 0$ means $\frac{\partial^2 u}{\partial \overline{z}^2} = 0$, where $\frac{\partial}{\partial \overline{z}} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$ and hence $\frac{\partial u}{\partial \overline{z}}$ is holomorphic in the unit circle. Hence $u = f + \overline{z}g$ where $f$ and $g$ are holomorphic in the unit circle. But $u = 0$ on the boundary $z\overline{z} = 1$ . Hence $0 = z\,u = z\,f + g$ on the boundary, and hence $g = -zf$ everywhere as $f$ and $g$ are holomorphic in the unit circle. Thus $u = (1 - z\overline{z})f(z)$ is a solution of the above problem for any holomorphic $f(z)$ which shows that the space of $u$ such that $Au = 0, u = 0$, on the boundary, is not finite dimensional.

For complementary results, see Schechter [15] and a forthcoming paper by Agmon, Douglis, Nirenberg.

# Lecture 15

## 9 Regularity at the boundary.

In the last lecture we dealt with the regularity in the interior or local regularity 73
of the solutions of the elliptic differential equations. Now we wish to consider the regularity of the solutions in $\bar{\Omega}$. In a sense such solutions can be extended to the boundary. These should not be confused with problems in which boundary values to be attained are given. These will be considered in a general set up under the name of Visik-Sobolev problems.

### 9.1

**Theorem 9.1.** *Let $\Omega$ be a bounded open set in $R^n$ with a boundary which is an $n-1$ dimensional $C^\infty$ manifold. Let*

$$a(u,v) = \sum_{|p|,|q|\leq m} \int a_{pq}\, D^q\, u\, \overline{D^p\, v}\, dx$$

*with $a_{pq} \in \mathscr{E}(\bar{\Omega})$ be given such that* Re $(a(u,u)) \geq \alpha \|u\|_m^2$ *for some $\alpha > 0$ and for all $u \in H^m(\Omega)$. Let $V = H^m(\Omega)$ and $Q = L^2(\Omega)$ and let $A$ and $N$ be as determined in theorem* 3.1. *If $f \in L^2(\Omega)$ and $u \in N$ is such that $Au = f$, then $u \in H^{2m}(\Omega)$.*

**Remark.** If we do not take any condition on the boundary (eg., $u \in N$) then we can assert only that $u \in \mathscr{L}^{2m}(\Omega)$ and cannot assert in general that $u \in H^{2m}(\Omega)$.

The proof of this theorem is fairly complicated and will be broken in several steps.

**Step 1.** *First we reduce the problem to one in a cube in the following way: Let $O_i$ be a finite covering by relatively compact open sets of the boundary $\Gamma$ such that there exists $C^\infty$ homeomorphisms $\psi_i$ of $O_i$ to $\bar{W} = \begin{cases} 0 < \epsilon_i < 1, \\ -1 < \epsilon_n < n \end{cases}$* 74
*$i = 1, \ldots n-1$ such that $\psi_i$ maps $0_i \cap \Omega$ onto $W_+ = \begin{cases} 0 < \epsilon_i < 1, \\ 0 < \epsilon_n < 1 \end{cases}$ $i = 1, \ldots, n-1$ and $\Gamma \cap O_i$ onto $W_0 = \{W \cap \{\xi_n = O\}\}$. Since the regularity is the interior of $u \in H^m(\Omega)$ has been already proved to prove that $u \in H^{2m}(\Omega)$, it remains only to prove the restrictions of $u$ to $O_i$, i.e., $u_{O_i} \in H^{2m}(O_i)$. The homeomorphisms $\psi_i$ define isomorphisms of $H^m(O_i \cap \Omega)$ onto $H^m(W_+)$. Let $u_1, v_1 \in H^m(W_+)$. Define $a_0(u, v) = a(\psi^{-1}(u_1), \psi^{-1}(v_1))$. (We drop $i$ from the suffix). This definition is possible as $A$ is an operator of local type, more precisely*

$$a(\psi^{-1}(u_1)), (\psi^{-1}(v_1)) = \int\limits_O a_{pq}(x)\, D^q(\psi^{-1}(u_1)\overline{D^p(\psi^{-1}(v_1)}dx.$$

*$a_0(u, v)$ is a continuous sesquilinear form on $H^m(W_+)$. Now by theorem 3.1, $a(u, v) = (f, v)_o$ for all $v \in H^m(\Omega)$. Let, in particular, $v$ vanish near the boundary of $O - \Gamma \cap O$, and have its support in $O$. Then $a(u, v) - a_o(u, v) = (f, v)_0$. Hence if $v_1$ is in $H^m(W_+)$, and vanishes, near the boundary of $W_+ - \Gamma$, then*

$$a(\psi(u), v) = (\psi(f), v)_0, \text{ when } \psi(f) \in L^2(W_+).$$

If we prove now that $\psi(u) \in H^{2m}(W^\epsilon)$ for every $\in > 0$, where $W^\epsilon = \begin{cases} 1- \in < \epsilon < \in \\ 0 < \epsilon_n < 1 \end{cases}$, then by an obvious shrinking argument, we will have proved the theorem.

## 9.2

**Step 2.** *Thus our problem is reduced to the following one. Let $\Omega = \{0 < x_i < 1\}$, $i = 1, \ldots, n$, be $n$-dimensional cube in $R^n$. Let $a(u, v) = \sum\limits_m \int a_{pq}(x) D^q u \overline{D^p u} dx$ with $a_{pq} \in \mathscr{E}(\bar{\Omega})$ be an elliptic form on $H^m(W)$. Let $f \in L^2(\Omega)$ and $u \in H^m(\omega)$ be such that for every $v \in H^m(\Omega)$ which is zero near $\partial\, \Omega - \sum$, we have* 75

$$a(u, v) = (f, v)_0. \tag{1}$$

Then we have to prove that $u \in H^{2m}(\Omega^\epsilon)$ for every $\in> 0$, where

$$\Omega^\epsilon = \left\{ \begin{matrix} \in < x_i < 1- \in, \; i = 1, \ldots, n-1 \\ 0 < x_n < 1 \end{matrix} \right\}.$$

We shall prove this in two steps. First we consider the derivatives of $u$ in the direction parallel to $x_n$ axis, which we call tangential derivatives and denote them by $D^p_\tau(u)$ with $p = (p_1, \ldots, p_{n-1}, 0)$. By an induction argument and considering difference quotients as in the previous lecture, we shall prove that $\underset{|p|=m}{D^p} \; u \in H^m(\Omega)$. In the next section we shall consider $D^m_{x_n} u$.

**Proposition 9.1.** *Under the hypothesis of the reduced problem* $\underset{|p|=m}{D^p_\tau} u \in H^m(\Omega^\epsilon)$.

*Proof.* If $u \in H^m(\Omega)$ is such that $v = 0$ near $\partial\Omega - \sum$, then we denote by $v^h(x) = \dfrac{1}{h}[v(x+h) - v(x)]$ which is defined for sufficiently small $h$, where $h = (h, 0, \ldots, 0)$. We note two simple identities relating to $v^h$.

1. $\int_\Omega u^h \, v \, dx + \int_\Omega uv^h \, dx = 0$ where $u$ and $v$ both vanish near $\partial\Omega - \sum$.
2. $(au)^{-h} = a \, u^{-h} + a^{-h} \, u(x-h)$.

Let $\phi$ be a function in $\mathscr{D}(\overline{\Omega})$ vanishing near $\partial\Omega - \sum$. $u$ is in $H^m(\Omega)$ and vanishes near the boundary. Using Leibnitz's formula, it is seen at once that to prove $u \in H^m(\Omega)$, it is enough to show that $D_\tau(\phi u) \in H^m(\Omega)$. We shall prove first that $(\phi \, u)^{-h}$ is bounded. □

Let 76

$$\begin{aligned} b(u, v) &= a(\phi \, u, v) - a(v, \phi u) \\ &= \sum_{|p| \leq m, |q| \leq m, |p|+|q| \leq 2m-1} \int b_{pq}(x) \, d^q u \, \overline{D^p v} \, dx \end{aligned} \tag{2}$$

where $b_{pq}(x)$ are products of derivatives of $\phi$ with $a'_{pq}$s, and so vanish near $\partial\Omega - \sum$ and are in $\mathscr{D}(\overline{\Omega})$. Using (1) and (2), we have

$$a(\phi \, u)^{-h}, v) = [a((\phi \, u)^{-h}, v) + a(\phi u, v^h)] - b(u, v^h) - (f, \phi v^h)_o. \tag{3}$$

Now we prove three lemmas.

**Lemma 9.1.** $| \, a((\phi \, u)^{-h}, v) + a(\phi \, u, v^h)| \leq c_1 \, ||u||_m$.

**Lemma 9.2.** $| b(u, v^h) | \leq c_2 ||v||_m$.

**Lemma 9.3.** $| (f, \phi v^h)_0 | \leq c_3 ||v||_m$.

Using these in (3) with $v = (\phi u)^{-h}$ and using ellipticity condition, we have

$$\alpha \| (\phi u)^{-h} \|_m^2 \leq c_4 \| (\phi u)^{-h} \|_m.$$

Hence $(\phi u)^{-h}$ is bounded in $H^m$. Since bounded sets in $H^m$ are weakly compact, there exists a sequence $h_i$ such that $(\phi u)^{-h_i}$ converges weakly to a function $g \in H^m$. However since $(\phi u)^{-h_i} \to \dfrac{\partial u}{\partial x_i}$ in $\mathscr{D}$, we have $\dfrac{\partial u}{\partial x_1} \in H^m$. This proves the proposition 9.1, that $D_T u \in H^m$. It remains to prove the above lemma 9.1, 9.2 and 9.3.

**Proof of Lemma 9.1.** $a((\phi u)^{-h}, v)) + a(\phi u, v^h)$ consists of sums of terms like

$$\begin{aligned}
X &= \int\limits_{|p|=m,|q|=m} a(x)\, D^q(\phi u)^{-h}\, \overline{D^p v}\, dx + \int\limits_{|p|=m,|q|=m} a(x) D^q(\phi u)\, \overline{D^p v^h}\, dx \\
&= \int a(x)\, D^q(\phi u)^{-h}\, \overline{D^p v}\, dx - \int ((a(x) D^q(\phi u))\, \overline{D^p v}\, dx \\
&= \int a(x) D^q(\phi u)^{-h}\, \overline{D^p v}\, dx - \int [a(x) D^q(\phi u)^{-h} \\
&\qquad\qquad + a^{-h}(x) D^q(\phi u)(x-h)]\, \overline{D^p v}\, dx \\
&= -\int a^{-h}\, D^q(\phi u)(x-h) D^p v\, dx.
\end{aligned}$$

Since $a^{-h}$ are bounded and translations are continuous in $H^m$ and $|q| \leq m$, 77
we have, by using Schwartz's lemma.

$$|X| \leq c|D^p v|_o \leq c_1 ||v||_m.$$

**Proof of Lemma 9.2.** By definition, $b(u, v^h) = \sum\limits_{|p|\leq m, |q|\leq m, |p|+|q|\leq 2m-1} \int b_{pq}(x) D^q u \overline{D^p v^n} dx$. If $|p| \leq m-1$, then as $v \in H^m, \overline{D^p v^h}$ is bounded in $L^2$. If $|p| = m$ we have $|q| \leq m-1$, and $\int b_{pq}(x) D^q u \overline{D^p v^h} dx = -\int (b_{pq} D^q u)^{-h} \overline{D^p v} dx$, and since $b_{pq} \in \mathscr{D}(\Omega)$ and $u \in H^m(\Omega)$, we have $(b_{pq} D^q u)^{-h}$ bounded in $L^2$; so that at any rate $|b(u, v^h)| \leq c||v||_m$.

**Proof of Lemma 9.3.** This follows easily, for as $h \to 0, v^h \to D_\tau v$ in $L^2$ and hence $(f, \phi v^h)_o \leq c||D_\tau v||_{L^2} \leq c||v||_m$.

# Lecture 16

## 9.3

We continue with the proof of theorem 9.1. Having proved that $D_\tau u \in H^m(\Omega^\epsilon)$ **78**
we proceed to prove now the

**Proposition 9.2.** *$D^p_\tau u \in H^m(\Omega^\epsilon)$ for $|p| \leq m$.*

We prove this by induction. Assume it to have been proved for $p = 1, \ldots,$ $r-1$; we prove it for $p = r$. Let $\phi$ be as before in $\mathscr{D}(\overline{\Omega})$ vanishing near $\partial\Omega - \sum$. To prove that $D^\mu_\tau u \in H^m(\Omega^\epsilon)$, it is enough to prove that $\phi D^\epsilon_\tau u \in H^m(\Omega)$, where $\phi$ is as in proposition 9.1. This will follow from the weak compactness argument used previously if we prove that $(\phi D^\mu_\tau u)^{-h}$ is bounded in $H^m(\Omega)$ and this itself will follow on account of ellipticity if we have proved that

$$|a((\phi D^\mu_\tau u)^{-h}, v)| \leq c||v||_m. \tag{1}$$

To prove (1), we write as before

$$a((\phi Du)^{-h}, v) = (a(\phi D^\mu_\tau u)^{-h}, v) + a((\phi D^\mu_\tau u), v^h) - a(D^\mu u, v^h) - b(D^\mu_\tau u v^h)$$
$$\text{where} \qquad b(u, v) = a(\phi u, v) - a(u\phi v).$$

We prove now in the following three lemmas saying that each of the terms above is bounded in $H^m(\Omega)$.

**Lemma 9.4.** $\left|a((\phi D^\mu_\tau u)^{-h}, v) + a(\phi D^\mu_\tau u^{-h}, v)\right| \leq c_1||v||_m$.

**Lemma 9.5.** $\left|a(\phi D^\mu_\tau u \phi v^h)\right| \leq c_1||v||_m$.

**Lemma 9.6.** $\left|b(\phi D^\mu_\tau u v^h)\right| \leq c_3||v||_m$.

We begin with 9.4. The expression to be estimated consists of sums of terms like

$$\begin{aligned} X &= \int (\alpha(x) D^p (\phi D_\tau^r u)^{-h} \overline{D^q v} + (x) d^p (\phi D_\tau^\mu u) \overline{D^q v^h} dx \\ &= \int ((\alpha[D^p(\phi D_\tau^r u)^{-h} - (\alpha D^p(\phi D_\tau^r u))^{-h}] \overline{D^q v} dx && \text{by using (1)} \\ &= \int \alpha^{-h} D^p \phi D_\tau^r u(x-h) \overline{D^q v} && \text{by using (2)} \end{aligned}$$

Since $|r| = k-1$, by induction hypothesis, $D_\tau^r u(x-h)$ is bounded in $H^{2m}(\Omega^\epsilon)$ **79**
and $\alpha^{-h} D^p (\phi D_\tau^r u(x-h)\Big)$ in $L^2$ as $|p| \leq m$, which proves 9.4.

Now we prove 9.5. We have $a(D_\tau^r u \phi v^h) = a(D_\tau^r u \phi v^h) - (-1)^{k-1} a(u$ $D_\tau^r (\phi u)^h) + (-1)^{k-1} a(u, D_\tau^r (\phi v)^h)$.

From 9.2, we have

$$a(u, D_\tau^r (\phi v)^h) = (f, D_\tau^r (\phi v)^h)_o, \text{ for } |r| = k-1 \leq m-1,$$

and hence $|a(u, D_\tau^r (\phi v)^h)| \leq c ||v||_m$.

It remains to consider the first difference, which consists of finite sum of terms

$$Z = \int_{|q| \leq m, |p| \leq m, r = k-1} \alpha D^q D_\tau^r u \overline{D^p \phi v^h} dx - (-1)^{k-1} \int_{|q| \leq m, |p| \leq m, |r| = k-1} \alpha D^q D_\tau^r u \overline{D^p \phi v^h dx}$$

By induction hypothesis, if $|q| \leq m-1$, and $|p| \leq m-1, D^p \phi v^h$ and $D^p (D_\tau^r \phi v^{-h})$ are bounded in $L^2$. So we consider the terms where $|p| = |q| = m$. Now

$$\int \alpha D^q u D^p (\overline{D_\tau^r \phi v^h}) dx = (-1)^{k-1} \int D_\tau^r (\alpha D^q u) \overline{D^q \phi v^h} dx$$

and terms in $Z$ with $|p| = |q| = m$ become

$$\sum_{j \geq 1} \int \beta D_\tau^{r-j} D^q u \overline{D^q \phi v^h} dx$$

which proves that $|Z| \leq c ||v||_m$.

Finally we prove 9.6 $b(D^r u, v^h)$ is a sum of terms like $\int \alpha D^q D_\tau^r u \overline{D^p \phi v^h} dx$. If $|p| \leq m-1$, since $v^h$ are bounded in $H^m$, $D^p v^h$ are bounded in $L^2$. If $|p| = m-1$, **80**

$$\int \beta D^q D_\tau^r u \overline{D^q \phi v^h} dx = -\int (\beta D^q D_\tau^r u)^{-h} \overline{D^p v} dx$$

and $v^h$ are bounded in $L^2$. Hence in any case

$$|b(D^r_\tau u, v^h)| \le c||v||_m.$$

Upto now we followed the proof given by Nirenberg [2]. In the following, the proof will be slightly more complicated than his, but will prove slightly more. Another proof is briefly indicated in Browder [5].

