Lectures on Sheaf Theory

by

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Lecture 1

Sheaves. 1

Definition. *A sheaf* $\mathscr{S} = (S, \tau, X)$ *of abelian groups is a map* $\pi : S \xrightarrow{\text{onto}} X$*, where* S *and* X *are topological spaces, such that*

1. *π is a local homeomorphism,*

2. *for each $x \in X$, $\pi^{-1}(x)$ is an abelian group,*

3. *addition is continuous.*

That π is a local homeomorphism means that for each point $p \in S$, there is an open set G with $p \in G$ such that $\pi|G$ maps G homeomorphically onto some open set $\pi(G)$.

Sheaves were originally introduced by Leray in Comptes Rendus 222(1946)p. 1366 and the modified definition of sheaves now used was given by Lazard, and appeared first in the Cartan Sem. 1950-51 Expose 14.

In the definition of a sheaf, X is not assumed to satisfy any separation axioms.

S is called the sheaf space, π the projection map, and X the base space.

The open sets of S *which project homeomorphically onto open sets of* X *form a base for the open sets of* S*.*

Proof. If p is in an open set H, there exists an open G, $p \in G$ such that $\pi|G$ maps G homeomorphically onto an open set $\pi(G)$. Then $H \cap G$ is open, $p \in H \cap G \subset H$, and $\eta|H \cap G$ maps $H \cap G$ homeomorphically onto $\pi(H \cap G)$ open in $\pi(G)$, hence open in X. $\square$

2 *π is a continuous open mapping*

Proof. Continuity of π follows from the fact that it is a local homeomorphism, and openness follows from the result proved above. $\square$

The set $S_x = \pi^{-1}(x)$ is called the stalk of S at x. S_x is an abelian group. If $\pi(p) \neq \pi(q)$, $p + q$ is not defined.

S_x has the discrete topology.

Proof. This is a consequence of the fact that π is a local homeomorphism. $\square$

Let $S \times S$ be the cartesian product of the space S with itself and let $S + S$ be the subspace consisting of those pairs (p, q) for which $\pi(p) = \pi(q)$. *Addition is continuous* means that $f : S + S \to S$ defined by $f(p, q) = p+q$ is continuous. In other words, if $p, q \in S$ and $\pi(p) = \pi(q)$, then given an open G containing $p + q$, there exist open sets H, K with $p \in H$, $q \in K$ such that if $r \in H$, $s \in K$ and $\pi(r) = \pi(s)$, then $r + s \in G$. We may write this as $H + K \subset G$.

Proposition 1. *Zero and inverse are continuous.*

(i) Writing O_x for the zero element of the group S_x, *zero is continuous* means that $f : X \to S$, where $f(x) = O_x$, is continuous.

Proof. Let $x \in X$ and let G be an open set containing $f(x) = O_x$. Then there is an open set G_1 such that $O_x \in G_1 \subset G$ and $\pi|G_1$ is a
3 homeomorphism of G_1 onto open $\pi(G_1)$. Since $O_x + O_x = O_x$, and addition is continuous, there exist open sets H, K with $O_x \in H$, $O_x \in K$ such that $H + K \subset G_1$. Let $L = G_1 \cap H \cap K$, then L is open, $O_x \in L$ and $\pi|L$ is a homeomorphism of L onto open $\pi(L)$. Then $x = \pi(O_x) \in \pi(L)$ and if $y \in \pi(L)$ there exists $q \in L$ with $\pi(q) = y$. Then $q \in H$, $q \in K$ and hence $q + q \in G_1$. But $q \in G_1$, and $\pi|G_1$ is $1-1$; hence $q = S_y \cap G_1$. Therefore, since $q+q \in S_y \cap G_1$, $q+q = q$, hence $q = O_y$. Thus if $y \in \pi(L)$, $f(y) = O_y \in L$. Thus x is in open $\pi(L)$, with $f(\pi(L)) = L \subset G$. Hence f is continuous. (Incidentally we have proved that each O_x is contained in an open set which

consists of zeros only and which projects homeomorphically onto an open set of X.) □

(ii) Writing-p for the inverse of p in the group $S_{\pi(p)}$ *inverse is continuous* means that $g : S \to S$ where $g(p) = -p$ is continuous.

Proof. Let $p \in S$, and let G be an open set containing-p. Then there exists open L containing $O_{\pi(p)}$ and consisting of zeros only. Since $p+(-p) = O_{\pi(p)}$ and addition is continuous, there exist open H, K, with $p \in H$, $-p \in K$ and $H + K \subset L$. Hence if $q \in H$, $r \in K$, $\pi(r) = \pi(q)$, then $q + r = O_{\pi(q)}$, i.e. $r = -q$. We may assume that $\pi|H$ is a homeomorphism. Let

$$H_1 = (\pi|H)^1(\pi(H) \cap \pi(K \cap G)),$$

then $p \in H_1$, and since π is open, continuous, H_1 is open. Then if $q \in H_1$, there exists $r \in K \cap G$ with $\pi(r) = \pi(q)$; then $r = -q = g(q)$. **4**
Thus H_1 is open and $g(H_1) \subset G$. Hence g is continuous. □

Corollary 1. *Subtraction is continuous.*

i.i. $f : S + S \to S$, where $f(p,q) = p - q$ is continuous.

Corollary 2. *The set of all zeros is an open set.*

Example 1. If X is a topological space, and G is an abelian group furnished with the discrete topology, let $\mathscr{S} = (X \times G, \pi, X)$ where $\pi(x,g) = x$ and $(x, g_1) + (x, g_2) = (x, g_1 + g_2)$. Each stalk S_x is isomorphic to G. Axioms a) and c) are easily verified. This sheaf is called the *constant sheaf* associated with G.

Example 2. Form the constant sheaf $(A \times Z, \pi, A)$ where A is the square $\{(x,y) : 0 \leq x \leq 1, 0 \leq 0 \leq y \leq 1\}$, and Z is the group of integers. Then identify $(x, 0)$ with $(1 - x, 1)$ in A to get a Möbius band X, and identify $(x, 0, n)$ with $(1 - x, 1, -n)$ in $A \times Z$ to get S. The resulting sheaf $\mathscr{S} = (S, \pi, X)$ is the sheaf of "twisted integers" over the Möbius band. Each S_x is isomorphic to the group of integers.

Example 3. Let X be the sphere of complex numbers. Let S_x be the additive group of function elements at x, each function element being a power series converging in some neighbourhood of x. Let $S = \cup_x S_x$
5 and define $\pi : S \rightarrow X$ by $\pi(S_x) = x$. If p is a function element, a neighbourhood of p in S is defined by analytic continuation. Then $\mathscr{S} = (S, \pi, X)$ is the sheaf of *analytic function elements*. Each component (maximal connected subset) of S is a Riemann surface without branch points. The sheaf space S is Hansdorff.

Lecture 2

Example 4. Let $S : \{(x, y) : x^2 + y^2 = 1, x < 1\}$ together with the group 6
of integers $Z\}$. The topology on the former set is the usual induced topology, and the neighbourhoods for an integer $n \in Z$ are given by $G_a(n) = \{n, (x, y) : x^2 + y^2 = 1, a < x < 1\}$; $X : \{(x, y) : x^2 + y^2 = 1\}$ and let $\pi : S \to X$ be defined by $\pi(n) = (1, 0)$, $\pi(x, y) = (x, y)$. $\pi^{-1}(1, 0)$ is the group Z, for other pints $(x, y) \in X$, $\pi^{-1}(x, y) = (x, y)$ is regarded as the zero group. It is easily verified that $\mathscr{S} = (s, \pi, X)$ is a sheaf. Here S is locally euclidean, and has a countable base. The set of all zeros is open (Corollary 2) and compact but not closed; its closure is not compact, S is T_1 but not Hausdorff though X is Hausdorff.

Exercise. If X is a T_o or a T_1 space, space, so is S

So far we have only defined sheaves of abelian groups. It is now quite clear how we can extend the definition to the case where the stalks are any algebraic systems.

A sheaf of rings is a local homeomorphism $\pi : S \to X$ such that each $\pi^{-1}(x)$ is a ring and addition and multiplication are continuous, i.e

$$f : S + S \to S \text{ where } f(p, q) = p + q,$$
$$g : S + S \to S \text{ where } g(p, q) = p \cdot q,$$

are continuous.

The sheaf of function elements (Example 3) where multiplication 7
of two function elements in the same stalk is defined to be the usual multiplication of power series is a sheaf of rings.

In the sheaf of twisted integers (Example 2) each S_x is isomorphic to the ring Z, but this sheaf is *not* a sheaf of rings.

A sheaf of rings with unit is a local homeomorphism $\pi : S \to X$ such that each $\pi^{-1}(x)$ is a ring with unit element 1_x and addition, multiplication and unit are continuous; i,e.,

$$\begin{aligned} f &: S+S \to S, f(p,q) = p+q, \\ g &: S+S \to S, f(p,q) = p \cdot q, \\ &h : X \longrightarrow S, h(x) = 1_x \text{ are continuous.} \end{aligned}$$

Example 5. Let A be the ring with elements 0, 1, b, c; where the rules of addition and multiplication are given by

$$\begin{aligned} &1+1 = b+b = c+c = 0; \\ &1+b = c, 1+c = b, b+c = 1; \\ &b^2 = b, c^2 = c, bc = cb = 0 \end{aligned}$$

[The ring A may be identified with the ring of functions defined on a set of two elements with values in the field Z_2.]

Let $X : \{x, k \leq x \leq 1\}, S$: subspace of $X \times A$ obtained by omitting the points $(k, 1)$, (k, b); and let $\pi : S \to X$, be defined by $\pi(x, a) = x$. Addition and multiplication in a stalk are defined by

$$\begin{aligned} (x, a_1) + (x, a_2) &= (x, a_1 + a_2), \\ (x, a_1) \cdot (x, a_2) &= (x, a_1 \cdot a_2). \end{aligned}$$

8 Then $\mathscr{S} = (S, \pi, X)$ is a sheaf of rings, each S_x is a ring with unit, but $\mathscr{S}$ is *not* sheaf of rings with unit.

A sheaf of (unitary left) $\mathfrak{a}$-modules, where $\mathfrak{a} = (A, \tau, X)$ is a sheaf of rings with unit, is a local homeomorphism $\pi : S \to X$ such that each $\pi^{-1}(x)$ is a unitary left A_x-module and addition, and multiplication by elements of A_x (for each x) are continuous; i.e.,

$$\begin{aligned} f &: S+S \to S, & f(p,q) &= p+q, \\ g &: A+S \to S, & g(a,p) &= ap \end{aligned}$$

are continuous.

$A+S$ is the subspace of $A \times S$ consisting of all pairs (a, p) for which $\tau(a) = \pi(p)$.

[If R is a ring with unit element, we say that M is a *unitary, left* R-module if it is a left R-module, and if $1 \cdot m = m$ for each $m \in M$.]

Example 6. Let $\mathfrak{a}$ denote the sheaf of function elements on the complex sphere X. Let S_x consist of all (p, q) of function elements at x. Let $S = \bigcup_{x \in X} S_x$. A neighbourhood of (p, q) is defined by analytic continuation of p and q. In each S_x addition is defined as $(p_1, q_1) + (p_2, q_2) = (p_1 + q_2, q_1 + q_2)$; and if a is a function element at x, define multiplication as $a(p, q) = (a.p, a.q)$. Define $\pi : S \to X$, by $\pi(p, q) = x$ if p, q are function elements at x. Then (S, π, X) is a sheaf of a-modules.

Any sheaf $\mathfrak{a}$ of rings with unit can be regarded as a sheaf of a-modules; the product ap for $a \in A_x$, $p \in A_x$ being defined as the product ap in A_x. 9

A sheaf of B-modules where B is a ring with unit element is a local homeomorphism $\pi : S \to X$ such that $\pi^{-1}(x)$ is a unitary left B-module and addition, and multiplication by elements of B are continuous; i.e.,

$$f : S + S \to S, \qquad f(p, q) = p + q,$$
$$g_b : S \to S, g_b(p) = b.p \text{ for each } b \in B$$

are continuous.

Let $\mathscr{S} = (S, \pi, X)$ be a sheaf of B-modules. This is equivalent to saying that $\mathscr{S}$ is a sheaf of $\mathbb{B}$-modules, where $\mathbb{B}$ is the constant sheaf $(X \times B, \tau, X)$.

Proof:

B is a ring with unit element.	$\mathfrak{B}(X \times B, \tau, X)$ is a constant sheaf.
$\mathscr{S} = (S, \pi, X)$ is a sheaf of abelian groups.	
1) $\mathscr{S}$ is a sheaf of B-modules. This means that $b \cdot p$ is defined such that S_x is a uni- left B-module.	1) $\mathscr{S}$ is a sheaf of $\mathfrak{B}$-modules. This means that $(\pi(p), b) \cdot p$ is defined such that S_x is a unitary left $x \times B$ module.
2) Addition is continuous.	2) Addition is continuous.
3) Multiplication is continuous means that $h : B \times S \to S$ $h(b, p) = b \cdot p$ is continuous.	3) Multiplication is continuous means that $g : (X \times B) + S \to S$ $g(\pi(p), b, p) = b.p$ is continuous.

10 To prove the assertion, it is enough to show that the continuity of h is equivalent to the continuity of g. To do this, define $\phi : B \times S \to (X \times B) + S$ as $\phi(b, p) = (\pi(p), b, p)$, then $g\phi = h.\phi$ is clearly 1-1. We show that ϕ is a homeomorphism and the result will follow from this. A base for $(X \times B) + S$ is formed by the sets $(U \times b) + G$ where G projects homeomorphically onto $\pi(G)$.

$$\begin{aligned}(U \times b) + G &= (V \times b) + G \text{ where } V = U \cap \pi(G)\\ &= (V \times b) + H \text{ where } H = (\pi|G)^{-1}V\\ &= (\pi(H) \times b) + H\\ &= \phi(b \times H).\end{aligned}$$

Since the sets $b \times H$ form a base for $B \times S$, it follows that ϕ is a homeomorphism.

Thus we may identify sheaves of $(X \times B, \tau, X)$-modules with sheaves if B-modules. By abuse of language, we write B for the ring as well as for the constant sheaf $(X \times B, \tau, X)$.

Example 7. Let C be the ring of complex numbers, $\mathscr{S} = (S, \pi, X)$ be the sheaf of function elements on the complex sphere X and for $c \in C$, $p \in S$ define $c \cdot p$ to be the usual product of a complex number with a
11 power series. Then $\mathscr{S}$ becomes a sheaf of C- modules.

Example 8. Let C be the ring of complex numbers, $\mathscr{S} = (S, \pi, X)$ be the sheaf of function elements on the complex sphere X. For $c \in C$ and $(p, q) \in S$ define $c(p, q) = (c \cdot p, c \cdot q)$.

Then $\mathscr{S}$ becomes a sheaf of C-modules.

Example 9. Let $\mathscr{S} = (S, \pi, X)$ be any sheaf of abelian groups and let Z be the ring of integers. For $n \in Z$ and $p \in S$ define $n \cdot p = p + \cdots + p$ (n times) if $n > 0$, $n \cdot p = -(-n)p$ if $n < 0$, and $0 \cdot p = 0_{\pi(p)}$. Thus $\mathscr{S}$ may be considered as a sheaf of Z-modules.

Thus sheaves of rings with unit, sheaves of B-modules and sheaves of abelian groups can be considered as special cases of sheaves of a-modules.

Lecture 3

Sections.

Definition. *A section of a sheaf* (S, π, X) *over an open set* $U \subset X$ *is a map* $f : U \to S$ *such that* $\pi \cdot f = 1|U$ *where* $1|U$ *denotes the identity function on* U. *(A map is a continuous function).* 12

By abuse of language, the image $f(U)$ is also called a section.

For each open set $U \subset X$ *the function* $f : U \to S$, *where* $f(x) = 0_x$ *is a section.*

Proof. Zero is continuous, and $\pi(0_x) = x$. □

This section will be called the 0-section (zero section).

Proposition 2. *If* $\mathscr{S} = (S, \pi, X)$ *is a sheaf of abelian groups, the set of all sections of* $\mathscr{S}$ *over a non-empty open set* U *forms an abelian group* $\Gamma(U, \mathscr{S})$. *If* $\mathscr{S}$ *is a sheaf of sings with unit,* $\Gamma(U, \mathscr{S})$ *is ring with unit element. If* $\mathscr{S}$ *is a sheaf of* $\mathfrak{a}$*-modules,* $\Gamma(U, \mathscr{S})$ *is a unitary left* $\Gamma(U, \mathfrak{a})$*-module. If* $\mathscr{S}$ *is a sheaf of* B*-modules,* $\Gamma(U, \mathscr{S})$ *is a unitary left* B*-module.*

Note. The operations are the usual ones for functions. If $f, g \in \Gamma(U, \mathscr{S})$, $a \in \Gamma(U, \mathfrak{a})$, $b \in B$, define

$$(f + g)(x) = f(x) + g(x), (fg)(x) = f(x) \cdot g(x),$$
$$(af)(x) = a(x) \cdot f(x), (bf)(x) = b \cdot f(x).$$

The zero element of $\Gamma(U, \mathscr{S})$ is the 0-section over U. If $\mathscr{S}$ is a sheaf of rings with unit, the unit element of $\Gamma(U, \mathscr{S})$ is the unit section $1 : U \to S$ where $1(x) = 1_x$.

13 *Proof.* The proposition follows from the fact that addition zero, inverse, unit and multiplication (ring multiplication as well as scalar multiplication) are continuous. ◻

Remark. If $\mathfrak{a} = (X \times B, \tau, X)$ is a constant sheaf of rings with units, for each open set $U \subset X$ we can identify B with the ring of constant sections f_b, $b \in B$ where $f_b(x) = (x, b)$ over U. Then $B \subset \Gamma(U, \mathfrak{a})$ is a subring, and by restricting the ring of scalars, each $\Gamma(U, \mathfrak{a})$-module becomes a B-module. (B need not be the whole of $\Gamma(U, \mathfrak{a})$).

Abuse of ϕ. If $\mathscr{S}$ is a sheaf of $\mathfrak{a}$-modules, we agree that the unique section over the empty set ϕ is the 0-section, and the set $\Gamma(\phi\mathscr{S}) = 0$.

Example 6. If $\mathscr{S} = (S, \pi, X)$ is the sheaf of function elements over the complex sphere X, $\Gamma(X, \mathscr{S})$ can be identified with the ring of functions, analytic in U. Then $\Gamma(X, \mathscr{S})$ is the ring of functions, analytic everywhere, hence is isomorphic to the ring of complex numbers C.

Note. Usually a sheaf $\mathscr{S} = (S, \pi, X)$ may be interpreted as describing some local property of the space X; then $\Gamma(X, \mathscr{S})$ gives the corresponding global property.

A section $f : U \to S$ is an open mapping

Proof. This is proved using the fact that f is continuous and that π is a local homeomorphism. ◻

14 We can now characterise the sections of S.

The necessary and sufficient condition that a set $G \subset S$ is a section $f(U)$ over some open set $U \subset X$ is that G is open and $\pi|G$ is a homeomorphism.

Proof. The sufficiency is easy to prove. To prove the necessity let f be a section over U, then since f is open $f(U)$ is open. Since $f : U \to f(U)$ is 1-1, open, continuous, $\pi|f(U) : f(U) \to U$ *is a homeomorphism* ◻

We have shown that if f is a section over U then f is a homeomorphism of U onto $f(U)$.

The sections $f(U)$ form a base for the open sets of S.

Proof. We have already proved that the open sets of S which project homeomorphically onto open sets of X form a base for the open sets of S □

The intersection $f(U) \cap g(V)$ of two sections is a section.

Proof. $f(U) \cap g(V)$ is open and projects homeomorphically onto an open set of X since each of $f(U)$ and $g(V)$ has this property. □

If $f : U \to S$ is a section, the set $\{x : f(x) = 0_x\}$ is open in U.

Proof. $\{x : f(x) = 0_x\} = \pi(f(U) \cap 0(U))$ (0 denotes the 0-section over U), hence is open in U. □

Definition. *If $f : U \to S$ is a section, the support of f, denoted as supp f, is the set $\{x : f(x) \neq 0_x\}$. This set is closed in U. If f is a section over* 15
X, supp f is a closedsubset of X

Note. Since the sections of S form a base for the open sets of S, the topology of $S = \cup S_x$ can be described by specifying the sections. (See appendix at the end of the lecture).

Let $\mathfrak{a} = (A, \tau, X)$ be a sheaf of rings with unit, and let $\mathscr{S} = (S, \pi, X)$ and $\mathscr{R} = (R, \rho, X)$ be sheaves of $\mathfrak{a}$-modules (all over the same base space X).

Definition. *A homomorphism $h : \mathscr{S} \to \mathscr{R}$ is a map $h : S \to R$ such that $\rho \cdot h = \pi$ and its restriction $h|S_x = h_x : S_x \to R_x$ is an A_x homomorphism for each $x \in X$*

This definition includes as a special case the definition of homomorphisms of sheaves of B-modules and sheaves of abelian groups.

If $h : \mathscr{S} \to \mathscr{R}$ is a homomorphism, the image of each section is a section.

Proof. If $f : U \to S$ is a section, then $hf : U \to R$, the image of the section f defined by $(hf)(x) = h(f(x))$, is continuous and $\rho(hf) = \Pi \cdot f = 1|U$.

(1) *h is an open mapping.*

(2) *h is a local homeomorphism.*

□

Proof. **(1)** If $G \subset S$ is open, then G is a union of sections, hence $h(G)$ is a union of sections of R, hence is open

16 **(2)** Each $p \in S$ is contained in some section $f(U)$. So $hf(U)$ is a section in R and $h|f(U) = hf\pi|f(U)$, and each of $\pi|f(U) : f(U) \to U$ and $hf : U \to hf(U)$ is a homeomorphism. □

Appendix.

A sheaf may be described by specifying its sections as follows: Suppose that we are given a space X and mutually disjoint abelian groups S_x, one for each point $x \in X$. Also suppose that we are given a family $\sum = \{s\}$ of functions with domain open in X and values in $\cup_x S_x$, $s : \mathrm{dom}(s) \to \cup_x S_x$, such that, if $x \in \mathrm{dom}(s)$, $s(x) \in S_x$. Suppose further that

(i) the images for all $s \in \sum$ cover $\cup S_x$,

(ii) if $s_1(x)$, $s_2(x)$ are defined then, for some open U with $x \in\subset \mathrm{dom}(s_1) \cap \mathrm{dom}(s_2)$, $(s_1 + s_2)|U \in \sum$,

(iii) if $s(x) = 0_x$ then, for some open U with $x \in U \subset \mathrm{dom}(s) s(U)$ consists entirely of zeros.

If $S = \cup S_x$ with $\{s(U)\}$, for all $s \in \sum$ and open $U \subset \mathrm{dom}(s)$, as base for open sets and if $\pi(p) = x$ for $p \in S_x$ then (S, π, X) is a sheaf.

Proof. Let $p \in s(U) \cap s_1(U_1)$ and let $x = \pi(p)$. By (i) there exists s_2 with $s_2(x) = -p$, by (ii) there exists a neighbourhood V of x with $(s + s_2)|V \in \sum$ and by (iii), since $(s + s_2)x = p - p = 0_x$ there is a smaller neighbourhood V' of x with $(s + s_2)(V')$ consisting of zeros.

Similarly there is a neighbourhood V_1' of x with $(s_1 + s_2)(V_1')$ consisting of zeros. $\square$

Let $W = V' \cap V_1'$, then $(s + s_1 + s_2)|W = s|W = s_1|W$. Then $s(W)$ is in the proposed base and $p \in s(W) \subset s(U) \cap s_1(U_1)$. Therefore the axioms for a base are satisfied. **17**

Then $s : U \to S$ is continuous. For if $x \in \mathrm{dom}(s)$ and $p = s(x)$, any neighbourhood G of p contains a neighbourhood $s_1(U_1)$ and again there is an open W with $p \in s(W) = s_1(W) \subset s(U_1) \subset G$. Hence $s : U \to s(U)$ is a homeomorphism, since s is clearly 1-1 and open. Then $\pi|s(U)$ is the inverse homeomorphism and its image U is open. Thus π is a local homeomorphism.

Addition is continuous. For if $p, q \in S_x$, with $p+q \in s_2(U_2)$ suppose $p \in s(U)$ and $q \in s_1(U_1)$. By (ii) there is a neighbourhood V of x with $(s_1+s_2)|V \in \sum$. Then $p+q \in s_2(U_2) \cap (s+s_1)(V)$ and hence for some W, with $x \in W \subset U_2 \cap V$, $s_2|W = (s + s_1)|W$. Thus, if $r \in s(W)$, $t \in s_1(W)$ and $\pi(r) = \pi(t)$, then $s(r) + s_1(t) = s_2(t) \in s_2(U_2)$.

Lecture 4

If (S', π', X) and (S, π, X) are two sheaves of a-modules with $S' \subset S$ and **18**
if the inclusion map $i : S' \to S$ is a homomorphism, then S' is an open subset of S since i is an open map; further the topology on S' is the one induced from S. Then $\pi' = \pi \cdot i = \pi | S'$ and since $i | S'_x : S'_x \to S_x$ is a homomorphism, S'_x is a sub - A_x - module of S_x.

This suggests the following definition of a subsheaf.

Definition. $(S', \pi | S', X)$ *is called a subsheaf of* (S, π, X) *if* S' *is open in* S *and, for each* x, $S'_x = S' \cap S_x$ *is a sub -* A_x*- module of* S_x.

A subsheaf is a sheaf and the inclusion map i is a homomorphism.

Proof. For each $p \in S'$ there exists an open set G, $p \in G \subset S$, with $\pi | G$ a homeomorphism. Then $G \cap S'$ is open in S' and $(\pi | S') | G \cap S' = \pi | G \cap S'$ is a homeomorphism. S'_x is an A_x-module and the operations which are continuous in S are continuous in the subspace S'. Therefore $(S', \pi | S', X)$ is a sheaf. ◻

Since $i : S' \to S$ is a map, and $\pi \cdot i = \pi | S'$ and $i | S'_x : S'_x \to S_x$ is the inclusion homomorphism of the submodule S'_x, it follows that i is a homomorphism.

The set of all zeros in S *is a subsheaf of* S.

Proof. The set of zeros is open in S, and 0_x is a sub-A_x-module of S_x. ◻

This sheaf is called the 0-sheaf (zero sheaf) and is usually identified **19**
with the constant sheaf $0 = (X \times 0, \pi, X)$.

If $h : \mathscr{S} \to \mathscr{R}$ is a homomorphism of sheaves, the set S' of $p \in S$ such that $h(p) = 0_{\pi(p)}$ forms a subsheaf $\mathscr{S}'$ of $\mathscr{S}$ called the kernel of $h(\mathscr{S}' = \ker h)$, and the image set $S'' = h(S)$ forms a subsheaf of $\mathscr{R}$ called the image of $h(\mathscr{S}'' = imh)$.

Proof. (1) Since h is continuous and 0 (0 denotes the set of zeros in R) is open, therefore $S' = h^{-1}(0)$ is open. Each $S'_x = S' \cap S_x$ is the kernel of $h|S_x : S_x \to R_x$, hence S'_x is a sub-A_x-module of S_x.

(2) Since h is an open map, S'' is open. Each $S''_x = S''_x \cap R$ is the image of the homomorphism $h|S_x : S_x \to R_x$, hence S''_x is a sub - A_x - module of R_x.

□

Definition. *A homomorphism $h : \mathscr{S} \to \mathscr{R}$ is called a monomorphism if* $\ker h = 0$, *an epimorphism if* $\operatorname{im} h = R$, *and and isomorphism if* $\ker h = 0$ *and* $\operatorname{im} h = R$.

Definition. *A sequence*

$$\cdots \to \mathscr{S}_{j-1} \xrightarrow{h_j} \mathscr{S}_j \xrightarrow{h_{j+1}} \mathscr{S}_{j+1} \to \cdots$$

of homomorphisms of sheaves is called exact at $\mathscr{S}_j$ *if* $\ker h_{j+1} = \operatorname{im} h_j$; *it is called* exact *if it is exact at each* $\mathscr{S}_j$.

If $h : \mathscr{S} \to \mathscr{R}$ is a homomorphism, the sequence

$$0 \to \ker h \xrightarrow{i} \mathscr{S} \xrightarrow{h} \operatorname{im} h \to 0$$

20 *is exact*.

Here $i : \ker h \to \mathscr{S}$ is the inclusion homomorphism, and $h' : \mathscr{S} \to \operatorname{im} h$ is defined by $h'(p) = h(p)$. It is a homomorphism. The other two homomorphisms are, of course, uniquely determined.

Proof. The statement is the composite of the three trivial statements:

(i) $i : \ker h \to \mathscr{S}$ is a monomorphism,

(ii) $\ker h' = \ker h$,

(iii) $h' : \mathscr{S} \to imh$ is an epimorphism.

$\square$

Definition. *A* directed set $(\Omega, <)$ *is a set* Ω *and a relation* $<$*, such that*

1) $\lambda < \lambda \quad (\lambda \in \Omega)$,

2) if $\lambda < \mu$ *and* $\mu < \nu$ *then* $\lambda < \nu (\lambda, \mu, \nu \in \Omega)$, 21

3) if $\lambda, \mu \in \Omega$*, there exists a* $\nu \in \Omega$ *such that* $\lambda < \nu$ *and* $\mu < \nu$.

That is, $<$ is reflexive and transitive and each finite subset of Ω has an upper bound. (We also write $\mu > \lambda$ for $\lambda < \mu$).

Example. Let Ω be the family of all compact subsets C of the plane let $C < D$ mean $C \subset D$. Ω is then a directed set.

Definition. *A* direct system $\{G_\lambda, \phi_{\mu\lambda}\}$ *of abelian groups, indexed by a directed set* Ω*, is a system* $\{G_\lambda\}_{\lambda\in\Omega}$ *of abelian groups and a system* $\{\phi_{\mu\lambda} : G_\lambda \to G_\mu\}_{\lambda<\mu}$ *of homomorphisms such that*

(i) $\phi_{\lambda\lambda} : G_\lambda \to G_\lambda$ *is identity,*

(ii) if $\lambda < \mu < \nu$, $\phi_{\nu\mu}\phi_{\mu\lambda} = \phi_{\nu\lambda} : G_\lambda \to G_\nu$.

Thus if $\lambda < \mu < \nu$ and $\lambda < k < \nu$, then $\phi_{\nu\mu}\phi_{\mu\lambda} = \phi_{\nu k}\phi_{k\lambda}$.

The definition of a direct system will be the same even when the G'_λ s are any algebraic systems.

Definition. *Two elements* $a \in G_\lambda$ *and* $b \in G_\mu$ *of* $\cup_{\lambda\in\Omega} G_\lambda$ *are said to be* equivalent $(a\sigma b)$ *if for some* ν, $\phi_{\nu\lambda} a = \phi_{\nu\mu} b$.

This relation is easily verified to be an equivalence relation. The equivalence class determined by a will be denoted by (a).

We will now define addition of equivalence classes. If (a) and (b) are equivalence classes, $a \in G_\lambda$, $b \in G_\mu$, choose a $\nu > \lambda$ and $> \mu$ and define $(a) + (b) = (\phi_{\nu\lambda} a + \phi_{\nu\mu} b)$.

$$\begin{tikzcd}
G_\lambda \arrow[d] \arrow[drr] & & G_\mu \arrow[d] \arrow[dll] \\
G_\nu \arrow[dr] & & G_{\nu_1} \arrow[dl] \\
& G_{\nu_2} &
\end{tikzcd}$$

To show that this does not depend on the choice of ν choose $\nu_1 > \lambda$ and $> \mu$, let $\nu_2 > \nu$ and $> \nu_1$. Then

$$\begin{aligned}
\phi_{\nu_2\nu}(\phi_{\nu\lambda} a + \phi_{\nu\mu} b) &= \phi_{\nu_2\lambda} a + \phi_{\nu_2\mu} b \\
&= \phi_{\nu_2\nu_1}(\phi_{\nu_1\lambda} a + \phi_{\nu_1\mu} b),
\end{aligned}$$

hence $\phi_{\nu\lambda} a + \phi_{\nu\mu} b \sim \phi_{\nu_1\mu} b$. Clearly the class $(\phi_{\nu\lambda} a + \phi_{\nu\lambda} b)$ is independent of the choice of a and b.

If $\{G_\lambda, \phi_{\mu\lambda}\}$ is a direct system of abelian groups, the equivalence classes form an abelian group G called the direct limit of the system.

Proof. That G is an abelian group follows easily from the fact that each G_λ is an abelian group. □

22 The zero element of G_λ is the class containing all the zeros of all the groups G_λ.

Clearly, if each G_λ is a ring, then G is a ring, and similarly for any other algebraic system.

The function $\phi_\lambda : G_\lambda \to G$, where $\phi_\lambda a = (a)$ is a homomorphism and if $\lambda < \mu$, $\phi_\mu \phi_{\mu\lambda} = \phi_\lambda$.

Proof.

$$\begin{aligned}
&\phi_\lambda(a + b) = (a + b) = (a) + (b) = \phi_\lambda a + \phi_\lambda b, \\
&\phi_\mu(\phi_{\mu\lambda} a) = (\phi_{\mu\lambda} a) = (a) = \phi_\lambda a.
\end{aligned}$$

□

Example. Let $(N, \leq)$ be the directed set of natural numbers. For each natural number n let $G_n = Z$ and if $n \leq m$, let $\phi_{mn} : G_n \to G_m$ be defined by $\phi_{mn} a = \dfrac{m!a}{n!}$. The direct limit is isomorphic to the group of rational numbers.

Example. Let $(N, \leq)$ be as before. For each natural number n, let G_n be the group of rational numbers modulo 1 and if $n \leq m$ let $\phi_{mn} : G_n \to G_m$ be defined by $\phi_{mn} a = \dfrac{m!a}{n!}$. The direct limit is zero.

If $\{G_\lambda, \phi_{\mu\lambda}\}$ *is a direct system of abelian groups and if* $\{f_\lambda : G_\lambda \to H\}$ *are homomorphisms into an abelian group* H *with* $f_\mu \phi_{\mu\lambda = f_\lambda}$, *there is a unique homomorphism* $f : G \to H$ *of the limit group* G *such that* $f \phi_\lambda = f_\lambda$.

Proof. Since $f_\mu \phi_{\mu\lambda} = f_\lambda$, all elements of an equivalence class have the same image in H. Then f is uniquely determined by $f(\phi_\lambda a) = f_\lambda a$. ◻

For any two equivalence classes, choose representatives a_1, b_1 in some G_ν. Then **23**

$$\begin{aligned} f(\phi_\nu a_1 + \phi_\nu b_1) &= f\phi_\nu(a_1 + b_1) \\ &= f\phi_\nu a_1 + f\phi_\nu b_1 \end{aligned}$$

since $f\phi_\nu = f_\nu$ is a homomorphism. Thus f is a homomorphism.

Lecture 5

Definition . *Let S be a unitary left A-module and R a unitary left B-module; a homomorphism $\phi : (A, S) \to (B, R)$ is a pair $(\phi' : A \to B, \phi'' : S \to R)$ where* 24

$$\phi'(a+b) = \phi'(a) + \phi'(b), \phi'(ab) = \phi'(a)\phi'(b), \phi'(1) = 1, a, b \in A$$
$$\phi''(s+t) = \phi''(s) + \phi''(t), \phi''(as) = \phi'(a)\phi''(s), s, t \in S, a \in A.$$

[**Remark**. For a homomorphism $\phi : (A, S) \to (B, R)$, we sometimes write only ϕ for both ϕ' and ϕ''.]

Direct systems and direct limits are defined for arbitrary algebraic systems. Thus if Ω is a directed set and $\{A_\lambda, S_\lambda, \phi_{\mu\lambda}\}\lambda$, $\mu \in \Omega$, where $\phi_{\mu\lambda} = (\phi'_{\mu\lambda : A_\lambda \to A_\mu, \phi''_{\mu\lambda} : S_\lambda \to S_\mu})$, is a direct system of unitary modules, the direct limit consists of a ring A with unit element, and a unitary left A-module S, and there are homomorphisms $\phi_\lambda : (A_\lambda, S_\lambda) \to (A, S)$ such that, if $\lambda < \mu, \phi_\mu \phi_{\mu\lambda} = \phi_\lambda$.

The unit element of A is the equivalence class containing all the unit elements of all A_λ, and the zero of A is the class containing all the zeros. Thus, if $1_\lambda = 0_\lambda$ in some A_λ, $1 = 0$ in A and hence for all $a \in A$, $a = 1 \cdot a = 0 \cdot a = 0$, and for all $s \in S$, $s = 0$, and the direct limit consists of the pair (0,0).

If $h_\lambda : (A_\lambda, S_\lambda) \to (B, R)$ are homomorphisms with $h_\mu \phi_{mu\lambda} = h_\lambda$, there is a unique homomorphism $h : (A, S) \to (B, R)$ such that $h\phi_\lambda = h_\lambda$ for each λ.

Proof. This is proved exactly as in the last lecture. □ 25

Definition. *If Ω is a directed set, a subset Ω' of Ω is said to be a subdirected set of Ω, if, with the induced order relation, it is a directed set.*

Definition. *A subset Ω' of a directed set Ω is said to be* cofinal *in Ω if, for any element $\lambda \cdot \in \Omega$, there exists a $\nu \in \Omega'$ with $\lambda < \nu$.*

If Ω' is cofinal in Ω, Ω' is a subdirected set.

Proof. This is simple. □

If $\sum = \{A_\lambda, S_\lambda, \phi_{\mu\lambda}\}_{\lambda,\mu\in\Omega}$ is a direct system and if Ω' is a subdirected set of Ω, then $\sum' = \{A_\lambda, S_\lambda, \phi_{\mu\lambda}\}$, $\lambda, \mu \in \Omega'$ is also a direct system. Let (A', S') be its direct limit and $\phi'_\lambda : (A_\lambda, S_\lambda) \to (A', S')$ for $\lambda \in \Omega'$. Since, for $\lambda < \mu \in \Omega'$ $\phi_\mu \phi_{\mu\lambda} = \phi_\lambda$; there is a unique induced homomorphism $i : (A', S') \to (A, S)$ with $i\phi'_\mu = \phi_\mu$.

$$\begin{tikzcd}
(A_\lambda, S_\lambda) \arrow[r, "\phi_{\mu\lambda}"] \arrow[drr, "\phi_\lambda"'] & (A_\mu, S_\mu) \arrow[r, "\phi'_\mu"] \arrow[dr, "\phi_\mu"] & (A', S') \arrow[d, "i"] \\
 & & (A, S)
\end{tikzcd}$$

If Ω' is cofinal in Ω, then $i : (A', S') \to (A, S)$ is an isomorphism.

Proof. Each class ϕ_λ a of $\bigcup\limits_{\Omega} A_\lambda$ has a representative in $\bigcup\limits_{\lambda\in\Omega'} A_\lambda$ and if $a, \in \bigcup\limits_{\lambda\in\Omega'} A_\lambda$ and $a \sim o$ in $\sum$ then $a \sim o$ in $\sum'$. Thus $i' : A' \to A$ is an isomorphism. Similarly $i'' : S' \to S$ is an *an isomorphism*. □

26 *If $\{A_\lambda, S_\lambda, \phi_{\mu\lambda}\}$ and $\{B_\lambda, R_\lambda, \theta_{\mu\lambda}\}$ are direct systems indexed by the same directed set Ω, and if $\{f_\lambda : (A_\lambda, S_\lambda) \to (B_\lambda, R_\lambda)$ are homomorphisms such that $f_\mu \phi_{\mu\lambda} = \theta_{\mu\lambda} f_\lambda : (A_\lambda, S_\lambda) \to (B_\mu, R_\mu)$, there is a unique homomorphism $f : (A, S) \to (B, R)$ of the limit modules such that $f\phi_\lambda = \theta_\lambda f_\lambda$.*

$$\begin{tikzcd}
(A_\lambda, S_\lambda) \arrow[r, "\phi_{\mu\lambda}"] \arrow[d, "f_\lambda"'] & (A_\mu, S_\mu) \arrow[d, "f_\mu"] \\
(B_\lambda, R_\lambda) \arrow[r, "\theta_{\mu\lambda}"'] & (B_\mu, R_\mu)
\end{tikzcd}
\qquad
\begin{tikzcd}
(A_\lambda, S_\lambda) \arrow[r, "\phi_\lambda"] \arrow[d, "f_\lambda"'] \arrow[dr, "h_\lambda"] & (A, S) \arrow[d, "f"] \\
(B_\lambda, R_\lambda) \arrow[r, "\theta_\lambda"'] & (B, R)
\end{tikzcd}$$

Proof. Let $h_\lambda = \theta_\lambda f_\lambda : (A_\lambda, S_\lambda) \to (B, R)$. Then $h_\mu \phi_{\mu\lambda} = \theta_\mu f_\mu \phi_{\mu\lambda} f_\lambda = \theta_\mu \theta_{\mu\lambda} f_\lambda = \theta_\lambda f_\lambda = h_\lambda$. Therefore there is a unique homomorphism $f : (A, S) \to (B, R)$ with $f\phi_\lambda = h_\lambda$, i.e. with $f\phi_\lambda = \theta_\lambda f_\lambda$. □

If $\{A_\lambda, S_\lambda, \phi_{\mu\lambda}\}$, $\{A_\lambda, R_\lambda, \theta_{\mu\lambda}\}$, $\{A_\lambda, Q_\lambda, \psi_{\mu\lambda}\}$ are direct systems of unitary left modules (with direct limits S, R, Q respectively) indexed by the same directed set Ω, with $\phi'_{\mu\lambda} = \theta'_{\mu\lambda} = \psi'_{\mu\lambda} : A_\lambda \to A_\mu$, and if, for each $\lambda \in \Omega$, $S_\lambda \xrightarrow{g_\lambda} R_\lambda \xrightarrow{f_\lambda} \varphi_\lambda$ is an exact sequence of homomorphisms of A_λ - modules, and if commutativity holds in

$$\begin{tikzcd}
(A_\lambda, S_\lambda) \arrow[r, "g_\lambda"] \arrow[d, "\theta_{\mu\lambda}"'] & (A_\lambda, R_\lambda) \arrow[r, "f_\lambda"] \arrow[d, "\theta_{\mu\lambda}"'] & (A_\lambda, Q_\lambda) \arrow[d, "\psi_{\mu\lambda}"'] \\
(A_\mu, S_\mu) \arrow[r, "g_\mu"'] & (A_\mu, R_\mu) \arrow[r, "f_\mu"'] & (A_\mu, Q_\mu)
\end{tikzcd}$$

then the sequence of induced homomorphisms of A-modules

$$S \xrightarrow{g} R \xrightarrow{f} Q$$

is exact. 27

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
(A_\lambda, S_\lambda) \arrow[r, "g_\lambda"] \arrow[d, "\phi_{\mu\lambda}"'] & (A_\lambda, R_\lambda) \arrow[r, "f_\lambda"] \arrow[d, "\theta_{\mu\lambda}"'] & (A_\lambda, Q_\lambda) \arrow[d, "\psi_{\mu\lambda}"'] \\
(A_\mu, S_\mu) \arrow[r, "g_\mu"] \arrow[d, "\phi_\mu"'] & (A_\mu, R_\mu) \arrow[r, "f_\mu"] \arrow[d, "\theta_\mu"'] & (A_\mu, Q_\mu) \arrow[d, "\psi_\mu"'] \\
(A, S) \arrow[r, "g"] & (A, R) \arrow[r, "f"] & (A, Q)
\end{tikzcd}$$

(i) $\operatorname{im} g \subset \ker f$. For if $s_1 \in S$, for some λ, $\phi_\lambda s = S_1$, $s \in S_\lambda$. Then $g(\phi_\lambda s) \in \operatorname{im} g$, and $f(g\phi_\lambda s) = \psi_\lambda(f_\lambda g_\lambda s) = \psi_\lambda(0) = (0)$. Hence $g\phi_\lambda s \in \ker f$.

(ii) ker $f \subset \operatorname{im} g$. For if $r_1 \in R$, then for some λ, $\theta_{\lambda r} = r_1$, $r \in R_\lambda$. Let $\theta_\lambda r \in \ker f$. Then $f(\theta_\lambda)r = 0 = \psi_\lambda f_\lambda r$, this means that $f_\lambda r \sim 0$, hence there exists a $\mu > \lambda$ such that

$$\psi_{\mu\lambda}(f_\lambda r) = 0 = f_\mu(\theta_{\mu\lambda} r), \text{i.e.} \theta_{\mu\lambda} r \in \ker f_\mu$$

and ker f_μ is equal to im g_μ by assumption, hence there must exist an $s \in S_\mu$ such that $\theta_{\mu\lambda} r = g_\mu s$, i.e.

(Q.e.d.) $$\theta_\lambda r = \theta_\mu \theta_{\mu\lambda} r = \theta_\mu g_\mu s = g\theta_\mu s \in \operatorname{im} g.$$

□

Definition. *A* bihomomorphism $f : (A, R, S) \to (B, T)$, *where A, B are* commutative rings with unit element, R, S, *are unitary A-modules and T is a unitary B-module, is a pair* $(f' : A \to B, f'' : R \times S \to T)$ *such that f'' is bilinear. More precisely,*

$$\begin{aligned} f'(a_1 + a_2) &= f'(a_1) + f'(a_2), f'(a_1 a_2) \\ &= f'(a_1) f'(a_2), f'(1) = 1, a_1, a_2 \in A, \\ f''(r, a_1 s_1 + a_2 s_2) &= f'(a_1) f''(r, s_1) + f'(a_2) f''(r, s_2), r \in R, s_1, s_2 \in S, \\ f''(a_1 r_1 + a_2 r_2, s) &= f'(a_1) f''(r_1, s) + f'(a_2) f''(r_2, s), r_1, r_2 \in R, s \in S. \end{aligned}$$

28 If $A = B$ and f' is the identity, we write $f : (R, S) \to T$.

Given (A, R, S) there exists a unitary A-module $R \otimes_A S$ and a bihomomorphism $\alpha : (R, S) \to R \otimes_A S$ where im α generates $R \otimes_A S$ such that, for any bihomomorphism $f : (A, R, S) \to (B, T)$ there is a unique homomorphism $\bar{f} : (A, R \otimes_A S) \to (B, T)$ with $\bar{f}\alpha = f$.

The module $R \otimes_A S$ together with the bihomomorphism α is called a *tensor product* of R and S over the ring A; it is uniquely determined upto isomorphism. Hence we can say 'the' tensor product of R and S.

Proof. For the proof of the existence of the tensor product, see Bourbaki, Algebre multilineaire. We give only the proof of uniqueness. □

$$\begin{tikzcd}
 & (A, R \otimes_A S) \arrow[dd, shift right, "\overline{\alpha}'"'] \\
(A, R, S) \arrow[ur, "\alpha"] \arrow[dr, "\alpha'"'] & \\
 & (A, R \otimes'_A S) \arrow[uu, shift right, "\overline{\alpha}"']
\end{tikzcd}$$

Let $R \otimes'_A S$ and $R \otimes_A S$ be two tensor products of R and S over A. Then by definition of the tensor product the bihomomorphism α' induces a homomorphism $\overline{\alpha}' : (A, R \otimes_A S) \to (A, R \otimes'_A S)$ such that $\overline{\alpha}' \cdot \alpha = \alpha'$. Similarly α induces a homomorphism $\overline{\alpha}$ such that $\overline{\alpha}\alpha' = \alpha$. Then $\overline{\alpha}' \cdot \overline{\alpha}$ is the identity of $R \otimes'_A S$, similarly $\overline{\alpha}\overline{\alpha}'$ is the identity of $R \otimes_A S$. Hence $\overline{\alpha}$ is an isomorphism.

If we identity $R \otimes'_A S$ with $R \otimes_A S$ under this isomorphism, α' will **29**
coincide with α

Example. Let A be a commutative ring with unit element, S a unitary A-module, and consider A itself as a unitary A-module. Define $\alpha : (A, A, S) \to (A, S)$ as $\alpha(a, s) = as$. Then α is verified to be a bihomomorphism. If $f : (A, A, S) \to (B, T)$ is any bihomomorphism, define the homomorphism $\overline{f} : (A, S) \to (B, T)$ as $\overline{f}'a = f'a$, $\overline{f}''s = f''(1, s)$. Then

$$\overline{f}''\alpha(a, s) * \overline{f}''(as) = f''(1, as) = f'(a) f''(1, s) = f''(a, s).$$

Thus $\overline{f}\alpha = f$, and S with the bihomomorphism $\alpha : (A, S) \to S$ is the tensor product $A \otimes_A S$.

Similarly $S \otimes_A A = S$ with $\alpha(s, a) = as$.

Lecture 6

Definition. *A homomorphism $\phi : (A, R, S) \to (B, P, Q)$ consists of homomorphisms $(A, R) \to (B, P)$ and $(A, S) \to (B, Q)$ (in the sense already defined), where the homomorphism $A \to B$ is the same in both cases.* 30

A homomorphism $\phi : (A, R, S) \to (B, P, Q)$ induces a map $\phi : R \times S \to P \times Q$. We now consider the following diagram:

$$\begin{tikzcd}
(A, R \times S) \arrow[r, "\phi"] \arrow[d, "\alpha"'] \arrow[dr, dash, "\beta\phi"] & (B, P \times Q) \arrow[r, "\theta"] \arrow[d, "\beta"'] & (C, T \times U) \arrow[d, "\gamma"'] \\
(A, R \otimes_A S) \arrow[r, "\bar{\phi}"'] & (B, P \otimes_B Q) \arrow[r, "\bar{\theta}"'] & (C, T \otimes_C U)
\end{tikzcd}$$

Here ϕ, θ are the induced maps and α, β, γ the bihomomorphisms included in the definition of tensor products. Further, the homomorphism ϕ induces a unique homomorphism $\bar{\phi}$ as indicated, such that $\bar{\phi}\alpha = \beta\phi$, and similarly $\bar{\phi}\beta = \gamma\theta$. From the uniqueness, we have $\overline{\theta\phi} = \delta bar\phi$. If ϕ is the identity then $\bar{\phi}$ also is the identity.

The operator of taking the tensor product commutes with the operation of taking the direct limit.

Proof. Let $\{A_\lambda, R_\lambda, S_\lambda, \phi_{\mu\lambda}\}_{\lambda,\mu\in\Omega}$ be a direct system, where each A_λ is a commutative ring with unit element, R_λ and S_λ are unitary A_λ-modules and $\phi_{\mu\lambda} : (A_\lambda, R_\lambda, S_\lambda) \to (A_\mu, R_\mu, S_\mu)$ are homomorphisms. Then, since $\bar{\phi}_{\lambda\lambda}$ is the identity and $\bar{\phi}_{\nu\lambda} = \overline{\phi_{\nu\mu}\phi_{\mu\lambda}} = \bar{\phi}_{\nu\lambda} \cdot \bar{\phi}_{\nu\lambda} (\lambda < \mu < \nu)$, the system $\{A_\lambda, R_\lambda \otimes_{A_\lambda} S_\lambda, \bar{\phi}_{\mu\lambda}\}$ is a direct system. Let its direct limit be denoted by (A, Q). ◻

$$\begin{tikzcd}
(A_\lambda, R_\lambda, S_\lambda) \arrow[r, "\phi_{\mu\lambda}"] \arrow[dd, "\alpha_\lambda"'] & (A_\mu, R_\mu, S_\mu) \arrow[r, "\phi_\mu"] \arrow[dd, "\alpha_\mu"'] & (A, R, S) \arrow[dd, "\beta"'] \arrow[dr, "f"] & \\
& & & (B, T) \\
(A_\lambda, R_\lambda \otimes_{A_\lambda} S_\lambda) \arrow[r, "\bar{\phi}_{\mu\lambda}"'] & (A_\mu, R_\mu \otimes_{A_\mu} S_\mu) \arrow[r, "\bar{\phi}_\mu"'] & (A, Q) \arrow[ur, "\bar{f}"'] &
\end{tikzcd}$$

31 We define a bihomomorphism $\beta : (A, R, S) \to (A, Q)$ as

$$\beta(r, s) = \bar{\phi}_\lambda \alpha_\lambda(r_\lambda, s_\lambda),$$

where r_λ and s_λ are respresentatives of $r \in R$, $s \in S$, for the same λ. Since

$$\bar{\phi}_\mu \alpha_\mu(\phi_{\mu\lambda} r_\lambda, \phi_{\mu\lambda} s_\lambda) = \bar{\phi}_\mu \bar{\phi}_{\mu\lambda} \alpha_\lambda(r_\lambda, s_\lambda) = \bar{\phi}_\lambda \alpha_\lambda(r_\lambda, s_\lambda),$$

$\beta : (R, S) \to Q$ is independent of the choice of representatives. For a suitable choice of representatives,

$$\begin{aligned}
\beta(r, bs + cs') &= \bar{\phi}_\lambda \alpha_\lambda(r_\lambda, b_\lambda s_\lambda + c_\lambda s'_\lambda) \\
&= \bar{\phi}_\lambda((b_\lambda \cdot \alpha_\lambda(r_\lambda, s_\lambda) + c_\lambda \cdot \alpha_\lambda(r_\lambda, s'_\lambda)) \\
&= \bar{\phi}_\lambda(b_\lambda) \cdot \bar{\phi}_\lambda \alpha_\lambda(r_\lambda, s_\lambda) + \phi_\lambda(c_\lambda) \bar{\phi}_\lambda \alpha_\lambda(r_\lambda, s'_\lambda) \\
&= b \cdot \beta(r, s) + c \cdot \beta(r, s')
\end{aligned}$$

and similarly

$$\beta(br + cr', s) = b \cdot \beta(r, s) + c \cdot \beta(r', s)$$

Thus $\beta : (R, S) \to Q$ is verified to be a bihomomorphism. Clearly im β generates Q, for, each $q \in Q$ has a representative q_λ in some $R_\lambda \otimes_{A_\lambda} S_\lambda$ and, since im α_λ generates $R_\lambda \otimes_{A_\lambda} S_\lambda$, $q_\lambda = \sum_{i=1}^{k} a_i.\ \alpha_\lambda(r_i, s_i)$, $a_i \in A_\lambda$, $r_i \in R_\lambda$, $s_i \in S_\lambda$. Then

$$q = \sum_{i=1}^{k} \phi_\lambda(a_i) \bar{\phi}_\lambda \alpha_\lambda(r_i, s_i) = \sum_{i=1}^{k} \phi_\lambda(a_i) \beta(\phi_\lambda r_i, \phi_\lambda s_i)$$

32 which proves that $\operatorname{im}\beta$ generates Q.