## 9.4

We still require a few preparatory lemmas before taking up the proof proper of the theorem.

**Lemma 9.7.** *Let* $\Omega = ]0,1[^n$ *be n-cube and* $u \in L^2(\Omega)$ *be such that* $D^m_i u \in L^2$ *where* $D^m_i = \dfrac{\partial^m}{\partial x^m_i}$*(exactly m-th derivatives in each variable). Then* $u \in H^m(\Omega)$.

**Remark.** This lemma is related to the theory of *coercive forms* of Aronszajn [1].

This lemma will be proved in two steps.

(a) We prove first $D^k_i \in L^2(\Omega)$for $|k| \le m-1$.

Let $K^m(\Omega)$ be the space of $u \in L^2$ such that $D^k_i u \in L^2(\Omega)$. This is a space of type $H(\Omega, A)$ and hence is a Hilbert space with its usual norm. By using Fourier transforms, we see that on $\mathscr{D}(\Omega)$ the $K^m$ metric and $H^m$ metric are equivalent. Hence the closure of $D(\Omega)$ in $K^m(\Omega)$ is $H^m_o(\Omega)$. From prop. 1.3, we have

$$K^m(\Omega) = H^m_o(\Omega) \oplus \mathscr{H},$$

where $f \in \mathscr{H}$ if and only if $\sum(-1)^m D^{2m}_i f + f = 0$. Now as the operator $\sum(-1)^m D^{2m}_i$ is uniformly elliptic, we have $f \in \mathscr{E}(\Omega)$.

To prove (a) we have to prove, say $D^k_1 f(x_1,\ldots,x_n) \in L^2(\Omega)$, $f \in \mathscr{E}(\Omega) \cap K^m$. From a classical inequality, we have **81**

$$\int\limits_\epsilon^{1-\epsilon} D^k_1 f(x_1,\ldots,x_n)|dx_1 \le c\int\limits_\epsilon^{1-\epsilon}(|f|^2 + |D^m_1 f|^2)dx_1,$$

where $c$ is independent of $\in$. Integrating over the remaining variables, we have

$$\int\limits_{\Omega\epsilon} D^k_1 f(x_1,\ldots,x_n)|^2 dx \le c\int\limits_{\Omega\epsilon}(|f|^2 + |D^m_1 f|^2)dx$$

for every $\in > 0$. Hence

$$\int\limits_{\Omega^{\epsilon}} D_1^k f(x_1, \ldots, x)|^2 dx \leq c||u||_K^2 m \text{ for } f \in \mathscr{H}.$$

We have proved then if $u \in K^m(\Omega)$, then $D_i^k u \in L^2$ for all $|k| \leq m-1$.

**Corollary.** *If $\mathscr{O} \in \mathscr{D}(\Omega)$ and $u \in \Re^m$, then $\mathscr{O}u \in \Re^m$.*

This follows at once from Leibnitz's formula, and (a). Now the second step is to prove

(b) Let

$$\Omega' = \begin{cases} -1 < x_1 < 1 \\ 0 < x_i < 1 \end{cases}, i = 2, \ldots, n.$$

Then for every $u \in K^m(\Omega)$, there exists $U \in K^m(\Omega')$ such that $U = u$a. e. on $\Omega$.

Assuming this for a moment, we finish the proof of the lemma, Applying $(b)$ to each of variables $x_i$, we get an open cube $Q$ such that $\Omega \cap Q$ and such that for every $u \in K^m(\Omega)$, there exists $U \in K^m(\Omega)$ with $U = u$ a.e. on $\Omega$. Let $\mathscr{O}$ be a function in $\mathscr{D}(Q)$ which is 1 on $\Omega$. Then by the corollary to $(a), \theta U \in K^m(Q)$ and having compact support is in $H_o^m(\Omega)$. Hence its restriction to $\Omega$ which is $u$ is in $H^m(\Omega)$.

Now to prove $(b)$, we require

(c) $\mathscr{D}(\bar{\Omega})$ is dense in $K^m(\Omega)$.

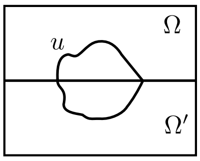
We may obviously assume $\Omega = ]-1, 1[^n$. Let $u \in K^m(\Omega)$ and define $v_t(x) = v(tx)$, for $t < 1$ for all $x \in \Omega$ such that $x \in \dfrac{1}{t}\Omega$.

Let $u_t$ be the restriction of $v_t$ to $\Omega$. Then it is easily seen that as $t \to$ **82**
$1, u_t(x) \to u(s)$ in $K^m(\Omega)$. Hence to prove $(c)$ it is enough to prove that each $u_t(x)$ can be approached by functions of $\mathscr{D}(\bar{\Omega})$. Since $\bar{\Omega} \subset \Omega'$, let $\theta$ be a function in $\mathscr{D}(\Omega)$ which is 1 on $\bar{\Omega}$. Then $\omega = \theta v_\tau(x) K^m(\Omega')$ and has compact support. Hence $\omega \in H_0^m(\Omega')$ and so $\omega = \lim \varphi_k$ in $H_o^m(\Omega')$ where $\varphi_k \in \mathscr{D}(\Omega')$. Hence restrictions of $\varphi_k$ to $\Omega$ converge to $u_\tau =$ restriction of $\omega$ to $\bar{\Omega}$ in $K^m(\Omega)$.

Now we prove $(b)$. It is enough to define a map from $\mathscr{D}(\bar{\Omega})$ to $K^m(\Omega')$ which is continuous in $\mathscr{D}(\bar{\Omega})$ with the topology of $K^m(\Omega)$. Let $u(x) \in \mathscr{D}(\bar{\Omega})$ and let $\Omega'$ be as in $(b)$. Define

$$U(x) = \begin{cases} u(x) \text{ in } \Omega. \\ \lambda_1 u(x', -x_n) + \cdots + \lambda_n u(x' - \frac{x_n}{n}) \end{cases}$$

where $x' = (x_1, \ldots, x_{n-1})$ and $(x', x_n) \in \Omega' - \Omega$. We find $\lambda_i$ suitably so that all the Derivatives of $U$ on $\Sigma$ are well defined. (See § 2.5). This mapping $u \to U$ of $\mathscr{E}(\Omega)$ with the topology of $K^m(\Omega)$ to $K^m(\Omega')$ is seen at once to be continuous. This finishes the poof of lemma 9.7.



# Lecture 17

## 9.5 Completion of the proof of theorem 9.1.

We are now in a position to complete the proof of theorem 9.1. Our problem **83**
is to prove that if $u$ is such that $Au = f \in H^m(\Omega^\epsilon)$, then $u \in H^{2m}(\Omega^\epsilon)$ for every $\in > 0$, where $A = \Sigma(-1)^{|p|} D^p(a_{pq} D^q)$ with $a_{pq} \in \mathscr{D}(\overline{\Omega})$. We have already proved that $D_\tau^p u \in L^2(\Omega)$ for $|p| \leq m$. We now have to consider $D^p u$. We denote the derivatives with respect to $x_n$ by $D_y$. In this part of proof only ellipticity of $A$ is required, and *boundary conditions will not be necessary*. We write also $\Omega$ for $\Omega^\epsilon$.

Now

$$Au = \sum (-1)^m D_y^m g + \sum_{r \leq m-1, |p| \leq 2m-r} (x) D_y^m D^p u \tag{1}$$

where

$$g = \sum_{|p| \leq m} (x) D^p u = \beta(x) D_y^m u + \cdots \tag{2}$$

We prove now

**Lemma 9.8.** $\operatorname{Re} \beta(x) \geq \alpha > 0$.

**Lemma 9.9.** $g \in H^m(\Omega)$.

**Lemma 9.10.** $D_y^{m+1} u \in L^2(\Omega)$.

Using these lemmas and lemmas 9.7, since already it is proved that $D_\tau^p u \in H^m(\Omega)$, for $|p| \leq m$, we have the

**Corollary 1.** $u \in H^{m+1}(\Omega)$.

**Proof of the lemma 9.8** It is easily checked that $\beta(x) = a_{\rho\rho}(x)$, $\rho = (0, \ldots, 0, m)$. Since $\operatorname{Re} \Sigma a_{pq}(x) \xi^p \xi^q \geq \alpha \xi^{2m}$, taking $\xi = (0, \ldots, 0, 1)$, we have $\operatorname{Re} \beta(x) = a_{pq}(x) \geq \alpha > 0$.

**Proof of the lemma 9.9** On account of lemma 9.7, it is enough to prove (a) $D^m_y g \in L^2(\Omega)$, (b) $D^\lambda_\tau g \in L^2(\Omega)$, $|\lambda| = m$.

a) follows from (1) for $Au \in L^2$ and $D^r_y D^p_\tau u = D^r_y D^q_\tau (D^{q'}_\tau u) \in L^2(\Omega)$, 84
$|r| \leq m-1, |p| = 2m - r|q| + r \leq m, |q'| \leq m$ for by proposition 9.2, $D^{q'}_\tau u \in H^m(\Omega)$ and $|q| + r \leq m$.

b) follows from (2) since we have $D^\lambda_\tau g = \sum\limits_{p \quad m, \quad m} \alpha D^p D^\lambda_\tau u$ and $D^\lambda_\tau u \in H^m(\Omega)$ by proposition 9.2.

**Proof of lemma9.10** From (2), we have

$$D_y g = \beta D^{m+1}_y u + (D_y\beta) D^m_y u + \sum_{|p| \leq m+1, |p_n| \leq m} \alpha(x) D^p u.$$

From lemma 9.9, $D_y g \in L^2$; $D^m_y u \in L^2$ as $u \in H^m(\Omega)$ and the last sum is in $L^2$ as seen in lemma 9.9. Now, by lemma 9.8, we have $D^{m+1}_y u \in L^2$.

Thus, having proved that $u \in H^{m+1}(\Omega)$. There are two ways in which we could possibly carry the induction. However, the easier one of proving that $u \in H^{m+k}(\Omega) \Rightarrow u \in H^{m+k+1}(\Omega)$ does not work for if we take $D^{k+1}_y g$ we get terms like $\sum\limits_{\rho \leq k+1, |p| \leq m, : p_n \leq m-1} \alpha D_y D^p_u$ about which we cannot say anything at once unless $k = 0$.

We proceed in a slightly different way. We prove first

a) $D^\lambda_\tau D^{m+1}_y u \in L^2(\Omega)$ with $|\lambda| = k$, and

b) assuming $D^\lambda_\tau D^{m+1}_y \rho_u \in L^2(\Omega)$ for $|\lambda| \leq k - \rho + 1$, we prove that $D^\mu_\tau K^{m+\rho+1}_y u \in L^2$ for $|\mu| \leq k - \rho$.

(a) From (2) we have

$$D^\lambda_\tau D_y g = \beta D^\lambda_\tau D^{m+1}_y u + \sum_{|p| \leq m+k+1, p_n \leq m} \alpha D^p u.$$

By lemma 9.9, $D^\lambda_\tau D_y g \in L^2$. Since $k+1 \leq m$, and $p_n \leq m$, $D^p u = D^\rho_\tau D^q u$ 85
with $|q| \leq m$. Hence the last sum is in $L^2$, and $D^\lambda_\tau D^{m+1}_y u \in L^2(\Omega)$ as $\mathrm{Re}\,\beta(x) \geq \alpha > 0$.

b) Again from (2),

$$D^\mu_\tau D^{\rho+1}_y g = \beta D^\mu_\tau D^{m+\rho+1}_y u + \sum_{|q| \leq m + |\mu| + \int +1, |q_n| \leq m+p} \alpha D^q u.$$

We have $|q| \leq m + 1k + 1, |q_n| \leq m + \rho$; hence by induction hypothesis, the sum is in $L^2(\Omega)$. Since $|\mu| + \rho + 1 \leq k + 1 \leq m, D^m_\tau D^{\rho+1}_y g \in L^2$. Hence $D^\mu_y D^{m+\rho+1}_y u \in L^2$. This proves $D^{2m}_y u \in L^2$.

Since we have already proved $D^p_\tau u \in L^2, |p| \leq 2m$, by lemma 9.7, we have $\underline{u \in H^{2m}(\Omega)}$.

## 9.6 Other results.

Theorem 9.1 is but a first step in considering the regularity at the boundary. We prove now the

**Theorem 9.2.** *Hypothesis being same as in theorem 9.1, if $f \in H^k(\Omega)$ and $Au = f$, then $u \in H^{2m+k}(\Omega)$.*

Theorem 9.1 corresponds to the case $k = 0$. The proof of this theorem is similar in its development to the proof of theorem 9.1. First by making use of local mappings we prove that it is enough to make the proof in the case of a cube $]0, 1[^n$.

Next to prove $u \in H^{2m+k}$ we have to prove $D^p_\tau u$ and $D^{p_n}_y u$ are in $H^m(\Omega)$ for $|p| \leq m + k$ and $p_n \leq m + k$ respectively. The third step not involving the boundary conditions is essentially the same as in the previous considerations. We consider briefly the second step by proving the

**Lemma 9.11.** *$D^p_\tau u \in H^m(\Omega^\epsilon)$ for $|p| \leq m + k$.* 86

We have proved this lemma for $k = 0$; we assume it to be true for $1, \ldots, k-1$, and prove it for $k$. As before we consider the difference quotients $(D^r_\tau u)^{-n}$ with $|r| = m + k + 1$, and prove that they are bounded in $H^m(\underline{\Omega}^\epsilon)$. It is actually enough to show that $(\phi D^r u)^{-h}$ is bounded where $\phi \in \mathscr{D}(\Omega)$ vanishing near $\partial\Omega - \sum$. As before, we consider the identity

$$a((\phi D^r_\tau u)^{-h} v) = a((\phi D^r_\tau u)^{-h}, v) + a(\phi D^r_\tau u, v^h) - b(D^r_\tau u, v) - a(D^r_\tau u, \phi v^h)$$

where $b(u, v) = a(\phi u, v) - a(u, \phi v)$.

By induction hypothesis we may assume that $D^p_\tau u \in H^m(\Omega^\epsilon)$ for $|p| \leq m + k - 1$. Using this and almost the same manipulation as in proposition 9.1, we prove that $a((\phi D^r_\tau u)^{-h}, v) + a(\phi D^r_\tau u, v^h)$ and $b(D^r_\tau u, v)$ are bounded in $H^m(\Omega^\epsilon)$ by $c||v||_m$. To prove $a(D^r_\tau u, \phi v^h)$ is bounded we write

$$a(D^r_\tau u, \phi v^h) = \Big[a(D^r_\tau u, \phi v^h) + (-1)^{|r|-1} a(u, D^r_\tau (\phi v)^h)\Big] + (-1)^{|r|-1} a(u, D^r (\phi v)^h).$$

The first sum is proved to be bounded again by the same methods as those used in proposition 9.1. However, to prove $e(u, D^r_\tau \phi v^h)$ is bounded, we cannot use at once $a(u, D^r_\tau \phi v^h) = (f, D^r_\tau \phi v^h)$, for $D^q_\tau v$ is not necessarily in $H^m(\Omega)$ for $|q| \leq r$. However, by regularization, it is seen that such $\underline{v}$ that $D^q_\tau v \in H^m(\Omega)$ for $|q| \leq r$ are dense in $H^m(\Omega)$. It is enough then to prove $\overline{a}(D^r_\tau u, \phi v^h)$ is bounded for such $v$'s and then we have $a(u, D^r_\tau(\phi v^h))_o, |r| = m + k - 1$. Now, since $f \in H^k$ we can integrate the last expression $k$-times by parts and obtain **87**

$$\begin{aligned} a(u, D^r_\tau(\phi v)^h) &= |(-1)^k (D^k_\tau f, D^{m-1}_\tau (\phi v)^h) \\ &\leq c|v|_o \leq c||v||_m. \end{aligned}$$

Having proved then that $a((\phi D^r_\tau u)^{-h}, v)$ is bounded by $c||v||_m$ in $H^m(\Omega)$ by putting $v = \phi(D^r_\tau u)^{-h}$, we obtain, as usual, by ellipticity, that $||\phi(D^r_\tau u)^{-h}||_m \leq c$ and by now standard arguments that $D^{m+1}_\tau u \in H^m(\Omega^\epsilon)$.

From theorem 4.4 we have $H^\rho(\Omega) \subset \mathscr{E}^\circ(\Omega)$ if $a\underline{\rho} > n$. Further if $\Omega$ has $\rho$-extension property from theorem 4.4 $H^\rho(\Omega) \subset \mathscr{E}^\circ(\underline{\Omega})$. Hence, by using theorem 9.2, we have the

**Theorem 9.3.** *Under the hypothesis of theorem 9.1, if $2k > n$, then $u$ is in $\mathscr{E}^{2m}(\overline{\Omega})$.*

In this case $u$ is a *usual* solution of Neumann problem.