We now show that Q together with the bihomomorphism $\beta : (R, S) \to Q$ is the tensor product $R \otimes_A S$. To do this, let $f : (A, R, S) \to (B, T)$ be any bihomomorphism, then for each λ, $f\phi_\lambda : (A_\lambda, R_\lambda, S_\lambda) \to (B, T)$ is also a bihomomorphism, hence it induces a unique homomorphism $\bar{f}_\lambda : (A_\lambda, R_\lambda \otimes_{A_\lambda} S_\lambda) \to (B, T)$ where for each $\lambda < \mu$, $\bar{f}_\mu \bar{\phi}_{\mu\lambda} \cdot \alpha_\lambda = \bar{f}_\mu \alpha_\mu \phi_{\mu\lambda} = f\phi_\mu \phi_{\mu\lambda} = f\phi_\lambda = \bar{f}_\lambda \alpha_\lambda$. Hence, since $\operatorname{im}\alpha_\lambda$ generates $R_\lambda \otimes_{A_\lambda} S_\lambda$, $\bar{f}_\mu \bar{\phi}_{\mu\lambda} = \bar{f}_\lambda$. Therefore there is a unique homomorphism $\bar{f} : (A, Q) \to (B, T)$ with $\bar{f}\bar{\phi}_\lambda = \bar{f}_\lambda$. Then

$$\bar{f}\beta(r, s) = \bar{f}\bar{\phi}_\lambda \alpha_\lambda(r_\lambda, s_\lambda) = \bar{f}_\lambda \alpha_\lambda(r_\lambda, s_\lambda) = f\phi_\lambda(r_\lambda, s_\lambda) = f(r, s);$$

and the proof of the statement is complete.

Presheaves.

Let Ω be the set of all open sets of X, with the order relation $\supset$, i.e. $U \supset V$ is equivalent to saying that $U < V$. then $U \supset U$ and if $U \supset V$ and $V \supset W$ then $U \supset W$ and given U, V there exists $W = U \cap V$ with $U \supset W$, $V \supset W$; hence Ω is a directed set.

Definition. *A presheaf of modules over a base space X is a direct system $\{A_U, S_U, \phi_{VU}\}$ indexed by Ω, such that $(A_\phi, S_\phi) = (0, 0)$, where ϕ is the empty set.*

For a presheaf over X, the index set Ω is always the family of all open sets of X.

The definition of a presheaf includes, as a special case, the definition 33
of a presheaf of B-modules, presheaf of rings with unit element and a presheaf of abelian groups.

Example 10. Let X be the complex sphere, A_U the ring of all functions analytic in U if U is a non-empty open set and $A_\phi = O$; and, if $f \in A_U$ and $U \supset V$, let $\phi_{VU} f = f|V$, i.e. ϕ_{VU} is the restriction homomorphism. Then $\{A_U, \phi_{VU}\}$ is a presheaf of rings with unit element.

Presheaf of sections. Let $\mathfrak{a}$ be a sheaf of rings *with unit* and $\mathscr{S}$ a sheaf of $\mathfrak{a}$-modules. For each open U, the ring $\Gamma(U, \mathscr{S})$ is a unitary $\Gamma(U, \mathfrak{a})$-module. If $V \subset U$, let

$$\phi_{VU} : (\Gamma(U, \mathfrak{a}), \Gamma(U, \mathscr{S})) \to (\Gamma(V, \mathfrak{a}), \Gamma(V, \mathscr{S}))$$

denote the restriction homomorphism. By convention $\Gamma(\phi, a) = o$, $\Gamma(\phi, \mathscr{S}) = o$ where ϕ denotes the empty set. Thus $\{\Gamma(U, \mathfrak{a}), \Gamma(U, \mathscr{S}), \phi_{VU}\}$ is a presheaf denoted by $(\overline{\mathfrak{a}, \mathscr{S}})$, and is called the presheaf of sections of $(\mathfrak{a}, \mathscr{S})$.

For each $x \in X$ let Ω_x denote the family of all open subsets of X containing x. Then Ω_x is a subdirected set of Ω. If $\{A_U, S_U, \phi_{VU}\}$ is a presheaf, let (A_x, S_x) denote the direct limit of the subsystem $\{A_U, S_U, \phi_{VU}\}$ indexed by Ω_x, and let $\phi_{xU} : (A_U, S_U) \to (A_x, S_x)$ be the homomorphism which sends each element into its equivalence class. If $a \in A_U$, its image $\phi_{xU} a = a_x$ is called the *germ* of a at x; similarly
34 for $s \in S_U$. We will denote by $\overline{a} : U \to \bigcup\limits_x A_x$ the function for which $\overline{a}(x) = a_x$, and similarly for $\overline{s} : U \to \bigcup\limits_x S_x$. For each $W \subset U$ we write $a_W = \overline{a}(W) = \{a_x : x \in W\}$, and similarly for s_W.

[For instance, in Example 10, if f is analytic in U, $x \in U$ the germ f_x is the class of those functions each of which coincides with f in some neighbourhood of x.]

Let $A = \bigcup_x A_x$, $S = \bigcup_x S_x$. Define $\tau : A \to X$, $\pi : S \to X$ by $\tau(A_x) = x$, $\pi(S_x) = x$. Then $\tau\overline{a} : U \to U$, $\pi\overline{s} : U \to U$ are the identity maps on U.

We can take $\{a_U\}_U$, $a \in A_U$ as a *base for open sets in A*. For, $\{a_U\}$ covers A and if $p \in a_U \cap b_V$, $x = \tau(p)$, we have $p = a_x = b_x$ with $a \in A_U$, $b \in A_V$ and $x \in U \cap V$. Then for some W with $x \in W \subset U \cap V$, $\phi_{WU}\, a = \phi_{WV}\, b = c$ say. Since

$$\phi_{xU} a = \phi_{xW}\phi_{WU} a = \phi_{xW} c = \phi_{xW}\phi_{WV} b = \phi_{xV} b \text{ for each } x \in W,$$

we have $c_W = a_W = b_W$. Then $p \in c_W = a_W = b_W \subset a_U \cap b_V$. Similarly the sets $\{s_U\}$ form a base for open sets in S.

Lecture 7

With the notations introduced in the last lecture, we prove 35

Proposition 3. *If* $\{A_U, S_U, \phi_{VU}\}$ *is a presheaf of modules over the space* X*, then* $\mathfrak{a} = (A, \tau, X)$ *is a sheaf of rings with unit and* $\mathscr{S} = (S, \pi, X)$ *is a sheaf of* $\mathfrak{a}$*-modules.*

Proof. If $a \in A_U$, $\overline{a} : U \to A$ is continuous. For, if $a_x \in b_V$ with $b \in A_V$, there exists $c \in A_W$ with $x \in W \subset U \cap V$ such that $\overline{a}(W) = a_W = c_W = b_W \subset b_V$. Also $\overline{a} : U \to A$ is an open mapping since, for open $V \subset U$, $a_V = (\phi_{VU} a)_V$ is open by definition. ◻

Hence $\overline{a} : U \to a_U$ being 1-1 is a homeomorphism and the inverse $\tau | a_U : a_U \to U$ is a homeomorphism of a_U onto the open set U.

Similarly $\overline{s} : U \to S$ is continuous and $\pi | s_U : s_U \to U$ is a homeomorphism. Thus τ and π are local homeomorphisms.

For each $x \in X$, $\tau^{-1}(x) = A_x$ is a ring with unit element, and $\pi^{-1}(x) = S_x$ is a unitary left A_x-module.

Addition is continuous, for if a_x, $b_x \in A_x$ (with $a \in A_U$, $b \in A_{U_1}$, $x \in U \cap U_1$) and $a_x + b_x \in c_{U_2}$ with $c \in A_{U_2}$, then for some W with $x \in W \subset U \cap U_2 \cap U_2$, $\phi_{WU} a + \phi_{WU_1} b = \phi_{WU_2} c$. Thus $a_x \in a_W$, $b_x \in b_W$ and for any $p \in a_W$, $q \in b_W$ with $\tau(p) = \tau(q) = y$ say, we have $p + q = a_y + b_y = c_y \in c_{U_2}$.

The unit is continuous, for if $1_x \in A_x$ is the unit element of A_x and $1_x \in b_U$, then for some V with $x \in V \subset U$, $\phi_{VU} b$ is the unit element of A_V. Then $b_V \subset b_U$ and consists of the unit elements 1_y for $y \in V$.

Similarly the other operations of $\mathfrak{a}$ and $\mathscr{S}$ are continuous. 36

We remark that a_U, $s_U \cdots$ are *sections* and that such sections form a base for open sets of A, S.

Remark. The "sheaves" introduced originally by Leray were actually presheaves with the indexing set Ω consisting of the family of all closed sets instead of the family of all open sets.

Example 11. Let X be the circle $|z| = 1$; for each open set U of X let S_U be the abelian group of all integer valued functions in U and let ϕ_{VU} be the restriction homomorphism. This system is a presheaf and the induced sheaf has Example 4 as a subsheaf.

Example 12. Let X be the real line, and S_U the R module (R denotes the ring of real numbers) of all real indefinitely differentiable functions in U and let ϕ_{VU} denote the restriction homomorphism. This system is a presheaf and the sheaf space S of the induced sheaf is not Hausdorff.

Let $(\mathfrak{a}, \mathscr{S})$ be a sheaf, $(\bar{\mathfrak{a}}, \bar{\mathscr{S}})$ its presheaf of sections and $(\mathfrak{a}', \mathscr{S}')$ the sheaf determined by $(\bar{\mathfrak{a}}, \bar{\mathscr{S}})$. We show that $(\mathfrak{a}', \mathscr{S}')$ and $(\mathfrak{a}, \mathscr{S})$ are canonically isomorphic.

$$\begin{tikzcd} & (\bar{\mathfrak{a}}, \bar{}) \arrow[dr, dashed] & \\ (\mathfrak{a},) \arrow[ur, dashed] & & (\mathfrak{a}', ') \arrow[ll, "h"'] \end{tikzcd}$$

If $x \in U$ and $f \in \Gamma(U, \mathfrak{a})$, let $h_{xU} f = f(x) \in A_x$. Similarly, if $s \in \Gamma(U, \mathscr{S})$, let $h_{xU} s = s(x) \in S_x$. Then

$$h_{xU} : (\Gamma(U, \mathfrak{a}), \Gamma(U, \mathscr{S})) \to (A_x, S_x)$$

37 is a homomorphism and, if $x \in V \subset U$, $h_{xV} \phi_{VU} = h_{xU}$. Then there is an induced homomorphism $h_x : (\cdots'_x, S'_x) \to (A_x, S_x)$ with $h_x \phi_{xU} = h_{xU}$.

In fact h_x is an isomorphism. For if $p \in A_x$, then there is some section $f : U \to A$ with $f(x) = p$, then

$$p = f(x) = h_{xU} f = h_x \phi_{xU} f \in im h_x,$$

and if $p' \in A'_x$ with $h_x p' = O$, choose a representative $f \in \Gamma(U, \mathfrak{a})$ for p'. Then $f(x) = h_{xU} f = h_x p' = O$. Hence, for some V, with $x \in V \subset U$, $f|V = o$. Therefore

$$p' = \phi_{xU} f = \phi_{xV} \phi_{VU} f = \phi_{xV} O_V = (O).$$

Thus $h_x : A'_x \to A_x$ is an isomorphism and similarly $h_x : S'_x \to S_x$ is an isomorphism.

Let $h : (A', S') \to (A, S)$ be given by $h|(A'_{\underline{x}}, S'_x) = h_x$. If $f \in \Gamma(U, \mathfrak{a})$ and f_U is the induced section in $\mathfrak{a}'$, given by $\overline{f}(x) = \phi_{xU} f$, then

$$h\overline{f}(x) = h_x \overline{f}(x) = h_x \phi_{xU} f = h_{xU} f = f(x)$$

and thus $h(\overline{f}(U)) = f(U)$. The same holds if $s \in \Gamma(U, \mathscr{S})$.

Thus h is an isomorphism of stalks for each x and, since it maps section f_U onto sections $f(U)$, h is a local homeomorphism and hence is continuous. Thus $h : (\mathfrak{a}'.\mathscr{S}) \to (\mathfrak{a}, \mathscr{S})$ is a sheaf isomorphism.

We identify $(\mathfrak{a}', \mathscr{S}')$ *with* $(\mathfrak{a}, \mathscr{S})$ *under this isomorphism.* **38**

If $a \in A$, $h^{-1}a$ is the class of all sections $f : U \to A$ where $f(U)$ contains a, and similarly for $h^{-1}s$.

Definition. *If* $\sum' = \{A'_U, S'_U, \phi'_{VU}\}$ *and* $\sum = \{A_U, S_U, \phi_{VU}\}$ *are presheaves over X, a homomorphism* $f := \sum' \to \sum$ *is a system* $\{f_U\}$ *of homomorphisms* $f_U : (A'_U, S'_U) \to (A_U, S_U)$ *such that* $f_V \phi'_{VU} = \phi_{VU} f_U$, *that is, the following diagram is commutative.*

$$\begin{tikzcd}
(A'_U, S'_U) \arrow[r, "\phi'_{VU}"] \arrow[d, "f_U"'] & (A'_V, S'_V) \arrow[d, "f_V"'] \\
(A_U, S_U) \arrow[r, "\phi_{VU}"'] & (A_V, S_V).
\end{tikzcd}$$

Let $(\mathfrak{a}', \mathscr{S}')$, $(\mathfrak{a}, \mathscr{S})$ *be the sheaves determined by* $\sum'$, $\sum$. *Then the homomorphism* $f : \sum' \to \sum$ *induces a sheaf homomorphism* $f : (\mathfrak{a}', \mathscr{S}') \to (\mathfrak{a}, \mathscr{S})$.

Proof. For each x, $\{f_U\}_{x \in U}$ induces a homomorphism $f_x : (A'_x, S'_x) \to (A_x, S_x)$, with $f_x \phi'_{xU} = \phi_{xU} f_U$ and these homomorphism f_x define a function $f : (A', S') \to (A, S)$. $\square$

If for $a' \in A'_U$, $f_U a' = a$, then

$$a_x = \phi_{xU} a = \phi_{xU} f_U a' = f_x \phi'_{xU} a' = f_x(a'_x).$$

Thus $f(a'_U) = a_U$ and f is a local homeomorphism, hence is continuous. Hence f is a sheaf homomorphism, and this completes the proof.

39 Let $\sum$ be a presheaf which determines the sheaf $(\mathfrak{a}, \mathscr{S}), (\bar{\mathfrak{a}}, \bar{\mathscr{S}})$ the presheaf of sections and $(\mathfrak{a}', \mathscr{S}')$ the sheaf determined by it. The functions $f_U : (A_U, S_U) \to \left(\Gamma(U, \mathfrak{a}), \Gamma(U, \mathscr{S})\right)$ (where $f_U a$ is the section $\bar{a} : U \to A$ determined by a, and similarly for $f_U s$), determine a homomorphism, $f = \{f_U\} : \sum \to (\bar{\mathfrak{a}}, \bar{\mathscr{S}})$. In general, the homomorphism f is neither an epimorphism nor a monomorphism, hence obviously not a isomorphism.

The induced homomorphism $f : (\mathfrak{a}, \mathscr{S}) \to (\mathfrak{a}', \mathscr{S}')$ *is the identifying isomorphism* h^{-1}.

Proof. Let $a \in A$ and suppose that $a = \phi_{xU} b$, $b \in A_U$. $f(a)$ is the class at x containing $f_U b$ which is a section with $(f_U b)(x) = \phi_{xU} b = a$. Thus $f(a)$ is the class h^{-1} a of all sections $g : U \to A$ with $g(x) = a$. $\square$

$$\begin{tikzcd}
\sum \arrow[r, "f"] \arrow[d, dashed] & (\bar{a}, \bar{s}) \arrow[d, dashed] \\
(a, s) \arrow[r, "f", "h^{-1}"'] \arrow[ur, dashed] & (a', s').
\end{tikzcd}$$

Lecture 8

Proposition 4. *If $\sum' \xrightarrow{g} \sum \xrightarrow{f} \sum''$ is an exact sequence of homomorphisms of presheaves, i.e. if each sequence* 40

$$(A_U, S'_U) \xrightarrow{g_U} (A_U, S_U) \xrightarrow{f_U} (A_U, S''_U)$$

is exact, then the induced sequence $\mathscr{S}' \xrightarrow{g} \mathscr{S} \xrightarrow{f} \mathscr{S}''$ of sheaves is also exact.

Proof. The sequences $(A_x, S'_x) \to (A_x, S_x) \to (A_x, S''_x)$ are exact by a property of the direct limit. ◻

Induced homomorphism of presheaves of sections. If $\mathscr{S}'$, $\mathscr{S}$ are sheaves of $\mathfrak{a}$-modules and $g : \underline{\mathscr{S}'} \to \underline{\mathscr{S}}$ is a homomorphism, there is a homomorphism $\overline{g} : \overline{\mathscr{S}'} \to \overline{\mathscr{S}}$ of the presheaves of sections with $\overline{g}_U : \Gamma(U, \mathscr{S}') \to \Gamma(U, \mathscr{S})$ defined by $\overline{g}_U(f) = gf$. This homomorphism takes all sections through $s' \in S'$ into sections through gs'. Thus, with the usual identification, $\overline{g}$ inducts the sheaf homomorphism $g : s' \to s$.
Quotient Sheaves.

Proposition 5. *If $\mathscr{S}$ is a sheaf of a-modules and $\mathscr{S}'$ is a subsheaf of $\mathscr{S}$, there is a unique sheaf $\mathscr{S}''$, whose stalks are the quotient modules $S''_x = S_x/S'_x$, such that $j : \mathscr{S} \to \mathscr{S}''$ where $j|S_x = j_x : S_x \to S''_x$ is the natural homomorphism, is a sheaf homomorphism.*

$\mathscr{S}''$ is the quotient sheaf $\mathscr{S}'' = \mathscr{S}/\mathscr{S}'$.

Proof. If $S'' = \bigcup_x S_x/S'_x$ is to have a topology such that $j : \mathscr{S} \to$ 41
$\mathscr{S}''$ is a sheaf homomorphism, S'' must be covered by sections which

are image $jf(U)$ of sections of $\mathscr{S}$, and this uniquely determines the topology of S''. This topology has the property that a set V of S'' is open if and only if $j^{-1}V$ is open. Thus $\mathscr{S}''$ if it exists, is unique. $\square$

$\Gamma(U, \mathscr{S}')$ is a sub-$\Gamma(U, \mathfrak{a})$-module of $\Gamma(U, \mathscr{S})$ and let $\phi_{VU}|\Gamma(U, \mathscr{S}') = \phi'_{VU}$. Let $S''_U = \Gamma(U, \mathscr{S})/\Gamma(U, \mathscr{S}')$ and let $j_U : \Gamma(U, \mathscr{S}) \to S''_U$ be the natural homomorphism. Let $\phi''_{VU} : S''_U \to S''_V$ denote the homomorphism induced by ϕ_{VU}. Then $\sum'' = \{A_U, S''_U, \phi''_{VU}\}$ is a presheaf and the sequence $0 \to (\bar{\mathfrak{a}}, \bar{\mathscr{S}'}) \xrightarrow{f} (\bar{\mathfrak{a}}, \bar{\mathscr{S}}) \xrightarrow{j} \sum'' \to 0$ is exact, where 0 is the presheaf $\{A_U, 0_U, \ldots\}$. Then the induced sequence of sheaves

$$0 \to \mathscr{S}' \xrightarrow{i} \mathscr{S} \xrightarrow{i} \mathscr{S}'' \to 0,$$

where $\mathscr{S}''$ is the sheaf determined by $\sum''$, is exact. That is, for each x, the sequence

$$0 \to S'_x \xrightarrow{i_x} S_x \xrightarrow{j_x} S''_x \to 0$$

is exact. Thus j_x induces an isomorphism $S_x/S'_x \to S''_x$, and if we identify S''_x with S_x/S'_x, $j_x : S_x \to S_x/S'_x$ is the natural homomorphism. Thus a sheaf $\mathscr{S}''$ having the required properties exists.

Definition. A homomorphism $f : (\mathfrak{a}, \mathscr{S}) \to (\mathscr{B}, \mathscr{R})$ *consists of maps*
42 $f' : A \to B$ *and and* $f'' : S \to R$, *commuting with the projections, such that the restrictions* $f'_x = f'|A_x \to B_x$ *and* $f''_x = f''|S_x : S_x \to R_x$ *give a homomorphism* $f_x = (f'_x, f''_x) : (A_x, S_x) \to (B_x, R_x)$.

If $f : (a, \mathscr{S}) \to (\mathcal{B}, \mathbb{R})$ *is a sheaf homomorphism and if* $\mathscr{S}'$, $\mathscr{R}'$ *are subsheaves of* $\mathscr{S}$, $\mathscr{R}$ *respectively such that* $f(S') \subset R'$, *there are induced homomorphisms*

$$f' : (a, \mathscr{S}') \to (\mathscr{B}, \mathscr{R}'),\ f'' : (a, \mathscr{S}/\mathscr{S}') \to (\mathscr{B}, \mathscr{R}/'_{\mathscr{R}})$$

with $fi = if'$, $f''j = jf$, *where* i *denotes the inclusion homomorphism and* j *the natural homomorphism of a sheaf onto a quotient sheaf.*

Proof. The result is clear for stalks, and the fact that f', f'' are homomorphisms follows from the fact they are continuous. $\square$

Example 13. Let X be the circle $|z| = 1$, let $\mathscr{S}$ be the constant sheaf $(X \times Z, \pi, X)$ of integers over X and let $\mathscr{S}'$ be the subsheaf obtained by omitting the points $(1, n)$ for $n \neq o$. Then $\mathscr{S}/\mathscr{S}'$ is isomorphic to the sheaf of Example 4.

Example 14. Let X be the complex plane. Let S_U be the additive abelian group of functions analytic in U, let S''_U be the multiplicative abelian group of non-vanishing analytic functions in U and let $j_U : S_U \to S''_U$ be the homomorphism defined by $j_U f = e^{2\pi i f}$. The system $\{j_U\}$ gives a homomorphism of presheaves and there is an induced exact sequence of sheaves

$$0 \to Z \to \mathscr{S} \to \mathscr{S}'' \to 0$$

where Z is the constant sheaf of integers. An element of $\Gamma(X, \mathscr{S})$ is a function analytic in the whole plane. **43**

More generally, in this example X can be replaced by a complex analytic manifold.

Tensor products of sheaves.

Let a, $\mathscr{B}$ be sheaves of *commutative rings with unit element* let $\mathscr{R}$, $\mathscr{S}$ be sheaves of $\mathfrak{a}$-modules and let $\mathcal{J}$ be a sheaf of $\mathscr{B}$-modules.

Definition . *A* bihomomorphism $f : (\mathfrak{a}, \mathscr{R}, \mathscr{S}) \to (\mathscr{B}, \mathcal{J})$ *consists of maps* $f' : A \to B$, $f'' : R + S \to T$, *which commute with the projections, such that, for each* $x \in X$, *the restriction* $f_x : (A_x, R_x, S_x) \to (B_x, T_x)$ *is a bihomomorphism.*

If $r \in \Gamma(U, \mathscr{R})$, $s \in \Gamma(U, \mathscr{S})$ there is a section $t : U \to T$ defined by $t(x) = f''(r(x), s(x))$. That t is a map follows from the fact that it is the composite of the two maps

$$U = U + U \xrightarrow{r,s} R + S \xrightarrow{f''} T;$$

where $U + U$ is the set of points (x, x), $x \in U$. We write $t = f''_U(r, s)$. Then $f_U : (A_U, R_U, S_U) \to (B_U, T_U)$ where $A_U = \Gamma(U, \mathfrak{a})$, etc., is a bihomomorphism, as follows from the property at each x, *e.g.*

$$f''_U(ar, s)(x) = f''((ar)(x), s(x)) = f''(a(x)r(x), s(x)$$

$$= f'(a(x)).f''(r(x), s(x)) = (f'_U(a))(x).f''_U(r,s)(x)$$
$$= (f'_U(a).f''_U(r,s))(x),$$

and similarly the other properties can be proved. Clearly f_U commutes
44 with the restriction of functions; $f_V\phi_{VU} = \theta_{VU} f_U$.

$$\begin{tikzcd}
(A_U, R_U, S_U) \arrow[r, "f_U"] \arrow[d, "\phi_{VU}"'] & (B_U, T_U) \arrow[d, "\theta_{VU}"'] \\
(A_V, R_V, S_V) \arrow[r, "f_V"'] \arrow[d, "\phi_{xV}"'] & (B_V, T_V) \arrow[d, "\theta_{xV}"'] \\
(A_x, R_x, S_x) \arrow[r, "f_x"'] & (B_x, T_x)
\end{tikzcd}$$

Also $f_x\phi_{xU}(r,s) = f(r(x), s(x)) = f_U(r,s)(x) = \theta_{xU} f_u(r,s)$; i.e., $f_x\phi_{xU} = \theta_{xU} f_U$. Thus the bihomomorphism f is determined by the system of bihomomorphisms $\{f_U\}$.

Proposition 6. *If $\mathfrak{a}$ is a sheaf of commutative rings with unit, and $\mathscr{R}$, $\mathscr{S}$ are sheaves of $\mathfrak{a}$-modules, there exists a sheaf $\mathscr{Q}$ of $\mathfrak{a}$-modules and a bihomomorphism $\alpha : (\mathscr{R}, \mathscr{S}) \to \mathscr{Q}$ with im α_x generating Q_x for each x, such that if $f : (\mathfrak{a}, \mathscr{R}, \mathscr{S}) \underline{\to} (\mathscr{B}, \mathscr{J})$ is any bihomomorphism there is a (unique) homomorphism $\bar{f} : (\mathfrak{a}, \mathscr{Q}) \to (\mathscr{B}, \mathscr{J})$ with $\bar{f}.\alpha = f$. The sheaf $\mathscr{Q}$ together with the bihomomorphism α is called the tensor product $\mathscr{R}\otimes_{\mathfrak{a}}\mathscr{S}$ and is unique upto isomorphism. Each Q_x together with $\alpha_x : (R_x, S_x) \to Q_x$ is the tensor product $R_x \otimes_{A_x} S_x$. The sections $q(U)$ where $q(x) = \sum_{i=1}^{k} \alpha_x(r_i(x), s_i(x))$ with $r_i \in \Gamma(U, a)$, $s_i \in \Gamma(U, \mathscr{S})$ and $o < k < \infty$, form a base for the open sets of $\Omega = \bigcup_x Q_x = \bigcup_x R_x \otimes_{A_x} S_x$.*

45 *Proof.* Let $\{A_U, R_U, S_U, \phi_{VU}\}$ be the presheaf $(\bar{\mathfrak{a}}, \bar{\mathscr{R}}, \bar{\mathscr{S}})$ i.e., $A_U = \Gamma(U, \mathfrak{a})$, etc. For each $\phi_{VU}, \underline{\phi_{VU}} : (A_U, R_U, S_U) \to (A_V, R_V, S_V)$ there is an induced homomorphism $\bar{\phi}_{VU} : (A_U, R_U \otimes_{A_U} S_U) \to (A_V, R_V \otimes_{A_V} S_V)$ and the system $\left\{A_U, R_U \otimes_{A_U} S_U, \bar{\phi}_{VU}\right\}$ is a presheaf determining some sheaf $(\mathfrak{a}, \mathscr{Q})$. □

Since tensor products and direct limits commute, for each x, there is a unique induced bihomomorphism $\alpha_x : (R_x, S_x) \to Q_x$ with $\alpha_x\phi_{xU} =$

$\overline{\phi}_{xU}\alpha_U$ and Q_x together with α_x is the tensor product $R_x \otimes_{A_x} S_x$. An arbitrary element $q = \sum_{i=1}^{k} \alpha_U(r_i, s_i) \in R_U \otimes_{A_U} S_U$ determines a section $q_U : U \to \Omega$ where

$$\begin{aligned} q_U(x) = \overline{\phi}_{xU} q &= \sum_{i=1}^{k} \overline{\phi}_{xU}\alpha_U(r_i, s_i) = \sum_{i=1}^{k} \alpha_x \phi_{xU}(r_i, s_i) \\ &= \sum_{i=1}^{k} \alpha_x(r_i(x), s_i(x)) \end{aligned}$$

and such sections from a base for Q.

If $f : (\mathfrak{a}, \mathscr{R}, \mathscr{S}) \to (\mathscr{B}, \mathscr{J})$ is any bihomomorphism, there is an induced homomorphism $\{f_U\} : (\overline{\mathfrak{a}}, \overline{\mathscr{R}}, \overline{\mathscr{S}}) \to (\overline{\mathscr{B}}, \overline{\mathscr{J}})$ of presheaves. Then, if $\overline{f}_U : (A_U, R_U \otimes_{A_U} S_U) \to (B_U, T_U)$ is the homomorphism induced by f_U,

$\left\{\overline{f}_U\right\} : \left\{A_U, R_U \otimes_{A_U} S_U, \phi_{VU}\right\} \to \left\{\overline{\mathscr{B}}, \overline{\mathscr{J}}\right\}$ is a homomorphism of presheaves which induces a homomorphism $\overline{f} : (\mathfrak{a}, \mathscr{Q}) \to (\mathscr{B}, \mathscr{J})$ of sheaves. Then

$$\begin{tikzcd}
(A_U, R_U, S_U) \arrow[r, "\phi_{VU}"] \arrow[d, "\alpha_U"'] \arrow[dd, bend right=60] & (A_V, R_V, S_V) \arrow[r, "\phi_{xV}"] \arrow[d, "\alpha_V"'] \arrow[dd, bend left, "f_V"] & (A_x, R_x, S_x) \arrow[d, "\alpha_x"'] \arrow[dd, bend left=60, "f_x"] \\
(A_U, R_U \otimes_{A_U} S_U) \arrow[r, "\phi_{VU}"] \arrow[d, "\overline{f}_U"'] & (A_V, R_V \otimes_{A_V} S_V) \arrow[r, "\phi_{xV}"] \arrow[d, "\overline{f}_V"'] & (A_x, Q_x) \arrow[d, "\overline{f}_x"'] \\
(B_U, T_U) \arrow[r, "\theta_{VU}"] & (B_V, T_V) \arrow[r, "\theta_{xV}"] & (B_x, T_x)
\end{tikzcd}$$

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$$\overline{f}_x \alpha_x \phi_{xU}(r, s) = \theta_{xU}\,\overline{f}_U \alpha_U(r, s) = \theta_{xU}\, f_U(r, s) = f_x \phi_{xU}(r, s);$$

thus $\overline{f}_x \alpha_x = f_x$ and hence $\overline{f} \cdot \alpha = f$. Since $\operatorname{im} \alpha_x$ generates $R_x \otimes_{A_x} S_x = Q_x$, $\overline{f}_x$ is uniquely determined by f_x; hence $\overline{f}$ is unique. That Q and α are unique upto isomorphism is proved in the usual manner.

Corollary. *If $\phi : (\mathfrak{a}, \mathscr{R}, \mathfrak{S}) \to (\mathscr{B}, \mathfrak{J}, \mathcal{U})$ is a homomorphism, there is a unique induced homomorphism $\overline{\phi} : (\mathfrak{a}, \mathscr{R} \otimes_{\mathfrak{a}} \mathscr{S}) \to (\mathscr{B}, \mathfrak{J} \otimes_{\mathscr{B}} \mathcal{U})$ with*

$\bar{\phi}\alpha = \beta\phi$

$$\begin{tikzcd}
(\mathfrak{a}, \mathscr{R},) \arrow[r, "\phi"] \arrow[d, "\phi"'] & (\mathscr{B}, \mathfrak{J}, \mathcal{U}) \arrow[d, "\beta"] \\
(\mathfrak{a}, \mathscr{R} \otimes_{\mathfrak{a}} \mathscr{S} \arrow[r, "\bar{\phi}"] & (\mathscr{B}, \mathfrak{J} \otimes_{\beta} \mathcal{U}),
\end{tikzcd}$$

Lecture 9

1 Cohomology groups of a space with coefficients in a presheaf

47

Definition. *A covering (an indexed covering) $\{U_i\}_{i\in I}$ of a space X is a system of open sets whose union is X.*

Definition. *If $\sum = \{S_U, \rho_{VU}\}$ is a presheaf of A-module where A is a fixed ring with unit element, a q-cochain $f(q = 0, 1, \ldots)$ of a covering $\mathscr{U} = \{U_i\}_{i\in I}$ with values in $\sum$ is an alternating function of $q+1$ indices with*

$$f(i_o, i_1, \ldots, i_q) \in S_{U_{i_o}} \cap \cdots \cap U_{i_q}$$

or more briefly $f(\sigma) \in S_{U_\sigma}$ where σ is the simplex $i_o, \ldots, i_q$. In particular $f(i_0, i_1, \ldots, i_q) = 0$ if $U_{i_0} \cap \cdots \cap U_{i_q} = \phi$. (A function f is called alternating if

(i) $f(i_0, i_1, \ldots, i_q) = 0$ if any two of $i_0, \ldots, i_q$ are the same,

(ii) $f(j_0, j_1, \ldots, j_q) = \pm f(i_0, i_1, \ldots, i_q)$ according as the permutation $j_0, \ldots, j_q$ of $i_0, \ldots, i_q$ is even or odd).

We will often write $\rho(V, U)$ for ρ_{VU}.

The q-cochains of $\mathscr{U}$ with values in $\sum$ form an A-module $C^q(\mathscr{U}, \sum)$. For $q < 0$, we define $C^q(\mathscr{U}, \sum) = 0$.

Definition. *The coboundary $\delta^{q+1} f$ (or simply δf) of $f \in C^q(\mathscr{U}, \sum)$ is the function on $(q+1)$-simplexes defined by*

$$(\delta^{q+1} f)(\sigma) = \sum_{j=0}^{q+1} (-1)^j \rho(U_\sigma, U_{\partial j\sigma}) f(\partial_j \sigma),$$

48 *where $\partial_j \sigma = i_0, \ldots, \hat{i}_j, \ldots, i_{q+1} = i_0, \ldots, i_{j-1}, i_{j+1}, \ldots, i_{q+1}$ is the j-th face of $\sigma = i_o, \ldots, i_{q+1}$.*

If $f \in C^q(\mathscr{U}, \sum)$, then $\delta f \in C^{q+1}(\mathscr{U}, \sum)$.

Proof. It is sufficient to verify that f is an alternating function, e.g.,

$$\begin{aligned}
\delta f(i_1, i_o, \ldots, i_{q+1}) &= \rho(U_\sigma, U_{\partial_1\sigma}) f(\partial_1 \sigma) - \rho(U_\sigma, U_{\partial_o\sigma}) f(\partial_o \sigma) \\
&\quad + \sum_{j=2}^{q+1} (-1)^j \rho(U_\sigma, U_{\partial j\sigma}) f(i_1, i_o, \ldots, \hat{i}_j, \ldots, i_{q+1}) \\
&= -\rho(U_\sigma, U_{\partial_o j\sigma}) f(\partial_o \sigma) + \rho(U_\sigma, U_{\partial_1\sigma}) f(\partial_1 \sigma) \\
&\quad - \sum_{j=2}^{q+1} (-1)^j \rho(U_\sigma, U_{\partial j\sigma}) f(i_o, i_1, \ldots, \hat{i}_j, \ldots, i_{q+1}) \\
&= -\delta f(i_o, i_1, \ldots, i_{q+1})
\end{aligned}$$

and, if $i_o = i_1$,

$$\begin{aligned}
\delta f(i_o, i_1, \ldots, i_{q+1}) &= \rho(U_\sigma, U_{\partial_o\sigma}) f(\partial_o \sigma) - \rho(U_\sigma, U_{\partial_1\sigma}) f(\partial_1 \sigma \partial) \\
&= 0,
\end{aligned}$$

where $\sigma = i_o, \ldots, i_{q+1}$. $\square$

It follows, since $\rho(U_\sigma, U_{\partial j\sigma})$ is a homomorphism, that

$$\delta^{q+1} : C^q(\mathscr{U}, \sum) \to C^{q+1}(\mathscr{U}, \sum)$$

is a homomorphism. One verifies by computation that $\delta^{q+1}\delta^q f = 0$ for $f \in C^{q-1}(\mathscr{U}, \sum)$, using the fact that for $j \leq k$

$$\partial_k \partial_j \sigma = \delta_k(i_o, \ldots, \hat{i}_j, \ldots, i_{k+1}, \ldots, i_{q+1}) = i_o, \ldots, \hat{i}_j, \ldots, \widehat{i_{k+1}}, \ldots, i_{q+1}$$

$$= \partial_j \partial_{k+1} \sigma.$$

49 (The computation is carried out at the end of the lecture).

Thus in $\delta^q \subset \ker \delta^{q+1}$ in the sequence

$$0 \to C^o(\mathscr{U}, \sum) \xrightarrow{\delta'} C'(\mathscr{U}, \sum) \to \cdots$$
$$\to C^{q-1}(\mathscr{U}, \sum) \xrightarrow{\delta^q} C^q(\mathscr{U}, \sum) \xrightarrow{\delta^{q+1}}$$

The quotient module $H^q(\mathscr{U}, \sum) = \ker \delta^{q+1} / \operatorname{im} \delta^q$ is called the q-th *cohomology module of $\mathscr{U}$ with coefficients in the presheaf $\sum$.*

The elements of the module $Z^q(\mathscr{U}, \sum) = \ker \delta^{q+1}$ are called *q-cocycles* and the elements of the module $B^q(\mathscr{U}, \sum) = \operatorname{im} \delta^q$ are called *q-coboundaries*. Since $B^o(\mathscr{U}, \sum) = 0$, we have $H^o(\mathscr{U}, \sum) \approx Z^o(\mathscr{U}, \sum)$.

Definition. *A covering $\mathscr{W} = \{V_j\}_{j \in J}$ is said to be a refinement of the covering $\mathscr{U} = \{U_i\}_{i \in I}$ if for each $j \in J$ there is some $i \in I$ with $V_j \subset U_i$.*

If $\mathscr{W}$ is a refinement of $\mathscr{U}$, choose a function $\tau : J \to I$ with $V_j \subset U_{\tau(j)}$. Then there is a homomorphism

$$\tau^+ : C^q(\mathscr{U}, \sum) \to C^q(\mathscr{W}, \sum)$$

defined by

$$\tau^+ f(\sigma) = \rho(V_\sigma, U_{\tau(\sigma)}) f(\tau\sigma)$$

where $\sigma = j_0, \ldots, j_q$; and $\tau\sigma = \tau(j_0), \ldots, \tau(j_q)$.

τ^+ commutes with δ since

$$\begin{aligned}
\delta^{q+1} \tau^+ f(\sigma) &= \sum_{k=0}^{q+1} (-1)^k \rho(V_\sigma, V_{\partial k \sigma}) \tau^+ f(\partial_k \sigma) \\
&= \sum_{k=0}^{q+1} (-1)^k \rho(V_\sigma, V_{\partial k \sigma}) \rho(V_{\partial k \sigma}, U_{\tau(\partial_k \sigma)}) f(\tau(\partial_k \sigma)), \\
&= \sum_{k=0}^{q+1} (-1)^k \rho(V_\sigma, U_{\tau(\partial_k \sigma)}) f(\tau(\partial_k \sigma),
\end{aligned}$$

$$\tau^+ \delta^{q+1} f(\sigma) = \rho(V_\sigma, U_{\tau\sigma}) \delta f(\tau\sigma)$$

$$= \sum_{k=0}^{q+1} (-1)^k \rho(V_\sigma, U_{\tau\sigma}) \rho(U_{\tau\sigma}, U_{\partial_k \tau\sigma}) f(\partial_k \tau\sigma),$$
$$= \sum_{k=0}^{q+1} (-1)^k \rho(V_\sigma, U_{\partial_k \tau\sigma}) f(\partial_k \tau\sigma)$$

50 and $\tau\partial_k\sigma = \partial_k\tau\sigma$. Hence there is an induced homomorphism,

$$\tau_{\mathscr{W}\mathscr{U}} : H^q(\mathscr{U}, \sum) \to H^q(\mathscr{W}, \sum).$$

The homomorphism $\tau_{\mathscr{W}\mathscr{U}} : H^q(\mathscr{U}, \sum) \to H^q(\mathscr{W}, \sum)$ *is independent of the choice of* τ.

Proof. Let $\tau : J \to I$, $\tau' : J \to I$ be two such choices. Let the set J be linearly ordered and define the function

$$k^{q-1} : C^q(\mathscr{U}, \sum) \to C^{q-1}(\mathscr{W}, \sum)$$

by

$$(k^{q-1} f)(\sigma) = \sum_{h=0}^{q-1} (-1)^h \rho(V_\sigma, U_{\tau_h\sigma}) f(\tau_h\sigma)$$

for $\sigma = j_0, \ldots, j_{q-1}$ with $j_0 < j_1 < \cdots < j_{q-1}$, where

$$\tau_h\sigma = \tau(j_0), \ldots, \tau(j_h), \tau'(j_h), \ldots, \tau'(j_{q-1}),$$

and let $k^{q-1} f$ be alternating. Then k^{q-1} is a homomorphism, since $\rho(V_\sigma, U_{\tau_h\sigma}) : S_{U_{\tau_h}\sigma} \to S_{V-\sigma}$ is a homomorphism. □

51 Using the facts that, for $\sigma = j_0, j_1, \ldots, j_q$,

$$\begin{aligned}
\tau_h\partial_i &= \partial_i\tau_{h+1} \qquad (0 \le i \le h \le q-1),\\
\tau_h\partial_i &= \partial_{i+1}\tau_h \qquad (0 \le h < i \le q),\\
\partial_h\tau_{h-1} &= \partial_h\tau_h \qquad (1 \le h \le q),\\
\partial_0\tau_0 &= \tau', \quad \partial_{q+1}\tau_q = \tau,
\end{aligned}$$

one finds that

$$\delta^q k^{q-1} f + k^q \delta^{q+1} f = \tau'^+ f - \tau^+ f.$$

(The computation is given at the end of the lecture.)

This holds for $q = 0$ with the obvious meaning of $k^{-1} : C^0(\mathscr{U}, \sum) \to 0$. Thus if $r \in H^q(\mathscr{U}, \sum)$ is represented by a cocycle f, $\tau'^{+} f - \tau^{+} f$ is a coboundary and $\tau_{\mathscr{W}\mathscr{U}} r$ is uniquely determined.

If the covering $\mathscr{W}$ is a refinement of $\mathscr{W}$ and $\mathscr{W}$ is a refinement of $\mathscr{U}$, then $\tau_{\mathscr{W}\mathscr{W}}\tau_{\mathscr{W}\mathscr{U}} = \tau_{\mathscr{W}\mathscr{U}}$ and $\tau_{\mathscr{U}\mathscr{U}}$ is the identity.

Proof. If $\mathscr{W} = \{W_k\}_{k\in K}$ is a refinement of $\mathscr{W}$, choose $\tau_1 : K \to J$ so that $W_k \subset V_{\tau_1 k}$. Then $W_k \subset V_{\tau_1 k} \subset U_{\tau\tau_1 k}$ and $\tau_2 : K \to I$ can be chosen to be $\tau\tau_1$. Then

$$\begin{aligned}(\tau_1^{+}\tau^{+} f)(\sigma) &= \rho(W_\sigma, V_{\tau_1\sigma})(\tau^{+} f)(\tau_1\sigma)\\ &= \rho(W_\sigma, V_{\tau_1\sigma})\rho(V_{\tau_1\sigma}, U_{\tau\tau_1\sigma}) f(\tau\tau_1\sigma)\\ &= \rho(W_\sigma, U_{\tau_2\sigma}) f(\tau_2\sigma)\\ &= (\tau_2^{+} f)(\sigma).\end{aligned}$$

Thus $\tau_1^{+}\tau^{+} = \tau_2^{+}$ and so for the induced homomorphisms, **52**

$$\tau_{\mathscr{W}\mathscr{W}}\tau\mathscr{W}\mathscr{U} = \tau\mathscr{W}\mathscr{U} : H^q(\mathscr{U}, \sum) \to H^q(\mathscr{W}, \sum).$$

Similarly, for the refinement $\mathscr{U}$ of $\mathscr{U}$, $\tau : I \to I$ can be chosen to be the identity, hence $\tau_{\mathscr{U}\mathscr{U}} : H^q(\mathscr{U}, \sum) \to H^q(\mathscr{U}, \sum)$ is the identity. □ **53**

(1) $\delta^{q+1}\delta^q = 0$. (2) $\delta^q k^{q-1} + k^q \delta^{q+1} = \tau'^{+} - \tau^{+}$.

(1) $$(\delta^{q+1}\delta^q f)(\sigma) = \sum_{j=0}^{q+1}(-1)^j \rho(U_\sigma, U_{\partial_j\sigma})(\delta^q f)(\partial_j\sigma)$$

$$= \sum_{j=0}^{q+1}(-1)^j \sum_{k=0}^{q}(-1)^k \rho(U_\sigma, U_{\partial_j\sigma})\rho(U_{\partial_j\sigma}, U_{\partial_k\partial_j\sigma}) f(\partial_k\partial_j\sigma)$$

$$= \sum_{j=0}^{k}\sum_{k=0}^{q}(-1)^{j+k}\rho(U_\sigma, U_{\partial_j\partial_{k+1}\sigma}) f(\partial_j\partial_{k+1}\sigma) +$$

$$\sum_{j=k+1}^{q+1}\sum_{k=0}^{q}(-1)^{j+k}\rho(U_\sigma, U_{\partial_k\partial_j\sigma}) f(\partial_k\partial_j\sigma)$$

$$= 0$$

$$(2) \quad (\delta^q k^{q-1} f)(\sigma) = \sum_{i=0}^{q} (-1)^i \rho(V_\sigma, V_{\partial_i\sigma})(k^{q-1} f)(\partial_i\sigma)$$

$$= \sum_{i=0}^{q} \sum_{h=0}^{q-1} (-1)^{i+h} \rho(V_\sigma, U_{\tau_h\partial_i\sigma}) f(\tau_h\partial_i\sigma)$$

$$= \sum_{i=0}^{h} \sum_{h=0}^{q-1} (-1)^{i+h} \rho(V_\sigma, U_{\partial_i\tau_{h+1}\sigma}) f(\partial_i\tau_{h+1}\sigma)$$

$$+ \sum_{i=h+1}^{q} \sum_{h=0}^{q-1} (-1)^{i+h} \rho(V_\sigma, U_{\partial_{i+1}\tau_h\sigma}) f(\partial_{i+1}\tau_h\sigma)$$

$$= \sum_{i=0}^{h-1} \sum_{h=0}^{q} (-1)^{i+h-1} \rho(V_\sigma, U_{\partial_i\tau_h\sigma}) f(\partial_i\tau_h\sigma)$$

$$+ \sum_{i=h+2}^{q+1} \sum_{h=0}^{q-1} (-1)^{i+h-1} \rho(V_\sigma, U_{\partial_i\tau_h\sigma}) f(\partial_i\tau_h\sigma)(k^q\delta^{q+1} f)(\sigma)$$

$$= \sum_{h=0}^{q} (-1)^h \rho(V_\sigma, U_{\tau_h\sigma})(\delta^{q+1} f)(\tau_h\sigma)$$

$$= \sum_{h=0}^{q} \sum_{i=0}^{q+1} (-1)^{i+h} \rho(V_\sigma, U_{\partial_i\tau_h\sigma}) f(\partial_i\tau_h\sigma), (\delta^q k^{q-1} f + k^q \delta^{q+1} f)(\sigma)$$

$$= \sum_{h=0}^{q} \rho(V_\sigma, U_{\partial_h\tau_h\sigma}) f(\partial_h\tau_h\sigma) - \sum_{h=0}^{q} \rho(V_\sigma, U_{\partial_{h+1}\tau_h\sigma}) f(\partial_{h+1}\tau_h\sigma)$$

$$= \sum_{h=0}^{q} \rho(V_\sigma, U_{\partial_h\tau_h\sigma}) f(\partial_h\tau_h\sigma) - \sum_{h=1}^{q+1} \rho(V_\sigma, U_{\partial_h\tau_{h-1}\sigma}) f(\partial_h\tau_{h-1}\sigma)$$

$$= \rho(V_\sigma, U_{\partial_0\tau_0\sigma}) f(\partial_0\tau_0\sigma) - \rho(V_\sigma, U_{\partial_{q+1}\tau_q\sigma}) f(\partial_{q+1}\tau_q\sigma)$$

$$= \rho(V_\sigma, U_{\tau'\sigma}) f(\tau'\sigma) - \rho(V_\sigma, U_{\tau\sigma}) f(\tau\sigma)$$

$$= (\tau'^{+} f)(\sigma) - (\tau^{+} f)(\sigma).$$

Lecture 10

Let $h = \{h_U\} : \Sigma' \to \sum$ be a homomorphism of presheaves, i.e., each 55
$h_U : S'_U \to S_U$ is a homomorphism and, if $V \subset U$, $h_V \rho'_{VU} = \rho_{VU} h_U$.
We define, for each $q \geq 0$, the mapping

$$h^+ : C^q(\mathscr{U}, \Sigma') \to C^q(\mathscr{U}, \sum)$$

by $(h^+ f)(\sigma) = h_{U_\sigma} f(\sigma)$. Then, h^+ is a homomorphism since each h_{U_σ} is a homomorphism.

h^+ commutes with δ

Proof.

$$\begin{aligned}(h^+ \delta^{q+1} f)(\sigma) &= h_{U_\sigma}(\delta^{q+1} f)(\sigma)\\ &= h_{U_\sigma} \sum_{j=0}^{q+1} (-1)^j \rho'(U_\sigma, U_{\partial_j \sigma}) f(\partial_j \sigma),\end{aligned}$$

and
$$\begin{aligned}(\delta^{q+1} h^+ f)(\sigma) &= \sum_{j=0}^{q+1} (-1)^j \rho(U_\sigma, U_{\partial_j \sigma})(h^+ f)(\partial_j \sigma)\\ &= \sum_{j=0}^{q+1} (-1)^j \rho(U_\sigma, U_{\partial_j \sigma}) h_{U_{\partial j \sigma}} f(\partial_j \sigma)\\ &= h_{U_\sigma} \sum_{j=0}^{q+1} (-1)^j \rho'(U_\sigma, U_{\partial_j \sigma}) f(\partial_j \sigma)\end{aligned}$$

Q. e. d.

Hence h^+ induces a homomorphism $h_{\mathscr{U}} : H^q(\mathscr{U}, \Sigma') \to H^q(\mathscr{U}, \sum)$. h^+ commutes with τ^+ ◻

Proof.

$$\begin{aligned}(h^+\tau^+ f)(\sigma) &= h_{V_\sigma}(\tau^+ f)(\sigma)\\ &= h_{V_\sigma}\rho'(V_\sigma, U_{\tau\sigma}) f(\tau\sigma)\\ &= \rho(V_\sigma, U_{\tau\sigma}) h_{U_{\tau\sigma}} f(\tau\sigma)\\ &= (\tau^+ h^+ f)(\sigma).\end{aligned}$$

56 *Hence $h_{\mathscr{W}}\tau_{\mathscr{W}\mathscr{U}} = \tau_{\mathscr{W}\mathscr{U}} h_{\mathscr{U}} : H^q(\mathscr{U}, \Sigma') \to H^q(\mathscr{W}, \sum)$,* i.e., the following diagram is commutative. ◻

$$\begin{tikzcd} H^q(\mathscr{U}, \sum') \arrow[r, "h_{\mathscr{U}}"] \arrow[d, "\tau_{\mathscr{W}\mathscr{U}}"'] & H^q(\mathscr{U}, \sum) \arrow[d, "\tau_{\mathscr{W}\mathscr{U}}"] \\ H^q(\mathscr{W}, \sum') \arrow[r, "h_{\mathscr{W}}"'] & H^q(\mathscr{W}, \sum). \end{tikzcd}$$

If $\sum' \xrightarrow{h} \sum \xrightarrow{g} \sum''$ is a sequence of homomorphisms of presheaves, then gh induces a homomorphism

$$(gh)^+ : C^q(\mathscr{U}, \sum{}') \to C^q(\mathscr{U}, \sum{}'')$$

such that $(gh)^+ = g^+h^+$ and, if h is the identity, h^+ is the identity.

Proof.

$$\begin{aligned}((gh)^+ f)(\sigma) &= (gh)_{U_\sigma} f(\sigma)\\ &= g_{U_\sigma} h_{U_\sigma} f(\sigma)\\ &= (g^+h^+ f)(\sigma).\end{aligned}$$

If h is the identity, i.e., if each h_U is the identity, then $(h^+ f)(\sigma) = h_{U_\sigma} f(\sigma) = f(\sigma)$; so h^+ is the identity, q.e.d. ◻

If one has the commutative diagram of homomorphisms of presheaves, i.e., $gh = h_1 g_1$, then

$$\begin{tikzcd} \sum \arrow[r, "h"] \arrow[d, "g_1"'] & \sum' \arrow[d, "g"] \\ \sum_1'' \arrow[r, "h_1"'] & \sum'' \end{tikzcd}$$

$$g^+h^+ = (gh)^+ = (h_1g_1)^+ = h_1^+g_1^+,$$

and hence $g_{\mathscr{U}} h_{\mathscr{U}} = h_{1\mathscr{U}} g_{1\mathscr{U}}$.

If $\mathscr{U} = \{U_i\}_{i \in I}$ is a covering and the sequence $\sum' \xrightarrow{h} \sum \xrightarrow{g} \sum''$ of homomorphisms of presheaves is exact, then the sequence **57**

$$C^q(\mathscr{U}, \sum{}') \xrightarrow{h^+} C^q(\mathscr{U}, \sum) \xrightarrow{g^+} C^q(\mathscr{U}, \sum{}'')$$

is also exact.

Proof. (i) If $f \in C^q(\mathscr{U}, \sum)$ is an element of $\operatorname{im} h^+$, clearly $f \in \ker g^+$, hence $\operatorname{im} h^+ \subset \ker g^+$.

(ii) Linearly order the index set I, and let $f \in C^q(\mathscr{U}, \sum)$ be an element of $\ker g^+$. Then $f(\sigma) \in \ker g_{U_\sigma} = imh_{U_\sigma}$ for each q-simplex σ, hence there is an element r in the module corresponding to the open set U_σ, of the presheaf Σ', such that $h_{U_\sigma}(r) = f(\sigma)$. If $\sigma = (i_0, \ldots, i_q)$ with $i_0 < \cdots < i_q$, define the function t on σ by $t(\sigma) = r$. If $\sigma' = (j_0, \ldots, j_q)$ is a permutation of $\sigma = (i_0, \ldots, i_q)$ define t on σ' by $t(\sigma') = \pm t(\sigma)$ according as σ' is an even or odd permutation of σ. If σ is a q-simplex in which two indices are repeated, define $t(\sigma)$ to be zero. It then follows that $t \in C^q(\mathscr{U}, \Sigma')$ and it is easily verified that $h^+(t) = f$, hence $\ker g^+ \subset imh^+$.

□

If the sequence $0 \to \sum' \xrightarrow{i} \sum \xrightarrow{j} \sum'' \to 0$ of homomorphisms of presheaves is exact, there is an induced homomorphism

$$\delta_{\mathscr{U}} : H^q(\mathscr{U}, \sum{}'') \to H^{q+1}(\mathscr{U}, \sum{}').$$

Proof. Since the homomorphisms i^+, j^+ commute with the homomorphism δ, there is commutativity in the following diagram:
$(*)$

$$\begin{tikzcd}
 & {} \arrow[d, "\delta"'] & {} \arrow[d, "\delta"'] & {} \arrow[d, "\delta"'] & \\
0 \arrow[r] & C^q(\mathscr{U},\sum') \arrow[r, "i^+"] \arrow[d, "\delta"'] & C^q(\mathscr{U},\sum) \arrow[r, "j^+"] \arrow[d, "\delta"'] & C^q(\mathscr{U},\sum'') \arrow[r] \arrow[d, "\delta"'] & 0 \\
0 \arrow[r] & C^{q+1}(\mathscr{U},\sum') \arrow[r, "i^+"] \arrow[d, "\delta"'] & C^{q+1}(\mathscr{U},\sum) \arrow[r, "j^+"] \arrow[d, "\delta"'] & C^{q+1}(\mathscr{U},\sum'') \arrow[r] \arrow[d, "\delta"'] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

□

58 Since the sequence $0 \to \sum' \to \sum \to \sum'' \to 0$ is exact, each row of the diagram is an exact sequence of homomorphisms. We will construct a homomorphism $\theta : Z^q(\mathscr{U},\sum'') \to H^{q+1}(\mathscr{U},\sum')$ which is zero on $B^q(\mathscr{U},\sum'')$, and hence θ will induce a homomorphism from $H^q(\mathscr{U},\sum'') \to H^{q+1}(\mathscr{U},\sum')$.

To do this, let $r \in Z^q(\mathscr{U},\Sigma'')$, and choose $s \in C^q(\mathscr{U},\sum)$ with $j^+s = r$. Since $\delta j^+s = j^+\delta s = \delta r = 0, \delta s \in \ker j^+$ and by exactness, there is a unique $t \in C^{q+1}(\mathscr{U},\Sigma')$ with $i^+t = \delta s$. Then $i^+\delta t = \delta i^+ t = \delta\delta s = 0$, hence $\delta t = 0$. Let $\tau \in H^{q+1}(\mathscr{U},\Sigma')$ be the element represented by t. To show that τ is unique. let s_1, t_1 be the result of a second such choice, then $j^+(s - s_1) = r - r = 0$ and $s - s_1 = i^+u$ for a unique $u \in C^q(\mathscr{U},\Sigma')$. Then since i^+ is a monomorphism and

$$i^+(t - t_1) = \delta(s - s_1) = \delta i^+ u = i^+\delta u,$$

hence $t - t_1 = \delta u$. Thus t and t_1 represent the same element $\tau \in H^{q+1}(\mathscr{U},\Sigma')$.

Let $\tau = \theta(r)$. If $r = ar_1 + br_2 \in Z^q(\mathscr{U},\Sigma'')$, suppose that $r_1 = j^+s_1$, $\delta s_1 = i^+t_1$ and that $r_2 = j^+s_2, \delta s_2 = i^+t_2$, and let $\tau_1 = \theta(r_1)$, $\tau_2 = \theta(r_2)$
59 be the elements represented by t_1 and t_2. Then since j^+, δ and i^+ are homomorphisms,

$$r = j^+(as_1 + bs_2), \delta(as_1 + bs_2) = i^+(at_1 + bt_2)$$

and, since $at_1 + bt_2$ represents $a\tau_1 + b\tau_2$, we have

$$\theta(r) = a\theta(r_1) + b\theta(r_2).$$

Thus $\theta : Z^q(\mathscr{U}, \Sigma'') \to H^{q+1}(\mathscr{U}, \Sigma')$ is a homomorphism.

If $r \in B^q(\mathscr{U}, \Sigma'')$, let $r = \delta_v$. For some $w \in C^{q-1}(\mathscr{U}, \sum)v = j^+w$. Then $j^+(\delta w) = \delta j^+ w = r$ and there exists a unique $t \in C^{q+1}(\mathscr{U}, \Sigma')$ with $i^+t = \delta(\delta w) = 0$, hence $t = 0$; i.e., $\theta(r) = 0$. Thus θ induces a homomorphism

$$\delta_{\mathscr{U}} : H^q(\mathscr{U}, \Sigma'') \to H^{q+1}(\mathscr{U}, \Sigma').$$

Q.e.d.

$\tau_{\mathscr{W},\mathscr{U}}$ commutes with $\delta_{\mathscr{U}}$, i.e., the following diagram is commutative.

$$\begin{tikzcd} H^q(\mathscr{U}, \sum'') \arrow[r, "\delta_{\mathscr{U}}"] \arrow[d, "\tau_{\mathscr{W}\mathscr{U}}"'] & H^{q+1}(\mathscr{U}, \sum') \arrow[d, "\tau_{\mathscr{W}\mathscr{U}}"] \\ H^q(\mathscr{W}, \sum'') \arrow[r, "\delta_{\mathscr{W}}"'] & H^{q+1}(\mathscr{W}, \sum') \end{tikzcd}$$

Proof. τ^+ commutes with j^+, δ, i^+. □

If $0 \to \sum' \xrightarrow{i} \sum \xrightarrow{j} \sum'' \to 0$ *is exact, then the sequence*

$$0 \to H^0\left(\mathscr{U}, \sum'\right) \to \cdots \to H^q\left(\mathscr{U}, \sum'\right) \xrightarrow{i_{\mathscr{U}}}$$
$$H^q\left(\mathscr{U}, \sum\right) \xrightarrow{j_{\mathscr{U}}} H^q\left(\mathscr{U}, \sum''\right) \xrightarrow{\delta_{\mathscr{U}}} H^{q+1}\left(\mathscr{U}, \sum'\right)$$

is exact.

Proof. The exactness of this sequence is the result of six properties of **60**
the form ker ⊂ im and im ⊂ ker. Each can be easily verified in (∗). (See Eilenberg-Steenrod, Foundations of Algebraic Topology, p. 128). □

If $0 \to \sum' \xrightarrow{i} \sum \xrightarrow{j} \sum'' \to 0$ *and* $0 \to \sum_1' \xrightarrow{i_1} \sum_1 \xrightarrow{j_1} \sum_1'' \to 0$ *are exact sequence, and if* $h : (\sum', \sum, \sum'') \to (\sum_1', \sum_1, \sum_1'')$ *is a homomorphism commuting with* i, j, i_1, *and* j_1, *then* $h_{\mathscr{U}}$ *commutes with* $\delta_{\mathscr{U}}$.

Proof. The homomorphism h^+ commutes with the homomorphisms j^+, δ and i^+, *q.e.d.* □

With the same assumptions as in the above statement, we then have the following commutative diagram, in which each row is exact.

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathscr{U}, \sum') \arrow[r] \arrow[d, "h_{\mathscr{U}}"'] & \cdots \arrow[r] & H^q(\mathscr{U}, \sum) \arrow[r, "i_{\mathscr{U}}"] \arrow[d, "h'_{\mathscr{U}}"'] & H^q(\mathscr{U}, \sum) \arrow[r, "j_{\mathscr{U}}"] \arrow[d, "h_{\mathscr{U}}"'] & H^q(\mathscr{U}, \sum'') \arrow[r, "\delta_{\mathscr{U}}"] \arrow[d, "h''_{\mathscr{U}}"'] & H^{q+1}(\mathscr{U}, \sum') \arrow[r] \arrow[d, "h'_{\mathscr{U}}"'] & \cdots \\
0 \arrow[r] & H^0(\mathscr{U}, \sum_1') \arrow[r] & \cdots \arrow[r] & H^q(\mathscr{U}, \sum_1') \arrow[r, "i_{1\mathscr{U}}"] & H^q(\mathscr{U}, \sum_1) \arrow[r, "j_{1\mathscr{U}}"] & H^q(\mathscr{U}, \sum_1'') \arrow[r, "\delta_{1\mathscr{U}}"] & H^{q+1}(\mathscr{U}, \sum_1') \arrow[r] & \cdots
\end{tikzcd}$$

Definition. *A proper covering of X is a set of open sets whose union is X.*

A proper covering $\mathscr{U} = \{U\}$ of X may be regarded as an indexed covering $\{U_U\}_{U \in \mathscr{U}}$ if each open set of the covering is indexed by itself. Every covering $\{U_i\}_{i \in I}$ has a refinement which is a proper covering, e.g., the set of all open sets U such that $U = U_i$ for some $i \in I$.

61 Let Ω be the set of all proper coverings of X and let $\mathscr{U} < \mathscr{W}$ mean that $\mathscr{W}$ is a refinement of $\mathscr{U}$. Then Ω *is a directed set*, for 1) $\mathscr{U} < \mathscr{U}$ 2) if $\mathscr{U} < \mathscr{W}$ and $\mathscr{W} < \mathscr{W}$ then trivially $\mathscr{U} < \mathscr{W}$ and 3) given $\mathscr{U}, \mathscr{W}$ there exists $\mathscr{W}$ with $\mathscr{U} < \mathscr{W}$, $\mathscr{W} < \mathscr{W}$, e.g, $\mathscr{W}$ may be chosen to consists of all open sets W with $W = \cup \cap V$ for some $U \in \mathscr{U}$, $V \in \mathscr{W}$.

(There is no set of *all* indexed coverings).

The system $\{H^q(\mathscr{U}, \sum), \tau_{\mathscr{W}\mathscr{U}}\}_{\mathscr{U}, \mathscr{W} \in \Omega}$ is then a direct system. Its direct limit $H^q(X, \sum)$ is called the *q-th cohomology module (over A) of X with coefficients in* $\sum$. Let $\tau_{\mathscr{U}} : H^q(\mathscr{U}, \sum) \to H^q(X, \sum)$ denote the usual homomorphism in to the direct limit.

If $h : \sum' \to \sum$ *is a homomorphism of presheaves, there is a induced homomorphism*

$$h^* : H^q(X, \sum{}') \to H^q(X, \sum) \text{with } h^* \tau_{\mathscr{U}} = \tau_{\mathscr{U}} h_{\mathscr{U}}.$$

Proof. This follows from the fact that $h_{\mathscr{U}}\tau_{\mathscr{W}\mathscr{U}} = \tau_{\mathscr{W}\mathscr{U}}h_{\mathscr{U}}$.

Q.e.d. □

If $O \to \sum' \xrightarrow{i} \sum \xrightarrow{j} \sum'' \to O$ is an exact sequence of presheaves, there is an induced exact sequence

$$0 \to H^0(X, \textstyle\sum') \to \cdots \to H^q(X, \textstyle\sum') \to H^q(X, \textstyle\sum) \to H^q(X, \textstyle\sum'') \to H^{q+1}(X, \textstyle\sum') \to \cdots$$

Proof. This is a consequence of the fact that the direct limits of exact sequences is again an exact sequence. □

If $h : (\sum', \sum, \sum'') \to (\sum'_1, \sum_1, \sum''_1)$ is a homomorphism of exact sequence of presheaves **62**

$$\begin{tikzcd}
0 \arrow[r] & \sum' \arrow[r, "i"] \arrow[d, "h'"'] & \sum \arrow[r, "j"] \arrow[d, "h"'] & \sum'' \arrow[r] \arrow[d, "h''"'] & 0 \\
0 \arrow[r] & \sum'_1 \arrow[r, "i_1"] & \sum_1 \arrow[r, "j_1"] & \sum''_1 \arrow[r] & 0
\end{tikzcd}$$

and h commutes with i, j, i_1, j_1 then the following diagram, where h^* is the homomorphism induced from h, is a commutative diagram.

$$\begin{tikzcd}
\cdots \arrow[r] & H^q(X, \sum') \arrow[r, "i^*"] \arrow[d, "h^*"'] & H^q(x, \sum) \arrow[r, "j^*"] \arrow[d, "h^*"'] & H^q(x, \sum'') \arrow[r, "\delta^*"] \arrow[d, "h^*"'] & H^q(x, \sum') \arrow[r] \arrow[d, "h^*"'] & \cdots \\
\cdots \arrow[r] & H^q(X, \sum'_1) \arrow[r, "i^*_1"] & H^q(X, \sum_1) \arrow[r, "j^*_1"] & H^q(X, \sum''_1) \arrow[r, "\delta^*_1"] & H^{q+1}(X, \sum'_1) \arrow[r] & \cdots
\end{tikzcd}$$

Proof. The result is a consequences of the fact that $h_{\mathscr{U}}$ commutes with $i_{\mathscr{U}}$, $j_{\mathscr{U}}$ and $\delta_{\mathscr{U}}$. □

Lecture 11

If the coefficient presheaf is the presheaf of sections of some sheaf $\mathscr{S}$ **63**
of A-modules, we write $C^q(\mathscr{U}, \mathscr{S})$ instead of $C^q(\mathscr{U}, \overline{\mathscr{S}})$ etc. Then, if $U_\sigma = U_{i_o} \cap \cdots \cap U_{i_q}$ is called the *support* of the simplex $\sigma = i_o, \ldots, i_q$, a q-cochain $f \in C^q(\mathscr{U}, \mathscr{S})$ is an alternating function which assigns to each q-simplex σ a section over the support of σ.

If $\mathscr{U} = \{U_i\} i \in I$ is any covering of X, $H^o(\mathscr{U}, \mathscr{S})$ is isomorphism to $\Gamma(X, \mathscr{S})$.

Proof. A 0-cochain belonging to $C^0(\mathscr{U}, \mathscr{S})$ is a system $(f_i)_{i \in I}$, each f_i being a section of $\mathscr{S}$ over U_i. In order that this cochain be a cocycle, it is necessary and sufficient that $f_i - f_j = O$ over $U_i \cap U_j$; in other words, that there exist a section $f \in \Gamma(X, \mathscr{S})$ which coincides with f_i on U_i for each $i \in I$. Thus there is an isomorphism $\phi_{\mathscr{U}} : \Gamma(X, \mathscr{S}) \to Z^0(\mathscr{U}, \mathscr{S}) \to H^0(\mathscr{U}, \mathscr{S})$. ◻

Proposition 7. *$H^0(X, \mathscr{S})$ can be identified with* $\Gamma(X, \mathscr{S})$.

Proof. Since $\tau_{\mathscr{W}\mathscr{U}} \phi_{\mathscr{U}} = \phi_{\mathscr{W}}$, there is an induced isomorphism ϕ : $\Gamma(X, \mathscr{S}) \to H^0(X, \mathscr{S})$ with $\tau_{\mathscr{U}} \phi_{\mathscr{U}} = \phi$. A homomorphism $h : \mathscr{S} \to \mathscr{S}_1$ of sheaves induces a homomorphism $\{h_U\}$ of the presheaves of section and hence induced homomorphism $h_{\mathscr{U}'}$, h^* with commutativity in ◻

$$\begin{tikzcd}
\Gamma(X, \mathscr{S}) \arrow[r, "\phi_{\mathscr{U}}"] \arrow[d, "h_X"'] & H^0(\mathscr{U}, \mathscr{S}) \arrow[r, "\tau_{\mathscr{U}}"] \arrow[d, "h_{\mathscr{U}}"'] & H^0(X, \mathscr{S}) \arrow[d, "h^*"] \\
\Gamma(X, \mathscr{S}_1) \arrow[r, "\phi_{\mathscr{U}}"] & H^0(\mathscr{U}, \mathscr{S}_1) \arrow[r, "\tau_{\mathscr{U}}"] & H^0(X, \mathscr{S}_1)
\end{tikzcd}$$

64 Thus we can identify $\Gamma(X, \mathscr{S})$ with $H^o(X, \mathscr{S})$ under ϕ if we also identify $h_x : \Gamma(X, \mathscr{S}) \to \Gamma(X, \mathscr{S}_1)$ with $h^* : H^o(X, \mathscr{S}) \to H^o(X, \mathscr{S}_1)$.

Definition. *A system $\{A_i\}_{i\in I}$ of subset of a space X is called finite if I is finite, countable if I is countable. The system is said to be locally finite if each point $x \in X$ has a neighbourhood V such that $V \cap A_i = \phi$ expect for a finite number of i. (This finite number may also be zero).*

We notice that a locally finite system $\{A_i\}_{i\in I}$ is always point finite. (A system $\{A_i\}_{i\in I}$ of subsets of X is said to be point finite if each point $x \in X$ belongs to A_i for only a finite number of i).

If $\{A_i\}_{i\in I}$ is locally finite, so is $\{B_j\}_{j\in J}$ if $J \subset I$ and each $B_j \subset A_j$. If $\{A_i\}_{i\in I}$ is locally finite, so is $\{\bar{A}_i\}_{i\in I}$, where $\bar{A}_i$ denotes the closure of A_i, and $\overline{\bigcup_{i\in I} A_i} = \bigcup_{i\in I} \bar{A}_i$. In particular, if each A_i is closed, so is $\bigcup_{i\in I} A_i$

Definition. *The* order *of a system $\{A_i\}_{i\in I}$ of subsets of X is -1 if A_i is the empty set for each $i \in I$. Otherwise the order is the largest integer n such that for $n+1$ values of $i \in I$, the A_i's have a non-empty intersection, and it is infinity if there exists no such largest integer.*

Definition. *The* dimension of X, denoted as $\dim X$, is the least integer n
65 such that every finite covering of X has a refinement of order $\leq n$, and the dimension is infinity if there is no such integer.

Definition. *A space X is called* normal, if for each pair E, F of closed sets of X with $E \cap F = \phi$, there are open sets G, H with $E \subset G$, $F \subset H$ and $G \cap H = \phi$.

Definition. *A covering $\mathscr{U} = \{U_i\}_{i\in I}$ of the space X is called* shrinkable if there is a refinement $\mathscr{W} = \{V_i\}_{i\in I}$ of $\mathscr{U}$ with $\bar{V}_i \subset U_i$ for each $i \in I$.

X is normal if and only if every locally finite covering of X is shrinkable.

Proof. See S. Lefschetz, Algebraic Topology, p.26. □

If X is normal, $\dim X \leq n$ *if and only if every locally finite covering of X has a refinement of order $\leq n$.*

Proof. See C.H.Dowker, Amer.Jour of Math. (1947), p.211. □

Definition. *A space X is called* paracompact if every covering of X has a refinement which is locally finite.

If X is paracompact and normal, $\dim X \leq n$ *if and only if every covering of X has a refinement of order $\leq n$.*

Proof. (i) Since X is paracompact, every covering of X has a locally finite refinement and since X is normal and $\dim X \leq n$, using the above result, every locally finite covering has a refinement of order **66**
$\leq n$, thus every covering has a refinement of order $\leq n$.

(ii) Since every covering of X has a refinement of order $\leq n$, in particular, every locally finite of X has a refinement of order $\leq n$, hence, since X is normal, using the above result, we obtained $\dim \leq n$.

□

Remark. Since a paracompact Hausdorff space is normal, (see J. Dieudonne, Jour. de Math. 23, (1944), p.(66), this result holds, in particular, when X is a paracompact Hausdorff space.

If a covering $\mathscr{U}$ of X has a refinement $\mathscr{W}$ of order $\leq n$, then it has a proper refinement $\mathscr{W}$ of order $\leq n$.

Proof. If $\mathscr{W} = \{V_j\}_{j\in J}$ has order $\leq n$, let $\mathscr{W}$ be the proper covering formed by all open sets W such that, for some $j \in j$, $W = V_j$. Then $\mathscr{W}$ has order $\leq n$ and is a refinement of $\mathscr{U}$. □

If X is paracompact and normal and $\dim X \leq n$, *then* $H^q(X, \sum) = 0$ *for $q > n$ and an arbitrary presheaf $\sum$.*

Proof. Replace the directed set Ω of all proper coverings of X by the cofinal sub-directed set Ω' of all proper covering of order $\leq n$. If $\mathscr{U} \in \Omega'$, $q > n$ and $f \in C^q(\mathscr{U}, \sum)$, then $f(U_0, U_1, \ldots, U_q) \in S_{U_o\cap\cdots\cap U_q} = S_\phi = 0$ for any $q+1$ distinct open sets of $\mathscr{U}$. If the open sets $U_0, U_1, \ldots, U_q$ are not all distinct, then $f(U_0, U_1, \ldots, U_q) = 0$ since f is alternating. Hence $C^q(\mathscr{U}, \sum) = 0$ and hence also $H^q(\mathscr{U}, \sum) = 0$. Therefore $H^q(X, \sum) = 0, q > n$. □

67 *If $\sum$ is a presheaf which determines the zero sheaf, then $H^o(X, \sum) =$* 0.

Proof. For any element $\eta \in H^0(X, \sum)$ choose a representative $f \in z^0(\mathscr{U}, \sum)\mathcal{S}H^0(\mathscr{U}, \sum)$, where $\mathscr{U}$ is some proper covering of X, so that $\tau_{\mathscr{U}} f = \eta$. For each $x \in X$ choose an open set $U = \tau(x)$ such that $x \in \tau(x) \in \mathscr{U}$. Since $\sum$ determines the 0-sheaf, one can choose an open set V_x such that $x \in V_x \subset \tau(x)$ and $\rho(V_x, \tau(x)) f(\tau(x)) = 0$. Then $\mathscr{W} = \{V_x\}_{x \in X}$ is a refinement of $\mathscr{U}$, and, for each x, $(\tau^+ f)(x) = \rho(V_x, \tau(x)) f(\tau(x)) = 0$, hence $\tau^+ f = 0$. If $\mathscr{W}$ is a proper refinement of $\mathscr{W}$, choose $\tau_1 : \mathscr{W} \to X$ so that each $W \subset V_{\tau_1(W)}$ and $(\tau\tau_1)^+ f = \tau_1^+ \tau^+ f = 0$. Thus $\tau_{\mathscr{W}\mathscr{U}} f = 0$ and hence $\eta = \tau_{\mathscr{U}} f = \tau_{\mathscr{W}} \tau_{\mathscr{W}\mathscr{U}} f = 0$. Hence $H^o(X, \sum) = 0$. □

This result is not true in general for the higher dimensional cohomology groups. However, if the space X is assumed to be *paracompact and normal*, we will prove the result to be true for the higher dimensional cohomology groups.

Lecture 12

Proposition 8. *If X is paracompact and normal and if $\sum$ is a presheaf which determines the zero sheaf, the $H^q(X, \sum) = 0$ for all $q \geq 0$.* 68

Proof. Let $f \in C^q(\mathscr{U}, \sum)$ where $\mathscr{U} = \{U_i\}_{i \in I}$ is any locally finite covering. Since X is normal, we can shrink $\mathscr{U}$ to $\mathscr{W}$, $\mathscr{W} = \{W_i\}_{i \in I}$ with $\overline{W}_i \subset U_i$. For each $x \in X$ choose a neighbourhood V_x of x such that the following conditions are satisfied: □

a) If $x \in U_i$, $V_x \subset U_i$,

b) If $x \in W_i$, $V_x \subset W_i$,

c) if $x \notin \overline{W}_i$, $V_x \cap W_i = \phi$,

d) if $x \in U_{i_o} \cap \cdots \cap U_{i_q} = U_\sigma$, $\rho_{V_x U_\sigma} f(\sigma) = 0$

Conditions a) and b) can be satisfied, for the coverings $\mathscr{U}$ and $\mathscr{W}$ being locally finite, each x is contained only in a finite number of sets of the coverings. To see that condition c) can be satisfied, consider all $\overline{W}_i$ for which $x \notin \overline{W}_i$. The union of these sets is the closed since $\mathscr{W}$ is locally finite, and x is in the open complement of this union. Next, by condition a), $V_x \subset U_\sigma$, and since $\sum$ determines the 0-sheaf, we can choose V_x small enough so that d) is satisfied. We can thus always choose V_x small enough so that the above conditions are fulfilled.

If the V_x are chosen as above, the covering $\{V_x\}_{x \in X}$ is a refinement of $\mathscr{U}$. Choose the function $\tau : X \to I$ so that $x \in W_{\tau(x)}$, then by b), $V_x \subset W_{\tau(x)} \subset U_{\tau(x)}$. Then 69

$$\tau^+ f(\sigma) = \tau^+ f(x_o, \ldots, x_q) = \rho(V_\sigma, U_\sigma) f(\tau(x_o), \ldots, \tau(x_q)).$$

If $V_\sigma = \phi$, $\tau^+ f(\sigma) = 0$. If $V_\sigma \neq \phi$ then V_{x_o} meets each V_{xj}, hence meets each $W_{\tau(x_j)}$ and hence by c), $x_o \in \overline{W}_{\tau(x_j)}$. Then since $x_o \in \overline{W}_{\tau(x_j)} \subset U_{\tau(x_i)}$, by a) $V_{x_o} \subset U_{\tau(x_i)}$ for each j and hence $V_{x_o} \subset U_{\tau(\sigma)}$. Hence

$$\begin{aligned}\tau^+ f(\sigma) = \rho_{V_\sigma U_{\tau(\sigma)}} f(\tau\sigma) &= \rho_{V_\sigma V_{x_0}} \rho_{V_{x_0}} U_{\tau(\sigma)} f(\tau\sigma)\\ &= 0 \text{ by } d).\end{aligned}$$

Thus $\tau^+ f(\sigma) = 0$ for all σ, hence $\tau^+ f = 0$.

If $\mathscr{U}_1$ is a proper covering and $f \in C^q(\mathscr{U}_1, \sum)$, there is a locally finite refinement $\mathscr{U}$ of $\mathscr{U}_1$ (since X is paracompact). Then there is a refinement $\mathscr{W}$ of $\mathscr{U}$ (found as above), a proper refinement $\mathscr{W}_1$ of $\mathscr{W}$ (existence of $\mathscr{W}_1$ is trivial) and a function $\tau_1 : \mathscr{W}_1 \to \mathscr{U}_1$ with $V \subset \tau_1(V)$ for $V \in \mathscr{W}_1$ such that $\tau_1^+ f = 0$. Hence every element of $\bigcup\limits_{(\mathscr{U}_1 \text{proper})} H^q(\mathscr{U}_1, \sum)$ is equivalent to zero, i.e., the direct limit $H^q(X, \sum)$ consists only of zero.

Example 15. Let X be the space with four points a, b, c, d and let a base for the open sets be the sets (a, c, d), (b, c, d), (c), (d). Let $\sum$ be the presheaf for which $S_U = z$, the group of integers, if $U = (c, d)$; and $S_U =$
70 0 otherwise. The homomorphisms ρ_{VU} are the obvious ones. Then $\sum$ determines the 0-sheaf, but $H^1(X, \sum) = Z$. The space X paracompact but not normal.

If $0 \to \mathscr{S}' \xrightarrow{i} \mathscr{S} \xrightarrow{j} \mathscr{S}''$ *is an exact sequence of sheaves, then*

$$0 \to \Gamma(U, \mathscr{S}') \xrightarrow{i_U} \Gamma(U, \mathscr{S}) \xrightarrow{j_U} \Gamma(U, \mathscr{S}'')$$

is exact and hence

$$0 \to \overline{\mathscr{S}} \to \overline{\mathscr{S}} \to \overline{\mathscr{S}}''$$

is exact

Proof. We will show that for $\ker j_U \subset imi_U$ (the rest is trivial). Let $f \in \ker j_U$. Then, $x \in U$, $jf(x) = (j_U f)(x) = 0_x$ and hence by exactness, $f(x) = ip'$ for some $p' \in S'_x$. Thus $f(U) \subset i(S')$. But $i : S' \to i(S')$ is homeomorphism. Then $g : U \to S'$, where $g(x) = 1^{-1} f(x)$, is a section of S' over U, and $f = i_U g$. □

One cannot in general complete the sequence

$$0 \to \Gamma(U, \mathscr{S}') \to \Gamma(U, \mathscr{S}) \to \Gamma(U, \mathscr{S}'')$$

by a zero on the right as the following example shows.

Example 16. Let X be the segment $\{x : 0 \leq x \leq 1\}$. Let G be the 4-group with elements 0, a, b, c. Let $\mathscr{S}$ be the subsheaf of the constant sheaf $G = (X \times G, \pi, X)$ formed by omitting the point $(0, a)$, $(0, c)$, $(1, b)$, $(1, c)$. Let $\mathscr{S}'$ be the subsheaf of $\mathscr{S}$ formed by omitting all the points (x, a), (x, b). Let $\mathscr{S}'' = (X \times Z_2 \pi, X)$ and let $j : \mathscr{S} \to \mathscr{S}''$ be the 71
homomorphism induced by $j : G \to Z_2$ where $j(a) = j(b) = 1$, $j(c) = j(0) = 0$. The the sequence

$$0 \to \mathscr{S}' \xrightarrow{i} \mathscr{S} \xrightarrow{j} \mathscr{S}'' \to 0$$

is exact, but the sequence

$$\begin{aligned} &0 \to \Gamma(X, \mathscr{S}') \to \Gamma(X, \mathscr{S}) \to \Gamma(X, \mathscr{S}'') \to O, \quad \textit{i.e.,} \\ &0 \longrightarrow O \longrightarrow O \longrightarrow Z_2 \longrightarrow O \end{aligned}$$

is not exact.

If $0 \to \sum' \xrightarrow{i} \sum \xrightarrow{j} \sum''$ *is an exact sequence of presheaves, there is an image presheaf* $\sum_0'' \subset \sum''$ *and a quotient presheaf* Q *such that the sequences*

$$O \to \sum{}' \xrightarrow{i} \sum \xrightarrow{j_o} \sum{}_0'' \to 0,$$

$$O \to \sum{}_0'' \xrightarrow{\bar{i}} \sum{}'' \xrightarrow{\bar{j}} Q \to 0$$

are exact. These sequences are 'natural' in the sense that if h is a homomorphism of exact sequences, commuting with i and j:

$$\begin{tikzcd} 0 \arrow[r] & \sum' \arrow[r, "i"] \arrow[d, "h"'] & \sum \arrow[r, "j"] \arrow[d, "h"'] & \sum'' \arrow[d, "h"'] \\ 0 \arrow[r] & \sum_1' \arrow[r, "i"] & \sum_1 \arrow[r, "j"] & \sum_1'' \end{tikzcd}$$

then there are induced homomorphisms h^* of the exact cohomology sequences

$$\begin{tikzcd}
\cdots \arrow[r] & H^q(X,\sum') \arrow[r, "i^*"] \arrow[d, "h^*"'] & H^q(X,\sum) \arrow[r, "j_0^*"] \arrow[d, "h^*"'] & H^q(X,\sum_0'') \arrow[r, "\delta_0^*"] \arrow[d, "h^*"'] & H^{q+1}(X,\sum') \arrow[r] \arrow[d, "h^*"'] & \cdots \\
\cdots \arrow[r] & H^q(X,\sum_1') \arrow[r, "i^*"] & H^q(X,\sum_1) \arrow[r, "j_0^*"] & H^q(X,\sum_{1\,0}'') \arrow[r, "\delta_0^*"] & H^{q+1}(X,\sum_1') \arrow[r] & \cdots
\end{tikzcd}$$

72 and

$$\begin{tikzcd}
\cdots \arrow[r] & H^q(X,\sum_0'') \arrow[r, "\bar{i}^*"] \arrow[d, "h^*"'] & H^q(X,\sum'') \arrow[r, "\bar{j}^*"] \arrow[d, "h^*"'] & H^q(X,Q) \arrow[r, "\bar{\delta}^*"] \arrow[d, "h^*"'] & H^{q+1}(X,\sum_0'') \arrow[r] \arrow[d, "h^*"'] & \cdots \\
\cdots \arrow[r] & H^q(X,\sum_{10}'') \arrow[r, "\bar{i}^*"] & H^q(X,\sum_1'') \arrow[r, "\bar{j}^*"] & H^q(X,Q_1) \arrow[r, "\bar{\delta}^*"] & H^{q+1}(X,\sum_{10}'') \arrow[r] & \cdots
\end{tikzcd}$$

commuting with i^*, j_0^*, δ_0^* and $\bar{i}^*$, $\bar{j}^*$, $\bar{\delta}^*$ respectively.

Proof. If $\sum'' = \{S_U'', \rho_{VU}''\}$, let $S_{oU}'' = im\, j_U$. Then since $j : \sum \to \sum''$ is a homomorphism, ρ_{VU}'' maps $im\, j_U$ into $im\, j_V$. Hence, writing $S_{oU}'' = im\, j_U$ and $Q_U = S_U''/S_{oU}''$, there are induced homomorphisms ρ_{oVU}'' and $\bar{\rho}_{VU}$ with comutativity in □

$$\begin{tikzcd}
S_U \arrow[r, "j_{oU}"] \arrow[d, "\rho_{VU}"'] & S_{oU}'' \arrow[r, "\bar{i}_U"] \arrow[d, "\rho_{oVU}''"'] & S_U'' \arrow[r, "\bar{j}_U"] \arrow[d, "\rho_{VU}''"'] & Q_U \arrow[d, "\bar{\rho}_{VU}"'] \\
S_V \arrow[r, "j_{oV}"] & S_{oV}'' \arrow[r, "\bar{i}_V"] & S_V'' \arrow[r, "\bar{j}_V"] & Q_V
\end{tikzcd}.$$

Clearly the systems $\sum_o'' = \{S_{oU}'', \rho_{oVU}''\}$ and $Q = \{Q_U, \bar{\rho}_{VU}\}$ are presheaves and the sequences

$$O \to \Sigma' \xrightarrow{i} \Sigma \xrightarrow{j} \Sigma_o'' \to O$$

and

$$O \to \Sigma_o'' \xrightarrow{\bar{i}} \Sigma'' \xrightarrow{\bar{j}} Q \to O$$

are exact. Since h commutes with i, j,

$$
\begin{tikzcd}
0 \arrow[r] & \Sigma' \arrow[r, "i"] \arrow[d, "h'"] & \Sigma \arrow[r, "j"] \arrow[d, "h"] & \Sigma'' \arrow[d, "h''"] \\
0 \arrow[r] & \Sigma'_1 \arrow[r, "i_1"] & \Sigma_1 \arrow[r, "j_1"] & \Sigma''_1
\end{tikzcd}
\qquad
\begin{tikzcd}
S_U \arrow[r, "j_U"] \arrow[d, "h_U"'] & S''_U \arrow[d, "h''_U"] \\
S_{1U} \arrow[r, "j_{1U}"] & S''_{1U}
\end{tikzcd}
$$

h''_U maps $S''_{oU} = im j_U$ into $S''_{1oU} = im j_{1U}$. Hence there are induced 73
homomorphisms h''_{oU}, $\bar{h}_U$ with comutativity in

$$
\begin{tikzcd}
S_U \arrow[r, "j_{oU}"] \arrow[d, "h_U"'] & S''_{oU} \arrow[r, "\bar{i}_U"] \arrow[d, "h''_{oU}"] & S''_U \arrow[r, "\bar{j}_U"] \arrow[d, "h''_U"] & Q_U \arrow[d, "\bar{h}_U"] \\
S_{1U} \arrow[r, "j_{oU}"] & S''_{1oU} \arrow[r, "\bar{i}_U"] & S''_{1U} \arrow[r, "\bar{j}_U"] & Q_{1u}
\end{tikzcd}
.
$$

Since h is a homomorphism of presheaves, h_U commutes with ρ_{VU} and h''_U with ρ''_{VU}. Hence, since j_{oU} and $\bar{j}_U$ are epimorphisms and j_{oU}, $\bar{j}_U$ commute with ρ and h, h''_{oU} and commutes with ρ''_{oVU} and $\bar{h}_U$ with $\bar{\rho}_{VU}$, i.e., the diagrams given below are commutative:

$$
\begin{tikzcd}
S''_{oU} \arrow[rrr, "\rho''_{oVU}"] \arrow[ddd, "h''_{oU}"'] & & & S''_{oV} \arrow[ddd, "h''_{oV}"] \\
& S_U \arrow[ul, "j_{oU}"] \arrow[r, "\rho_{VU}"] \arrow[d, "h_U"'] & S_V \arrow[ur, "j_{oV}"'] \arrow[d, "h_V"] & \\
& S_{1U} \arrow[dl, "j_o"'] \arrow[r, "\rho_{VU}"] & S_{1V} \arrow[dr, "j_o"] & \\
S''_{1oU} \arrow[rrr, "\rho''_{oVU}"] & & & S''_{1oV}
\end{tikzcd}
\qquad
\begin{tikzcd}
Q_U \arrow[rrr, "\bar{\rho}_{VU}"] \arrow[ddd, "\bar{h}_U"'] & & & Q_V \arrow[ddd, "\bar{h}_V"] \\
& S''_U \arrow[ul, "\bar{j}_U"] \arrow[r, "\rho''_{VU}"] \arrow[d, "h''_U"'] & S''_V \arrow[ur, "\bar{j}_V"'] \arrow[d, "h''_V"] & \\
& S''_{1U} \arrow[dl, "\bar{j}"'] \arrow[r, "\rho''_{VU}"] & S''_{1V} \arrow[dr, "\bar{j}"] & \\
Q_{1U} \arrow[rrr, "\bar{\rho}_{VU}"] & & & Q_{1V}
\end{tikzcd}
$$

Thus h, h''_o, h'', $\bar{h}$ are homomorphisms of presheaves commuting with j_o, $\bar{i}$, $\bar{j}$.

$$
\begin{tikzcd}
\Sigma \arrow[r, "j_o"] \arrow[d, "h"'] & \Sigma''_o \arrow[r, "\bar{i}"] \arrow[d, "h''_o"'] & \Sigma'' \arrow[r, "\bar{j}"] \arrow[d, "h''"] & Q \arrow[d, "\bar{h}"] \\
\Sigma_1 \arrow[r, "j_o"] & \Sigma''_{1o} \arrow[r, "\bar{i}"] & \Sigma''_1 \arrow[r, "\bar{j}"] & Q_1
\end{tikzcd}
.
$$

74 Thus $\{h\}$ is a homomorphism of the exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \sum' \arrow[r, "j"] \arrow[d, "h'"'] & \sum \arrow[r, "j_o"] \arrow[d, "h"'] & \sum''_o \arrow[r] \arrow[d, "h''_o"'] & 0 \\
0 \arrow[r] & \sum'_1 \arrow[r, "i"] & \sum_1 \arrow[r, "j_o"] & \sum''_{1o} \arrow[r] & 0
\end{tikzcd}$$

commuting with i and j_o, and a homomorphism of the exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \sum''_o \arrow[r, "\bar{i}"] \arrow[d, "h''_o"'] & \sum'' \arrow[r, "\bar{j}"] \arrow[d, "h''"'] & Q \arrow[r] \arrow[d, "\bar{h}"'] & 0 \\
0 \arrow[r] & \sum''_{1o} \arrow[r, "\bar{i}"] & \sum''_1 \arrow[r, "\bar{j}"] & Q_1 \arrow[r] & 0
\end{tikzcd}$$

commuting with $\bar{i}$ and $\bar{j}$. Therefore the induced homomorphisms of the exact cohomology sequences commute with i^*, j^*_o, δ^*_o and $\bar{i}^*$, $\bar{j}^*$, $\bar{\delta}^*$ respectively.

Lecture 13

Example 17. Let X consist of the natural numbers together with two special points p and q. Each natural number forms an open set. A neighbourhood of p (resp. q) consists of p (resp; q) together with all but a finite number of the natural numbers. Let $S_U = Z$ if U consists of all but a finite number of the natural numbers and if $V \subset U$ is another such set, let $\rho_{VU} : Z \to Z$ be the identity. If U is an open set not containing all but a finite number of the natural numbers or if it contains either p or q, let $S_U = O$ and let ρ_{VU}, ρ_{UW} be the zero homomorphisms. Then $\sum = \{S_U, \rho_{VU}\}$ is a presheaf determining the 0-sheaf, but $H^1(X, Z) = Z$. The space X is T_1 and paracompact but not normal. **75**

Example 18. Let R be a set with cardinal number $\mathscr{N}_1$ let $S = 2^R$ be the set of all subsets of R and let $T = 2^S$ be the set of all subsets of S. If $r \in R$, let $r' \in T$ be the largest subset of S, which is such that, each of its elements considered as a subset of R contains the elements r. Let $R' \subset T$ consists of all r, for all $r \in R$ and let $T_1 = T - R'$.

Let X be a space consisting of (1) all elements $r \in R$ and (2) all triples (t, r_1, r_2) with $t \in T_1$, r_1, $r_2 \in R$ and $r_1 \neq r_2$. Each point (t, r_1, r_2) is to form an open set. Neighborhoods of points r of the first kind are sets $N(r; s_1, \ldots, s_k)$, where $o \leq k < \infty$ and $s_1, \ldots, s_k \in S$, consisting of r together with all points (t, r_1, r_2) with $r \in (r_1, r_2)$ and, for each **76**
$i = 1, \ldots, k$, either $r \in s_i \in t$ or $r \notin s_i \notin t$. [cf. Bing's Example G, Canadian Jour, of Math. 3 (1951) p.184].

For sets $U \subset X$ of cardinal number ≥ 2 and consisting of points of the second kind, let $S_U = Z$ and, if $V \subset U$ is another such set, let

$\rho_{VU} : Z \to Z$ be the identity. If U is an open set containing any point of the first kind or consisting of at most one point, let $S_U = 0$. Then $\sum = \left\{S_U, \rho_{VU}\right\}$ is a presheaf determining the 0-sheaf, but $H'(X, \sum) \neq 0$ (although dim $X = 0$). X is a completely normal, Hausdorff space but is not paracompact. (A space X is said to be completely normal if each subspace of X is normal).

If $0 \to \mathscr{S}' \xrightarrow{i} \mathscr{S} \xrightarrow{j} \mathscr{S}'' \to 0$ *is an exact sequence of sheaves, let* $\overline{\mathscr{S}''}_o$, $\overline{Q}$, *be the image and quotient presheaves in* $0 \to \overline{\mathscr{S}'} \xrightarrow{i} \mathscr{S} \xrightarrow{j} \mathscr{S}''$ *for which the sequences*

$$0 \to \overline{\mathscr{S}} \xrightarrow{i} \overline{\mathscr{S}} \xrightarrow{j_o} \overline{\mathscr{S}}''_o \to 0,$$

$$0 \to \overline{\mathscr{S}}''_0 \xrightarrow{\bar{i}} \overline{\mathscr{S}''} \xrightarrow{\bar{j}} \overline{Q} \to 0,$$

are exact. Then $\overline{Q}$ *determines the zero sheaf.*

Proof. An arbitrary element of a stalk Q_k of the induced sheaf has the form $\overline{\rho}_{xU}\,\overline{j_U} f$ where $x \in U$ and $f \in \Gamma(U, \mathscr{S}'')$. Since j maps $\mathscr{S}$ onto $\mathscr{S}$, there is an open set $V, x \in V \subset U$, for which $f|V \operatorname{im} j_V$. Then $\rho''_{VU} f = f|V \in \operatorname{im} j_V = \operatorname{im} \overline{i_V}$ and by exactness $\overline{j_V} \rho''_{VU} f = 0$. Hence

$$\overline{\rho}_{xU}\,\overline{j_U} f = \overline{\rho}_{xV} \overline{\rho}_{VU}\,\overline{j_U} f = \overline{\rho}_{xV}\,\overline{j_V} \rho''_{VU} f = 0.$$

77 Therefore the sheaf determined by $\overline{Q}$ is the 0-sheaf. □

Note . In example 16, if $\Gamma_o(U, \mathscr{S}'') = im\, j_U$, we have $\Gamma_o(U, \mathscr{S}'') = \Gamma(U, \mathscr{S}'')$ for all U expect X, but $\Gamma_o(X, \mathscr{S}'') = 0$, $\Gamma_o(X, \mathscr{S}'') = Z_2$. Thus $\overline{Q}_X = Z_2$, $\overline{Q}_U = 0$ for all smaller U, and thus $\overline{Q}$ determine the 0-sheaf.

Proposition 9. *If* X *is paracompact and normal and if* $0 \to \mathscr{S}' \xrightarrow{i} \mathscr{S} \xrightarrow{j} \mathscr{S}'' \to 0$ *is an exact sequence of sheaves, there is an exact cohomology sequence*

$$0 \to H^o(X, \mathscr{S}') \to \cdots \to H^q(X, \mathscr{S}') \xrightarrow{i^*}$$

$$H^q(X, \mathscr{S}) \xrightarrow{j^*} H^q(X, \mathscr{S}'') \xrightarrow{\delta^*} H^{q+1}(X, \mathscr{S}') \to .$$

If $h : (\mathscr{S}', \mathscr{S}, \mathscr{S}'') \to (\mathscr{S}'_1, \mathscr{S}_1, \mathscr{S}''_1)$ is a homomorphism of exact sequences, commuting with i and j,

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{S}' \arrow[r, "i"] \arrow[d, "h"'] & \mathscr{S} \arrow[r, "j"] \arrow[d, "h"'] & \mathscr{S}'' \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & \mathscr{S}'_1 \arrow[r, "i"] & \mathscr{S}_1 \arrow[r, "j"] & \mathscr{S}''_1 \arrow[r] & 0
\end{tikzcd}$$

the induced homomorphisms S^* of the cohomology sequences commute with i^*, j^* and δ^*, i.e. the following diagram is commutative:

$$\begin{tikzcd}
\cdots \arrow[r] & H^q(x, \mathscr{S}') \arrow[r, "i^*"] \arrow[d, "h^*"'] & H^q(X, \mathscr{S}) \arrow[r, "j^*"] \arrow[d, "h^*"'] & H^q(x, \mathscr{S}'') \arrow[r, "\delta^*"] \arrow[d, "h^*"'] & H^{q+1}(X, \mathscr{S}') \arrow[r] \arrow[d, "h^*"'] & \cdots \\
\cdots \arrow[r] & H^q(X, \mathscr{S}'_1) \arrow[r, "i^*"] & H^q(X, S_1) \arrow[r, "j^*"] & H^q(X, \mathscr{S}''_1) \arrow[r, "\delta^*"] & H^{q+1}(X, \mathscr{S}'_1) \arrow[r] & \cdots
\end{tikzcd}$$

Proof. As before, if $\bar{\mathscr{S}}''_o$, $\bar{Q}$ denote the image and quotient presheaves in the exact sequence of presheaves of sections

$$0 \to \bar{\mathscr{S}}' \xrightarrow{i} \bar{\mathscr{S}} \xrightarrow{j} \bar{\mathscr{S}}'',$$

we obtain the exact sequence of presheaves **78**

$$0 \to \bar{\mathscr{S}}' \xrightarrow{i} \bar{\mathscr{S}} \xrightarrow{j_0} \bar{\mathscr{S}}''_0 \to 0,$$

and

$$0 \to \bar{\mathscr{S}}'' \xrightarrow{\bar{i}} \bar{\mathscr{S}}'' \xrightarrow{\bar{j}} \bar{Q} \to 0,$$

□

From these exact sequences of presheaves we obtain the following exact cohomology sequences:

$$
\begin{tikzcd}
 & & & {} \arrow[d] & & \\
 & & & H^{q-1}(X,\bar{Q})=0 \arrow[d, "\bar{\delta}^*"] & & \\
\cdots \arrow[r] & H^q(X,\mathscr{S}') \arrow[r, "i^*"] & H^q(X,\mathscr{S}) \arrow[r, "j^*_o"] \arrow[dr, "j^*"'] & H^q(X,\bar{\mathscr{S}}''_o) \arrow[r, "\delta^*_o"] \arrow[d, "\bar{i}^*"] & H^{q+1}(X,\mathscr{S}') \arrow[r] & \cdots \\
 & & & H^q(X,\mathscr{S}'') \arrow[d, "\bar{j}^*"] \arrow[ur, "\delta^*"'] & & \\
 & & & H^q(X,\bar{Q})=0 \arrow[d] & & \\
 & & & {} & &
\end{tikzcd}
$$

Since $\bar{Q}$ determine the 0-sheaf, by Proposition 8, $H^q(X,\bar{Q}) = 0$ for all $q \geq 0$ and hence, by exactness, $\bar{i}^*$ is an isomorphism. Hence if $\delta^* = \delta^*_o(\bar{i}^*)^{-1} : H^q(X,\mathscr{S}'') \to H^{q+1}(X,\mathscr{S}')$, the cohomology sequence

$$
\begin{gathered}
0 \to H^o(X,\mathscr{S}') \to H^o(X,\mathscr{S}) \to H^o(X,\mathscr{S}'') \to \cdots \\
\cdots \to H^q(X,\mathscr{S}') \to H^q(X,\mathscr{S}) \to H^q(X,\mathscr{S}'') \to H^{q+1}(X,\mathscr{S}') \to \cdots
\end{gathered}
$$

is exact.

79 Next, since the homomorphism h commutes with i and j, the induced homomorphism h of presheaves also commutes with i and j:

$$
\begin{tikzcd}
0 \arrow[r] & \bar{\mathscr{S}}' \arrow[r, "i"] \arrow[d, "h"'] & \bar{\mathscr{S}} \arrow[r, "j"] \arrow[d, "h"'] & \bar{S}'' \arrow[d, "h"'] \\
0 \arrow[r] & \bar{\mathscr{S}}'_1 \arrow[r, "i"] & \bar{\mathscr{S}}_1 \arrow[r] & \bar{\mathscr{S}}''_1
\end{tikzcd}
\quad .
$$

Hence, the induced homomorphism h^* of the cohomology modules commutes with i^*, j^*_o, $\bar{i}^*$ and δ^*_o. Thus in the exact cohomology sequences, h^* commutes with i^*, j^* and δ^*.

Note. If X is not paracompact and normal, in general, $\overline{i}^*$ is not an isomorphism, to be precise, the cohomology sequence is not defined. One does, however, have the exact sequence

$$0 \to H^o(X, \mathscr{S}') \to H^o(X, \mathscr{S}) \to H^o(X, \mathscr{S}'') \to \\ H^1(X, \mathscr{S}') \to H^1(X, \mathscr{S}) \to H^1(X, \mathscr{S}'')$$

as one sees from the exact sequences

$$\begin{tikzcd}
 & & & 0 \arrow[d] & & & 0 = H^o(X, \overline{Q}) \arrow[d] \\
0 \arrow[r] & H^o(X, \mathscr{S}') \arrow[r, "i^*"] & H^o(X, \mathscr{S}) \arrow[r, "j_o^*"] \arrow[dr, "j^*"'] & H^o(X, \mathscr{S}_o'') \arrow[r, "\delta_o^*"] \arrow[d, "\overline{i}^*"'] & H^1(X, \mathscr{S}') \arrow[r, "i^*"] & H^1(X, \mathscr{S}) \arrow[r, "j_o^*"] \arrow[dr, "j^*"'] & H^1(X, \mathscr{S}_o'') \arrow[d, "\overline{i}^*"'] \\
 & & & H^o(X, \mathscr{S}'') \arrow[ur] \arrow[d] & & & H^1(X, \mathscr{S}'') \\
 & & & H^o(X, \overline{Q}) = 0. & & &
\end{tikzcd}$$

The following examples show that Proposition 9 is not true, in general,unless the space is both paracompact and normal. **80**

Example 19. Let X consist of the unit segment I with the usual topology and of two points p and q. A neighbourhood of p (resp. q) consists of p (resp. q) together with the whole of I. Let $\mathscr{S}'$, $\mathscr{S}$, $\mathscr{S}''$ be the sheaves $\mathscr{S}'$, $\mathscr{S}$, $\mathscr{S}''$ of Example 16 over I together with zeros at p and q. A neighbourhood of 0_p (resp. 0_q) consists of the zeros over a neighbourhood of p (resp. q). Then there is an exact sequence

$$0 \to \mathscr{S}' \xrightarrow{i} \mathscr{S} \xrightarrow{j} \mathscr{S}'' \to 0$$

where i, j correspond to those in Example 16. Since $H^1(X, \mathscr{S}) = H^2(X, \mathscr{S}') = 0$ and $H^1(X, \mathscr{S}'') = Z_2$, there is no exact cohomology sequence. The space X is paracompact but not normal.