**Remarks.** Analogous proof applies for Dirichlet's problem. Now the question arises for what spaces $V$ such that $H^n_o \subset V \subset H^m$ can we apply the above methods for proving regularity at the boundary. One of the crucial steps in above proof was the manipulation of difference quotients $v^h$ and hence the subspace of $V$ consisting of functions which vanish near the boundary $\partial\Omega - \Sigma$ must be invariant for translations. For spaces $V$ given by conditions like $\left\{u, \dfrac{\partial u}{\partial \eta}, \ldots, \dfrac{\partial^k u}{\partial_\eta k} = 0, k \leq n-1\right\}$, this condition is satisfied. However, for spaces $V \subset H^m, m \geq 2$, determined by conditions like $\alpha(x)u + \beta(x)\dfrac{\partial u}{\partial x_n} = 0$ on $\Sigma$, this condition is obviously not satisfied. Nevertheless by changing a little the method of proof the regularity theorems have been proved by Aronsza jn-Smith for the spaces $V$ given by conditions like $\alpha(x)u + \beta(x)\dfrac{\partial u}{\partial x_n} = 0$. We shall consider these methods in later lectures. **88**

## 9.7 An application of theorem 9.1.

Let $A$ and $N$ be as in theorem 9.1. On $N$ we consider the metric $|u|_N = ||u||_m + |Au|_o$ and on $H^{2m} \cap N$, the metric $||u||_{2m} + ||u||_m + |Au|_o$ which defines on $H^{2m} \cap N$

the upper bound topology. The inclusion mapping $H^{2m} \cap N \to N$ is continuous and since from theorem 9.1 it is onto, it is an isomorphism. Hence

$$||u||_{2m} + ||u||_m + |Au|_o \geq c(||u||_o + |Au|_o).$$

i.e. ,
$$|Au|_o + ||u||_m \geq \gamma' ||u||_{2m}.$$

This is equivalent to

$$|Au|_o + |u|_o \geq \gamma' ||u||_{2m},$$

in the case of strongly $m$-regular open sets (which is the case in theorem 9.1).

This is proved directly by Ladyzenskya for the case $m = 2$, and by Guseva for the general case. [ ]. To obtain the regularity at the boundary from these inequalities, one has to prove moreover a non-trivial density theorem.

# Lecture 18

## 9.8 Regularity at the boundary in the case of problem of *oblique type*

Let $\Omega = \{x_n > 0\}$. In § 2.4, we have defined map $\gamma$ of $H^1(\Omega)$ onto $H^{\frac{1}{2}}(\Gamma)$. Let **89** $\Lambda u = \Sigma\alpha_i \dfrac{\partial u}{\partial x_i}$ where $\alpha_i$ are *real* constants, and let $a(u,v) = (u,v)_1 + \lambda(u,v)_o + <\Lambda\gamma u, \gamma\overline{v}>$ with $\lambda > 0$ be a sesquilinear form on $H^1(\Omega)$. In § 6.7, we have proved that the form $a(u,v)$ is $H^1(\Omega)$ elliptic and that the operator $A$ associated with it is $-\triangle + \lambda$. We gave there a formal interpretation of the space $N$. Now we prove some regularity theorems justifying the formal interpretation in regular cases.

**Theorem 9.4.** *If $f \in L^2$ and $u \in N$ is such that $Au = f$, then $u \in H^1(\Omega)$.*

As it is by now usual we consider the difference quotients $u^h(x) = \dfrac{u(x+h) - u(x)}{h}$ and prove that they are bounded in $H^1(\Omega)$. This will imply that $\dfrac{\partial_u}{\partial x_i} \in H^1$ for $i = 1, \ldots, n-1$. Next we consider $\dfrac{\partial u}{\partial x_n}$. We know $a(u,v) = (f,v)_o$ for all $v \in H^1(\Omega)$. Hence $a(u,v^h) = (f,v^h)_o$ for all $v \in H^1(\Omega)$. Since $a(u,v)$ has constant coefficients we have $a(u,v^h) = -a(u^{-h},v)$. Hence $a(u^{-h},v) = (f,v^h)_o$ and so $\left|a(u^{-h},v)\right| \leq c|v^h|_o \leq c||v||_1$. Taking $v = u^{-h}$ we have $\alpha||u^{-h}||^2 \leq \left|a(u^{-h},u^{-h})\right| \leq c||u^{-h}||_1$ . Hence $||u^{-h}||_1 \leq c$. Next $-\Delta u + \lambda u = f$ and $\Delta u = \dfrac{\partial^2 u}{\partial x_n^2} +$ tangential derivatives. Since $\Delta u \in L^2$, $u \in L^2$, $f \in L^2$ and as has been proved the tangential derivative are in $L^2$ we have $\dfrac{\partial^2 u}{\partial x_n^2} \in L^2$, this complete the proof that $u \in H^1(\Omega)$.

The same proof can be adopted to prove the **90**

**Corollary.** *If $f \in H^k$, then $u \in H^{k+2}$.*

If $k$ is large enough, say $2k > n$, then we have proved in that $H^k(\Omega) \subset \mathscr{E}^o(\Omega)$. Hence for $2k > n+2$, $u \in \mathscr{E}^2(\Omega)$. Hence the formal interpretation given § 6.7 for $u \in N$ is a genuine one and we have if $2k > n+2$ and if $f \in H^k$ and $u \in N$ is such that $Au = f$, then $u$ satisfies $\Lambda\gamma u = \gamma \dfrac{\partial u}{\partial x_n}$.

## 9.9 Regularity at the boundary for some more problems.

In the § 6.86.8 we have considered the case where $V$ consists of $u \in H^1(\Omega)$ such that $\gamma u \in H^1(\Gamma)$, the topology on $V$ being given by the norm $||u||_1 + ||\gamma u||_1$. If $a(u,v) = (u,v)_1 + \lambda(u,v)_o + (\gamma u, \gamma v)$ with $\lambda > 0$, then we have proved that $a(u,v)$ is $V$-elliptic, that the operator defined by $a(u,v)$ is $-\Delta + \lambda$ and that the boundary value problem solved formally was $\dfrac{\partial u}{\partial x_n}(x', 0) = \Delta_\Gamma u$. We prove now the

**Theorem 9.5.** *If $f \in L^2$ and $u \in N$ is such that $Au = f$, then $u \in H^1(\Omega)$ and $\gamma u \in H^1(\Gamma)$.*

*Proof.* First of all we observe that $V$ is closed for translations, i.e., $u \in V \Rightarrow v^h \in V$ for sufficiently small $h$. Now we know $a(u,v) = (f,v)_o$ for all $v \in V$ and hence $a(u,v^h) = (f,v^h)_o$. Since $a(u,v)$ is with constant coefficients $-a(u^{-h},v) = +a(u,v^h) = (f,v^h)_o$. Hence $\left|a(u^{-h},v)\right| \le c||v||_V$. Putting $v = u^{-h}$ we obtain $||u^{-h}||_1^2 + ||\gamma u^{-h}||_1^2 \le c||u^{-h}||_1$. Hence $u^h$ and $\gamma u^h$ are bounded so that as usual, $D_\tau u \in H^1(\Omega)$ and $D_\tau \gamma u \in H^1(\Gamma)$. Further since $-\Delta u \in L^2$ and $D_\tau u \in H^1(\Omega)$, we have $\dfrac{\partial^2 u}{\partial x_n} \in L^2$. Hence $u \in H^2(\Omega)$. □

**Corollary.** *If $f \in H^k$, then $u \in H^{k+}$.* 91

For sufficiently large $k$, *e.g.*, $2k > n$, we have $H^k \subset \mathscr{E}^o$. Hence for $k > \dfrac{n}{2} + 1$, the formal boundary condition becomes a genuine one, and we have

$$\frac{\partial u}{\partial x_n}(x', 0) - \Delta\Gamma u = 0.$$

# 10 Visik-Soboleff Problems

## 10.1

In a sense these problems generalize non-homogeneous boundary value problems, e.g., such ones in which solutions of $Au = f$ are sought which would attain in some sense boundary values given a priori. However, since not until late this aspect of the problem will be evident from the way we shall formulate the problem, and since the hypothesis we shall have to assume in order to ensure the existence and the uniqueness of solutions will not be obvious, in this lecture we prefer to discuss the development of the problem and deduce theorems as consequences thereof.

Let $\Omega$ be an open set in $R^n$ and $V$ be such that $H^m_o(\Omega) \subset V \subset H^m(\ )$. Let $Q = L^2(\Omega)$ and $a(u,v) = \sum_{|p|,|q|\leq m} \int \Omega a_{pq} D^q_u \overline{D^p v} dx+$ some surface integrals for $u, v \in V$. (However in the sequel we shall drop surface integrals as their inclusion only complicates the technical details. ). As in theorem 3.1, we define the spaces $N$ and the operator $\Lambda = \sum_{|p|,|q|\leq m} (-1)^p D^p(a_{pq}D^q)$.

We shall assume $a(u,v)$ to be V-elliptic, i. e. , $|a(u,u)| \geq \alpha ||u||^2_m$ . for some $\alpha > 0$ and all $u \subset V$. In this case it is known that $A$ is an isomorphism of $N$ onto $L^2$. Let $a^*(u,v) = a(\overline{v,u})$. Then $\left|a^*(u,u)\right| \geq \alpha ||u||^2_m$ for all $u \in V$ and the **92**
operator $A^* = \sum (-1)^{|p|}(D^p \overline{a_{qp}(x)} D^q)$ it defines is an isomorphism of $N^*$ onto $Q = L^2$.

Suppose now there exists $\mathscr{A}_{qp} \in \mathscr{D}_{L^\infty}(R^n)$ such that $\mathscr{A}_{pq} = a_{pq}$ on $\Omega$ and let $\mathscr{A} = \sum (-1)^{|p|} D^p(\mathscr{A}_{pq}(x) D^q)$. We remark that though $A$ is elliptic, $\mathscr{A}$ need not be elliptic. Let for $f \in L^2(\Omega), u \in N$ be such that $Au = f$. Let $\tilde{f}$ and $\tilde{u}$ be the extensions of $f$ and $u$ respectively obtained by defining them to be zero outside $\Omega$. Of course, we do not have $\mathscr{A}\tilde{u} = \tilde{f}$. The difference $\mathscr{A} u - f$ is given by the

**Proposition 10.1.** *If $u \in N$ be such that $Au = f$, then for every $v \in H^{2m}(R^n)$ such that $v_\Omega \in N^*$ we have $< \mathscr{A} u - f, \overline{v} >= 0$, where $v_\Omega$ is the restriction of $v$ to $\Omega$.*

*Proof.* $< \mathscr{A} u - f, \overline{v} >=< \tilde{u}, \overline{\mathscr{A}^* v} > - < \tilde{f}, \overline{v} >$ for $v \in H^{2m}(R^n)$. ◻

Now since $u$ vanishes outside $\Omega$, we have

$$< \tilde{u}, \mathscr{A}^* v >= (u, A^* v_\Omega)_o = (\overline{A^* v_\Omega, u_o}).$$

Since $v_\Omega \in N^*$ we have $\overline{(A^* v_\Omega, u)_o = \phi a^*(v_\Omega, u)} = a(u, v_\Omega) = (Au, v_\Omega)_o$. Further $< \tilde{f}, \overline{v} >= (f, v_\Omega)$ as $f$ vanishes outside $\Omega$. Hence

$$< \mathscr{A}\tilde{u} - \tilde{f}, \overline{v} >=< Au - f, \overline{v} >= 0, \text{ for } v \in H^{2m}(R^n)$$

such that $v_\Omega \in N^*$.

Now arises the converse problem. Let $w \in L^2(R^n)$ be such that the support of $w$ is contained in $\overline{\Omega}$ and let there exist $f \in L^2(\Omega)$ such that $< \mathscr{A} w - f, \overline{v} >= 0$ for all $v \in H^m(R^n)$ such that $v_\Omega \in N^*$. Does there exist $u \in N$ such that $w = \tilde{u}$ and $Au = f$. Let $u_o \in N$ be the solution of $Au_o = f$. By proposition 10.1, $<$

$\mathscr{A}\tilde{u}_o - \tilde{f}, \bar{v} >= 0$ for $v \in H^m(R^n)$ such that $v_\Omega \in N^*$. Hence $< \mathscr{A} - w - \tilde{u}_o, \bar{u} >= 0$, 93
i.e., $< -(w - \tilde{u}_o), \mathscr{A}^* v >= 0$. Since and $\tilde{u}_o$ have their support in $\Omega$, the above means $(w - \tilde{u}_o, \mathscr{A} * v_\Omega) = 0$.

In order to have $w - \tilde{u}_o = 0$, we have to secure that $\Lambda^* v_\Omega$ be dense in $L^2(\Omega)$. $A^*$ being an isomorphism of $N^*$ onto $L^2(\Omega)$ we must consider when the solution $x \in N^*$ of $Ax = g$ is restriction of a $v \in H^{2m}(R^n)$. This would follow $(a)$ if we should apply the theory of § 9. Then it would follow that $x \in H^{2m}(\Omega)$, and $(b)$ if $\Omega$ had $2m$-extension property, then there would exist $x \in H^{2m}(R^n)$, such that $(\pi x)_\Omega = x$.

In other words, for every $g \in L^2(\Omega)$ there exists $v_\Omega$ such that $A^* v_\Omega = g$ if the above two conditions are satisfied. We have proved then the

**Proposition 10.2.** *Besides the hypothesis of Proposition 10.1. , assume*

1) *$A^* u = g$ with $u \in N^*$ and $g \in H^o$ implies $u \in H^{2m}(\Omega)$,*

2) *$\Omega$ has $2m$-extension property, and*

3) *there is given $\in H^o(R^n)$ such that the support of $\bar{\Omega}$ contained in $\bar{\Omega}$ and $< \mathscr{A} w - f, \bar{v} >= 0$ for all $v \in H^{2m}(R^n)$ such that $v_\Omega \in N^*$. Then $wu_o, u_o \in N$ being the solution of $Au_o = f$.*

**Remarks.** Since $\mathscr{D}$ is dense in $L^2(\Omega)$ instead of assuming the theory of § 9, it would be enough to assume that $A^* x = g, g \in \mathscr{D}(\Omega)$ implies $x \in H^{2m}(\Omega)$.

(2) It is not known whether (1) and (2) in proposition 10.2 are independent or not, or whether (2) is a consequence of (1) . The condition (3) can be put more succinctly by making the following

**Definition 10.1.** *$M^o$ is the subspace of $H^{-2m}(R^n)$ consisting of distribution $T$ such that $< T, \bar{v} >= 0$ for all $v \in H^{2m}(R^n)$ such that $v_\Omega \in N^*$.* 94

It is easily seen that $M^o$ is a closed subspace of $H^{-2m}(R^n)$ and that the support of $T \in M^o$ is contained in $\lceil$. We may summarize the proposition 10.1 and 10.2 in the following.

**Theorem 10.1.** *Under the hypothesis of Proposition 10.1 and 10.2 the boundary value problem "Given $f \in L^2(\Omega)$, find $u \in N$ such that $Au = f''$ is equivalent to "Given $f \in L^2(\Omega)$, find $w \in H^o(R^n)$ such that $\mathscr{A} w - \tilde{f} \in M^o$".*

## 10.2

Now the second formulation has an advantage over the first one that it can be generalized. In the first instance we notice that instead of $f$ we could take any

$T \in H_{\Omega}^{-2m}$[1] and raise the problem

**Problem 10.1.** Given $T \in H_{\Omega}^{-2m}$ does there exist $w \in L^2(R^n)$ with the support in $\Omega$ such that $\mathscr{A}w - T \in M^o$.

Similarly a much general problem could be formulated by defining new spaces $M^k$.

**Definition 10.2.** *$M^k$ is the subspace of $H^{-(k+2m)}(R^n)$, $k$ being a non-negative integer, such that $< T, \overline{v} >= 0$ for all $v \in H^{k+2m}(\Omega)$ with $v_\Omega \in N^*$.*

**Lemma 10.1.** *$M^k$ is a closed subspace of $H^{-k+2m}$.*

*Proof.* We prove only that the support of $\underline{T} \in M^k$ is contained in the other assertion being then obvious. If $\varphi \in \mathscr{D}(\sigma\overline{\Omega})$, take $v = \tilde{\varphi}$. Then $v_\Omega = 0$, and hence is in $N^*$. Then $< T, \overline{\varphi} >=< T, v_\Omega >= 0$. Hence the support of $T$ is contained in $\overline{\Omega}$. If now $\varphi \in \mathscr{D}(\Omega)$, then again let $v = \tilde{\varphi}$. Now $v \in H^{k+2m}(R^n)$ **95**
and $v_\Omega = \varphi \in N^*$. Hence $< T, \varphi >= 0$. This proves that the support of $T$ is contained in $\Gamma$. ◻

We have now the

**Problem 10.2.** Given $T \in H^{-(k+2m)}$ does there exist $U \in H^{-k}(R^n)$ with support in $\overline{\Omega}$ such that $\mathscr{A}U - T \in M^k$. For $K = 0$ we get the problem 10.1

[1] $H^{-m}(\Omega)$ consists of $u \in H^{-m}(R^n)$ such that the support of $u \subset \overline{\Omega}$

# Lecture 19

The problem of formulated in the last lecture would loose its interest if $\mathscr{A}\cup$ **96**
were not independent of the extension $\mathscr{A}$ of $A$ that we have chosen. We prove that in fact this is the case for some kinds of domains.

Let $\mathscr{A}$ and $\mathscr{A}'$ be two extensions of $A$. Let $\in H^{-k}_{\bar{\Omega}}$. We have then $\langle \mathscr{A}\cup -\mathscr{A}\cup, \bar{v}\rangle = \langle \cup., \overline{(\mathscr{A}^* - \mathscr{A}^*)}v >$ for $v \in H^{k+2m}(R^n)$. Since $\mathscr{A}^* = \mathscr{A}^*$ on $\Omega$, $w = (\mathscr{A} * - \mathscr{A}^*)v$ is such that $w_\Omega = 0$. Now in order that $\mathscr{A}\cup = \mathscr{A}'\cup$ it is sufficient to assume some sort of density of $(\mathscr{A}^* - \mathscr{A}^{*'})v$, for $v \in H^{k+2m}(R^n)$ in $H^k(R^n)$, i. e. , of $w \in H^k(R^n)$ such that $w_\Omega = 0$. This can be done by having the following definition.