Example 20. Let X consists of a sequence of copies I_n of the unit segment together with two special points p and q. A neighbourhood of p (resp q) consists of p (resp. q) together with all but a finite number of the segments I_n. Let G be the 4-group and let $\mathscr{S}$ be the subsheaf of $(X \times G, \pi, X)$ consisting of zero at p and q and on each I_n a copy of the sheaf $\mathscr{S}$ of Example 16. Let $\mathscr{S}''$ be the subsheaf of $(X \times Z_2, \pi, X)$ formed by omitting the points $(p, 1)$, $(q, 1)$ and let the homomorphism $j : \mathscr{S} \to \mathscr{S}''$ be induces by $j : G \to Z_2$ as defined in Example 16. Then there is exact sequence

$$0 \to \mathscr{S}' \xrightarrow{i} \mathscr{S} \xrightarrow{j} \mathscr{S}'' \to 0$$

but $H^1(X, \mathscr{S}) = H^2(X, \mathscr{S}') = 0$ while $H^1(X, \mathscr{S}'') \neq 0$. Thus there is **81**
no exact cohomology sequence. The space X is paracompact and T_1 but not normal.

Example 21. Let R, S, T_1 be as in Example 18. Let X be the space consisting of (1) the elements $r \in R$ and (2) segments $I_{ntr_1r_2}$ where n is a natural number, $t \in T_1$, r_1 and r_2 are in R, and $r_1 \neq r_2$. Neighbourhoods of the points r are sets $N(r; n, s_1, ., s_k)$ where n is a natural number and $s_1, \ldots, s_k \in S$, consisting of r together with all segments $I_{mtr_1r_2}$ with

$m > n$, $r \in (r_1, r_2)$ and, for each $i = 1, \ldots, k$, either $r \in s_i \in t$ or $r \notin s_i \notin t$.

Let G be the 4-group and let $\mathscr{S}$ be the subsheaf of $(X \times G, \pi, X)$ consisting of zero at each r and a copy of the sheaf $\mathscr{S}$ of Example 16 on each $I_{ntr_1r_2}$. Let $\mathscr{S}''$ be the subsheaf of $(X \times Z_2, \pi, X)$ formed by omitting $(r, 1)$ for all r, and let the homomorphism $j : \mathscr{S} \to \mathscr{S}''$ be induced by $j : G \to Z_2$ as mentioned before. Then there is an exact sequence

$$0 \to \mathscr{S}' \to \mathscr{S} \to \mathscr{S}'' \to 0$$

but $H^1(X, \mathscr{S}) = H^2(X, \mathscr{S}') = 0$ while $H^1(X, \mathscr{S}'') \neq 0$. Thus again, there is no exact cohomology sequence. X is a perfectly normal Hausdorff space but is not paracompact. (A space X is said to be perfectly normal if, for each closed set C of X there is a continuous real valued function defined on X and vanishing on C but not at point $x \in X - C$. Perfectly normal spaces are completely normal.)

Lecture 14

Definition. *A* resolution *of a sheaf $\mathscr{G}$ of A-modules is an exact sequence of sheaves A-modules* 82

$$0 \to \mathscr{G} \xrightarrow{e} \mathscr{S}^o \xrightarrow{d^1} \mathscr{S}^1 \to \cdots \to \mathscr{S}^{q-1} \xrightarrow{d^q} \mathscr{S}^q \to \cdots$$

such that $H^q(X, \mathscr{S}^q) = 0$, $p \geqq 1$, $q \geqq 0$.

There are than induced homomorphisms

$$0 \xrightarrow{d^o} \Gamma(X, \mathscr{S}^o) \xrightarrow{d^1} \cdots \to \Gamma(X, \mathscr{S}^{q-1}) \xrightarrow{d^q} \Gamma(X, \mathscr{S}^q) \xrightarrow{d^{q+1}} \Gamma(X, \mathscr{S}^{q+1}) \to \cdots$$

for which $\operatorname{im} d^q \subset \ker d^{q+1}$, i.e., $d^{q+1}d^q = 0$. The A-modules $\Gamma(X, \mathscr{S}^k)$ ($k \geqq 0$) together with the homomorphisms d^q form a formal cochain complex denoted by $\Gamma(X, \mathscr{S})$. Let the $q-th$ cohomology module of the complex $\Gamma(X, \mathscr{S})$ be denoted by $H^q\Gamma(X, \mathscr{S}) = \ker d^{q+1} / \operatorname{im} d^q$.

Example 22. Let X be the unit segment $\left\{x : o \leqq x \leqq 1\right\}$, and let $\mathscr{G}$ be the subsheaf of the constant sheaf Z_2 formed by omitting the points (0,1) and (1,1). A resolution

$$0 \to \mathscr{G} \xrightarrow{e} \mathscr{S}^o \xrightarrow{d^1} \mathscr{S}^1 \to 0 \tag{1}$$

of $\mathscr{G}$ is obtained by identifying $\mathscr{G}$ with the sheaf $\mathscr{S}'$ of Example 16 and taking $\mathscr{S}, \mathscr{S}''$, i, j for $\mathscr{S}^o$, $\mathscr{S}^1$, e, d^1. (That $H^p(X, \mathscr{S}^q) = 0$, $p \geqq 1$, $q \geqq 0$ can be verified.) The induced sequence

$$0 \to \Gamma(X, \mathscr{S}^o) \to \Gamma(X, \mathscr{S}') \to 0$$

is 83

$$0 \to 0 \to Z_2 \to 0.$$

Another resolution

$$(2) \qquad 0 \to \mathscr{G} \xrightarrow{e} \mathscr{S}^o \xrightarrow{d^1} \mathscr{S}^1 \xrightarrow{d^2} \mathscr{S}^2 \to 0$$

of $\mathscr{G}$ is obtained by taking e, $\mathscr{S}^o$ as before, $\mathscr{S}^1 = Z_2 + Z_2$, $\mathscr{S}^2 = Z_2$, $d^1 = kj$ where $k(x, 1) = (x, (1, 0))$, and d^2 with $d^2(x, (1, 1)) = d^2(x, (0, 1)) = (x, 1)$.

Another resolution

$$(3) \qquad 0 \to \mathscr{G} \xrightarrow{e} \mathscr{S}^o \xrightarrow{d^1} \mathscr{S}^1 \to 0$$

of $\mathscr{G}$ is obtained by taking $\mathscr{S}^o$ to be subsheaf of Z_2 formed by omitted (0,1), with $e : \mathscr{G} \to \mathscr{S}^o$ as the inclusion homomorphism and with d^1 as the natural homomorphism onto the quotient sheaf $\mathscr{S}^1 = \mathscr{S}^o/\mathscr{G}$.

Yet another resolution

$$(4) \qquad 0 \to \mathscr{G} \xrightarrow{e} \mathscr{S}^o \xrightarrow{d^1} \mathscr{S}^1 \to 0$$

of $\mathscr{G}$ is obtained by taking $\mathscr{S}^o = Z_2$ and $\mathscr{S}^1 = \mathscr{S}^o/\mathscr{G}$.

In each case $H^1\Gamma(X, \mathscr{S}) = Z_2$, $H^p\Gamma(X, \mathscr{S}) = 0$, $p > 1$.

Example 23. Let X be the sphere $x^2 + y^2 + z^2 = 1$, and let $\mathscr{G}$ be the constant sheaf Z_2. Let $\mathscr{R}$ denote the constant sheaf $Z_2 + Z_2$ with $i : \mathscr{G} \to$
84 $\mathscr{R}$ defined by $i(1) = (1, 1)$. Let $\mathscr{R}' \subset \mathscr{R}$ consist of all zeros together with $((x, y, z), (0, 1))$ for $z < 0$; let $\mathscr{S}^o = \mathscr{R}/\mathscr{R}'$ and let $j : \mathscr{R} \to \mathscr{S}^o$ be the natural homomorphism. Let $e = ji : \mathscr{G} \to \mathscr{S}^o$.

Let $\mathfrak{I}_o$ be the quotient sheaf $\mathscr{S}^o/e(\mathscr{G})$ and let $h : \mathscr{S}^o \to \mathfrak{I}_o$ be the natural homomorphism. The stalks of $\mathfrak{I}_o$ are Z_2 on the equator and 0 elsewhere.

Let $\mathfrak{I}$ be the quotient sheaf of $\mathscr{R}$ consisting of $Z_2 + Z_2$ on the equator and 0 elsewhere. Identify $\mathfrak{I}_o$ with the subsheaf of $\mathfrak{I}$ consisting of all zeros and all $((x, y, 0), (1, 1))$, and let $k : \mathfrak{I}_o \to \mathfrak{I}$ be the inclusion homomorphism. Let $\mathfrak{I}'$ be for $y > 0$ and $((x, y, 0), (1, 0))$ for $y < 0$. Let

$\mathscr{S}' = \Im/\Im'$ and let $l : \Im \to \mathscr{S}^1$ be the natural homomorphism. Let $d^1 = 1kh : \mathscr{S}^o \to \mathscr{S}^1$. Let $\mathscr{S}^2 = \mathscr{S}^1/_{d^1}\mathscr{S}^o$ and let d^2 be the natural homomorphism. Then from the diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
 & \mathscr{R}' \arrow[d] & & \Im' \arrow[d] & \\
 & \mathscr{R} \arrow[d, "j"] & \Im_o \arrow[r, "k"] & \Im \arrow[d] & \\
\mathscr{G} \arrow[ur, "i"] \arrow[r, "e"] & \mathscr{S}^o \arrow[ur, "h"] \arrow[rr, "d'"] & & \mathscr{S}' \arrow[r, "d^2"] & \mathscr{S}^2
\end{tikzcd}$$

we see that

$$0 \to \mathscr{G} \xrightarrow{e} \mathscr{S}^o \xrightarrow{d^1} \mathscr{S}^1 \xrightarrow{d^2} \mathscr{S}^2 \to 0$$

is a resolution of $\mathscr{G}$ and the induced sequence

$$0 \to \Gamma(X, \mathscr{S}^o) \to \Gamma(X, \mathscr{S}^1) \to \Gamma(X, \mathscr{S}^2) \to 0$$

is **85**

$$0 \to Z_2 + Z_2 \to Z_2 + Z_2 \to Z_2 + Z_2 \to 0$$

and

$$H^o\Gamma(X, \mathscr{S}) = H^2\Gamma(X, \mathscr{S}) = Z_2, H^1\Gamma(X, \mathscr{S}) = 0$$

Proposition 10. *If X is paracompact normal and if*

$$0 \to \mathscr{G} \xrightarrow{e} \mathscr{S}^o \to \cdots \to \mathscr{S}^{q-1} \xrightarrow{d^q} \mathscr{S}^q \to \cdots$$

is a resolution of $\mathscr{G}$, there is a uniquely determined isomorphism η : $H^q\Gamma(X, \mathscr{S}) \to H^q(X, \mathscr{G})$.

If

$$0 \to \mathscr{G}_1 \xrightarrow{e} \mathscr{S}^o_1 \xrightarrow{d^1} \mathscr{S}^1_1 \to \cdots \to \mathscr{S}^{q-1}_1 \xrightarrow{d^q} \mathscr{S}^q_1 \to \cdots$$

is a resolution of another sheaf $\mathscr{G}_1$ and if

$$h : (\mathscr{G}, \mathscr{S}^o, \mathscr{S}^1, \ldots) \to (\mathscr{G}_1, \mathscr{S}^o_1, \mathscr{S}^1_1, \ldots)$$

is a homomorphism commuting with $e, d^1, d^2, \ldots$, then the induced homomorphism H^* commutes with η.

$$\begin{tikzcd} H^q\Gamma(X, \mathscr{S}) \arrow[r, "\eta"] \arrow[d, "h^*"] & H^q(X, \mathscr{G}) \arrow[d, "h^*"] \\ H^q\Gamma(X, \mathscr{S}_1) \arrow[r, "\eta"'] & H^q(X, \mathscr{G}_1) \end{tikzcd} \quad .$$

86 *Proof.* The homomorphism $h^* : H^q(X, \mathscr{G}) \to H^q(X, \mathscr{G}_1)$ is the usual induced homomorphism. Now, since h commutes with d^q, $q \geqq 1$, h also commutes with the homomorphisms

$$d^q : \Gamma(X, \mathscr{S}^{q-1}) \to \Gamma(X, \mathscr{S}^q) \ (q \geqq 1),$$

and hence there is an induced homomorphism

$$h^* : H^q\Gamma(X, \mathscr{S}) \to H^q\Gamma(X, \mathscr{S}_1).$$

$\square$

Let $\mathfrak{z}^q = \operatorname{im} d^q = \ker d^{q+1} \subset \mathscr{S}^q$; then there are exact sequences

$$\left.\begin{aligned} &0 \to \mathscr{G} \xrightarrow{e} \mathscr{S}^o \xrightarrow{d^1_o} \mathfrak{z}^1 \to 0 \\ &0 \to \mathfrak{z}^q \xrightarrow{i^q} \mathscr{S}^q \xrightarrow{d^{q+1}_o} \mathfrak{z}^{g+1} \to 0 \ \ (q \geqq 1) \end{aligned}\right\} \tag{1}$$

where i^q is the inclusion homomorphism and d^q_o is the homomorphism induced by $d^q : i^q d^q_o = d^q$. Since h commutes with d, h maps $\mathfrak{z}^q$ in $\mathfrak{z}^q_1$, and commutes with i, d_o.

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{G} \arrow[r, "e"] \arrow[d, "h"] & \mathscr{S}^o \arrow[r, "d^1_o"] \arrow[d, "h"] & \mathfrak{z}^1 \arrow[r] \arrow[d, "h"] & 0, \\ 0 \arrow[r] & \mathscr{G}_1 \arrow[r, "e"] & \mathscr{S}^o_1 \arrow[r, "d^1_o"] & \mathfrak{z}^1_1 \arrow[r] & 0, \end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{z}^q \arrow[r, "i^q"] \arrow[d, "h"'] & \mathscr{S}^q \arrow[r, "d_o^{q+1}"] \arrow[d, "h"'] & \mathfrak{z}^{q+1} \arrow[r] \arrow[d, "h"'] & 0, \\
0 \arrow[r] & \mathfrak{z}_1^q \arrow[r, "i^q"] & \mathscr{S}_1^q \arrow[r, "d_o^{q+1}"] & \mathfrak{z}_1^{q+1} \arrow[r] & 0.
\end{tikzcd}$$

Hence the induced homomorphism h^* of the corresponding exact cohomology sequences also commutes also commutes with e^*, d_o^*, δ and i^*, d_o^*, δ^* respectively.

Case 1. $\underline{q = 0}$. Since $0 \to \mathscr{G} \xrightarrow{e} \mathscr{S}^o \xrightarrow{d^1} \mathscr{S}^1$ is exact, so is the sequence $0 \to \Gamma(X, \mathscr{G}) \xrightarrow{e} \Gamma(X, \mathscr{S}^o) \xrightarrow{d^1} \Gamma(X, \mathscr{S}^1)$. Then $H^o(X, \mathscr{S}) = \ker d^1 = ime$, but e is a monomorphism and $\Gamma(X, \mathscr{G}) = H^o(X, \mathscr{G})$, hence $e : H^o(X, \mathscr{G}) \to H^o\Gamma(X, \mathscr{S})$ is an isomorphism commuting with h^*. Let $\eta = e^{-1}$. **87**

Case 2. $\underline{q > 0}$. The exact cohomology sequence corresponding to that exact sequences (1) for $q-1$ (where $\mathfrak{z}^o = \mathscr{G}$) is

$$0 \to \Gamma(X, \mathfrak{z}^{q-1}) \xrightarrow{i^{q-1}} \Gamma(X, \mathscr{S}^{q-1}) \xrightarrow{d_o^q} \Gamma(X, \mathfrak{z}^q) \xrightarrow{\delta^*} H^1(X, \mathfrak{z}^{q-1}) \to 0 \to \cdots$$

since $H^1(X, \mathscr{S}^{q-1}) = 0$. Thus δ^* induces an isomorphism

$$\delta^* : \Gamma(X, \mathfrak{z}^q)/imd_o^q \to H^1(X, \mathfrak{z}^{q-1}) \ (q \geqq 1).$$

Since $\operatorname{im} i^q = \ker d_o^{q+1} = \ker d^{q+1}$, the monomorphism i^q induces an isomorphism

$$\begin{aligned}
i^{q^*} : \Gamma(X, \mathfrak{z}^q)/\operatorname{im} d_o^q &\to \operatorname{im} i^q/\operatorname{im} d^q \\
&= \ker d^{q+1}/\operatorname{im} d^q \\
&= H^q\Gamma(X, \mathscr{S}).
\end{aligned}$$

Thus we have an isomorphism

$$\delta^*(i^{q^*})^{-1} : H^q\Gamma(X, \mathscr{S}) \to H^1(X, \mathfrak{z}^{q-1}) \ (q \geqq 1)$$

commuting with h^*, since h^* commutes with δ^* and $(i^*)^{-1}$.

$$\begin{tikzcd}
& 0 \arrow[d] & & \\
\Gamma(X,\mathscr{S}^{q-1}) \arrow[r,"d_o^q"] \arrow[dr,"d^q"'] & \Gamma(X,\mathfrak{z}^q) \arrow[r,"\delta^*"] \arrow[d,"i^q"'] & H^1(X,\mathfrak{z}^{q-1}) \arrow[r] & 0 \\
& \Gamma(X,\mathscr{S}^q) \arrow[d,"d_o^{q+1}"'] \arrow[dr,"d^{q+1}"] & & \\
0 \arrow[r] & \Gamma(X,\mathfrak{z}^{q+1}) \arrow[r,"i^{q+1}"'] & \Gamma(X,\mathscr{S}^{q+1}). &
\end{tikzcd}$$

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Also, the exact cohomology sequences corresponding to (1) contain

$$0 \xrightarrow{(d_o^{q-p+1})^*} H^{p-1}(X,\mathfrak{z}^{q-p+1}) \xrightarrow{\delta^*} H^p(X,\mathfrak{z}^{q-p}) \xrightarrow{(i^{q-p})^*} 0$$

for $1 < p < q$ and

$$0 \xrightarrow{d_o^{1*}} H^{q-1}(X,\mathfrak{z}^1) \xrightarrow{\delta^*} H^q(X,\mathscr{G}) \xrightarrow{e^*} 0 \quad \text{for} \quad p = q.$$

Thus we have isomorphisms ($q \geqq 1$),

$$H^q\Gamma(X,\mathscr{S}) \xrightarrow{\delta^*(i^{q^*})^{-1}} H^1(X,\mathfrak{z}^{q-1}) \to \cdots \to H^{q-1}(X,\mathfrak{z}^1) \xrightarrow{\delta^*} H^q(X,\mathscr{G})$$

commuting with h^*. Let η be the composite of these isomorphisms, $\eta : H^q\Gamma(X,\mathscr{S}) \to H^q(X,\mathscr{G})$.

Theorem 1 (Uniqueness theorem)**.** *If X is paracompact normal, and if*

$$0 \to \mathscr{G} \xrightarrow{e} \mathscr{S}^o \xrightarrow{d^1} \mathscr{S}^1 \to \cdots \to \mathscr{S}^{q-1} \xrightarrow{d^q} \mathscr{S}^q \to \cdots,$$
$$0 \to \mathscr{G} \xrightarrow{e} \mathscr{S}_1^o \xrightarrow{d^1} \mathscr{S}_1^1 \to \quad \to \mathscr{S}_1^{q-1} \xrightarrow{d^q} \mathscr{S}_1^q \to \cdots,$$

89 *are two resolutions of the same sheaf $\mathscr{G}$ of A-modules, there is a canon-*

ical isomorphism

$$\phi : H^q\Gamma(X, \mathscr{S}) \to H^q\Gamma(X, \mathscr{S}_1).$$

Moreover, if $h : (\mathscr{S}^o, \mathscr{S}^1, \mathscr{S}^2, \ldots) \to (\mathscr{S}_1^o, \mathscr{S}_1^1, \mathscr{S}_1^2, \ldots)$ *is a homomorphism commuting with* $e, d^1, d^2, \ldots$*,*

$$\begin{tikzcd}
 & & \mathscr{S}^o \arrow[r, "d^1"] \arrow[dd, "h"] & \mathscr{S}^1 \arrow[r, "d^2"] \arrow[dd, "h"] & \cdots \arrow[r] & \mathscr{S}^{q-1} \arrow[r, "d^q"] \arrow[dd, "h"] & \mathscr{S}^q \arrow[r] \arrow[dd, "h"] & \cdots \\
0 \arrow[r] & \mathscr{G} \arrow[ur, "e"] \arrow[dr, "e"'] & & & & & & \\
 & & \mathscr{S}_1^o \arrow[r, "d^1"] & \mathscr{S}_1^1 \arrow[r, "d^2"] & \cdots \arrow[r] & \mathscr{S}_1^{q-1} \arrow[r, "d^q"] & \mathscr{S}_1^q \arrow[r] & \cdots
\end{tikzcd}$$

then the induced homomorphism

$$h^* : H^q\Gamma(X, \mathscr{S}) \to H^q\Gamma(X, \mathscr{S}_1)$$

is the isomorphism ϕ*.*

Proof. We have the canonical isomorphisms η, η_1,

$$H^Q\Gamma(X, \mathscr{S}) \xrightarrow{\eta} H^q(X, \mathscr{G}) \xleftarrow{\eta_1} H^q\Gamma(X, \mathscr{S}_1);$$

let $\phi = \eta_1^{-1}\eta$. ◻

There is commutativity in the diagram:

$$\begin{tikzcd}
H^q\Gamma(X, \mathscr{S}) \arrow[r, "\eta"] \arrow[d, "h^*"'] & H^q(X, \mathscr{G}) \arrow[d, "h^*"'] \\
H^q\Gamma(X, \mathscr{S}_1) \arrow[r, "\eta_1"] & H^q(X, \mathscr{G}),
\end{tikzcd}$$

where the homomorphism h^* on the right is the identity. Hence the homomorphism h^* on the left is equal to $\eta_1^{-1}\eta = \phi$.

Lecture 15

We now given example to show that the uniqueness theorem fails in more general spaces. 90

Example 24. Let X consist of the unit segment $I = \{x : o \leqq x \leqq 1\}$ together with two points p, q. A neighbourhood of p (resp. q) consists of p (resp. q) together with all of I. Let $\mathscr{G}$ be the subsheaf of the constant sheaf Z_2 formed by omitting the points $(p, 1)$, $(q, 1)$, $(0, 1)$, $(1, 1)$. Let $\mathscr{S}^o$ be the subsheaf of the constant sheaf $Z_2 + Z_2$ formed by omitting (p, a), (q, a) for all $a \neq 0$ and $(0, (1, 0))$, $(0, (1, 1))$, $(1, (0, 1))$, $(1, (1, 1))$. Let $\mathscr{S}^1$ be the subsheaf of Z_2 formed by omitting $(q, 1)$. Let $\mathscr{S}^2$ have the stalk Z_2 at p and 0 elsewhere; a neighbourhood of $(p, 1)$ consists of $(p, 1)$ together with all the zeros over I. Then there is a resolution

$$0 \to \mathscr{G} \xrightarrow{e} \mathscr{S}^o \xrightarrow{d^1} \mathscr{S}^1 \xrightarrow{d^2} \mathscr{S}^2 \xrightarrow{d^3} 0$$

and the corresponding sequence

$$0 \to \Gamma(X, \mathscr{S}^o) \xrightarrow{d^1} \Gamma(X, \mathscr{S}^1) \xrightarrow{d^2} \Gamma(X, \mathscr{S}^2) \xrightarrow{d^3} 0$$

is

$$0 \to 0 \to 0 \to Z_2 \to 0;$$

so $H^2\Gamma(X, \mathscr{S}) = Z_2$.

There is also a resolution 91

$$0 \to \mathscr{G} \xrightarrow{e} \mathscr{R}^o \xrightarrow{d^1} 0$$

where e is an isomorphism; then $H^p\Gamma(X,\mathscr{R}) = 0$ for all p. There is even a homomorphism h, commuting with $e, d^1, \ldots$,

$$\begin{tikzcd}
 & & \mathscr{R}^o \arrow[r, "d^1"] \arrow[dd, "h"] & 0 \arrow[r] \arrow[dd] & 0 \arrow[r] \arrow[dd] & 0 \\
0 \arrow[r] & \mathscr{G} \arrow[ur, "e"] \arrow[dr, "e"'] & & & & \\
 & & \mathscr{S}^o \arrow[r, "d^1"] & \mathscr{S}^1 \arrow[r] & \mathscr{S}^2 \arrow[r] & 0.
\end{tikzcd}$$

[The space X is not normal.]

Definition. *A sheaf $\mathscr{S}$ of A-modules is called fine if for every closed set E in X and open set G in X with $E \subset G$, there is a homomorphism $h : \mathscr{S} \to \mathscr{S}$ such that*

i) $h(s) = s$ *if* $\pi(s) \in E$,

ii) $h(s) = 0_{\pi(s)}$ *if* $\pi(s) \notin \overline{G}$

example of a fine sheaf.

Example 25. For each open subset U of X, let S_U be the A-module of all functions $f : U \to A$. If $V \subset U$, define ρ_{VU} to be restriction homomorphism. Let $\mathscr{S}$ be the sheaf of germs of functions determined by the presheaf $\sum = \{S_U, \rho_{VU}\}$. If $E \subset G$ with E closed and G open, let $h_U : S_U \to S_U$ be defined by

$$(h_U f)(x) = f(x)\,\chi_G(x)$$

92 where $x \in U$ and χ_G is the characteristic function of G.

$$(\chi_G(x) = 1 \in A \quad \text{if} \quad x \in G, \chi_G(x) = 0 \in A \quad \text{if} \quad x \notin G).$$

Then $\{h_U\} : \sum \to \sum$ is a homomorphism. If $h : \mathscr{S} \to \mathscr{S}$ is the induced homomorphism, $h(s) = s$ if $\pi(s) \in E$ and $h(s) = 0$ if $\pi(s) \notin \overline{G}$, hence the sheaf is fine.

Exercise. If M is any non-zero A-module and the space X is normal, the constant sheaf $(X\chi M, \pi, X)$ is fine if and only if $\dim X \leq 0$.

Note. The set of all endomorphisms $h : \mathscr{S} \to \mathscr{S}$ forms an A-algebra, in general, non commutative, where $h_1 \cdot h_2$ is the composite endomorphism. The identity $1 : \mathscr{S} \to \mathscr{S}$ is the unit element of the algebra.

If $\mathscr{S} = (S, \pi, X)$ is a sheaf and X_1 is a subset of X, let X_1 and $S_1 = \pi^{-1}(X_1)$ have the induced topology. Then $(S_1, \pi|S_1, X_1)$ is a sheaf called the *restriction* of $\mathscr{S}$ to X_1.

If X is normal, the restriction of a fine sheaf $\mathscr{S}$ to any closed set C is fine.

Proof. Let E be any closed subset of C and G any open subset of C with $E \subset G$. Extend G to an open set H of X, $G = H \cap C$. Then, since X is normal, E closed in X, H open in X with $E \subset H$, there is an open subset V of X with $E \subset V \subset \overline{V} \subset H$. Since $\mathscr{S}$ is fine, there is a homomorphism $h : \mathscr{S} \to \mathscr{S}$ with

$$\begin{aligned} h(s) &= s && \text{if } \pi(s) \in E, \\ &= 0_{\pi(s)} && \text{if } \pi(s) \in X - \overline{V}. \end{aligned}$$

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Then if $\mathscr{S}_1$ is the restriction of $\mathscr{S}$ to C, $h|S_1 : S_1 \to S_1$ is a homomorphism $h_1 : \mathscr{S}_1 \to \mathscr{S}_1$ and we have

$$\begin{aligned} h_1(s) = h(s) &= s && \text{if } \pi(s) \in E \\ &= 0_{\pi(s)} && \text{if } \pi(s) \in C - \overline{G} \subset C - G \subset X - H \subset X - \overline{V}. \end{aligned}$$

□

Proposition 11. *If X is normal, $\mathscr{U} = \{U_i\}_{i\varepsilon I}$ a locally finite covering of X, and if the restriction of $\mathscr{S}$ to each $\overline{U_i}$ is fine (in particular, if $\mathscr{S}$ is fine), there is a system $\{l_i\}_{i\in I}$ of homomorphisms $l_i : \mathscr{S} \to \mathscr{S}$ such that*

i) for each $i \in I$ there is a closed set $E_i \subset U_i$ such that $1_i(S_x) = 0_x$ if $x \notin E_i$, *ii)* $\sum_{i \in I} 1_1 = 1$.

(1 denotes the identity endomorphism $\mathscr{S} \to \mathscr{S}$).

Proof. Using the normality of X, we shrink the locally finite covering $\mathscr{U} = \{U_i\}_{i \in I}$ to the covering $\mathscr{U}\{V_i\}_{i \in I}$ with $\overline{V}_i \subset U_i$ and we further shrink the locally finite covering $\mathscr{U}$ to the covering $\mathscr{U} = \{W_i\}_{i \in I}$ with $\overline{W}_i \subset V_i$. Since the restriction $\mathscr{S}_i$ of $\mathscr{S}$ to $\overline{U}_i$ is fine, there is a homomorphism $g_i : \mathscr{S}_i \to \mathscr{S}_i$ with

$$\begin{aligned} g_i(s) &= s & &\text{if} \quad \pi(s) \in \overline{W}_i, \\ &= o_{\pi(s)} & &\text{if} \quad \pi(s) \in \overline{U}_i - \overline{V}_i. \end{aligned}$$

94 Let the homomorphism $h_i : \mathscr{S} \to \mathscr{S}$ be defined by

$$\begin{aligned} h_i(s) &= g_i(s) & &\text{if} \quad \pi(s) \in \overline{U}_i, \\ &= 0_{\pi(s)} & &\text{if} \quad \pi(s) \in X - U_i. \end{aligned}$$

(This definition is consistent, since $g_i(s) = 0_{\pi(s)}$ on $\overline{U}_i - U_i$). This $h_i : \mathscr{S} \to \mathscr{S}$ is continuous and is a homomorphism with

$$\begin{aligned} h_i(s) &= s & &\text{if} \quad \pi(s) \in \overline{W}_i, \\ &= 0_{\pi(s)} & &\text{if} \quad \pi(s) \in X - \overline{V}_i. \end{aligned}$$

Let the set I of indices be well ordered and define the homomorphisms $l_i : \mathscr{S} \to \mathscr{S}$ by

$$1_i \left(\prod_{j<i} (1 - h_j) \right) h_i,$$

where the product is taken in the same order as that of the indices. □

Each point $x \in X$ has a neighbourhood N_x meeting U_i for only a finite number of i, say $i_1, i_2, \ldots, i_q$ with $i_1 < i_2 < \cdots < i_q$. If $\pi(s) \in N_x$,

$$\begin{aligned} l_i(s) &= (1 - h_{i1}) \cdots (1 - h_{i_{k-1}}) h_{i_k}(s), i = i_k, k = 1, \ldots, q, \\ &= 0_{\pi(s)} \text{for all other } i. \end{aligned}$$

Clearly $l_i(S_x) \subset S_x$ and $l_i|S_x : S_x \to S_x$ is a homomorphism. The function l_i is continuous on each $\pi^{-1}(N_x)$ and coincides on the overlaps of two such neighbourhoods, hence $l_i : \mathscr{S} \to \mathscr{S}$ is continuous. Thus $l_i : \mathscr{S} \to \mathscr{S}$ is a homomorphism, and

$$h_i(S_x) = 0_x, x \notin \overline{V}_i,$$

hence
$$l_i(S_x) = 0_x, x \notin \overline{V}_i.$$

Take $E_i = \overline{V}_i \subset U_i$. Let $\pi(s) \in N_x$; then for some i_k, $1 \leqq k \leqq q$, $\pi(s) \in W_{i_k}$ and hence $h_{i_k}(s) = s$. Hence 95

$$(1 - h_{i_1}) \cdots (1 - h_{i_q})(s) = 0.$$

Therefore

$$\begin{aligned}\sum_{i \in I} 1_i(s) &= h_{i_1}(s) + (1 - h_{1_1})h_{i2}(s) + \cdots + (1 - h_{i_1}) \cdots (1 - h_{i_{q-1}})h_{i_q}(s) \\ &= s - (1 - h_{i_1}) \cdots (1 - h_{i_q})(s) \\ &= s.\end{aligned}$$

Note. The homomorphisms l_i are usually not uniquely determined and they cannot therefore be expected to commute with other given homomorphisms.

Lecture 16

Let $\{l_i\}_{i\in I}$ be a system of endomorphisms of a fine sheaf $\mathscr{S}$ correspond- **96**
ing to a locally finite covering $\{U_i\}_{i\in I}$ of a normal space X. Each l_i gives a homomorphism $l_i(U) : \Gamma(U, \mathscr{S}) \to \Gamma(U, \mathscr{S})$ for each open U and $\sum\limits_{i\in I} l_i(U)$ has a meaning and is the identity endomorphism of $\Gamma(U\mathscr{S})$. Also l_i determines a homomorphism

$$\overline{l_i}(U) : \Gamma(U_i \cap U, \mathscr{S}) \to \Gamma(U, \mathscr{S})$$

defined by

$$\begin{aligned}(\overline{l_i}(U)g)(x) &= l_i(g(x)) &&\text{if } x \in U_i \cap U,\\ &= 0 &&\text{if } x \in (X - U_i) \cap U.\end{aligned}$$

One verifies that the following diagrams are commutative.

$$\begin{tikzcd}
\Gamma(U, \mathscr{S}) \arrow[r, "l_i(U)"] \arrow[d, "\rho(U_i\cap U{,}\mathscr{S})"'] & \Gamma(U, \mathscr{S}) \arrow[d, "\rho(U_i\cap U{,}\mathscr{S})"] \\
\Gamma(U_i \cap U, \mathscr{S}) \arrow[r, "l_i(U_i\cap U)"'] \arrow[ur, "\overline{l_i}(U)"] & \Gamma(U_i \cap U, \mathscr{S})
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma(U_i \cap U, \mathscr{S}) \arrow[r, "\overline{l_i}(U)"] \arrow[d, "\rho(U_i\cap V{,}U_i\cap U)"'] & \Gamma(U, \mathscr{S}) \arrow[d, "\rho_{VU}"] \\
\Gamma(U_i \cap V, \mathscr{S}) \arrow[r, "\overline{l_i}(V)"] & \Gamma(V, \mathscr{S})
\end{tikzcd}$$

If X is normal, $\mathscr{S}$ is fine and $\mathscr{U} = \{U_i\}_{i\in I}$ is a locally finite covering of X, then $H^q(\mathscr{U}, \mathscr{S}) \to 0$ for $q \geqq 1$.

Proof. Let $k^{q-1} : C^q(\mathscr{U}, \mathscr{S}) \to C^{q-1}(\mathscr{U}, \mathscr{S})$ for $q \geqq 1$ be the homomorphism defined by

$$(k^{q-1} f)(\sigma) = \sum_{i\in I} \overline{l_i}(U_\sigma)(f(i\sigma)),$$

where $i\sigma = i, i_o, \ldots, i_{q-1}$ if $\sigma = i_o, \ldots, i_{q-1}$. (This infinite sum of sec-
97 tions is finite neglecting zeros, in some neighbourhood of each point.) Using the fact that $\partial_o(i\sigma) = \sigma$ and $\partial_j(i\sigma) = i\partial_{j-1}(\sigma)$ for $j > 0$, one verifies that

$$\delta^q k^{q-1} f + k^q \delta^{q+1} f = f.$$

(The computation is given at the end of the lecture.)

Hence each cocycle f is a coboundary *q.e.d.* □

Proposition 12. *For a fine sheaf $\mathscr{S}$ over a paracompact, normal space X, $H^q(X, \mathscr{S}) = 0$ for $q \geqq 1$*

Proof. $H^q(X, \mathscr{S}) = 0$ $q \geqq 1$ for each locally finite covering $\mathscr{U}$ of the space X. Since the space X is paracompact, this means that $H^q(X, \mathscr{S}) = 0, q \geqq 1$. □

Corollary . *If X is paracompact and normal, any exact sequence of sheaves*

$$0 \to \mathscr{G} \xrightarrow{e} \mathscr{S}^o \xrightarrow{d^1} \cdots \to \mathscr{S}^{q-1} \xrightarrow{d^q} \mathscr{S}^q \to \ldots,$$

where each $\mathscr{S}^q (q \geqq 0)$ is fine, is a resolution of $\mathscr{G}$.

Definition. *A sheaf $\mathscr{S}$ is called locally fine, if for each open U and each $x \in U$, there is an open V with $x \in V \subset U$ such that the restriction of $\mathscr{S}$ to $\overline{V}$ is fine.*

If X normal, a fine sheaf $\mathscr{S}$ is locally fine.

Proof. The restriction of $\mathscr{S}$ to an arbitrary closed set is fine. □

98 **Proposition 13.** *If X is paracompact normal, a locally fine sheaf $\mathscr{S}$ is fine.*

Proof. Let $E \subset G_1$, with E closed and G_1 open. If $G_2 = X - E$ then $\{G_i\}_{i=1,2}$ is a covering of X. Since $\mathscr{S}$ is locally fine, for each $x \in G_i$, there is an open V_x with $x \in V_x \subset G_i$ such that the restriction of $\mathscr{S}$ to $\overline{V}_x$ is fine. Since X is paracompact, there is a locally finite refinement $\mathscr{U}\{U_j\}_{j \in J}$ of $\{V_x\}_{x \in X}$, hence $\mathscr{U}$ is also a refinement of $\{G_i\}$. If $U_j \subset V_x$, then $\bar{U}_j \subset \bar{V}_x$, and $\bar{V}_x$, being closed in X, is normal. Since the restriction of $\mathscr{S}$ to $\bar{V}_x$ is fine, the restriction of $\mathscr{S}$ to $\bar{U}_j$ is also fine. Now by proposition 11, there exist endomorphisms l_j such that $\sum_{j \in J} l_j = 1$ and l_j is zero outside a closed set $E_j \subset U_j$. Choose the function $\tau : J \to (1, 2)$ so that $U_j \subset G_{\tau(j)}$ and let

$$l_i = \sum_{\tau(j)=i} l_j, \quad i = 1, 2.$$

Then $I_1 + I_2 = l$ and

$$l_i(s) = 0 \text{ if } \pi(s) \in X - \bigcup_{\tau(j)=i} U_j \supset X - G_i.$$

Hence

$$l_1(s) = 0 \quad \text{if} \quad \pi(s) \in X - G_1$$

and

$$l_1(s) = s \quad \text{if} \quad \pi(s) \varepsilon X - G_2 = E.$$

l_1 thus gives the required function, and this completes the proof. □

Corollary . *If X is paracompact and normal, any exact sequence of sheaves*

$$0 \to \mathscr{G} \xrightarrow{e} \mathscr{S}^o \xrightarrow{d^1} \ldots \mathscr{S}^{q-1} \xrightarrow{d^q} \mathscr{S}^q \cdots,$$

where each $\mathscr{S}^q (q \geqq 0)$ is locally fine, is a resolution of $\mathscr{G}$. **99**

The following examples shows that, in more general spaces, fineness need not coincide with local fineness.

Example 26. Let X have points $a, b, \ldots, h$ with base for open sets consisting of (f), (g), (h), (d, f, h), (e, g, h), (c, f, g), (b, e, g, h), (a, d, e, f, g, h). Let $\mathscr{S}$ be the subsheaf of the constant sheaf Z_2 formed by omitting $(c, 1)$, $(f, 1)$. Then $\mathscr{S}$ is fine but not locally fine. In fact, $V = (c, f, g)$ is the least open set containing c and the restriction of $\mathscr{S}$ to $\bar{V} = X - (h)$ is not fine. (X is not normal.)

Example 27. Let T be the space of ordinal numbers $\leqq \omega_1$ with the usual topology induced by the order. Let A be the space or ordinal numbers $\leqq \omega_o$ and let X be the subspace of TXA formed by omitting the point (ω_1, ω_o). Let $\mathscr{S}$ be the constant sheaf Z_2 over X. Then $\mathscr{S}$ is locally fine, for every point has a closed neighbourhood which is normal and zero dimensional. But $\mathscr{S}$ is not fine. If B is the set of even numbers, then $B \subset A$. Let $E = \omega_1 \times B$ and $G = T \times B$. Then $E \subset G \subset X$ with E closed and G open. There is no endomorphism of $\mathscr{S}$ which is the identity on E and is zero outside $\overline{G}$. (X in neither paracompact nor normal.)

100 **Example 28.** The space M of Quart. Jour. Math. 6 (1955), p. 101 is normal and locally zero dimensional but not zero dimensional. Therefore
101 the constant sheaf Z_2 is locally fine but not fine. (M is not paracompact).

$$\delta^q k^{q-1} f + k^q \delta^{q+1} f = f.$$

$$\delta^q k^{q-1} f(\sigma) = \sum_{j=0}^{q} (-1)^j \rho(U_\sigma, U_{\partial_j}\sigma)(k^{q-1} f)(\partial_j \sigma)$$

$$= \sum_{j=o}^{q} (-1)^j \rho(U_\sigma, U_{\partial_j}\sigma) \sum_i \overline{I}_i(U_{\partial_j}\sigma) f(i\partial_j\sigma)$$

$$= \sum_{j=0}^{q} (-1)^j \sum_i \overline{l}_1(U_\sigma) \rho(U_{i\sigma}, U_{j\partial_j}\sigma) f(i\partial_j\sigma).$$

$$k^q \delta^{q+1} = \sum_i \overline{l}_1(U_\sigma)(\delta^{q+1} f)(i\sigma)$$

$$= \sum_i \overline{l}_i(U_\sigma) \sum_{j=0}^{q+1} (-1)^j \rho(U_i\sigma, U_{\partial_j} i\sigma) f(\partial_j i\sigma)$$

$$= \sum_i \overline{l}_i(U\sigma) \rho(U_i\sigma, U_\sigma) f(\sigma)$$

$$+ \sum_i \overline{l}_i(U\sigma) \sum_{j=1}^{q+1} (-1)^j \rho(U_j\sigma, U_i\partial_{j-1}\partial) f(i\partial_{j-1}\sigma)$$

$$= \sum_i l_i(U_\sigma) f(\sigma) \sum_i \overline{l_i}(U_\sigma) \sum_{j=0}^{q} (-1)^{j+1} \rho(U_{i_\sigma}, U_{i\partial_j\sigma}) f(i\partial_j\sigma).$$

$$\delta^q k^{q-1} f(\sigma) + k^q \delta^{q+1} = \sigma_i l_i(U_\sigma) f(\sigma) = f(\sigma)$$

Lecture 17

In this and the next lecture, we shall give a proof of de Rham's theorem. **102**

Let X be an indefinitely differentiable (C^∞) manifold of dimension n, which is countable at infinity (i.e) a countable union of compact sets); we assume that X is a Hausdorff space. Then X is paracompact and normal. (Dieudonne, Jour. de Math. 23 (1944)). The set $\mathscr{E}^p(U)$ of all $\underline{C^\infty}$ (alternating) differential p-forms on an open set U forms a vector space over the field of real numbers. Exterior differentiation gives a homomorphism d^p,

$$d^p : \mathscr{E}^{p-1}(U) \to \mathscr{E}^p(U)$$

with $d^{p+1}d^p = 0$. In particular, there is a sequence

$$0 \xrightarrow{d^o} \mathscr{E}^o(X) \xrightarrow{d^1} \mathscr{E}^1(X) \to \cdots \xrightarrow{d^p} \mathscr{E}^p(X) \to \cdots \xrightarrow{d^n} \mathscr{E}^n(X) \xrightarrow{d^{n+1}} 0$$

with $\operatorname{im} d^p \subset \ker d^{p+1}$. Let

$$H^p(\mathscr{E}(X)) = \ker d^{p+1} / \operatorname{im} d^p.$$

This vector space of the closed p-forms modulo the derived p-forms is called the p-th *de Rham cohomology vector space* of the manifold X.

If $V \subset U$, the inclusion map $i : V \to U$ induces a homomorphism

$$\rho_{VU} = i^{-1} : \mathscr{E}^p(U) \to \mathscr{E}^p(V)$$

which commutes with d. Thus the system $\mathscr{E}^p = \{\mathscr{E}^p(U), \rho_{VU}\}$ is a **103** presheaf which determines a sheaf Ωp, called the sheaf of germs of p-forms, and

$$d^p : \mathscr{E}^{p-1} \to \mathscr{E}^p$$

is a homomorphism of presheaves which induces a homomorphism

$$d^p : \Omega^{p-1} \longrightarrow \Omega^p.$$

There is a constant presheaf $\{R, \rho_{VU}\}$ where R is the filed of real numbers and $\rho_{VU} : R \to R$ is the identity. This presheaf determines the constant sheaf R. There is a homomorphism $e : R \to \mathscr{E}^o(U)$, ($\mathscr{E}^o(U)$ is the space of C^∞ functions on U) where $e(r)$ is the function on U with the constant value r, and further e commutes with ρ_{VU}. Thus, there is an induced sheaf homomorphism $e : R \to \Omega^o$. Hence, we have a sequence of homomorphisms of sheaves

$$0 \to R \xrightarrow{e} \Omega^o \xrightarrow{d^1} \cdots \to \Omega^{p-1} \xrightarrow{d^p} \Omega^p \to \ldots,$$

with $d^{p+1} d^p = 0$.

There is a homomorphism

$$\{f_U\} : \{\mathscr{E}^p(U), \rho_{VU}\} \to \bar{\Omega}^p,$$

104 ($\bar{\Omega}^p$ denotes the presheaf of sections of Ω^p), where the image of an element of $\mathscr{E}^p(U)$ is the section over U which it determines in Ω^p. Then d commutes with f_U and, in particular, with f_x. Thus we have the commutative diagram:

$$\begin{tikzcd}
\mathscr{E}^{p-1}(X) \arrow[r, "d^p"] \arrow[d, "f_X"'] & \mathscr{E}^p(X) \arrow[d, "f_X"] \\
\Gamma(X, \Omega^{p-1}) \arrow[r, "d^p"] & \Gamma(X, \Omega^p).
\end{tikzcd}$$

If a p-form $\omega \in \mathscr{E}^p(X)$ is not zero, there is some point $x \in X$ at which it does not vanish, and hence $\rho_{xX}\omega \neq 0$. Thus $f_X\omega \neq 0$, i.e., f_X is a monomorphism. (In the same way, f_U is a monomorphism for each open set U.) f_X is also onto, hence an isomorphism. For, if $g \in \Gamma(X, \Omega^p)$, then since Ω^p is the sheaf of germs of p-forms, for each $x \in X$ there is a neighbourhood U_x of x and a p-form ω_x defined on U_x such that section g and the section determined by ω_x coincide on U_x. Then $\{U_x\}$ forms a covering for X, and since the section determined by ω_x and ω_y coincide on $U_x \cap U_y$, using the fact that f_U is a monomorphism for each open set U, we see that the forms ω_x and ω_y themselves coincide on $U_x \cap U_y$; hence they define a global from ω, such that $f_X(\omega) = g$. Thus f_X gives an isomorphism of the sequences:

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{E}^o(X) \arrow[r, "d^1"] \arrow[d, "f_X"'] & \cdots \arrow[r] & \mathscr{E}^{p-1}(X) \arrow[r, "d^p"] \arrow[d, "f_X"'] & \mathscr{E}^p(X) \arrow[r] \arrow[d, "f_X"'] & \cdots \\
0 \arrow[r] & \Gamma(X, \Omega^o) \arrow[r, "d^1"] & \cdots \arrow[r] & \Gamma(X, \Omega^{p-1}) \arrow[r, "d^p"] & \Gamma(X, \Omega^p) \arrow[r] & \cdots
\end{tikzcd}$$

Hence there is an induced isomorphism of the cohomology vector spaces: **105**

$$f_X^* : H^q(\mathscr{E}(X)) \to H^q\Gamma(X, \Omega) \qquad (q \geqq 0).$$

Poincare's lemma. The sequence

$$0 \to R \to \Omega^o \xrightarrow{d^1} \cdots \to \Omega^{p-1} \xrightarrow{d^p} \Omega^p \to \cdots$$

is exact.

Proof. We have to prove that for each point $a \in X$, the sequence

$$0 \to R_a \xrightarrow{e} \Omega_a^o \xrightarrow{d^1} \cdots \to \Omega_a^{p-1} \xrightarrow{d^p} \Omega_a^p \cdots$$

is exact, where the subring R_a of Ω_a^o, consisting of the germs of constant functions at $\underline{a}$, is identified with the filed R of real number. Choose a coordinate neighbourhood W of $\underline{a}$ with coordinates $(x_1, \ldots, x_n)$ and suppose that $a = (0, \ldots, 0)$. Then Ω_a^p is the direct limit of the system $\{\mathscr{E}^p(U), \rho_{VU}\}_{a \varepsilon U'}$ where U belongs to the cofinal set of those spherical neighbourhoods $x_1^2 + \cdots + x_n^2 < r^2$ which are contained in W. □

For each such U, let

$$h : \mathscr{E}^o(U) \to R$$

$$\text{and} \qquad k^{p-1} : \mathscr{E}^p(U) \to \mathscr{E}^{p-1}(U) \qquad (p \geqq 1)$$

be the homomorphisms defined by

$$h(f) = f(0, \ldots, 0),$$

and

$$k^{p-1}(f(x_1, \ldots, x_n) dx_i \cdots dx_{i_p})$$
$$= (\int_0^1 f(tx_1, \ldots, tx_n) t^{p-1} dt) \cdot \sum_{j=1}^{P} (-1)^{j-1} x_{i_j} dx_{i_1} \ldots \hat{dx}_{i_j} \ldots dx_{i_p}$$

106 respectively. (The formula on the right is an alternating function of $i_1, \ldots, i_p$; h and k^{p-1} are then extended by linearity to $\mathscr{E}^o(U)$ and $\mathscr{E}^p(U)$ respectively.) One now verifies that

$$eh + k^0 d^1 = 1,$$
$$d^p k^{p-1} + k^p d^{p+1} = 1 \ (p \geqq 1),$$

where 1 denotes the identity map.

(The computation is carried out at the end of the lecture.)

Thus $f \in \ker d'$ implies that $f \in \operatorname{im} e$ and $\omega \in \ker d^{p+1}$ implies that $\omega \in \operatorname{im} d^p$. Hence $\ker d^1 = \operatorname{im} e$ and $\ker d^{p+1} = imd^p$, since already $\operatorname{im} e \subset \ker d^1$ and $\operatorname{im} d^p \subset \ker d^{p+1}$. Hence the sequence

$$0 \to R_U \xrightarrow{e} \mathscr{E}^o(U) \xrightarrow{d^1} \ldots \mathscr{E}^{p-1}(U) \xrightarrow{d^p} \mathscr{E}^p(U) \to \ldots$$

is exact, and since exactness is preserved under direct limits, therefore the limit sequence

$$0 \to R_a \xrightarrow{e} \Omega_a^o \xrightarrow{d^1} \cdots \to \Omega_a^{p-1} \xrightarrow{d^p} \Omega_a^p \to \ldots$$

is exact, *q.e.d*.

The sheaf Ω^p is fine.

Proof. Since the space X is paracompact and normal, by Proposition 13 (Lecture 16), it is enough to prove that the sheaf Ω^p is locally fine. Let U be an open set of X, and let $a \in U$. □

107 We may assume that $\overline{U}$ is compact and that it is contained in some coordinate neighbourhood N of a. Let V be an open subset with $a \in V$ and $\overline{V} \subset U$. We will now prove that the restriction $\Omega^p_{\overline{V}}$ of Ω^p to $\overline{V}$ is fine.

Let $E \subset G$ with E closed and G open in $\overline{V}$. Extend G to an open set $H \subset U$, so that $G = \overline{V} \cap H$. Then $\overline{U}$ is covered by a finite number of spherical neighbourhoods S_i contained in N, such that $\overline{S}_i$ either does not meet E or is contained in H.

For each i, choose an indefinitely differentiable function f_i which is positive inside S_i and vanishes out side S_i. We construct one such function as follows: For the spherical neighbourhood $\sum\limits_{j=1}^{n}(x_j - b_{ij})^2 < r_i^2$, let

$$\begin{aligned} g_i(r) &= 0 && (r \geqq r_i), \\ &= \int_r^{r_i} \exp\left\{\frac{1}{(t-\frac{r_i}{2})(t-r_i)}\right\} dt && (\frac{r_i}{2} \leqq r \leqq r_i), \\ &= \int_{r_{i/2}}^{r_i} \exp\left\{\frac{1}{(t-\frac{r_i}{2})(t-r_i)}\right\} dt && (0 \leqq r \leqq \frac{r_i}{2}) \end{aligned}$$

and define f_i by

$$f_i(x_1, \ldots, x_n) = g_i(\sqrt{\sum_{j=1}^{n}(x_j - b_{ij})^2}.$$

Let $\varphi_1(x) = \sum f_i(x)$, summed for all i for which $\overline{S}_i$ meets E and let $\varphi_2(x) = \sum f_i(x)$, summed for all the remaining i. Then $\varphi_1 + \varphi_2$ is positive in U and, if

$$\theta(x) = \varphi_1(x)/(\varphi_1(x) + \varphi_2(x)),$$

θ is indefinitely differentiable in U, is zero outside H and is constant, 108
equal to 1, in a neighbourhood of E.

Let $h : \mathscr{E}^p(W) \to \mathscr{E}^p(W)$, for open $W \subset U$ be defined by $h(\omega) = \theta \cdot \Omega$. Then h is a homomorphism commuting with $\rho_{_{YW}}$, Y open in U, $Y \subset W$. Hence h induces a homomorphism $h : \Omega^p(U) \to \Omega^p(U)$ for which

$$h(\omega_b) = \omega_b \text{ if } b \in E,$$

$$h(\omega_b) = 0_b \text{ if } b \in \overline{V} - \overline{G} \subset U - H.$$

(ω_b denotes the germ determined by ω at $b \in U$.)

Proposition 14. *There is an isomorphism*

$$\eta f_X^* : H^p(\mathscr{E}(X)) \to H^P(X, R).$$

Proof. By the corollary to Proposition 12, the exact sequence

$$0 \to R \xrightarrow{e} \Omega^o \xrightarrow{d^1} \cdots \to \Omega^{p-1} \xrightarrow{d^p} \Omega^p \to \ldots$$

is a resolution of the constant sheaf R and hence, by Proposition 10, there is an isomorphism

$$\eta : H^p\Gamma(x, \Omega) \to H^p(X, R);$$

but we already have (as proved in the earlier part of this lecture) an isomorphism

$$f_X^* : H^p(\mathscr{E}(X)) \to H^p\Gamma(X, \Omega).$$

□

109 $\underline{(1) eh + k^0 d^1 = 1}$. $\underline{(2) d^p k^{p-1} + k^p d^{p+1} = 1 \ (p \geqq 1)}$.

(1) If $f(x) = f(x_1, \ldots, x_n) \in \mathscr{E}^0(U)$,

$$ehf(x) = f(0, \ldots, 0) \text{ and } d^1 f(x) = \sum_{i=1}^{n} D_i f(x) dx_i, (*),$$

hence $k^o d^1 f(x) = \sum_{i=1}^{n} \int_0^1 D_i f(tx) dt \cdot x_i = \int_0^1 \dfrac{d}{dt} f(tx) dt = f(x) - f(0)$, thus $ehf(x) + k^o d^1 f(x) = f(x)$.

(2) If $\omega = f(x_1, \ldots, x_n) dx_{i_1} \cdots dx_{i_p}$,

$$d^p k^{p-1} = d^p(\Big\{\int_0^1 f(tx) t^{p-1} dt\Big\} \cdot \sum_{j=1}^{p} (-1)^{j-1} x_{i_j} dx_{i_1} \cdots d\hat{x}_{i_j} \cdots dx_{i_p}$$

$$= \left\{ (\sum_{i=1}^{n} (\int_0^1 D_i f(tx) t^p dt)) \cdot \sum_{j=1}^{p} (-1)^{j-1} x_{i_j} dx_{i_1} \cdots d\hat{x}_{i_j} \ldots dx_{i_p} \right\}$$

$$+ (\int_0^1 f(tx) t^{p-1} dt) \cdot p dx_{i_1} \cdots dx_{i_p}.$$

$$k^p d^{p+1} \omega = k^p (\sum_{i=1}^{n} Di f(x) dx_i dx_{i_1} \cdots dx_{i_p})$$

$$\sum_{i=1}^{n} (\int_0^i D_i f(tx) t^p \cdot dt) \Big\{ x_i dx_{i_1} \cdots dx_{i_p}$$

$$- \sum_{j=1}^{p} (-1)^{j-1} x_{i_j} dx_i dx_{i_1} \cdots d\hat{x}_{i_j} \cdots dx_{i_p} \Big\}.$$

(∗) D_i denotes partial derivation with respect to the i - th variable concerned.

Thus $d^p k^{p-1} \omega + k^p d^{p+1} \omega$ **110**

$$= (\int_0^1 f(tx) p \cdot t^{p-1} dt) dx_{i_1} \ldots dx_{i_p}$$

$$+ \sum_{i=1}^{n} \int_0^1 D_i f(tx) t^p \cdot dt \cdot x_i dx_{i_1} \ldots dx_{i_p}$$

$$= \left\{ f(tx) t^p \Big]_0^1 - \int_0^1 \sum_{i=1}^{n} D_i f(tx) \cdot x_i t^p dt \right\} dx_{i_1} \ldots dx_{i_p}$$

$$+ \sum_{i=1}^{n} \int_0^1 D_i f(tx) \cdot t^p dt \cdot x_i dx_{i_1} \ldots dx_{i_p}$$

$$= f(x) dx_{i_1} \ldots dx_{i_p}$$

$$= \omega.$$

Lecture 18

Let s^p be a fixed p-simplex in Euclidean p-space R^p, with vertices **111**
$a_0, a_1, \ldots, a_p$, i.e. s^p is the convex set spanned by points $a_0, \ldots, a_p$ which are in general position. We may assume that a_o is the origin and $a_1, \ldots, a_p$ are unit points of a coordinate axes in R^p, and that s^{p-1} is the face opposite a_p in s^p.

Definition. *A differentiable singular p-simplex in a C^∞ manifold X is a C^∞− map $t : s^p \to X$. The image,* $\operatorname{im} t$*, is called the support of the singular simplex t. The j-th face $\partial_j t$ is the composite map $td_j : s^{p-1} \to X$ where $d_j : s^{p-1} \to s^p$ is the linear map which maps $a_o, \ldots, a_{p-1}$ into $a_o, \ldots, \hat{a}_j, \ldots, a_p$.*

The support of $\partial_j t$ is contained in the support of t.

Definition. *A differentiable singular p-cochain in an open set $U \subset X$ is a real valued function of differentiable p-simplexes with supports in U; $f(t) \in R$ if $suppt \subset U$.*

The set S^p_U of of all differentiable $P-$ cochains in U forms a real vector space. There is a restriction homomorphism $\rho_{VU} : S^P_U \to S^p_V$ for $V \subset U$ and a coboundary homomorphism $d^p : S^{p-1}_U \to S^p_U$ defined by

$$(d^p f)(t) = \sum_{j=0}^{p} (-1)^j f(\partial_j t).$$

The homomorphisms ρ_{VU} and d^p commute, and $imd^o \subset \ker d^{p+1}$.

In particular, there is a sequence

$$0 \to S^o_X \xrightarrow{d_1} S^1_X \to \cdots \to S^{p-1}_X \xrightarrow{d^p} S^p_X \to \ldots$$

112 with $d^{p+1}d^p = 0$, i.e., $\operatorname{im} d^p \subset \ker d^{p+1}$. Let

$$H^p(S_X) = \ker d^{p+1}/imd^p.$$

This vector space is called the $p-$th *cohomology vector space of X based on differentiable singular cochains.*

Since a singular 0-simplex may be identified with the point which is its support, S^o_U can be identified with the vector space of all functions $f : U \to R$. The vector space $\mathscr{E}^o(U)$ of $C^\infty-$ functions on U is a subspace of S^o_U, and the space R of constant functions on U is a subspace of $\mathscr{E}^o(U)$, i.e., $R \subset \mathscr{E}^o(U) \subset S^o_U$.

The presheaf $\left\{S^p_U, \rho_{VU}\right\}$ determines a sheaf $\mathscr{S}^p$ and since ρ_{VU} commutes with d^p and with the inclusion homomorphism $e : R \to S^0_U$, there are induced homomorphisms

$$0 \to R \xrightarrow{e} \mathscr{S}^0 \xrightarrow{d_1} \cdots \to \mathscr{S}^{p-1} \xrightarrow{d^p} \mathscr{S}^p \to \ldots$$

Here the constant sheaf R is identified with the sheaf of germs of constant functions.

There is a homomorphism

$$\{g_U\} : \left\{S^p_U, \rho_{VU}\right\} \to \bar{\mathscr{S}}^p,$$

where the image of an element of S^p_U is the section which it determines. Then d commutes with g_U and, in particular, with g_X. Then we have the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & S^o_X \arrow[r] \arrow[d, "g_X"] & \cdots \arrow[r] & S^{p-1}_X \arrow[r, "d^p"] \arrow[d, "g_X"] & S^p_X \arrow[r] \arrow[d, "g_X"] & \cdots \\
0 \arrow[r] & \Gamma(X, \mathscr{S}^o) \arrow[r] & \cdots \arrow[r] & \Gamma(X, \mathscr{S}^{p-1}) \arrow[r, "d^p"] & \Gamma(X, \mathscr{S}^p) \arrow[r] & \cdots
\end{tikzcd}$$

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The induced homomorphism $g^*_X : H^p(S_X) \to \Gamma(X, \mathscr{S})$is an isomorphism for each p.

Proof. For each covering $\mathscr{U}$ of X, let $S^p_{\mathscr{U}}$ denote the vector space of all real valued functions of differentiable $p-$simplexes, which are defined for each simplex t with supp t contained in some open set of $\mathscr{U}$. $d^p maps S^{p-1}_{\mathscr{U}}$ into $S^p_{\mathscr{U}}$ and, if $\mathscr{W}$ is a refinement of $\mathscr{U}$, there is the obvious restriction homomorphism $g_{\mathscr{W}\mathscr{U}} : S^p_{\mathscr{U}} \to S^p_{\mathscr{W}}$ commuting with d^p. Then $\left\{S^p_{\mathscr{U}}, g_{\mathscr{W}\mathscr{U}}\right\}$ is a direct system and it can be proved, using a method similar to the one used in the proof of Proposition 8 (Lecture 12), that its direct limit is $\Gamma(X, \mathscr{S}^p)$. □

The induced homomorphisms $g^*_{\mathscr{W}\mathscr{U}} : H^p(S_{\mathscr{U}\mathscr{U}}) \to H^p(S_{\mathscr{W}})$ are isomorphisms. (See Cartan Seminar, 1948-49, Exposé 8, §3.). Hence $g^*_{\mathscr{U}} : H^p(S_{\mathscr{U}}) \to H^p\Gamma(X, \mathscr{S})$ is an isomorphism and, in particular, taking $\mathscr{U}$ as the covering by one open set X,

$$g^*_X : H^p(S_X) \to H^p\Gamma(X, \mathscr{S})$$

is an isomorphism, *q.e.d.*

The sequence

$$0 \to R \xrightarrow{e} \mathscr{S} \to \cdots \to \mathscr{S}^{p-1} \xrightarrow{d^p} \mathscr{S}^p \to \dots$$

is exact.

Proof. It is sufficient to show that **114**

$$0 \to R \xrightarrow{e} S^0_U \to \cdots \to S^{p-1}_U \xrightarrow{d^p} S^p_U \to \dots$$

is exact for a cofinal system of neighbourhoods U of each point $a \in X$. For this system, take the spherical neighbourhoods U contained in a coordinate neighbourhood N of $\underline{a}$. The result is proved using the conical homotopy operator. (See Cartan Seminar, 1948-49, Expose 7, §6 ; his formula should be replaced by

$$\begin{aligned} y(\lambda_0, \dots\dots \lambda_{p+1}) &= \phi(\lambda_0)x \ \ (\lambda_1/(1-\lambda_0), \dots) \qquad && \lambda_0 \neq 1, \\ &= 0 && \lambda_0 = 1; \end{aligned}$$

where the indefinitely differentiable function ϕ is chosen so that $0 \leqq \phi(\lambda)_0) \leqq 1$ for $0 \leqq \lambda_0 \leqq 1$, $\phi(0) = 1$ and $\phi(\lambda_0) = 0$ for λ_o in some neighbourhood of 1.) □

The sheaf $\mathscr{S}^p$ is fine.

Proof. Let $E \subset G$ with E closed and G open. Define $h_U : S^p_U \to S^p_U$ by

$$\begin{aligned}(h_U f)(t) &= f(t) \quad \text{if supp } t \subset G\\ &= 0 \quad \text{otherwise.}\end{aligned}$$

Then h_U is a homomorphism commuting with ρ_{VU} and induces a homomorphism $h : \mathscr{S}^p \to \mathscr{S}^p$ such that $h_x : S^p_x \to S^p_x$ is the identity if $x \in G$, and is zero if $x \in X - \bar{G}$. Thus $\mathscr{S}^p$ is fine, *q.e.d.* □

115 Now let $h_U : \mathscr{E}^p(U) \to S^p_U$ be the homomorphism defined by

$$(h_U\omega)(t) = \int\limits_{s^p} t^{-1}\omega,$$

where $t^{-1}\omega$ is the inverse image of the form ω by t. Clearly h_U commutes with ρ_{VU}, hence induces a homomorphism $h : \Omega^p \to \mathscr{S}^p$ with commutativity in

$$\begin{array}{ccc}\mathscr{E}^p(U) & \xrightarrow{\rho_{xU}} & \Omega^p_x\\ {\scriptstyle h_U}\downarrow & & \downarrow{\scriptstyle h_x}\\ S^p_U & \xrightarrow{\rho_{xU}} & S^p_x.\end{array}$$

Hence there is an induced homomorphism $h : \Gamma(X, \Omega^p) \to \Gamma(X, \mathscr{S}^p)$ with commutativity in

$$\begin{array}{ccc}\mathscr{E}^p(X) & \xrightarrow{f_X} & \Gamma(X, \Omega^p)\\ {\scriptstyle h_X}\downarrow & & \downarrow{\scriptstyle h}\\ S^p_X & \xrightarrow{g_X} & \Gamma(X, \mathscr{S}^p).\end{array}$$

h_U commutes with d^p.

Proof.

$$(h_U\, d^p\omega)t = \int\limits_{s^p} t^{-1}(d^p\omega)$$

$$= \int_{s^p} d^p(t^{-1}\omega) \text{ (since} d^p \text{ commutes with } t^{-1})$$
$$= \sum_{j=0}^{p} (-1)^j \int_{d_j s^p} t^{-1}\omega \text{(by Stokes' theorem for } s^p),$$
$$= \sum_{j=0}^{p} (-1)^j (h_U\omega)\partial_j t$$
$$= (d^p h_U \omega)t.$$

Thus h_X induces a homomorphism $h_X^* : H^p(\mathscr{E}(x)) \to H^p(S_X)$. Also **116**
the homomorphisms $h : \Omega^p \to \mathscr{S}^P$ and hence the induced homomorphisms $h : \Gamma(X, \Omega^p) \to \Gamma(X, \mathscr{S}^P)$ commute with d^p, and thus there are induced homomorphisms $h^* : H^p\Gamma(X, \Omega) \to H^p\Gamma(X, \mathscr{S})$. There is commutativity in

$$\begin{tikzcd} H^p(\mathscr{E}(X)) \arrow[r, "f_X^*"] \arrow[d, "h_X^*"'] & H^p\Gamma(X, \Omega) \arrow[d, "h^*"] \\ H^p(S_X) \arrow[r, "g_X^*"] & H^p\Gamma(X, \mathscr{S}). \end{tikzcd}$$

The homomorphism h^* is an isomorphism. □

Proof. We have the two resolutions

$$\begin{tikzcd} & \Omega^0 \arrow[r] \arrow[dd, "h"] & \cdots \arrow[r] & \Omega^{p-1} \arrow[r, "d^p"] \arrow[dd, "h"] & \Omega^p \arrow[r] \arrow[dd, "h"] & \cdots \\ 0 \arrow[r] & R \arrow[u, "e"] \arrow[d, "e"'] & & & & \\ & \mathscr{S}^0 \arrow[r] & \cdots \arrow[r] & \mathscr{S}^{p-1} \arrow[r, "d^p"] & \mathscr{S}^p \arrow[r] & \cdots \end{tikzcd}$$

of R. The homomorphism h commutes with d^p, and commutativity in the triangle follows from the fact that $R \subset \Omega^o \to \mathscr{S}^o$ and e, h and e are inclusion homomorphisms. Hence h^* is the isomorphism of the uniqueness theorem, *q.e.d.* □

Theorem 2 (de Rham)**.** *The homomorphism*

$$h_X^* : H^p(\mathscr{E}(X)) \to H^p(S_X)$$

is an isomorphism.

Proof. The following diagram is commutative :

$$\begin{tikzcd} H^p(\mathscr{E}(X)) \arrow[r, "f_X^*"] \arrow[d, "h_X^*"'] & H^p\Gamma(X, \Omega) \arrow[d, "h^*"] \\ H^p(S_X) \arrow[r, "g_X^*"] & H^p\Gamma(X, \mathscr{S}). \end{tikzcd}$$

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Since f_X^*, g_X^*, and h^* are isomorphisms, and the above diagram is commutative, we have $h_X^* = g_X^{*-1} h^* f_X^*$. Therefore, h_X^* is an isomorphism. □

Lecture 19

Definition. *A double complex K is a system of $A-$ modules (A is a com-* 118
mutative ring with unit element $\left\{K^{p,q}\right\}$, indexed by pairs (p,q) of integers, together with homomorphisms d_1 and d_2 with

$$\begin{gathered}
d_1^{p,q} : K^{p-1,q} \to K^{p,q}, \quad d_2^{p,q} : K^{p,q-1} \to K^{p,q}, \\
d_1^{p+1,q} \cdot d_1^{p,q} = 0, \quad d_2^{p,q+1} d_2^{p,q} = 0, \\
d_2^{p+1,q} d_1^{p+1,q-1} + d_1^{p+1,q} d_2^{p,q} = 0,
\end{gathered}$$

(i.e., d_1 and d_2 are differential operators of bi degree (1, 0) and (0, 1) respectively, which anticommute. Usually we omit the superscripts attached to d_1 and d_2.) We have then the anticommutative diagram:

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
\cdots \arrow[r] & K^{p-1,q-1} \arrow[r, "d_2"] \arrow[d, "d_1"'] & K^{p-1,q} \arrow[r, "d_2"] \arrow[d, "d_1"'] & K^{p-1,q+1} \arrow[r] \arrow[d, "d_1"'] & \cdots \\
\cdots \arrow[r] & K^{p,q-1} \arrow[r, "d_2"] \arrow[d] & K^{p,q} \arrow[r, "d_2"] \arrow[d] & K^{p,p+1} \arrow[r] \arrow[d] & \cdots \\
 & {} & {} & {} &
\end{tikzcd}$$

(Each row and column of a double complex forms a (single) complex with the homomorphisms d_2 and d_1 respectively.)

Definition. *A subcomplex L of K is a system of submodules $L^{p,q} \subset K^{p,q}$*
stable under d_1 and d_2; thus $d_1(L^{p-1,q}) \subset L^{p,q}$ and $d_2(L^{p,q-1}) \subset L^{p,q}$. 119

If L is a subcomplex of a double complex K, then clearly $K/L = \left\{K^{p,q}/L^{p,q}\right\}$ with the homomorphisms induced by d_1 and d_2 is again a double complex.

Let $Z_1^{p,q}(K)$ be the kernel of $d_1 : K^{p,q} \to K^{p+1,q}$ and let $B_1^{p,q}(K)$ be the image of $d_1 : K^{p-1,q}K^{p,q}$. Since $d_1^2 = 0$, $B_1^{p,q} \subset Z_1^{p,q} \subset K^{p,q}$.

Now $d_1(Z^{P-1,Q})_1 = 0 \in Z_1^{p,q}$ and, since $d_1 d_2(Z_1^{p,q-1}) = -d_2 d_1(Z_1^{p,q-1}) = 0$, $d_2(Z_1^{p,q-1}) \subset Z_1^{p,q}$. Thus $Z_1(K) = \left\{Z_1^{p,q}\right\}$ is a subcomplex of K.

Also $d_1(B_1^{P-1,q}) = 0 \in B_1^{p,q}$ and $d_2(B_1^{p,q-1}) = d_2 d_1(K^{p-1,q-1}) = -d_1 d_2(K^{p-1,q-1}) \subset B_1^{p,q}$. Thus $B_1(K) = \left\{B_1^{p,q}\right\}$ is a subcomplex of $Z_1(K)$. Let $H_1(K) = Z_1(K)/B_1(K)$ with $H_1(K) = \left\{H_1^{p,q}(K) = Z_1^{p,q}/B_1^{p,q}\right\}$. In the double complex $H_1(K)$, the homomorphism induced by d_1 is the trivial (zero) homomorphism.

$$\cdots \longrightarrow H_1^{p-1,q-1} \xrightarrow{d_2} H_1^{p-1,q} \xrightarrow{d_2} H_1^{p-1,q+1} \longrightarrow \cdots$$

$$\cdots \longrightarrow H_1^{p,q-1} \xrightarrow{d_2} H_1^{p,q} \xrightarrow{d_2} H_1^{p,q+1} \longrightarrow \cdots$$

Similarly if $Z_2^{p,q} = \ker d_2$ and $B_2^{p,q} = \operatorname{im} d_2$ there is a double complex $H_2(K)$ with $H_2(K) = \left\{H_2^{p,q}(K) = Z_2^{p,q}|B_2^{p,q}\right\}$. In $H_2(K)$, the homomorphism induced by d_2 is the trivial homomorphism.

$$\begin{tikzcd}
{} \arrow[d] & {} \arrow[d] & {} \arrow[d] \\
H_2^{p-1,q-1} \arrow[d, "d_1"'] & H_2^{p-1,q} \arrow[d, "d_1"'] & H^{p-1,q+1} \arrow[d, "d_1"'] \\
H_2^{p,q-1} \arrow[d] & H_2^{p,q} \arrow[d] & H_2^{p,q+1} \arrow[d] \\
{} & {} & {}
\end{tikzcd}$$

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In particular, there is a double complex $H_2(H_1(K))$, which we write as $H_{12}(K) = \left\{H_{12}^{p,q}(K)\right\}$, where $H_{12}^{p,q} = \left\{Z_{12}^{p,q} B_{12}^{p,q}\right\}$ and $Z_{12}^{p,q} = \ker d_2 : H_1^{p,q} \to H_1^{p,q+1}$; $B_{12} = \operatorname{im} d_2 : H_1^{p,q-1} \to H_1^{p,q}$. In the double complex $H_{12}(K)$, the induced homomorphisms d_1 and d_2 are the trivial homomorphisms.

Similarly there is a double complex

$$H_{21}(K) = \left\{H_{21}^{p,q}(K)\right\} = H_1(H_2(K)).$$

Notations. In terms of the more usual notation, $H_{12}^{p,q}(K) = H_{II}^q(H_I^p(K))$ and $H_{21}^{p,q}(K)) = H_I^p(K)(H_{II}^q(K))$.

To the double complex $K = \left\{K^{p,q}, d_1, d_2\right\}$ we can now associate the (single) complex $\left\{K^n, d\right\} K^n$ being the direct sum $K^n = \sum\limits_{p+q=n} K^{p,q}$ (each K^n is an $A-$ module) with the differential operator $d = d_1 + d_2 : K^{n-1} \to K^n$. ($d$ is a homomorphism and $d^2 = d_1^2 + d_1 d_2 + d_2 d_1 + d_2^2 = 0$).

$$\cdots \to K^{n-1} \xrightarrow{d^n} K^n \xrightarrow{d^{n+1}} K^{n+1} \to \cdots$$

Thus $\operatorname{im} d^n \subset \ker d^{n+1}$ and there are cohomology modules $H^n(K) = \ker d^{n+1}/imd^n$.

Definition. *A* homomorphism $f : K \to L$ *(of bidegree* (r, s)*) of double complexes is a system of homomorphisms* $f : K^{p,q} \to L^{p+r,q+s}$. 121

Definition. A map $f : K \to L$ *of double complexes is a homomorphism of bidegree (0, 0), which commutes with* d_1 *and* d_2.

Clearly a map $f : K \to L$ induces homomorphisms

$$f^+ : H_1^{p,q}(K) \to H_1^{p,q}(L),\ f^* : H_{12}^{p,q}(K) \to H_{12}^{p,q}(L)$$

and $$f^* : H_{21}^{p,q}(K) \to H_{21}^{p,q}(L)$$

Also, f determines obvious homomorphisms $f : K^n \to L^n$ which commute with $d = d_1 + d_2$ and there are induced homomorphisms $f^* : H^n(K) \to H^n(L)$

Definition. *A sequence*

$$\cdots \to K_{r-1} \xrightarrow{h_r} K_r \xrightarrow{h_{r+1}} K_{r+1} \to \cdots$$

of homomorphisms of bidegree (0, 0) of double complexes is called exact *if, each pair* (p, q)*, the sequence*

$$\cdots \to K^{p,q}_{r-1} \to K^{p,q}_{r} \to K^{p,q}_{r-1} \to \cdots$$

is exact.

Given an exact sequence of maps of double complexes

$$0 \to K' \xrightarrow{i} K \xrightarrow{j} K'' \to 0,$$

there is an exact cohomology sequence

$$\cdots \to H^n(K') \xrightarrow{i^*} H^n(K) \xrightarrow{j^*} H^n(K'') \xrightarrow{d^*} H^{n+1}(K') \to \cdots$$

122 *Proof.* The sequences

$$0 \to K'^n \xrightarrow{i} K^n \xrightarrow{j} K''^n \to 0$$

are clearly exact for each each n, and d commutes with i and j. Then, using the standard arguments of Lecture 10, we obtain the result. ◻

Definition. *Two maps of double complexes,* $f : K \to L$ *and* $g : K \to L$ *are called* homotopic *(*$f \circ g$*) if there exist homomorphisms* $h_1 : K^{p+1,q} \to L^{p,q}$ *and* $h_2 : K^{p,q+1} \to L^{p,q}$ *(i.e.,* h_1 *and* h_2 *are homomorphisms* $K \to L$ *of bidegree* $(-1, 0)$ *and* $(0, -1)$ *respectively) such that*

$$d_1h_1 + h_1d_1 + d_2h_2 + h_2d_2 = g - f,$$
$$d_1h_2 = -h_2d_1, \qquad d_2h_1 = -h_1d_2.$$

(*Homotopy of maps is obviously an equivalence relation.*)

Homotopic maps $f :\to L$ *and* $g : K \to L$ *induce the same homomorphisms*

(i) $f^* = g^* : H_{12}^{p,q}(K) \to H_{12}^{p,q}(L)$,

(ii) $f^* = g^* : H_{21}^{p,q}(K) \to H_{21}^{p,q}(L)$,

(iii) $f^* = g^* : H^n(K) \to H^n(L)$.

Proof. Let (h_1, h_2) be the pair of homomorphisms $K \to L$ which express the homotopy between f and g. Since h_2, like d_2, anticommutes with d_1, there are induced homomorphisms

$$h_2^+ : H_1^{p,q+1}(K) \to H_1^{p,q}(L).$$

Further, since **123**

$$d_1h_1 + h_1d_1 = g - f - d_2h_2 - h_2d_2,$$

h_1 expresses the homotopy of g and $f + d_2h_2 + h_2d_2$ from a column complex of K to the corresponding column complex of L. □

Hence

$$f^+ + d_2h_2^+ + h_2^+d_2 = g^+ : H_1^{p,q}(K) \to H_1^{p,q}(L),$$

i.e., $$d_2h_2^+ + h_2^+d_2 = g^+ - f^+.$$

Thus h_2^+ expresses the homotopy of g^+ and f^+ from a row complex of $\left\{H_1^{p,q}(K)\right\}$ to the corresponding row complex of $\left\{H_1^{p,q}(L)\right\}$.

Hence

$$f^* = g^* : H_{12}^{p,q}(K) \to H_{12}^{p,q}(L). \text{ This proves } (i).$$

The proof of (ii) is carried out in a similar manner, using the other anti-commutativity $d_2h_1 = -h_1d_2$.

To prove (iii), let $h = h_1 + h_2 : K^{n+1} \to L^n$. Then

$$\begin{aligned} dh + hd &= (d_1 + d_2)(h_1 + h_2) + (h_1 + h_2)(d_1 + d_2) \\ &= (d_1h_1 + h_1d_1 + d_2h_2 + h_2d_2) + (d_1h_2 + h_2d_1) + (d_2h_1 + h_1d_2) \\ &= g - f. \end{aligned}$$

Thus h is a homotopy of the complexes $\{K^n\}$ and $\{L^n\}$, and we obtain $f^* = g^* : H^n(K) \to H^n(L)$.

Note. In the double complexes which occur in the usual applications, one has commutativity $d_1 d_2 = d_2 d_1$, $d_1 h_2 = h_2 d_1$ and $d_2 h_1 = h_1 d_2$ 124
rather than anti-commutativity. The commutative case can be transformed into the anti-commutative case and vice versa by replacing d_2 by $(-1)^P d_2 : K^{p,q-1} \to K^{p,q}$ and h_2 by $(-1)^P h_2 : K^{p,q-1} \to K^{p,q}$. These substitutions do not change $\ker d_2$, $\operatorname{im} d_2$, etc., and so the cohomology modules $H_1^{p,q}$, $H_{12}^{p,q}$, etc., remain unchanged. But if K is a commutative double complex, K^n, d, $H^n(K)$ are understood to refer to the associated anticommutative double complex.

Lecture 20

Definition. *If Ω is a directed set, a* direct system of double complexes 125
$\left\{K_\lambda, \phi_{\mu\lambda}\right\}_{\lambda,\mu\in\Omega}$ *is a system of double complexes K_λ and maps*

$$\phi_{\mu\lambda} : K_\lambda \to K_\mu \quad (\lambda < \mu),$$

such that

(i) $\phi_{\lambda\lambda}$ is the identity,

(ii) $\phi_{\nu\mu}\phi_{\mu\lambda}$ is homotopic *to $\phi_{\nu\lambda}$ for $\lambda < \mu < \nu$.*

If $\left\{K_\lambda, \phi_{\mu\lambda}\right\}$ is a direct system of double complexes, there are uniquely determined direct limits:

(i) $H_{12}^{p,q}(K) =$ direct limit $\{H_{12}^{p,q}(K_\lambda), \phi^*_{\mu\lambda}\}$,

(ii) $H_{21}^{p,q}(K) =$ direct limit$\{H_{21}^{p,q}(K_\lambda), \phi^*_{\mu\lambda}\}$,

(iii) $H^n(K) =$ direct limit$\{H^n(K_\lambda), \phi^*_{\mu\lambda}\}$.

Proof. (i) The system $\left\{H_{12}^{p,q}(K_\lambda), \phi^*_{\mu\lambda}\right\}$ is a direct system, as

(a) $\phi^*_{\lambda\lambda}$ is the identity, since $\phi_{\lambda\lambda}$ is the identity,

(b) $\phi^*_{\nu\mu}\phi^*_{\mu\lambda} = (\phi_{\nu\mu}\phi_{\mu\lambda})^* = \phi^*_{\nu\lambda}$, since homotopic maps induce the same homomorphism on the cohomology groups.

∎

The proofs of (ii) and (iii) are carried out in a similar manner.

Definition. *We say that a system $\{K_\lambda\}$ of double complexes is bounded*
126 on the right *if there is some integer m, such that $K_\lambda^{p,q} = 0$ whenever $q > m$ (for all p and λ). The system is said* to be bounded on the left *(resp.* above, below*) if there is an integer m such that $K_\lambda^{p,q} = 0$ whenever $q < m$ (resp. $p < m, p > m$).*

Proposition 15. *If $\left\{K_\lambda, \phi_{\mu\lambda}\right\}$ is a direct system of double complexes which is bounded above or on the right and if $H_{12}^{p,q}(K) = 0$ for all p and q, then $H^n(K) = 0$ for all n.*

Proof. Let $\alpha \in H^n(K)$, let α_λ be its representative in some $H^n(K_\lambda)$ and let $a_\lambda \in Z^n(K_\lambda)$ represent the class α_λ. Let $a_\lambda = a_\lambda^{p,q} + a_\lambda^{p-1,q+1} + \ldots$, where $p + q = n$ and the sum terminates with $a_\lambda^{m,n-m}$ (resp. $a_\lambda^{n-m,m}$). □

Since $a_\lambda \in Z^n(K_\lambda)$, $da_\lambda = (d_1 + d_2)a_\lambda = 0$, i.e.,

$$da_\lambda = d_1 a_\lambda^{p,q} + (d_2 a_\lambda^{p,q} + d_1 a_\lambda^{p-1,q+1}) + \cdots ,$$

and the sum being direct, we have $d_1 a_\lambda^{p,q} = 0$ and $d_2 a_\lambda^{p,q} + d_1 a_\lambda^{p-1,q+1} = 0$. Thus $a_\lambda^{p,q} \in Z_1^{p,q}(K_\lambda)$ and $d_2 a_\lambda^{p,q} \in B_1^{p,q+1}(K_\lambda)$. Therefore, $a_\lambda^{p,q}$ represents an element of $Z_{12}^{p,q}(K_\lambda)$. Since $H_{12}^{p,q} = 0$, there is some $\mu > \lambda$ such that $a_\mu^{p,q} = \phi_{\mu\lambda} a_\lambda^{p,q}$ represents an element of $B_{12}^{p,q}(K_\mu)$. Thus, for some $b \in Z_1^{p,q-1}(K_\mu)$, $a_\mu^{p,q} - d_2 b \in B_1^{p,q}(K_\mu)$, and hence $a_\mu^{p,q} = d_2 b + d_1 c$ for some $c \in K_\mu^{p-1,q}$.

$$\begin{aligned}
\text{Let } a_\mu &= \phi_{\mu\lambda} a_\lambda \text{ and let} \\
e_\mu &= a_\mu - d(b + c) \\
&= a_\mu - d_1 b - d_2 b - d_1 c - d_2 c \\
&= a_\mu - a_\mu^{p,q} - d_2 c \quad (\text{ since } d_1 b = 0) \\
&= (a_\mu^{p-1,q+1} - d_2 c) + a_\mu^{p-2,q+1} + \cdots \\
&= e_\mu^{p-1,q+1} + a_\mu^{p-2,q+2} + \cdots
\end{aligned}$$

127 where $e_\mu^{p-1,q+1} = a_\mu^{p-1,q+1} - d_2 c$. Then since e_μ and a_μ represent the same

class $\alpha_\mu \in H^n(K_\mu)$, they represent the same element $\alpha \in H^n(K)$. The reason for choosing this representative e_μ is the fact that its $(p,q)-$ th component is zero. Continuing thus, after a finite number k of steps for a suitable ρ, α is represented by $e_\rho = -d_2 c'$, consisting of a single term in $K_\rho^{p-k,q+k}$. Continuing the construction still further, since the system of double complexes is bounded above or to the right, after a finite number of steps, we obtain a representative $e_\nu = -d_2 c'' = 0$, i.e., α is represented by $0 \in Z^n(K_\nu)$ for a suitable ν. Hence $\alpha = 0$.

Proposition 15-a. *If $\left\{K_\lambda, \phi_{\mu\lambda}\right\}$ is a direct system of double complexes which is bounded below or on the left, and if $H_{21}^{p,q}(K) = 0$ for all p and q then $H^n(K) = 0$ for all n.*

Proof. This is carried out exactly as in Proposition 15, except that we eliminate the component of a_λ with highest second degree q instead of the one with highest first degree, and $-d_1 c$ plays the role of $-d_2 c$. ◻

Proposition 16. *If $\left\{K_\lambda, \phi_{\mu\lambda}\right\}$ is a direct system of double complexes which is bounded above or on the right and if $H_{12}^{p,q}(K) = 0$ except (at most) for $p = 0$, then there exist isomorphisms $\theta : H^q(K) \to H_{12}^{o,q}(K)$ for all q.*

If another direct system $\left\{K_lambda, \phi_{\mu\lambda}\right\}$ is bounded above or on the right with $H_{12}^{p,q}(K') = 0$ except for $p = 0$ and if $h_\lambda : K'_\lambda \to K_\lambda$ are maps with $h_\mu \phi'_{\mu\lambda} = \phi_{\mu\lambda} h_\lambda$, then there are induced homomorphisms **128**

$$h^* : H^q(K') \to H^q(K) \text{ and } h^* : H_{12}^{o,q}(K') \to H_{12}^{o,q}(K)$$

commuting with the isomorphisms θ.

$$\begin{tikzcd}
H^q(K') \arrow[r, "\theta"] \arrow[d, "h^*"'] & H_{12}^{0,q}(K') \arrow[d, "h^*"] \\
H^q(K) \arrow[r, "\theta"] & H_{12}^{0,q}(K).
\end{tikzcd}$$

Proof. Let L_λ be the subcomplex

$$L_\lambda : \quad \begin{tikzcd} & {} \arrow[d] & {} \arrow[d] & \\ \cdots \arrow[r] & K_\lambda^{-1,q-1} \arrow[r] \arrow[d] & K_\lambda^{-1,q} \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & Z_1^{0,q-1} \arrow[r] \arrow[d] & Z_1^{0,q} \arrow[r] \arrow[d] & \cdots \\ & 0 & 0 & \end{tikzcd}$$

of K_λ, with $L_\lambda^{p,q} = K_\lambda^{p,q}$ for $p < 0; L_\lambda^{p,q} = 0$ for $p > 0$ and $L_\lambda^{o,q} = Z_1^{o,q}(K_\lambda)$. Since L_λ is stable under d_1 and d_2, it is a subcomplex of K_λ.

Let M_λ be the subcomplex

$$M_\lambda : \quad \begin{tikzcd} & {} \arrow[d] & {} \arrow[d] & \\ \cdots \arrow[r] & K_\lambda^{-1,q-1} \arrow[r] \arrow[d] & K_\lambda^{-1,q} \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & B_1^{0,q-1} \arrow[r] \arrow[d] & B_1^{0,q} \arrow[r] \arrow[d] & \cdots \\ & 0 & 0 & \end{tikzcd}$$

129 of L_λ, with $M_\lambda^{p,q} = L_\lambda^{p,q} = K_\lambda^{p,q}$ for $p < 0; M_\lambda^{p,q} = L_\lambda^{p,q} = 0$ for $p > 0$ and $M_\lambda^{o,q} = B_\lambda^{o,q}(K_\lambda)$. Since M_λ is stable d_1 and d_2, it is a subcomplex of L_λ. $\square$

Since $h_\lambda : K'_\lambda \to K_\lambda$ commutes with d_1 and d_2, we have $h_\lambda : L'_\lambda \to L_\lambda$ and $h_\lambda : M'_\lambda \to M_\lambda$. Thus there are induced maps $h_\lambda : K'_\lambda/L'_\lambda \to K_\lambda/L_\lambda$ and $h_\lambda : L'_\lambda/M'_\lambda \to L_\lambda/M_\lambda$ which commute with i and j in the exact sequences

$$\begin{tikzcd}
0 \arrow[r] & L'_\lambda \arrow[r, "i"] \arrow[d, "h_\lambda"'] & K'_\lambda \arrow[r, "j"] \arrow[d, "h_\lambda"'] & K'_\lambda/L'_\lambda \arrow[r] \arrow[d, "h_\lambda"'] & 0 \\
0 \arrow[r] & L_\lambda \arrow[r, "i"] & K_\lambda \arrow[r, "j"] & K_\lambda/L_\lambda \arrow[r] & 0,
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & M'_\lambda \arrow[r, "i"] \arrow[d, "h_\lambda"'] & L'_\lambda \arrow[r, "j"] \arrow[d, "h_\lambda"'] & L'_\lambda/M'_\lambda \arrow[r] \arrow[d, "h_\lambda"'] & 0 \\
0 \arrow[r] & M_\lambda \arrow[r, "i"] & M_\lambda \arrow[r, "j"] & L_\lambda/M_\lambda \arrow[r] & 0.
\end{tikzcd}$$

Hence h^*_λ commutes with d^*, i^* and j^* in the exact cohomology sequences:

$$\begin{tikzcd}
\cdots \arrow[r] & H^{n-1}(K'_\lambda/L'_\lambda) \arrow[r, "d^*"] \arrow[d, "h^*_\lambda"'] & H^n(L'_\lambda) \arrow[r, "i^*"] \arrow[d, "h^*_\lambda"'] & H^n(K'_\lambda) \arrow[r, "j^*"] \arrow[d, "h^*_\lambda"'] & H^n(K'_\lambda/L'_\lambda) \arrow[r] \arrow[d, "h^*_\lambda"'] & {} \\
\cdots \arrow[r] & H^{n-1}(K_\lambda/L_\lambda) \arrow[r, "d^*"] & H^n(L_\lambda) \arrow[r, "i^*"] & H^n(K_\lambda) \arrow[r, "j^*"] & H^n(K_\lambda/L_\lambda) &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\cdots \arrow[r] & H^n(M'_\lambda) \arrow[r, "i^*"] \arrow[d, "h^*_\lambda"'] & H^n(L'_\lambda) \arrow[r, "j^*"] \arrow[d, "h^*_\lambda"'] & H^n(L'_\lambda/M'_\lambda) \arrow[r, "d^*"] \arrow[d, "h^*_\lambda"'] & H^{n+1}(M'_\lambda) \arrow[r] \arrow[d, "h^*_\lambda"'] & \cdots \\
\cdots \arrow[r] & H^n(M_\lambda) \arrow[r, "i^*"] & H^n(L_\lambda) \arrow[r, "j^*"] & H^n(L_\lambda/M_\lambda) \arrow[r, "d^*"] & H^{n+1}(M_\lambda) \arrow[r] & \cdots
\end{tikzcd}$$

In the direct limit, we have the following commutative diagram **130**
where each row is exact.

$$\begin{tikzcd}
(A_1)\cdots \arrow[r] & H^{n-1}(K'/L') \arrow[r, "d^*"] \arrow[d, "h^*"'] & H^n(L') \arrow[r, "i^*"] \arrow[d, "h^*"] & H^n(K') \arrow[r, "j^*"] \arrow[d, "h^*"'] & H^n(K'/L') \arrow[r] \arrow[d, "h^*"'] & \cdots \\
(A_2)\cdots \arrow[r] & H^{n-1}(K/L) \arrow[r, "d^*"] & H^n(L) \arrow[r, "i^*"] & H^n(K) \arrow[r, "j^*"] & H^n(K/L) \arrow[r] & \cdots
\end{tikzcd}$$

and

$$\begin{tikzcd}
(B_1)\cdots \arrow[r] & H^n(M') \arrow[r, "i^*"] \arrow[d, "h^*"'] & H^n(L') \arrow[r, "j^*"] \arrow[d, "h^*"'] & H^n(L'/M') \arrow[r, "d^*"] \arrow[d, "h^*"'] & H^{n+1}(M') \arrow[r] \arrow[d, "h^*"'] & \cdots \\
(B_2)\cdots \arrow[r] & H^n(M) \arrow[r, "i^*"] & H^n(L) \arrow[r, "j^*"] & H^n(L/M) \arrow[r, "d^*"] & H^{n+1}(M) \arrow[r] & \cdots
\end{tikzcd}$$

The quotient double complex K_λ/L_λ is

$$K_\lambda/L_\lambda : \quad \begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & \\ \cdots \arrow[r] & K_\lambda^{0,q-1}/Z_1^{0,q-1} \arrow[r] \arrow[d] & K^{0,q}/Z_1^{0,q} \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & K_\lambda^{1,q-1} \arrow[r] \arrow[d] & K_\lambda^{1,q} \arrow[r] \arrow[d] & \cdots \\ & {} & {} & \end{tikzcd}$$

and since the sequence $0 \to K_\lambda^{o,q}/Z_1^{o,q} \to K_\lambda^{1,q}$ is exact, we have $H_1(K_\lambda/L_\lambda)$:

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & \\ \cdots \arrow[r] & H_1^{1,q-1}(K_\lambda) \arrow[r] \arrow[d] & H_1^{1,q}(K_\lambda) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & H_1^{2,q-1}(K_\lambda) \arrow[r] \arrow[d] & H_1^{2,q}(K_\lambda) \arrow[r] \arrow[d] & \cdots \\ & {} & {} & \end{tikzcd}$$

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Thus $H_{12}^{p,q}(K_\lambda/L_\lambda) = 0$ for $p \leqq 0$, and is equal to $H_{12}^{p,q}(K_\lambda)$ for $p > 0$, hence $H_{12}^{p,q}(K/L) =$ direct limit $\left\{H_{12}^{p,q}(K_\lambda/L_\lambda)\right\} = 0$ for $p \leqq 0$, and by hypothesis is also zero for $p > 0$, hence is zero for all pairs (p, q). Since K_λ/L_λ is bounded above or to the right, by Proposition 15, we have $H^n(K/L) = 0$ for all n. Thus, in the sequence (A_2), we see that $i^* : H^n(L) \to H^n(K)$ is an isomorphism.

Again, since the sequence $K_\lambda^{-1,q} \to B_1^{o,q} \to 0$ is exact, we have for

$$H_1(M_\lambda) : \quad \begin{tikzcd} & {} \arrow[d] & {} \arrow[d] & \\ \cdots \arrow[r] & H_1^{-1,q-1}(K_\lambda) \arrow[r] \arrow[d] & H_1^{-1,q}(K_\lambda) \arrow[r] \arrow[d] & \cdots \\ & 0 & 0 & \end{tikzcd}$$

Thus, $H_{12}^{p,q}(M_\lambda) = 0$ for $p > 0$ and is equal to $H_{12}^{p,q}(K_\lambda)$ for $p < 0$, hence $H_{12}^{p,q}(M) =$ direct limit $\left\{H_{12}^{p,q}(M_\lambda)\right\}$ is equal to zero for $p \geq 0$, and by hypothesis, is also zero for $p < 0$, hence is zero for all pairs (p, q). As before, the conditions of Proposition 15 being satisfied, we **132**
have $H^n(M) = 0$ for all n. Thus, in the sequence (B_2), we see that $j^* : H^n(L) \to H^n(L/M)$ is an isomorphism.

The quotient double complex L_λ/M_λ is given by

$$L_\lambda/M_\lambda : \quad \begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & \\ \cdots \arrow[r] & H_1^{0,q-1}(K_\lambda) \arrow[r] \arrow[d] & H_1^{0,q}(K_\lambda) \arrow[r] \arrow[d] & \cdots \\ & 0 & 0 & \end{tikzcd}$$

Thus $(L_\lambda/M_\lambda)^q = H_1^{o,q}(K_\lambda)$ and $d = d_2 : H_1^{o,q-1}(K_\lambda) \to H_1^{o,q}(K_\lambda)$. Hence $H^q(L_\lambda/M_\lambda) = H_{12}^{o,q}(K_\lambda)$; similarly $H^q(L'_\lambda/M'_\lambda) = H_1^{o,q}(K'_\lambda)$. Furthermore, in the limit we have $H^q(L/M) = H_{12}^{o,q}(K)$ and $H^q(L'/M') = H_1^{o,q}(K')$.

From the sequences (A_1), (A_2), (B_1), (B_2), we have the commutative diagram :

$$\begin{tikzcd} H^q(K') & H^q(L') \arrow[l, "i^*"'] \arrow[r, "j^*"] \arrow[d, "h^*"'] & H^q(L'/M') \arrow[d, "h^*"'] & = & H^{0,q}_{12}(K') \arrow[d, "h^*"'] \\ H^q(K) \arrow[from=u, "h^*"'] & H^q(L) \arrow[l, "i^*"'] \arrow[r, "j^*"] & H^q(L/M) & = & H^{0,q}_{12}(K). \end{tikzcd}$$

Then there is an isomorphism

$$\theta : H^q(K) \to H^{o,q}_{12}(K),$$

where $\theta = j^*(i^*)^{-1} : H^q(K) \to H^q(L) \to H^q(L/M) = H^{o,q}_{12}(K)$, θ
133 being an isomorphism since we have proved that each of i^* and j^* is an isomorphism.

Further, form (I), we have obviously commutativity in the following diagram :

$$\begin{tikzcd} H^q(K') \arrow[r, "\theta"] \arrow[d, "h^*"'] & H^{0,q}_{12}(K') \arrow[d, "h^*"] \\ H^q(K) \arrow[r, "\theta"] & H^{0,q}_{12}(K). \end{tikzcd}$$

Proposition 16-a. *If $\left\{K_\lambda, \phi_{\mu\lambda}\right\}$ is a direct system of double complexes which is bounded below or on the left and if $H^{p,q}_{21}(K) = 0$ except for $q = 0$, then there exist isomorphisms $\theta : H^p(K) \to H^{p,o}_{21}(K)$ for all p.*

If another direct system $\left\{K'_\lambda, \phi'_{\mu\lambda}\right\}$ is bounded below or on the left with $H^{p,q}_{21}(K') = 0$ except for $q = 0$, and if $h_\lambda : K'_\lambda \to K_\lambda$ are maps with $h_{\mu\phi_{\mu\lambda}} = \phi_{\mu\lambda} h_\lambda$, then there are induced homomorphisms

$$h^* : H^p(K') \to H^p(K) \text{ and } h^* : H^{p,o}_{21}(K') \to H^{p,o}_{21}(K)$$

which commute with θ.

Remark . In particular, all the propositions proved in this lecture are true for a double complex $K = \left\{K^{p,q}\right\}$ satisfying the conditions stated in the propositions. We have only to replace the $\phi_{\mu\lambda}$ by the identity map $K \to K$.

Example 29. Let 134

$$K^{p,q} = Z \text{ (ring of integers) if } q \geqq 0 \text{ and } p = -q \text{ or } -q-1,$$
$$= 0 \text{ otherwise.}$$

For $q \geqq 0$, let $d_1 : K^{-q-1,q} \to K^{-q,q}$ and

$$d^2 : K^{-q-1,q} \to K^{-q-1,q+1}$$

be the identity isomorphisms of Z onto itself. The other homomorphisms are all the trivial homomorphisms. Then $K = \left\{K^{p,q}\right\}$ is a double complex with

$$\begin{aligned} H^{p,q}_{21}(K) = H^{p,q}_{2}(K) &= Z \text{ if } (p,q) = (0,0), \\ &= 0 \text{ otherwise ;} \\ H^{p,q}_{12}(K) = H^{p,q}_{1}(K) &= 0 \text{ for all } (p,q), \end{aligned}$$

and

$$\begin{aligned} H^n(K) &= Z \text{ if } n = 0, \\ &= o \text{ otherwise .} \end{aligned}$$

This double complex is bounded below and on the left, but is unbounded above and on the right.

Lecture 21

Introduction of the family Φ

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Let Φ be a family of paracompact normal closed subsets of a topological space X such that

(1) if $F \in \Phi$, then every closed subset of F is in Φ,

(2) if $F_1, F_2 \in \Phi$, then $F_1 \cup F_2 \in \Phi$,

(3) if $F \in \Phi$, there is an open U with $F \subset U$ and $\overline{U} \in \Phi$.

For example, if X is paracompact and normal, Φ can be taken to the family of all closed subsets of X, and, if X is locally compact and Hausdorff, then Φ can be taken to be the family of all compact sets of X.

Sections with supports in the family Φ. If $\mathscr{S}$ is a sheaf of $A-$ modules, the set of all sections $f \in \Gamma(X, \mathscr{S})$ such that supp $f = \left\{x : f(x) \neq 0_x\right\}$ is in Φ, forms an $A-$ module (if supp $f_1 \in \Phi$ and supp $f_2 \in \Phi$ then supp $(f_1 \pm f_2) \subset$ (supp $f_1 \cup supp f_2$) is in Φ), a submodule of $\Gamma(X, \mathscr{S})$, which we denote by $\Gamma\Phi(X, \mathscr{S})$.

Any homomorphism $h : \mathscr{S}^1 \to \mathscr{S}$ of two sheaves of $A-$ modules induces a homomorphism

$$h : \Gamma\overline{\Phi}(X, \mathscr{S}') \to \Gamma\overline{\Phi}(X, \mathscr{S}),$$

since a homomorphism of sheaves decreases supports (i.e., supp $hf \subset$ supp f).

Definition. *A* Φ- covering of X *is a locally finite proper covering* $\mathscr{U}$ **136**
such that, if $X \notin \Phi$, *there is a* special *open set* $U_* \in \mathscr{U}$ *with* $\bigcup\limits_{U \in \mathscr{U}-(U_*)} \bar{U} \in \Phi$.

Remark. If $X \notin \Phi$, U_* is unique, and is not the empty set, for otherwise, in each case, X would belong to Φ. If $X \in \Phi$, a Φ- covering is just a locally finite proper covering of X.

The Φ- *coverings of* X *form a subdirected set* Ω_* *of the directed set of all locally finite proper coverings of the space* X.

Proof. (i) If $X \in \Phi, \Omega_*$ is the directed set of all locally finite proper coverings of X.

(ii) If $X \notin \Phi$ and $\mathscr{U}, \mathscr{W}$ are any two Φ- coverings of X, let $\mathscr{W} = \left\{W : W = U \cap V\right\}$ for some $U \in \mathscr{U}$ and some $V \in \mathscr{W}$ with $W_* = U_* \cap V_*$. Then $\mathscr{W}$ is a locally finite proper covering of X and

$$\bigcup_{W \in \mathscr{W}-(W_*)} \bar{W} = (\bigcup_{U \in \mathscr{U}-(U_*)} \bar{U}) \cup (\bigcup_{V \in \mathscr{W}-(V_*)} \bar{V})$$

is in Φ, since each set contained in brackets is in Φ. Thus $\mathscr{W}$ is a Φ-covering which is a common refinement of $\mathscr{U}$ and $\mathscr{W}$.

□

Remark. If $\mathscr{U}$ and $\mathscr{W}$ are two Φ coverings with special sets U_* and V_* then $U_* \cap V_*$ is not empty, and if $\mathscr{W}$ is refinement of $\mathscr{U}$, then $V_* \subset U_*$ and V_* is not contained in any other $U \in \mathscr{U}$. In particular, if $\mathscr{W}$ is equivalent to $\mathscr{U}$, then $V_* = U_*$.