**Definition 10.3.** *$\Omega$ is k-sufficiently regular if $w \in H^k(R^n)$ is such that $w_\Omega = 0$. Then there exists $g \in H^k_o(\sigma\bar{\Omega})$ such that $= \tilde{g}$.*

Assuming then $\Omega$ to be sufficiently regular, we have $\mathscr{A}^* - \mathscr{A}^{*'}v = w = \lim \varphi_i$ in $H^k_o([\bar{\Omega})$ with $\varphi_j \in \mathscr{D}([\bar{\Omega})$. But since $\cup = 0$ on $[\bar{\Omega}$, $\langle \cup, \bar{\varphi}_j\rangle = 0$. Hence $\langle (\mathscr{A} - \mathscr{A}^*)\cup, \bar{v}\rangle = 0$ for all $v \in H^{k+2m}(R^n)$, and so $\mathscr{A}u = \mathscr{A}'\cup$.

**Definition 10.4.** *If $\Omega$ is k-sufficiently regular, the problem will be called Visik-Soboleff problems.*

## 10.3

We prove now the uniqueness and existence theorem for the Visik-Soboleff problems.

**Theorem 10.2.**

*(1) Let $\Omega$ be a domain in $R^n$ such that*

*(a) $\Omega$ has $k$ and $k + 2m$ extension property;*

*(b)* $\Omega$ *is* $k$ *and* $k + 2m$ *sufficiently regular.*

(2) *Let* $V$ *be such that* $H_o^m(\Omega) \subset V \subset H^m(\Omega)$ *and* $a(u, v)$ *be a sesquilinear form on* $V$ *satisfying* $a(u, v) \geq \alpha \|u\|_m^2$ *and such that there exists* $\mathscr{A}_{pq} \in \mathscr{D}_{L\infty}(\Omega)$ *such that* $\mathscr{A}_{pq} = O_{pq}$ *on* $\Omega$. 97

(3) *Let the operator* $A^*$ *defined by* $a^*(u, v) = \overline{a(v, u)}$ *be such that* $A^* u \in H^k(\Omega)$ *imply* $u \in H^{k+2m}$. *Then Visik-Soboleff problem admits a unique solution i. e. , given* $T \in H^{-k+2m}(\bar{\Omega}$, *there exists a unique* $u \in H^{-k}(\Omega)$ *such that* $Au - T \in M^k$.

*Proof.* To prove this theorem we shall require some lemmas. Let $(H^k(\Omega))'$ be the dual of $H^k(\Omega)$. We do not identify $(H^k(\Omega))'$ with any space of distributions for $\mathscr{D}(\Omega)$ is not dense in general in $H^k(\Omega)$. We know the restriction map $v \to v_\Omega$ of $H^k(R^n)$ into $H^k(\Omega)$ is continuous. The transpose of this mapping is a mapping of $H^k(\Omega))'$ into $(H^k(R^n)) = (H_o^k(R^n)) = (H_o^k(R^n)) = H^{-k}(R^n)$, given explicitly by $\langle \pi_k, T, \bar{v} \rangle = (T \bar{v}_\Omega)$ for $v \in H^k(R^n)$. Further if $v_\Omega = 0, \langle \pi_k T, \bar{v} \rangle = 0$, i. e. , $\pi_k T = 0$ on $[\bar{\Omega}$ so that the support of $\pi_k$ is contained in $\bar{\Omega}$. Hence $\pi_k$ is a continuous mapping of $(H^k(\Omega))'$ into $H_{\bar{\Omega}}^{-k}$. □

Using (1) - (a), and (b) of the theorem we prove the fundamental

**Lemma.** *The mapping* $T \to \pi_k T$ *of* $(H^k(\Omega))'$ *into* $H_{\bar{\Omega}}^{-k}$ *is an isomorphism.*

*Proof.* We build explicitly the inverse. On account of the $m$-extension property of $\Omega$, there exists a continuous mapping $u \to P(u)$ of $H^k(\Omega)$ into $H^k(R^n)$, with $P\mu = u$ a. e. on $\Omega$. Let $S \in H_{\bar{\Omega}}^{-k}$. Then the semi-linear form $u \to \langle S, \bar{Pu} \rangle$ is continuous on $H^k(\Omega)$ and hence defines an element $\bar{\omega} S \in (H^k(\Omega))'$ so that $\langle S, \overline{P(u)} \rangle = (\bar{\omega}_k S)\, (\bar{u})$

The mapping $S \to \bar{\omega}_k S$ is obviously continuous. The lemma will be proved if we prove 98

$a)$ $\bar{\omega}_k -_{-k} T = T$, and $b)$ $\pi_k \bar{\omega}_k S = S$. □

a) We have $\bar{\omega}_k \pi_k T(\bar{u}) = \langle \pi_k T, \overline{Pu} \rangle = T((\overline{Pu})_\Omega) = T(\bar{u})$.

b) For $v \in H^k(R^n)$ we have

$$\langle \pi_k \bar{\omega}_k S, \bar{v} \rangle = (\bar{\omega}_k S)(\bar{v}_\Omega) = \langle S, \bar{P}(v_\Omega) \rangle.$$

Let $w = P(v)$ we have to prove

$$\langle S, \bar{w} \rangle = \langle S, \bar{v} \rangle.$$

Let $g = w - v$; we have $g \in H^k(R^n)$ and $g_\Omega = W_\Omega = 0$. By 1$b$) we have $g = \tilde{h}$ with $h \in H^q_o(\sum(\overline{\Omega})$. Hence

$$\langle S, \overline{g}\rangle = \langle S, \tilde{h}\rangle = \lim_{0}\langle S, \varphi_j\rangle, \varphi_j \in \mathscr{D}(\in \overline{\Omega})$$

since $\zeta \in H^{-k}_{\overline{\Omega}}$.

Hence $\langle \overline{S}, w\rangle = \langle S, \overline{v}\rangle$ which completes the proof of lemma. To complete the proof of the theorem, given $T \in H^{-k+2m}(\overline{\Omega})$ we have to determine $\cup \in H^{-k})_{\overline{\Omega}}$ such that $\mathscr{A}V - T \in M^k$; $\langle \mathscr{A}u - T, \overline{v}\rangle = 0$ for $v \in H^{k+2m}(R^n)$ such that $v_\Omega \in N^*$; or such $aU$ that

$$\langle U, \overline{\mathscr{A}^* v}\rangle = \langle T, \overline{v}\rangle. \tag{1}$$

By hypothesis 1) ($a$), ($b$), on account of the above lemma, there exist isomorphisms $\overline{\omega}_k, \overline{\omega}_{k+2m}$ of $H^{-k}_{\overline{\Omega}}$ and $H^{-(k+2m)}_{\overline{\Omega}}$ into $(H^k(\Omega))'$ and $(H^{k+2m}(\Omega))'$ respectively. Let $\overline{\omega}_k U = u, \overline{\omega}_{k+2m} T = t$. Then

$$\langle U, \overline{\mathscr{A}^* v}\rangle = \langle \pi_k u, \overline{\mathscr{A}^* v}\rangle = u((\overline{\mathscr{A}^* v})_\Omega) = u(\overline{A^* v_\Omega})$$

and $\langle T, \overline{v}\rangle = \langle \pi_{k+2m} t, \overline{v}\rangle = t(\overline{v}_\Omega)$, so that from (1) our problem will be solved if given $t = \overline{\omega}_{k+2m} T$ we can determine $u \in (H^k(\Omega))'$ such that

$$u(\overline{A^* v_\Omega}) = t(\overline{v}_\Omega), \text{ for all } v \in H^{k+2m}(R^n) \text{ such that } v_\Omega \in N^*. \tag{2}$$

We prove that the problem can be still simplified in as much as we need prove (2) only for $w \in H^{k+m}(\Omega) \in N*$. Indeed on account of $(k+2m)$ extension property of $\Omega$, $w \in H^{k+2m}(\Omega) \cap N^*$ is a $v_\Omega$ when $v = P(w)$. Hence our problem is reduced to: Given $t = \overline{\omega}_{k+2m} T$ determine $u \in (H^k(\Omega))'$ such that **99**

$$u(\overline{A * v}) = t(\overline{v}) \text{ for all } v \in H^{k+2m}(\Omega) \cap N^*.$$

Now we use (3). Let $f \in H^k$. Then $A^* v = f$ has a unique solution $G^* f \in N^*$ which on account of the hypothesis (3) of the theorem is in $H^{k+2m}(\Omega)$. If $f \to 0$ in $H^k(\Omega), G^* f \to 0$ in $H^{k+2m}(\Omega)$. Hence by the closed graph theorem, $G^*$ is a continuous mapping of $H^k$ into $H^{k+2m}(\Omega) \cap N^*$ in the topology of $H^{k+2m}(\Omega)$. Since $t$ $(H^{k+2m}(\Omega))' f \to t(\overline{G^* f})$ is a continuous semi-linear mapping on $H^k(\Omega)$ and hence there exists a unique $u \in (H^k(\Omega))'$ such that

$$u(\overline{A^* v}) = u(\overline{f}) = t(G^* f) = t(\overline{v}).$$

**Remark.** $U$ depends continuously on $T$.

# Lecture 20

## 10.4 Application.

We now consider some applications of the above theory bringing out how the usual non-homogeneous boundary value problems are particular case of Visik-Soboleff problems. 100

Let $V$ be such that $H^1_o(\Omega) \subset V \subset H^1(\Omega)$. and $a(u, v) = (u, v)_1 + \lambda(u, v)_o$ for $\lambda > 0$. The operator $A$ associated with $a(u, v)$ is then by § 3.5 $\mathcal{F} - \triangle + \lambda$. Let $\mathscr{A} = -\triangle + \lambda$. Since $a(u, v)$ is hermitian $A = A^*$ and $n = N^*$. Visik-Soboleff problem reads now for $M^o$ as: Given $T \in H^{-2}_{\overline{\Omega}}$ determine $U \in L^2(\overline{\Omega})$ such that $-\triangle u + \lambda v - T \in^\circ$. From theorem 10.2, it follows that this problem admits a solution, say for example, if $\Omega$ has smooth boundary.

Now we take a particular $T = \tilde{f} + S$ where $f \in L^2(\Omega)$ and $S \in H^{-2}_{\Gamma}$. Since the support of $\mathscr{A}u - T$ is in $\Gamma$ restricting to $\Omega$ we see $Au = f$ where $u = U_\Omega$. Further $\langle \mathscr{A} U - T, \overline{v}\rangle = 0$ for all $v \in H^2(R^n)$ such that $v_\Omega \in N^*$. Hence for such $v$,

$$\begin{aligned}\langle -\triangle U + \lambda U, \overline{v}\rangle &= \langle T, \overline{v}\rangle = \langle \tilde{f}, \overline{v}\rangle + \langle S, \overline{v}\rangle\\ &= (-(\triangle \widetilde{U) + \lambda u}), \overline{v}\rangle + \langle S, \overline{v}\rangle.\end{aligned}$$

Formally, by Green's formula,

$$\langle -\triangle U + \lambda U, \overline{v}\rangle = \int_\Omega -(\triangle + \lambda) U\overline{v} dx = \int_\Omega U(-\triangle + \lambda)\overline{v} dx$$
$$= \int_\Omega u.(-\triangle + \lambda)\overline{v} dx = -\int u \frac{\partial \overline{v}_\Omega}{\partial \eta} d\sigma + \int \frac{\partial u}{\partial \eta}\overline{v}_\Omega d\sigma + \int_\Omega (-\triangle + \lambda) u\overline{v}_\Omega dx$$

and $\langle(-\triangle \widetilde{u + \lambda u}), \overline{v}\rangle = \int_\Omega (-\triangle + \lambda) u\overline{v} dx$.

Hence the original problem is formally equivalent to: given $S \in H^{-2}$, find $u \in L^2(\Omega)$ such that $\int_\Gamma \left(\frac{\partial u}{\partial \eta}\bar{v} - u\frac{\partial \bar{v}}{\partial \bar{\eta}}\right) d\sigma = \langle S, \bar{v}\rangle$ where $v \in H^2(R^n)$ such that $v_\Omega \in N^*$.

We now take some particular cases of $S$. 101

1) Let $g$ and $h \in L^2(\Gamma)$. If $\varphi \in \mathscr{D}(R^n)$ the mapping $\varphi \to \int_\Gamma g\bar{\varphi} d\sigma - \int h\frac{\partial\bar{\varphi}}{\partial \eta} d$ is a continuous linear mapping on $\mathscr{D}(R^n)$ with the topology of $H^2(R^n)$. For if $\varphi \to 0$ in $H^2(R^n)$, $\frac{\partial \varphi}{\partial \eta} \to 0$ in $H^1(R^n)$ and since the mapping $\gamma$ from $H^1(R^n)$ to $L^2(\Gamma)$ is continuous, $\int_\Gamma g\bar{\varphi} d\sigma$ and $\int_\Gamma h(\frac{\partial\bar{\varphi}}{\partial \eta}) d\sigma$ tend to zero in $L^2(\Omega)$. Hence this mapping defines $S \in H^{-2}(\Gamma)$. Then (1) reads

$$\int_\Gamma \left(\frac{\partial u}{\partial n}\bar{v} - u\frac{\partial \bar{v}}{\partial n}\right) d\sigma = \int_\Gamma g\bar{v} d - \int_\Gamma h\frac{\partial \bar{v}}{\partial n} d\sigma \tag{2}$$

Let now

a) $V = H^1(\Omega)$. Then $\frac{\partial v}{\partial n} = 0$ and (1) means $\int_\Gamma \frac{\partial u}{\partial n}\bar{v} d\sigma = \int_\Gamma g\bar{v} d\sigma$ for all $v$. Hence $\frac{\partial u}{\partial n} = g$, on $\Gamma$. Hence the problem solved is

$$\triangle \quad \lambda \quad -u + u = f, \frac{\partial u}{\partial n} = g \text{ on } \Gamma. \tag{3}$$

b) $V = H^1_o(\Omega)$. Then $\gamma v = 0$, and (2) becomes

$$-\triangle u + \lambda u = f, \; u = h \text{ on } \Gamma. \tag{4}$$

2) Another example would be to take $g \in H^{-3/2}(\Omega)$ and $h \in H^{-\frac{1}{2}}(\Gamma)$. Then the mapping $\varphi \to \langle g, \bar{\varphi}\rangle - \left\langle h, \gamma\left(\frac{\partial\bar{\varphi}}{\partial n}\right)\right\rangle$ is continuous on $\mathscr{D}(R^n)$ with the topology of $H^2(R^n)$ for as we shall prove later on $\gamma\varphi \to 0$ in $H^{3/2}(\Omega)$ and $\gamma\left(\frac{\partial \varphi}{\partial_n}\right) \to 0$ in $H^{\frac{1}{2}}(\Omega)$, as $\varphi \to 0$ in $H^2(R^n)$. This defines a $S \in H^{-2}(\Gamma)$. With this $S$ and

a) $V = H^1$, the formal problem solved is $-\triangle u + \lambda u = f, \frac{\partial u}{\partial n} = g \in H^{-\frac{3}{2}}(\varphi)$.

b) $V = H^{\frac{1}{2}}, \ldots \; - \Delta u + \lambda u = f, u = h \in H^{-\frac{1}{2}}(\Omega)$.

These are the problems studied by the Italian School. The principal problem is to give a precise meaning to (3), (4) and so on. (See Magenes [11].

# 11 Aronszajn and Smith Problems[1]

## 11.1 Complements on $H^m(\Omega)$

In § 2.4, we have defined a mapping $\gamma$ of $H^1(\Omega)$ onto $H^{\frac{1}{2}}(\Gamma)$ where $\overline{\Omega} = \{x_n >$ 102
$o\}$ and $\Gamma = \{x_n \ = 0\}$. Now we prove the

**Proposition 11.1.** *Let $\Omega = \{x_n > 0\}$ and $\Gamma = \{x_n = 0\}$. Then the mapping $\gamma$ maps $H^m(\Omega)$ onto $H^{m-\frac{1}{2}}(\Gamma)$ for all m.*

*Proof.* We denote $x' = (x_1, \ldots, x_{n-1}), x_n = y, \xi' = (\xi_1, \ldots, \xi_{n-1})$. □

Here we shall prove that the mapping is into. That it is onto will follow from a more general theorem to be proved later on. Since $\Omega$ is $m$-extendible and since $\gamma$ on $\mathscr{D}(\overline{\Omega})$, the restrictions of functions $u(x', y)$ of $\mathscr{D}(R^n)$ is $u(x', 0)$, it is enough to prove that the mapping $u \to u(x', 0)$ is a continuous mapping of $\mathscr{D}(R^n)$ with the topology of $H^m_o(R^n)$, into $H^{m-\frac{1}{2}}(\Gamma)$. This we do by using Fourier transform.

Let $\mathscr{F}(u(x)) = \overline{u}(\xi) = \int e^{-2\pi i x \cdot} u(x)dx$ be the Fourier transform of $u$. Then $u(x) = \int e^{2\pi i x \cdot} v(\xi)d\xi$ and so $u(x', 0) = \int e^{2\pi\xi' \cdot}\ v(\xi, \xi_n)d\xi d\xi_n$. Hence

$$\mathscr{F}'_{x'} u(x', 0) = \int v(\xi' \xi_n)d\xi_n. \tag{1}$$

We have now to prove that the mapping $\mathscr{F}(u) = v\mathscr{F}_{x'}(u(x', 0))$ is continuous from $\mathscr{F}(\mathscr{D})$ with the topology of $\hat{H}(m)$ into $\mathscr{F}(H^{m-\frac{1}{2}}(\Gamma))$ or that $v \to (1 + |\xi|^{m-\frac{1}{2}})\mathscr{F}_{x'}(u(x', 0))$ is continuous from $\mathscr{F}(\mathscr{D})$ with the topology of $\hat{H}^m$ into $L^2$.

Hence we have to prove, using (1), that

$$\int (1 + |\xi|^{2m-1} d\xi' \Big| \int v(\xi', \xi_n)d\xi_n \Big|^2 \leq c \int (1 + |\xi|^{2m})|v(\xi)\Big|^2 d\xi.$$

Now 103

$$\begin{aligned}
\Big| \int v(\xi', \xi_n)d\xi_n \Big|^2 &= \Big| \int v(\xi', \xi_n)(1 + |\xi|^m)(1 + |\xi|^m) - 1d\xi_n|^2 \\
&\leq \int |v(\xi', \xi_n)|^2(1 + |\xi|^m)^2 d\xi_n \frac{d\xi_n}{(1 + |\xi|^m)^2} \\
&\leq c \int \frac{d\xi_n}{(1 + |\xi|^2)^m} \int |v|^2(1 + |\xi|^m)^2 d\xi_n.
\end{aligned} \tag{2}$$

[1]The author's thanks are due to Professors Aronszajn and Smith for lending him an unpublished manuscript concerning these problems.

Putting

$$\xi_n = \sqrt{1+\xi'^2}t, \int \frac{d\xi_n}{(1+\overline{|\xi|}^2)^{m-\frac{1}{2}}} = \int \frac{dt}{(1+t^2)^m} \tag{3}$$

Hence from (2) and (3),

$$\begin{aligned}\int (1+|\xi'|^{2m-1})\left|\int v(\xi',\xi_n)d\xi_n\right|^2 \\ &\leq c\int (1+|\xi'|^{2m-1})\frac{d\xi'}{(1+|\xi'|^2)^{m-\frac{1}{2}}}\int |v|^2(1+|\xi|^m)^2 d\xi_n \\ &\leq c\int |v|^2(1+|\xi|^m)^2 d\xi\end{aligned}$$

as was to be proved.

Now, if $u \in H^m(\Omega)$, we have $\dfrac{\partial u}{\partial x_n} \in H^{m-1}(\Omega)$ and by the above proposition, $\gamma\left(\dfrac{\partial u}{\partial x_n}\right) \in H^{m-\frac{1}{2}}(\Gamma)$. Hence we have the

**Corollary.** $\gamma_\partial(u) = \gamma(D^j_y u) \in H^{m-j-\frac{1}{2}}(\Gamma)$ *for* $j = 1,\ldots,m-1$.

Now, let $\vec{\gamma}(u) = (\gamma_o u,\ldots,\gamma_{m-1}u)$ and $F = H^{m-\frac{1}{2}}(\Gamma)\times\ldots\times H^{m-j-\frac{1}{2}}(\Gamma)$ with the product Hilbertia an structure.

**Theorem 11.1.** *The mapping* $u \to \vec{\gamma}(u)$ *of* $H^m(\Omega)$ *onto* $F$ *with kernel* $H^m_o(\Omega)$.

From the above proposition, it follows that $\vec{\gamma}$ maps $H^m(\Omega)$ into $F$ and that its kernel is $H^m_o(\Omega)$. To prove that $\vec{\gamma}$ is onto it is enough to show that if $\vec{f} = (0,\ldots,f_j,\ldots,0) \in F$ with $f_j \in H^{m-j-\frac{1}{2}}(\Gamma)$, then there exists $u \in H^m(\Omega)$ such that $\gamma_j u = f_j$ and $\gamma_k u = 0$ for $k \neq j$ and $k \leq m-1$.

Taking Fourier transforms in $x'$, with $\xi' = (\xi_1,\ldots,\xi_{n-1})$ we have to find **104** $v(\xi',y)$ such that

1) $(1+|\xi|^m)v(\xi',y) \in L^2(,y)$.

2) $D^m_y v(\xi',y) \in L^2(\xi',y)$, and

3) (a) $D^j_y v(\xi',0) = \hat{f}_j(\xi')$

(b) $D^k_y v(\xi',0) = 0$ for $k \neq j, k \leq m-1$.

Put $\quad \phi(\xi', y) = \dfrac{1}{j!} y^j e^{-1+|\xi'|y} \hat{f}_j(\xi').$

Let $t = (1 + |\xi'|)y$. Then put

$$v(\xi', y) = \phi(\xi', y)(1 + \alpha_1 t + \cdots + \alpha_{m-j-1} t^{m-j-1}).$$

By direct computation, we have

$$D_y^k \phi(\xi', 0) = 0 \text{ for } k \leq j - 1, \; D_y^j \phi(\xi', 0) = \hat{f}_j(\xi'), \text{ and}$$
$$D_y^{j+1}(\xi', 0) = \frac{1}{j!} C_{j+1}^j D^j (y^j)(-1)^l (1 + |\xi'|)^l \hat{f}_j(\xi').$$

Hence $D_y^k v(\xi', 0) = 0$, for $k \leq j - 1$, $D_y^j(\xi', 0) = \hat{f}_j(\xi)$, and

$$D_y^{j+1} v(\xi', 0) = C_{j+1}^j (-1)^l (1 + |\xi'|)^l F_j(\,) + (j + 1) C_{j+l-1}^j (1 + |\xi'|)^{l-1} ({}_1 1 + |\xi'|) \hat{f}_j(\xi').$$

$\alpha_1, \ldots, \alpha_{m-j-1}$ are determined by $m - j - 1$ conditions that $D_y^{j+1} v(\xi', 0) = 0$ for $l = 1, \ldots, m - j - 1, .\alpha_1, \ldots, \alpha_{m-j-1}$ are then well-determined independent of $\xi'$, e. g. , $(j + 1)\alpha_1 = C_{j+1}^j$ and so on.

It remains to verify

(a) $(1 + |\xi'|^m) t^k \phi(\xi', y) \in L^2$ for $k \leq m - j - 1$, and

(b) $D_y^m (t^k \phi(\xi', y) \in L^2$.

(a) We have to consider $\Big|(1 + |\xi'|^m t^k \phi(\xi', y)\Big|_o$.

$$\int \Big(1 + |\xi'|^{2m}\Big) \Big|\hat{f}_j(\xi')\Big|^2 (1 + |\xi'|^{2k}) d\xi' \int_o^\infty y^{2j+2k} e^{-(1+|\xi'|)y} dy$$
$$= \int (1 + |\xi'|^{2m} |\hat{f}_j(\xi')|^2 (1 + |\xi'|^{2k}) d\xi' \int_o^\infty \frac{t^{j+2k_e - tdt}}{(1 + |\xi'|^{2j+2k+1}}$$
$$\text{by putting } (1 + |\xi'|)y = t.$$
$$\leq c \int (1 + |\xi'|^{2m-2j-1} |\hat{f}_j(\xi')|^2 d\xi' < \infty, \text{ since } f_j \in H^{m-j-\frac{1}{2}}(\Gamma).$$

b) We have to consider $D_y^m (t^k y^j e^{-(1+|\xi'|)y} \hat{f}_j(\xi'))$. This is a sum of terms **105**

$$t^{k-r}(1 + |\xi|^r) y^{j-1} (1 + |\xi|^{m-r-1} e^{(1+|\xi'|)y} \hat{f}_j(\xi') \text{ for } r \underset{1=1,\ldots,j.}{=} l, \ldots, k.$$

Hence we have to consider

$$\int (1+|\xi'|)^{2k-2r}(1+|\xi'|)^{2m-2l}|\hat{f}_j|^2 d\xi' \int y^{2j+2l+2k-2r} e^{-2(1+|\xi'|y)} dy$$

$$= \int (1+|\xi|)^{2k-2r}(1+|\xi'|)^{2m-2l}|\hat{f}_j|^2 d\xi' \frac{1}{(1+|\xi'|)^{2j-2l+2k-2r+1}}$$

$$t^{2j-2l+2k-2r} e^{-2t} d \text{ by putting } t = (1+|\xi'|)y$$

$$\leq c \int (1+|\xi'|)^{2m-2j-1}|\hat{f}_j|^2 d\xi' < \infty,$$

which proves the theorem.

# Lecture 21

## 11.2 Aronszajn-Smith Problems

We prove a lemma which will be required often. 106

**Lemma 11.1.** *Let* $\Omega = ]0, 1L^n$ *and* $\Gamma = \{\Omega \cap \{x_n = 0\}\}$. *Let* $F = H^{m-\frac{1}{2}}(\Gamma) \times \ldots \times H^{\frac{1}{2}}(\Gamma)$. *Let* $f_\alpha$ *be a bounded set in* $F$ *such that all* $f_\alpha$ *have their support in a fixed compact set* $\sum$ *in* $\Gamma$. *Then there exists a bounded set* $v_\alpha$ *in* $H^m(\Omega)$ *such that* $\vec{\gamma} v_\alpha = f_\alpha, \vec{\gamma}$ *being as defined in 11.1 above, and* $v_\alpha \equiv 0$ *near* $\partial\Omega - \sum$.

*Proof.* By Theorem 11.1, $\vec{\gamma} : H^m(\Omega) \to F$ is onto with kernel $H^m_\circ$. Hence $\vec{\gamma}$ induces an isomorphism $\vec{\gamma}_1$ of $H^m/H^m_o$ onto $F$. Since $f_\alpha$ is a bounded set in $F$, $\vec{\gamma}^{-1}(f_\alpha)$ is bounded in $H^m/H^m_\circ$. Therefore we can choose a bounded set $\omega_\alpha \in H^m$ such that $\gamma\omega_\alpha = f$. Next let $\varphi \in \mathscr{D}(\bar{\Omega})$ be zero near $\partial\Omega - \sum$ and $\varphi = 1$ on $\sum$. Then $v_\alpha = \varphi v_\alpha$ are bounded, vanish near $\partial\Omega - \sum$ and $\vec{\gamma} v_\alpha = \vec{\gamma}(\varphi)\vec{\gamma}(\omega_\alpha) = \vec{f}_\alpha$. ◻

In § 9, we considered regularity at the boundary of some problems related to the operator $A$ defined by a sesquilinear form $a(u, v)$ on $V$ such that $H^m_o \subset V \subset H^m$. Now we take up a particular example of a different space $V$. In this case the technique used in § 9 is not at once applicable. Since the preliminary step of using local maps is at any rate permissible we assume $\Omega = ]0, 1[^n$. Further to avoid technical details, we assume that $m = 2$. Let $\vartheta$ be the subspace of $H^2(\Omega)$ consisting of functions

a) vanish near $\partial\Omega - \sum$,

b) $Bu = 0$ on where

$$Bu = \frac{\partial u}{\partial x_n}(x', 0) + \sum_{i=1}^{n} \alpha_i(x') \frac{\partial u}{\partial x_i}(x', 0) + \alpha_o(x') u(x', 0)$$

with $\alpha_o, \ldots, \alpha_{n-1} \in \mathscr{D}(\xi)$. **107**

Let $a(u, v) = \sum\limits_{|p|,|q|\leq} \int_\Omega a_{pq}(x) D^q u \overline{D^p v} dx$ with $a_{pq} \in \mathscr{E}(\bar{\Omega})$.

Let Re $a(u, u) \geq \alpha ||u||_2^2$ for all $u \in \vartheta$. In this case according to the theory of § 3, as transformed by local maps as in §, for a given $f \in L^2(\Omega)$, there exists $u \in N$ such that $a(u, v) = (f, v)_o$ for all $v \in \vartheta$. We prove now

**Theorem 11.2.** *Let $u \in H^2(\Omega)$ with $Bu = 0$ and $a(u, v) = (f, v)_o$ for all $v \in \vartheta$, $f \in L^2$. Then $u \in H^4(\Omega^\epsilon)$ for every $\in > 0$.*

*Proof.* Though a shorter proof by induction is possible in order to bring out the significance of the method we give a direct proof. Since after having proved that $D_T^p u \in H^2$ for $|p| \leq 2$, to prove $D_y^m u \in H^2$ no use of boundary conditions need be made as in §9, to prove the theorem, we have to prove $D_\tau^P u \in H^2$ for $|p| \leq 2$. Further if $\phi \in \mathscr{D}\underline{\Omega}), \varphi \equiv 0$, near $\partial^- - \sum$, to prove $D_\tau^p u \in H^2$ for $|p| \leq 2$, it is enough to prove $D_\tau^p(\phi u) \in H^2$ for $|p| \leq 2$. We break this in two steps. □

**Step 3.** $D_\tau^1(\phi u) \in H^2$.

As usual we need prove $(\phi u^{-h})$ is bounded in $H^2(\Omega)$ by $c||u||_2$, and for this we consider $a((\phi u)^{-h}, v)$.

**Lemma 11.2.** $\left|a(\phi u)^{-h}, v)\right| \leq c||v||_2$.

We write

$$a((\phi u)^{-h}, v) = [a((\phi u)^{-h}, v) + a(\phi u, v^h)] - b(u, v^h) - a(u, \phi v^h)$$

where $b(u, v) = a(\phi u, v) - a(u, \phi v)$.

As in § 9, we can estimate $a((\phi u)^{-h}, v) + a(\phi u, v^{-h})$ and $b(u, v)$ by almost the same methods. It remains to be proved that $\left|a(u, \phi v^h\right| \leq c||v||_2$. We cannot put **108** $a(u, \phi v^h) = (f, \phi v^h)_o$ as $\vartheta$ is not necessarily closed for translations. However by "correcting" $\phi v^h$ with a "compensating" function $\omega^h$ we prove that $\left|a(u, \phi v^h)\right| < c||v||_2$. More precisely we prove the

**Lemma 11.3.** *There exists $w_h$ in $H^2(\Omega)$ such that*

*(a)* $\phi v^h - w_h \in \vartheta$

*(b)* $||w_h||_2 \leq c||v||_2$.

Assuming for a moment the lemma 11.3, we prove lemma 11.2. We have

$$a(u, \phi v^h) = a(u, \phi v^h - w_h) + a(u, w_h).$$

Since $\phi v^h - w_h \in \vartheta, a(u, \phi v^h - w_h) = (f, \phi v^h - w_h)_o$.

Hence $\left|a(u, \phi v^h - w_h)\right| \leq |f|_o |\phi v^h - w_h|_o \leq c(||v||_1 + |w_h|_o) \leq c||v||_2$.

Further $\left|a(u, w_h)\right| \leq c||w_h||_2 \leq c||v||_2$, whence the lemma 11.2.

Now we prove lemma 11.3. We have to find $w_h$ such that

$$\phi v^j - w_h \in \vartheta, \text{ i. e. }, B(\phi v^h - w_h) = 0,$$

i.e., $\dfrac{\partial w_h}{\partial x_n}(x', 0) + \sum \alpha_i(x') \dfrac{\partial w_h}{\partial x_i}(x', 0) + \alpha_o(x') w_h(x', 0) = B(\phi v^h)$.

This holds if $\dfrac{\partial wh}{\partial x_n}(x', 0) = B(\phi v^h)$, and $w_h(x', 0) = 0$.

If we prove that $B(\phi v^h)$ is bounded in $H^{\frac{1}{2}}$ by $c'||v||_2$, by using lemma 11.1, we can find $w_h$ bounded by $c||v||_2$, such that $\gamma w_h = 0$ and $\gamma_1 w_h = B(\phi v^h)$ which will prove the lemma. Now

$$B(\phi v^h) = g_h + k_h$$

where $\quad g_h(x', 0) = \phi(x', 0) B v^h$,

and $\quad k_h(x', 0) = \dfrac{\partial \varphi}{\partial x_n}(x', 0) v^h(x', 0) + \sum \dfrac{\partial \varphi}{\partial x_i}(x', 0) \alpha_i v^h(x', 0)$.

Since $\varphi$ has compact support all $k_h$ have support in a fixed compact.

Further since $v \in H^2$, we have $v^h(x', 0) \in H^{3/2}(\Gamma\Gamma)$ and since $\dfrac{\partial \varphi}{\partial x_h}$ are **109**
smooth, we have $k_h(x) \in H^{3/2}(\Gamma)$. Now as $h \to 0, v^h(x', 0) \to D_\tau v(x', 0)$, hence $k_h$ is bounded by $c||v||_2$ in $H^{\frac{1}{2}}(\Gamma)$.

It remains to see that $g_h$ is bounded in $H^{\frac{1}{2}}(\Gamma)$ by $c||v||_2$. Since $Bv = 0$, $(Bv)^h = 0$, and since

$$(Bv)^h = Bv^h + \sum \alpha_i^h \frac{\partial v}{\partial x_i}(x' + h, 0) + \alpha_o^h v(x' + h, 0)$$

we have $g_h = \varphi B v^h = -\phi(x', 0)(\sum \alpha_i^h \dfrac{\partial v}{\partial x_i}(x' + h, 0) + \alpha_o^h v(x' + h, 0))$.