137 *Cohomology groups with supports in the family* Φ. If $\mathscr{U}$ is a Φ-covering and $\sum$ a presheaf of A- modules, we define

$$C^p_\Phi\left(\mathscr{U}, \sum\right) = C^p\left(\mathscr{U}, \sum\right) \text{ for } p > 0,$$
$$\text{and} \qquad C^p_\Phi\left(\mathscr{U}, \sum\right) \subset C^o\left(\mathscr{U}, \sum\right) \text{ for } p = 0,$$

where$C^p_\Phi(\mathscr{U},\sum)$ is the submodule of $C^o(\mathscr{U},\sum)$ consisting of those zero cochains which assign to U_* the zero of S_{U_*}. Then we have a mapping

$$\delta^p : C^{p-1}_\Phi\left(\mathscr{U},\sum\right) \to C^p_\Phi\left(\mathscr{U},\sum\right).$$

Let **138**

$$H^p_\Phi\left(\mathscr{U},\sum\right) = \ker\delta^{p+1}/\operatorname{im}\delta^p.$$

Then $H^p_\Phi(\mathscr{U},\sum)$ is called the *$p-$ th cohomology module of the covering $\mathscr{U}$ with coefficients in the presheaf $\sum$ and supports in the family* Φ.

If a Φ- covering $\mathscr{W}$ is a refinement of $\mathscr{U}$, for each choice of the function $\tau : \mathscr{W} \to \mathscr{U}, \tau(V_*) = U_*$. We then have the mapping (Lecture 9)

$$\tau^+ : C^p_\Phi\left(\mathscr{U},\sum\right) \to C^p_\Phi\left(\mathscr{W},\sum\right) \qquad (p \geqq 0).$$

τ^+ induces the homomorphism

$$\tau_{\mathscr{W}\mathscr{U}} : H^p_\Phi\left(\mathscr{U},\sum\right) \to C^p_\Phi\left(\mathscr{W},\sum\right)$$

with $\tau_{\mathscr{U}\mathscr{U}}$ = identity, and $\tau_{\mathscr{W}\mathscr{W}}\tau_{\mathscr{W}\mathscr{U}} = \tau_{\mathscr{W}\mathscr{U}}$ if $\mathscr{U} < \mathscr{W} < \mathscr{W}$. Thus $\{H^p_\Phi(\mathscr{U},\sum), \tau_{\mathscr{W}\mathscr{U}}\}$ is a direct system of A-modules.

Let

$H^p_\Phi(X,\sum) =$ direct limit $\left\{H^p_\Phi(\mathscr{U},\sum), \tau_{\mathscr{W}\mathscr{U}}\right\}_{\mathscr{U},\mathscr{W}\in\Omega_*}$. $H^p_\Phi(X,\sum)$ is called the *p-th cohomology module of the space X with coefficients in the presheaf $\sum$ and supports in the family* Φ.

The result analogous to Proposition 7 (Lecture 11) is true in this case.

Proposition 7-a *If $\mathscr{S}$ is a sheaf of A-modules, $H^o_\Phi(X,\mathscr{S}) = \Gamma_\Phi(X,\mathscr{S})$.*

Proof. We can identify $H^o_\Phi(\mathscr{U},\mathscr{S}) = Z^o_\Phi(\mathscr{U},\mathscr{S})$(see Lecture 11) with the submodule of $\Gamma_\Phi(X,\mathscr{S})$ consisting of the sections f with supp $f \subset X - U_* \bigcup\limits_{U\in\mathscr{S}-(U_*)} \bar{U} \in \Phi$; then supp $f \in \Phi$. If $\mathscr{W}$ is a refinement of $\mathscr{U}$, since $V_* \subset U_*$, we have $X - U_* \subset X - V_*$, hence

$$\tau_{\mathscr{W}\mathscr{U}} : H^o_\Phi(\mathscr{U},\mathscr{S}) \to H^o_\Phi(\mathscr{W},\mathscr{S})$$

is the inclusion homomorphism. Thus $H^o_\Phi(X, \mathscr{S})$ can be identified with a submodule of $\Gamma_\Phi(X, \mathscr{S})$ i.e., $H^o_\Phi(X, \mathscr{S}) \subset \Gamma_\Phi(X, \mathscr{S})$.

If $f \in \Gamma_\Phi(X, \mathscr{S})$, let $U_* = X - \operatorname{supp} f$, and let U be an open set containing $\operatorname{supp} f$ with $\overline{U} \in \Phi$. Clearly $\mathscr{U} = \{U, U_*\}$ is a Φ-covering (with special set U_*) and the cochain g defined by

$$g(U) = f|U; g(U_*) = 0$$

is the cocycle in $Z^o_\Phi(\mathscr{U}, \mathscr{S}) = H^o_\Phi(\mathscr{U}, \mathscr{S})$ which is identified with $f \in \Gamma_\Phi(X, \mathscr{S})$. Thus $\Gamma_\Phi(X, \mathscr{S}) \subset H^o_\Phi(X, \mathscr{S})$, hence $H^o_\Phi(X, \mathscr{S}) = \Gamma_\Phi(X, \mathscr{S})$. ◻

139 *Given a sequence of homomorphisms of presheaves*

$$\cdots \to \sum^{q-1} \xrightarrow{d^q} \sum^{q} \xrightarrow{d^{q+1}} \sum^{q+1} \to \cdots$$

with im $d^q \subset \ker d^{q+1}$ $(i.e., d^2 = 0)$ *for each* Φ*-covering* $\mathscr{U}$ *the system* $\{k^{p,q}_{\mathscr{U}} = C^p_\Phi(\mathscr{U}, \sum^q)$ *with the homomorphisms*

$$\delta : C^p_\Phi(\mathscr{U}, \sum^q) \to C^p_\Phi\left(\mathscr{U}, \sum^q\right)$$

$$\text{and} \qquad d : C^p_\Phi\left(\mathscr{U}, \sum^{q-1}\right) \to C^p_\Phi\left(\mathscr{U}, \sum^q\right)$$

forms a double complex denoted by $K_{\mathscr{U}} = C_\Phi(\mathscr{U}, \sum^q)$.

Proof. Any homomorphism $\sum' \xrightarrow{d} \sum$ of two presheaves induces a homomorphism of $C^p_\Phi(\mathscr{U}, \sum') \xrightarrow{d} C^p_\Phi(\mathscr{U}, \sum)$ commuting with the coboundary operator δ. We have $\delta^2 = 0$, and by hypothesis $d^2 = 0$. Further $d\delta = \delta d$; so we have the commutative case of a double complex, *q.e.d*. ◻

For each pair $\mathscr{U}, \mathscr{W}$ of Φ- coverings for which $\mathscr{W}$ is a refinement of $\mathscr{U}$ choose $\tau : \mathscr{W} \to \mathscr{U}$ with $V \subset \tau(V)$; if $\mathscr{W} = \mathscr{U}$, let $\tau : \mathscr{U} \to \mathscr{U}$ be the identity and let

$$\phi_{\mathscr{W}\mathscr{U}} = \tau^+ : C^p_\Phi(\mathscr{U}, \sum^q) \to C^p_\Phi\left(\mathscr{W}, \sum^q\right).$$

$$C_\Phi(\sum) = \left\{C_\Phi\left(\mathscr{U}, \sum\right), \phi_{\mathscr{W}\mathscr{U}}\right\}_{\mathscr{U},\mathscr{W}\in\Omega_*}$$

is a direct system of double complexes.

(If $\mathscr{U} < \mathscr{W}$, $\phi_{\mathscr{W}\mathscr{U}} = \tau^+$, for an arbitrary but *fixed* choice of $\tau : \mathscr{W} \to \mathscr{U}$ if $\mathscr{W} \neq \mathscr{U}$, and $\tau : \mathscr{U} \to \mathscr{U}$ is the identity.) **140**

Proof. Since δ and d commute with $\phi_{\mathscr{W}\mathscr{U}}, \phi_{\mathscr{W}\mathscr{U}} : C_\Phi(\mathscr{W}, \sum) \to C_\Phi(\mathscr{U}, \sum)$ is a map of double complexes, and by construction $\phi_{\mathscr{U}\mathscr{U}}$ is the identity. □

If $\mathscr{W}$ is a Φ- refinement of $\mathscr{W}, \phi_{\mathscr{W}\mathscr{U}}\phi_{\mathscr{W}\mathscr{U}}$ corresponds to a possible choice of $\tau : \mathscr{S} \to \mathscr{U}$. Since for all possible choices of τ, $\tau(W_*) = U_*$, the homotopy operator k (see Lecture 9)

$$k : C^p\left(\mathscr{U}, \sum^q\right) \to C^{p-1}\left(\mathscr{W}, \sum^q\right)$$

maps $C^1_\Phi(\mathscr{U}, \sum^q) = C^1(\mathscr{U}, \sum^q)$ into $C^o_\Phi(\mathscr{W}, \sum^q)$. Thus we have two homomorphisms $k : C^p_\Phi(\mathscr{U}, \sum^q) \to C^{p-1}_\Phi(\mathscr{W}, \sum^q)$ and the trivial homomorphism $C^p_\Phi(\mathscr{U}, \sum^q) \to C^p_\Phi\left(\mathscr{W}, \sum^{q-1}\right)$ such that

$$\delta k + k\delta = \phi_{\mathscr{W}\mathscr{W}}\phi_{\mathscr{W}\mathscr{U}} - \phi_{\mathscr{W}\mathscr{U}},$$

and further, d commutes with k. Hence $\phi_{\mathscr{W}\mathscr{U}}\phi_{\mathscr{W}\mathscr{U}}$ is homotopic to $\phi_{\mathscr{W}\mathscr{U}}$, hence $\left\{C_\Phi(\mathscr{U}, \sum), \phi_{\mathscr{W}\mathscr{U}}\right\}$ is a direct system.

Lecture 22

Proposition 8-a. *If $\sum$ is a presheaf which determines the zero sheaf, then $H^p_\Phi(X,\sum) = 0$ for all $p \geqq 0$.* **141**

Proof. Let $f \in C^p_\Phi(\mathscr{U},\sum)$. Then $\bigcup\limits_{U\in\mathscr{U}-(U_*)} \bar{U} \in \Phi$, and hence has neighbourhoods G, H with $\bar{G} \subset H$; $\bar{G}, \bar{H} \in \Phi$. Shrink the covering $\mathscr{U}' = \left\{\mathscr{U} - (U_*), U_* \cap \bar{H}\right\}$ of $\bar{H}$ to a covering $\mathscr{W}' = \{W_U\}_{U\in\mathscr{U}'}$ with $\bar{W}_U \subset U$. For each $x \in H$ choose a neighbourhood V_x of x such that

a) if $x \in U$, $V_x \subset U \cap H$,

b) if $x \in W_U$, $V_x \subset W_U \cap H$,

c) if $x \notin \bar{W}_U$, $V_x \cap W_U = \phi$,

d) if $x \in U_o \cap \ldots \cap U_p = U_\sigma$, $\rho_{V_x U_\sigma} f(\sigma) = 0$,

and let $V_* = X - \bigcup\limits_{U\in\mathscr{U}-(U_*)} \bar{U}$. □

Then $\left\{V_*, V_x\right\}_{x\in H}$ is a refinement of $\mathscr{U}$. Choose $\tau : H \cup (*) \to \mathscr{U}$ such that $x \in W_{\tau(x)}$ and $\tau(*) = U_*$. Then it can be verified that $\tau^+ f = 0$.

The covering $\left\{V_x \cap \bar{G}\right\}_{x\in H}$ of $\bar{G}$ has a locally finite refinement $\{Y_i\}_{i\in I}$. Let $\mathscr{U}_1$ be the proper covering consisting of V_* together with all V such that $V = Y_i \cap G$ for some $i \in I$. Then $\mathscr{W}_1$ is a Φ- covering which is a refinement of $\left\{V_*, V_x\right\}_{x\in H}$. Hence there is a function $\tau_1 : \mathscr{W}_1 \to \mathscr{U}$ with $V \subset \tau_1(V)$ and such that $\tau_1^+ f = 0$. Therefore $H^p_\Phi(X,\sum) = 0$, *q.e.d.*

142 Let

$$\cdots \to \mathscr{S}^{q-1} \xrightarrow{d^q} \mathscr{S}^q \xrightarrow{d^{q+1}} \mathscr{S}^{q+1} \to \cdots$$

be a sequence of homomorphisms of sheaves of A-modules with $\operatorname{im} d^q \subset \ker d^{q+1}$. Let $\mathbb{B}^q = \operatorname{im} d^q$, $\mathfrak{z}^q = \ker d^{q+1}$ and $\mathscr{H}^q = \mathfrak{z}^q/\mathbb{B}^q$.

There is an induced sequence of homomorphisms of presheaves

$$\cdots \to \overline{\mathscr{S}}^{q-1} \xrightarrow{d^q} \overline{\mathscr{S}}^q \xrightarrow{d^{q+1}} \overline{\mathscr{S}}^{q+1} \to \cdots$$

with $\operatorname{im} d^q = \overline{\mathbb{B}}_o^q \subset \overline{\mathbb{B}}^q$ and $\ker d^{q+1} = \overline{\mathfrak{z}}^q$. Also there is an induced sequence of homomorphisms

$$\cdots C_\Phi^p\left(\mathscr{U}, \mathscr{S}^{q-1}\right) \xrightarrow{d^q} C_\Phi^p\left(\mathscr{U}, \mathscr{S}^q\right) \xrightarrow{d^{d+1}} C_\Phi^p\left(\mathscr{U}, \mathscr{S}^{q+1}\right) \to \cdots$$

where $C_\Phi^p(\mathscr{U}, \mathscr{S}^q) = C_\Phi^p\left(\mathscr{U}, \overline{\mathscr{S}}^q\right)$, with $\operatorname{im} d^q = C_\Phi^p\left(\mathscr{U}, \overline{\mathbb{B}}^q\right)$ and $\ker d^{q+1} = C_\Phi^p(\mathscr{U}, \mathfrak{z}^q)$. Then $H_2^{p,q} C_\Phi(\mathscr{U}, \mathscr{S}) = C_\Phi^p(\mathscr{U}, \mathfrak{z}^q)/C_\Phi^p\left(\mathscr{U}, \overline{\mathbb{B}}_o^q\right)$.

Let $\psi : \mathfrak{z}^q \to \mathscr{H}^q$ be the natural homomorphism. There is an induced homomorphism $\psi : \overline{\mathfrak{z}}^q \to \mathscr{H}^q$ with $\psi(\overline{\mathbb{B}}_o^q) = 0$.

Hence there is an induced homomorphism

$$\psi : C_\Phi^p(\mathscr{U}, \mathfrak{z}^q) \to C_\Phi^p(\mathscr{U}, \mathscr{H}^q),$$

which commutes with δ and $\phi_{\mathscr{W}\mathscr{U}}$ such that $\psi(C_\Phi^p(\mathscr{U}, \overline{\mathbb{B}}^q)) = 0$.

Hence there are induced homomorphisms :

$$\begin{aligned}
&\psi : H_2^{p,q} C_\Phi(\mathscr{U}, \mathscr{S}) \to C_\Phi^p(\mathscr{U}, \mathscr{H}^q),\\
&\psi* : H_{21}^{p,q} C_\Phi(\mathscr{U}, \mathscr{S}) \to H_\Phi^p(\mathscr{U}, \mathscr{H}^q),\\
&\psi* : H_{21}^{p,q} C_\Phi(\mathscr{S}) \to H_\Phi^p(X, \mathscr{H}^q).
\end{aligned}$$

143 *The homomorphism $\psi* : H_{21}^{p,q} C_\Phi(\mathscr{S}) \to C_\Phi^p(X, \mathscr{H}^q)$ is an isomorphism.*

Proof. The exact sequence

$$0 \to \overline{\beta}_o^q \to \overline{\mathfrak{z}}^q \to \overline{\mathfrak{z}}^q/\overline{\mathbb{B}}_o^q \to 0$$

gives rise to an exact sequence

$$0 \to C^p_\Phi(\mathscr{U}, \overline{\mathbb{B}}^q_o) \to C^p_\Phi(\mathscr{U}, \mathfrak{z}^q) \to C^p_\Phi(\mathscr{U}, \overline{\mathfrak{z}}^q/\overline{\mathbb{B}}^q_o) \to 0.$$

Hence the induced homomorphism

$$H^{p,q}_{21} C_\Phi(\mathscr{U}, \mathscr{S}) = C^p_\Phi(\mathscr{U}, \mathfrak{z}^q)/C^p_\Phi(\mathscr{U}, \overline{\mathbb{B}}^q_o) \to C^{p\Phi}(\mathscr{U}, \overline{\mathfrak{z}}^q/\overline{\mathbb{B}}^q_o)$$

is an isomorphism. Therefore

$$H^{p,q}_{21} C_\Phi(\mathscr{U}, \mathscr{S}) \to H^p_\Phi(\mathscr{U}, \overline{\mathfrak{z}}^q/\overline{\mathbb{B}}^q_o)$$

and hence the homomorphisms

$$(1) \qquad H^{p,q}_{21} C_\Phi(\mathscr{S}) \to H^p_\Phi(X, \overline{\mathfrak{z}}^q/\overline{\mathbb{B}}^q_o) \cdots$$

are isomorphisms. □

The exact sequence

$$0 \to \overline{\mathbb{B}}^q_o \to \overline{\mathbb{B}}^q \to \overline{\mathbb{B}}^q/\overline{\mathbb{B}}^q_o \to 0$$

gives rise to an exact sequence

$$0 \to C^p_\Phi(\mathscr{U}, \overline{\mathbb{B}}^q_o) \xrightarrow{i} C^p_\Phi(\mathscr{U}, \overline{\mathbb{B}}^q) \xrightarrow{j} C^p_\Phi(\mathscr{U}, \overline{\mathbb{B}}^q/\overline{\mathbb{B}}^q_o) \to 0$$

and i, j commute with δ. Hence there is an exact cohomology sequence **144**

$$\cdots \to H^{p-1}_\Phi(\mathscr{U}, \overline{\mathbb{B}}^q/\overline{\mathbb{B}}^q_o) \xrightarrow{\delta^*} H^p_\Phi(\mathscr{U}, \overline{\mathbb{B}}^q_o) \xrightarrow{i^*}$$
$$H^p_\Phi(\mathscr{U}, \mathbb{B}^q) \xrightarrow{i^*} H^p_\Phi(\mathscr{U}, \overline{\mathbb{B}}^q/\overline{\mathbb{B}}^q_o) \to .$$

Since i, j and δ commute with $\phi_{\mathscr{W}\mathscr{U}}$, there is an exact cohomology sequence of the direct limits

$$\cdots \to H^{p-1}_\Phi(X, \overline{\mathbb{B}}^q/\overline{\mathbb{B}}^q_o) \to H^{p-1}_\Phi(X, \overline{\mathbb{B}}^q_o) \to$$
$$H^p_\Phi(X, \overline{\mathbb{B}}^q) \to H^p_\Phi(X, \overline{\mathbb{B}}^q/\overline{\mathbb{B}}^q_o) \to \cdots$$

The presheaf $\overline{\mathbb{B}}^q/\overline{\mathbb{B}}^q_o$ determines the 0-sheaf and hence $H^p_\Phi(X, \overline{\mathbb{B}}^q/\overline{\mathbb{B}}^q_o) = o$ for all p. Hence, by exactness,

$$i^* : H^p_\Phi(X, \overline{\mathbb{B}}^q_o) \to H^p_\Phi(X, \overline{\mathbb{B}}^q_o)$$

is an isomorphism.

From the exact sequences of homomorphisms

$$\begin{tikzcd}
0 \arrow[r] & \overline{\mathbb{B}}^q_0 \arrow[r] \arrow[d] & \overline{\mathfrak{z}}^q \arrow[r] \arrow[d] & \overline{\mathfrak{z}}^q/\overline{\mathbb{B}}^q_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overline{\mathbb{B}}^q \arrow[r] & \overline{\mathfrak{z}}^q \arrow[r] & \overline{\mathfrak{z}}^q/\overline{\mathfrak{B}}^q \arrow[r] & 0,
\end{tikzcd}$$

one obtains exact sequences of homomorphisms

$$\begin{tikzcd}
H^p_\Phi(X, \overline{\mathbb{B}}^q_0) \arrow[r] \arrow[d] & H^p_\Phi(X, \mathfrak{z}^q) \arrow[r] \arrow[d] & H^p_\Phi(X, \overline{\mathfrak{z}}^q/\overline{\mathbb{B}}^q_0) \arrow[r] \arrow[d, dashed] & H^{p+1}_\Phi(X, \overline{\mathbb{B}}^q_0) \arrow[r] \arrow[d] & H^{p+1}_\phi(X, \mathfrak{z}^q) \arrow[d] \\
H^p_\Phi(X, \overline{\mathbb{B}}^q) \arrow[r] & H^p_\Phi(X, \mathfrak{z}^q) \arrow[r] & H^p_\Phi(X, \overline{\mathfrak{z}}^q/\overline{\mathbb{B}}^q) \arrow[r] & H^{p+1}_\Phi(X, \overline{\mathbb{B}}^q) \arrow[r] & H^{p+1}_\Phi(X, \mathfrak{z}^q)
\end{tikzcd}$$

where four of the vertical homomorphisms are isomorphisms. Hence, by the "five" lemma (see Eilenberg-Steenrod, Foundations of Algebraic Topology, p. 16), the homomorphism

(2) $$H^p_\Phi(X, \mathfrak{z}\overline{\mathbb{B}}^q_o) \to H^p_\Phi(X, \mathfrak{z}^q\overline{\mathbb{B}}^q)\cdots\cdots$$

145 is an isomorphism.

Next, the exact sequence

$$0 \to \mathbb{B}^q \to \mathfrak{z}^q \to \mathscr{H}^q \to 0$$

gives rise to an exact sequence

$$0 \to \overline{\mathbb{B}}^q \to \overline{\mathfrak{z}}^q \to \overline{\mathscr{H}}^q.$$

Let $\overline{\mathscr{H}}^q_o$ be the image of $\overline{\mathfrak{z}}^q$ in $\overline{\mathscr{H}}^q$. Then there is an exact sequence

$$0 \to \overline{\mathfrak{z}}^q/\overline{\mathbb{B}}^q \to \overline{\mathscr{H}}^q \to \overline{\mathscr{H}}^q/\overline{\mathscr{H}}^q_o \to 0$$

and the presheaf $\overline{\mathscr{H}}^q/\overline{\mathscr{H}}_o^q$ determines the 0-sheaf. Hence $H_\Phi^p(X, \overline{\mathscr{H}}^q/\overline{\mathscr{H}}_o^q) = 0$, and in the exact cohomology sequence, the homomorphism

$$H_\Phi^p(X, \overline{\mathscr{H}}^q/\overline{\mathbb{B}}^q) \to H_\Phi^p(X, \overline{\mathscr{H}}^q) \cdots\cdots \tag{3}$$

is an isomorphism.

Then ψ^* is the composite isomorphism

$$H_{21}^{p,q} C_\Phi(\mathscr{S}) \to H_\Phi^p(X, \overline{\mathfrak{z}}^q/\overline{\mathbb{B}}_o^q) \to H_\Phi^p(X, \overline{\mathfrak{z}}^q/\overline{\mathbb{B}}^q) \to H_\Phi^p(X, \overline{\mathscr{H}}^q).$$

Definition. *We say that the degrees of* $\{\mathscr{S}^q\}$ *are bounded below if there is an integer* m *such that* $\mathscr{S}^q = 0$ *for* $q < m$.

Lecture 23

Definition. *The* Φ*- dimension of a space* X*,* $\Phi - \dim X$*, is* $\sup\limits_{F \in \Phi} \dim F$. **146**

$\Phi - \dim X \leqq n$ *if and only if every* Φ*- covering has a* Φ*- refinement of order* $\leqq n$*.*

Proof. Necessity. Let $\Phi - \dim X \leqq n$ and let $\mathscr{U}$ be a Φ-covering of X. Let G be a neighbourhood of $\bigcup\limits_{U \in \mathscr{U} - (U_*)} \overline{U}$ with $\overline{G} \in \Phi$. Then $\mathscr{U}' = \{(\mathscr{U} - (U^*)) \cap \overline{G}, U_* \cap \overline{G}\}$ forms a locally finite covering of $\overline{G}$. Since $\overline{G}$ is normal and $\dim \overline{G} \leq n$, the covering $\mathscr{U}'$ (see Lecture 11) has a (locally finite) proper refinement $\mathscr{W}'$ of order $\leq n$. Let V_* be the union of $X - \overline{G}$ and those elements $\mathscr{W}'$ which are contained in $U_* \cap \overline{G}$ together with V_* form a Φ- covering $\mathscr{W}$ of order $\leq n$ which is a Φ- refinement of $\mathscr{U}$. □

Sufficiency. Let $F \in \Phi$ and let $\mathscr{U}$ be a finite proper covering of F. Let G be an open set with $F \subset G$ and $\overline{G} \in \Phi$. Extend each $U \in \mathscr{U}$ to an open set V of G with $V \cap F = U$. These sets together with $V_* = X - F$ form a Φ- covering $\mathscr{W}$ of X. Then $\mathscr{W}$ has a Φ– refinement $\mathscr{W}$ of order $\leqq n$ and $\{W \cap F\}_{W \in \mathscr{W}}$ is a refinement of $\mathscr{U}$ of order $\leqq n$. Thus $\dim F \leqq n$, and hence $\Phi - \dim X \leqq n$.

Note. The paracompactness of the sets of Φ was not used in this proof.

Example. In Example M (see C. H. Dowker, Quart. Jour. Math 6 (1955), p. 115) let Φ be the family of all paracompactsets of M (the space is also denoted by M). Then $\Phi - \dim M = 0$ and $\dim M = 1$. **147**

Remark. It is always true that $\Phi - \dim X \leq \dim x$.

Proposition 17. *Let*

$$\cdots \to \mathscr{S}^{q-1} \xrightarrow{d^q} \mathscr{S}^q \xrightarrow{d^{q+1}} \mathscr{S}^{q+1} \to \cdots$$

be a sequence of homomorphisms of sheaves of $A-$ modules with $\operatorname{im} d^q = \ker d^{q+1}$ *for* $q \neq 0$ *and* $\operatorname{im} d^0 \subset \ker d^1$ *and let* $\mathscr{G} = \ker d^1 / \operatorname{im} d^0$. *If* $\Phi - \dim X$ *is finite or if the degrees of* $\{\mathscr{S}^q\}$ *are bounded below there is an isomorphism* $\eta : H^p C_\Phi(\mathscr{S}) \to H^p_\Phi(X, \mathscr{G})$.

If

$$\cdots \to \mathscr{S}'^{q-1} \xrightarrow{d^q} \mathscr{S}'^{q} \xrightarrow{d^{q+1}} \mathscr{S}'^{q+1} \to \cdots$$

is another such sequence (with $\mathscr{G}'$ isomorphic to $\mathscr{G}$ and identified with $\mathscr{G}$) and if $h : \mathscr{S}'^q \to \mathscr{S}^q$ are homomorphisms commuting with d^q such that the induced homomorphism $h : \mathscr{G}' \to \mathscr{G}$ is the identity, then there are induced homomorphisms $h^* : H^P C_\Phi(\mathscr{S}') \to H^P C_\Phi(\mathscr{S})$ with commutativity in

$$\begin{tikzcd} H^p C_\Phi(\mathscr{S}') \arrow[r, "\eta"] \arrow[d, "h^*"'] & H^p_\Phi(X, \mathscr{G}) \arrow[d, "h^* = \text{identity}"] \\ H^p C_\phi(\mathscr{S}) \arrow[r, "\eta"] & H^p_\phi(X, \mathscr{G}). \end{tikzcd}$$

Proof. If the degrees of $\{\mathscr{S}^q\}$ are bounded below, then $C^p_\Phi(\mathscr{U}, \mathscr{S}^q) = 0$ for $q < n$ for some n, and so the system $\left\{C_\Phi(\mathscr{U}, \mathscr{S}), \Phi_{\mathscr{W}\mathscr{U}}\right\}_{\mathscr{U}, \mathscr{W} \in \Omega_*}$ is
148 bounded on the left. If $\Phi - \dim X \leq m$, then there is a cofinal directed set Ω'_*, consisting of $\Phi-$coverings of order $\leqq m$. If $\mathscr{U} \in \Omega'_*$, $C^P_\Phi(\mathscr{U}, \mathscr{S}^q) = 0$ for $p > m$; thus the system $\left\{C_\Phi(\mathscr{U}, \mathscr{S}), \Phi_{\mathscr{W}\mathscr{U}}\right\}_{\mathscr{U}, \mathscr{W} \in \Omega'_*}$ is bounded below. ◻

If $q \neq 0$, we have $\mathscr{H}^q = 0$, and by a result in Lecture 22, we have $H^{p,q}_{21} C_\Phi(\mathscr{S}) \approx H^p_\Phi(\chi, \mathscr{H}) = 0$, hence $H^{p,q}_{21} C_\Phi(\mathscr{S}) = 0$. Therefore by Proposition 16-a, there is an isomorphism

$$\theta : H^p C_\Phi(\mathscr{S}) \to H^{p,o}_{21} C_\Phi(\mathscr{S}).$$

Also for $q = 0$, there is an isomorphism (see Lecture 22)

$$\psi^* : H_{21}^{p,o} C_\Phi(\mathscr{S}) \to H_\Phi^p(x, \mathscr{G}).$$

Let η be the composite isomorphism

$$\eta = \psi^{*\theta} : H^P C_\Psi(\mathscr{S}) \to H_\Psi^P(X, \mathscr{G}).$$

Next, the homomorphisms $h : \mathscr{S}'^q \to \mathscr{S}^q$ induce homomorphisms $h : C_\Phi^p(\mathscr{U}, \mathscr{S}'^q) \to C_\Phi^p(\mathscr{U}, \mathscr{S}^q)$ which commute with d, δ and $\phi_{\mathscr{W}\mathscr{U}}$, and hence give rise to maps $h : C_\Phi(\mathscr{U}, \mathscr{S}') \to C_\Phi(\mathscr{U}, \mathscr{S})$ which commute with $\phi_{\mathscr{W}\mathscr{U}}$. Therefore, there are induced homomorphisms h^* which commute with θ,

$$\begin{tikzcd}
H^p C_\Phi(\mathscr{S}') \arrow[r, "\theta"] \arrow[d, "h^*"'] & H_{21}^{p,0} C_\Phi(\mathscr{S}') \arrow[d, "h^*"] \\
H^p C_\Phi(\mathscr{S}) \arrow[r, "\theta"] & H_{21}^{p,0} C_\Phi(\mathscr{S}).
\end{tikzcd}$$

Since h commutes with d^q, h maps $\mathfrak{z}'^q$ into $\mathfrak{z}^q$ and $\mathbb{B}'^q$ into $\mathbb{B}^q$, hence induces a homomorphism $h : \mathscr{H}'^q \to \mathscr{H}^q$, and there is commutativity in **149**

$$\begin{tikzcd}
C_\Phi^p(\mathscr{U}, \mathfrak{z}'^q) \arrow[r, "\psi"] \arrow[d, "h"'] & C_\Phi^p(\mathscr{U}, \mathscr{H}'^q) \arrow[d, "h"] \\
C_\Phi^p(\mathscr{U}, \mathfrak{z}^q) \arrow[r, "\psi"] & C_\Phi^p(\mathscr{U}, \mathscr{H}^q),
\end{tikzcd}$$

where ψ is the homomorphism induced by the natural homomorphisms $\mathfrak{z}'^q \to \mathscr{H}'^q$ and $\mathfrak{z}^q \to \mathscr{H}^q$. Therefore there is commutativity in

$$\begin{tikzcd}
H_{21}^{p,q} C_\Phi(\mathscr{S}') \arrow[r, "\psi^*"] \arrow[d, "h^*"'] & H_\Phi^p(X, \mathscr{H}'^q) \arrow[d, "h^*"] \\
H_{21}^{p,q} C_\Phi(\mathscr{S}) \arrow[r, "\psi^*"] & H_\Phi^p(X, \mathscr{H}^q).
\end{tikzcd}$$

Therefore, taking $q = 0$, we see that h^* commutes with $\eta = \psi^*\theta$.

$$\begin{tikzcd}
H^p C_\Phi(\mathscr{S}') \arrow[r, "\theta"] \arrow[d, "h^*"'] & H_{21}^{p,0} C_\Phi(\mathscr{S}') \arrow[r, "\psi^*"] \arrow[d] & H_\Phi^p(X, \mathscr{G}) \arrow[d, "h^* = \text{identity}"] \\
H^p C_\Phi(\mathscr{S}) \arrow[r, "\theta"] & H_{21}^{p,0} C_\Phi(\mathscr{S}) \arrow[r, "\psi^*"] & H_\Phi^p(X, \mathscr{G}).
\end{tikzcd}$$

Lecture 24

Every Φ *covering is shrinkable* as is shown by the following result. **150**

Let $\{U_i\}_{i\in I}$ *be a locally finite covering of space* X *with some* $i_* \in I$ *such that* $\overline{U}_i$ *is normal for* $i \in I - (i_*)$*. Then there is a refinement* $\{V_i\}_{i\in I}$ *with* $\overline{V}_i \subset U_i$.

Proof. The union of the locally finite system $\{\overline{U}_i\}_{i\in I-(i_*)}$ of normal closed sets is normal and closed and $X - U_{i_*} \subset \bigcup\limits_{i\neq i_*} U_i \subset \bigcup\limits_{i\neq i_*} \overline{U}_i$ with $X - U_i*$ closed and $\bigcup\limits_{i\neq i_*} U_i$ open. Hence there are open sets G, H with $X - U_{i_*} \subset G$, $\overline{G} \subset H$, $\overline{H} \subset \bigcup\limits_{i\neq i_*} U_i$. Let $V_{i_*} = X - \overline{G}$, then $\overline{V}_{i_*} \subset X - G \subset U_{i_*}$. $\square$

Since $\{\overline{H} \cap U_i\}_{i\in I-(i_*)}$ is a covering of $\overline{H}$ and the closed subset $\overline{H}$ of $\bigcup\limits_{i\neq i_*} \overline{U}_i$ is normal, there is a covering $\{P_i\}_{i\in I-(i_*)}$ of $\overline{H}$ with $\overline{P}_i \subset \overline{H} \cap U_i$. Let $V_i = H \cap P_i$ for $i \in I - (i_*)$. Then V_i is open, $\overline{V}_i \subset U_i$ and $\bigcup\limits_{i\neq i_*} V_i = H$. Then $\{V_i\}_{i\in I}$ is a covering of X and $\overline{V}_i \subset U_i$ for all $i \in I$ q.e.d.

If $\mathscr{S}$ *is a fine sheaf and if* $C \subset U \subset X$ *with* C *closed,* U *open and* $\overline{U}$ *normal, then the restriction of* $\mathscr{S}$ *to* C *is fine.*

Proof. This result is proved in the same way as in the case (see Lecture 15) that X is normal except that the open set H is to be replaced by its intersection with U if necessary, so that $H \subset U$. $\square$

Proposition 11-a *If* $\{U_i\}_{i\in I}$ *is a locally finite covering of a space* X *with some* $i_* \in I$ *such that* $\overline{U}_i$ *is normal for* $i \in I - (i_*)$*, and if* $\mathscr{S}$ *is a sheaf whose restriction to each closed subset* C *of each* U_i*,* $i \in I - (i_*)$ *is fine* **151**

(in particular this is true if $\mathscr{S}$ is fine), then there is a system $\{l_i\}_{i\in I}$ of homomorphisms $l_i : \mathscr{S} \to \mathscr{S}$ such that

(i) *for each $i \in I$ there is a closed set $E_i \subset U_i$ such that $1_i(S_x) = 0_x$ if $x \notin E_i$,*

(ii) $\sum\limits_{i\in I} l_i = 1$.

Proof. Shrink to a covering $\{W_i\}_{i\in I}$ with $\bar{W_i} \subset V_i$, $\bar{V_i} \subset G_i$, $\bar{G_i} \subset U_i$, where W_i, V_i and G_i, are open. Using the fineness of the restriction of $\mathscr{S}$ to $\bar{G_i}$. one constructs homomorphisms $h_i : \mathscr{S} \to \mathscr{S}$ $i \neq i_*$, (actually the homomorphisms are $h_i : \mathscr{S}_{\bar{G_i}} \to \mathscr{S}_{\bar{G_i}}$, and we extend these by zero outside $\bar{G_i}$; $\mathscr{S}_{\bar{G_i}}$ denotes the restriction of $\mathscr{S}$ to $\bar{G_i}$) with

$$\begin{aligned} h_i(s) &= s \quad \text{ if } \pi(s) \in \bar{W_i}, \\ &= 0_{\pi(s)} \text{ if } \pi(s) \in X - \bar{V_i}. \end{aligned}$$

Let the set $I - (i_*)$ be well-ordered and define

$$\begin{aligned} l_i &= \Big(\prod_{j<i}(1 - h_j)\Big)h_i \quad (i \neq i_*), \\ l_{i_*} &= \prod_{\{j \,\in\, I-(i_*)\}} (1 - h_j). \end{aligned}$$

152 (In a neighbourhood of each point of X, l_i, $i \in I$, is only a finite product.) Then $l_i : \mathscr{S} \to \mathscr{S}$ is a homomorphism. Let $E_i = \bar{V_i}$ for $i \neq i_*$; then if $\pi(s) \in X - \bar{V_i}$, we have $l_i(s) = 0_{\pi(s)}$ since $h_i(s) = 0_{\pi(s)}$. Let $E_{i_*} = X - \bigcup\limits_{i\in I-(i_*)} W_i$; then $E_{i_*} \subset W_{i_*} \subset U_{i_*}$. If $\pi(s) \in X - E_{i_*}$, then, for some $i \in I - (i_*)$, $\pi(s) \in W_i$ and hence $h_i(s) = s$; so $l_{i_*}(s) = 0_{\pi(s)}$. □

If $\pi(s) = x$, choose a neighbourhood N_x of x meeting at most a finite number of the sets U_i, $i \in I-(i_*)$, say, for $i = i_1, \ldots, i_q$ with $i_1 < \cdots < i_q$. Then

$$\begin{aligned} \sum_{i\in I} l_i(s) &= h_{i_1}(s) + (1 - h_{i_1})h_{i_2}(s) \\ &\quad + (1 - h_{i_1}) \ldots (1 - h_{i_{q-1}})h_{i_1}(s) \end{aligned}$$

$$+ (1 - h_{i_1}) \dots (1 - h_{i_q})\,(s)$$
$$= s,$$

and this completes the proof.

Let

$$\cdots \to \mathscr{S}^{q-1} \xrightarrow{dq} \mathscr{S}^q \xrightarrow{d^{q+1}} \mathscr{S}^{q+1} \to \cdots$$

be a sequence of homomorphisms of sheaves with $d^{q+1} d^q = 0$. Such a sequence of sheaves is called a *complex* of sheaves.

Definition. *A complex of sheaves* $\{\mathscr{S}^q\}$ *is called* homotopically fine, *if, for each locally finite covering* $\{U_i\}_{i \in I}$ *with some* $i_* \in I$ *such that* $\overline{U}_i$ *is normal for* $i \in I - (i_*)$*, there exist homomorphisms* $h^{q-1} : \mathscr{S}^q \to \mathscr{S}^{q-1}$ **153**
and a family $\{l_i^q\}_{i \in I}$ *of endomorphisms of* $\mathscr{S}^q$ *such that*

(i) for each $i \in I$ *there is a closed set* $E_i^q \subset U_i$ *such that* $l_i^q(S_x^q) = 0_x$ *if* $x \notin E_i^q$,

(ii) $\sum_{i \in I} l_i^q = 1 + d^q h^{q-1} + h^q d^{q+1}$.

If each $\mathscr{S}^q$ *is fine, then the sequence* $\{\mathscr{S}^q\}$ *is homotopically fine.*

Proof. Taking $h^q = 0$, this result follows immediately from Proposition 11-a. □

If the sequence $\{\mathscr{S}^q\}$ *is homotopically fine, and* $\mathscr{U}$ *is a locally finite covering satisfying the conditions of the previous definition, then* $H_{12}^{p,q} C_\Phi(\mathscr{U}, \mathscr{S}) = 0$ *for all* $p > 0$. *(This result is trivially true for* $p < 0$*)*

Proof. As in the proof of Proposition 12, there are induced homomorphisms $l_i^q(U) : \Gamma(U, \mathscr{S}^q) \to \Gamma(U, \mathscr{S}^q)$ induced by l_i^q, and homomorphisms

$$k^{p-1} : C_\Phi^p(\mathscr{U}, \mathscr{S}^q) \to C_\Phi^{p-1}(\mathscr{U}, \mathscr{S}^q) \quad (p > 0)$$

such that

$$\begin{aligned} \delta^p k^{p-1} f(\sigma) + k^p \delta^{p+1} f(\sigma) &= \sum_{i \in I} l_i^q(U_\sigma) f(\sigma) \\ &= f(\sigma) + d^q h^{q-1} f(\sigma) + h^q d^{q+1} f(\sigma) \end{aligned}$$

Thus $\delta k f + k\delta f = f + dhf + hdf$. and hence **154**

$$\delta k + k\delta = 1 + dh + hd : C^p_\Phi(\mathscr{U}, \mathscr{S}^q) \to C^p_\Phi(\mathscr{U}, \mathscr{S}^q).$$

Since d and h commute with δ, there are induced homomorphisms

$$\begin{aligned} d^q &: H^p_\Phi(\mathscr{U}, \mathscr{S}^{q-1}) \to H^p_\Phi(\mathscr{U}, \mathscr{S}^q), \\ h^{+q} &: H^p_\Phi(\mathscr{U}, \mathscr{S}^{q+1}) \to H^p_\Phi(\mathscr{U}, \mathscr{S}^q). \end{aligned}$$

Now, $H_1^{p,q} C_\Phi(\mathscr{U}, \mathscr{S}) = H^p_\Phi(\mathscr{U}, \mathscr{S}^q)$ and, from the homotopy k we have

$$dh^+ + h^+ d = 0 - 1 : H_1^{p,q} C_\Phi(\mathscr{U}, \mathscr{S}) \to H_1^{p,q} C_\Phi(\mathscr{U}, \mathscr{S})$$

is homotopic to zero and hence $H_{12}^{p,q} C_\Phi(\mathscr{U}, \mathscr{S}) = 0$. $\square$

Lecture 25

Proposition 18. *If the complex of sheaves $\{\mathscr{S}^q\}$ is homotopically fine, there is an isomorphism $\rho : H^q C_\Phi(\mathscr{S}) \to H^q \Gamma_\Phi(X, \mathscr{S})$.* **155**

If the complex of sheaves $\{\mathscr{S}'^q\}$ is also homotopically fine, and $h : \mathscr{S}'^q \to \mathscr{S}^q$ are homomorphisms commuting with d^q, then h^* commutes with ρ.

$$\begin{tikzcd}
H^q C_\Phi(\mathscr{S}') \arrow[r, "\rho"] \arrow[d, "h^*"'] & H^q \Gamma_\Phi(X, \mathscr{S}') \arrow[d, "d^*"] \\
H^q C_\Phi(\mathscr{S}) \arrow[r, "\rho"] & H^q \Gamma_\Phi(X, \mathscr{S}).
\end{tikzcd}$$

Proof. The system $C_\Phi(\mathscr{U}, \mathscr{S}) = \{(C^p_\Phi(\mathscr{U}, \mathscr{S}^q))\}$ of double complexes is bounded above by $p = 0$. Since (see Lecture 24) $H^{p,q}_{12} C_\Phi(\mathscr{S}) = 0$ for $p > 0$, and $H^{p,q}_{12} C_\Phi(\mathscr{S}) = 0$ trivially for $p < 0$, by Proposition 16, there is an isomorphism

$$\theta : H^q C_\Phi(\mathscr{S}) \to H^{o,q}_{12} C_\Phi(\mathscr{S}).$$

Since $h : C^p_\Phi(\mathscr{U}, \mathscr{S}'^q) \to C^p_\Phi(\mathscr{U}, \mathscr{S}^q)$ commutes with d, δ and $\phi_{\mathscr{W}\mathscr{U}}, h : C_\Phi(\mathscr{U}, \mathscr{S}') \to C_\Phi(\mathscr{U}, \mathscr{S})$ is a map of double complexes which commutes with $\phi_{\mathscr{W}\mathscr{U}}$. Therefore h^* commutes with θ. $\square$

Since $\Gamma_\Phi(X, \mathscr{S}^q) = H^o_\Phi(X, \mathscr{S}^q) = \text{dir lim}\, H^o_\Phi(\mathscr{U}, \mathscr{S}^q)$ and the homomorphism $\tau_{\mathscr{U}} : H^o_\Phi(\mathscr{U}, \mathscr{S}^q) \to \Gamma_\Phi(X, \mathscr{S}^q)$ commutes with d^q, there are induced homomorphisms

$$\tau^*_{\mathscr{U}} : H^q\, H^o_\Phi(\mathscr{U}, \mathscr{S}) \to H^q(\Gamma_\Phi(X, \mathscr{S}),$$
$$\tau^* : \text{dir lim}\, H^q\, H^o_\Phi(\mathscr{U}, \mathscr{S}) \to H^q \Gamma_\Phi(X, \mathscr{S}).$$

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Since the operation of forming cohomology groups commutes with the operation of forming direct limits, (see Cartan - Eilenberg, Homological Algebra, Proposition 9.3*, p. 100), τ^* is an isomorphism, and since h commutes with $\tau_{\mathscr{U}}$, h^* commutes with τ^*.

Now,

$$\begin{aligned} H_1^{o,q} C_\Phi(\mathscr{U}, \mathscr{S}) &= H_\Phi^o(\mathscr{U}, \mathscr{S}^q), \\ H_{12}^{o,q} C_\Phi(\mathscr{U}, \mathscr{S}) &= H^q H_\Phi^o(\mathscr{U}, \mathscr{S}), \\ H_{12}^{o,q} C_\Phi(\mathscr{S}) &= \text{ dir } \lim H^q\, H_\Phi^o(\mathscr{U}, \mathscr{S}). \end{aligned}$$

Thus we have an isomorphism

$$\tau^* : H_{12}^{o,q} C_\Phi(\mathscr{S}) \to H^q \Gamma_\Phi(X, \mathscr{S})$$

which commutes with h^*. Let $\rho = \tau^* \theta$ be the composite isomorphism then ρ commutes with h^*.

$$\begin{array}{ccccc} H^q C_\Phi(\mathscr{S}') & \xrightarrow{\theta} & H_{12}^{o,q} C_\Phi(\mathscr{S}') & \xrightarrow{\tau^*} & H^q \Gamma_\Phi(X, \mathscr{S}') \\ \downarrow{\scriptstyle h^*} & & \downarrow{\scriptstyle h^*} & & \downarrow{\scriptstyle h^*} \\ H^q C_\Phi(\mathscr{S}) & \xrightarrow{\theta} & H_{12}^{0,q} C_\Phi(\mathscr{S}) & \xrightarrow{\tau^*} & H^q \Gamma_\Phi(X, \mathscr{S}). \end{array}$$

Theorem 3 (Uniqueness Theorem)**.** *Let*

$$\cdots \to \mathscr{S}^{q-1} \xrightarrow{d^q} \mathscr{S}^q \xrightarrow{d^{q+1}} \mathscr{S}^{q+1} \to \cdots$$

157 *be a homotopically fine complex of sheaves of $A-$ modules with* $\operatorname{im} d^q = \ker d^{q+1}$ *for $q \neq 0$ and* $\operatorname{im} d^o \subset \ker d^1$ *and let* $\mathscr{H}^o = \ker d^1 / \operatorname{im} d^o$. *Let* $\{\mathscr{S}'^q\}$ *be another such complex with $\mathscr{H}'^o$ isomorphic to $\mathscr{H}^o$.*

If $\Phi - \dim X$ *is finite or the degrees of* $\{\mathscr{S}'^q\}$ *and* $\{\mathscr{S}^q\}$ *are bounded below, then any isomorphism $\lambda : \mathscr{H}'^o \to \mathscr{H}^o$ induces an isomorphism*

$$\phi_\lambda : H^q \Gamma_\Phi(X, \mathscr{S}') \to H^q \Gamma_\Phi(X, \mathscr{S})$$

and if $\{\mathscr{S}''^q\}$ *is another such complex and $\mu : \mathscr{H}^o \to \mathscr{H}''^o$ is an isomorphism, then*

$$\phi_{\mu\lambda} = \phi_\mu \phi_\lambda : H^q \Gamma_\Phi(X, \mathscr{S}') \to H^q \Gamma_\Phi(X, \mathscr{S}'').$$

If $h : \mathscr{S}'^q \to \mathscr{S}^q$ are homomorphisms (for each q) commuting with d^q, and if the induced homomorphism $h : \mathscr{H}'^o \to \mathscr{H}^o$ is an isomorphism, then the homomorphism $h^* : H^q\Gamma_\Phi(X, \mathscr{S}') \to H^q\Gamma_\Phi(X, \mathscr{S}$ is the isomorphism ϕ_h.

Proof. Since the hypotheses of Propositions 17 and 18 are satisfied, there are isomorphisms

$$\eta : H^q C_\Phi(\mathscr{S}) \to H^q_\Phi(X, \mathscr{H}^o),$$
$$\rho : H^q C_\Phi(\mathscr{S}) \to H^q\Gamma_\Phi(X, \mathscr{S}).$$

Since $\lambda : \mathscr{H}'^o \to \mathscr{H}^o$ is an isomorphism, so is

$$\lambda^* : H^q_\Phi(X, \mathscr{H}'^o) \to H^q_\Phi(X, \mathscr{H}^o).$$

Let ϕ_λ be the isomorphism $\rho\eta^{-1}\lambda^*\eta\rho^{-1}$. Since $(\mu\lambda)^* = \mu^*\lambda^*$, $\phi_{\mu\lambda} =$ **158**
$\phi_\mu\phi_\lambda$.

$$\begin{tikzcd}
H^q\Gamma_\Phi(X, \mathscr{S}') \arrow[d, "\phi_\lambda"] & H^q C_\Phi(\mathscr{S}') \arrow[l, "\rho"'] \arrow[r, "\eta"] & H^q_\Phi(X, \mathscr{H}'^0) \arrow[d, "\lambda^*"] \\
H^q\Gamma_\Phi(X, \mathscr{S}) \arrow[d, "\phi_\mu"'] & H^q C_\Phi(\mathscr{S}) \arrow[l, "\rho"'] \arrow[r, "\eta"] & H^q_\Phi(X, \mathscr{H}^0) \arrow[d, "\mu^*"] \\
H^q\Gamma_\Phi(X, \mathscr{S}'') & H^q C_\Phi(\mathscr{S}'') \arrow[l, "\rho"'] \arrow[r, "\eta"] & H^q_\Phi(X, \mathscr{H}''^0).
\end{tikzcd}$$

If $h : \mathscr{S}'^q \to \mathscr{S}^q$ are homomorphisms commuting with d^q and inducing an isomorphism $h : \mathscr{H}'^o \to \mathscr{H}^o$, then it follows from the commutative diagram:

$$\begin{tikzcd}
H^q\Gamma_\Phi(X, \mathscr{S}') \arrow[d, "\phi_h"', "h^*"] & H^q C_\Phi(\mathscr{S}') \arrow[l] \arrow[r] \arrow[d, "h^*"] & H^q_\Phi(X, \mathscr{H}'^0) \arrow[d, "h^*"] \\
H^q\Gamma_\Phi(X, \mathscr{S}) & H^q C_\Phi(\mathscr{S}) \arrow[l] \arrow[r] & H^q_\Phi(X, \mathscr{H}^0),
\end{tikzcd}$$

that $\phi_h = h^* : H^q\Gamma_\Phi(X, \mathscr{S}') \to H^q\Gamma_\Phi(X, \mathscr{S})$. ◻

Lecture 26

Singular chains

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Definition. *Let T_p be the set of all singular p- simplexes in X. A* singular p-chain *with integer coefficients is a function $c : T_p \to Z$ such that $c_t = c(t)$ is zero for all but a finite number of $t \in T_p$. It is usually written as a formal sum $\sum\limits_{t \in T} c_t \cdot t$. The* support *of c is the union of the supports $t(s_p)$ of those simplexes t for which $c_t \neq 0$. The boundary of the p - chain c is the $(p-1)$-chain*

$$\partial_{p-1} c = \sum_{t \in T_p} c_t \sum_{j=o}^{p} (-1)^j \partial_j t.$$

The singular p-chains of X form a free abelian group $C_p(X, Z)$ and $\partial_{p-1} : C_p(X, Z) \to C_{p-1}(X, Z)$ is a homomorphism with $\partial_{p-1}\partial_p = 0$; ∂_{p-1} decreases support, i.e., $\operatorname{supp} \partial_{p-1} c \subset \operatorname{supp} c$.

If $\mathscr{W} = \{V_i\}_{i \in I}$ is a covering of X, let $C_p(X, Z, \mathscr{W})$ be the subgroup of $C_p(X, Z)$ consisting of chains c such that $c_t = 0$ unless $\operatorname{supp} t \subset V_i$ for some $i \in I$. Let $j : C_p(X, Z, \mathscr{W}) \to C_p(X, Z)$ be the inclusion homomorphism. Since ∂_{p-1} decreases supports, there is an induced homomorphism $\partial_{p-1} : C_p(X, Z, \mathscr{W}) \to C_{p-1}(X, Z, \mathscr{W})$ which commutes with j.

It is known (Cartan Seminar, 1948-49, Exposé 8, §3) that there is a *subdivision* consisting of homomorphisms $r : C_p(X, Z) \to C_p(X, Z, \mathscr{W})$ such that

(i) $rc = c$ if $c \in C_p(X, Z, \mathscr{W})$, 160

(ii) $\operatorname{supp} rc = \operatorname{supp} c$.