As $h \to 0, \alpha_i^h$ are uniformly bounded and $\dfrac{\partial v}{\partial x_i}$ are bounded in $H^{\frac{1}{2}}(\Gamma)$ as translations are continuous.

This proves then that $B(\phi v^h)$ is bounded in $H^{\frac{1}{2}}(\Gamma)$ and the proof of lemma 11.3 and hence that of lemma 11.2 is complete.

Now we are in a position to prove the

**Lemma 11.4.** $||(\phi u)^{-h}||_2 \leq c$.

We cannot prove this as in § 9 by taking $v = (\phi u)^{-h}$ in lemma 11.2, and using ellipticity for $(\phi u)^{-h}$ does not necessarily belong to $\vartheta$. We again correct this by

**Lemma 11.5.** *There exists $w_h \in H^2(\Omega)$ such that*

$$a)(\phi u)^{-h} - w_h \in \vartheta \text{ and } b)||w_h||_2 \leq c||u||_2 = c'$$

*(since u is fixed).*

To prove this we note $(\phi u)^{-h} = \phi u^{-h} + \phi u^{-h}(x-h)$ and from lemma 11.3, there exists $w'_h$ such that $\phi u^{-h} - w_h \in \vartheta$, and $||w_h||_2 \leq c||u||_2$. We have only to look then for $w^{(2)}_h$ such that

$$\phi^{-h}u(x-h) - w^{(2)}_h \in \vartheta, \text{ and}$$
$$||w^{(2)}_h||_2 \leq c||u||_2.$$

We have to find $w^{(2)}_h$ bounded in $H^2(\Omega)$ by $c||u||_2$ and such that **110**

$$w^{(2)}_h(x', 0) = \phi^{-h}u(x'-h, 0)$$
$$\frac{\partial w_h}{\partial x_n}(x', 0) = \frac{\partial}{\partial x_n}(\phi^{-h}u(x'-h, x_n)_{x_n} = 0.$$

Hence, by lemma 11.1 such $w^{(2)}_h$ as required above exist and the lemma 11.5 is proved.

To prove lemma 11.4 consider now

$$\begin{aligned} a((\phi u)^{-h} - w'_h, (\phi u)^{-h} - w'_h) &= a((\varphi u)^{-h}, (\phi u)^{-h} - w'_h).a(w'_h, (\phi u)^{-h} - w'_h), \\ &= X_h - Y_h. \end{aligned}$$

By lemma 11.2, we have

$$|X_h| \leq c||(\phi u)^{-h} - w'_h||_2$$
$$|Y_h| \leq c||w'_h||_2||(\phi u)^{-h} - w'_h||_2 \leq c'||(\phi u^{-h} - w'_h||^2_2.$$

On account of ellipticity, $|a((\phi u)^{-h} - w'_h, (\phi u)^{-h} - w'_h)| \geq \alpha||(\phi u)^{-h} - w'_h||^2_2$.

Hence $||(\phi u)^{-h} - w'_h||_2 \leq c$ and since $||w'_h||_2 \leq c$ we get the lemma. This completes the first step of the proof, viz. $D^p_\tau u \in H^2(\Omega), |p| = 1$.

# Lecture 22

## 11.3

Now we come to the 111
**Second step.** We wish to prove $(\phi D_\tau u)^{-h}$ is bounded in $H^2(\Omega)$. To do this we consider $a((\phi D_\tau u)^{-h}, v)$ and prove the

**Lemma 11.6.** $\left| a((\phi D_\tau u)^{-h}, v) \right| \leq c \|v\|_2$ *for* $v \in \vartheta$ *such that* $D_\tau \overline{u} \in H^2$.

*Proof.* We write

$$\begin{aligned} a((\phi D_\tau u)^{-h}, v) = a((\phi D_\tau u)^{-h}, v + a(\phi D_t auu, v^h) - b(D_\tau u, v^h) \\ - a(D_\tau u, \phi v^h). \end{aligned}$$

As in the previous cases, we have straight forward estimates except for $a(D_\tau u, \phi v^h)$. Now

$$a(D_\tau u, \phi v^h) = a(D_\tau u, \phi v^h) + a(u, D_\tau(\phi v^h)) - a(u, D_\tau(\phi v^h))$$

which exists since $D_\tau v \in H^2$. Again as in the previous cases, only non-trivial part is to prove that $\left| a(u, D_\tau(\phi v^h)) \right| \leq c \|v\|_2$. □

To do this we have to correct $D_\tau(\phi v^h)$ by the

**Lemma 11.7.** *There exists* $w_h \in H^2(\Omega), w_h = 0$ *near* $\partial\Omega - \sum$ *such that*

i) $D_\tau(\phi v^h) - w_h \in \vartheta$.

ii) $\|w_h\|_1 \leq c \|v\|_2$.

iii) $\left| a(u, w_h) \right| \leq c \|v\|_2$.

Admitting this for a moment, we have

$$\begin{aligned} a(u, D_\tau(\phi v^h)) &= a(u, w_h) + a(u, D_\tau(\phi v^h) - w_h) \\ &= a(u, w_h) + (f, D_\tau(\phi v^h) - w_h) \end{aligned}$$

and we have the lemma 11.6 as usual.

So we have to prove now lemma 11.7. If $w_h$ have to verify

i) Then we can write :

$$B(w_h) = BD_\tau(\phi v^h) = f_h + \sum_{i=1}^{n-1} D_i g_h^i, D_i = \frac{\partial}{\partial x_i}, \text{ say .}$$

We shall prove that one can choose $f_h$ and $g_h^i$ such that 112

a) $f_h$ have their support in a fixed compact and are bounded in $H^{\frac{1}{2}}(\Gamma)$ by $c\|v\|_2$.

b) $g_h^i \in H^{3/2}(\Omega)$, and are bounded in $H^{\frac{1}{2}}(\Omega)$ by $c\|v\|_2$ with support in a fixed compact.

Assuming (a) and (b) we prove that $w_h$ satisfying (i), (ii), and (iii) can be found. For by (a) and lemma. 11.1, there exists $w_h^o \in H^2(\Omega)$ such that $w_h^o(x'.0) = 0, \dfrac{\partial}{\partial x_n} w_h^o(x', 0) = f_h, w_h^o$ are bounded in $H^2(\Omega)$ by $c\|v\|_2$ and $w_h^o$ vanish near $\partial\Omega - \sum$. Similarly on account of (b) and lemma 11.1, there exists $w'_h \in H^2(\Omega)$ with $w_h^i(x', 0) = 0, \dfrac{\partial}{\partial x_n} w_h^i(x', 0) = g_h^i, w_h^i$ bounded in $H^2(\Omega)$ by $c\|v\|_2$ and $w_h^i \equiv 0$ near $\partial\Omega - \sum$. Setting $w_h = w_h^o + \sum_{i=1}^{n-1} D_i w_h^i$ we see that (i) and (ii) are at once satisfied. Further to verify (iii) we have $\left|a(u, w_h^o\right| \leq c\|v\|_2$ and it remains to estimate $a(u, D_i w_h^i)$. But since $D_\tau u \in H^2$ and since $w_h^i = 0$ near $\partial\Omega - \sum$ by integration by parts, we get $\left|a(u, D_i w_h^i)\right| \leq c\|v\|_2$.

We have still to verify (a) and (b), we indicate which parts of $B(D_\tau(\phi v^h))$ are to be taken as $f_h$ and which as $g_h^i$ and prove each time that they are bounded by $c\|v\|_2$ in the appropriate spaces. We have

$$\begin{gathered} B(D_\tau(\phi v^h) = B((D_\tau \phi) v^h) + B(\phi D_\tau v^h) \\ B((D_\tau \phi) v^h) = (D_\tau \phi) B v^h + \left(\frac{\partial}{\partial x_n} D_\tau \phi\right) v^h + \sum \alpha_i \left(\frac{\partial}{\partial x_i} (D_\tau \phi)\right) v^h, \end{gathered}$$

and

$$B v^h = -\sum \alpha_i^h \frac{\partial v}{\partial x_i}(x' + h, 0), \text{ since } (Bv)^h = 0.$$

Then we take $B((D_\tau\phi)v^h)$ as part of $f_h$; it is $H^{\frac{1}{2}}$ and is bounded by $c\|v\|_2$.

Now $\quad B(\phi(D_\tau v^h)) = \phi B(D_\tau v^h) + \dfrac{\partial\phi}{\partial x_n}(x', 0)D_\tau v^h + \sum \alpha \dfrac{\partial\phi}{i\partial x_i} D_\tau v^h.$ **113**

We consider each of the summands separately :

$$\frac{\partial\phi}{\partial x_n}(D_\tau v^h) = D_\tau\left(\frac{\partial\phi}{\partial x_n}v^h\right) - \left(D_\tau\frac{\partial\phi}{\partial x_n}\right)v^h.$$

We take $D_\tau\dfrac{\partial\phi}{\partial x_n})v^h$ as part of $f_n$ and it is seen that it satisfies ($a$). For $D_\tau\left(\dfrac{\partial\phi}{\partial x_n}v^h\right)$ we take it as part of $D_\tau g_h^i$. It is also seen that $g_h^i$ satisfies (b). Similarly we consider $\displaystyle\sum_i \frac{\partial\phi}{\partial x_i\partial_{x_i}} D_\tau v^h$. It remains only to consider $\phi B(D_\tau v^h)$. Now since $(Bv)^h = 0$, we have

$$\phi B(D_\tau v^h) = -\phi\left(\sum D_\tau\alpha_i\frac{\partial v^h}{\partial x_i} + D_\tau\alpha_o\right)v^h),$$

and they are to be taken as parts of $f_h$. This completes the proof of lemma 11.6.

To complete the proof of the theorem, we require the

**Lemma 11.8.** $\|D_\tau(\phi_u)^{-h}\|_2 \leq c$.

Again we require corrections $w'_h$ as follows:

**Lemma 11.9.** *There exists $w'_h \in H^2(\Omega)$ vanishing near $\partial\Omega - \sum$ and such that*

*(1)* $D_\tau(\phi u)^{-h} - w'_h \in \vartheta$

*(2)* $\|w'_h\|_2 \leq c$.

Assuming the existence of such $w'_h$ and considering

$$a(D_\tau(\phi u)^{-h} - w'_h, D_\tau(\phi u^{-h} - w'_h)$$

and using the ellipticity of $a(u, u)$ we obtain lemma 11.8, as in the lemma 11.4, after observing that lemma 11.6 can be applied though $v = D_\tau(\phi u)^{-h} - w'_h$ is such that $D_\tau v \notin |H^2(\Omega)$. To see this last point we prove that in $\vartheta$, $v's$ such that $D_\tau v \in H^2(\Omega)$ are dense.

Let $v \in \vartheta$ and $v(x', 0) = f$. Let $\psi \in \mathscr{D}(\Omega)$ be such that $\varphi_\alpha f$ in $H^{3/2}(\Gamma)$, and let $\psi_\alpha = -\sum \alpha\dfrac{\partial\varphi}{\partial x_i} - \alpha_o\varphi_\alpha$.

Now it is possible to find $v_\alpha \in H^2(\Omega) \cap \vartheta(\bar{\Omega})$ such that $v_\alpha(x', 0) = \varphi_\alpha, \dfrac{\partial v}{\partial x_n} =$ **114**
$\varphi_\alpha, v \equiv 0$ near $\partial\Omega - \sum$ and $v_\alpha \to v$ in $H^2$. By the choice of $\psi_\alpha, v$ belongs to $\vartheta$ and the result follows.

To prove lemma 11.9, we observe

$$\begin{gathered}(\phi u)^{-h} = \phi u^{-h} + \phi^{-h} u(x' - h, x_n), \text{ and}\\ D_\tau(\phi u)^{-h} = (D_\tau\phi)u^{-h} + \phi D_\tau u^{-h} + (D_\tau\phi^{-h} u(x' - h, x_n) + \phi^{-h} D_\tau u(x - h).\end{gathered}$$

We first define $w_h^1$ such that

a) $(D_\tau\phi^{-h})u(x - h) + \phi^{-h} D_\tau u(x - h) - w_h^1 \in \vartheta$, and

b) $\|w_h^1\|_2 \leq c$.

To verify $(a)$ we choose $w_h^1$ so that

$$w_h^1(x', 0) = ((D_\tau\phi)^{-h} u(x - h) + \phi^{-h} D_\tau u(x - h))_{x_n} = 0.$$

$$\text{and} \qquad \frac{\partial \omega_h'(x', 0)}{0x_n} = \left(\left(\frac{\partial}{\partial x_n}\right)^{-h} u(x - \lambda) + \phi^{-h} \frac{\partial}{\partial x_n} u(x - \lambda)\right) x_n = \partial$$

Since $D_\tau u \in H^2(\Omega)$ the right hand side in the first expression ie in $H^{3/2}(\Gamma)$, and is bounded. Similarly the one in the second expression is in $H^{\frac{1}{2}}(\Gamma)$, and is bounded, and by lemma 11.1 the existence of $w_h^1$ is proved. Next we find $w_h^2$ so that

a) $(D_\tau\phi)u^{-h} - w_h^2 \in \vartheta$ and

b) $\|w_h^2\|_2 \leq c$.

The existence of such $w_h^2$ is assured by lemma 11.4. Finally we define $w_h^3$ so that

a) $\sigma D_\tau u^{-h} - w_h^3 \in \vartheta$ and

b) $\|w_h^3\|_2 \leq c$.

To verify (a), we should have $B(\phi(D_T u^{-h})) = Bw_h^3$. But

$$B(\phi(D_\tau u^{-h})) = \phi B(D_T u^{-h}) + \frac{\partial\phi}{\partial x_n} D_\tau u^{-h} + \sum \alpha_i \frac{\partial\phi}{\partial x_i} D_\tau u^{-h}.$$

Since $Bu = 0$ and $D_\tau Bu = 0$, we have **115**

$$B(D_\tau u) + \sum_{i=1}^{n-1}(D_\tau \alpha_i)D_i u + (D_\tau \alpha_o)u = 0.$$

Hence on the support of $\phi$,

$$(B(D_\tau u))^{-h} + \sum((D_\tau \alpha_i)D_i u)^{-h} + ((D_\tau \alpha_o)u)^{-h} = 0, \text{ i.e.,}$$
$$(B(D_\tau u^{-h} + \sum \alpha_i^{-h} D_\tau u(x-h) + \sum(D_\tau \alpha_i D_i u)^{-h} + ((D_\tau \alpha_o)u)^{-h} = 0;$$

so that we have to find $w_h^3$ satisfying

$$Bw_h^3 = -\phi \sum \alpha_i^{-h} D_\tau u(x-h) - \phi \sum((D_\tau \alpha_i)D_i u)^{-h} - \phi((D_\tau \alpha_o)u)^{-h} + \frac{\partial \phi}{\partial x_n} D_\tau u^{-h} + \sum \alpha_i + \frac{\partial \phi}{\partial x_i} D_\tau u^{-h} = g_h.$$

Hence it is enough to find $w_h^3$ such that

$$\begin{cases} w_h^3(x', 0) = 0 \\ \frac{\partial w_h^3}{\partial x_n}(x', 0) = g_h \text{ defined above.} \end{cases}$$

Since $D_\tau u \in H^2(\Omega), g_h \in H^{3/2}(\Gamma)$ and is bounded in $H^{\frac{1}{2}}(\Gamma)$. Hence by lemma 11.1 we have the existence of such $w_h^3$. This completes the proof of the theorem 11.2.

**Final Remarks.**

(1) By using Stampanhia [17] and Lions [6], Campanato [6] has proved the regularity at the boundary for Picone problems.

(2) For another method for Dirichlet conditions with constant coefficients in two dimensions, and very general conditions on the boundary, see Agmon [1].

# Lecture 23

## 12 Regularity of Green's Kernels

### 12.1

In § 3.5 we have defined Green's kernel of the operator $A$ associated with an elliptic sesquilinear from $a(u,v)$ on $V$ such that $H_o^m(\Omega) \subset V \subset H^m(\Omega)$, $Q$ being $L^2(\Omega)$ say. We recall that $a(u,v)$ begin $V$ elliptic, $A$ is an isomorphism of $N$ onto $Q'$. Hence $A^{-1} = G$ is an isomorphism of $Q'$ onto $N$. Since $\mathscr{D}(\Omega)$ is dense in $Q'$, by Schwartz's kernel theorem $A^{-1} = G$ is given by $G_{x,y} \in \mathscr{D}'(\Omega_x \times \Omega_y).G_{x,y}$ is called the kernel of the operator $A$. 116

Let $a^*(u,v) = \overline{a(v,u)}$; $a^*(u,v)$ is also $V$ elliptic and defines a space $N^*$ and an operator $A^*$. Let its kernel be $G^*_{x,y}$. If $T_{x,y} \in \mathscr{D}'(\Omega_x, \Omega_y)$, then $T_{y,x}$ will be defined by setting on the everywhere dense set $\mathscr{D}(\Omega_x) \times \mathscr{D}(\Omega_y)$ in $\mathscr{D}(\Omega_x \times \Omega_y)$.

$$T_{y,x}(\varphi(x) \cdot \psi(y)) = T_{x,y}(\psi(x)\varphi(y)).$$

We denote $\mathscr{D}(\Omega_x)$ by $D_x$, $\mathscr{D}(\Omega_x \times \Omega_y)$ by $\mathscr{D}_{x,y}$ and so on. We have the

**Proposition 12.1.** $G_{xy} = G^*_{y,x}$.

Let $\varphi, \psi \in \mathscr{D}(\Omega_x)$. We have to verify that $\langle G\varphi, \overline{\psi}\rangle = \langle \varphi, G^*\psi\rangle$. Let $G\varphi = u \in N$ and $G^*\psi = w \in N^*$. Then $\varphi = Au$ and $\psi = A^*w$. Hence we have to verify that $\overline{\langle u, A^*w\rangle} = \langle A\psi, \bar{\omega}\rangle$. This follows since $\langle u, \overline{A^*w}\rangle = \overline{a^*(w,u)} = a(u,w) = \langle Au, \overline{w}\rangle$.

**Definition 12.1.** *An element $G_{x,y}$ in $\mathscr{Q}'(\Omega_x \times \Omega_y)$ will be called a* kernel.

**Definition 12.2.** *A kernel is semi-regular, with respect to $x$, if $G_{x,y}$ is given by a $C^\infty$ function of $\Omega_x$ into $\mathscr{D}'_y$. We write it then as $G(x)_y$ or $G_y(x)$.* 117

**Definition 12.3.** *A kernel is regular if it is semi-regular with respect to $x$ and $y$.*

**Definition 12.4.** *A kernel is very regular if it is regular and a $C^\infty$ function outside the diagonal.*

If $G_{x,y}$ is semi-regular it is an element of $\mathscr{E}_x \hat{\otimes} \mathscr{D}'_y = \mathscr{E}(x, \mathscr{D}'_y)$. Hence it defines a mapping $G : \mathscr{D}_y \to \mathscr{E}_z$ given by $\int G(x)_y \varphi(y) dy \in \mathscr{E}(x)$ for $\varphi(y) \in \mathscr{D}_y$. Conversely by Schwartz's kernel theorem, any linear mapping $G : \mathscr{D}_y \to \mathscr{E}_x$ is given by a semi-regular kernel.

We now with to consider conditions on $a(u, v)$ so that the kernel $G_{x,y}$ be very regular.

**Definition 12.5.** *A partial differential operator $A$ defined in $\Omega$ with $C^\infty$ coefficients is said to by* hypo-elliptic *if for $s \in \mathscr{D}'(\Omega)(AS)_\theta \in \mathscr{E}(\theta)$ implies that $S \in \mathscr{O}$, for every $0 \subset \Omega$.*

For example, if $a(u, v)$ is $V$ elliptic, $\Omega$ is bounded with smooth boundary, then by the results of § 8, it follows that $A$ is hypo-elliptic.

**Theorem 12.1.** *Let $|a(u, u)| \geq \alpha |u|^2_V$ and the operators $A$ and $A^*$ be hypo-elliptic. Then $G_{x,y}$ is very regular.*

*Proof.* First we prove $G_{x,y}$ is regular. We know $G$ is an isomorphism of $Q'$ onto $N$. Let $\varphi \in \mathscr{D}(\Omega)$. Then $G\varphi = u \in N$ and $Au = \varphi$. By hypo-ellipticity of $A$ it follows that $u \in \mathscr{E}$. Hence $G$ defines a mapping of $\mathscr{D}(\Omega)$ onto $N \cap \mathscr{E}$. By closed graph theorem this mapping is continuous and hence Schwartz's kernel theorem is given by a semi-regular kernel $G(x)_y$. Hence $G_{x,y} = G^*(x)_y$. Similarly, since $A^*$ is hypo-elliptic $G^*_{x,y} = G^*(x)_y$. But by proposition 12.1, 118
$G_{x,y} = G^*_{y,x}$ and hence $G_{x,y} = G^*_x(y)$. This shows that $G_{x,y}$ is regular. □

To complete the proof we have to show that outside the diagonal it is a $C^\infty$ function.

Let $O_1$ and $O_2 \subset \Omega$ be two open sets such that $O_1 \cap O_2 = \phi$. Let $T \in \mathscr{E}'(\mathscr{O}_2)$. Then $\int G_x(y) T_y d\mu$ is defined and is an element of $\mathscr{D}_x$ say $S_x$ such that $AS = T$. Restricting $A, S, T$ to $O_1$ since $T = 0$ on $O_1$ by hypo-ellipticity of $A$ on $O_1$, we have $SO_1 \in \mathscr{E}(O_2)$. Hence $T \to \int G_x(y) T_y d\mu$ is a mapping of $\mathscr{E}'(O_2)$ onto $\mathscr{E}(O_1)$, which on account of the closed graph theorem, is continuous. Hence

$$G \in L(\mathscr{E}'(O_2); \mathscr{E}(O_1) \simeq \mathscr{E}(O_1) \hat{\otimes} \mathscr{E}(O_1 = \mathscr{E}(\theta_1 \times \theta_2).$$

That is to say $G$ is $C^\infty$ in $O_1 \times O_2$. Since $O_1$ and $O_2$ are any two open set such that $O_1 \cap O_2 = \phi, G$ is a $C^\infty$ function outside the diagonal.

**Remark.** For a more detailed study, see Malgrange [12].

2. The extension of the mapping $G : Q' \to N$.

Under the hypothesis of theorem 12.1, $G$ defines an algebraic isomorphism of $Q' \cap \mathscr{E}$ onto $N \cap \mathscr{E}$. For if $f \in Q' \cap \mathscr{E}$ and $Gf = u\mathscr{E}N$ . Then $Au = f$. Hence by hypo-ellipticity of $A$ it follows that $u \in \mathscr{E}$. Conversely if $u \in N \cap \mathscr{E}$, then $Au = f \in Q' \cap \mathscr{E}$ and $Gf = u$. If we could apply closed graph theorem, then it would follow that $G$ is a topological isomorphism of $Q' \cap \mathscr{E}$ onto $N \cap \mathscr{E}$, the intersections being given as usual the upper bound topology. This is so, for example, if $Q$ is a Banach space. We have then the

**Theorem 12.2.** *If the closed graph theorem is applicable $G \in \mathscr{L}(Q' \cap \mathscr{E}, N \cap \mathscr{E})$ and is an isomorphism. Similarly $G^* \in \mathscr{L}(Q' \cap \mathscr{E}, N \cap \mathscr{E})$ and is an isomorphism.* **119**

## 12.2

Now we wish to consider the transpose $G$. Let us first consider the Dirichlet problem so that $V = H_o^m = Q \cdot \mathscr{D}(\Omega)$ is dense in $V$ and $N = V$. Hence by transposing $G$ we have an isomorphism ${}^tG : V' + \mathscr{E}' \to V' + \mathscr{E}'$. However since $\mathscr{D}(\Omega)$ is not always dense in $N$, the dual of $N$ is not a space of distributions and hence we do not consider directly the transport of $G$. Here the sums of locally convex topological vector space $A$ and $B$ subspaces of an algebraic vector space $F$ is topologized as follows : we consider the mapping $(a, b) \to a + b$ of $A \times B$ onto $A + B$ and put on $A + B$ the finest locally convex topology such that this mapping is continuous. If $Z$ is the kernel, then $A + B \approx A \times B/Z$.

**Theorem 12.3.** *Under the hypothesis of theorem 12.1, if further, for every $S \in Q' \cap \mathscr{E}'$ there exists a sequence $\varphi_n \in \mathscr{D}(\Omega)$ such that $\varphi_n \to S$ in $Q' \cap \mathscr{E}'$, then $G : Q' \to N$ can be extended by continuity to $G : Q' + \mathscr{E}' \to N + \mathscr{E}'$.*

Proof due to L. Schwartz (unpublished). We define first $G$ on $Q + \mathscr{E}'$. $G$ is already defined on $Q'$. By theorem 12.1 is very regular and is given by $\int G(x)_y \varphi(\varphi) dy \varphi(\varphi) \in Q'$ We cannot use this at once to define it on $\mathscr{E}'$, for then the integral itself is not in $\mathscr{E}'$. We proceed then as follows : Let $\alpha(x, y) \in \mathscr{E}(\Omega_x \times \Omega_y)$ be a function with support in a neighbourhood of diagonal and equal to 1 in another neighbourhood of diagonal. Let

$$H_{x,y} = \alpha(x, y) G_{x,y} = H(x)_y = H_x(y).$$

Hence $H_{x,y}$ is regular. It is easily seen, since $G_{x,y}$ is a $C^\infty$ function outside **120**
the diagonal that $A_x H_{x,y} - \delta(x)_y = L(x, y) \in \mathscr{E}(\Omega_x \times \Omega_y)$. Now let $T \in \mathscr{E}'(\Omega)$

with compact support $K$ say. Since $\int \rho(x)_y T = T$, we have $T = LT + AHT$ where

$$LT_x = \int L(x,y) T_y dy \in \mathscr{E}.$$
$$(H,T)_x = \int H_x(y) T_y dy \in \mathscr{D}'.$$

But the supports of the mapping $y \to H_x(y)$ and $y \to L(x,y)$ are contained in the support of $\alpha(x,y)$. By choosing the support of near enough the diagonal, we may have the support of $HT$ and $LT$ in any arbitrary neighbourhood of the support of $T$. Hence $HT \in \mathscr{E}'$ and $LT \in \mathscr{D}$. We define $\tilde{G}T = HT + GLT \in \mathscr{E}' + N$. We have to verify that if $\varphi \in \mathscr{D}(\Omega)$, then $\tilde{G}(\varphi) = G(\varphi)$. This follows as in general

$$A\tilde{G}T = AHT + AGLT = T - LT + LT = T.$$

If $\varphi \in \mathscr{D}(\Omega), \tilde{G}\varphi = H\varphi + GL\varphi \in \mathscr{D} +_N \subset_N$ and $A\tilde{G}\varphi = \varphi$. Since $A$ is an isomorphism, $\tilde{G}\varphi = G\varphi$.

Now $G$ is continuous from $\xi_k'(\Omega)$ into $\mathscr{E}' + N$. This proves that $\tilde{G}$ does not depend on $\alpha$ and $\tilde{G}$ can be extended to $\mathscr{E}'(\Omega)$ so that $\tilde{G} : \mathscr{E}'(\Omega) \to \mathscr{E}'(\Omega) + N$ is continuous.

We denote now $\tilde{G}$ be $G$ itself. $G$ defines then a continuous mapping $\theta$ from $Q' \times \epsilon' \to N + \epsilon'$ by $\theta(f,s) = Gf + GS$. If we prove $\theta$ is zero on the kernel of $Q' \times \mathscr{E}' \to Q' + \mathscr{E}'$, we shall have proved that $\theta$ defines a mapping $G$ of $Q' + \epsilon' \to N + \mathscr{E}'$ as required. Let $f \in Q'$ and $S \in \mathscr{E}'$ such that $f + S = 0$. Hence $f \in Q' \cap \mathscr{E}'$. By assumption (2), there exists $\varphi_n \in \mathscr{D}(\Omega)$ which converges in $Q'$ and $\mathscr{E}'$ to $f$. Hence - $\varphi_n$ converges to $S$ in $\mathscr{E}'$ and we have $Gf + GS = \lim(G\varphi_n + G(\varphi_n)) = 0$. **121**

**Corollary.** *We have $A.G = I_{Q'+\mathscr{E}'}$ and $G.A = I_{N+\mathscr{E}'}$ where $I_{Q'+\mathscr{E}'}$ and $I_{N+\mathscr{E}'}$ are the identity maps $Q' + \mathscr{E}'$ and $N + \mathscr{E}'$ respectively.*

For
$$A(Gf + GS) = f + AGS = f + S$$
and $G.A(u + S) = GAu + G\,A\,S = u + G\,A\,S$.

Since $GAS = S$ on $\mathscr{D}(\Omega)$ it is so on $\mathscr{E}'$. This proves the

**Theorem 12.4.** *Under the hypothesis of theorem 12.3, $A$ is a topological isomorphism from $N + \mathscr{E}'$ to $Q' + \mathscr{E}'$.*

The uniqueness of $G_x(y)$ is given by the following

**Theorem 12.5.** *Under the hypothesis of theorem 12.3, for $y \in \Omega$, $G_x(y)$ is defined as the solution of*

$$A_x(G_x(y) = \delta_x(y)$$
$$G_x(y) \in N + \mathscr{E}'.$$

Consider $y \to \delta_x(y) \in \mathscr{E}(\Omega_y, \mathscr{E}'(\Omega_x))$. Let $G(\delta_x(y)) = G_x(y)$. Then $y \in G_x(y)$ is a $C^\infty$ function from $\Omega \to N + \epsilon'$ we have $A_x(G_x(y)) = \delta_x(y)$, and $G_x(y)$ is the only distribution to verify the equation in $N + \mathscr{E}'$.

# Lecture 24

## 12.3 Study at the boundary.

**Definition 12.6.** *We say that $a(u, v)$ is regular at the boundary if $u \in N$ is such that $a(u, v) = 0$ for every $v \in V$ vanishing outside a neighbourhood of some compact $K \subset \Omega$ then $u$ is $C^\infty$ in a neighbourhood of $\Gamma$.* **122**

If $V = H_o^m(\Omega)$ or $H^m(\Omega)$ and $a(u, v) = \sum a_{pq} D^q \overline{D^p} v$ then the results on regularity at the boundary of § 9 state that under the conditions specified in theorem 9.1, $a(u, v)$ is regular at the boundary.

**Theorem 12.6.** *Under the hypothesis of theorem 12.3, if further $a(u, v)$ is regular at the boundary, then for fixed $y, G_x(y)$ is $C^\infty$ in a neighbourhood of $\Gamma$.*

This means in this case $G(x, y)$ for fixed $y$ is a usual function in a neighbourhood of $\Gamma$ satisfying usual boundary conditions.

By theorem 12.5, $G_x(y) = G(\delta_x(y)) = S + u$ with $S \in \mathscr{E}'$ and $u \in N$. Hence $Au + AS = \delta_x(y)$, i.e., $Au = \delta_x(y) - AS = T$ for $T \in \mathscr{E}'$. Let $K$ be the support of $T$, which on account of the splitting proved in theorem 12.1 can be taken in any arbitrary neighbourhood of $y$.

Let $\in V$ such that $v = 0$ in neighbourhood of $K$. Now by regularization we can find $\varphi_n$ vanishing on the support of $v$ such that $T = \lim \varphi_n$ in $\mathscr{E}'$. Then $\langle Au, \overline{v} \rangle = \lim \langle \varphi_n, \overline{v} \rangle$. Hence $a(u, v) = 0$ for all $v \in V$ vanishing in a neighbourhood of $K$. By regularity at the boundary of $K, u$ is $C^\infty$ in a neighbourhood of $\Gamma$. This completes the proof of the theorem.

# 13 Regularity at the Boundary Problems for General Decompositions.

## 13.1

Hitherto we considered boundary value problems for differential operators in the space $H^m(\Omega)$. For this we obtained $A$ as the operator associated with a form $a(u, v)$ on $H^m(\Omega)$ is the space of type $H(\{A\}, \Omega)$ where $\{A\}$ stands for the system $D^{(p)}$. More generally we consider now what problems are solved by considering $A$ as the operator associated with sesquilinear forms $a(u, v)$ on spaces $H(\{A_i\}, \Omega)$. That this solves now problems can be seen from the following example. Let $A = \Delta^2 + 1$. Consider on $H(\Delta, \Omega)$ the sesquilinear form $a(u, v) = (\Delta u, \Delta v)_o + (u, v)_o$. The operator $A$ associated with $a(u, v)$ is $\Delta^2 + 1.a(u, v)$ is $H(\Delta, \Omega)$ elliptic and hence for $f \in Q'$ where $Q$ is such that $H(\Delta, \Omega)$ is dense in $Q$, we have $u \in N$ such that $Au = f$. 123

Firstly we observe that $H^2(\Omega)$ may be contained in $H(\Delta, \Omega)$ strictly. For example, if $\Omega$ is a domain such that for a given $T \in H^{-2}_{\Omega}$, there exists $\in H^o$ such that $-\Delta U - T \in M^o$, i.e., for which Visik-Soboleff problem is soluble, then there exists $u \in H(\Delta, \Omega)$ such that $u \notin H^1(\Omega)$. For, let $T \in H^{-2}_{\Omega}$ be defined by $\langle T, \overline{\varphi}\rangle = \int_r f(\gamma\overline{\varphi})d\sigma$ for $f \in L^2(\Gamma)$ and such that $f \notin H^{\frac{1}{2}}(\Gamma) = \gamma(H^1(\Omega))$.

Now if $\cup$ is the corresponding solution, let $\underline{u}$ be its restriction to $\Omega$. We have $u \in L^2(\Omega)$ and $-\Delta u = u$ by § 10. Hence $u \in H(\Delta, \Omega)$ : If $u$ were in $H^1(\Omega)$, then $\gamma u = f$ would be in $H^{\frac{1}{2}}(\Gamma)$ contrary to the assumption. Another more elementary example can be given for a circle. It is easy to construct examples such that $u \in L^2$ and $\Delta u = 0$, but $\gamma u \notin H^1$. Thus Hadamard's classical example with $u = \sum a_n r^n e^{in\theta}$ with suitable $a_n$ is of this type.