Further, there is a *homotopy* $h_{p+1} : C_p(X, Z) \to C_{p+1}(X, Z)$ such that

(iii) $\partial_p h_{p+1} c + h_p \partial_{p-1} c = jrc - c$,

(iv) $h_{p+1} jc = 0$,

(v) $\operatorname{supp} h_{p+1} c \subset \operatorname{supp} c$.

Let I be well-ordered, let $j_i(t) = t$ if $\operatorname{supp} t \subset V_i$ but $\operatorname{supp} t \not\subset V_k$ for all $k < i$ and let $j_i(t) = 0$ otherwise. This defines a homomorphism

$$j_i : C_p(X, Z, \mathscr{W}) \to C_p(X, Z),$$

with $j = \sum\limits_{i \in I} j_i$ and $\operatorname{supp} u_i c \subset V_i \cap \operatorname{supp} c$.

Let $l_i = j_i r : C_p(X, Z) \to C_p(X, Z)$. Then $\operatorname{supp} l_i c \subset V_i \cap \operatorname{supp} c$. Let $1 = \sum\limits_{i \in I} l_i = \sum j_i r = jr$. Since $rj = 1 : C_p(X, Z, \mathscr{W}) \to C_p(X, Z, \mathscr{W})$, $l^2 = j(rj)r = jr = l$. Further, since $jr = l$, we have

$$\partial_p h_{p+1} + h_p\, \partial_{p-1} = l - 1,$$

and $\operatorname{supp} lc \subset \operatorname{supp} c$.

If U is open, let $C_p(X, Z)_U$ be the set of chains $c \in C_p(X, Z)$ such that U does not meet $\operatorname{supp} c$. Then $C_p(X, Z)_U$ is a subgroup of $C_p(X, Z)$; let $S_{pU} = C_p(X, Z)/C_p(X, Z)_U$. Since ∂_{p-1}, h_{p+1}, l_i and l decrease supports, there are induced homomorphisms

$$\partial_{p-1} : S_{pU} \to S_{p-1,U'} \qquad h_{p+1} : S_{pU} \to S_{p+1,U'}$$
$$l_i : S_{pU}, \quad \text{and } l : S_{pU} \to S_{pU}.$$

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If $V \subset U$ then $C_p(X, Z)_U \subset C_p(X, Z)_V$ and there is an induced epimorphism $\rho_{VU} : S_{pU} \to S_{pV}$ which commutes with ∂_{p-1}, h_{p+1}, l_i and l. Then $\{S_{pU}, \rho_{VU}\}$ is a presheaf which determines a sheaf $\mathscr{S}_p$ called the sheaf of singular p-chains. There are induced sheaf homomorphisms

$$\partial_{p-1} : \mathscr{S}_p \to \mathscr{S}_{p-1}, h_{p+1} : \mathscr{S}_p \to \mathscr{S}_{p+1},$$

$$l_i : \mathscr{S}_p \to \mathscr{S}_p, \text{ and } l : \mathscr{S}_p \to \mathscr{S}_p,$$

with

$$1 + \partial_p h_{p+1} + h_p \partial_{p-1} = l = \sum_{i \in I} l_i.$$

If $c \in C_p(X, Z)$ and $U = X - \overline{V}_i$ then, since supp $l_i(c) \subset V_i$, $l_i(c) \in C_p(X, Z)_U$. Therefore $l_i : S_{pW} \to S_{pW}$ is the zero homomorphism for each open $W \subset U$. Hence $l_i(S_{px}) = 0_x$ for all $x \in X - \overline{V}_i$.

Definition. *If $\mathscr{G}$ is a sheaf of $A-$ modules, the sheaf $\mathscr{S}_p \otimes_Z \mathscr{G}$ is a sheaf of A-modules called the* sheaf of singular p - chains with coefficients in $\mathscr{G}$.

Let $\mathscr{S}^p = \mathscr{S}_{k-p} \otimes_Z \mathscr{G}$ for some fixed integer k and let $d^{p+1} : \mathscr{S}^p \to \mathscr{S}^{p+1}$, $h^{p-1} : \mathscr{S}^p \to \mathscr{S}^{p-1}$, $l_i : \mathscr{S}^p \to \mathscr{S}^p$ and $l : \mathscr{S}^p \to \mathscr{S}^p$ be the homomorphisms induced by $\partial_{k-p-1}, h_{k-p+1}, l_i$ and l. Then **162**

$$1 + d^p h^{p-1} + h^p d^{p+1} = l = \sum l_i,$$

and $l_i(S^p_x)0_x$ for all $x \in X - \overline{V}_i$.

The sequence $\cdots \to \mathscr{S}^{p-1} \xrightarrow{d^p} \mathscr{S}^p \xrightarrow{d^{p+1}} \mathscr{S}^{p+1} \to \cdots$ *is homotopically fine.*

Proof. Let $\mathscr{U} = \{U_i\}_{i \in I}$ be a locally finite covering with $i_* \in I$ such that $\overline{U_i}$ is normal for $i \in I - (i_*)$. Shrink $\mathscr{U}$ to a covering $\mathscr{W} = \{V_i\}_{i \in I}$ with $\overline{V}_i \subset U_i$. Construct the homomorphisms $h^{p-1} : \mathscr{S}^p \to \mathscr{S}^{p-1}$, $l_i : \mathscr{S}^p \to \mathscr{S}^p$ (as above) such that

$$1 + d^p h^{p-1} + h^p \, d^{p+1} = \sum_{i \in I} l_i,$$

and $l_i(S^p_x) = 0_x$ if $x \notin \overline{V}_i$. Thus we can take $E_i = \overline{V}_i$. $\square$

Definition. *Let $C^\Phi_p(X, \mathscr{G}) = \Gamma_\Phi(X, \mathscr{S}_p \otimes_Z \mathscr{G})$; this $A-$ module is called the module of* singular p - chains of x with coefficients in $\mathscr{G}$. *Let $H^\Phi_p(X, \mathscr{G}) = \ker \partial_{p-1}/im\, \partial_p$ in the sequence*

$$\cdots \to C^\Phi_{p+1}(X, \mathscr{G}) \xrightarrow{\partial_p} C^\Phi_p(X, \mathscr{G}) \xrightarrow{\partial_{p-1}} C^\Phi_{p-1}(X, \mathscr{G}) \to \cdots$$

where the homomorphism $\partial_p : C^{\Phi}_{p+1}(X,\mathscr{G}) \to C^{\Phi}_p(X,\mathscr{G})$ is the one induced by the homomorphism $\partial_p : \mathscr{S}_{p+1} \to \mathscr{S}_p$. The A− module $H^{\Phi}_p(X,\mathscr{G})$ is called the p-th singular homology module of the space X with coefficients in the sheaf $\mathscr{G}$, and supports in the family Φ.

163 An element of a stalk S_{px} can be written uniquely in the form $\sum\limits_{t\in T_p} c_t\cdot t$ where $c_t \in Z$, $c_t = 0$ if x is not in the closure of the support of t, $\overline{\operatorname{supp} t}$, and $c_t = 0$ except for a finite number of t.

An element of a stalk $S_{qx} \otimes_Z G_x$ can be written uniquely in the form $\sum\limits_{x\in T_p} g_t \cdot t$ where $g_t \in G_x$, $g_t = 0$ if $x \notin \overline{\operatorname{supp} t}$ and $g_t = 0$ except for a finite number of t.

An element of $C^{\Phi}_p(X,\mathscr{G})$ can be written uniquely in the form $\sum\limits_{t\in T_p} \gamma_t \cdot t$ where γ_t is a section of $\mathscr{G}$ over $\overline{\operatorname{supp} t}$, $\gamma_t = 0$ except for a set of simplexes whose supports form a locally finite system and the set of points x such that, for some t, $\gamma_t(x)$ is defined and $\neq 0_x$ is contained in a set of Φ. (A section over a closed set E is a map $\gamma : E \to G$ such that $\pi\gamma : E \to E$ is the identity.)

Definition . *An n - manifold is a Hausdorff space X which is locally euclidean, i.e., each point $x \in X$ has a neighbourhood which is homeomorphic to an open set in R^n.*

If X is an n-manifold, then $\Phi - \dim X = n$.

Proof. Since X can be covered by open sets whose closures are homeomorphic to subsets of R^n, X is locally n-dimensional. Then each closed set $E \in \Phi$ is locally of dimension $\leq n$ and E is paracompact and normal, hence $\dim E \leq n$. Further, any non-empty set $E \in \Phi$ has a closed
164 neighbourhood $\bar{V} \in \Phi$ and $\bar{V}$ contains a closed set homeomorphic to the closure of an open set in R^n. Hence $\dim \bar{V} \geqq n$; thus $\Phi - \dim X = n$, and this completes the proof. □

In the sequence $\cdots \to \mathscr{S}_{p+1} \xrightarrow{\partial_p} \mathscr{S}_p \xrightarrow{\partial_{p-1}} \mathscr{S}_{p-1} \to \cdots$

Let $\mathscr{H}_p = \ker \partial_{p-1} / \operatorname{im} \partial_p$; $\mathscr{H}_p$ is called the p - *the singular homology sheaf in X*.

If X is an n-manifold, the p-th singular homology sheaf in X is locally isomorphic with the p-th singular homology sheaf in R^n.

Proof. Let $x_0 \in U_1 \subset X$ where U_1 is open in X and let $f : U_1 \to U'_1$ be a homeomorphism onto an open set $U'_1 \subseteq R^n$. Choose an open set V with $x_o \in V$, $\bar{V} \subset U_1$ and let $U_2 = X - \bar{V}$, $U'_2 = R^n - f(\bar{V})$. Then $\{U_1, U_2\}$ is a covering of X, and there is the homotopy defined above,

$$\partial h + h\partial = l_1 + l_2 - t,$$

with $l_2(S_{px}) = 0$ for $x \in V$. Hence $\mathscr{H}_p(\mathscr{S})$ is isomorphic with $\mathscr{H}_p(l_1\mathscr{S})$ in V. But $f : U_1 \to U'_1$ takes $l_1(\mathscr{S}_p)$ into $l'_1(\mathscr{S}'_p)$ where $\mathscr{S}'_p$ is the sheaf of singular $p-$ chains in R^n and l'_1 is the corresponding homomorphism for the covering $\{U'_1, U'_2\}$ of R^n. In $f(V)$, $\mathscr{H}_p(l'_1\mathscr{S}')$ is with $\mathscr{H}_p(\mathscr{S}')$. $\square$

Using triangulations of R^n, one can verify, for R^n, that $\mathscr{H}_p = o$ for $p \neq n$ and $\mathscr{H}_n$ is isomorphic with the constant sheaf $(R^n \times Z, \pi, R^n)$. One uses a homotopy which does not decrease supports and which does 165
not induce a sheaf homotopy. The isomorphism is not a natural one but depends on the choice of an orientation for R^n.

In an n-manifold X, $\mathscr{H}_p = 0$ for $p \neq n$ and $\mathscr{H}_n$ is locally isomorphic with Z. Let $\mathcal{J} = \mathscr{H}_n$; if $\mathcal{J}$ is isomorphic with Z the manifold is called *orientable*, otherwise the manifold is said to be *non-orientable* and $\mathcal{J}$ is called the sheaf of twisted integers over X. Example 2 is the restriction ot the Möbius band of the sheaf of twisted integers over the projective plane.

If $\mathscr{S}^p = \mathscr{S}_{n-p} \otimes_Z \mathscr{G}$ on an n-manifold X, then $\mathscr{H}^p(\mathscr{S})$ for $p \neq 0$ and $\mathscr{H}^o(\mathscr{S}) = \mathcal{J} \otimes_Z \mathscr{G}$.

Proof. Since S_{px} is a free abelian group, so are the subgroups Z_{px} and B_{px}. Also, H_{px} being either 0 or Z, is free. It is known (Cartan Seminar, 1948-49, Expose 11) that if

$$0 \to F \to F \to F'' \to 0$$

is an exact sequence of abelian groups and F'' is without torsion and if G is an abelian group, then the induced sequence

$$0 \to F' \otimes G \to F \otimes G \to F'' \otimes G \to 0$$

is exact. From the following commutative diagram of exact sequences, one can see that

$$\mathscr{H}_p(\mathscr{S}\otimes\mathscr{G}) = \ker\partial_{p-1}/\operatorname{im}\partial_p \approx \mathfrak{z}_p\otimes\mathscr{G}/\mathscr{B}_p\otimes\mathscr{G} \approx \mathscr{H}_p\otimes\mathscr{G}.$$

166 Thus $\mathscr{H}^p(\mathscr{S}) = \mathscr{H}_{n-p}\otimes\mathscr{G} = O$ for $p\neq O$ and $\mathscr{H}^p(\mathscr{S}) = \mathcal{J}\otimes\mathscr{G}$ for $p = O$. □

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
0 & \mathbb{B}_p\otimes\mathscr{G} \arrow[l] \arrow[dd] & \mathscr{S}_{p+1}\otimes\mathscr{G} \arrow[l] \arrow[dd, "\partial_p"'] & \mathfrak{z}_{p+1}\otimes\mathscr{G} \arrow[l] & 0 \arrow[l] \\
 & & & 0 \arrow[dl, bend right] \arrow[d] & \\
 & \mathfrak{z}_p\otimes\mathscr{G} \arrow[r] \arrow[dd] & \mathscr{S}_p\otimes\mathscr{G} \arrow[r] \arrow[ddd, "\partial_{p-1}"'] & \mathbb{B}_{p-1}\otimes\mathscr{G} \arrow[r] \arrow[dd] & 0 \\
 & & & & 0 \arrow[dl] \\
 & \mathscr{H}_p\otimes\mathscr{G} \arrow[d] & & \mathfrak{z}_{p-1}\otimes\mathscr{G} \arrow[dl] \arrow[dd] & \\
 & 0 & \mathscr{S}_{p-1}\otimes\mathscr{G} \arrow[dl] & & \\
 & \mathbb{B}_{p-2}\otimes\mathscr{G} \arrow[dl] & & \mathscr{H}_{p-1}\otimes\mathscr{G} \arrow[d] & \\
0 & & & 0 &
\end{tikzcd}$$

Q.e.d.

Proposition 19. *If X is an n- manifold, there is an isomorphism:*

$$\eta^{-1} \quad : H^{\Phi}_{n-p}(X, \mathscr{G}) \to H^{p}_{\Phi}(X, \mathcal{J} \otimes_{Z} \mathscr{G}).$$

Proof. Since $C^{\Phi}_{n-p}(X, \mathscr{G}) = \Gamma_{\Phi}(X, \mathscr{S}_{n-p} \otimes_z \mathscr{G}) = \Gamma_{\bar{\Phi}}(X, \mathscr{S}^p)$, $H^{\bar{\Phi}}_{n-p}(X, \mathscr{G}) = H^p\Gamma_{\bar{\Phi}}(X, \mathscr{S})$. And, since $\mathscr{H}^o(\mathscr{S}) = \mathcal{J} \otimes_z \mathscr{G}$, $H^p_{\Phi}(X, \mathcal{J} \otimes_z \mathscr{G}) = H^p_{\Phi}(X, \mathscr{H}^o)$. By proposition 17 and 18, there are isomorphisms

$$\begin{aligned} \eta &: H^p C_{\Phi}(\mathscr{S}) \to H^p_{\Phi}(X, \mathscr{H}^o), \\ \rho &: H^p C_{\Phi}(\mathscr{S}) \to H^p \Gamma_{\Phi}(X, \mathscr{S}). \end{aligned}$$

Thus $\eta\rho^{-1}$ is the required isomorphism. ◻

This proposition is part of the Poincare duality theorem.

Lecture 27

Given any sheaf $\mathscr{G}$ of A-modules, there exists an exact sequences of sheaves **167**

$$O \to \mathscr{G} \xrightarrow{e} \mathscr{S}^o \xrightarrow{d^1} \mathscr{S}^1 \to \cdots \to \mathscr{S}^{q-1} \xrightarrow{d^q} \mathscr{S}^q \to \cdots$$

where each $\mathscr{S}^q (q \geq O)$ is finite.

Proof. For each open set U of X, let $S_U^q (q = O, 1, \ldots)$ be the abelian group of integer valued functions $f(x_0, \ldots, x_q)$ of $q + 1$ variables $x_0, \ldots$ $x_q \in U$. If V is an open set with $V \subset U$, the restriction of the functions f gives a homomorphism $\rho_{VU} : S_U^q \to S_V^q$. There is a homomorphism $d_U^{q+1} : S_U^q \to S_U^{q+1}$ defined by

$$d^{q+1} f(x_o, \ldots, x_{q+1}) = \sum_{j=o}^{q+1} (-1)^j f(x_o, \ldots, \hat{x}_j, \ldots, x_{q+1}).$$

If $e : Z \to S_U^o$ is the inclusion homomorphism of the constant functions on U, the sequence

$$O \to Z \xrightarrow{e} S_U^o \xrightarrow{d^1} \cdots \to S_U^{q-1} \xrightarrow{d^q} S_U^q \to \cdots$$

is exact (Cartan Seminar 1948-49, Expose 7, §8). Clearly $\rho_{VU} d^{q+1} f = d^{q+1} \rho_{VU} f$ and ρ_{VU} commutes with e. The presheaves $\{S_U^q, \rho_{VU}\} (q = 0, 1, \ldots)$ determine sheaves $\mathscr{S}^q$ and there is an induced exact sequence

$$O \to Z \xrightarrow{e} S_U^o \xrightarrow{d^1} \cdots \to S_U^{q-1} \xrightarrow{d^q} S^q \to \cdots$$

It is easily verified that each abelian group S^p_U is without torsion, and this property is preserved in the direct limit. Hence each stalk $S^q_x =$ 168
$dir \lim\{S^q_U, \rho_{VU}\}_{x\in U}$ is without torsion, i.e., the sheaves $\mathscr{S}^q$ are without torsion. Therefore the sequences of sheaves of A-modules

$$O \to Z \otimes_Z \mathscr{G} \xrightarrow{e} \mathscr{S}^o \otimes_Z \mathscr{G} \xrightarrow{d^1} \cdots \to \mathscr{S}^{q-1} \otimes_Z \mathscr{G} \xrightarrow{d^q} \mathscr{S}^q \otimes_Z \mathscr{G} \to \cdots$$

is exact, (this follows from the fact that if $O \to F' \to F \to F'' \to O$ is an exact sequence of abelian group, F'' is without torsion, and G is an abelian group, then the sequence

$$O \to F' \otimes G \to F \otimes G \to F'' \otimes G \to O$$

is exact), that is, the sequence

$$(1) \quad 0 \to \mathscr{G} \xrightarrow{e} \mathscr{S}^o \otimes_Z \mathscr{G} \xrightarrow{d^1} \cdots \to \mathscr{S}^{q-1} \otimes_Z \mathscr{G} \xrightarrow{d^q} \mathscr{S}^q \otimes_Z \mathscr{G} \to \cdots$$

is exact.

We now show that *each of the sheaves* $\mathscr{S}^q \otimes_Z \mathscr{G}$ *is a fine sheaf.* To do this, let $E \subset G$ with E closed and G open and let $h : S^q_U \to S^p_U$ be the homomorphism defined by

$$\begin{aligned} hf(x_o, \ldots, x_q) &= f(x_o, \ldots, x_q) \text{ if } x_o, \ldots, x_q \in U \cap G, \\ &= 0 \text{ otherwise.} \end{aligned}$$

Then h commutes with ρ_{VU} and induces a homomorphism $h : \mathscr{S}^q \to \mathscr{S}^q$ for which $h_x : S^q_x \to S^q_x$ is the identity if $x \in \bar{G}$, and $h(S^q_x) = 0_x$ if $x \in X - \bar{G}$. There is then an induced homomorphism $h : \mathscr{S}^q \otimes_Z \mathscr{G} \to \mathscr{S}^q \otimes_Z$ which is the identity on the stalks $S^q_x \otimes_Z G_x$ for $x \in G$ and zero
169 on $S^q_x \otimes_Z G_x$ for $x \in X - \bar{G}$. Thus $\mathscr{S}^q \otimes_Z \mathscr{G}$ is fine, and the sequence (1) is a fine resolution of $\mathscr{G}$.

We now give a definition of the cochains of a covering of a space X coefficients in a sheaf $\mathscr{G}$ and support in a Φ-family and also give an alternative definition of the cohomology groups of X with coefficients in $\mathscr{G}$ and supports in the family Φ. We then prove Leray's theorem on acyclic coverings. □

Definition. *Let $C^p_\Phi(\mathscr{U},\mathscr{G})$, for an arbitrary covering $\mathscr{U}=\{U_i\}_{i\in I}$ be the $C^p(\mathscr{U},\mathscr{G})$ consisting of those cochains f such that the closure of the set $supp f=\{x : f(i_o,\ldots,i_p)(x)$ is defined for some $\sigma=(i_o,\ldots,i_p)$ and $\neq 0_x\}$, belongs to the family Φ. (If $p=0$, this definition does not agree with the previous one even when $\mathscr{U}$ is a $\bar{\Phi}$-covering.)*

Then, since the homomorphisms

$$\delta^{p+1}: C^p(\mathscr{U},\mathscr{G})\to C^{p+1}(\mathscr{U},\mathscr{G})$$
$$\tau^+ : C^p(\mathscr{U},\mathscr{G})\to C^p(\mathscr{W},\mathscr{G})$$

decrease supports, they induce homomorphisms

$$\delta^{p+1}: C^p_\phi(\mathscr{U},\mathscr{G})\to C^{p+1}_\Phi(\mathscr{U},\mathscr{G})$$

and

$$\tau^+ : C^p_\Phi(\mathscr{U},\mathscr{G})\to C^p_\phi(\mathscr{W},\mathscr{G})$$

respectively. Hence there are cohomology module $H^p_\Phi(\mathscr{U},\mathscr{G})$ and homomorphisms $\tau_{\mathscr{W}\mathscr{U}} : H^p_\Phi(\mathscr{U},\mathscr{G})\to H^p_\Phi(\mathscr{W},\mathscr{G})$. For $p=0$, $H^o_\Phi(\mathscr{U},$ **170**
$\mathscr{G})=\Gamma_\Phi(X,\mathscr{G})$ for every covering $\mathscr{G}$, and

$$\tau_{\mathscr{W}\mathscr{U}} : \Gamma_\Phi(X,\mathscr{G})\to\Gamma_\Phi(X,\mathscr{G})$$

is the identity. Using the directed set Ω of all proper covering of $X,\{H^p_\Phi(\mathscr{U},\mathscr{G}),\tau_{\mathscr{W}\mathscr{U}}\}_{\mathscr{U},\mathscr{W}\in\Omega}$ is a direct system. The direct limit of this system will be denoted by $H^p_\Phi(X,\mathscr{G})$. This module also is called the *p- th cohomology module of the space X with coefficients in the sheaf $\mathscr{G}$ and supports in the family* Φ. (This cohomology module is isomorphic with that previously defined by means of Φ-coverings, Lecture 21.) There are homomorphisms into the direct limits, $\tau_{\mathscr{U}} : H^p_\Phi(\mathscr{U},\mathscr{G})\to H^p_\Phi(X,\mathscr{G})$.

Let X be a *paracompact normal* space, $\mathscr{G}$ a sheaf of A-modules over X, and let $\mathscr{U}=\{U\}$ be locally finite proper covering of X where each $U\in\mathscr{U}$ is an F_σ set. ($\mathscr{U}$ is indexed by itself.) Let Φ_σ be the set of all intersection $E\cap U_\sigma$ with $E\in\Phi$. Since U_σ is an open F_σ set in X, U_σ is paracompact and normal. Hence each $E\cap U_\sigma\in\Phi_\sigma$ is paracompact and normal. One easily verifies that Φ_σ is a Φ-family in U_σ. We now as sume the following conditions on the family Φ and the covering $\mathscr{U}$.

(i) For some infinite cardinal number m, the union of fewer than m elements of Φ is contained in a set belonging to Φ. (If $X \in \Phi$, choose m greater than the number of closed sets of X; if Φ is family of compact sets, let $m = \mathscr{N}_o$.)

171 (ii) Each set in Φ meets fewer than m sets of the covering $\mathscr{U}$.

(iii) Each $U \in \mathscr{U}$ is an F_σ sets, i.e., is a countable union of closed subsets of X.

(iv) For each $U_\sigma = U_o \cap \cdots U_p$, $H^p_{\Phi_\sigma}(U_\sigma, \mathscr{G}) = 0$ for $q > 0$; here $\mathscr{G}$ denotes the restriction of $\mathscr{G}$ to U_σ

(A covering $\mathscr{U}$ is called *acyclic* is conditions (iv) is satisfied.)

Under the conditions (i), (ii), (iii), (iv) *stated above, the homomorphism* $\tau_{\mathscr{U}} : H^p_\Phi(\mathscr{U}, \mathscr{G}) \to H^p_\Phi(X, \mathscr{G})$ *is an isomorphism.*

Proof. Let

$$O \to \mathscr{G} \xrightarrow{e} \mathscr{S}^o \xrightarrow{d^1} \cdots \to \mathscr{S}^{q-1} \to d^q \mathscr{S}^q \to \cdots$$

be any fine resolution of $\mathscr{G}$. Then there is a system $\{(C^p_\Phi(\mathscr{U}, \mathscr{S}^q))\}_{\mathscr{U} \in \Omega'}$ of double complexes, where Ω' is the cofinal directed set of all locally finite proper coverings of X. This system is bounded above by $p = 0$ and on the left by $q = 0$. ◻

Since X is normal, $\mathscr{S}^q$ is fine and $\mathscr{U}$ is locally finite, there is a homotopy (see Lecture 16)

$$k^{p-1} : C^p(\mathscr{U}, \mathscr{S}^q) \to C^{p-1}(\mathscr{U}, \mathscr{S}^q) \qquad (p > 0).$$

Since k^{p-1} decreases supports, it induces a homomorphism

$$k^{p-1} : C^p_\Phi(\mathscr{U}, \mathscr{S}^q) \to C^{p-1}(\mathscr{U}, \mathscr{S}^q) \quad (p > 0),$$

172 with $\delta^p k^{p-1} + k^p \delta^{p+1} = 1$. Hence $H^p_\Phi(\mathscr{U}, \mathscr{S}^q) = 0$ for $p > 0$, and for $p = 0$, $H^o_\Phi(\mathscr{U}, \mathscr{S}^q) = \Gamma_\Phi(X, \mathscr{S}^q)$. Hence

$$H^{O,q}_{12} C_\Phi(\mathscr{U}, \mathscr{S}) = H^q \Gamma_\Phi(X, \mathscr{S}).$$

Thus we have the isomorphisms indicated below (see Lecture 20):

$$\begin{tikzcd}
H^q\Gamma_\Phi(X,\mathscr{S}) \arrow[d, "\approx"'] & H^{0,q}_{12}C_\Phi(\mathscr{U},\mathscr{S}) \arrow[l, "\approx"'] \arrow[d] & H^qC_\Phi(\mathscr{U},\mathscr{S}) \arrow[l, "\approx"'] \arrow[d] \\
H^q\Gamma_\Phi(X,\mathscr{S}) & H^{0,q}_{12}C_\Phi(\mathscr{S}) \arrow[l, "\approx"'] & H^qC_\Phi(\mathscr{S}). \arrow[l, "\approx"']
\end{tikzcd}$$

Since $\mathscr{S}^q$ is fine and X is normal, $\mathscr{S}^q$ is locally fine, hence its restriction to U_σ is locally fine. But U_σ is paracompact and normal, so that the restriction of $\mathscr{S}^q$ to U_σ is fine. Hence there is an isomorphism (see Proposition 17 and 18, lectures 23 and 25 respectively),

$$\eta\rho^{-1} : H^q\Gamma_{\Phi_\sigma}(U_\sigma,\mathscr{S}) \to H^q_{\Phi_\sigma}(U_\sigma,\mathscr{G}).$$

Hence by condition (iv), $H^q\Gamma_{\Phi_\sigma}(U_\sigma,\mathscr{S}) = 0$ for $q > 0$, $H^o\Gamma_\Phi(U_\sigma,\mathscr{S}) \approx H^o_{\Phi_\sigma}(U_\sigma,\mathscr{G}) = \Gamma_{\Phi_\sigma}(U_\sigma,\mathscr{G})$.

If $f \in C^p_\Phi(\mathscr{U},\mathscr{S}^q)$ $(q > O)$ and $d^{q+1}f = O$, then $(d^{q+1}f)(U_o,\ldots,U_p) = O$ in each $U_\sigma = U_o \cap \cdots \cap U_p$. Since $H^q\Gamma_{\Phi_\sigma}(U_\sigma,\mathscr{S}) = 0(q > 0)$, there is a section $g(U_o,\ldots,U_p) \in \Gamma_{\Phi_\sigma}(U_\sigma,\mathscr{S}^{q-1})$ with $dg(U_o,\ldots,U_p) = f(U_o,\ldots,U_p)$ (choose $g(U_o,\ldots,U_p) = 0$ if $f(U_o,\ldots,U_p) = 0$. There is then a cochain $g \in C^p(\mathscr{U},\mathscr{S}^{q-1}))$ with $dg = f$, (see p.57). Since $f \in C^o_\Phi(\mathscr{U},\mathscr{S}^q)$, supp f is contained in a set belonging to Φ and **173**
hence $f(\sigma)$ is different from zero on fewer than m sets U_σ. Then $g(\sigma)$ is different from zero on fewer than m sets U_σ and hence supp g is the union of fewer than m set $\{x \in U_\sigma : g(\sigma)(x) \neq 0$, each of which is in Φ_σ and hence has its closure in Φ. Hence supp g is contained in a set belonging to Φ and $g \in C^p_\Phi(\mathscr{U},\mathscr{S}^{q-1})$. Hence $H^{p,q}_2C_\phi(\mathscr{U},\mathscr{S}) = 0(q > 0)$.

Since the sequences

$$0 \to C^p(\mathscr{U},\mathscr{G}) \xrightarrow{e} C^p(\mathscr{U},\mathscr{S}^o) \xrightarrow{d^1} C^p(\mathscr{U},\mathscr{S}^1)$$

is exact, if $f \in C^p_\Phi(\mathscr{U},\mathscr{S}^O)$ and $d^1f = 0$, then $f = e(g)$ for some $g \in C^p(\mathscr{U},\mathscr{G})$ and clearly $g \in C^p_\Phi(\mathscr{U},\mathscr{G})$. Thus

$$H^{p,o}_2C_\Phi(\mathscr{U},\mathscr{S}) \approx C^p_\Phi(\mathscr{U},\mathscr{G}) \text{ and } H^{p,0}_{21}C_\Phi(\mathscr{U},\mathscr{S}) \approx H^p_\Phi(\mathscr{U},\mathscr{G}).$$

Thus we have the isomorphism indicated below:

$$\begin{array}{ccccc}
H^pC_\Phi(\mathscr{U},\mathscr{S}) & \xrightarrow{\approx} & H_{21}^{p,0}C_\Phi(\mathscr{U},\mathscr{S}) & \xrightarrow{\approx} & H^p_\Phi(\mathscr{U},\mathscr{G}) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \tau_{\mathscr{U}}} \\
H^pC_\Phi(\mathscr{S}) & \xrightarrow{\approx} & H_{21}^{p,0}C_\Phi(\mathscr{S}) & \xrightarrow{\approx} & H^p_\Phi(X,\mathscr{G}).
\end{array}$$

Combining this diagram with the previous one, we see that the homomorphism $\tau : H^p_\Phi(\mathscr{U},\mathscr{G}) \to H^p_\Phi(X,\mathscr{G})$ is an isomorphism. Q.e.d

In particular, we have proved the following proposition (Cartan Seminar, 1953-54, Expose 17, p.7).

174 **Proposition 20.** *If $\mathscr{U}$ is a locally finite proper covering of a paracompact normal space X by open F_σ sets, and if $\mathscr{G}$ is a sheaf of A-modules such that $H^q(U_\sigma,\mathscr{G}) = O(q > O)$ for every $U_\sigma = U_o \cap \cdots \cap U_k(k = O, 1, \ldots)$, then*

$$\tau_{\mathscr{U}} : H^p(\mathscr{U},\mathscr{G}) \to H^p(X,\mathscr{G})$$

is an isomorphism.

In the case that Φ is the family of all compact sets of X, we write H^p_* instead of H^p_Φ.

Proposition 20 (-a). *If $\mathscr{U}$ is a locally finite proper covering of a locally compact and paracompact Hausdorff space by open F_σ sets with compact closures, and if $\mathscr{G}$ is a sheaf of A-modules such that $H^p(U_\sigma,\mathscr{G}) = O(q > O)$ for every*

$$U_\sigma = U_o \cap \cdots \cap U_k(k = 0, 1, \ldots)$$

then

$$\tau_\sigma : H^p_*(\mathscr{U},\mathscr{G}) \to H^p_*(X,\mathscr{G})$$

is an isomorphism.

Note. It is no restriction to assume that $\mathscr{U}$ is a proper covering. If $\mathscr{W}$ is any covering, there is an equivalent proper covering $\mathscr{U}$ with the open sets. Then $\tau_{\mathscr{W}\mathscr{U}}$ and $\tau_{\mathscr{U},\mathscr{W}}$ are isomorphisms.

Lecture 28

Direct sum of modules

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Definition. *The* direct sum *of a system $\{M^i\}_{i\in I}$ of A-modules is an A-module whose elements are systems $\{m^i\}_{i\in I}$, usually written as formal sums $\sum_i m^i$ where $m^i \in M^i$ and $m^i = Q$ for all but a finite number of i. The operations in M are defined by*

$$\sum_i m_1^i + \sum_i m_2^i = \sum_i (m_1^i + m_2^i), \qquad a\sum_i m^i = \sum_i am^i,$$

where $\sum_i m_1^i \in M$, $\sum_i m_2^i \in M$, $\sum_i m^i \in M$ and $a \in A$.

Clearly there is a homomorphism $p^j : \sum_i M^i \to M^j$ defined by $p^j(\sum_i m^i) = m^j$ and a homomorphism $h^j : M^j \to \sum_i M^i$ defined by

$$p^i h^j(m^j) = 0 \text{ for } i \neq j, p^j h^j(m^j) = m^j, m^j \in M^j.$$

A system of homomorphism $(g', g^i) : (A, M^i) \to (B, N^i)$, $i \in I$, induces a homomorphism $(g', g) : (A, \sum_i M^i) \to (B, \sum_i N^i)$ where $g(\sum_i m^i) = \sum_i g^i(m^i)$. There is commutativity in

$$\begin{tikzcd}
(A, M^j) \arrow[r, "h^j"] \arrow[d, "{(g', g^j)}"'] & (A, \sum_i M^i) \arrow[r, "p^j"] \arrow[d, "{(g', g)}"'] & (A, M^j) \arrow[d, "{(g', g^j)}"'] \\
(B, N^j) \arrow[r, "h^j"] & (B, \sum_i N^i) \arrow[r, "p^j"] & (B, N^j).
\end{tikzcd}$$

Proof. Let $\sum_i m_1^i \in \sum_i M^i$, $\sum_i m_2^i \in \sum_i M^i$, $\sum_i m^i \in \sum_i M^i$ and let $a \in A$. Then

$$g(\sum_i m_1^i + \sum_i m_2^i) = g(\sum_i (m_1^i + m_2^i) = \sum_i g^i(m_1^i + m_2^i)$$

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$$\begin{aligned}
&= \sum_i (g^i(m_1^i) + g^i(m_2^i)) = \sum_i g^i(m_1^i) + \sum_i g^i(m_2^i)\\
&= g(\sum_i m_1^i) + g(\sum_i m_2^i)\\
g(a\sum_i m^i) &= g(\sum_i am^i) = \sum_i g^i(am^i)\\
&\sum_i g'(a)g^i(m^i) = g'(a)\sum_i g^i(m^i)\\
&= g'(a)g(\sum_i m^i).\\
g^i p^j(\sum_i m^i) &= g^j m^j = p^j(\sum_i g^i m^i)\\
&= p^j g(\sum_i m^i)\\
p^k g h^j(m^j) &= g^k p^k h^j(m^j) = g^k(0) = 0 \text{ for } k \neq j
\end{aligned}$$

and

$$p^j g h^j(m^j) = g^j p^j h^j(m^j) = g^j(m^j),$$

hence

$$g h^j(m^j) = h^j g^j(m^j), \qquad m^j \in M^j.$$

The operation of forming the direct limit commute with the operation of forming the direct sum. ◻

Proof. Let $\{A_\lambda, M_\lambda^i, \phi'_{\mu\lambda}, \phi^i_{\mu\lambda}\}_{\lambda,\mu\in\Omega}$ (Ω a directed set) be a direct system for each $i \in I$. Let the direct limits be (A, M^i) with homomorphisms $(\phi'_\lambda, \phi^i_\lambda) : (A_\lambda, M^i_\lambda) \longrightarrow (A, M^i)$. There are induced homomorphisms

$$(\phi'_{\mu\lambda}, \phi_{\mu\lambda}) : (A_\lambda, \sum_i M^i_\lambda) \to (A_\mu, \sum_i M^i_\mu)$$

177 for which $\phi_{\nu\mu}\phi_{\mu\lambda} = \phi_{\nu\lambda}, \lambda < \mu < \nu$ and $\phi_{\lambda\lambda}$ is the identity. Thus $\{A_\lambda, \sum_i M^i_\lambda, \phi'_{\mu\lambda}, \phi'_{\mu\lambda}\}_{\lambda\in\Omega}$ is direct system. Let its direct limit be (A, M) with homomorphisms

$$(\phi'_\lambda, \phi_\lambda) : (A_\lambda, \sum_i M^i_\lambda) \to (A, M).$$

$$\begin{tikzcd}
(A_\lambda, M^i_\lambda) \arrow[r, "\phi^i_{\mu\lambda}"] \arrow[d, "h^i_\lambda"'] & (A_\mu, M^i_\mu) \arrow[r, "\phi^i_\mu"] \arrow[d, "h^i_\mu"] & (A, M^i) \arrow[d] \arrow[dd, bend left, "h^i"] \\
(A_\lambda, \sum_i M^i_\lambda) \arrow[r, "\phi_{\mu\lambda}"] \arrow[drr, "\phi''_\lambda"'] & (A_\mu, \sum_i M^i_\mu) \arrow[r, "\phi_\mu"] \arrow[dr, "\phi''_\mu"] & (A, M) \arrow[d, "\phi''"] \\
 & & (A, \sum_i M^i)
\end{tikzcd}$$

The system of homomorphisms $(\phi^i_\lambda, \phi^i_\lambda) : (A_\lambda, M^i_\lambda) \to (A, M^i)$ induces a homomorphism $(\phi'_\lambda, \phi''_\lambda) : (A_\lambda, \sum_i M^i_\lambda) \to (A, \sum_i M^i)$. Since for $\lambda < \mu$,

$$\begin{aligned}
\phi''_\mu\phi_{\mu\lambda}(\sum_i m^i_\lambda) = \phi''_\mu(\sum_i \phi^i_{\mu\lambda}m^i_\lambda) = \sum_i \phi^i_\mu\phi^i_{\mu\lambda}m^i_\lambda \\
= \sum_i \phi^i_\lambda m^i_\lambda = \phi''_\lambda(\sum_i m^i_\lambda),
\end{aligned}$$

therefore $\phi''_\mu\phi_{\mu\lambda} = \phi''_\lambda$ and there is an induced homomorphism $\phi'' : M \to \sum_i M^i$. □

If $\sum_i m^i \in \sum_i M^i$ then $m^i = 0$ except for a finite number of i, say $i_1, \ldots, i_k$. Then, for some λ, each m^{i_j} has a representative $m^{i_j}_\lambda \in M^{i_j}_\lambda$; let $m^i_\lambda = 0$ for $i \notin (i_1, \ldots, i_k)$. Then $\sum_i m^i_\lambda \in \sum_i M^i_\lambda$ and $\sum_i m^i = \phi''_\lambda \sum_i m^i_\lambda =$
$\phi''\phi_\lambda \sum_i m^i_\lambda$. Thus ϕ'' is an epimorphism. 178

To show that Φ'' is a monomorphism, let $m \in M$ and $\phi''(m) = 0$. Choose a representative $\sum_i m^i_\lambda$ of m. Then $m^i_\lambda = 0$ except for a finite number of i, say $i = i_1, \ldots, i_k$. Since

$$0 = \phi''(m) = \phi''_\lambda(\sum_i m^i_\lambda) = \sum_i \phi^i_\lambda m^i_\lambda,$$

therefore each $\phi^{ij}_{\lambda} m^{i_j}_{\lambda} = 0$. Now choose μ so that each $\phi^{i_j}_{\mu\lambda} m^{i_j}_{\lambda} = 0$. Then m is represented by 0 in $\sum_i M^i_\mu$ and hence $m = 0$. Thus ϕ'' is an isomorphism. We identity M with $\sum_i M^i$ under the isomorphism ϕ''.

Direct sum of sheaves

Let $\mathfrak{a}$ *be a sheaf of rings with unit and let* $\{\mathscr{S}^i\}_{i\in I}$ *be a system of sheaves of* $\mathfrak{a}$*-modules. Then there is a unique sheaf* $\mathscr{S}$ *whose stalks are the direct sums* $\sum\limits_{i\in I} S^i_x$ *such that the homomorphisms* $h^j_x : S^j_x \to S_x$ *determine a sheaf homomorphism* $h^j : \mathscr{S}^j \to \mathscr{S}$.

This $\mathscr{S}$ is called the *direct sum* of the sheaves $\mathscr{S}^i$ and is denoted by $\mathscr{S} = \sum\limits_{i\in I} \mathscr{S}^i$.

Proof. Uniqueness. If $s = \sum_i s^i \in \sum_i S^i_k$ with $s^i = 0$ except for $i_1, \ldots, i_k$, choose a neighbourhood U of x and sections $f_j : U \to S^{i_j}$, $j = 1, \ldots, k$ such that $f_j(x) = S^{i_j}$. Let $f : U \to S (= \bigcup\limits_x \sum\limits_{i\in I} S^i_x)$ be defined by $f(y) = \sum_j h^{i_j} f_j(y)$. Since h^{i_j} is a sheaf homomorphism, the composite function

$$U \xrightarrow{f_j} S^{i_j} \xrightarrow{h^i j} S$$

179 is a section and hence f is a section. Since

$$f(x) = \sum_j h^{i_j} f_j(x) = \sum_j h^{i_j} s^{i_j} = \sum_i s^i = s,$$

the section goes through s. Thus, since such section cover S, they uniquely determine the topology of S. ◻

Existence From the presheaves of sections $(\overline{a}, \overline{\mathscr{S}^i}) = \{A_U, S^i_U, \rho'_{VU}, \rho^i_{VU}\}$ where $A_U = \Gamma(U, a)$ and $S^i_U = \Gamma(U, \mathscr{S}^i)$. Then $\{A_U, \sum_i S^i_U, \rho'_{VU}, \rho_{VU}\}$ (where U runs through the directed set of all the open sets of X) is a presheaf. It determines a sheaf $(a, \mathscr{S})$ and there is a homomorphism

$h^j_U : S^j_U \to \sum_i S^i_U$ with commutativity in

$$\begin{tikzcd}
(A_U, S^j_U) \arrow[r, "h^j_U"] \arrow[d, "(\rho'_{VU}, \rho^j_{VU})"'] & (A_U, \sum_i S^i_U) \arrow[d, "(\rho'_{VU}, \rho_{VU})"] \\
(A_V, S^j_V) \arrow[r, "h^j_V"] & (A_V, \sum_i S^i_V).
\end{tikzcd}$$

Hence there is an induced sheaf homomorphism $h^j : \mathscr{S}^j \to \mathscr{S}$. **180**
The stalk (A_x, S_x) of $(\mathfrak{a}, \mathscr{S})$ is the direct limit of the direct system $\{A_U, \sum_i S^i_U, \rho'_{VU}, \rho_{VU}\}_{x \in U}$ which is identified with the direct sum of the direct limits $(A_x, S^i_x) = \operatorname{dir\,lim}\{A_U, S^i_U, \rho'_{VU}, \rho^i_{VU}\}_{x \in U}$. Thus $S_x = \sum_i S^i_x$ and the required sheaf exists, *q.e.d*.

There is also a homomorphism $p^i_U : \sum_i S^i_U \to S^j_U$ (as defined in the beginning of the lecture) with commutativity in

$$\begin{tikzcd}
(A_U, \sum_i S^i_U) \arrow[r, "p^j_U"] \arrow[d, "(\rho'_{VU}, \rho_{VU})"'] & (A_U, S^j_U) \arrow[d, "(\rho'_{VU}, \rho^j_{VU})"] \\
(A_V, \sum_i S^i_V) \arrow[r, "p^j_V"] & (A_V, S^j_V)
\end{tikzcd}$$

Hence there is a sheaf homomorphism $p^j : \mathscr{S} \to \mathscr{S}^j$ which is clearly onto. If $s = \sum_i s^i \in S$, it is easily verified that $p^j(s) = s^j$, hence $p^j h^j(s^j) = s^j$ and $p^j h^i(s^j) = 0$ if $i \neq j$.

Lecture 29

Given homomorphisms $g_i : \mathscr{S}^i \to \mathscr{T} (i \in I)$, there is an induced homomorphism $g : \sum_{i \in I} \mathscr{S}^i \to \mathscr{T}$ with $gh^i = g_i$. 181

Proof. If $s = \sum_i s^i \in S_x$, let $g(s) = \sum_{i g_i}(s^i) \in T_x$.Then $g|S_x : S_x \to T_x$ is clearly a homomorphism. Choose an open set U and sections $f_j : U \to S^{i_j}$ so that the section defined by $f(y) = \sum h^{i_j} f(y)$ goes through s. Then $gf(y) = \sum gi_j f_j(y)$ and gf being the sum of a finite number of sections, is a section. Thus g is continuous and is a sheaf homomorphism. □

Note. Since $\mathfrak{a}$ itself is a sheaf of $\mathfrak{a}$-modules, there is, for any I, a direct sum $\sum_{i \in I} \mathfrak{a}$ where each direct summand is $\mathfrak{a}$. This is again a sheaf of $\mathfrak{a}$ -modules.

If $\mathscr{S}$ is a sheaf of $\mathfrak{a}$-modules over a space X, and $\mathfrak{a}_Y$ is the restriction of $\mathfrak{a}$ to a subset Y, then clearly the restriction $\mathscr{S}_Y$ of $\mathscr{S}$ to Y is a sheaf of $\mathfrak{a}_Y$-modules.

Definition. *The following properties of sheaves $\mathscr{S}$ of $\mathfrak{a}$- modules over a space X are called property (a_1) and property (a) :*

Property (a_1). There is a covering $\{U_j\}_{j \in J}$ of X, and for each $j \in J$, there is an index set I_j and an epimorphism $\phi_j : \sum_{i \in I_j} \mathfrak{a}_{U_j} \to \mathscr{S}_{U_j}$.

Property (a). There is a covering $\{U_j\}_{j \in J}$ of X, and for each $j \in J$, there is a natural number k_j and an epimorphism $\phi_j : \sum_{i=1}^{k_j} \mathfrak{a}_{U_j} \to \mathscr{S}_{U_j}$. 182

$\mathscr{S}$ has property (a_1) if and only if each point $x \in X$ has a neighbourhood U such that the sections in $\Gamma(U, \mathscr{S})$ generate $\mathscr{S}_U$. (That is, for each $y \in U$ and $s \in S_y$, $s = \sum_{j=1}^{k} a_y^j f_j(y)$ for some finite number k of elements $a_j^j \in A_y$ and sections $f_j \in \Gamma(U, \mathscr{S})$.)

Proof. Necessity. Let $\mathscr{S}$ have property (a_1). Since $\{U_j\}$ is a covering, $x \in U_j$ for some j; let $U = U_j$. Then there is an index set I and an epimorphism $\phi : \sum\limits_{i\in I} \mathfrak{a}_U \to \mathscr{S}_U$. ◻

Let $f^i = \phi h^i 1 : U \to S$ where 1 is the unit section in $\mathfrak{a}_U$,

$$U \xrightarrow{1} \mathfrak{a}_U \xrightarrow{h^i} \sum_{i\in I} \mathfrak{a}_U \xrightarrow{\phi} \mathscr{S}_U.$$

Then if $s \in \mathscr{S}_U$, $s = \phi(\sum_i a_y^i)$ for some $a_y^i \in A_y$ with $a_y^i = O_y$ except for a finite number of i, say $i = i_1, i_2, \ldots, i_k$. Then

$$\begin{aligned}\sum_{j=1}^{k} a_y^{i_j} f_{i_j}(y) &= \sum_{j=1}^{k} a_y^{i_j} \phi h^{i_j} 1_y = \sum_{j=1}^{k} \phi h^{i_j} a_y^{i_j} 1_y \\ &= \phi(\sum_{j=1}^{k} h^{i_j} a_y^{i_j}) = \phi(\sum_{i\in I} a_y^i) = s.\end{aligned}$$

Sufficiency. For each $x \in X$, there is a neighbourhood U_x of x such that the sections over U_x generate $\mathscr{S}_{U_x}$. Then $\{U_x\}_{x\in X}$ is a covering of X. Let I_x be the set of sections $\Gamma(U_x, \mathscr{S})$. For each section $i \in I_x$, there is a sheaf homomorphism $\phi_x^i : a_{U_x} \to \mathscr{S}_{U_x}$ given, for $a \in A_y$, by $\phi_x^i(a) = a \cdot i(\pi a)$.

183 Then there is an induced homomorphism

$$\phi_x : \sum_{i\varepsilon I_x} a_{U_x} \to \mathscr{S}_{U_x}.$$

Then for $s \in S_y$ and $y \in U_x$,

$$s = \sum_{j=1}^{k} a_y^j i_j(y) = \sum_{j=1}^{k} \phi_x^{i_j}(a_y^j) \in \phi_x(\sum_{i\in I_x} a_{U_x}).$$

Thus ϕ_x is an epimorphism.

$\mathscr{S}$ has property (a) if and only if each point x has a neighbourhood U such that some finite number of sections $f_i \in \Gamma(U, \mathscr{S})$ $(i = 1, \ldots, k)$ generate $\mathscr{S}_U$.

Proof. Similar to the proof given above. □

It is clear that (a) implies (a_1), i.e., each sheaf with property (a) has property (a_1). The sheaf $\sum\limits_{i \in I} \mathfrak{a}$ has property (a_1) and $\sum_{i=1}^{k} \mathfrak{a}$ has property (a). In particular, the sheaf $\mathfrak{a}$ of $\mathfrak{a}$-modules has property (a).

If $\mathscr{S}_i, i = 1, \ldots, k$ are sheaves of $\mathfrak{a}$- modules with property (a_1) (resp. (a)), then the direct sum $\sum_{i=1}^{k} \mathfrak{a}_i$ has property (a_1) (resp (a)).

Proof. Clear.

Statements (1), (2), (3), (4), (5), (6), (Lectures 29, 30, 31,) are required to prove Serre's theorem on coherent sheaves, (see the next lecture for the definition of coherent sheaves), i.e., if $0 \to \mathscr{S}' \to \mathscr{S} \to \mathscr{S}'' \to 0$ is an exact sequence of sheaves, and if two of them are coherent, then the third is also coherent. **184**

(1) *If $f : \mathscr{S}' \to \mathscr{S}$ is an epimorphism and $\mathscr{S}'$ has property (a_1) (resp (a)), then $\mathscr{S}$ has property (a_1)* (resp (a)). □

Proof. Clear. □

Example . If M is a finitely generated A- module, the constant sheaf M has property (a) with respect to the constant sheaf A. If a constant sheaf is a sheaf of $\mathfrak{a}$- modules, then it has property (a_1). If X is the unit segment $0 \leq x \leq 1$, the subsheaf $\mathscr{S}$ of the constant sheaf Z_2 obtained by omitting (1, 1) does not have property (a_1) either as a sheaf of Z-modules or as a sheaf of Z_2 -modules. With the same X, the sheaf $\mathscr{S}$ of germs of functions $f : X \to Z_2$, considered as a sheaf of Z_2 -modules, has property (a_1) but not property (a). But, considering $\mathscr{S}$ as a sheaf of rings with unit, it has property (a) with respect to itself, The sheaf $\mathfrak{a}$ of germs of analytic functions in the complex plane has property (a) as a sheaf of $\mathfrak{a}$- modules, but as a sheaf of C- modules (where C is the field of complex numbers) it does not even have property (a_1). For, there are natural boundaries for analytic functions, e.g., let f be an analytic function in $|z| < 1$, with $|z| = 1$ as natural boundary. If $\mathfrak{a}$ has property (a_1) as a sheaf of C-modules, then by considering a point on the boundary, we see that f can be continued to a neighbourhood of this boundary point and this is a contradiction.

Lecture 30

Definition. *The following properties of sheaves $\mathscr{S}$ of $\mathfrak{a}$- modules over a space X are called property (b_1) and property (b).* **185**

Property (b_1). *For each open set U of X and each homomorphism $f : \sum\limits_{i \in I} \mathfrak{a}_U \to \mathscr{S}_U$, ker f has property (a_1) as a sheaf of $\mathfrak{a}_U$- modules.*

Property (b). *For each open set U of X and each homomorphism $f : \sum_{i=1}^{k} a_U \to \mathscr{S}_U$,* ker f *has property (a) as a sheaf of a_U- modules.*

Note. Since (a_1) and (a) are local properties, properties (b_1) and (b) are equivalent to the following properties (b'_1) and (b') respectively.