However it is true that $H^2_o(\Omega) = H(\Delta, \Omega)$. For, by Plancherel's formula, the two norms are equivalent on $\mathscr{D}(\Omega)$. This raises in fact the question : To determine the conditions on $A_i$ and $\Omega$ so that $H(A; \Omega) = H^m(\Omega)$ where $m =$ highest of orders of the operators $A_i$. 124

Now we interpret formally the boundary value problems that are solved on $V \in H(\Delta, \Omega)$. We write first of all Green's formula

$$\int \Delta^2 u \cdot \overline{v} dx = \int_\Gamma \frac{\partial \Delta u}{\Delta n}.\overline{v} d\sigma - \int_\Gamma \Delta u.\frac{\partial \overline{v}}{\partial n} d\sigma + \int \Delta u.\overline{\Delta v} dx \tag{1}$$

a) Let $V = H_o(\Delta, \Omega)$. Since $H_2(\Delta, \Omega) = H^2_o(\Omega)$ no new problem is solved.

b) $V = H(\Delta, \Omega)$. Given $f \in Q'$ there exists $u \in N$ such that $a(u, v) = (f, v)_o$ for all $v \in V$. Further

$$(\Delta^2 + 1)u = f \tag{2}$$

Hence $\int_\Omega (\Delta^2 u - \overline{\Delta} v| dx + \int_\Omega u, \overline{v} dx = \int (f - \overline{v}) dx$

Using (1) and (2),

$$\int_\Gamma \frac{\partial \Delta u}{\partial n} . \overline{v} d\sigma + \rho \in \Delta u . \frac{\partial v}{\partial n} d = 0 \text{ for all } v \in V.$$

Formally this means $\Delta u = 0$, and $\dfrac{\partial \Delta u}{\partial n} = 0$.

c) $V$ = Closure in $H(\Delta, \Omega)$ of continuous function with $u = 0$. Then $u \in N$ implies $u_\Gamma = 0$ and $\Delta u_\Gamma = 0$.

d) $V$ = Closure in $H(\Delta, \Omega)$ of continuous function with $\dfrac{\partial u}{\partial n}_\Gamma = 0$ Then $u \in N$ implies $\dfrac{\partial u}{\partial n}_\Gamma = 0$ and $\dfrac{\partial \Delta u}{\partial n}_\Gamma = 0$.

However, problems in which $\dfrac{\partial u}{\partial n} = 0$ and $\Delta u = 0$ or $u = 0$ and $\dfrac{\partial \Delta u}{\partial n} = 0$ are not solved by this method.

# Lecture 25

Now we consider the regularity at the boundary of solutions so determined. **125**
This means we want to determine whether if $f \in H^k(\Omega)$ implies $u \in H^k(\Omega)$. The solution of this problem in full generality is not known though it would be desirable to know it, for in that case, for large $k$ weak solutions would be usual ones. We shall show that this is the case with certain kind of operators in $\Omega = \{x_n > 0\}$ with constant coefficients. Let

$$\Omega = \{x_n > 0\} \text{ and } Bu = D_y^m u + D_y^{m-1} \wedge_1 u + D_y^{m-2} \wedge_2 u + \cdots + \wedge_m u,$$

where $\wedge_m$ are partial differential in $x_1, \ldots, x_{n-1}$ operators with constant coefficients of order $\leq k$. Let $V = H(B, \Omega)$ and $a(u, v) = (Bu, Bv)_o + (u, v)_o$ be a sesquilinear form on $V, a(u, v)$ is V-elliptic. Let $Q = L^2(\Omega)$. If $f \in L^2(\Omega)$, by § 3, there exists $u \in N$ such that $a(u, v) = (f, v)_o$ for all $v \in V$. Further $(B^* B + 1)u = f$. To consider the regularity of $\underline{u}$ we consider first its tangential derivatives and next the normal ones.

**Proposition 13.1.** *Let $D_T^p f \in L^2$ for all $|p| \leq \mu$ for any positive integer $\mu$. Then $D_\tau^p u$ and $BD_\tau^p u$ are in $L^2$ for $|p| \leq \mu$. Let $v^h(x) = \dfrac{1}{h}(v(x+h) - v(x))$ where $h = (0, \ldots, h, \ldots, 0), h_n = 0$. Since $B$ is with constant coefficients, $v^h \in V$ if $v \in V$. Hence $a(u, v^h) = (f, v^h)_o$, i.e., $(Bu, Bv^h)_o + (u, v^h)_o = (f, v^h)_o$ for all $v \in V$. Since $B$ is with constant coefficients*

$$(Bh^h, Bv) + (u^{-h}, v)_o = (f^{-h}, v)_o. \tag{1}$$

*Putting $v = u^{-h}$,*

$$|Bu^{-h}|_o^2 + |u^{-h}|_o^2 \leq C |f^{-h}|_o |u^{-h}|_o.$$

If $D_\tau f$ in $L^2$, $Bu^{-h}$ and $u^{-h}$ are bounded in $L^2$ which means $BD_\tau u$ and $D_\tau u \in$ **126**

$L^2$. Letting $h \to 0$ in (1),

$$(B(D_\tau u), Bv)_o + (D_\tau u, v)_o = (D_\tau f, v)_o.$$

If now $D_t^{(\mu)} f \in L^2$ we can repeat the process proving that if $D_\tau^\mu f \in L^2$, then $D_\tau^\mu u$ and $BD_\tau^\mu u \in L^2$. Now we consider normal derivatives.

**Theorem 13.1.** *Let $\mu = m$. Under the hypothesis of proposition 13.1 $u \in H^m(\Omega)$.*

We use the

**Lemma 13.1.** *Let $\Omega = \{y > 0\}$. Consider $\mu$ such that*

$$(1) \quad \begin{cases} D_\tau^p u \in L^2 & \text{for } |p| \le k \\ D_y D_\tau^p u \in L^2 & \text{for } |p| \le k-1 \\ \quad \vdots & \quad \vdots \\ D_y^{k-1} D_\tau^p u \in L^2 & \text{for } |p| \le 1 \end{cases} \qquad \textit{and}$$

*(2)* $\quad D_y^m u \in H^{-m+k}$.

Then $D_y^k \in L^2$.

If we denote by $E(\Omega)$ the space defined by all the conditions above, the lemma means $E(\Omega) = H^k(\Omega)$. In general, i.e., for arbitrary $\Omega, H^k(\Omega) \subset E(\Omega)$. This lemma should hold for $\Omega$ with smooth boundary, though as yet it is not proved.

Assuming the lemma for a moment, we complete the proof of the theorem.

We have $Bu = D_y^m u + D_y^{m-1} + D_y^{m-1} \wedge_1 u + \cdots + \wedge_m u$.

From proposition 13.|, $D_\tau^p u \in L^2$ and $D_\tau^p Bu \in L^2$ for $|p| \le m$. Hence $\wedge_k u \in H^o(\Omega)$ and $D^{m-k} \wedge_k u \in H^{-m+1}$. Hence by lemma 13.1

$$D_y u \in L^2.$$

Next we consider $D_\tau Bu$ which is in $L^2$. 127

$$D_\tau Bu = D_y^m D_\tau u + D^{m-1} \wedge_1 D_\tau u + \cdots$$

This gives $D_y^m D_\tau u \in H^{-m+1}$. But $D^p (D_\tau u) L^2, |p| \le 1$. By lemma 13.1 again $D_y D_\tau u \in L^2$.

Proceeding similarly we obtain $D_y^k u \in L^2$. Hence $u \in H^k(\Omega)$. Now we prove if $\Omega = \{x_n > 0\}$, then $E(\Omega) = H^k(\Omega)$.

**Lemma 13.2.** *If $\Omega = R^n, H^k = E(R^n)$.*

Let by Fourier transformation $x_1, \ldots, x_{n-1}$ go into $\xi_1, \ldots, \xi_{n-1}$ and $x_n$ into $\xi_n$. Actually we need use only $D^p_\tau u \in L^2|p| \leq k$ and $D^m_y u \in H^{-m+k}(\Omega)$;

i.e.,

$$(1 + |\xi|^k)\hat{u} \in L^2 \quad \text{and} \quad \frac{|\eta|^m \hat{u}}{1 + |\xi|^{m-k} + |\eta|^{m-k}} \in L^2 \tag{1}$$

We may also assume $m > k$. We have to conclude that $|\eta|^k \hat{u} \in L^2$. Now we use the following inequality

$$|\eta|^k \leq c_1(1 + |\xi|^k) + c_2 \frac{|\eta|^m}{1 + |\xi|^{m-k} + |\eta|^{m-k}}.$$

For then $|\eta|^k \hat{u} \in L^2$ by (1). To prove the inequality we have to prove that

$$|\eta|^k + |\xi|^{m-k}|\xi|^k + |\eta|^m \leq c_1(1 + |\xi|^k)(1 + |\xi|^{m-k} + |\eta|^{m-k}) + c_2|\eta|^m.$$

Since $m > k, |\eta|^k \leq c_3|\eta|^m 1 + c$. Hence we need prove

$$|\eta|^m + |\xi|^{m-k}|\eta|^k \leq c_1(1 + |\xi|^k)(1 + |\xi|^{m-k} + |\eta|^{m-k}) + c_2|\eta|^m.$$

But $|\xi|^{m-k}|\eta|^k \leq \dfrac{|\eta|^{kp}}{p} + \dfrac{|\eta|^{(m-k)q}}{q} \left(ab \leq \dfrac{a^p}{p} + \dfrac{b^q}{q}, \dfrac{1}{p} + \dfrac{1}{q} = 1\right)$.

Hence $|\eta|^m + |\xi|^{m-k}|\eta|^k \leq |\eta|^m + |\xi|^m$ . This is trivially less than right hand side of the inequality.

**Lemma 13.3.** *If $\rho \in D_{L_\infty}(\Omega)$ and $u \in E(\Omega)$, then $\rho u \in E(\Omega)$. This follows from the definition of $E(\Omega)$ itself.*

**Lemma 13.4.** *$E(R^n)\Omega$, i.e., restrictions of $E(R^n)$ to $\Omega$ is dense in $E(\Omega)$.* **128**

*Let $u_t(x) = u(x', y + t)$ for $t > 0$. Let $v_t = u_t(x)|\Omega$. $v_\tau \to u$ in $E(\Omega)$. Let*

$$\rho(y) = \begin{cases} 0 & \text{for } y < -t \\ 1 & \text{for } y > 1 \\ 0 < y < 1 & \text{elsewhere.} \end{cases} \qquad \rho(y) \in \mathscr{D}_{L\infty}(\Omega)$$

*$\rho u_\tau \in V$ by lemma 13.3, and $(\rho u_\tau)\Omega = v_\tau$. The extension $\widetilde{\rho u}_c$ of $\delta u_t$ are in $E(R^n)$ and their restrictions $v_\tau$ are dense in $E(\Omega)$.*

**Lemma 13.5.** *$E(\Omega) \cap \mathscr{D}(\bar{\Omega})$ is dense in $E(\Omega)$.*

From lemma 13.4, we need prove that if $u = v_\Omega$ with $u \in E(R^n)$ then $u$ can be approached by function from $E(\Omega) \cap \mathscr{D}(\bar{\Omega})$. For $\cup = \lim \cup \times \rho_n$ with $\rho_n \to \delta$. The restrictions of $\cup * \rho_n \to u$.

To complete the proof of the lemma 13.1, then, we prove

**Lemma 13.6.** *Let $u \in E(\Omega) \cap \mathscr{D}(\overline{\Omega})$. Then*

$$U(x) = \begin{cases} u(x), & x_n \geq 0 \\ \lambda_1 u(x', \ldots, y) + \lambda_2 u(x', \ldots, -\frac{y}{2}) + \cdots + \lambda_n (x', \ldots, -\frac{y}{n}) x_n < 0 \end{cases}$$

*is in $E(R^n)$ for suitable $\lambda' s$.*

If we prove this since $E(\Omega) \cap \mathscr{D}(\overline{\Omega})$ is dense in $E(\Omega)$, we have a continuous mapping $\pi : E(\Omega) \to E(R^n) = H^k(R^n)$. Hence $E(\Omega) \subset H^k(\Omega)$, which proves that $H^k(\Omega) = E^k(\Omega)$. $\lambda' s$ are determined so that on $y = 0$, $\dfrac{\partial^R \cup}{\partial y}$ should be equal from above and below. A simple argument shows that $\cup \in \mathscr{E}(R^n)$.

# 14 Systems

## 14.1

We shall consider briefly systems. We shall denote in this article $H^m(\Omega)$ by $H^m$. Let $H^{(m)} = H^{m_1} \times \cdots \times H^{m_\nu}$ with the usual product Hilbert structures. In $H^{(m)}$ the closure of $(\mathscr{D}(\Omega))^\nu$ is $H_o^{m_1} \times \cdots \times H_o^m$. An element of $H^{(m)}(\Omega)$ we denote by $\vec{u} = (u_1, \ldots, u_\nu)$ with $u_i \in H^{mi}$. Let $V$ be such that $H_o^{(m)} \subset V \subset H^{(m)}$. Let 129

$$a(u, v) = \sum \int a_{pq,\lambda u}(x) D^q u_\mu D^p v_\lambda dx \ \lambda = \underset{\mu=1,\ldots,\nu}{1, \ldots, \nu}$$

be a sesquilinear form with $a_{pq,\lambda\mu} \in L^\infty(\Omega)$ and

$$a_{pq,\lambda u} = 0, \text{ if } |p| \geq m_\lambda \text{ or } |q| > m_\mu.$$

This last condition assures that $a(u, v)$ is continuous on $V \times V$.

If $a(\vec{u}, \vec{u}) \geq \alpha ||u||^2$ for all $u \in V$ and for $\alpha > 0$ and if $Q = L^2$ then from the general theory of § 3, there exists a space $N$ and an operator $A$ which establishes an isomorphism of $N$ onto $Q'$ so that

$$\langle A\vec{u}, \overline{\vec{\varphi}} \rangle = a(\vec{u}, \vec{\varphi}) \text{ for } \vec{\varphi} \in (\mathscr{D}(\Omega))^\nu,$$

i.e., $\langle A_1 \vec{u} \overline{\vec{\varphi}}_1 \rangle + \cdots + \langle A_\nu \vec{u} \overline{\vec{\varphi}}_1 \rangle = \sum \int a_{pq\lambda\mu} D^q u_\mu D^p_\lambda dx.$

Hence

$$A_\lambda(\vec{u}) = \sum_{p,q,\mu} (-1)^p D^p (a_{pq\lambda\mu} D^q u_\mu)$$

Hence the theory solves the differential systems

$$A_\lambda \vec{u} = \vec{f}.$$

The variety of boundary value problems solved is much larger; e.g., if $\nu = 2, m_1 = m_2$ and $V$ may be defined as consisting of $(u_1, u_2)$ such that $\gamma u_1 = \gamma u_2$.

## 14.2

We now give, following Nirenberg [13] an example which presents a little strange behaviour. We take $n = 2, \nu = 2, m_1 = 1, m_2 = 3$. We write $x_1 = x$ and $x_2 = y$, so that $V = H^1 \times H^3$. Let $L_1, M_2, L_3, M_3, N_z$ be the differential operators the order of which is equal to the index. Let **130**

$$\begin{aligned} a(u, v) = {} & (D_x u_1, D_x v_1) + (D_y u_1, D_y v_1) + (u_1, L_1^* v_1) + \\ & + (-D_y^3 u_2, D_y v_1) + (L_3 u_2, v_1) + (D_y u_1, D_y^3 v_2) + \\ & + (u_1, M_3^* v_2) + (D_x^3 u_2, D_x^3 v_2) + (D_y^3 u_2, D_y^3 v_2) + \\ & + 3(D_x^2 D_y u_2, D_x^2 D_y v_2) + 3(D_x D_y^2 u_2, D_x D_y^2 v_2) \\ & + (M_2 u_2, N_3 v_2). \end{aligned}$$

**Lemma 14.1.** *If $\Omega$ is three strongly regular,*

$$a(\vec{u}, \vec{v}) + \lambda(\vec{u}, \vec{v})$$

*elliptic for $\lambda$ large enough. The system $A$ associated with $a(u, v)$ is*

$$\begin{aligned} A_1(\vec{u}) &= -(D_x^2 + D_y^2) u_1 + L_1 u + D_y^4 u_3 + L_3 u_2 \\ A_2(\vec{u}) &= -\underline{D_y^4 u_1} + M_3 u_1 - (D_x^6 + D_y^6 + 3 D_x^2 D_y^4) u_2 + N_3 M_2 u_1. \end{aligned}$$

From the underlined term in the operator it would look like as if we have to assume $v_1 \in H^2$ and $u_2 \in H^2$. While existence and uniqueness in ensured in $H^1 \times H^3$ itself, i.e., we require four conditions on boundary while from the differential equation it looks as if we require five conditions. Further $a(\vec{u}, \vec{v})$ is not elliptic on $H^2 \times H^3$. This happens because in computation of the real part of $a(\vec{u}, \vec{v})$ the terms involving $D_y^2 u_1$, *e.g.*, $(-D_y^3 u_2, D_y v_1) + (D_y u_1, D_y^3 u_2) = e$, give zero real part as they are of the form $z - \overline{z}$. To see that the form is $H^1 \times H^3$ is straight forward by using the definition of strong regularity and the above remarks.

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