Property (b'_1). *For each neighbourhood V of each point $x \in X$, there exists an open set U, $x \in U \subset V$, such that for each homomorphism $f : \sum\limits_{i \in I} \mathfrak{a}_U \to \mathscr{S}_U$,* ker f *has property (a_1) as a sheaf of $\mathfrak{a}_U$- modules.*

Property (b'). For each neighbourhood V of each point $x \in X$, there exists an open set U, $x \in U \subset V$ such that for each homomorphism $f : \sum\limits_{i=1}^{k} \mathfrak{a}_U \to \mathscr{S}_U$, ker f has property (a) as a sheaf of $\mathfrak{a}_U$-modules.

Thus (b_1) and (b) are also local properties. The sheaf ker f is called the *sheaf of relations* between the sections $f_i : U \to \mathscr{S}_U$, where $f_i = fh^i 1$, **186**

$$U \xrightarrow{1} \mathfrak{a}_U \xrightarrow{h^i} \sum_{i \in I} \mathfrak{a}_U \xrightarrow{f} \mathscr{S}_U.$$

The sheaf of relations between the sections f_i is described by the following result.

The element of $(\ker f)_x$ *for* $x \in U$ *are the elements* $\sum_i a_i$ *of which* $\sum_i a_i f_i(x) = 0$.

Proof. For each element $\sum a_i$, only a finite number of the a_i being different from zero, we have

$$\sum a_i = \sum_{j=1}^{q} h^{i_j} a_{i_j} \text{ and}$$

$$\begin{aligned} f(\sum a_i) &= f(\sum_{j=1}^{q} h^{i_j} a_{i_j}) = \sum_{j=1}^{q} f h^{i_j}(a_{i_j}) \\ &= \sum_{j=1}^{q} a_{i_j} f h^{i_j}(l_x) = \sum_{j=1}^{q} a_{i_j} f_{i_j}(x) \\ &= \sum_i a_i f_i(x), \end{aligned}$$

and this completes the proof. □

If we start with a system $\{f_i\}_{i\in I}$ of sections of $\mathscr{S}$ over an open set U, then each f_i defines a homomorphism (again denoted by f_i) $f_i : \mathfrak{a}_U \to \mathscr{S}_U$ where $f_i(a) = a \cdot f_i(x)$, $a \in A_x$. Then the system $\{f_i\}$ of homomorphisms defines a homomorphism $f : \sum_{i\in I} \mathfrak{a}_U \to \mathscr{S}_U$, and the sheaf $\ker f$ is called the *sheaf of relations* between the sections $f_i : U \to \mathscr{S}_U$.

187 (2) *If* $\mathscr{S}$ has property (b_1) (resp (b)), then every subsheaf of $\mathscr{S}$ has property (b_1) (resp (b)).

Proof. Clear. □

Definition. *A sheaf* $\mathscr{S}$ *of* $\mathfrak{a}$- *modules is called coherent if it has properties* (a) *and* (b).

Note. If $\mathscr{S}$ is a coherent sheaf, then $\mathscr{S}_U$ is coherent for each open set U. Coherence is a local property, i.e., if each point has a neighbourhood U such that $\mathscr{S}_U$ is coherent, then $\mathscr{S}$ is coherent. If we define a sheaf $\mathscr{S}$ to be of *finite type* if, for each $x \in X$, there is an open set U, $x \in U$, such that each stalk of $\mathscr{S}_U$ is generated by the same *finite* number of sections

$f_1, \ldots, f_k$ over U; it is then easily verified that conditions (a) and (b) for coherence are equivalent to the following conditions:

(i) The sheaf $\mathscr{S}$ is of finite type.

(ii) If $f_1, \ldots, f_k$ are any finite number of sections of $\mathscr{S}$ over any open set U, then the sheaf of relations between these finite number of sections is of finite type.

Example 30. Let $\mathfrak{a}$ be the constant sheaf Z_2 on $0 \leq x \leq 1$, let $\mathscr{R}$ be the subsheaf obtained by omitting (1, 1) and let $\mathscr{S}$ be the quotient sheaf with stalks Z_2 at 1 and zero elsewhere. Then the natural homomorphism $\mathfrak{a} \to \mathscr{S}$ has kernel $\mathscr{R}$ and $\mathscr{R}$ does not have property (a_1). Hence $\mathscr{S}$ has neither (b_1) nor (b).

Example 31. Let A be the ring of Example 5, with elements 0, 1, b, c, **188**
such that $b^2 = b$, $c^2 = c$, $bc = cb = 0$. Let $\mathfrak{a}$ be the subsheaf of the constant sheaf A on $0 \leq x \leq 1$ obtained by omitting $(1, b)$ and $(1, c)$. Let $\mathscr{S}$ be the constant sheaf Z_2 on which 1 and c of A operate as the identity and b operates as zero. The sheaf $\mathscr{R}$ of relations for the section 1 of Z_2 consists of 0 and (x, b) for $x < 1$. If U is a connected neighbourhood of 1, the only homomorphism of $\mathfrak{a}_U$ into $\mathscr{R}_U$ is the zero homomorphism. Thus $\mathscr{R}$ does not have (a_1) and $\mathscr{S}$ has neither (b_1) nor (b).

Example 32. Let A be the ring $Z[y_1, y_2, \ldots]$ of polynomials in infinitely many variables with integer coefficients. Let $\mathfrak{a}$ be the constant sheaf A on $0 \leq x \leq 1$ and let $\mathscr{S}$ be the constant sheaf Z on which all the indeterminates $y_1, y_2, \ldots$, operate as zero. The sheaf of relations for the section 1 of Z is the constant sheaf formed by the ideal of all polynomials without constant terms. This ideal is not finitely generated, hence $\mathscr{S}$ does not have (b). However, for every homomorphism $f : \sum\limits_{i \in I} \mathfrak{a}_U \to \mathscr{S}_U$, ker f is constant on each component of U and hence has property (a_1). Thus $\mathscr{S}$ has (b_1) but not (b).

(3) *If* $0 \to \mathscr{S}' \xrightarrow{f} \mathscr{S} \xrightarrow{g} \mathscr{S}'' \to 0$ *is an exact sequence of sheaves of* $\mathfrak{a}$ *- modules such that* $\mathscr{S}$ *has* (a_1) *(resp* (a)*) and* $\mathscr{S}''$ *has* (b_1) *(resp* (b)*), then* $\mathscr{S}'$ *has* (a_1) *(resp* (a)*).*

189 *Proof.* If $\mathscr{S}$ has (a_1), each $x \in X$ has a neighbourhood U for which there is an epimorphism $\phi : \sum\limits_{i\in I} \mathfrak{a}_U \to \mathscr{S}_U$. Since $g\phi : \sum\limits_{i\in I} \mathfrak{a}_U \to \mathscr{S}''_U$ is a homomorphism and $\mathscr{S}''$ has (b_1), $\ker g\phi$ has (a_1). Hence for some open set V with $x \in V \subset U$, there is a homomorphism

$$\psi : \sum_{j\in J} \mathfrak{a}_V \to \sum_{i\in I} \mathfrak{a}_V$$

such that $\operatorname{im}\psi = (\ker g\phi)_V$. Hence, since ϕ is an epimorphism $\operatorname{im}\phi\psi = (\ker g)_V$, and therefore $\operatorname{im}\phi\psi = (\operatorname{im} f)_V$. Then, since f is a monomorphism, there is an epimorphism

$$f^{-1}\phi\psi : \sum_{j\in J} \mathfrak{a}_V \to \mathscr{S}'_V.$$

Thus $\mathscr{S}'$ has (a_1). Similarly if $\mathscr{S}$ has (a) and $\mathscr{S}''$ has (b), then $\mathscr{S}'$ has (a). $\square$

$$\begin{tikzcd}
 & \sum\limits_{j\in J}\mathfrak{a}_V \arrow[r, "\psi"] & \sum\limits_{i\in I}\mathfrak{a}_V \arrow[d, "\phi"'] \arrow[dr, "g\phi"] & & \\
0 \arrow[r] & \mathscr{S}' \arrow[r, "f"] & \mathscr{S} \arrow[r, "g"] & \mathscr{S}'' \arrow[r] & 0
\end{tikzcd}$$

Q.e.d.

(4) *If* $0 \to \mathscr{S}' \xrightarrow{f} \mathscr{S} \xrightarrow{g} \mathscr{S}'' \longrightarrow 0$ *is an exact sequence of sheaves of* $\mathfrak{a}$*- modules such that* $\mathscr{S}'$ *and* $\mathscr{S}''$ *have* (b_1) *(resp* (b)*) then* $\mathscr{S}$ *also has* (b_1) (resp (b)).

Proof. Let $\mathscr{S}'$ and $\mathscr{S}''$ have (b_1) and let $\phi : \sum\limits_{i\in I} \mathfrak{a}_U \to \mathscr{S}_U$ be a given
190 homomorphism. Since $\mathscr{S}''$ has (b_1), $\ker g\phi$ has (a_1) and hence for each $x \in U$, there is an open set V with $x \in V \subset U$ and a homomorphism,

$$\psi : \sum_{j\in J} \mathfrak{a}_V \to \sum_{i\in I} \mathfrak{a}_V$$

such that $\operatorname{im}\psi = (\ker g\phi)_V$. Then $\operatorname{im}\phi\psi \subset (\ker g)_V = (\operatorname{im} f)_V$ and hence there is a homomorphism

$$\theta : \sum_{j\in J} \mathfrak{a}_V \to \mathscr{S}'_V$$

with $f\theta = \phi\psi$. Since $\mathscr{S}'$ ahs (b_1), $\ker\theta$ has (a_1) and there is an open set W with $x \in W \subset V$, and a homomorphism

$$\eta : \sum_{k\in K} \mathfrak{a}_W \to \sum_{j\in J} \mathfrak{a}_W$$

such that $\operatorname{im}\eta = (\ker\theta)_W$. □

$$\begin{tikzcd}
 & & & & \sum\limits_{k\in K} \mathfrak{a}_W \arrow[dll, "\eta"'] \arrow[ddl, "\psi\eta"] & \\
 & & \sum\limits_{j\in J} \mathfrak{a}_V \arrow[dr, "\psi"] \arrow[ddl, "\theta"'] & & & \\
 & & & \sum\limits_{i\in I} \mathfrak{a}_U \arrow[dl, "\phi"'] \arrow[dr, "g\phi"] & & \\
0 \arrow[r] & \mathscr{S}' \arrow[r, "f"] & \mathscr{S} \arrow[rr, "g"] & & \mathscr{S}'' \arrow[r] & 0
\end{tikzcd}$$

To show that $\mathscr{S}$ has (b_1) it is enough to show that $\operatorname{im}\psi\eta = (\ker\phi)_W$. For any element $r \in \sum\limits_{k\in K} a_W$,

$$\phi\psi\eta(r) = f\phi\eta(r) = f(0) = 0.$$

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Thus $\operatorname{im}\psi\eta \subset (\ker\phi)_W$. Next, for any element $p \in (\ker\phi)_W$, we have $p \in (\ker g\phi)_W = (\operatorname{im}\psi)_W$. choose $q \in \sum\limits_{j\in J} a_W$ such that $\psi(q) = p$; then

$$\theta(q) = f^{-1}\phi\psi(q) = f^{-1}\phi(p) = f^{-1}(0) = o.$$

Thus $q \in (\ker\theta)_W = \operatorname{im}\eta$, and hence $p = \psi(q) \in \operatorname{im}\psi\eta$. Thus $\operatorname{im}\psi\eta = (\ker\phi)_W$ and $\mathscr{S}$ has (b_1). Similarly if $\mathscr{S}'$ and $\mathscr{S}''$ have (b), it can be proved that $\mathscr{S}$ also has (b).

Lecture 31

If $f : \mathscr{S} \to \mathscr{S}''$ is an epimorphism, $\phi : \sum_{i=1}^{k} \mathfrak{a}_U \to \mathscr{S}''_U$ a homomorphism, and if x is a point of the open set U, then there exists an open set V with $x \in V \subset U$ and a homomorphism 192

$$\psi : \sum_{i=1}^{k} a_V \to \mathscr{S}_V \text{ such that } f\psi = \phi| \sum_{i=1}^{k} a_V.$$

Proof. Since $f : \mathscr{S} \to \mathscr{S}''$ is an epimorphism, each of the sections $\theta_i = \phi h^i 1 : U \to \mathscr{S}''$, $i = 1, \ldots, k$, is locally the image of a section in $\mathscr{S}$. Hence there is an open set V_i with $x \in V_i \subset U$ and a section $\eta_i : V_i \to \mathscr{S}$ such that $f\eta_i\ \theta_i|V_i$. Let $\psi_i : \mathfrak{a}_{V_i} \to \mathscr{S}_{V_i}$ be the homomorphism defined by $\psi_i(a) = a \cdot \eta_i \pi(a)$ These homomorphisms ψ_i induce a homomorphism

$$\psi : \sum_{i=1}^{k} \mathfrak{a}_V \to \mathscr{S}_V$$

where $V = \bigcap_{i=1}^{k} V_i$. Then, if $\sum_{i=1}^{k} a_i \in \sum_{i=1}^{k} \mathfrak{a}_V$, we have

$$\begin{aligned} f\psi(\sum_{i=1}^{k} a_i) = f(\sum_{i=1}^{k} a_i \cdot \eta_i \pi(a_i)) &= \sum_{i=1}^{k} a_i \cdot \theta_i \cdot \pi(a_i) \\ &= \sum_{i=1}^{k} a_i \cdot \phi h^i 1 \pi(a_i) = \sum_{i=1}^{k} \phi h^i a_i \\ &= \phi(\sum_{i=1}^{k} a_i), \end{aligned}$$

i.e., $f\psi = \phi| \sum_{i=1}^{k} \mathfrak{a}_V$. □

193 (5) *If* $0 \to \mathscr{S}' \xrightarrow{f} \mathscr{S} \xrightarrow{g} \mathscr{S}'' \to 0$ *is an exact sequence of sheaves of* $\mathfrak{a}$- *modules such that* $\mathscr{S}'$ *has property* (a) *and* $\mathscr{S}$ *has property* (b)*, then* $\mathscr{S}''$ *has property* (b)*.*

Proof. Let U be an open set and let $\phi : \sum_{i=1}^{k} \mathfrak{a}_U \to \mathscr{S}''_U$ be a homomorphism. Since $g : \mathscr{S} \to \mathscr{S}''$ is an epimorphism, by the result proved above, if $x \in U$, there is an open set V with $x \in V \subset U$ and a homomorphism $\psi : \sum_{i=1}^{k} \mathfrak{a}_V \to \mathscr{S}_V$ such that $g\psi = \phi | \sum_{i=1}^{k} \mathfrak{a}_V$. Since $\mathscr{S}'$ has property (a), there is an open set W with $x \in W \subset V$ and an epimorphism $\eta : \sum_{i=k+1}^{1} \mathfrak{a}_W \to \mathscr{S}'_W$. Then ψ and $f\eta$ induce a homomorphism $\theta : \sum_{i=1}^{1} \mathfrak{a}_W \to \mathscr{S}_W$. □

We also have the projection homomorphisms $p : \sum_{i=1}^{l} \mathfrak{a}_W \to \sum_{i=1}^{k} \mathfrak{a}_W$ and $p' : \sum_{i=1}^{1} \mathfrak{a}_W \to \sum_{i=k+1}^{l} \mathfrak{a}_W$ such that $\theta = \psi p + f\eta p'$.

$$\begin{tikzcd}
 & \sum_{i=k+1}^{l} \mathfrak{a}_W \arrow[d, "\eta"'] \arrow[dr, "f\eta"] & \sum_{i=1}^{l} \mathfrak{a}_W \arrow[l, "p'"'] \arrow[r, "p"] \arrow[d, "\theta"] & \sum_{i=1}^{k} \mathfrak{a}_W \arrow[d, "\phi"] \arrow[dl, "\psi"'] & \\
0 \arrow[r] & \mathscr{S}'_W \arrow[r, "f"] & \mathscr{S}_W \arrow[r, "g"] & \mathscr{S}''_W \arrow[r] & 0
\end{tikzcd}$$

The second square forms a commutative diagram since $g\theta = g\psi p + gf\eta p' = g\psi p = \phi p$ and hence p maps $\ker\theta$ into $\ker\phi$. Actually, p maps $\ker\theta$ *onto* $\ker\phi$, for, if $a \in \ker\phi$, then $g\psi a = \phi a = 0$ and by exactness there exists $b \in \mathscr{S}'_W$ such that $fb = \psi a$. Since η is an epimorphism, there exists $c \in \sum_{i=k+1}^{l} \mathfrak{a}_W$ such that $\eta c = -b$. Then $\theta(a + c) = \psi a + f\eta c = 0$
194 and $p(a + c) = a$. Thus p maps $\ker\phi$. Since $\mathscr{S}$ has (b) $\ker\theta$ has (a), and since $p|\ker\theta : \ker\theta \to \ker\phi$ is an epimorphism, $\ker\phi$ has (a). Hence $\mathscr{S}''$ has (b).

The corresponding statement, with (a_1) and (b_1) in place of (a) and (b), is not true as the following example shows.

Example 33. Let X be the union of the sequence of circles $C_n = \left\{(x, y) : x^2 + y^2 = x/n\right\}$, $n = 1, 2, \ldots$. let each stalk of $\mathfrak{a}$ be the ring $Z[x_1, x_2, \ldots]$ of polynomials in infinitely many variables, with coefficients in Z, and let $\mathfrak{a}$ be constant except that, on going around the circle C_n, x_n and -x_n interchange.

More precisely, let T_n be the automorphism of the ring $Z[x_1, x_2,$. which interchanges x_n and $-x_n$. If U is open in X and $U \subset C_n$, let A_U be the ring of functions f defined on U and with values in $Z[x_1, x_2, \ldots]$ such that f is constant on each component of U not containing $\left(\frac{1}{n}, 0\right)$ and, if a component W of U contains $\left(\frac{1}{n}, 0\right)$, $f(x, y) = f\left(\frac{1}{n}, 0\right)$ for $(x, y) \in W$ and $y < 0$ and $f(x, y) = T_n f\left(\frac{1}{n}, 0\right)$ for $(x, y) \in W$ and $y > 0$. If U is not contained in any C_n let A_U be the ring of functions, constant on each component of U, with values in

$$Z[x_{n_1}, x_{n_2}, \ldots] \subset Z[x_1, x_2, \ldots]$$

where $n_1, n_2, \ldots$, are those values of n for which $\left(\frac{1}{n}, 0\right) \notin U$.

If $V \subset U$ let $\rho_{VU} : A_U \to A_V$ be given by $\rho_{VU} f = f|V$. Let $\mathfrak{a}$ be the sheaf of rings determined by the presheaf $\{A_U, \rho_{VU}\}$.

Let $\mathscr{I}$ be the sheaf of ideals formed by polynomials with even coef- **195**
ficients, then $\mathscr{I}$ is generated by the section given by the polynomial 2. Let

$$\mathscr{S}'' = \mathfrak{a}/\mathscr{I} = Z_2[x_1, x_2, \ldots],$$

then

$$0 \to \mathscr{I} \to \mathfrak{a} \to \mathscr{S}'' \to 0$$

is exact. Then, as sheaves of $\mathfrak{a}$-modules, $\mathscr{I}$ has properties (a) and (a_1), $\mathfrak{a}$ has (b) and (b_1) and $\mathscr{S}''$ has (b) but *not* (b_1).

(6) *If* $0 \to \mathscr{S}' \xrightarrow{f} \mathscr{S} \xrightarrow{g} \mathscr{S}'' \to 0$ *is an exact sequence of sheaves of* $\mathfrak{a}$*-modules such that* $\mathscr{S}'$ *and* $\mathscr{S}''$ *have property* (a)*, then* $\mathscr{S}$ *has property* (a).

Proof. Since $\mathscr{S}''$ has property (a), for each point x there is a neighbourhood U of x and an epimorphism $\phi : \sum_{i=1}^{k} \mathfrak{a}_U \to \mathscr{S}''_U$. There is an open

set W with $x \in W \subset U$, a homomorphism $\psi : \sum_{i=1}^{k} \mathfrak{a}_W \to \mathscr{S}_W$, an epimorphism $\eta : \sum_{i=k+1}^{k} \mathfrak{a}_W \to \mathscr{S}'_W$ and homomorphisms θ, p, p' as in the previous proof. If $s \in \mathscr{S}_W$, there is some $a \in \sum_{i=1}^{k} \mathfrak{a}_W$ such that $\phi a = gs$. Then $g(s - \psi a) = 0$, and by exactness, for some $b \in \mathscr{S}'_W$, $s - \psi a = fb$. Then for some $c \in \sum_{i=k+1}^{k} \mathfrak{a}_W$, $b = \eta c$ and $s = \psi a + f\eta c = \theta(a + c)$.
196 Hence θ, $\sum_{i=1}^{l} \mathfrak{a}_W \to \mathscr{S}_W$ is an epimorphism, and hence $\mathscr{S}$ has property (a). □

The corresponding statement, with (a_1) in place of (a), is not true as the following example shows.

Example 34. Let $X = \bigcup_{n=1}^{\infty} C_n$ as in Example 33. Let $\mathfrak{a} = Z$ and let $\mathscr{S}_n$ be a sheaf which is locally Z_4, but on going around the circle C_n, 1 and 3 interchange. Let $\mathscr{S}'_n$ be the subsheaf with stalks consisting of 0 and 2; it is the constant sheaf Z_2. Let $\mathscr{S}''_n = \mathscr{S}_n/\mathscr{S}'_n$; this is also Z_2. Then the sequence

$$0 \to \mathscr{S}'_n \to \mathscr{S}_n \to \mathscr{S}''_n \to 0$$

is exact. Let

$$\mathscr{S}' = \Sigma_{n=1}^{\infty} \mathscr{S}'_n, \mathscr{S} = \Sigma_{n=1}^{\infty} \mathscr{S}_n, \mathscr{S}'' = \Sigma_{n=1}^{\infty} \mathscr{S}''_n.$$

Then the sequence

$$0 \to \mathscr{S}' \to \mathscr{S} \to \mathscr{S}'' \to 0$$

is exact. Since $\mathscr{S}'$ and $\mathscr{S}''$ are constant sheaves, they have property (a_1) but $\mathscr{S}$ does not have property (a_1).

Statements (1), (2), (3), (4), (5), (6) (Lectures 29, 30, 31) prove the following proposition due to Serre.

Proposition 21. *If* $0 \to \mathscr{S}' \xrightarrow{f} \mathscr{S} \xrightarrow{g} \mathscr{S} \to 0$ *is an exact sequence of sheaves of* $\mathfrak{a}$*-modules and if two of them are coherent (i.e., possess property (a) and (b)), then the third is also coherent.*

197 **Corollary.** *If* $\mathscr{S}_i$, $i = 1, \ldots, k$ *are coherent sheaves of* $\mathfrak{a}$*- modules, then* $\sum_{i-1}^{k} \mathscr{S}_i$ *is coherent*

Proof. Since the sequence of sheaves

$$0 \to \mathscr{S}_k \to \Sigma_{i-1}^{k} \mathscr{S}_i \to \Sigma_{i=1}^{k-1} \mathscr{S}_i \to 0$$

is exact, the result follows by induction. $\square$

Lecture 32

Let $\mathscr{S}'$ and $\mathscr{S}$ be sheaves of $\mathfrak{a}$- modules and let $\mathscr{S}'$ have property (a). If f and g are two homomorphisms of $\mathscr{S}' \to \mathscr{S}$, the set of points x for which $f|S'_x = g|S'_x$ is an open set. **198**

Proof. Let W be the set of points x for which $f|S'_x = g|S'_x$ and let $x_o \in W$. Since $\mathscr{S}'$ has property (a), there is an epimorphism

$$\phi : \sum_{i=1}^{k} \mathfrak{a}_U \to \mathscr{S'}_U$$

for some neighbourhood U of x_o. Then since $x_o \in W$, $f\phi h^j l_{x_o} = g\phi h^j l_{x_o}$ $(j = 1, \ldots, k)$, and hence for some open set V_j, with $x_0 \in V_j \subset U$,

$$f\phi h^j 1 = g\phi h^j 1 : V_j \to S.$$

$$\begin{tikzcd} V \arrow[r, "1"] & \mathfrak{a}_V \arrow[r, "h^j"] & \sum_{i=1}^{k} \mathfrak{a}_V \arrow[r, "\phi"] & \mathscr{S}'_V \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & \mathscr{S}_V \end{tikzcd}$$

□

Let $V = \bigcap_{j=1}^{k} V_j$. If $\sum_{i-1}^{k} a_i \in \sum_{i-1}^{k} A_i$ with $x \in V$, then

$$\begin{aligned} f\phi(\sum_{i=1}^{k} a_i) = f\phi(\sum_{i=1}^{k} h^i a_i) &= \sum_{i=1}^{k} a_i f\phi h^i 1_x \\ &= \sum_{i=1}^{k} a_i g\phi h^i 1_x = g\phi(\sum_{i=1}^{k} a_i). \end{aligned}$$

Hence $f\phi|\sum_{i=1}^{k}\mathfrak{a}_V = g\phi|\sum_{i=1}^{k}\mathfrak{a}_V$ and, since ϕ is an epimorphism, $f|\mathscr{S}'_V = g|\mathscr{S}'_V$. Thus $x_o \in V \subset W$ with V open. Hence W is an open set.

199 The following result deals with the extension of stalk homomorphisms.

Let $\mathscr{S}', \mathscr{S}$ be sheaves of $\mathfrak{a}$- modules such that $\mathscr{S}'$ is coherent and $\mathscr{S}$ has property (a). At a point $x \in X$, let $f : S'_x \to S_x$ be a homomorphism of A_x modules. Then there is a neighbourhood U of x and a homomorphism $f : \mathscr{S}'_U \to \mathscr{S}_U$ whose restriction to S'_x is f_x.

Proof. Since $\mathscr{S}'$ and $\mathscr{S}$ have property (a), there is a neighbourhood W of x and epimorphisms

$$\phi : \sum_{j=1}^{p}\mathfrak{a}_W \to \mathscr{S}'_W,$$

$$\text{and} \qquad \theta : \sum_{j=1}^{q}\mathfrak{a}_W \to \mathscr{S}_W.$$

For each $j = 1, \ldots, p$, choose an element $\sum_{i=1}^{q} a_i^j \in \sum_{i=1}^{q} A_x$ such that

$$\theta\left(\sum_{i=1}^{q} a_i^j\right) = f_x\phi h^j l'_x$$

and choose sections

$$\eta_j : V_j \to \sum_{i=1}^{q}\mathfrak{a}_{V_j}$$

with $x \in V_j \subset W, V_j$ open, such that $\eta_j(x) = \sum_{i=1}^{q} a_i^j$, $j = 1, \ldots, p$. Let the homomorphism

$$g_j : \mathfrak{a}_{v_j} \to \sum_{i=1}^{q}\mathfrak{a}_{v_j}$$

be defined by $g_j(a) = a \cdot \eta_j(\pi(a))$. Then, for $a \in A_x$,

$$\theta g_j(a) = \theta(a \cdot \eta_j(x)) = a \cdot \theta(\sum_{i=1}^{q} a_i^j)$$

$$= a \cdot f_x \phi h^j 1_x = f_x \phi h^j(a).$$

Then the homomorphisms $\{g_j\} j = 1, \ldots, p$, induce a homomorphism 200

$$g : \sum_{i=1}^{p} \mathfrak{a}_V \to \sum_{i=1}^{q} \mathfrak{a}_V$$

where $V = \cap_{j=1}^{p} V_j$, such that $\theta_g | \sum_{i=1}^{p} A_x = f_x \phi$.

$$\begin{tikzcd}
\sum_{k=1}^{r} \mathfrak{a}_U \arrow[r, "\psi"] & \sum_{j=1}^{p} \mathfrak{a}_U \arrow[r, "\phi"] \arrow[d, "g"'] & \mathscr{S}'_U \arrow[r] \arrow[d, dashed, "f_x \mid f"] & 0 \\
& \sum_{i=1}^{q} \mathfrak{a}_U \arrow[r, "\theta"] & \mathscr{S}_U &
\end{tikzcd}$$

Since $\mathscr{S}'$ has property (b), there is an open set Y with $x \in Y \subset V$, and homomorphism

$$\psi : \sum_{k=1}^{r} \mathfrak{a}_Y \to \sum_{j=1}^{p} \mathfrak{a}_Y$$

such that $\operatorname{im} \psi = ker \phi$. Then

$$\theta_g \psi \left| \sum_{k=1}^{r} A_x = f_x \phi \psi \right| \sum_{k=1}^{r} A_x = 0$$

by exactness. Hence by the previous result, there is an open set U with $x \in U \subset Y$, such that the homomorphism $\theta_g \psi \Big| \sum_{k=1}^{r} \mathfrak{a}_U$ coincides with the zero homomorphism. Therefore θ_g induces a homomorphism

$$f : \left(\sum_{j=1}^{p} \mathfrak{a}_U \right) / \operatorname{im} \psi \to \mathscr{S}_U.$$

We can identify this quotient $(\sum_{j=1}^{p} \mathfrak{a}_U) / \operatorname{im} \psi$ with $\mathscr{S}'_U$ so that ϕ 201
becomes the natural homomorphism; then f is a homomorphism f :

$\mathscr{S'}_U \to \mathscr{S}_U$ with $f\phi = \theta_g \Big| \sum_{j=1}^p \mathfrak{a}_u$. If $s \in S'_x$ since ϕ is an epimorphism there is some element $\sum_{j=1}^p a_j \in \sum_{j=1}^p A_x$ such that $s = \phi(\sum_{j=1}^p a_j)$. Then

$$\begin{aligned} f(s) &= f\phi(\sum_{j=1}^p a_j) = \theta_g(\sum_{j=1}^p a_j) \\ &= f_x\phi(\sum_{j=1}^p a_j) = f_x(s); \end{aligned}$$

i.e., the restriction of f to S'_x is f_x, *q.e.d.* □

Let $Y \subset X$ and either, let x be paracompact and normal and Y closed, or, let X be hereditarily paracompact and normal. Let $\mathscr{S}'$, $\mathscr{S}$ be sheaves of $\mathfrak{a}$- modules over X such that $\mathscr{S}'$ is coherent and $\mathscr{S}$ has property (a). If $f : \mathscr{S'}_y \to \mathscr{S}_y$ is a homomorphism of sheaves of $\mathfrak{a}_y$ modules, there exists an open set U with $Y \subset U$, and a homomorphism $g : \mathscr{S}'_{\mathscr{U}} \to \mathscr{S}_U$ such that $g|S'_y = f$.

Proof. By the previous result, for each point $y \in Y$ there is an open set V_y in X with $x \in V_y$ and a homomorphism $\phi_y : \mathscr{S'}_{v_y} \to \mathscr{S}_{V_y}$ such that $\phi_y \Big| S'_y = f \Big| S'_y$. Then by the first result of this lecture, the set of points of $V_y \cap Y$ at which $\phi_y = f$ is open in Y. Hence there is a set W_y open in X with $y \in W_y \subset V_y$ such that $\phi_y \Big| \mathscr{S'}_{W_y \cap Y} = f \Big| \mathscr{S'}_{W_y \cap Y}$. □

202 We now show that there are systems $\{G_i\}_{i\in I}$ and $\{H_i\}_{i\in I}$ of open sets of X such that, if $G = \bigcup\limits_{i\in I} G_i$,

(i) $\bar{H_i} \cap G \subset G_i$,

(ii) the system $\{G_i\}$ is locally finite in G,

(iii) $Y \subset \bigcup\limits_{i\in I} H_i$,

(iv) each G_i is contained in some W_y.

(1) If X is paracompact and normal and Y is a closed subset of X, then the covering $\{X - Y, W_y\}_{y\in Y}$ of X has a locally finite refinement $\{G_j\}_{j\in j}$, and the covering $\{G_j\}_{j\in J}$ can be shrunk to a covering $\{H_j\}_{j\in J}$ with $\bar{H}_j \subset G_j$. Let $I \subset J$ be the set of indices for which $G_j \cap Y$ is not empty. Then, for $i \in I$, G_i is not contained in $X - Y$ and hence is contained in some W_y. Clearly $Y \subset \bigcup_{i\in I} H_i$ and conditions (i), (ii), (iii), (iv) are satisfied.

(2) If X is hereditarily paracompact and normal, then $G = \bigcup_{y\in Y} W_y$ is paracompact and normal. Then there is a locally finite refinement $\{G_i\}_{i\in I}$ of the covering $\{W_y\}_{y\in Y}$ of G. Since G is normal, the covering $\{G_i\}_{i\in I}$ of G can be shrunk to a covering $\{H_i\}_{i\in I}$ with $\bar{H}_i \cap G \subset G_i$. The sets G_i, H_i being open in G are open in X. Conditions (i), (ii), (iii), (iv) are thus satisfied.

Since G_i is contained in some W_y, there are homomorphisms $\psi_i : \mathscr{S}'_{G_i} \to \mathscr{S}_{G_i}$ such that $\psi_i \Big| \mathscr{S}'_{G_i\cap Y} = f \Big| \mathscr{S}'_{G_i\cap Y}$. Let E_{ij} be the set of **203**
points $x \in \bar{H}_i \cap \bar{H}_j \cap G$ at which $\psi_i | S'_x \neq \psi_j | S'_x$; then E_{ij} is closed in G. Let $E = \bigcup_{i,j} E_{ij}$; it is the union of a locally finite system of closed sets in G, hence is closed in G. Let $U = G - E$, then U is open in G, hence open in X, and $Y \subset U$.

Let $g : \mathscr{S}'_U \to \mathscr{S}_U$ be defined as follows: For $x \in \bar{H}_i \cap U$ let $g|S'_x = \psi_i | S'_x$. This gives a consistent definition of g, and g is continuous on each closed set $S'_{\bar{H}_i\cap U}$ of a locally finite system in S'_U (These closed sets cover S'_U). Thus g is a sheaf homomorphism and $g|\mathscr{S}'_Y = f$.

The above result for the case when X paracompact and normal is more useful in applications. In particular, the above results are when both $\mathscr{S}'$ and $\mathscr{S}$ are coherent sheaves of $\mathfrak{a}$–modules.

Example 35. Let T be the space of ordinal numbers $\leq \omega_1$ with the topology induced by the order, let Q be the space of ordinal numbers $\leq \omega_o$ and let $X = T \times Q$. Then X is compact Hausdorff and hence paracompact normal. Let $Y_1 = (T - (\omega_1)) \times \omega_o$, $Y_2 = \omega_1 \times (Q - (\omega_o))$ and $Y = Y_1 \cup Y_2$. Let $\mathfrak{a} = \mathscr{S}' = \mathscr{S} = Z_2$, then $\mathscr{S}$ is coherent. Let $f : \mathscr{S}_Y \to \mathscr{S}_Y$ be the homomorphism which is the identity on Y_1 and is zero on Y_2, There is no extension of f over an open set containing Y.

Let $Y \subset X$, and let $\mathscr{S}$ be a coherent sheaf of $\mathfrak{a}$ modules over X. Then the restriction $\mathscr{S}_Y$ is a coherent sheaf of $\mathfrak{a}_y$ modules.

204 *Proof.* Since $\mathscr{S}$ has property (a), for each $y \in Y$ there is an open set W of X with $y \in W$ and an epimorphism $\theta : \sum_{i=1}^{p} \mathfrak{a}_W \to \mathscr{S}_W$. Then the restriction $\theta \Big| \sum_{i=1}^{p} \mathfrak{a}_{W\cap Y} : \sum_{i=1}^{p} \mathfrak{a}_{W\cap Y} \to \mathscr{S}_{W\cap Y}$ is an epimorphism. Thus $\mathscr{S}_Y$ has property (a). ◻

To prove that $\mathscr{S}_Y$ has property (b), let $\theta; \sum_{i=1}^{k} \mathfrak{a}_{W\cap Y} \to \mathscr{S}_{W\cap Y}$ be a homomorphism where W is an open set in X, and let $y \in W \cap Y$. Choose a section $f_i : V_i \to S, (i = 1, \ldots, k) y \in V_i$, through $\phi h^i 1_y \in S_y$ and let $V = \bigcap_{i=1}^{k} V_i$. There is a homomorphism $\phi' : \sum_{i=1}^{k} \mathfrak{a}_V \to \mathscr{S}_V$ defined by $\phi' \sum_{i=1}^{k} a_i = \sum_{i=1}^{k} a_i f_i \pi(a_i)$. If $\sum_{i=1}^{k} a_i \in \sum_{i=1}^{k} A_y$, then $\phi' \sum_{i=1}^{k} a_i = \sum_{i=1}^{k} a_i f_i(y) = \sum_{i=1}^{k} a_i \phi h^i 1_y \phi \sum_{i=1}^{k} h^i a_i = \phi \sum_{i=1}^{k} a_i$.

Since the set of points of Y where $\phi' = \phi$ is open in Y, there is an open set G of X with $y \in G \subset V \cap W$, such that $\phi' \Big| \sum_{i=1}^{k} \mathfrak{a}_{G\cap Y} = \phi \Big| \sum_{i=1}^{k} \mathfrak{a}_{G\cap Y}$. Since $\mathscr{S}$ has property (b), there is an open set U with $y \in U \subset G$, and a homomorphism

$$\psi' : \sum_{i=1}^{1} \mathfrak{a}_U \to \sum_{i=1}^{k} \mathfrak{a}_U$$

such that the sequence

$$\sum_{i=1}^{1} \mathfrak{a}_U \xrightarrow{\psi'} \sum_{i=1}^{k} \mathfrak{a}_U \xrightarrow{\phi'} \mathscr{S}_U$$

is exact. Then if $\psi = \psi' \Big| \sum_{i=1}^{1} \mathfrak{a}_{U\cap Y}$, the sequence

$$\sum_{i=1}^{1} \mathfrak{a}_{U\cap Y} \xrightarrow{\psi} \sum_{i=1}^{k} \mathfrak{a}_{U\cap Y} \xrightarrow{\phi} \mathscr{S}_{U\cap Y}$$

is exact. Thus $\mathscr{S}_Y$ has property (b).

Lecture 33

Definition . *A sheaf* $\mathfrak{a}$ *of rings with unit is called a* coherent sheaf of rings *if it is coherent as a sheaf of -modules, i.e., it has property* (b) *(Property (a) is trivially satisfied.)* 205

If $\mathfrak{a}$ *is coherent and* $\mathscr{S}$ *is a sheaf of* $\mathfrak{a}$*- modules, then* $\mathscr{S}$ *is coherent if and only if for each point x there is an open set U with* $x \in U$*, and an exact sequence*

$$\sum_{i=1}^{l} \mathfrak{a}_U \xrightarrow{\psi} \sum_{i=1}^{k} \mathfrak{a}_U \xrightarrow{\phi} \mathscr{S}_U \to 0.$$

Proof. Necessity. If $\mathscr{S}$ is coherent, property (a) implies the existence of ϕ and property (b) implies the existence of ψ. □

Sufficiency. Since $\mathfrak{a}$ is coherent, $\mathfrak{a}_U$ is coherent in U for each open set U and so are $\sum_{i=1}^{l} \mathfrak{a}_U$ and $\sum_{j=1}^{k} \mathfrak{a}_U$. As the image of $\sum_{i=1}^{l} \mathfrak{a}_U$, im ψ has property (a), and as a subsheaf of $\sum_{j=1}^{k} \mathfrak{a}_U$, im ψ has property (b). Thus im ψ is coherent, and there is an exact sequence

$$0 \to \operatorname{im}\psi \to \sum_{i=1}^{k} \mathfrak{a}_U \to \mathscr{S}_U \to 0.$$

Hence, since two of the sheaves are coherent, the third, $\mathscr{S})U$, is coherent for a neighbourhood of each point x and hence $\mathscr{S}$ is coherent.

Example 36. In the ring $B = Z[y, x_1, x_2, \ldots]$ of polynomials in infinitely many variables with integer coefficients, let I be the ideal generated 206
by $yx_1, yx_2, \ldots$ and let $A = B/I$. Then multiplication by y gives a

homomorphism $f : A \to A$ whose kernel C consists of polynomials in $Z[x_1, x_2, \ldots]$ without constant terms. Then the A- module C is not finitely generated. Hence if X is a consisting of one point, the constant sheaf A is not a coherent sheaf of rings.

Example 37. Let $X = \{x : 0 \leq x \leq 1\}$ and let F be the ring of functions $f : X \to Z_4$ for which $f(x) = f(1)$ for $x > 0$ and $f(0) = f(1)$ mod 2. There are eight functions in F. Let a be the sheaf of germs of function in F. Let g be the (constant) global section defined by the function $g : X \to Z_4$ where $g(0) = 2$ and $g(x) = 0$ for $x > 0$. The sheaf $\mathscr{R}$ of relations for this section is obtained by omitting from $\mathfrak{a}$ the germs at 0 of functions f with $f(0) = 1$ or 3. Then the sections of $\mathscr{R}$ over any connected neighbourhood U of 0 contain only the germs of even valued functions, hence do not generate $\mathscr{R}_U$. Thus $\mathscr{R}$ does not have (a_1) and hence a has neither (b_1) nor (b).

The sheaf of germs of analytic functions in the complex plane is a coherent sheaf of rings.

Proof. Let $\mathfrak{a}$ be the sheaf of germs of analytic functions in the complex place, and let $f_1, \ldots, f_k$ be sections of $\mathfrak{a}$ over a neighbourhood U of a point z_o, i.e., f_i is an analytic function in U. We can write $f_i(z) =$
207 $(z - z_o)^n g_i(z)$ where g_i does not vanish at z_o and hence does not vanish in a neighbourhood V_i of z_o, $z_o \in V_i \subset U$. Let $V = \bigcap_{i=1}^{K} V_i$. Let $\mathscr{R}$ be the sheaf of relations between the sections $f_i|V$, $i = 1, \ldots, k$. We will show that $\mathscr{R}$ is finitely generated in V. □

Let $P = C[z]$ be the ring of polynomials in z with complex coefficient and let M be the submodule (over P) of the direct sum $\sum_{i=1}^{k} P$ consisting of elements $(p_1, \ldots, p_k)$ for which $\sum_{i=1}^{k} p_i(z)(z - z_o)^{n_i} = 0$. Since P is a euclidean ring and $\sum_{i=1}^{k} P$ is finitely generated over P, the submodule M is finitely generated over p. (See van der Waerden , Modern Algebra, Vol, I, p. 106). Let $(p_1^j, \ldots, p_k^j)$, $j = 1, \ldots, l$, be a system of generators for M. Let $r_i^j(z) = p_i^j(z)/g_i(z)$; then each $r_i^j(z)$, $i = 1, \ldots, k$, is an analytic function in V, and

$$\sum_{i=1}^{k} r_i^j(z) f_i(z) = \sum_{i=1}^{k} p_i^j(z)(z-z_o)^{n_i} = 0 \quad (j = 1, \ldots, l).$$

Thus for each j, the section determined by $(r_i^j, \ldots, r_k^j)$ is in $\mathscr{R}$. We will show that these sections generate $\mathscr{R}$.

Let $\sum_{i=1}^{k} t_{iz_1}, f_{iz_1} = 0$ be a relation between germs at $z_1 \in V$, i.e., let $t_i(z)$, $i = 1, \ldots, k$, be analytic functions at z_1 such that $\sum_{i=1}^{k} t_i(z) f_i(z) = 0$. Then $\sum_{i=1}^{k} t_i(z) g_i(z)(z-z_o)^{n_i} \equiv 0$. Let $n = \max(n_1, \ldots, n_k)$ and suppose that $n_k = n$. We can write

$$t_i(z) g_i(z) = (z-z_o)^n q_i(z) + s_i(z), \quad (i = 1, \ldots, k-1)$$

where $q_i(z)$ is analytic at z_1 and $s_i(z) \equiv 0$ of $z_1 \neq z_o$ and is a polynomial **208**
of degree less than n if $z_1 = z_o$. Then

$$\begin{aligned} &(t_1(z)g_1(z), \ldots, t_k(z)g_k(z)) = \\ &\quad q_1(z)((z-z_o)^n, 0, \ldots, 0, -(z-z_o)^{n_1} + \cdots \\ &+ \quad q_{k-1}(z)(0, \ldots, (z-z_o)^n, -(z-z_o)^{n_{k-1}} + (s_1(z), \ldots, s_k(z)) \end{aligned}$$

where $s_k(z)$ is the analytic function defined by

$$s_k(z) = t_k(z) g_k(z) + \sum_{i=1}^{k-1} q_i(z)(z-z_0)^{n_i}.$$

Now, $((z-z_0)^n, 0, \ldots, 0, -(z-z_0)^{n_1}$, etc. are in M and by direct verification we have $\sum_{i=1}^{k-1} s_i(z)(z-z_0)^{n_i} \equiv 0$. Since $s_1(z), \ldots, s_{k-1}(z)$, are polynomials, it follows that $s_k(s)(z-z_o)^n$, is a polynomial.

(i) If $z_1 \neq z_0$, then $s_1(z), \ldots, s_{k-1}(z)$ are all zero and

$$s_k(z) = t_k(z) g_k(z) + \sum_{i=1}^{k-1} q_i(z)(z-z_0)^{n_i},$$

$$(z-z_0)^n s_k(z) = t_k(z) g_k(z)(z-z_0)^n + \sum_{i=1}^{k-1} q_i(z)(z-z_0)^n (z-z_0)^{n_i}$$

$$= t_k(z)g_k(z)(z-z_0)^n + \sum_{i=1}^{k-1} t_i(z)g_i(z)(z-z_0)^{n_i}$$

$$\equiv 0,$$

$$\text{hence } s_k(z) \equiv 0,$$

(ii) If $z_1 = z_0$, then $s_k(z)$ has a series expansion $\sum_{r=0}^{\infty} a_r(z-z_0)^r$ and on multiplication by $(z-z_o)^n$ this series has a finite number of terms. Hence $s_k(z)$ is already a polynomial.

209 In either case, $(t_1(z)g_1(z), \ldots, t_k(z)g_k(z))$ is a linear combination of elements of M with coefficients analytic at z_1. Hence

$$(t_1(z)g_1(z), \ldots, t_k(z)g_k(z)) = \sum_{j=1}^{l} h_j(z)(p_1^j(z), \ldots, p_k^j(z)),$$

where $h_j(z)$ is analytic at z_1. Then

$$(t_1(z), \ldots, t_k(z)) = \sum_{j=1}^{l} h_j(z)(r_1^j(z), \ldots, r_k^j(z)).$$

Thus the sheaf $\mathscr{R}$ of relations is generated by a finite number of sections, hence the sheaf $\mathfrak{a}$ is coherent.

This result is a special case of Oka's theorem, Cartan Seminar, 1951-52, Expose 15, §5. The following proposition on the extension of coherent sheaves is based on Expose 19, §1, of the same seminar.

Proposition 22. *Let $Y \subset X$ and either, let X be paracompact and normal with Y closed, or, let$X$ be hereditarily paracompact and normal. Let $\mathfrak{a}$ be a coherent sheaf of rings with unit over X and let $\mathscr{S}$ be a coherent sheaf of $\mathfrak{a}_Y$- modules over Y. Then there is an open set U with $Y \subset U$, and a coherent sheaf $\mathcal{J}$ of $\mathfrak{a}_U$- modules over U whose restriction to Y is isomorphic to $\mathscr{S}$.*

Proof. Since $\mathscr{S}$ is coherent, there is a covering $\{V_j \cap Y\}_{j\in J}$ of Y, where V_j is open in X, and for each $j \in J$ an exact sequence

$$\sum_{i=1}^{l_j} \mathfrak{a}_{V_{j\cap Y}} \xrightarrow{\psi_j} \sum_{i=1}^{k_j} \mathfrak{a}_{V_j\cap Y} \xrightarrow{\phi_j} \mathscr{S}_{V_j\cap Y} \to 0.$$

□

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From the properties of X there exist system $s\{G_i\}_{i\in I}$ and $\{H_i\}_{i\in I}$ of open sets of X such that, if $G = \cup_i G_i$,

(i) $\bar{H_i} \cap G \subset G_i$,

(ii) the system $\{G_i\}_{i\in I}$ is locally finite in G,

(iii) $Y \subset \cup_i H_i$,

(iv) each G_i is contained in some V_j,

For the first case, we can assume that each G_i is an F_σ set in X, hence G and all intersections $\cap_{r=1}^{k} G_{i_r}$ are F_σ-sets and hence are paracompact and normal. In the second case, all subsets of X are paracompact and normal.

Since each G_i is contained in some V_j, there are exact sequences

$$\sum_{i=1}^{l_j} \mathfrak{a}_{G_i\cap Y} \xrightarrow{\psi_i} \sum_{r=1}^{k_i} \mathfrak{a}_{G_i\cap Y} \xrightarrow{\phi_j} \mathscr{S}_{G_j\cap Y} \to 0.$$

Either G_i is paracompact and normal with $G_i \cap Y$ closed in G_i or G_i is hereditarily paracompact and normal, and the sheaves $\sum_{r=1}^{l_i} \mathfrak{a}_{G_i}$ and $\sum_{r=1}^{k_i} \mathfrak{a}_{G_i}$ are coherent. Hence (see Lecture 32) there is an open set G'_i with $G_i \cap Y \subset G'_i \subset G_i$ and an extension

$$\psi' : \sum_{r=1}^{l_i} \mathfrak{a}_{G'_i} \to \sum_{r=1}^{k_i} \mathfrak{a}_{G'_i}$$

of ψ_i. For the first case we may assume that G'_i is also an F_σ- set. Let $\mathscr{S}^i = (\sum_{r=1}^{k_i} \mathfrak{a}_{G'_i})/\operatorname{im}\psi'$. Then, if ϕ' is the natural homomorphism, the sequence

$$\sum_{r=1}^{l_i} \mathfrak{a}_{G'_i} \xrightarrow{\psi'} \sum_{r=1}^{k_i} \mathfrak{a}_{G'_i} \xrightarrow{\phi'} \mathscr{S}^i \to 0$$

is exact, i.e., 211

$$0 \to \operatorname{im}\psi' \to \sum_{r=1}^{k_i} \mathfrak{a}_{G'_i} \to \mathscr{S}^i \to 0$$

is exact and hence $\mathscr{S}^i$is coherent. There is an isomorphism

$$g_i : \mathscr{S}^i_{G'_i \cap Y} \to \mathscr{S}_{G'_i \cap Y}.$$

There are open sets H'_i with $\bar{H}_i \cap Y \subset H'_i$, $\bar{H}'_i \cap G' \subset G'_i$ where $G' = \bigcup\limits_{i \in I} G'_i$. For the first case, G' and all intersections $\cap_{r=1}^{q} G'_i$ are F_σ sets, hence are paracompact and normal.

For i, $j \in I$, there is an open set G_{ij}, $G'_i \cap G'_j \cap Y \subset G_{ij} \subset G'_i \cap G'_j$, and a homomorphism $f_{ij} : \mathscr{S}^j_{G_{ij}} \to \mathscr{S}^i_{G_{ij}}$ such that $f_{ij}\Big|\mathscr{S}^j_{G_{ij}\cap Y} = g_i^{-1} g_j\Big|\mathscr{S}^j_{G_{ij}\cap Y}$. For the first case, we may assume that G_{ij} is an F_σ set. Then there is an open set G'_{ij} with $G'_i \cap G'_j \cap Y \subset G'_{ij} \subset G_{ij} \cap G_{ji}$ such that $f_{ij} f_{ji}\Big|\mathscr{S}^j_{G'_{ij}\cap Y}$ is the identity and $f_{ji} f_{ij}\Big|\mathscr{S}^j_{G_{ij}\cap Y}$. is the identity. Let $E_{ij} = \bar{H'}_i \cap \bar{H'}_j \cap (G' - G'_{ij})$, then E_{ij} is closed in G'.

For i, j, $k \in I$ there is an open set G_{ijk}, $G'_i \cap G'_j \cap G'_k \cap Y \subset G_{ijk} \subset G'_i \cap G'_j \cap G'_k$, such that $f_{ij} f_{jk} | G_{ijk} = f_{ik} \| G_{ijk}$. Let $E_{ijk} = \bar{H}'_i \cap \bar{H}'_i \cap \bar{H}'_k \cap (G' - G_{ijk})$, then E_{ijk} is closed in G'.

212 Let $E = (\bigcup_{i,j} E_{ij}) \bigcup (\bigcup_{i,j,k} E_{ijk})$ and let $U = (G' - E) \cap (\bigcup_i H'_i)$. Then E is closed in G' and U is open with $Y \subset U$. Over each $H'_i \cap U$ there is a sheaf $\mathscr{S}^i_{H'_i \cap U}$; over each $H'_i \cap H'_j \cap U$ there is an isomorphism

$$f'_{ij} = f_{ij}\Big|\mathscr{S}^i_{H'_i \cap H'_j \cap U} : \mathscr{S}^j_{H'_i \cap H'_j \cap U} \to \mathscr{S}^i_{H'_i \cap H'_j \cap U}$$

and $f'_{ij} = (f'_{ij})^{-1}$. Further, over each $H'_i \cap H'_j \cap H'_k \cap U$ these isomorphisms are consistent, i.e., $f'_{ij} f'_{jk} = f'_{ik}$. Then by identification there is determined a sheaf $\mathcal{J}$ over U such that $\mathcal{J}_{H'_i \cap U}$ is identified with $\mathscr{S}^i_{H'_j \cap U}$. Then the isomorphisms $g_i : \mathscr{S}^i_{H'_i \cap Y} \to \mathscr{S}_{H'_i \cap Y}$ induce an isomorphism $g : \mathcal{J}_Y \to \mathscr{S}$. Since each $\mathscr{S}^i_{H'_i \cap U}$ is coherent. $\mathcal{J}$ is coherent.

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